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PhD Thesis

Non-uniqueness of Leray-Hopf weak solutions to the fractional Navier-Stokes equations

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Abstract

This doctoral thesis is about the density of wild initial data for the hypodissipative Navier-Stokes equations in the set of L^2 divergence-free vector fields defined on the three-dimensional torus $\mathbb{T}^3$.

The motivation for considering the fractional Navier-Stokes equations is to study the effect of a fractional dissipation term on the Euler equations, and specifically on uniqueness or non-uniqueness of weak solutions to the Cauchy problem. On one hand, in the case of the Euler equations, for any $\beta < 1/3$, there exist $\mathcal{C}^\beta$ initial data which generate infinitely many $\mathcal{C}^\beta$ weak solutions which satisfy the energy inequality. On the other hand, for the Navier-Stokes equations, as long as the initial datum has at least L^3 regularity, the Cauchy problem is locally well-posed. The case of the fractional laplacian with exponent $\theta \in (0,1)$ studied in this thesis should represent an intermediate case between these two situations.

The context wherein our work places itself is a relatively broad literature regarding the properties of solutions to the Euler, Navier-Stokes, and fractional Navier-Stokes equations. In this thesis, we briefly review the known uniqueness and non-uniqueness properties, as well as some other results about these systems, most notably Onsager's conjecture for the Euler equations and the regularity results for solutions of the Navier-Stokes equations.

The method used to tackle the problem studied in this thesis is the technique of Convex Integration. To introduce it, we discuss the development of its application to the Euler equations, and how it led to both the proof of Onsager's conjecture, and the proof of the density in L^2 divergence-free vector fields on $\mathbb{T}^3$ of $\mathcal{C}^\beta$ wild initial data for the Euler equations. We also discuss the applications of this technique to the fractional Navier-Stokes equations for a laplacian exponent $\theta < \frac{1}{3}$.

The above-mentioned work provided the motivation for the main result of this thesis, as well as the main ideas for its proof. This result states that, for any $T > 0$, if we consider the set of divergence-free L^2 vector fields on $\mathbb{T}^3$ which generate infinitely many $L^2([0,T];H^\theta(\mathbb{T}^3))$ solutions of the fractional Navier-Stokes equations with exponent θ, this set is dense in the set of divergence-free L^2 vector fields on $\mathbb{T}^3$.

The general strategy for the proof features four steps. In the first step, three kinds of "subsolutions" are defined, i.e. approximate solutions to the hypodissipative Navier-Stokes equations. An existence result for subsolutions of the "weakest" kind is then proved. Convex Integration is then used to prove that weak solutions are approximated by subsolutions of the strongest kind, which in turn are approximated by subsolutions of the weakest kind. These approximation results, combined with the existence result, finally lead to the proof of the theorem.

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Contents

Chapter 1

Introduction

In fluid dynamics, the Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{1.1}$$

and the Navier-Stokes equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \nu \Delta v \\ \operatorname{div} v = 0 \end{cases}, \tag{1.2}$$

both describe the motion of a Newtonian fluid: a non-viscous one in the former case, and a viscous one in the latter. In both systems, v is the velocity of the fluid and p is the pressure. In (1.2), ν is the viscosity.

The presence of the dissipation term $\nu \Delta v$ has allowed several regularity results to be proved for solutions of the Navier-Stokes equations. However, it is as of now not entirely clear how much this term affects the existence and (non-)uniqueness of solutions to the system. Another puzzle is the fact that, while formally the Euler equations are the limit of the Navier-Stokes equations for $\nu \to 0$, it has not yet been proved that the solutions satisfy a similar limit relation in general.

The fractional Navier-Stokes equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^{\theta} v = 0 \\ \operatorname{div} v = 0 \end{cases}, \tag{1.3}$$

are a way of modulating the dissipation term to investigate how its presence affects the uniqueness or non-uniqueness of weak solutions to these two systems of PDEs.

In this chapter, we introduce the key steps in the understanding of some of the features of the solutions to these system, eventually leading to the main result of this thesis.

1.1 The Euler equations

The Euler equations (1.1) first appeared in print in the 1757 paper [31] by Leonhard Euler. The general properties of the solutions to these equations were studied during the XIX century, and it was well known that the kinetic energy

$$E_E(t) := \frac{1}{2} \int |v|^2(t,x) \mathrm{d}x$$

is a conserved quantity in the case of classical solutions. This is not the case, however, for weak (i.e. distributional) solutions. Indeed, halfway through the XX century, Onsager wrote his famous note [58]. In this note, he conjectured the existence of solutions to the Euler equations which did not conserve the kinetic energy. More specifically, he formulated the following conjecture (cfr. Chapter 2).

Onsager's Conjecture. *Let* (v, p) *be a weak solution of* (1.1). *If* $v \in \mathcal{C}^\beta$ *for* $\beta > 1/3$, *then the energy* E_E *is a conserved quantity, i.e.* $E_E(t) \equiv E_E(0)$.

By contrast, for any $\beta < 1/3$, *there exist* $\mathcal{C}^\beta$ *weak solutions of* (1.1) *which do not conserve the energy.*

The first part was proved in the 1994 paper [13].

The second part of the conjecture was the object of a long series of papers in the first two decades of the XXI century, starting with the 2009 paper [21], where the first form of convex integration was applied to the Euler equations by De Lellis and Székelyhidi, and culminating in the 2018 paper [40], which finally provided a proof of the conjecture. These papers used successive refinements of the Convex Integration technique (cfr. Chapter 3) first introduced by Nash in the context of differential geometry in [56].

The Euler equations arise from Physics. Hence, it is of interest to consider physical solutions, i.e. admissible solutions. These are defined by the fact that they satisfy the energy inequality:

$$\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x \le \int_{\mathbb{T}^3} |v(0,x)|^2 \mathrm{d}x. \tag{1.1.1}$$

Such solutions satisfy what is known as *weak-strong uniqueness*, i.e., if there exists a smooth ($\mathcal{C}^1$) solution with a certain initial datum, it is the only admissible one with that datum. The question therefore presented itself as to whether admissible solutions, given an initial datum, were unique. A negative answer for general initial data was given in the paper [18]. In it, the authors considered the so-called "$\mathcal{C}^\beta$-wild" initial data, i.e. initial data $w \in \mathcal{C}^\beta(\mathbb{T}^3)$ which generate infinitely many $\mathcal{C}^0([0,T]; \mathcal{C}^\beta(\mathbb{T}^3))$ solutions of the Euler equations which satisfy (1.1.1). They proved that, for every $\beta < 1/3$, the set of $\mathcal{C}^\beta$-wild initial data is dense in the set of divergence-free $L^2(\mathbb{T}^3)$ vector fields.

1.2 The Navier-Stokes equations

The Navier-Stokes equations (1.2) first appeared in published form in his 1822 paper [57] by Claude-Louis Marie Henri Navier. They were given their final form by George Gabriel Stokes in his 1845 paper [66]. Since these equations describe the motion of a viscous fluid, it is not surprising that the kinetic energy E_E is not conserved by classical solutions. Such solutions do have a conserved quantity, however, which is E_E plus a dissipation term:

$$E_{NS}(t) := \frac{1}{2} \int |v|^2(t,x) \mathrm{d}x + \nu \int_0^t \int |\nabla v|^2(s,x) \mathrm{d}x \mathrm{d}s.$$

The presence of this dissipation term in E_{NS} played an important role in the proof, given by Leray in the 1934 paper [51], of the existence, for any initial datum, of at least one weak solution satisfying the following energy inequality:

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x + \nu \cdot \int_0^t \int_{\mathbb{T}^3} |\nabla v(s,x)|^2 \mathrm{d}x\mathrm{d}s \leq \frac{1}{2}\int_{\mathbb{T}^3} |v(0,x)|^2 \mathrm{d}x \qquad \text{a.e. } t > 0. \quad (1.2.1)$$

Such solutions are called *Leray solutions* or *Leray-Hopf solutions*; we will call them *admissible solutions of* (1.2).

Note that this result by Leray does not have an analogue in the case of the Euler equations. Indeed, the existence (or non-existence) of admissible solutions to the Euler equations (i.e. solutions to (1.1) satisfying (1.1.1)) for any initial data is a long-standing open problem.

The dissipative term in (1.2) has a smoothing effect on the solutions. On the one hand, this provides the solutions with higher regularity, as discussed in the regularity results of Chapter 2. On the other hand, for the Navier-Stokes equations, an equivalent of the wild initial data result of [18] discussed in the previous subsection remains, until now, unproven. Indeed, the uniqueness or non-uniqueness of solutions to (1.2) satisfying (1.2.1) is still a long-standing open problem.

Several non-uniqueness results have been obtained for non-admissible solutions, or solutions that are not proved admissible. For an overview, see Chapter 2.

1.3 The fractional Navier-Stokes equations

Introducing a fractional dissipative term in (1.1) is a way to study how much the presence of dissipation affects the uniqueness or non-uniqueness of solutions to the Cauchy problem. More specifically, we consider the fractional Navier-Stokes equations (1.3) with exponent $\theta < \frac{1}{3}$ and admissibility condition

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v(s,x)\right|^2 \mathrm{d}x\mathrm{d}s \leq \frac{1}{2}\int_{\mathbb{T}^3} |v(0,x)|^2 \mathrm{d}x. \qquad (1.3.1)$$

On one side, the presence of the fractional laplacian allows one to adapt the strategy of Leray in [51] to prove any initial datum generates an admissible solution for (2.1.3.1), for any exponent $\theta > 0$. This is done for instance in [15]. Calculations entirely analogous to those employed in the Euler and Navier-Stokes cases prove that smooth solutions of (1.3) satisfy the energy equality, i.e. conserve the following quantity:

$$E_{FNS}(t) \coloneqq \frac{1}{2}\int |v|^2(t,x)\mathrm{d}x + \int_0^t \int \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 (s,x)\mathrm{d}x\mathrm{d}s.$$

On the other side, in [15] and [26] it was proved that there exist infinitely many $\mathcal{C}^\beta$ initial data for $\theta < \beta < {}^1/_3$ which generate infinitely many $\mathcal{C}^0([0,T];\mathcal{C}^\beta(\mathbb{T}^3))$ solutions which, by [16], are in fact $C^\beta([0,T];C^\beta(\mathbb{T}^3))$. In fact, [15] produces such data for θ up to $^1/_2$, with solutions that can only be proved to be admissible (i.e. satisfy (1.3.1)) for $\theta < {}^1/_5$, and the solutions in the range $[^1/_5, {}^1/_2]$ are not C^θ.

Inspired by the wild initial data result of [18] discussed in Section 1.1, in this thesis we investigate the existence of an L^2-dense class of C^β wild initial data (namely data for which non-uniqueness holds) for admissible solutions to (1.3) in $L^2([0,T];H^\theta(\mathbb{T}^3))$.

Notice the two main differences between our result and that of [26]:

- While they obtain infinitely many initial data, we prove the existence of an L^2-dense set of wild data, which is a topologically stronger statement;

- In order to obtain this stronger property, we must pay the price of the $C^\beta_{x,t}$ regularity, and be content with $L^2_t H^\theta_x$ regularity on the whole of $[0,T]$, and $C^\beta_{x,t}$ only locally near $t = 0$.

Concerning the second point, note that, if we fix an L^2 vector field w and an $\eta > 0$, we can find wild data w_η which are η-close to w in L^2 which have $C^\beta_{x,t}$ solutions on $[0,T]$ which we can only prove admissible on $[0,T(\eta)]$, where $T(\eta) \to 0$ as $\eta \to 0$. This arises from the necessity of controlling the dissipation term in the energy. Therefore, proving the density of the wild data requires us to continue these solutions by means of Leray solutions, which are only $L^2_t H^\theta_x$.

Here, we explore and extend the strategy of [18] to the fractional Navier-Stokes equations. The main issue with respect to the Euler setting is to control the dissipative term in the energy. Our main results are the following.

Theorem 1.3.1 (C^β weak solutions with data close to L^2 functions and time of admissibility). *Let $\theta < \beta < {}^1/_3, w \in L^2(\mathbb{T}^3)$. Then, for all $\eta > 0$, there exist a time $T_\eta > 0$, an initial datum $w_\eta \in C^\beta(\mathbb{T}^3)$ such that $\|w_\eta - w\|_{L^2} < \eta$, and infinitely many weak solutions $v_\eta \in C^0([0,T], C^\beta(\mathbb{T}^3))$, with initial datum $v_\eta|_{t=0} = w_\eta$, which satisfy* (1.1.1) *on $[0,T]$, but can be proved to satisfy* (1.3.1) *(i.e to be admissible) only on $[0,T_\eta]$. Moreover*

$$\lim_{\eta \to 0} T_\eta = 0.$$

Definition 1.3.1 (Wild initial data). *Let X be a function space. A divergence-free vector field $w \in L^2(\mathbb{T}^3)$ is a (θ, X, T)-*wild initial datum *for* (1.3) *if there exist infinitely many weak solutions $v : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ of* (1.3) *such that $v \in X$, $v(0,x) = w(x)$ a.e. in $\mathbb{T}^3$, and the admissibility condition* (1.3.1) *holds on $[0,T]$. The set of such data is denoted by $W_{\theta,X,T}$. If $X = L^\infty([0,T];C^\beta(\mathbb{T}^3))$, we will speak of (θ,β,T)-wild data, and of the set $W_{\theta,\beta,T}$.*

As a consequence of **Theorem 1.3.1**, we obtain the following corollary.

Corollary 1.3.1 (Density of wild initial data – Hölder solutions). *The set $\bigcup_T W_{\theta,\beta,T}$ is dense in the set of divergence-free L^2 vector fields, for all $\theta < \beta < {}^1/_3$.*

Moreover, by taking a solution v_η as given by **Theorem 1.3.1**, and continuing it with a Leray solution $\tilde{v}_\eta : [T_\eta, \infty] \times \mathbb{T}^3 \to \mathbb{R}^3$ with datum $\tilde{v}_\eta(T_\eta) = v_\eta(T_\eta)$, as provided by **Theorem 2.4.2.1**, we obtain the following.

Theorem 1.3.2 (Density of wild initial data – Sobolev solutions). *$W_{\theta,L^2_t H^\theta_x,T}$ is dense in the sef ot divergence-free L^2 vector fields, for all $\theta < {}^1\!/_3, T > 0$.*

The general strategy of the proof is to define suitable relaxations of the notion of solution (the so-called "subsolutions" of Chapter 4), and approximating one kind of subsolution with another one which is closer to the notion of solution. This is done by constructing sequences of subsolutions that converge, in an appropriate sense, to a "stronger" subsolution. We will need two such approximations, and therefore two convex integration schemes: the first one will converge to a so-called "adapted subsolution", i.e. a subsolution such that $\mathring{R}(\cdot,0) \equiv 0$ which has the C^1 norm of the velocity blowing up at a controlled rate at $t = 0$ (cfr. **Definition 4.2.2**). As in [18], these adapted subsolutions are the basis for a quantitative criterion for non-uniqueness. The second approximation will lead to a weak solution.

The thesis is organized as follows. Chapter 2 gives the context in which this thesis places itself, i.e. the state of the art of existence and (non-)uniqueness results for the Euler, Navier-Stokes, and fractional Navier-Stokes equations. Chapter 3 gives a survey of the applications of Convex Integration to the Euler equations, with a brief conclusion on previous results which apply Convex Integration to fractional Navier-Stokes. Chapter 4 introduces three kinds of subsolutions, namely strict, strong, and adapted subsolutions, proves an existence result for strict subsolutions, and gives a guide to the following chapters. Chapter 5 shows how one can approximate strict subsolutions with strong ones. Chapter 6 contains the two substeps of each convex integration step, namely a gluing step and a perturbation step. Chapter 7 state and prove the other approximation results, namely the approximation of strong subsolutions with adapted ones, and that of adapted subsolutions with weak solutions. Finally, Chapter 8 deduces **Theorem 1.3.1** from those approximation results.

1.4 Notations

The following notations are used throughout the rest of this thesis:

- $\mathcal{S}^{3\times3}$ are the symmetric 3-by-3 matrices; within this set, $\mathcal{S}^{3\times3}_+$ are the positive definite ones, $\mathcal{S}^{3\times3}_0$ are the traceless ones, and $\mathcal{S}^{3\times3}_{\geq 0}$ are the positive semidefinite ones.
- If $R \in \mathcal{S}^{3\times3}$, we decompose it as

$$R = \frac{1}{3}\operatorname{tr} R \operatorname{Id} + \mathring{R} = \rho \operatorname{Id} + \mathring{R},$$

  where $\mathring{R} \in \mathcal{S}^{3\times3}_0$ is the *traceless part of* R.
- For scalar functions f, we write $\nabla f := (\partial_1 f, \partial_2 f, \partial_3 f) =: \mathfrak{D} f$;
- However, for vector fields v, we define $\mathfrak{D} v$ so that $(\mathfrak{D} v)_{ij} = \partial_j v_i$, whereas $\nabla v = (\mathfrak{D} v)^T$; with these choices, $(v \cdot \nabla) v = \mathfrak{D} v \cdot v = v \cdot \nabla v$;

- In a similar fashion, for tensor fields S, $\mathfrak{D}S$ is defined by $(\mathfrak{D}S)_{ijk} = \partial_k S_{ij}$, whereas ∇S is defined by $(\nabla S)_{ijk} = \partial_i S_{jk}$;

- The Hölder norms are defined as follows:

$$\|f\|_0 \coloneqq \sup |f(x)|, \qquad [f]_k \coloneqq \max_{\substack{\beta\in\mathbb{N}^3 \\ |\beta|=k}} \sup \left|\partial_\beta f\right|, \qquad [f]_\alpha \coloneqq \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

$$\|f\|_k \coloneqq \|f\|_0 + \sum_{i=1}^k [f]_i, \qquad \|f\|_{k+\alpha} \coloneqq \|f\|_k + \max_{\substack{|\beta|=k \\ \beta\in\mathbb{N}^3}} [\partial_\beta f]_\alpha,$$

  for $k \in \mathbb{N}, \alpha \in (0,1)$;

- We will denote the time slices of a function v defined on $[0,T]\times\mathbb{T}^3$ by the notation $v_t(x) \coloneqq v(t,x)$;

- Instead of writing $\mathcal{C}^A([0,T],\mathcal{C}^B(\mathbb{T}^3))$, or $L^p([0,T];L^q(\mathbb{T}^3))$, or similar notations, we will write $\mathcal{C}^A_t\mathcal{C}^B_x([0,T],\mathbb{T}^3)$ and $L^p_tL^q_x([0,T];\mathbb{T}^3)$ respectively, often with the domains $[0,T]$ and $\mathbb{T}^3$ left implied.

Chapter 2

General features of Euler, Navier-Stokes, and fractional Navier-Stokes equations and their solutions

In this chapter we present some general features of the Euler, the Navier-Stokes, and the fractional Navier-Stokes equations. We first define the equation systems, the associated energy functionals, and the corresponding notions of weak, strong, and classical solutions. We then illustrate the main properties of solutions to these systems with regard to existence, uniqueness, and regularity.

2.1 The equation systems and the associated energy functionals

2.1.1 The Euler equations

The Euler equations describe the motion of a non-viscous Newtonian fluid, with velocity v and pressure p:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases}. \tag{2.1.1.1}$$

The total kinetic energy of the fluid is given by:

$$E_E(t) \coloneqq \frac{1}{2} \int_{\mathbb{T}^d} |v(t,x)|^2 \mathrm{d}x.$$

A *classical* solution of the Euler equations is a pair $(v,p) \in C^1([0,T] \times \mathbb{T}^d; \mathbb{R}^d \times \mathbb{R})$ which solves the equations in the classical sense.
A *strong* solution of the Euler equations is a weak solution v (as defined below) with the additional regularity $v \in L^\infty_t H^m_x$ for some integer $m \geq 1$.

A *weak* solution of the Euler equations with initial datum $v_0 \in L^2$ is a function $v \in L_t^\infty L_x^2([0,T] \times \mathbb{T}^d)$ which solves the equations in the sense of distributions, i.e., for every $\varphi \in C_c^\infty([0,\infty) \times \mathbb{T}^d)$ such that $\operatorname{div} \varphi = 0$ and every $\psi \in C_c^\infty(\mathbb{T}^d)$, the following hold for a.e. $t \in [0,\infty)$:

$$\int_0^t \int_{\mathbb{T}^d} \partial_t \varphi(s,x) v(s,x) \mathrm{d}s \mathrm{d}x$$

$$+ \int_0^t \int_{\mathbb{T}^d} (v \otimes v)(s,x) : \nabla \varphi(s,x) \mathrm{d}s \mathrm{d}x = \int_{\mathbb{T}^d} v(t,x) \cdot \varphi(t,x) - v_0(x) \cdot \varphi(0,0) \mathrm{d}x$$

$$\int_{\mathbb{T}^d} v(t,x) \cdot \nabla \psi(x) \mathrm{d}x \equiv 0.$$

Note that classical solutions are also weak solutions. Vice versa, C^1 weak solutions are classical solutions.

2.1.2 The Navier-Stokes equations

The Navier-Stokes equations describe the motion of a viscous Newtonian fluid with viscosity ν:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \nu \Delta v = 0 \\ \operatorname{div} v = 0 \end{cases}. \tag{2.1.2.1}$$

The total kinetic energy and dissipation term for the Navier-Stokes equations are given by:

$$E_{NS}(t) := \frac{1}{2} \int_{\mathbb{T}^d} |v(t,x)|^2 \mathrm{d}x + \nu \cdot \int_0^t \int_{\mathbb{T}^d} |\nabla v|^2(s,x) \mathrm{d}x \mathrm{d}s.$$

We will see that classical solutions of the Navier-Stokes equations conserve E_{NS}, whereas classical solutions of the Euler equations conserve E_E. The same is true of strong solutions of the Navier-Stokes equations, in accordance with [60, Theorem 6.5]. This means that the *kinetic energy* E_E of the fluid is dissipated by sufficiently regular Navier-Stokes solutions.
A *classical* solution of the Navier-Stokes equations is a pair $(v,p) : \mathbb{T}^d \to \mathbb{R}^d \times \mathbb{R}$ such that $v \in C_t^0 C_x^2 \cap C_t^1 C_x^0, p \in C_t^0 C_x^1$ which solves the equations in the classical sense.
A *strong* solution of the Navier-Stokes equations is a weak solution v (as defined below) with the additional regularity $v \in L_t^\infty H_x^1 \cap L_t^2 H_x^2$.
A *weak* solution of the Navier-Stokes equations with initial datum $v_0 \in L^2$ is a function $v \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ which solves the equations in the sense of distributions, i.e., for every

$\varphi \in C_c^\infty(\mathbb{T}^d) : \operatorname{div}\varphi = 0$ and every $\psi \in \varphi \in C_c^\infty([0,\infty)\times\mathbb{T}^d)$, the following hold:

$$\int_0^t \int_{\mathbb{T}^d} \partial_t \varphi(s,x) v(s,x) + (v\otimes v)(s,x) : \nabla\varphi(s,x) \mathrm{d}s\mathrm{d}x$$
$$+ \int_0^t \int_{\mathbb{T}^d} \nu v(s,x)\cdot\Delta\varphi(s,x)\mathrm{d}s\mathrm{d}x = \int_{\mathbb{T}^d} v(t)\cdot\varphi(t,x) - v_0\cdot\varphi(0,x)\mathrm{d}x$$
$$\int_{\mathbb{T}^d} v(t,x)\cdot\nabla\psi(x)\mathrm{d}x = 0.$$

2.1.3 The fractional Navier-Stokes equations

It is of mathematical interest to study how different levels of diffusion affect the behaviour of the solutions. This leads to the fractional Navier-Stokes equations, which are obtained by replacing the laplacian from the Navier-Stokes equations with a fractional laplacian:

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p + (-\Delta)^\theta v = 0 \\ \operatorname{div} v = 0 \end{cases}. \tag{2.1.3.1}$$

These equations are termed *hypodissipative* when $\theta < 1$ and *hyperdissipative* when $\theta > 1$. In this thesis, unless otherwise stated, we shall study solutions of the hypodissipative range, and more specifically solutions for $\theta < {}^1/_3$.

As described for instance in [54], these equations also model the behaviour of a fluid with internal friction interaction when $\theta \in [{}^1/_2, 1]$.

The total kinetic energy and dissipation terms for the fractional Navier-Stokes equations are given by:

$$E_{FNS}(t) \coloneqq \frac{1}{2}\int_{\mathbb{T}^d} |v(t,x)|^2 \mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 (s,x)\mathrm{d}x\mathrm{d}s.$$

A *classical* solution of the fractional Navier-Stokes equations is a pair $(v,p) \in (C_t^0 C_x^{2\theta} \cap C_{t,x}^1)([0,T]\times\mathbb{T}^d;\mathbb{R}^d\times\mathbb{R})$ which solves the equations in the classical sense.

A *weak* solution of the fractional Navier-Stokes equations with initial datum $v_0 \in L^2$ is a function $v \in (L_t^\infty L_x^2 \cap L_t^2 H_x^\theta)([0,T]\times\mathbb{T}^d)$ which solves the equations in the sense of distributions, i.e., for every $\forall\varphi\in C_c^\infty([0,\infty)\times\mathbb{T}^d) : \operatorname{div}\varphi = 0$ and every $\psi \in C_c^\infty(\mathbb{T}^d)$ the following hold for a.e. $t \in [0,\infty)$:

$$\int_0^t \int_{\mathbb{T}^d} \partial_t \varphi(s,x) v(s,x) + (v \otimes v)(s,x) : \nabla \varphi(s,x) \mathrm{d}x\mathrm{d}s$$

$$- \int_0^t \int_{\mathbb{T}^d} (v \cdot (-\Delta)^\theta \varphi)(s,x) \mathrm{d}s\mathrm{d}x = \int_{\mathbb{T}^d} v(t,x) \cdot \varphi(t,x) - v_0(x) \cdot \varphi(0,x) \mathrm{d}x$$

$$\int_{\mathbb{T}^d} v(t,x) \cdot \nabla \psi(x) \mathrm{d}x \equiv 0.$$

2.1.4 Conservation of energy vs. dissipation: admissibility conditions

For all three of these systems, a straightforward computation shows that classical solutions satisfy energy balances, i.e.

$$E_E(t) \equiv E_E(0)$$

for the Euler equations,

$$E_{NS}(t) \equiv E_{NS}(0)$$

for the Navier-Stokes equations, and

$$E_{FNS}(t) \equiv E_{FNS}(0)$$

for the fractional Navier-Stokes equations. The proof is especially simple for the Euler equations (2.1.1.1). By multiplying the equation by v, we obtain:

$$0 = v \cdot \partial_t v + v \cdot \mathrm{div}(v \otimes v) + v \cdot \nabla p = \frac{1}{2} \partial_t |v|^2 + v \cdot \mathrm{div}(v \otimes v) + v \cdot \nabla p = 0.$$

If we now integrate in space, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_E(t) + \int_{\mathbb{T}^d} v \cdot (\mathrm{div}(v \otimes v) + \nabla p) \mathrm{d}x.$$

Now, $\int v \cdot \nabla p \mathrm{d}x = -\int p \,\mathrm{div}\, v = 0$ because $\mathrm{div}\, v = 0$. As for the other term:

$$\int_{\mathbb{T}^d} v \cdot \mathrm{div}(v \otimes v) \mathrm{d}x = \sum_{ij} \int_{\mathbb{T}^d} v_i \partial_j (v_i v_j) \mathrm{d}x = -\sum_{ij} \int_{\mathbb{T}^d} v_i v_j \partial_j v_i \mathrm{d}x.$$

On the other hand:

$$\int_{\mathbb{T}^d} v \cdot \mathrm{div}(v \otimes v) \mathrm{d}x = \sum_{ij} \int_{\mathbb{T}^d} v_i (\partial_j v_i) v_j + v_i (\partial_j v_j) \partial_i \mathrm{d}x = \sum_{ij} \int_{\mathbb{T}^d} v_i (\partial_j v_i) v_j \mathrm{d}x + \int_{\mathbb{T}^d} |v|^2 \mathrm{div}\, v \mathrm{d}x,$$

where the second term is zero because $\mathrm{div}\, v = 0$, and the first term is the opposite of what we found before. Therefore, $\int v \cdot \mathrm{div}(v \otimes v) = 0$, and conservation of energy is proved.

However, experiments [29, 64] and numerical simulations [65, 41] have shown that, in some turbulent viscous régimes, the kinetic energy dissipation does not approach zero for very small viscosities. This suggests that even the Euler equations may admit nonconservative solutions, a phenomenon known as *anomalous dissipation*. From a physical standpoint, it is natural to require that, if the energy is not conserved, at least it does not increase in time. Thus, the natural admissibility condition for these systems is:

$$E_E(t) \leq E_E(0) \qquad \text{for a.e. } t \in [0,T]. \tag{2.1.4.1}$$

If the dissipation does not approach zero while the viscosity does, E_{NS} will also not be conserved. Analogously to the Euler case, the natural admissibility condition for Navier-Stokes will then be:

$$E_{NS}(t) \leq E_{NS}(0) \qquad \text{for a.e. } t \in [0,T] \tag{2.1.4.2}$$

An analogous condition is considered for the fractional Navier-Stokes case as well:

$$E_{FNS}(t) \leq E_{FNS}(0) \qquad \text{for a.e. } t \in [0,T]. \tag{2.1.4.3}$$

More explicitly, the admissibility conditions for the three systems, required to hold for a.e. $t \in [0,T]$, read:

$$\frac{1}{2}\int_{\mathbb{T}^d} |v(t,x)| \mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{T}^d} |v(0,x)|^2 \mathrm{d}x \tag{Euler}$$

$$\frac{1}{2}\int_{\mathbb{T}^d} |v(t,x)| \mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} |\nabla v(s,x)|^2 \mathrm{d}x\mathrm{d}s \leq \frac{1}{2}\int_{\mathbb{T}^d} |v(0,x)|^2 \mathrm{d}x \tag{Navier-Stokes}$$

$$\frac{1}{2}\int_{\mathbb{T}^d} |v(t,x)| \mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \left|(-\Delta)^{\frac{\theta}{2}} v(s,x)\right|^2 \mathrm{d}x\mathrm{d}s \leq \frac{1}{2}\int_{\mathbb{T}^d} |v(0,x)|^2 \mathrm{d}x. \tag{Fractional Navier-Stokes}$$

We will refer to solutions satisfying the appropriate above condition as *admissible* or *dissipative* solutions.

2.2 Existence and (non-)uniqueness of solutions to the Euler equations

Much of the research regarding PDEs concerns the existence and uniqueness properties of their solutions. In this section, we briefly review the main results for the cases of classical, strong, and weak solutions to the Euler equations.

2.2.1 Smooth solutions: local well-posedness and the problem of finite-time singularities in three dimensions

A local well-posedness result holds for solutions belonging to Sobolev spaces. The result is drawn from [53, Theorems 3.4-3.5].

Theorem 2.2.1.1 (Local well-posedness). *Given a divergence-free initial condition* $v_0 \in H^m(\mathbb{T}^d)$*,* $m > \lfloor d/2 \rfloor + 1$*, then the following holds. There exists a time* T *with the rough upper bound:*

$$T \leq \frac{1}{c_m \|v_0\|_{H^m}},$$

such that there exists the unique solution $v \in \mathcal{C}([0,T], \mathcal{C}^2(\mathbb{T}^3)) \cap \mathcal{C}^1([0,T], \mathcal{C}(\mathbb{T}^3))$ *to the Euler equations.*([1]) *In fact, the solution* $v \in \mathcal{C}([0,T), H^m(\mathbb{T}^3)) \cap \mathcal{C}^1([0,T), H^{m-1}(\mathbb{T}^3))$.

A very similar result holds for general compact manifolds, as obtained by combining [70, Theorem 2.1 and Proposition 2.2], giving solutions of regularity $L^\infty_t H^m_x \cap \mathrm{Lip}_t H^{m-1}_x$.

An important consequence of this result is the following continuation property, which is [53, Corollary 3.2].

Corollary 2.2.1.1 (Maximal time of existence). *Given a divergence-free initial condition* $v_0 \in H^m(\mathbb{T}^d), m > \lfloor 1/2 \rfloor + 1$*, there exists a maximal time of existence* T^**, possibly infinite, and a unique solution* $v \in \mathcal{C}([0,T^*), H^m(\mathbb{T}^d)) \cap \mathcal{C}^1([0,T^*), H^{m-2}(\mathbb{T}^d))$ *to the Euler equations. Moreover, if* $T^* < \infty$*, then necessarily:*

$$\lim_{t \to T^*} \|v(t,\cdot)\|_{H^m} = \infty.$$

Proof.

Assume v exists up to time $T^* < \infty$. Either it exists up to some $\tilde{T} > T^*$, or it does not. If it does, then T^* is not the maximal time. If it does not, then, by **Theorem 2.2.1.1**, it must mean that, for any $t < T^*$, $\|v(t,\cdot)\|_{H^m} \geq c_m^{-1}(T^* - t)^{-1}$. Indeed, if there was $t^* < T^*$ such that $\|v(t^*,\cdot)\|_{H^m} < c_m^{-1}(T^* - t^*)^{-1}$, we could consider the solution v^* defined on $[t^*, t^* + (c_m\|v(t^*,\cdot)\|_{H^m})^{-1})$ given by **Theorem 2.2.1.1**. This would have the desired regularity, it would coincide with v on $[t^*, T^*]$ by the uniqueness part of **Theorem 2.2.1.1**, and $t^* + (c_m\|v(t^*,\cdot)\|_{H^m})^{-1} > T^*$ by choice of t^*, thus contradicting the fact that v cannot be continued past T^*.

Therefore, if v cannot be prolonged past $T^* < \infty$, then $\lim_{t \to T^*} \|v(t,\cdot)\|_{H^m} = +\infty$, completing the proof. ◇

While the local-in-time existence of H^m solutions has been established, the global-in-time existence of such solutions on $\mathbb{T}^d$ is a long-standing open problem. The main result in this regard, proved in [2] and reported as [53, Theorem 3.6], asserts an equivalent condition for the global-in-time existence.

Theorem 2.2.1.2 (Beale-Katô-Majda criterion). *Let the initial velocity* $v_0 \in H^m_{\mathrm{div}=0}(\mathbb{T}^d)$*,* $m > \lfloor d/2 \rfloor + 1$*, so that there exists a classical solution* $v \in \mathcal{C}^1([0,T], (\mathcal{C}^2 \cap H^m_{\mathrm{div}=0})(\mathbb{T}^d))$ *to the Euler or Navier-Stokes equations. Then:*

(i) *If for any* $T > 0$ *there exists* $M_1 > 0$ *such that the vorticity* $\omega = \mathrm{curl}\, v$ *satisfies*

$$\int_0^T |\omega(\tau,\cdot)|_{L^\infty} \mathrm{d}\tau \leq M_1,$$

then the solution v *exists globally in time;*

[1]For solutions on $\mathbb{R}^d$, this result holds in H^s, for every $s > d/2 + 1$, as proved by Katô and Ponce in [43].

(ii) *If the maximal time T of existence of the solution in H^m is finite, then necessarily the vorticity accumulates so rapidly that*

$$\lim_{t\uparrow T}\int_0^t |\omega(\tau,\cdot)|_{L^\infty}\mathrm{d}\tau = \infty.$$

To conclude the subsection, a general theorem ([3, Theorem 1.1]) exists, which states that, for some $\mathcal{C}^m$ data, there is no $C_t^0\mathcal{C}_x^m$ solution, m being any strictly positive integer. In other words, the solution does not preserve the regularity of the initial datum in its time evolution. This loss of regularity phenomenon is somewhat akin to what happens in the fractional Navier-Stokes case for the solutions provided by **Theorem 1.3.2**, which are $\mathcal{C}^\beta$ for small times, and then only remain $L_t^2H_x^\theta$. Whether or not a stronger but more delayed form of regularity loss happens for Euler solutions is also a long-standing open problem, related to the question of uniqueness of time-global admissible solutions for sufficiently regular data.

Theorem 2.2.1.3. *Let $m \geq 1$ be an integer. For any given velocity $v^{(g)} \in C_c^\infty(\mathbb{T}^3)$ and any $\varepsilon > 0$, we can find a $\mathcal{C}^\infty$ perturbation $v^{(p)} : \mathbb{T}^3 \to \mathbb{T}^3$ such that the following hold true:*

(1) $\|v^{(p)}\|_{L^1} + \|v^{(p)}\|_{H^m} + \|v^{(p)}\|_{\mathcal{C}^m} < \varepsilon$;

(2) *Let $v_0 := v^{(g)} + v^{(p)}$ and*
$$T_1 := \frac{c_d}{2+2\left\|v^{(g)}\right\|_{H^{\frac{7}{2}}}};$$
there exists a unique strong solution $v = v(t,x)$ to the Euler equations
$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla p & 0 < t \leq T_1, x\in\mathbb{T}^3 \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0 \end{cases}$$
satisfying
$$\max_{0\leq t\leq T_1}\left(\|\omega(t,\cdot)\|_{L^1} + \|\omega(t,\cdot)\|_{\mathcal{C}^{m-1}}\right) < \infty,$$
$\omega = \operatorname{curl} v$ being the vorticity; furthermore, $v\in C_t^0H_x^m$ and $v(t,\cdot)\in\mathcal{C}^\infty(\mathbb{T}^3)$ for each $0 \leq t \leq T_1$;

(3) *For any $0 < t_0 \leq T_1$, we have*
$$\text{ess-sup}_{0<t\leq t_0}\|u(t,\cdot)\|_{\mathcal{C}^m} = +\infty;$$
more precisely, for any $n = 1,2,3,\ldots$ there exist $0 < t_n^1, t_n^2 < n^{-1}$ and open balls $B_n = B(x_n,1)\subset\mathbb{T}^3$ such that $u(t,\cdot)\in\mathcal{C}^\infty(B_n)$ for $t\in[t_n^1,t_n^2]$, but
$$\|u(t,\cdot)\|_{\mathcal{C}^m(B_n)} > n \qquad \forall t\in[t_n^1,t_n^2].$$

As remarked in [3], $\mathcal{C}^k \hookrightarrow \mathcal{C}^{k-1,1}$ and $\|\cdot\|_{\mathcal{C}^k} \sim \|\cdot\|_{\mathcal{C}^{k-1,1}}$, so the above result implies an analogous result for $\mathcal{C}^{m-1,1}$, for any $m \geq 1$. It is striking that, looking between $\mathcal{C}^k$ and $\mathcal{C}^{k,1}$, one finds $\mathcal{C}^{k,\alpha}$, where a local well-posedness result holds, as stated in the 3D case $k = 1$ in [52] and in the general case in [37].

2.2.2 Global-in-time existence of smooth solutions in two dimensions

For solutions defined on $\mathbb{R}^2$ or $\mathbb{T}^2$, the known properties are remarkably different. All the results presented in the 3D case hold for the 2D case as they are or with minor modifications. However, a stronger result holds in the 2D case, namely the following global-in-time existence result ([53, Corollary 3.3] adapted to $\mathbb{T}^2$) for smooth solutions to the 2D Euler equations.

Theorem 2.2.2.1 (Global well-posedness in two dimensions). *Given an initial 2D divergence-free velocity field $v_0 \in H^m(\mathbb{T}^2), m > 3$, there exists for all time a unique smooth solution $v \in C^0_t H^m_x$ to the 2D Euler equations.*

In order to prove this result, it is necessary to lay some groundwork. We begin by noting that, in two dimensions, the vorticity $\omega = \operatorname{curl} v = \partial_{x_1} v_2 - \partial_{x_2} v_1$ satisfies the following equation:

$$\partial_t \omega + (v \cdot \nabla)\omega = 0. \tag{2.2.2.1}$$

By computations entirely analogous to those by which we deduced that solutions of the Euler equations conserve E_E in the previous section, one finds that equation (2.2.2.1) implies the following balance:

$$\frac{1}{2}\|\omega\|^2_{L^2} = \|\omega_0\|^2_{L^2}. \tag{2.2.2.2}$$

The final elements we need in order to prove **Theorem 2.2.2.1** are an a priori bound for Sobolev norms of H^m Euler solutions, and a potential theory estimate. The former can be proved by computations entirely analogous to those used for the proof of [53, Proposition 3.7], while the latter is an adaptation of [53, Proposition 3.8] to $\mathbb{T}^2$.

Proposition 2.2.2.1. *If $u \in C^0_t H^m \cap \{\operatorname{div} u = 0\}, m \in \mathbb{Z}^+ \cup \{0\}$ is a solution of the Euler equations, then it satisfies the following estimates:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m \leq c_m\{|\nabla u|_{L^\infty}\}\|u\|_m.$$

Proposition 2.2.2.2. *Let v be a smooth, $L^2 \cap L^\infty(\mathbb{T}^2)$, divergence-free velocity field, and let $\omega = \operatorname{curl} v$. Then*

$$|\nabla v|_{L^\infty} \leq c\big(1 + \ln^+ \|v\|_3 + \ln^+ \|\omega\|_0\big)(1 + |\omega|_{L^\infty}).$$

We are finally ready to prove **Theorem 2.2.2.1**.

Proof. (**Theorem 2.2.2.1**)

Combining **Proposition 2.2.2.1** and Grönwall's lemma, we have

$$\|v(T,\cdot)\|_m \leq \|v_0\|_m e^{\int_0^T c_m(|\nabla v|_{L^\infty})\mathrm{d}t}. \tag{2.2.2.3}$$

Hence, bounding ∇v in L^∞ will prove the existence of the solution in V^m.

To obtain such a bound, we will pass through the vorticity, so we recall that, as stated above, we have:

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \tag{2.2.2.4}$$

$$\|\omega\|_{L^2} \leq \|\omega_0\|_{L^2}. \tag{2.2.2.5}$$

Using **Proposition 2.2.2.2**, we obtain that

$$|\nabla v|_{L^\infty} \le c(1+\ln^+\|v\|_3+\ln^+\|\omega\|_0)(1+|\omega|_{L^\infty}) \le K(1+\ln(\|v\|_3+e)) =: K(1+\ln y(t)),$$

where $\ln^+$ is the positive part of the logarithm, and the last inequality used (2.2.2.4)-(2.2.2.5).

Inserting (2.2.2.3) into this, we conclude that

$$y(t) \le y(0)e^{K\int_0^t (1+\ln y(s))\mathrm{d}s},$$

which implies that

$$\ln y(t) \le \ln y(0) + Kt + K\int_0^t \ln y(s)\mathrm{d}s,$$

so that, by Grönwall's lemma

$$\ln y(t) \le (\ln y(0)+Kt)e^{Kt}.$$

This implies that $|\nabla v|_{L^\infty}$ is similarly bounded, and therefore $\|v\|_m$ is bounded by $y(0)e^{Kte^{Kt}}$, thus it will not blow up in finite time, completing the proof. $\diamond$

2.2.3 Weak solutions: existence, (non-)uniqueness, and Onsager's conjecture

The first non-uniqueness results for weak solutions were given by Scheffer [61] and Shnirelman [63], on $\mathbb{R}^2$ and $\mathbb{T}^2$ respectively. The combination of their results is the following statement.

Theorem 2.2.3.1. *Let Ω be $\mathbb{T}^2$ or $\mathbb{R}^2$. There exists a weak solution $u:[0,T]\times\Omega\to\mathbb{R}^2$ of the Euler equations such that $u\equiv 0$ for $|x|^2+|t|^2>1$ and u is nonzero on a set of positive Lebesgue measure.*

Note that the initial datum of these solutions is $u_0\equiv 0$, so even for the simplest initial datum there is non-uniqueness. In fact, Wiedemann [72] proved the non-uniqueness of weak solutions for any L^2 initial data.

Theorem 2.2.3.2. *Call $H(\mathbb{T}^d) := \{u\in L^2(\mathbb{T}^d) : \operatorname{div} u = 0, \int_{\mathbb{T}^d} u(t,x)\mathrm{d}x = 0\}$, and let $v_0\in H(\mathbb{T}^d)$. Then there exist infinitely many weak solutions $v\in \mathcal{C}^0([0,\infty),H_w(\mathbb{T}^d))$ of the Euler equations with $v(0)=v_0$. Moreover, the kinetic energy $E_E(t)$ is bounded and satisfies $E(t)\to 0$ as $t\to+\infty$.*

The proof of this theorem in [72] also yields the following.

Remark 2.2.3.1. *The solutions of [72] have a jump discontinuity at $t=0$. Thus, for every initial datum $v\in\mathcal{C}^\infty(\mathbb{T}^3)$, there exist non-admissible weak solutions.*

In other words, for C^∞ initial data, we have at most one C^∞ solution which, if it exists (and it does for small enough times), is the only admissible solution (cfr. **Theorem 2.2.3.3** below), but there are also non-admissible solutions.

It is then natural to ask whether admissible solutions are unique. The following weak-strong uniqueness result ([73, Theorem 2.1]) gives a partial answer: if there is a classical solution, it is the only admissible one.

Theorem 2.2.3.3 (Weak-strong uniqueness). *Let $u \in L^\infty((0,T);L^2(\mathbb{T}^3))$ be an admissible weak solution of* (2.1.1.1) *and $U \in C^1([0,T]\times\mathbb{T}^3)$ be a strong solution of* (2.1.1.1)*, and assume u, U share the same initial datum u^0. Then $u(t,x) = U(t,x)$ for a.e. $(t,x) \in (0,T)\times\mathbb{T}^3$.*

Note now that the local well-posedness of the initial value problem for such data was proved in [52] in 3D for $C^{1,\alpha}$ data, and in [37] for the general case. The local solution preserves the $C^{k,\alpha}$ regularity of the datum, if $\alpha \in (0,1)$. If $\alpha = 0$ or $\alpha = 1$, it may instead lose it instantly, as stated in [3, Theorem 1.1] (**Theorem 2.2.1.3**). At any rate, by weak-strong uniqueness, this means that, if an initial datum is $C^{k,\alpha}$ for some $k \geq 1, \alpha > 0$, then locally it admits a unique admissible solution.
Admissible solutions to the 3D Euler equations may or may not conserve the kinetic energy. Generally speaking, energy dissipation depends on the features of the space-time fluctuations of the solutions. In physics, fluctuations in fluids can have different properties depending on the underlying instabilities. In 1941, Kolmogorov ([44, 45, 46]) provided a general mathematical framework for the description of the so-called "fully developed turbulence", which is verified experimentally in many physical systems. For the case of fully developed turbulence, as explained in [33], the fluctuations of the velocity satisfy the following relation, known as "Kolmogorov's four-fifths law":

$$\left\langle (v(x+\ell)-v(x))^3 \right\rangle = -\frac{4}{5}\varepsilon|\ell|. \tag{2.2.3.1}$$

In this relation, v is the usual velocity of the fluid, ℓ is the (small) increment of the position vector x, ε is the so-called "mean energy dissipation per unit mass" (see [33]), and $\langle\cdot\rangle$ denotes the spatial average.

This result can be compared with the estimate on $\left\langle (v(x+\ell)-v(x))^3 \right\rangle$ which holds in the case where v is Hölder-continuous with exponent β, i.e. if

$$|v(x+\ell)-v(x)| \leq C|\ell|^\beta \qquad \forall x,\ell \in \mathbb{T}^3,$$

which gives

$$\left|\left\langle (v(x+\ell)-v(x))^3 \right\rangle\right| \leq \left\langle (C|\ell|^\beta)^3 \right\rangle \leq C|\ell|^{3\beta}$$

for some positive constant $C > 0$. For $|\ell|$ sufficiently small, this estimate is compatible with the four-fifths law (2.2.3.1) only if $\beta < {}^1/_3$. In other words, one should expect stronger turbulence, and hence stronger dissipation of energy, when the conditions of fully developed turbulence can be achieved, i.e. $\beta < {}^1/_3$.

This led Onsager to his famous conjecture in the 1949 paper [58], which can be formulated as follows.

Onsager's Conjecture. *Let v be a weak solution of* (2.1.1.1) *and define the total kinetic energy as*

$$E(t) := \frac{1}{2}\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x$$

- *If* $v \in C^\beta$ *for* $\beta > 1/3$*, then the energy is a conserved quantity, i.e.* $E(t) \equiv E(0)$*.*
- *By contrast, for any* $\beta < 1/3$*, there exist* C^β *weak solutions of* (2.1.1.1) *which do not conserve the energy.*

A subcase of the first statement was proved in by Eyink in [32], with the stronger condition $\sum_k |\hat{v}_k|^{1/3+\varepsilon}|k| < \infty$. The exact statement by Onsager was proved in [13] by Constantin, E, and Titi, and sharper results can be found for instance in [10] and [28].

The efforts to prove the second statement led to the development of the convex integration techniques which form the basis the main original result presented in this thesis ([35]). A number of papers were produced, culminating in the proof of the conjecture by Isett in [40].

The first steps towards proving this second part of the conjecture were made by Scheffer [61] on $\mathbb{R}^2$ and Shnirelman [63] on $\mathbb{T}^2$, proving the existence of nontrivial compactly supported Euler solutions as seen at the start of this subsection. De Lellis and Székelyhidi then, in [21], used methods reminiscent of Nash's convex integration technique ([56]) in order to construct anomalously dissipative solutions in L^∞. They thus pointed out an analogy between Nash's theorem in the context of isometric embeddings and Onsager's conjecture. Successive refinements of the convex integration technique led to continuous dissipative solutions in [24], then $C^{1/10-\varepsilon}$ ones in [25], $C^{1/5-\varepsilon}$ ones in [38] (refined to have prescribed kinetic energy profiles in [6]), a.e. $C^{1/3-\varepsilon}$ ones in [4], $C^{1/3-\varepsilon}$ ones with L^1-in-time Hölder norm in [7], and finally the full conjecture in [40].

There are further results about C^β solutions to the 3D Euler equations with $\beta < 1/3$. Paper [8] shows that, for any $\beta < 1/3$, there exist C^β solutions with arbitrary energy profiles $E_E(t)$. More explicitly, the result below is stated in [8, Theorem 1.1].

Theorem 2.2.3.4. *Assume* $e : [0,T] \to \mathbb{R}$ *is a strictly positive smooth function. Then, for any* $0 < \beta < 1/3$*, there exists a weak solution* $v \in C^\beta([0,T] \times \mathbb{T}^3)$ *to* (2.1.1.1) *such that*

$$\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x = e(t).$$

This means that the energy may be conserved, be strictly dissipated, and be increased in time. In other words, even up to $C^{1/3-\varepsilon}$ regularity, we have both non-admissible and, most importantly, strictly dissipative solutions.

All these results produce infinitely many solutions. However, they have no control on the initial data. The question of the existence or non-existence of wild initial data was tackled in another series of works, producing infinitely many wild initial data for $C^{1/16-\varepsilon}$-Hölder solutions in [29], and a dense subset of L^2 consisting of wild initial data for $C^{1/5-\varepsilon}$-Hölder solutions in [19], and for $C^{1/3-\varepsilon}$-Hölder solutions in [18]. We conclude this subsection by citing the last of these three, and specifically its main theorem, [18, Theorem 1.1], which states that, for every $\beta < 1/3$, there are infinitely many $\mathcal{C}^\beta$ initial data that give rise to infinitely many $\mathcal{C}^\beta$ admissible solutions. In the next chapter, we will be reviewing the techniques used in all these works, and highlighting the main innovations that allowed the various regularity jumps as well as the passage to the question of wild initial data.

Theorem 2.2.3.5. *For any $0 < \beta < 1/3$, the set of divergence-free fields $v_0 \in \mathcal{C}^\beta(\mathbb{T}^3, \mathbb{R}^3)$ which admit infinitely many $\mathcal{C}^\beta$ admissible weak solutions of the 3D Euler equations is a dense subset of $L^2(\mathbb{T}^3, \mathbb{R}^3) \cap \{\operatorname{div} v = 0\}$.*

This is the main result obtained so far regarding the non-uniqueness of admissible solutions to the 3D Euler equations. In the case of the 3D fractional Navier-Stokes equations, an analogous result is the main theorem of this thesis, **Theorem 1.3.2**.

2.3 Existence and (non-)uniqueness results for solutions to the Navier-Stokes equations

Some of the existence and uniqueness properties for solutions to the Navier-Stokes equations are analogous to their Euler counterparts. There are, however, important differences due to the regularizing effect of the Laplacian term, as shown in this section.

2.3.1 Local existence and uniqueness of smooth solutions in three dimensions

H^m solutions to the Navier-Stokes equations share the same local well-posedness properties as their Euler equations counterparts. In fact, [53, Theorems 3.4-3.5, Corollary 3.2] (**Theorem 2.2.1.1** and **Corollary 2.2.1.1**) are stated simultaneously both for Euler, and for Navier-Stokes for every viscosity $0 \le \nu < \infty$. With a suitable interpretation of the differential operators, a very similar result also holds true on any compact Reimannian manifold, giving solutions of regularity $L^\infty_t H^m_x \cap \operatorname{Lip}_t H^{m-2}_x$, as can be obtained by combining [70, Theorem 5.1 and Proposition 4.2].

Again like in the Euler case, we have local well-posedness for $\mathcal{C}^{k,\alpha}$ initial data, as proved in the book [49].

For the Navier-Stokes equations, however, lower regularity is enough to ensure local well-posedness. In fact, we have local well-posdeness as soon as all the terms in the equations can be understood as functions rather than just distributions. The minimal regularity for which this happens is $L^\infty_t H^1_x \cap L^2_t H^2_x$, as stated in the theorem below, which combines two results from [60], namely Lemma 6.2 and Theorem 6.4.

Theorem 2.3.1.1. *Suppose v is a strong solution of the Navier-Stokes equations on $[0,T]$. Then $\partial_t v, \Delta v, (v\cdot\nabla)v$ are all elements of $L^2_t L^2_x$. Moreover, there exists a function $p \in L^2_t H^1_x$ such that, for a.e. $(t,x) \in [0,T] \times \mathbb{T}^3$, we have*

$$\partial_t v(t,x) + [(v\cdot\nabla)v](t,x) + \nabla p(t,x) - \Delta v(t,x) = 0.$$

In order to prove local well-posedness for $L^\infty_t H^1_x \cap L^2_t H^2_x$ solutions, one proceeds in three steps, the first being a proof of local existence, provided by [60, Theorem 6.8] reported below.

Theorem 2.3.1.2. *There exists a constant $c > 0$ such that any $v_0 \in H^1$ with zero divergence gives rise to a strong solution $v \in L^\infty([0,T], H^1(\mathbb{T}^3)) \cap L^2([0,T], H^2(\mathbb{T}^3))$ of the Navier-Stokes equations* (2.1.2.1)*, where:*

$$T = c\|\nabla v_0\|_{L^2}^{-4}.$$

The second step is the proof that strong solutions are admissible. In fact, as stated in [60, Theorem 6.5] reported below, they conserve E_{NS}.

Theorem 2.3.1.3. *If v is a strong solution of* (2.1.2.1) *on $[0,T]$, then it conserves E_{NS}, i.e. it satisfies the energy equality:*

$$\frac{1}{2}\|v_{t_1}\|_{L^2}^2 + \int_{t_0}^{t_1} \|\nabla v_s\|^2 \mathrm{d}s = \frac{1}{2}\|v_{t_0}\|_{L^2}^2,$$

for all $0 \leq t_0 < t_1 \leq T$.

Therefore, if we prove that, once a strong solution exists, all admissible solutions coincide with it, we can conclude that strong solutions are unique. The following "weak-strong uniqueness" result, which is [60, Theorem 6.10], is thus the last step towards local well-posedness for H^1 initial data.

Theorem 2.3.1.4. *If $u \in L^\infty_t H^1_x \cap L^2_t H^2_x$ and $v \in L^\infty_t L^2_x \cap L^2_t H^1_x$ are a strong solution and a weak solution on the interval $[0,T]$, with the same initial data, and v is admissible (i.e. satisfies* (2.1.4.2)*), then $u = v$ on $[0,T]$.*

The global existence and regularity properties of solutions to the Navier-Stokes equations are generally stronger than their counterparts for Euler solutions, as can be seen from **Theorem 2.3.3.4**, which has no analogue in the Euler case, as well as the following three theorems. The first theorem ([60, Theorem 6.12]) states that, for initial data whose H^1 norm is sufficiently small, the $L^\infty_t H^1_x \cap L^2_t H^2_x$ strong solution exists globally in time.

Theorem 2.3.1.5. *There exists a constant $C > 0$ such that, if $\|v_0\|_{H^1} < C$, then the strong solution arising from v_0 exists globally in time.*

The existence and regularity properties of solutions with H^m data which are small in H^1 can be summarized as follows: there exists a unique global $L^\infty_t H^1_x \cap L^2_t H^2_x$ solution which is $C^0_t H^m_x$ locally, and which is also the only Leray solution generated by its initial datum.

For H^m data which are small in H^1, an even stronger result holds, namely that the higher regularity $L^\infty_t H^m_x \cap L^2_t H^{m+1}_x$ is found on any finite time interval, as stated in [60, Theorem 7.1] reported below.

Theorem 2.3.1.6. *If v is a strong solution of the Navier-Stokes equations on $[0,T]$ with initial data $v_0 \in H^m$, then $v \in L^\infty_t H^m_x \cap L^2_t H^{m+1}_x$.*

In fact, [60, Theorem 7.5], which is reported below, states that, for positive times, strong solutions are much more regular than their initial data.

Theorem 2.3.1.7. *If v is a strong solution on $[0,T]$ then $v \in \mathcal{C}^\infty([\varepsilon,T] \times \mathbb{T}^3)$ for every $0 < \varepsilon < T$.*

With the results presented so far, we know that, for $\mathcal{C}^2$ data, we have a unique global-in-time $L_t^\infty H_x^2 \cap L_t^2 H_x^3$ solution that is smooth for positive times. Therefore, the only obstruction to the existence and uniqueness of $\mathcal{C}_t^0 \mathcal{C}_x^2$ solutions is if the $\mathcal{C}^2$ norm of that unique solution is not continuous at $t = 0$. This instantaneous loss of regularity was proven for the Euler equations (cfr. **Theorem 2.2.1.3**).

2.3.2 Global-in-time existence of smooth solutions in two dimensions

Analogously to the Euler case, all the results presented in the 3D case hold for the 2D case as they are or with minor modifications. In addition, the following stronger result holds in the 2D case in relation to the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. The theorem below is, in fact, almost identical to its Euler counterpart (**Theorem 2.2.2.1**).

Theorem 2.3.2.1 (Global well-posedness in two dimensions). *Given an initial 2D velocity field v_0 with locally finite-energy decomposition $v_0 = u_0 + \tilde{v}_0$ with $u_0 \in H^m(\mathbb{T}^2), m > 3$ and $\operatorname{curl} \overline{v} = \omega_0(r) \in \mathcal{C}^\infty(\mathbb{T}^2) \cap L^2(\mathbb{T}^2)$, then there exists for all time a unique smooth solution*

$$v(t,x) = u(t,x) + \overline{v}(t,x)$$

to the 2D Navier-Stokes equations, with $u(t,x) \in L_t^\infty H_x^m \cap L_t^2 H_x^{m+1}$ on any time interval $[0,T]$ and $\overline{v}(t,x)$ an exact solution.

Compared to the global existence result in the 3D case (**Theorem 2.3.1.5**), this result is stronger in that it holds for all H^m data, whereas the result in the 3D case only holds for data which are small in H^1. The additional regularity $L_t^2 H_x^{m+1}$ in the above statement with respect to its Euler counterpart is a consequence of **Theorem 2.3.1.6**.

The proof of **Theorem 2.3.2.1** can be straightforwardly derived from that of **Theorem 2.2.2.1**, with the observation that (2.2.2.1) is replaced by

$$\partial_t \omega + (v \cdot \nabla)\omega = \nu \Delta \omega,$$

which means that (2.2.2.2) and (2.2.2.4) are replaced by inequalities. More precisely, equation (2.2.2.2) is replaced by the following balance:

$$\|\omega\|_{L^2}^2 + \nu \int_0^t \int_{\mathbb{T}^2} |\nabla \omega|^2 \mathrm{d}x\mathrm{d}s = \|\omega_0\|_{L^2}^2.$$

2.3.3 Weak solutions: existence, (non-)uniqueness, weak-strong uniqueness, and regularity

A first non-uniqueness result follows from [9, Theorem 1.2], which asserts the existence of weak solutions with $\mathcal{C}^0$ regularity in time and arbitrary smooth kinetic energy profiles.

Theorem 2.3.3.1. *There exists $\beta > 0$ such that, for any non-negative smooth function $e(t) : [0,T] \to \mathbb{R}_{\geq 0}$, there exists a weak solution $v \in \mathcal{C}^0_t([0,T]; H^\beta_x(\mathbb{T}^3))$ to* (2.1.2.1) *such that:*

$$\int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x = e(t).$$

Moreover, the associated vorticity $\nabla \times v \in \mathcal{C}^0_t([0,T]; L^1_x(\mathbb{T}^3))$.

A consequence of this result is the non-uniqueness of solutions with zero initial data, since any solution associated to a kinetic energy profile E_E such that $E_E(0) = 0$ will have that initial condition. Any nonzero solution with $E_{NS}(0) = 0$ must be non-admissible.

The construction used to prove **Theorem 2.3.3.1** in [9] also yields the following remark.

Remark 2.3.3.1. *The solutions constructed in **Theorem 2.3.3.1** lie in $\mathcal{C}^\gamma_t L^2_x$ for some regularity index $\gamma > 0$.*

For weak solutions with $\mathcal{C}^0$ regularity in time, the following gluing result ([5, Theorem 1.1]) also holds.

Theorem 2.3.3.2. *There exists a $\beta > 0$(²) such that the following holds. For $T > 0$, let $v^{(1)}, v^{(2)} \in \mathcal{C}^0([0,T]; \dot{H}^3(\mathbb{T}^3))$ be two strong solutions of the Navier-Stokes equations* (2.1.2.1) *on $[0,T]$, with data $v^{(1)}(0,x)$ and $v^{(2)}(0,x)$ of zero mean. There exists a weak solution $\overline{v}$ of the Cauchy problem to* (2.1.2.1) *on $[0,T]$ with initial datum $\overline{v}|_{t=0} = u^{(1)}|_{t=0}$, which has the additional regularity $\overline{v} \in \mathcal{C}^0([0,T]; H^\beta(\mathbb{T}^3) \cap W^{1,1+\beta}(\mathbb{T}^3))$, and such that $\overline{v} \equiv v^{(1)}$ on $[0, {}^T\!/_3]$ and $\overline{v} \equiv v^{(2)}$ on $[{}^2\!/_3 T, T]$. Moreover, for every such $\overline{v}$, there exists a zero Lebesgue measure set of times $\Sigma_T \subset [0,T]$ with Hausdorff (in fact box-counting) dimension less than $1 - \beta$ such that $\overline{v} \in \mathcal{C}^\infty(((0,T] \setminus \Sigma_T) \times \mathbb{T}^3)$. In particular, $\overline{v}$ is smooth almost everywhere.*

The above result implies that, for all $\dot{H}^3$ initial data, $\mathcal{C}^0_t H^\beta_x$ solutions are non-unique.

The paper does not quantify β. However, the strategy employed therein allows β to be at most slightly higher than 10^{-3}.

It is also known (see e.g. [14]) that $\beta = {}^1\!/_2$ is enough to guarantee weak-strong uniqueness. It is, in fact, a threshold for it in general dimensions, given the result in Terence Tao's blog post [68]. This result states that, at least for the zero initial datum, for any $s < {}^1\!/_2$ there exists a sufficiently high dimension $d(s)$ such that non-uniqueness holds for $\mathcal{C}^0_t H^s_x$ solutions on $\mathbb{T}^{d(s)}$.

Lowering the time regularity to L^1 substantially increases the space regularity for which non-uniqueness holds. This is the content of the following theorem ([11, Theorem 1.7]).

²The maximal β_{max} for which this holds can be quantified as $\beta_{max} \approx 10^{-3}$.

Theorem 2.3.3.3. *Let $d \geq 2$ be the dimension and $1 \leq p < 2, q < \infty$, and $\varepsilon > 0$. For any smooth divergence-free vector field $u \in C^\infty([0,T] \times \mathbb{T}^3)$ with zero spatial mean for each $t \in [0,T]$, there exists a weak solution v of* (2.1.2.1) *and a set*

$$\mathcal{I} = \bigcup_{i=1}^{\infty} (a_i, b_i) \subset [0,T]$$

such that the following hold.

(1) *The solution v satisfies*

$$v \in (L_t^p L_x^\infty \cap L_t^1 W_x^{1,q})([0,T] \times \mathbb{T}^3);$$

(2) *v is a smooth solution on (a_i, b_i) for every i, i.e.*

$$v|_{\mathcal{I} \times \mathbb{T}^3} \in C^\infty(\mathcal{I} \times \mathbb{T}^3);$$

(3) *The Hausdorff dimension of the residue set $\mathcal{S} = [0,T] \setminus \mathcal{I}$ satisfies*

$$d_{\mathcal{H}}(\mathcal{S}) \leq \varepsilon;$$

(4) *The solution v and the given vector field are ε-close in $L_t^p L_x^q \cap L_t^1 W_x^{1,q}$, i.e.*

$$\|u - v\|_{L_t^p L_x^q \cap L_t^1 W_x^{1,q}} \leq \varepsilon.$$

Theorem 2.3.3.3 implies that, for all $\dot{H}^3$ data, $L_t^p L_q^\infty \cap L_t^1 W_x^{1,q}$ solutions are non-unique. To obtain this, we note that, assuming $v^{(1)}, v^{(2)}$ are strong solutions as in **Theorem 2.3.3.2**, we can apply **Theorem 2.3.3.3** to an opportune gluing of $v^{(1)}, v^{(2)}$, i.e. $\chi v^{(1)} + (1-\chi)v^{(2)}$ with χ a smooth cutoff in time. If we proceed as in the proof of **Theorem 2.3.3.3** (which does not modify u where it already is a solution) only on $[{}^1/_3 T, {}^2/_3 T]$, we obtain a gluing theorem in the spirit of **Theorem 2.3.3.2**.

A simple adaptation of the proof in [11] allows one to extend **Theorem 2.3.3.3** to $L_t^p L_x^q \cap L_t^s W_x^{1,q}$, for $s < 2$ and $q < q_{max}(p,s)$ such that $q_{max}(p,s) \to 1$ whenever $p \to 2$ or $s \to 2$. Therefore, non-uniqueness also holds for $L_t^s W_x^{1,q}$ solutions.

Analogously to what is done in the Euler case, one can attempt to achieve existence and uniqueness of the solutions to the Navier-Stokes equations by imposing the condition of admissibility. Admissible solutions to (2.1.2.1) are known to exist for all L^2 initial data by Leray's theorem in [51], which is the reason why admissible solutions are also called *Leray solutions*. We report the theorem below.

Theorem 2.3.3.4 (Global-in-time existence of Leray solutions for the Navier-Stokes equations). *For any $v_0 \in L^2(\mathbb{T}^3)$ with $\operatorname{div} v_0 = 0$ there is a weak solution $v \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}^+, H^1(\mathbb{T}^3))$ of* (2.1.2.1) *such that $v(0,\cdot) = v_0$ and* (2.1.4.2) *holds. In fact, the following form of energy inequality also holds:*

$$\frac{1}{2} \int_{\mathbb{T}^3} |v|^2(t,x)\mathrm{d}x + \int_s^t \int_{\mathbb{T}^3} |\nabla v|^2(x,\tau)\mathrm{d}x\mathrm{d}\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |v|^2(s,x)\mathrm{d}x \qquad a.e.\ s, \forall t > s.$$

This also holds on general compact manifolds, as stated in [70, Theorem 4.6]. An analogous existence result for the Euler equations remains so far elusive. The presence of the dissipation term in the energy inequality plays an important role in the proof. In effect, some form of control of the gradient of the velocity also allows to prove an existence criterion for the Euler equations: the Beale-Katô-Majda criterion of **Theorem 2.2.1.2** has a control on the vorticity as its condition, and this is a form of control on the gradient of the velocity since $\omega = \operatorname{curl} v$.

There are also quite a few regularity results about admissible solutions to the Navier-Stokes equations. Firstly, if an admissible solution with H^1 datum is $L_t^1 \dot{W}_x^{1,\infty}$, then it is strong, as stated in [60, Theorem 7.6] below.

Theorem 2.3.3.5. *Let v be an admissible solution of the Navier-Stokes equations arising from a divergence free initial datum $v_0 \in H^1$. Then, if*

$$\int_0^T \|\nabla v(s)\|_\infty \mathrm{d}s < \infty,$$

v is strong on $[0,T]$.

Notice how this, again, parallels the Beale-Katô-Majda criterion (**Theorem 2.2.1.2**), this time controlling the same $L_t^1 L_x^\infty$ norm of the ∇v as the Beale-Katô-Majda criterion controls for ω.

Moreover, all global admissible solutions are eventually strong, as given by [60, Theorem 8.1] below.

Theorem 2.3.3.6. *Any global-in-time admissible weak solution v is $C^\infty((T^*,\infty)\times\mathbb{T}^3)$ for some $T^* > 0$.*

The statement of the next theorem requires the following definition of singular and regular times of an admissible solution.

Definition 2.3.3.1. *Let v be an admissible solution with initial datum v_0. We say that t_0 is a* regular time *of v if $\|\nabla v_t\|_{L^2}$ is essentially bounded in some neighrborhood of t_0, i.e. if $\nabla v \in L^\infty([t_0-\varepsilon, t_0+\varepsilon]; L^2(\mathbb{T}^3))$ for some $\varepsilon > 0$. t_0 is instead a* singular time *of v if it is not regular. We denote the set of strictly positive regular times by $\mathcal{R}$ and the set of singular times by $\mathcal{T}$ ([3]).*

On the open set $\mathcal{R}\times\mathbb{T}^3$, where $\mathcal{R}$ has just been defined, admissible solutions can be shown to be smooth (C^∞), as stated in [60, Lemma 8.4], which is reported below with $\mathcal{R}$ written in the form given by [60, Theorem 8.14].

Theorem 2.3.3.7. *An admissible weak solution is smooth on the open set*

$$\mathcal{R}\times\mathbb{T}^3 = \bigcup_{i=1}^\infty (a_i,b_i)\times\mathbb{T}^3,$$

where $a_i, b_i \in \mathbb{R}$ for all i.

[3] By **Theorem 2.3.3.6**, $\mathcal{T}$ is bounded. Since $\mathcal{R}$ is open by definition, we conclude ([60, Lemma 8.3]) that $\mathcal{T}$ must be compact.

On general 3-dimensional compact manifolds, the same result holds, with $\mathscr{R}$ simply being an open dense subset with Lebesgue-null complement. This is stated in [70, Proposition 4.7].

A very low regularity requirement, namely $v \in L^r_t L^s_x$ for suitable choices of r, s, is enough to ensure smoothness of admissible solutions for positive times. This was established by Ladyzhenskaya [48], and is the content of [60, Lemma 8.16] reported below.

Theorem 2.3.3.8 (Ladyzhenskaya regularity theorem). *If v is an admissible weak solution for some $v_0 \in H^1$, and*

$$v \in L^r_t L^s_x \qquad \frac{2}{r} + \frac{3}{s} = 1 \qquad 2 \le r < \infty, \tag{2.3.3.1}$$

then v is smooth on $(0,T]$.

For the endpoint case $(r,s) = (\infty, 3)$, one has a local well-posedness result, as per [30] and [60, Theorem 11.4].

Theorem 2.3.3.9. *If $v_0 \in L^3 \cap L^2$ is a divergence-free initial datum, then there exists a time $T > 0$ such that the equations have a unique local solution $v \in L^\infty_t L^3_x([0,T] \times \mathbb{T}^3)$. Moreover, there exists an absolute constant $C > 0$ such that, if in addition*

$$\|v_0\|^3_{L^3} \int_0^T \int_{\mathbb{T}^3} \left|\nabla e^{t\Delta} v_0\right|^2 \left|e^{t\Delta} v_0\right| \mathrm{d}x\mathrm{d}t < C,$$

then the solution v is global, meaning there exists a unique global solution which which is smooth for all positive times.

Moreover, such regularity is sufficient for weak-strong uniqueness, as proved by Prodi [59] and [62]. This result is known as the Prodi-Serrin criterion.

Theorem 2.3.3.10 (Prodi-Serrin criterion). *Let $v_0 \in L^2$ be a divergence-free initial condition. Assume that, for v_0, there exists a solution $v_1 \in L^2_t H^1_x \cap L^\infty_t L^2_x \cap L^p_t L^q_x$ to the Navier-Stokes initial-value problem, where ${}^2/_p + {}^3/_q = 1$. Then, if v_2 is a Leray solution, we have $v_2 = v_1$.*

Note that both the Ladyzhenskaya regularity theorem **Theorem 2.3.3.8** and the Prodi-Serrin criterion **Theorem 2.3.3.10** hold on $\mathbb{R}^3$ as well as $\mathbb{T}^3$. In the latter domain, due to the scale of L^p spaces, we have these results for ${}^2/_p + {}^3/_q < 1$ too. Indeed, if this relation holds, then it is possible to find $p' \le p, q' \le q$ such that ${}^2/_{p'} + {}^3/_{q'} = 1$, and thus we apply the above versions of the results for such a choice of p', q'.

It should be noted that the Prodi-Serrin criterion can be extended to other dimension, but the relation changes to ${}^2/_p + {}^d/_q$, where d is the dimension. Thanks to this, since compact Riemannian manifolds have the same scale of L^p spaces as $\mathbb{T}^3$, the following result ([70, Proposition 4.3]) holds.

Theorem 2.3.3.11 (Local well-posedness on manifolds). *If $\operatorname{div} v_0 = 0$ and $v_0 \in L^p(M)$, where M is a compact Riemannian manifold and $p > n = \dim M$, and if $\nu > 0$, then (2.1.2.1) has a unique short-time solution on an interval $I = [0,T]$, and this solution $u = u_\nu \in C^0_t L^p_x([0,T] \times M) \cap C^\infty((0,T) \times M)$.*

Moreover, in the book [49], global well-posedness for L^2 data is proved in two dimensions.

In the book [50], it is proved (see [50, Theorem 2.4]) that, for weak-strong uniqueness, it is sufficient for the velocity to belong to a space of multipliers which, as stated in [50, Proposition 12.3], includes the spaces of the Prodi-Serrin criterion.

Theorem 2.3.3.12 (Lemarié-Rieusset weak-strong uniqueness). *Let $v_0 \in L^2$ be a divergence-free initial condition. Assume the Navier-Stokes problem with initial condition v_0 has a solution $v_1 \in X \cap \mathbb{X}_0^{(T)}$, where:*

- *X is $L^2_t H^1_x \cap L^\infty_t L^2_x$;*
- *$\mathbb{X}_T$ is the space of pointwise multipliers from X to $L^2_t L^2_x$, normed with:*
$$\|u\|_{\mathbb{X}_T} = \sup_{\|w\|_X \le 1} \|uw\|_{L^2_t L^2_x};$$
- *$\|w\|_X = \|w\|_{L^\infty_t L^2_x} + \|w\|_{L^2_t H^1_x}$;*
- *$\mathbb{X}_T^{(0)}$ is the space of multipliers $u \in \mathbb{X}_T$ such that, for every $t \in [0,T)$,*
$$\lim_{t_1 \to t_0^+} \left\| 1_{(t_0,t_1)}(t) u(t,x) \right\|_{\mathbb{X}_T} = 0.$$

Then, if v_2 is an admissible solution of the same Navier-Stokes initial value problem, we have $v_2 = v_1$.

The above result is formulated for solutions on $(0,T) \times \mathbb{R}^3$, but the arguments of the proof easily adapt to $(0,T) \times \mathbb{T}^3$.

We have one last local existence and uniqueness result, stated in [60, Corollary 10.2].

Theorem 2.3.3.13. *Suppose that $v_0 \in L^2 \cap \dot{H}^{1/2}$ is divergence-free. Then there exists a time $T = T(v_0) > 0$ such that the Navier-Stokes equations have a solution $v \in L^\infty_t \dot{H}^{1/2}_x([0,T] \times \mathbb{T}^3) \cap L^2_t \dot{H}^{3/2}_x([0,T] \times \mathbb{T}^3)$ which is unique in the class of admissible solutions and is smooth on $(0,T]$. Moreover, there exists an absolute constant C' such that, if $\|v_0\|_{\dot{H}^{1/2}} < C'$, then the solution exists globally in time and hence is the unique global admissible solution of the Navier-Stokes equations.*

Despite the wealth of results concerning regularity and weak-strong uniqueness reported above, the question of uniqueness for admissible solutions to the Navier-Stokes equations is still an open problem. The most recent result in the direction of non-uniqueness is found in [1]. In this paper, the authors consider the forced Navier-Stokes equations on $\mathbb{R}^3$:

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = f \\ \operatorname{div} v = 0 \end{cases}, \tag{2.3.3.2}$$

with the following energy inequality:

$$\frac{1}{2} \int_{\mathbb{R}^3} |v(t,x)|^2 \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} |\nabla v(t,x)|^2 \mathrm{d}x \mathrm{d}\tau \le \frac{1}{2} \int_{\mathbb{R}^3} |v(t,x)|^2 \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} (f \cdot v)(s,x) \mathrm{d}x \mathrm{d}s, \tag{2.3.3.3}$$

and by a suitable choice of f are able to exhibit two solutions with zero initial datum. More specifically, they prove the following result.

Theorem 2.3.3.14 (ABC Non-uniqueness). *There exist $T > 0, f \in L^1_t L^2_x((0,T) \times \mathbb{R}^3)$, and two distinct solutions $v, \bar{v} : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ of* (2.3.3.2) *satisfying* (2.3.3.3) *with body force f and initial condition $v_0 \equiv 0$.*

2.4 Properties of solutions to the fractional Navier-Stokes equations

The fractional Navier-Stokes equations are a relatively recent topic of study, and therefore the available literature on the properties of their solutions is more limited. As shown below, the known properties are more similar to those of the Euler or Navier-Stokes equations depending on the exponent of the Laplacian.

2.4.1 Local existence and uniqueness of smooth solutions

The local well-posedness for H^m solutions of the Euler and Navier-Stokes equations also holds in the case of the fractional Navier-Stokes equations. This is stated in [26, Theorem 3.4] reported below.

Theorem 2.4.1.1. *For any Laplacian exponent $\theta > 0, m \geq 3$ there exists a constant $c_m = c(m)$ such that the following holds. Given any divergence-free initial condition $v_0 \in H^m$ and $T_m := c_m \|v_0\|^{-1}_{H^m}$, there exists a unique solution $v \in (C^0_t H^m_x \cap C^1_t H^{m-2}_x)([0,T_m] \times \mathbb{T}^3)$ of* (2.1.3.1)*. Moreover we have the estimate*

$$\|v(t)\|_{H^m} \leq \|v_0\|_{H^m} e^{c_m \int_0^t \|\nabla v(s)\|_0 \mathrm{d}s} \qquad \forall t \in [0, T_m]. \tag{2.4.1.1}$$

A local well-posedness result can also be obtained in the case of $C^{k,\alpha}$ spaces. We saw that, in the Euler case, local well-posedness holds in $C^{k,\alpha}$ for every $k \geq 1, \alpha \in (0,1)$. In the fractional Navier-Stokes case, local well-posedness holds for every $k \geq 3, \alpha \in (0,1)$.

Theorem 2.4.1.2. *For any $\nu > 0$, $0 < \alpha < 1$, and $k \geq 3$, there exists a constant $c = c(\alpha) > 0$ with the following property. Given any initial datum $v_0 \in C^{k,\alpha}$, and $T \leq c\|v_0\|^{-1}_{1+\alpha}$, there exists a unique solution $v \in C^0([0,T], C^{k,\alpha}(\mathbb{T}^3, \mathbb{R}^3))$ of* (2.1.3.1)*. Moreover, v obeys the bounds*

$$\|v\|_{N+\alpha} \leq C(N,\alpha)\|v_0\|_{N+\alpha} \qquad \forall 1 \leq N \leq k. \tag{2.4.1.2}$$

The proof of this result is an adaptation of that of [26, Proposition 3.5], and requires the use of the Schauder estimates reported in **Lemma A.4**. (see e.g. the book [34]), as well as some estimates on the transport-diffusion equations, which are [26, Proposition 3.3] and are reported in **Proposition B.2**. In the course of this proof, we will use the notation $A \lesssim B$ to mean $A \leq CB$ for some positive constant $C > 0$.
Proof. (**Theorem 2.4.1.2**)

We first show that all solutions given by **Theorem 2.4.1.1** (which applies since $v_0 \in \mathcal{C}^{k,\alpha} \hookrightarrow H^3$) exist in the interval $[0,T]$, for any $T \lesssim \|u_0\|_{1+\alpha}^{-1}$. Fix any $\alpha \in (0,1)$ and let T^* be the maximal time such that

$$T^* \sup_{0 \le t \le T^*} [v(t)]_1 \le 1.$$

Suppose $T^* < c\|u_0\|_{1+\alpha}^{-1}$, for some constant $c = c(\alpha)$ to be fixed later (we will see that this contradicts the assumption on the maximality of T^*, in particular $T^* \ge c\|u_0\|_{1+\alpha}^{-1}$). Using **Lemma A.4** on $-\Delta p = \mathrm{tr}(\nabla v \nabla v)$, we have

$$\|p(t,\cdot)\|_{2+\alpha} \lesssim \|v(t,\cdot)\|_{1+\alpha}^2,$$

thus, differentiating the equation in the x variable, we get

$$\left\|(\partial_t + v\cdot\nabla + \nu(-\Delta)^\theta)Dv\right\|_\alpha \lesssim \|v(t,\cdot)\|_{1+\alpha}^2.$$

By **Proposition B.2**, for any $0 \le t \le T^*$, we have

$$\|v(t,\cdot)\|_{1+\alpha} \lesssim \|u_0\|_{1+\alpha} + \int_0^t \|v(s,\cdot)\|_{1+\alpha}^2 \mathrm{d}s.$$

Finally, using Grönwall's inequality, we get the estimate

$$\|v(t,\cdot)\|_{1+\alpha} \lesssim \|u_0\|_{1+\alpha} < \frac{1}{T^*} \qquad \forall t \in [0,T^*],$$

where in the last inequality we have chosen the constant $c = c(\alpha)$ so as to absorb the implicit constants and get a strict inequality. Obviously, this contradicts the hypothesis on the maximality of T^*, and also gives the a priori estimate (2.4.1.2) for $N = 1$, which together with (2.4.1.1) gives the existence of a $\mathcal{C}^{1,\alpha}$ solution in the interval $[0,T]$, for any $T \le c\|u_0\|_{1+\alpha}^{-1}$.

We are left with the higher-order bounds (2.4.1.2) for $N \ge 2$. For any multi-index μ with $|\mu| = N$, we have

$$\partial_t \partial^\mu v + v\cdot\nabla\partial^\mu v + \nu(-\Delta)^\theta \partial^\mu v + [\partial^\mu, v\cdot\nabla]v + \nabla\partial^\mu p = 0.$$

Using again **Lemma A.4** for the pressure, we obtain

$$\|\nabla\partial^\mu p\|_\alpha \lesssim \|\mathrm{tr}(\nabla v \nabla v)\|_{N-1+\alpha} \lesssim \|v\|_{1+\alpha}\|v\|_{N+\alpha}.$$

Therefore

$$\left\|(\partial_t + v\cdot\nabla + \nu(-\Delta)^\theta)\partial^\mu v\right\|_\alpha \lesssim \|v\|_{1+\alpha}\|v\|_{N+\alpha},$$

and (2.4.1.2) follows by applying **Proposition B.2** and the Grönwall inequality. ◇

2.4.2 Weak solutions: existence, non-uniqueness, and weak-strong uniqueness

As is the case for the Navier-Stokes equations, admissible solutions of the fractional Navier-Stokes equations are also known as *Leray solutions*. This is because, as in the case of **Theorem 2.3.3.4**, the proof from [51] can be adapted to prove the existence of admissible solutions of the fractional Navier-Stokes equations.

Theorem 2.4.2.1. *For any divergence-free $\bar{v} \in L^2(\mathbb{T}^3)$ and every $\theta > 0$, there is a weak solution $v \in (L^\infty_t L^2_x \cap L^2_t H^\theta_x)(\mathbb{R}^+ \times \mathbb{T}^3)$ of* (2.1.3.1) *such that $v(0,\cdot) = \bar{v}$ and* (2.1.4.3) *holds, i.e.*

$$\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(t,x)\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2(s,x)\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\int_{\mathbb{T}^3} |\bar{v}|^2(x)\mathrm{d}x \qquad \forall t \ge 0.$$

For the proof of this result, see the Appendix of [15]. That paper states the result for $\theta \in (0,1)$, but the argument used in the proof also works for $\theta \ge 1$.

Moreover, a weak-strong uniqueness theorem holds for the fractional Navier-Stokes, similarly to what was seen for the Euler and Navier-Stokes equations. In the statement below, we impose the same conditions as **Theorem 2.2.3.3** in the Euler case.

Theorem 2.4.2.2 (Weak-strong uniqueness). *Let $u \in L^\infty((0,T); L^2(\mathbb{T}^3)) \cap L^2_t H^\theta_x$ be an admissible weak solution of* (2.1.3.1) *and $U \in \mathcal{C}^1([0,T] \times \mathbb{T}^3)$ be a strong solution of* (2.1.3.1)*, and assume u, U share the same initial datum u^0. Then $u(t,x) = U(t,x)$ for a.e. $(t,x) \in (0,T) \times \mathbb{T}^3$.*

The proof is a simple adaptation of the one given in [73] for the Euler case.

Proof.

Define $E^{u,U}_{FNS,rel}$ as the functional E_{FNS} computed on the difference $u - U$, and observe that

$$\begin{aligned}
E^{u,U}_{FNS,rel}(t) &= \frac{1}{2}\int_{\mathbb{T}^3} |u-U|^2(t,x)\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}(u-U)\right|^2(s,x)\mathrm{d}x\mathrm{d}s \\
&= \frac{1}{2}\int_{\mathbb{T}^3} |u|^2(t,x) + |U|^2(t,x)\mathrm{d}x - \int_{\mathbb{T}^3}(u\cdot U)(t,x)\mathrm{d}x + \\
&\quad + \int_0^t\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}u\right|^2(s,x) + \left|(-\Delta)^{\frac{\theta}{2}}U\right|^2(s,x)\mathrm{d}x\mathrm{d}s \\
&\quad - 2\int_0^t\int_{\mathbb{T}^3} ((-\Delta)^{\frac{\theta}{2}}u \cdot (-\Delta)^{\frac{\theta}{2}}U)(s,x)\mathrm{d}x\mathrm{d}s \\
&\le \int_{\mathbb{T}^3} |u_0|^2\mathrm{d}x - \int_{\mathbb{T}^3}(u\cdot U)(t,x)\mathrm{d}x - 2\int_0^t\int_{\mathbb{T}^3} ((-\Delta)^{\frac{\theta}{2}}u \cdot (-\Delta)^{\frac{\theta}{2}}U)(s,x)\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_{\mathbb{T}^3} (-\partial_t u \cdot U - u \cdot \partial_t U)(s,x) - 2((-\Delta)^{\frac{\theta}{2}} u \cdot (-\Delta)^{\frac{\theta}{2}} U)(s,x) \mathrm{d}x\mathrm{d}s = \\
&= -\int_0^t \int_{\mathbb{T}^3} (u \otimes u) : \nabla U - ((-\Delta)^{\frac{\theta}{2}} u \cdot (-\Delta)^{\frac{\theta}{2}} U) + u \cdot \partial_t U(t,x) \mathrm{d}x\mathrm{d}s \\
&\quad -\int_0^t \int_{\mathbb{T}^3} 2((-\Delta)^{\frac{\theta}{2}} u \cdot (-\Delta)^{\frac{\theta}{2}} U) \mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{T}^3} \mathrm{div}(U \otimes U) \cdot u - (u \otimes u) : \nabla U(t,x) \mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{T}^3} [(U-u) \cdot \nabla] U \cdot u(t,x) \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{T}^3} [(U-u) \cdot \nabla] U \cdot (u-U) \mathrm{d}x\mathrm{d}s \\
&\leq \int_0^t \int_{\mathbb{T}^3} |U-u|^2 \|\nabla U\|_{L^\infty} \mathrm{d}x\mathrm{d}s = 2\int_0^t \|\nabla U(t)\|_{L^\infty_x} E^{U,u}_{E,rel}(t) \mathrm{d}s \\
&\leq 2\int_0^t \|\nabla U(t)\|_{L^\infty_x} E^{U,u}_{FNS,rel}(t) \mathrm{d}s.
\end{aligned}
$$

Grönwall's lemma then implies that

$$E^{U,u}_{FNS,rel}(t) \leq 0,$$

but since it has to be non-negative, we conclude it is zero. This means that for all t

$$\|U(t) - u(t)\|^2_{L^2} = 0,$$

hence $U = u$. $\diamond$

The uniqueness or non-uniqueness properties of weak solutions depend on the value of the exponent θ. In the hyperdissipative range $\theta > 1$, we note that **Theorem 2.3.3.2** holds for $\theta \in [1, {}^5\!/\!_4)$, but $\beta_{max} \to 0$ as $\theta \to {}^5\!/\!_4$. As seen in the Navier-Stokes case, this implies that, for any $\dot{H}^3$ initial datum, there exist non-unique $C^0_t(H^\beta \cap W^{1,1+\beta})_x$ solutions to the fractional Navier-Stokes equations, for $\theta \in [0, {}^5\!/\!_4)$.

In the hypodissipative range, the following theorem ([15, Theorem 2.1]) tells us that solutions with arbitrary kinetic energy profiles $E_E(t)$ exist for $\theta \in (0, {}^1\!/\!_2)$.

Theorem 2.4.2.3. *Assume $e : [0,1] \to \mathbb{R}$ is a positive smooth function with ${}^1\!/\!_2 \leq e(t) \leq 1$ and $\varepsilon > 0$ a positive number. For any $\theta \in (0, {}^1\!/\!_2)$ there is a solution $(v,p) \in C^0([0,1] \times \mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R})$ of* (2.1.3.1) *such that*

$$e(t) = \int_{\mathbb{T}^3} |v|^2(t,x) \mathrm{d}x \qquad \forall t \in [0,t]$$

and:

(i) $v \in C^{1/5-\varepsilon}, p \in C^{2/5-2\varepsilon}$ *if* $\theta \leq 1/4$;

(ii) $v \in C^{1-2\theta/3-2\theta-\varepsilon}, p \in C^{2^{1-2\theta}/3-2\theta-2\varepsilon}$ *if* $1/4 < \theta < 1/2$.

We can compare this result with its Euler counterpart **Theorem 2.2.3.4** and its Navier-Stokes counterpart **Theorem 2.3.3.1**. In doing that, we find that the $C^{1/3-\varepsilon}$ regularity given by the Euler counterpart is always higher than the one of **Theorem 2.4.2.3**, whereas for the Navier-Stokes counterpart the situation is more complex. Indeed, for $\theta < 1/4$, the $C^{1/5-\varepsilon}$ regularity of the solutions given in **Theorem 2.4.2.3** is higher than the H^β regularity of the Navier-Stokes counterpart. For the case $1/4 < \theta < 1/2$, however, **Theorem 2.4.2.3** only gives $C^{1-2\theta/3-2\theta-\varepsilon}$, which tends to 0 as $\theta \to 1/2$. This means that, eventually, $1-2\theta/3-2\theta < \beta$, at which point the two regularities are not comparable.

As seen for the Euler and Navier-Stokes counterparts, this means that weak solutions can conserve, dissipate, or increase their kinetic energy (the last of which makes them non-admissible whenever E_{FNS} is defined.

In fact, thanks to [15, Proposition 2.2], there exist infinite families of profiles associated to solutions with the same initial data. In other words, we have non-uniqueness. The following theorem ([15, Theorem 1.3 and Corollary 2.3]) makes this more precise, asserting the existence of C^β wild initial data for $\theta < 1/5$, with $\theta < \beta < 1/5$, and the existence of C^0 data generating infinitely many weak solutions up to $\theta < 1/2$.

Theorem 2.4.2.4. *If we choose* $\theta < 1/5$, *there are initial data* $\overline{v} \in L^2(\mathbb{T}^3)$ *with* $\operatorname{div} \overline{v} = 0$ *such that:*

(a) $\overline{v}$ *belongs to some Hölder space* $C^\beta(\mathbb{T}^3)$ *for* $\theta < \beta < 1/5$;

(b) *There is a positive time* T *and infinitely many solutions* $v \in C^\beta([0,T] \times \mathbb{T}^3)$ *of* (2.1.3.1) *with* $v(0,\cdot) = \overline{v}$;

(c) *Such solutions are admissible, and in fact satisfy the weak energy inequality* (2.1.4.3) *for* all *times* $0 \leq s \leq t \leq T$.

If instead we choose $1/5 \leq \theta < 1/2$, *then there are divergence-free initial data* $\overline{v} \in C(\mathbb{T}^3)$ *for which there exist infinitely many weak solutions* $v \in L^\infty([0,\infty), L^2(\mathbb{T}^3))$ *of* (2.1.3.1) *with* $v(0,\cdot) = \overline{v}$.

Note that the solutions obtained for $\theta \geq 1/5$ may be too irregular for the admissibility condition to even make sense, since they are not necessarily $L^2([0,\infty), H^\theta)$.

The non-uniqueness of admissible solutions can be extended to $\theta < 1/3$ with the higher regularity of $C^{1/3-\varepsilon}$. This is the content of [26, Theorem 1.2] reported below.

Theorem 2.4.2.5. *Let* $\theta < 1/3$. *Then there are initial data* $\overline{v} \in L^2(\mathbb{T}^3)$ *with* $\operatorname{div} \overline{v} = 0$ *for which there exist infinitely many Leray solutions* v *of* (2.1.3.1) *in* $[0,+\infty) \times \mathbb{T}^3$. *More precisely, if* $\theta < \beta < 1/3$, *there are divergence-free initial data* $\overline{v} \in C^\beta(\mathbb{T}^3)$ *and a positive time* T *such that:*

(a) *There are infinitely many Leray-Hopf solutions of* (2.1.3.1) *and moreover* $v \in \mathcal{C}^{\beta}([0,T] \times \mathbb{T}^3)$*;*

(b) *Such solutions strictly dissipate the total energy in* $[0,T]$*, i.e. the function* E_{FNS} *defined in Section 2.1 is strictly decreasing on* $[0,T]$.

A final result regarding non-uniqueness of admissible solutions is **Theorem 1.3.2** stated in Chapter 1, which concerns the L^2-density of initial data for which infinitely many solutions exist, and is the main result of this thesis. Both this result and the theorems in this section produce $\mathcal{C}^{\beta}$ admissible solutions up to a time T depending only on the initial datum. The proof of **Theorem 1.3.2**, when approaching a given L^2 vector field with wild data, is unable to maintain the admissibility of the regular solution up to a fixed time, and therefore has to give the regularity up in order to restore the admissibility on this fixed time interval.

Chapter 3

Convex Integration and the Euler equations

3.1 The general idea

Convex integration is a powerful technique for addressing problems of existence and (non-)uniqueness of solutions to systems of PDEs. First introduced by Nash in [56], it was extended by Kuiper in [47] in the context of differential geometry. It was then formalized by Gromov [36] in the more general setting of partial differential relations, of which PDEs and the isometric embeddings of the Nash-Kuiper theorem are special cases.

The general idea is that, given a system of PDEs, one tries to do the following:

1. Modify the system of PDEs by introducing an additional "error" term, thus obtaining the so-called "relaxed system";
2. Prove the existence of "suitable" solutions to the modified system, also known as "approximate solutions" or "subsolutions";
3. Prove that the existence of one such approximate solution implies the existence of infinitely many solutions to the original system of PDEs.

Perhaps the most important achievement of convex integration applied to the Euler equations is the proof of Onsager's conjecture. In this chapter, we give an overview of the steps that led to the proof of this result. We also present the adaptations of the technique that made it possible to prove the L^2-density of initial data which generate infinitely many solutions for the Euler equations, and the existence of such data for the fractional Navier-Stokes equations. This was the starting point for the application of the convex integration technique to prove this thesis's main result, **Theorem 1.3.2**.

An example of "relaxed system", which will be used in sections 3.3-3.7, is the *Euler-Reynolds system*, used in fluid dynamics (cfr. [39, Chapter 1]) to describe turbulent phenomena. In this approach, a solution to the Euler equations $v = \overline{v} + w$ is separated into the sum of a coarse-grained flow $\overline{v}$, i.e. an "averaged" solution, and a perturbation term w which accounts for fluctuations. If $\bar{\cdot}$ indicates an averaging process, applying it to the Euler equations we get:

$$\begin{cases} \partial_t \overline{v} + \operatorname{div} \overline{v \otimes v} + \nabla \overline{p} = 0 \\ \operatorname{div} \overline{v} = 0 \end{cases}.$$

Introducing the Reynolds stress tensor R:

$$R \coloneqq \overline{v\otimes v} - \overline{v}\otimes\overline{v},$$

we obtain the so-called *Euler-Reynolds system*:

$$\begin{cases} \partial_t \overline{v} + \operatorname{div}(\overline{v}\otimes\overline{v}) + \nabla\overline{p} = -\operatorname{div} R \\ \operatorname{div}\overline{v} = 0 \end{cases}. \tag{3.1.1}$$

By Jensen's inequality, R is positive semidefinite. System (3.1.1) is a relaxed system where the error term is $-\operatorname{div} R$ and the triple $(\overline{v},\overline{p},R)$ is a subsolution.

3.2 The first steps to Onsager's conjecture

3.2.1 Bounded solutions with compact support

The first applications of the general idea expressed above to the Euler equations was in [21] and [22], producing $L^\infty(\mathbb{R}\times\mathbb{R}^n)$ solutions and $C^0([0,T];L^2_w(\mathbb{R}^n))$ solutions (i.e. such that $t\mapsto v(t,\cdot)\in L^2(\mathbb{T}^3)$ is continuous w.r.t. the weak topology), respectively. As noted in the previous section, oscillations play a major role in the development of turbulence. Hence, it was only natural to attempt to insert the Euler equations into the Tartar [69] framework for the analysis of oscillations in systems of linear PDEs coupled with nonlinear constraints.

In this framework, one writes the system of equations in the form

$$\sum_i A_i\partial_i z = 0,$$

where z is the "state variable" (in our case $z=(v,u,q)$) and the $A_i\in\mathbb{R}^M$ are constant coefficient vectors, and then considers "plane wave solutions" of the system as oscillatory building blocks, i.e. solutions of the form

$$z(x) \coloneqq ah(x\cdot\xi),$$

for h a smooth function. Such plane waves are the simplest oscillatory solutions to the system, so it makes sense to use them as a starting point to construct the perturbations we need to prove an existence theorem for solutions.

The set of directions a such that, for some ξ, the functions above are solutions for any profile h is called *wave cone* and is denoted by

$$\Lambda \coloneqq \left\{ a\in\mathbb{R}^M : \exists\xi\in\mathbb{R}^m : \sum_i \xi_i A_i a = 0 \right\}.$$

To use this framework, we recast the Euler equations as follows:

$$\begin{cases} \partial_t v + \operatorname{div} u + \nabla p = 0 \\ \operatorname{div} v = 0 \\ u = v\mathring{\otimes} v = v\otimes v - \frac{1}{n}|v|^2\operatorname{Id} \end{cases}, \tag{3.2.1.1}$$

where $a\mathring{\otimes}b \coloneqq a\otimes b - {}^1/_n\langle a,b\rangle$ denotes the traceless part of the tensor product, and Id is the $n\times n$ identity matrix.

The goal of this subsection is to sketch the proof of the following theorem given in [21].

Theorem 3.2.1.1. *Let $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ be a bounded open domain. There exist a $v\in L^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ and a $p\in L^\infty(\mathbb{R}\times\mathbb{R}^n)$ which solve the Euler equations in the sense of distributions, such that $|v|=1$ a.e. in Ω and $v=0$ a.e. outside Ω.*

To this end, one fixes a bounded spacetime domain $\Omega\subset\mathbb{R}\times\mathbb{R}^n$, then proves a "perturbation property", and then concludes by a Baire category argument in a suitable function space. Note that the solutions provided by **Theorem 3.2.1.1** cannot conserve the energy. Indeed, for any $t\in\mathbb{R}$ such that $\Omega\cap(\{t\}\times\mathbb{R}^n)=\varnothing$, one has $E_E(t)=0$, yet $\int_{-\infty}^{\infty}E_E(t)\mathrm{d}t = 2^{-1}|\Omega|\neq 0$.

To prove such a perturbation property, one needs oscillatory perturbations supported in smaller and smaller regions of Ω. One also needs to preserve a certain property of what is being perturbed, which is achieved by keeping the perturbation "sufficiently close" to a line segment with one endpoint in the wave cone. This is what [21, Proposition 2.2] provides.

The next step is to set up a suitable functional framework for the Baire category argument. The goal is to show the existence of solutions (v,u,q) of (3.2.1.1) supported in Ω such that $(v(t,x),u(t,x))\in K$ for a.e. $(t,x)\in\Omega$, where

$$K \coloneqq \left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}_0^{n\times n} : u = v\otimes v - {}^1/_n|v|^2\,\mathrm{Id}, |v|=1\right\}.$$

Defining

$$\mathcal{U} \coloneqq \operatorname{int}(K^{co}\times[-1,1]),$$

where int denotes the topological interior, we see that $(v,u)\in K$ is equivalent to (v,u,q) taking values in the convex extremal points of $\overline{\mathcal{U}}$. We can thus give a definition of subsolutions.

Definition 3.2.1.1 (Subsolution). *Let X_0 be the following space:*

$$X_0 \coloneqq \left\{(v,u,q)\in\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n) : \begin{cases}\operatorname{supp}(v,u,q)\subset\Omega\\ (v,u,q) \textit{ solves } (3.2.1.1) \textit{ in } \mathbb{R}\times\mathbb{R}^n\\ (v,u,q)(\mathbb{R}\times\mathbb{R}^n)\subseteq\mathcal{U}\end{cases}\right\}.$$

A subsolution *of* (3.2.1.1) *is a triple (v,u,q) such that $(v,u,q)\in X_0$.*

To compare with the previous subsection, the error term, which was $\operatorname{div} R$ there, takes here the form of $v\otimes v - {}^1/_n|v|^2\,\mathrm{Id} - u$.

By [21, Lemma 4.2], $0\in\mathcal{U}$, so X_0 is nonempty. Therefore, X_0 is a bounded nonempty subset of L^∞, and thus its weak-* closure X will be a compact nonempty metrizable space, as stated in [21, Lemma 4.4].

This result states a further property: if $(v,u,q)\in X$ and $|v|=1$ a.e. in Ω, then (v,p) is a weak solution of the Euler equations, where $p\coloneqq q - {}^1/_3|v|^2$. Therefore, if we can show that there are infinitely many elements of X with $|v|=\mathbb{1}_\Omega$ a.e., the existence result is proved. To do so, as said above, one looks for a suitable Baire-1 map defined on X.

Definition 3.2.1.2 (Baire-1 map). *A map $f:(X,d_X)\to(Y,d_Y)$ between two metric spaces is called a* Baire-1 map*, or a* map of Baire class 1*, if it is the pointwise limit of continuous maps.*

Since it is known ([67, Theorem 4.6]) that the set of continuity points of Baire-1 maps between complete metric spaces is dense in (X,d_X), the proof will be completed by finding a complete metric space Y and a Baire-1 map $f:X\to Y$ such that the continuity points of f satisfy the condition $|v|=1$ a.e.. Indeed, since X is nonempty, a residual set in X is also nonempty, proving the existence of the desired "wild solutions". Proving the density of a set by proving it contains the continuity points of a Baire-1 map (or the shared continuity points of a countable family of such maps) is what is known as a *Baire category argument*.

To make such an argument, we first note that any element $(v,u,q)\in X$ is an L^∞ map with compact support, and therefore is in $L^2(\mathbb{R}\times\mathbb{R}^n)$. If d^*_∞ is a metric on X inducing the weak-* topology of L^∞, then the identity map $I:(X,d^*_\infty)\to L^2$ is a Baire-1 map. Indeed, let φ be a convolution kernel in spacetime and $\varphi_r(x):=r^{-n}\varphi(r^{-1}x)$. Then, $I(f)=\lim_{r\to 0}\varphi_r(f)$ for any $f\in X\subseteq L^2$, and the map $\varphi_r:(X,d^*_\infty)\to(L^2,\|\cdot-\cdot\|_{L^2})$ is continuous for all $r>0$.

To conclude the argument, one proves that the identity map's continuity points must satisfy $|v|=1$ a.e. in Ω. Doing this requires the "perturbation property" of [21, Lemma 4.6], reported below.

Lemma 3.2.1.1. *There exists a dimensional constant $\kappa>0$ with the property that, given $(v_0,u_0,q_0)\in X_0$, there exists a sequence $(v_k,u_k,q_k)\in X_0$ such that*

$$\|v_k\|_{L^2(\Omega)}\geq\|v_0\|^2_{L^2(\Omega)}+\kappa(|\Omega|-\|v_0\|^2_{L^2(\Omega)})^2,$$

and $(v_k,u_k,q_k)\overset{*}{\to}(v_0,u_0,q_0)$ in $L^\infty(\Omega)$.

Assume now that $(v,u,q)\in X$ is such that $|v|$ is not 1 a.e. in Ω. Since X is the closure of X_0, there is a sequence $\{(v_k,u_k,q_k)\}\subset X_0$ approximating (v,u,q). By the above lemma and a standard diagonal argument, there exists $(\tilde{v}_k,\tilde{u}_k,\tilde{q}_k)$ a sequence in X_0 converging to (v,u,q) weakly-* in L^∞ but such that

$$\liminf_{k\to\infty}\|\tilde{v}_k\|^2_{L^2}\geq\liminf_{k\to\infty}\left(\|v_k\|^2_{L^2}+\kappa(|\Omega|-\|v_k\|^2_{L^2})^2\right).$$

If I were continuous at (v,u,q), one would have that both $v_k,\tilde{v}_k\to v$ in L^2, and thus

$$\|v\|_2^2\geq\|v\|_2^2+\kappa\left(|\Omega|-\|v\|_2^2\right)^2,$$

which would imply

$$\|v\|_2^2=|\Omega|.$$

Then again, $v=0$ a.e. outside Ω and $|v|\leq 1$ a.e., therefore $|v|=1$ a.e., which is a contradiction.

The perturbation property is proved, roughly speaking, by covering "a sufficiently large portion" of Ω with balls of small radius, applying [21, Proposition 2.2] on each ball, and summing all the perturbations constructed on those balls together with the starting

(v,u,q). By repeating the construction with smaller and smaller radii, we get a sequence (v_k,u_k,q_k) with the desired properties.

The above arguments lead to the proof of **Theorem 3.2.1.1**. This was a remarkable generalization of the result by Scheffer [61], namely the existence of nontrivial Euler solutions with compact support in space and time in two space dimensions. Indeed, Scheffer's result is a weaker version of the case $n=2$ of the above statement: the statement above yields solutions in L^∞, and thus in L^2 by the boundedness of their supports, whereas Scheffer's result had no control over the energy, meaning that the solutions which it provides may not be L^2.

3.2.2 Infinitely many admissible solutions

Theorem 3.2.1.1 left open the question of whether one might achieve the uniqueness of weak solutions by imposing a form of the energy inequality. This question was addressed in [22], where the following result ([22, Theorem 1.1]) was proved.

Theorem 3.2.2.1. *Let $n\geq 2$. There exist bounded and compactly supported divergence-free vector fields v^0 for which there are*

(a) *Infinitely many solutions of the Cauchy problem for* (2.1.1.1) *satisfying both the strong and the local energy inequalities;*

(b) *Weak solutions satisfying the strong energy inequality but not the energy equality;*

(c) *Weak solutions satisfying the weak energy inequality but not the strong energy inequality.*

In this statement:

- The weak energy inequality is our admissibility condition, i.e. $E_E(t)\leq E_E(0)$ for every $t>0$;
- The strong energy inequality is $E_E(t)\leq E_E(s)$ for all $s.t : t>s$;
- The energy equality is the conservation of E_E, i.e. $E_E(t)\equiv E_E(0)$;
- The local energy inequality reads

$$\int_0^\infty\int_{\mathbb{R}^n}|\nabla v|^2\varphi \mathrm{d}x\mathrm{d}t \leq \int_0^\infty\int_{\mathbb{R}^n}\frac{|v|^2}{2}(\partial_t\varphi+\nu\Delta\varphi)+\left(\frac{|v|^2}{2}+p\right)v\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t,$$

for any nonnegative $\varphi\in C_c^\infty((0,\infty)\times\mathbb{R}^n)$.

This theorem is obtained by proving the following proposition, and then constructing suitable triples to apply it to.

Proposition 3.2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set (not necessarily bounded) and let*

$$\overline{e} \in \mathcal{C}((0,T)\times\overline{\Omega}) \cap \mathcal{C}([0,T],L^1).$$

Assume there exists (v_0,u_0,q_0) smooth solution of (3.2.1.1) on $(0,T)\times\mathbb{R}^n$ with the following properties:

$$\begin{aligned} v_0 &\in \mathcal{C}([0,T],L^2_w) \\ \operatorname{supp}(v_0(t,\cdot),u_0(t,\cdot)) &\subset\subset \Omega \qquad \forall t\in(0,T) \\ e(v,u) := \frac{n}{2}\lambda_{max}(v_0(t,x)\otimes v_0(t,x) - u_0(t,x)) &< \overline{e}(t,x) \qquad \forall (t,x)\in\Omega\times(0,T), \end{aligned}$$

where λ_{max} denotes the maximum eigenvalue and L^2_w is the space L^2 endowed with the weak topology. Then there exist infinitely many weak solutions v of the Euler equations (2.1.1.1) in $[0,T)\times\mathbb{R}^n$ with pressure

$$p = q_0 - \frac{1}{n}|v|^2$$

such that

$$\begin{aligned} v &\in \mathcal{C}([0,T];L^2_w) \\ v(t,x) &= v_0(t,x) \qquad t\in\{0,T\},\ a.e.\ x\in\mathbb{R}^n \\ \frac{1}{2}|v(t,x)|^2 &= \overline{e}(t,x)\mathbb{1}_\Omega \qquad \forall t\in(0,T),\ a.e.\ x\in\mathbb{R}^n. \end{aligned}$$

The strategy to prove **Proposition 3.2.2.1** is similar to the one in the previous subsection: find a suitable complete metric space, and prove that the desired solutions are residual by using one or more Baire-1 maps whose continuity points are among those solutions.

In this case, the space of subsolutions will be the following:

$$X_0 := \left\{ (v,u,q)\in\mathcal{C}^\infty((0,T)\times\mathbb{R}^n)\cap\mathcal{C}^0([0,T];L^2_w) : \begin{array}{c} (v,u,q) \text{ solves } (3.2.1.1) \\ \operatorname{supp}(v,u)(t,\cdot)\subset\Omega \quad \forall t\in(0,T) \\ v(0,x) = v_0(0,x) \\ v(T,x) = v_0(T,x) \\ e(v,u) < \overline{e} \quad \forall (t,x)\in\Omega\times[0,T] \end{array} \right\}.$$

Also in this case, the role of the error term is played by $v\otimes v - {}^1/_n|v|^2\,\mathrm{Id} - u$.

Since we assumed $\overline{e}\in\mathcal{C}^0_t L^1_x$ and $\int_\Omega |v|^2 \mathrm{d}x \le \int_\Omega \overline{e}\mathrm{d}x$, the functions in X_0 take value in a bounded subset $B\subset L^2$, which can be metrized under the weak topology. Thus, $X_0\subset Y := \mathcal{C}^0([0,T],B)$, and since Y is a complete metric space under the uniform norm, one concludes that X, defined as the closure of X_0 in Y, is a complete metric space.

We then introduce a family of Baire-1 maps. For any $\varepsilon>0$ and bounded $\Omega_0\subset\Omega$, we define

$$I_{\varepsilon,\Omega_0}(v) := \inf_{t\in[\varepsilon,T-\varepsilon]} \int_{\Omega_0} \left[\frac{1}{2}|v(t,x)|^2 - \overline{e}(t,x)\right]\mathrm{d}x.$$

These are all Baire-1 maps on X, they take values in the non-positive real numbers, and, whenever v is such that $I_{\varepsilon,\Omega_0}(v) = 0$ for all ε, Ω_0, v is a weak solution of the Euler equations of the type provided by **Proposition 3.2.2.2**. It is therefore of interest to investigate the relation between the continuity points of these maps and their zeroes. The relation is provided by the perturbation property [22, Proposition 4.5] reported below.

Proposition 3.2.2.2. *Let Ω_0 and $\varepsilon > 0$ be given. For all $\alpha > 0$ there exists $\kappa = \kappa(\alpha, \Omega_0, \varepsilon) > 0$ such that, whenever $v \in X_0$ with $I_{\varepsilon,\Omega_0}(v) \leq -\alpha$, there exists a sequence $\{v_k\} \subset X_0$ with $v_k \to v$ in Y such that*

$$\liminf_{k\to\infty} I_{\varepsilon,\Omega_0}(v_k) \geq I_{\varepsilon,\Omega_0}(v) + \kappa.$$

In other words, whenever $I_{\varepsilon,\Omega_0}(v) < 0$, we can find a sequence converging to v in Y such that $I_{\varepsilon,\Omega_0}(v_k)$ stays above $I_{\varepsilon,\Omega_0}(v)$ by at least a certain positive quantity. Thus, in particular, if v is a continuity point for I_{ε,Ω_0}, then $I_{\varepsilon,\Omega_0}(v) = 0$.

To conclude the proof of **Proposition 3.2.2.1**, consider an exhausting sequence Ω_k for Ω, and the maps I_{k^{-1},Ω_k}. The set C of points where all of these are continuous is a countable intersection of residual sets, and is thus residual. Moreover, it is clear that, for $\varepsilon' < \varepsilon$ and $\Omega_0' \subset \Omega_0$, we have $I_{\varepsilon',\Omega_0'} \leq I_{\varepsilon,\Omega_0}$. Thus, for any ε, Ω_0, we have $I_{k^{-1},\Omega_k} \leq I_{\varepsilon,\Omega_0}$ whenever $k^{-1} < \varepsilon$ and $\Omega_0 \subset \Omega_k$, and since $k^{-1} \to 0, \Omega_k \uparrow \Omega$, we can always find such a K. If $v \in C$, it means $I_{k^{-1},\Omega_k}(v) = 0$ for all k, and therefore, for any ε, Ω, we can find a k such that $0 = I_{k^{-1},\Omega_k} \leq I_{\varepsilon,\Omega_0} \leq 0$, meaning C is formed by triples such that $I_{\varepsilon,\Omega_0}(v) = 0$ for all ε, Ω_0. Thus, the points of C provide the solutions from the statement of **Proposition 3.2.2.2**, completing its proof.

3.3 Analogy with a Nash theorem

By comparing the first equations of the two systems (3.2.1.1) and (3.1.1), we see that they are equivalent if we set

$$u = \overline{v} \otimes \overline{v} + R - \frac{1}{n}(|\overline{v}|^2 + \operatorname{tr} R)\,\mathrm{Id}\,.$$

Recalling the $e(v,u)$ introduced in the previous subsection, we note that

$$\overline{v} \otimes \overline{v} - u = -\mathring{R} + \frac{1}{n}\overline{|v|^2}\,\mathrm{Id} \leq \frac{2}{n}\overline{e}\,\mathrm{Id},$$

where $\overline{e} = {}^1/_2\overline{|v|^2}$. This is equivalent to $e(v,u) \leq \overline{e}$, whose strict form $e(v,u) < \overline{e}$ was one of the defining conditions of the triples in X_0 in the previous subsection. Moreover, we see that $e(v,u) = \overline{e}$ is equivalent to $R = 0$, and thus to v being a solution of the Euler equations.

This brings to light a striking analogy between **Proposition 3.2.2.2** and the famous Nash-Kuiper theorem of [56] and [47]. In order to state this theorem, we need to define short immersions.

Definition 3.3.1. *Let (Σ, g) be a Riemannian manifold. An immersion $v : \Sigma \to \mathbb{R}^N$ is* short *if it reduces the lengths of curves, i.e. $\ell(v * \gamma) \leq \ell_g(\gamma)$ for any curve γ. For $\mathcal{C}^1$ immersions and in local coordinates, this condition is equivalent to the inequality*

$$(\partial_i v \cdot \partial_j v) w^i w^j \leq g_{ij} w^i w^j \qquad \forall w \in T\Sigma.$$

Theorem 3.3.1 (Nash-Kuiper). *Let (Σ, g) be a smooth closed n-dimensional Riemannian manifold and $v : \Sigma \to \mathbb{R}^N$ a $\mathcal{C}^\infty$ short immension with $N \geq n+1$. Then, for any $\varepsilon > 0$ there exists a $\mathcal{C}^1$ isometric immersion $u : \Sigma \to \mathbb{R}^N$ such that $\|u - v\|_{\mathcal{C}^0} \leq \varepsilon$. If v is, in addition, an embedding, then u can be assumed to also be an embedding.*

Indeed, the isometric embedding problem can be formulated for the gradient $A \coloneqq \mathfrak{D}u$ of the immersion, and produces a linear PDE

$$\operatorname{curl} A = 0 \tag{3.3.1}$$

coupled with a nonlinear constraint

$$A^T A = g. \tag{3.3.2}$$

Short immersions satisfy the inequality

$$A^T A < g. \tag{3.3.3}$$

Nash's theorem is thus an analogue of the existence theorem for the Euler equations: if there exists a solution of (3.3.1) (resp. (3.2.1.1)) satisfying the strict inequality (3.3.3) (resp. $u - v \otimes v < n^{-1}|v|^2$), then there exist infinitely many solutions satisfying the equality (3.3.2) (resp. $u = v \otimes v + n^{-1}|v|^2$).

However, Nash's theorem has one additional property: the continuity of the gradients it provides. Indeed, the solutions proved to exist for the Euler equations are only L^∞_{loc}, not continuous. To have a "perfect analogue" of Nash's theorem, it would thus be desirable to show the existence of infinitely many continuous Euler solutions.

This was done by adopting an approach more similar to that of Nash, i.e. by iteratively adding highly oscillatory corrections to the "subsolutions" in order to absorb the error, i.e. the Reynolds stress, one bit at a time. Following [20, Section 7], we give an outline of the iteration scheme, which also shows what kind of Hölder regularity we may expect to get from this argument.

The aim is to construct a sequence of subsolutions of (3.1.1), i.e. triples (v_q, p_q, R_q) solving

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = -\operatorname{div} R_q \\ \operatorname{div} v_q = 0 \end{cases},$$

such that the error $R_q \geq 0$ is gradually removed. We note that, to measure the error from being an Euler solution, only the traceless part $\mathring{R}_q$ matters, since if we write

$$R_q = \rho_q \operatorname{Id} + \mathring{R}_q,$$

we have $\operatorname{div}(\rho_q \operatorname{Id}) = \nabla \rho_q$, and we can thus absorb ρ_q into the pressure term. This means that, if $\mathring{R}_q = 0$, v_q is an Euler solution, perhaps with a pressure different from p_q.

Recalling that we wish to prescribe the kinetic energy profile, we choose the trace ρ_q to be

$$\rho_q(t) \coloneqq \frac{1}{3(2\pi)^3} \left(E_{q+1}(t) - \frac{1}{2} \int_{\mathbb{T}^3} \left| v_q(t,x) \right|^2 \mathrm{d}x \right).$$

The aim is to construct a sequence $(v_q, p_q, \mathring{R}_q) \to (v, p, 0)$ uniformly. We will henceforth mostly focus on the velocity v. Such a sequence is achieved iteratively by adding suitable perturbations. We set

$$w_q \coloneqq v_q - v_{q-1}.$$

The size of w_q is controlled by two parameters: an amplitude $\delta_q^{1/2}$ and a frequency λ_q. Indeed, we will assume a bound on the uniform norm:

$$\|w_q\| \lesssim \delta_q^{\frac{1}{2}}.$$

We will choose perturbations that oscillate at frequency λ_q, hence obtaining a bound on the uniform norm of the gradient:

$$\|\nabla w_q\|_0 \lesssim \delta_q^{\frac{1}{2}} \lambda_q. \tag{3.3.4}$$

Naturally, we will let $\delta_q \to 0$. Since we expect that removing the error completely may require unbounded frequencies, we will let $\lambda_q \to \infty$. We will, in fact, require at least an exponential rate. For the upcoming exposition, for the sake of definiteness, we imagine

$$\lambda_q \coloneqq \lambda^q \qquad \delta_q \coloneqq \lambda_q^{-2\beta_0}, \tag{3.3.5}$$

for some $\lambda > 1$. The actual proofs (as well as the proof of this thesis's main result, which uses a very similar method) actually require super-exponential growths. By interpolation, we see that

$$\|v_q - v_{q-1}\|_\alpha = \|w_q\|_\alpha \lesssim \delta_q^{\frac{1}{2}} \lambda_q^\alpha \lesssim \lambda_q^{\alpha - \beta_0},$$

meaning $\{v_q\}$ is a Cauchy sequence in C^α for any $\alpha < \beta_0$.

The perturbations are meant to absorb the error R_q, so we would expect $R_q \sim w_{q+1} \otimes w_{q+1}$. We will see that this is indeed the case, and thus we have

$$\begin{aligned} \|\mathring{R}_q\|_0 &\leq c_0 \delta_{q+1} \\ \|\nabla \mathring{R}_q\|_0 &\lesssim \delta_{q+1} \lambda_q. \end{aligned} \tag{3.3.6}$$

Ideally, we would choose the main part of the perturbation w_{q+1} to satisfy an Ansatz of the type

$$w_0(t,x) = W(v_q(t,x), R_q(t,x), \lambda_{q+1} x, \lambda_{q+1} t),$$

where the "profile" $W = W(v, R, \xi, \tau)$ is a suitable function to be specified later. The pressure p_{q+1} will be defined similarly, but we omit the details.

Since the perturbation must be oscillatory, we require that W be periodic in $\xi \in \mathbb{T}^3$. We then observe that we need $\operatorname{div} v_{q+1} = 0$, and since v_q is divergence-free, w_{q+1} is also required to be. However, w_o as defined above is unlikely to be divergence-free, so we will need to add a suitable correction w_c such that $\operatorname{div}(w_o + w_c) = 0$. To this end, consider a vector potential for v_q, i.e. a smooth z_q such that $\operatorname{curl} z_q = v_q$. We would like to perturb z_q in a similar way:

$$z_{q+1}(t,x) = z_q(t,x) + \frac{1}{\lambda_{q+1}} Z(v(t,x), R(t,x), \lambda_{q+1} x, \lambda_{q+1} t).$$

Computing v_{q+1}, we obtain

$$v_{q+1}(t,x) = \operatorname{curl} z_{q+1}(t,x) = v_q(t,x) + \operatorname{curl}_\xi Z(v(t,x),R(t,x),\lambda_{q+1}x,\lambda_{q+1}t) + O\left(\frac{1}{\lambda_{q+1}}\right).$$

The second term would ideally be w_o. We thus need to find a ξ-periodic potential Z for W, which implies $\operatorname{div}_\xi W = 0$ and $\langle W \rangle = 0$, i.e. W is average-free in the ξ variable.

Similar considerations, as illustrated for instance in [67], lead to the following conditions for W:

(H1) $\xi \mapsto W(v,R,\xi,\tau)$ is 2π-periodic with vanishing average;

(H2) The average stress is R, i.e.
$$\langle W \otimes W \rangle = R;$$

(H3) The "cell problem" is satisfied:
$$\begin{cases} \partial_\tau W + (v \cdot \nabla_\xi)W + \operatorname{div}_\xi(W \otimes W) + \nabla_\xi P = 0 \\ \operatorname{div}_\xi W = 0 \end{cases},$$
where $P = P(v,R,\xi,\tau)$ is a suitable pressure;

(H4) W is smooth in all its variables and satisfies the estimates
$$|W| \lesssim |R|^{\frac{1}{2}} \quad |\partial_v W| \lesssim |R|^{\frac{1}{2}} \quad |\partial_R W| \lesssim |R|^{-\frac{1}{2}}.$$

When computing $|v_{q+1}|^2$, we see that we have

$$\left|v_{q+1}\right|^2 = \left|v_q\right|^2 + 2v_q \cdot w_o + |w_o|^2 + O\left(\frac{1}{\lambda_{q+1}}\right),$$

since $w_o = W(v_q,R_q,\lambda_{q+1}x,\lambda_{q+1}t) = \operatorname{curl}_\xi Z(v_q,R_q,\lambda_{q+1}x,\lambda_{q+1}t)$. If we then integrate, the second term also becomes $O(\lambda_{q+1}^{-1})$, since w_{q+1} is fast-oscillating and v_q is slow-oscillating. This, combined with (H1)-(H2) above, implies that

$$\int_{\mathbb{T}^3} \left|v_{q+1}\right|^2 \mathrm{d}x \sim \int_{\mathbb{T}^3} \left|v_q\right|^2 + \left|W(v_q,R_q,\lambda_{q+1}x,\lambda_{q+1}t)\right|^2 \mathrm{d}x \sim \int_{\mathbb{T}^3} \left|v_q\right|^2 \mathrm{d}x + 3(2\pi)^3 \rho_q(t),$$

thus the total kinetic energy of v_{q+1} is, up to small errors, e_{q+1}.

Having defined the pair (v_{q+1}, p_{q+1}), we must find a suitable Reynolds tensor $\mathring{R}_{q+1}$. An important remark is that one can select a good "antidivergence operator" solving $\operatorname{div} \mathring{R} = f$, as stated in the following technical lemma.

Lemma 3.3.1 (Antidivergence). *There exists a homogeneous Fourier-multiplier operator of order* -1*, denoted*
$$\operatorname{div}^{-1} : C^\infty(\mathbb{T}^3,\mathbb{R}^3) \to C^\infty(\mathbb{T}^3,\mathcal{S}_0^{3\times 3})$$
such that, for any $f \in C^\infty(\mathbb{T}^3,\mathbb{R}^3)$ *with zero average* $⨍_{\mathbb{T}^3} f = 0$*, we have:*

(a) $\operatorname{div}^{-1} f(x)$ *is a symmetric trace-free matrix for any* $x \in \mathbb{T}^3$*;*

(b) $\operatorname{div}\operatorname{div}^{-1} = f$.

With this operator div^{-1}, assuming the existence of an ideal profile W, the next stress tensor $\mathring{R}_{q+1}$ is defined as

$$\begin{aligned}
\mathring{R}_{q+1} &:= -\operatorname{div}^{-1}[\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1}] \\
&= -\operatorname{div}^{-1}\left[\partial_t w_{q+1} + (v_q \cdot \nabla) w_{q+1}\right] \\
&\quad -\operatorname{div}^{-1}\left[\operatorname{div}(w_{q+1} \otimes w_{q+1} - \mathring{R}_q) + \nabla(p_{q+1} - p_q)\right] \\
&\quad -\operatorname{div}^{-1}[(w_{q+1} \cdot \nabla) v_q] \\
&=: \mathring{R}^{(1)}_{q+1} + \mathring{R}^{(2)}_{q+1} + \mathring{R}^{(3)}_{q+1}.
\end{aligned}$$

Since we are assuming that the size of the corrector w_c is negligible compared to w_o, we will discuss the corresponding terms where w_o replaces w_{q+1}.

The main issues we have are thus:

- To show that indeed it is possible to send $\delta_q \to 0$ as $q \uparrow \infty$ (so that the scheme converges);

- To obtain a relation between δ_q and λ_q in the form of (3.3.5).

If we were able to find a profile W satisfying (H1)-(H2)-(H3)-(H4), the above iteration would lead to a proof of the Onsager conjecture. To see this, we first expand W as a Fourier series in ξ. We then compute

$$\mathring{R}^{(3)}_{q+1} = -\operatorname{div}^{-1}[(w_0 \cdot \nabla) v_q] = \operatorname{div}^{-1} \sum_{\substack{k \in \mathbb{Z}^3 \\ k \neq 0}} c_k(t,x) e^{i\lambda_{q+1} k \cdot x},$$

where the coefficients c_k vary much slower than the rapidly oscillating exponentials. Applying the antidivergence operator, we can therefore treat the c_k as constants and gain a factor λ_{q+1}^{-1} in the outcome: a typical "stationary phase argument". Note that it is crucial that c_0 vanishes, which is the content of (H1).

Using (H4), we can estimate the size of each term c_k:

$$\|c_k\|_0 \lesssim \|W\|_0 \|\nabla v_q\|_0 \lesssim \|R_q\|_0^{\frac{1}{2}} \|\nabla v_q\|_0.$$

Applying (3.3.4) and (3.3.6), we arrive at

$$\left\|\mathring{R}^{(3)}_{q+1}\right\|_0 \lesssim \frac{\delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}}.$$

In fact in our computations so far we are ignoring a lot of technical issues: the relevant estimates are much more complicated and affected by several other terms which we are neglecting.

Similar arguments for the two other error tensors $\mathring{R}^{(1)}_{q+1}, \mathring{R}^{(2)}_{q+1}$ lead to an estimate like the one below:

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q}{\lambda_{q+1}}.$$

This is just one of the estimates for $(v_{q+1}, p_{q+1}, R_{q+1})$, and similar ones should be obtained for all the other quantities (and for other norms). However, this estimate already implies a relation between δ_q, λ_q. Indeed, comparing it with (3.3.6), the inductive step requires

$$\delta_{q+2} \sim \frac{\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q}{\lambda_{q+1}}.$$

Assuming $\lambda_q \sim \lambda^q$ for some fixed $\lambda \gg 1$, this yields

$$\delta_q^{\frac{1}{2}} \sim \lambda^{-\frac{q}{3}} \sim \lambda_q^{-\frac{1}{3}},$$

which gives $\beta_0 = {}^1\!/_3$ as the critical Hölder regularity.

3.4 Beltrami flows and the first Hölder regularity

To start the search for a suitable profile W, we consider first the case where we set $v = 0$. Since we wish to fulfil (H1), we will assume that

$$W(0,R,\xi,\tau) = W_s(R,\xi) := \sum_k a_k(R) B_k e^{ik\cdot\xi},$$

for some directions B_k, requiring that $a_0 = 0$. Since v is constant, we also eliminate the dependence on the time τ. This entails that (H3) reduces to

$$\begin{cases} \operatorname{div}_\xi(W_s \otimes W_s) + \nabla_\xi P = 0 \\ \operatorname{div}_\xi W_s = 0 \end{cases},$$

and condition (H4) reduces to $|a_k| \lesssim |R|^{1/2}$ and $|\mathfrak{D} a_k| \lesssim |R|^{-1/2}$.

Condition (H2) reads as follows:

$$\sum_{k,k'} \fint_{\mathbb{T}^3} a_k(R) a_{k'}(R) B_k \otimes B_{k'} e^{i(k+k')\cdot\xi} \,\mathrm{d}\xi = R.$$

Naturally, only for $k' = -k$ do those integrals not vanish. Since we will choose real-valued a_k, and the B_k from **Proposition 3.4.1** below satisfy $\overline{B}_k = B_{-k}$, we thus need

$$\sum_k \int_{\mathbb{T}^3} |a_k(R)|^2 (R) B_k \otimes B_k = R.$$

This suggests that, in choosing the set Λ of indices k to sum over, we will want to ensure that $-\Lambda \subseteq \Lambda$.

Condition (H3) tells us we are looking for stationary solutions of the Euler equations. It seems thus natural to recall a well-known class of such solutions, namely the Beltrami flows. These are summarized in [24, Proposition 3.1] reported below.

Proposition 3.4.1 (Beltrami flows). *Let $\lambda_0 \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that*

$$A_k \cdot k = 0 \quad |A_k| = \frac{1}{\sqrt{2}} \quad A_{-k} = A_k$$

for $k \in \mathbb{Z}^3$ with $|k| = \lambda_0$. Furthermore, let

$$B_k := A_k + i\frac{k}{|k|} \times A_k \in \mathbb{C}^3.$$

For any choice of a_k with $\overline{a_k} = a_{-k}$ the vector field

$$W(\xi) = \sum_{|k|=\lambda_0} a_k B_k e^{ik\cdot\xi}$$

is divergence-free and satisfies

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}.$$

Furthermore

$$\langle W \otimes W \rangle = \fint_{\mathbb{T}^3} W \otimes W \mathrm{d}\xi = \frac{1}{2} \sum_{|k|=\lambda_0} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right).$$

The choice of the set (in fact, any number of suitable sets) of indices k is given by [24, Lemma 3.2], reported below.

Lemma 3.4.1. *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\lambda_0 > 1$ with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\} \qquad j \in \{1, \dots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id})) \qquad j \in \{1, \dots, N\}, k \in \Lambda_j$$

such that:

(a) *$k \in \Lambda$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;*

(b) *For each $R \in B_{r_0}(\mathrm{Id})$ we have the identity*

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(R) \right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right) \qquad \forall R \in B_{r_0}(\mathrm{Id}).$$

We therefore have the following profiles for $v = 0$:

$$W_s(R, \xi) = \sum_{k \in \Lambda_j} a_k(R) B_k e^{ik\cdot\xi},$$

for any j, where the Λ_j are as prescribed in the above lemma.

The natural extension to nonzero v would then be to choose the solution W to

$$\partial_\tau W + v \cdot \nabla_\xi W = 0$$

with $W|_{\tau=0} = W_s$, leading to the formula

$$W(v,R,\xi,\tau) = W_s(R,\xi - v\tau) = \sum_{k\in\Lambda_j} a_k(R) B_k e^{i(k-v\tau)\cdot\xi}.$$

However, this fails to satisfy (H4), since

$$|\partial_v W(v,R,\xi,\tau)| \sim |R|^{\frac{1}{2}}|\tau|,$$

which is estimated as required by (H4) only for bounded τ. This is a serious problem, since we will eventually set $\tau = \lambda t$, leading to an additional factor λ, which prevents the iterative scheme from converging.

To overcome this, paper [25] introduces a "phase function" to deal with the transport part of the cell problem. In other words, the chosen profile is

$$W(v,t,\xi,\tau) = \sum_{|k|=\lambda_0} a_k(R)\varphi_k(v,\tau) B_k e^{ik\cdot\xi}.$$

With this Ansatz, condition (H3) gives us

$$\partial_\tau \varphi_k + i(v\cdot k)\varphi_k = 0.$$

The exact solution is $\varphi_k(v,\tau) = e^{-i(v\cdot k)\tau}$, but as seen above it is incompatible with (H4), so (H3) was required to hold only approximately:

$$\partial_\tau \varphi_k + i(v\cdot k)\varphi_k = O(\mu^{-1}) \qquad |\partial_v \varphi_k| \lesssim \mu,$$

for some new parameter μ.

More specifically, we choose a suitable cutoff function $\varphi \in C_c^\infty(B_1(0))$ which is identically one on $B_{\sqrt{3}/2}$, and define:

$$\varphi_k^{(j)}(v,\tau) \coloneqq \sum_{\ell\in\mathcal{C}_j} \alpha_\ell(\mu v) e^{-i(k\cdot\frac{\ell}{\mu})\tau},$$

where

$$\alpha_\ell(x) \coloneqq \frac{\varphi(x-\ell)}{\sqrt{\psi(x)}} \qquad \psi(x) = \sum_{k\in\mathbb{Z}^3} \phi_k^2(v) \qquad \phi_k(x) \coloneqq \varphi(x-k),$$

and the $\mathcal{C}_j$ are the equivalence classes of $\mathbb{Z}^3$ modulo $(2\mathbb{Z})^3$, i.e. with respect to the relation $k \sim \ell \iff k-\ell \in (2\mathbb{Z})^3$. Since ψ can be seen to be bounded away from 0, bounded, and smooth, the functions α_k are smooth and bounded, and $\sum_k \alpha_k^2 = 1$. Since $\alpha_\ell, \alpha_{\ell'}$ have disjoint supports for $\mathcal{C}_j \ni \ell \neq \ell' \in \mathcal{C}_j$, this means that $\{|\varphi_k^{(j)}|^2\}_{j,k}$ is a partition of unity. This fact means that, when we compute $\langle W \otimes W\rangle$, the $|\varphi_k^{(j)}|^2$ will sum to 1 and not affect (H2).

It is simple enough to see that, for any $m = 0, 1, 2, \ldots$

$$\sup_{v,\tau} \left| D_v^m \varphi_k^{(j)}(v,\tau) \right| \lesssim \mu^m.$$

Next, fix any $(v,\tau), j$. There is at most one $\ell \in \mathcal{C}_j$ such that $\alpha_\ell(\mu v) \neq 0$ and it satisfies $|\mu v - \ell| < 1$. Thus, in a neighborhood of (v,τ), we have

$$\partial_\tau \varphi_k^{(j)} + i(k \cdot v)\varphi_k^{(j)} = ik \cdot \left(v - \frac{\ell}{\mu} \right) \varphi_k^{(j)}.$$

Given $|\mu v - \ell| < 1$, we conclude $|v - \ell \mu^{-1}| < \mu^{-1}$, so that

$$\left\| D_v^m (\partial_\tau \varphi_k^{(j)} + i(k \cdot v)\varphi_k^{(j)}) \right\|_{\mathcal{C}^0_{v,\tau}} \lesssim |k| \mu^{m-1}.$$

Thus, the approximate version of (H3) is satisfied. This means that $|\partial_v W| \lesssim \mu |R|^{1/2}$ holds, which replaces the second estimate in (H4).

With the above, the paper [25] provides the first examples of Hölder solutions of the Euler equations with prescribed energy. More precisely, the authors show the following theorem.

Theorem 3.4.1. *Given any positive smooth function E on $[0,T]$ and any $\beta < 1/10$, there is a pair $(v,p) : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}$ of $\mathcal{C}^\beta$ functions which solves the Euler equations* (2.1.1.1) *in the distributional sense and satisfies $1/2 \int_{\mathbb{T}^3} |v|^2(t,x) \mathrm{d}x = E(t)$.*

3.5 A new profile and 1/5-Hölder regularity

A new Ansatz for the profile was introduced in [38], and the scheme was refined in [6] to produce not only nontrivial compactly supported flows, but flows with prescribed energy. This new Ansatz was the following:

$$w_o(t,x) = W_s(R_q(t,x), \lambda_{q+!}\Phi_q(t,x)) = \sum_{k \in \Lambda} a_k(R_q(t,x)) B_k e^{i\lambda_{q+1}\Phi_q(t,x)},$$

where Φ_q solves the transport equation

$$(\partial_t + v_q \cdot \nabla)\Phi_q = 0.$$

One term in the new Reynolds stress would then be

$$S := \operatorname{div}^{-1}[(\partial_t + v_q \cdot \nabla)(w_0 + w_c)] = \sum_{k \in \Lambda} \nabla a_k(R_q)(\partial_t + v_q \cdot \nabla) R_q e^{i\lambda_{q+1}\Phi_q}.$$

If $D\Phi_q$ is not too far from the identity, using the Stationary Phase Lemma (**Lemma C.3**), we would conclude that

$$\|S\|_0 \lesssim \frac{\delta_{q+1}^{\frac{3}{2}} \delta_q^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}}.$$

In fact, the above estimate also requires the assumption, justified by Isett in [38], that the advective derivative $(\partial_t + v_q \cdot \nabla)R_q$ satisfies a better bound than the regular derivative $\mathfrak{D}R_q$, a property that also holds for v_q.

However, $\|\mathfrak{D}v\|_0 \to \infty$, which would lead us to expect $\mathfrak{D}\Phi_q$ to be controllable only for short times. More precisely, by a well-known elementary estimate on ODEs, if $\Phi_q(t,x_0) = x$, then

$$\left\|\mathfrak{D}\Phi_q(t,\cdot) - \mathrm{Id}\right\|_0 \lesssim \left\|\nabla v_q\right\|_0 |t-t_0| \lesssim \delta_q^{\frac{1}{2}}\lambda_q|t-t_0|$$

for $|t-t_0| \lesssim (\delta_q^{1/2}\lambda_q)^{-1}$.

The strategy employed in [38, 6] to handle this problem was to consider a partition of unity $\{\chi_j\}_j$ on the time interval $[0,T]$ such that supp χ_j is an interval I_j of size $|I_j| = \mu_q^{-1}$ for some large parameter μ_q. In each such interval, we consider the solution $\Phi_{q,j}$ of the transport equation above with initial condition

$$\Phi_{q,j}(t,x_j) = x,$$

where t_j is the center of I_j. Recalling that $\|\mathfrak{D}v_q\|_0 \lesssim \delta_q^{1/2}\lambda_q$, the above estimate on $\mathfrak{D}\Phi_q - \mathrm{Id}$ leads to

$$\left\|\mathfrak{D}\Phi_{q,j}\right\|_0 = O(1) \qquad \text{and} \qquad \left\|\mathfrak{D}\Phi_{q,j} - \mathrm{Id}\right\|_0 \lesssim \frac{\delta_q^{\frac{1}{2}}\lambda_q}{\mu_q},$$

provided we assume, as we henceforth will, that

$$\mu_q \geq \delta_q^{\frac{1}{2}}\lambda_q.$$

Note also that $|\partial_t \chi_j| \lesssim \mu_q$.

Our new principal perturbation will be

$$w_o = \sum_j \chi_j(t) \sum_{k\in\Lambda^{i(j)}} a_k(R_q)B_k e^{i\lambda_{q+1}k\cdot\Phi_{q,j}},$$

where $i(j)$ is 1 if j is odd and 2 if j is even, and $\Lambda^{(i)}$ are two disjoint families as in the Lemma above. This new Ansatz yields the following estimate:

$$\left\|\mathring{R}_{q+1}\right\|_0 \lesssim \delta_{q+1}^{\frac{1}{2}}\mu_q\lambda_{q+1}^{-1} + \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\mu_q^{-1}.$$

The aim is to bound this with δ_{q+2}. Let $\tilde{\mu}_q$ be the choice making the two terms equal, and let any other choice be $\tilde{\mu}_q\hat{\mu}_q$. If $\hat{\mu}_q > 1$, then the first term will be larger than it would be with the choice $\tilde{\mu}_q$. If $\hat{\mu}_q < 1$, the second term would be in the same situation. Thus, the optimal choice for μ_q is the one making the two terms equal:

$$\delta_{q+1}^{\frac{1}{2}}\mu_q\lambda_{q+1}^{-1} = \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\mu_q^{-1} \iff \mu_q^2 = \delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q\lambda_{q+1} \iff \mu_q = \delta_{q+1}^{\frac{1}{4}}\delta_q^{\frac{1}{4}}\lambda_q^{\frac{1}{2}}\lambda_{q+1}^{\frac{1}{2}}.$$

With this choice, the estimate becomes

$$\left\|\mathring{R}_{q+1}\right\|_0 \lesssim \delta_{q+1}^{\frac{3}{4}}\delta_q^{\frac{1}{4}}\lambda_q^{\frac{1}{2}}\lambda_{q+1}^{-\frac{1}{2}},$$

which gives us the following relation:

$$\delta_{q+2} \gtrsim \delta_{q+1}^{\frac{3}{4}} \delta_q^{\frac{1}{4}} \lambda_q^{\frac{1}{2}} \lambda_{q+1}^{-\frac{1}{2}}.$$

With our assumption that $\lambda_q = \lambda^q, \delta_q = \lambda_q^{-2\beta_0}$, this means that

$$\lambda^{-2\beta_0(q+2)} \gtrsim \lambda^{-\frac{3}{2}\beta_0(q+1)-\frac{\beta_0}{2}q+\frac{q}{2}-\frac{q+1}{2}} = \lambda^{-2\beta_0 q-\frac{3}{2}\beta_0-\frac{1}{2}},$$

that is

$$-2\beta_0 q - \frac{3}{2}\beta_0 - \frac{1}{2} < -2\beta_0(q+2) \iff \beta_0 < \frac{1}{5},$$

leading to the following theorem.

Theorem 3.5.1. *Given any positive smooth function $e : [0,T] \to \mathbb{R}^+$ and any $\alpha < 1/5$, there is a pair $(v,p) : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}$ of C^α functions which solves the Euler equations in the distributional sense and satisfies $1/2 \int_{\mathbb{T}^3} |v|^2(t,x)\mathrm{d}x = e(t)$.*

3.6 Mikado flows and Onsager-critical regularity

In [4], Buckmaster observed that, by a clever choice of the cut-off functions χ_i from the previous section, it is possible to show that the solution produced in proving the previous theorem is $C^{1/3-\varepsilon}$ at almost every time slice. The idea is to make the cut-offs flat on large portions of their supports, paying the price of a very steep time derivative on the remaining small portions. This causes the global Hölder regularity to be much weaker, making the solution only C^η for some very small $\eta(\varepsilon)$.

In [7], Buckmaster, De Lellis, and Székelyhidi exploited a quantitative version of this idea to reach the first Onsager-critically regular non-conservative flows, proving the existence of nontrivial $L^1_t C_x^{1/3-\varepsilon}$ continuous compactly supported solutions, i.e. continuous pairs $(v,p) : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}$ satisfying

$$|v(t,x) - v(y,t)| \leq C(t)|x-y|,$$

for every $t \in \mathbb{R}, x, y \in \mathbb{T}^3$, and for some L^1 function $C : \mathbb{R} \to \mathbb{R}^+$.

To reach global $C^0_t C^{1/3-\varepsilon}$ regularity and fully prove the Onsager conjecture, better estimates for the various error terms were needed. These were reached with a key ingredient introduced in [19]: Mikado flows. These are a family of stationary flows whose existence is given by the following lemma.

Lemma 3.6.1 (Mikado flows). *For any compact subset $\mathcal{N}$ consisting of positive definite 3×3 matrices, there exists a smooth vector field $W_s : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3$ such that, for every $R \in \mathcal{N}$, we have*

$$\begin{cases} \operatorname{div}_\xi (W_s(R,\xi) \otimes W_s(R,\xi)) = 0 \\ \operatorname{div}_\xi W_s(R,\xi) = 0 \end{cases},$$

and

$$\langle W_s \rangle_\xi = R$$
$$\langle W_s \otimes W_s \rangle_\xi = R.$$

While these flows do indeed improve the estimates on several error terms, they are incompatible with the "patching strategy" described in the previous section. This is because, while single Mikado flows give better control, there appears to be no way of controlling the interference terms where distinct Mikado flows interact with each other. Indeed, the work [19] only used them for one perturbation step, an initial perturbation used to constrain the Reynolds stress in an opportune cone of tensors, and then used Beltrami flows for the rest of its iterations.

This problem was overcome by Isett in [40], where he introduced the final key ingredient: the "gluing argument". This consists in first modifying the subsolution (v_q, p_q, R_q) to a new $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$, satisfying essentially the same estimates, but such that the Reynolds stress is identically zero on several disjoint "stripes". This is obtained by partitioning the time interval $[0,T]$ as before, but finding exact smooth solutions of the Euler equations in each of these intervals (which need to be small enough so that the solution is guaranteed to exist) whose initial data coincide with the time-slices of v_q, and then gluing those together, to obtain $\sum_i \chi_i v_i =: \overline{v}_q$.

With that, there will be stripes where $\overline{v}_q$ coincides with one of the exact solutions, implying $\overline{R}_q = 0$. This means that we can use one Mikado flow for each of the remaining time regions, and those Mikado flows will have disjoint support in time and thus never interact.

With this strategy, Isett was able to prove the following theorem.

Theorem 3.6.1. *For every $\beta < {}^1/_3$ there is a nontrivial continuous compactly supported solution $(v,p) \in C^\beta(\mathbb{R} \times \mathbb{T}^3)$ of the Euler equations* (2.1.1.1).

The reason Isett did not find such solutions with arbitrary prescribed kinetic energy profile was that, not being able to obtain good enough estimate with the usual definition of $\overline{R}_q$, he had to generate this Reynolds stress by a different and more complicated strategy. The satisfactory estimate were then obtained in [8], where the following theorem was proved.

Theorem 3.6.2. *For every $\beta < {}^1/_3$ and every positive smooth $E : [0,T] \to \mathbb{R}$, there exists a solution $(v,p) \in C^\beta([0,T] \times \mathbb{T}^3)$ of the Euler equations such that*

$$\frac{1}{2} \int_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x = E(t).$$

In fact, the use of the Mikado flows allowed the authors of [8] to prove a stronger statement, of which the previous theorem is a corollary. This stronger statement is the below *h*-principle, the true analogue of the Nash-Kuiper theorem.

Theorem 3.6.3. *Let $(\tilde{v}, \tilde{p}, \tilde{R})$ be a smooth solution of* (3.1.1) *on $[0,T] \times \mathbb{T}^3$ such that $\tilde{R}(t,x)$ is positive definite for all t,x. Then for any $\alpha < {}^1/_3$ there exists a sequence $\{(v_k, p_k)\} \subset C^\alpha$ of weak solutions of the Euler equations such that*

$$v_k \overset{*}{\rightharpoonup} \tilde{v} \qquad \text{and} \qquad v_k \otimes v_k \overset{*}{\rightharpoonup} \tilde{v} \otimes \tilde{v} + \tilde{R} \qquad \text{in } L^\infty$$

uniformly in time, and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} v_k \otimes v_k \mathrm{d}x = \int_{\mathbb{T}^3} (\tilde{v} \otimes \tilde{v} + \tilde{R}) \mathrm{d}x.$$

3.7 Wild initial data and their "quantity"

While the above results were being proved, a parallel trilogy of papers was published in [17], [19], and [18], namely Daneri, Daneri-Székelyhidi, and Daneri-Runa-Széhelyhidi. Their concern was to measure the "quantity" of what they called "wild initial data", i.e. those initial data which generated infinitely many admissible solutions of the Euler equations. The main theorems of these papers are reported below.

Theorem 3.7.1 (Daneri). *Let $e : [0,t] \to \mathbb{R}$ be a positive smooth function. Then, for any $\beta < {}^1/_{16}$ there exist infinitely many $v_0 \in C^{0,\beta}(\mathbb{T}^3;\mathbb{R}^3)$ satisfying $e(0) = \int_{\mathbb{T}^3} |v_0|^2$ and each being the initial datum of infinitely many $(v,p) \in C^0([0,1] \times \mathbb{T}^3)$ solving* (2.1.1.1) *and satisfying*

$$\left|v(t,x) - v(x',t)\right| \leq C\left|x - x'\right|^\beta \qquad \forall x,x' \in \mathbb{T}^3, t \in [0,T]$$

$$\int\limits_{\mathbb{T}^3} |v(t,x)|^2 \mathrm{d}x = e(t) \qquad \forall t \in [0,T].$$

Theorem 3.7.2 (Daneri-Székelyhidi). *For any $\beta < {}^1/_5$ the set of divergence-free vector fields $v_0 \in C^\beta(\mathbb{T}^3;\mathbb{R}^3)$ which are wild initial data in C^β (i.e. which generate infinitely many admissible C^β solutions) is a dense subset of the divergence-free vector fields in $L^2(\mathbb{T}^3;\mathbb{R}^3)$.*

Theorem 3.7.3 (Daneri-Runa-Székelyhidi). *For any $0 < \beta < {}^1/_3$, the set of divergence-free vector fields $v_0 \in C^\beta(\mathbb{T}^3;\mathbb{R}^3)$ which are wild initial data in C^β (i.e. which generate infinitely many admissible C^β solutions) is a dense subset of the divergence-free vector fields in $L^2(\mathbb{T}^3;\mathbb{R}^3)$.*

The main ingredients towards these results, aside from the innovations described in the previous sections, were the localization of the perturbations, and a double convex integration scheme. More precisely, the idea was the following:

- First, perform a convex integration scheme where the perturbation is localized in time near $t = 0$;
- Arrive at a subsolution which is a solution (i.e. has zero Reynolds stress) at $t = 0$, and satisfies suitable additional properties required for a second convex integration: a so-called "C^β-adapted subsolution";
- Finally, obtain a solution by performing another convex integration scheme where the perturbation is localized in time away from $t = 0$, thus leaving the initial datum untouched.

Since all convex integration schemes provide a sequence of solutions approximating the starting point in some sense (in this case, in the C^β norm), this led to the conclusion that the initial data of adapted subsolutions are automatically wild, assuming they satisfy an appropriate "admissibility condition" (cfr. [18, Corollary 3.1]).

The techniques used in these papers form a good parallel with those of [25], [6], and [8], with a few things to be noted aside from the localization discussed above. Two things immediately hit the eye when looking at the statements:

- Firstly, the fact that, from [19] onwards, the prescription of an arbitrary kinetic energy profile is abandoned; however, that does not mean the energy is not prescribed; indeed, what happens in [19] and [18] is the conservation of the generalized energy of the subsolutions, namely $\int_{\mathbb{T}^3} |v|^2(t,x) + \operatorname{tr} R(t,x) \mathrm{d}x$, across the whole iterations;

  Secondly, while the papers [19] and [18] reach the same thresholds as [6] and [8], paper [17] does not reach $1/_{10}$ as in [25], but the smaller $1/_{16}$.

Concerning the second one, looking at the proofs in [17], we see that the first iteration does reach the $1/_{10}$ threshold, and it is in passing from adapted subsolutions (which they call admissible subsolutions) to weak solutions with the second iteration that the threshold is lowered. The paper itself notes how the method actually gives a threshold of $3/_{47}$, which is slightly larger. A similar loss of regularity in the second iteration occurs in [18, Proposition 3.2], but it can be made arbitrarily small.
The last remark we wish to make is that, from [19], a second intermediate step (besides the one of adapted subsolutions mentioned above) is introduced: strong subsolutions. Essentially, these are subsolutions where the C^0 norm of the traceless part of the Reynolds stress is controlled by the trace. The introduction of such a control is motivated by the construction in [12], where the first h-principle for the Euler equations was proved, which is reported below.

Theorem 3.7.4 (Choffrut h-principle). *Assume $d = 2$ or $d = 3$. Let $e : [0,T] \to \mathbb{R}^+$ be a smooth positive profile. Let (u, π, S) be a strong subsolution, i.e. a subsolution such that*

$$\frac{e - \fint_{\mathbb{T}^d} |v(t,x)|^2 \mathrm{d}x}{d} \operatorname{Id} - \mathring{R}(x,t) > 0 \qquad x \in \mathbb{T}^d, t \in [0,T].$$

Let $0 < \beta < \frac{1}{10}$ and $\sigma > 0$. Then:

(1) *There exists a vector field $v \in C^0_t C^\beta_x([0,T] \times \mathbb{T}^d)$ and a function $p \in C^0([0,T] \times \mathbb{T}^d)$ which solve the Euler equations (2.1.1.1) in the weak sense, and such that*

$$\sup_{t\in[0,T]} \|v(t,\cdot) - u(t,\cdot)\|_{H^{-1}(\mathbb{T}^d)} < \sigma;$$

(2) *The solution can be constructed so that, for all $t \in [0,T]$,*

$$\left| \fint_{\mathbb{T}^3} \big(v(t,x) \otimes v(t,x) - u(t,x) \otimes u(t,x) + \mathring{R}(t,x) \big) \mathrm{d}x - \frac{e(t) - \fint_{\mathbb{T}^d} |u(t,x)|^2 \mathrm{d}x}{d} \right| < \sigma.$$

Starting from here, the paper [19] introduced the Mikado flows mentioned in the previous subsection to obtain a satisfactory passage from strict to strong subsolutions. The paper [12] was missing not only this step, but also an existence result for strict or strong subsolutions relative to arbitrary profiles. In the absence of such a result, the authors of [19] were able to obtain the admissibility of their solutions by imposing the conservation the generalized energy of the subsolutions, as pointed out above.

3.8 Adaptations to hypodissipative Navier-Stokes

To the best of our knowledge, two papers exist which adapt the convex integration technique to the hypodissipative Navier-Stokes equations (specifically $\theta < {}^1/_2$): [15] and [26].

In [15], **Theorem 2.4.2.4** is proved. The technique employed therein is an adaptation to the hypodissipative Navier-Stokes equations of the one used in [6] for the Euler equations in the case of Hölder exponent $\beta < {}^1/_5$. This yields the existence of solutions with arbitrary kinetic energy profiles up to $\theta < {}^1/_2$, and the existence of wild initial data for $\theta < {}^1/_5$. This adaptation features the following new elements with respect to the strategy of [6]:

- A new term in the definition of the new Reynolds stresses, added to account for the laplacian term in the equation, whose relaxed form is the Navier-Stokes-Reynolds system:
$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\theta v = -\operatorname{div} R \\ \operatorname{div} v = 0 \end{cases};$$
- Estimates on the fractional laplacian to estimate both the aforementioned term and the dissipation term in the admissibility condition;
- A control of the $\mathcal{C}^\beta$ norm, for $\beta = \theta + \varepsilon$ for some small ε, by the $\mathcal{C}^1$ and $\mathcal{C}^2$ norms of the kinetic energy profile;
- Choosing the kinetic energy profile in such a way that the above control on the dissipation guarantees admissibility.

The estimates mentioned in the second item are the contents of [26, Theorem B.1 and Corollary B.1], summarized in the theorem below.

Theorem 3.8.1 (Fractional laplacian and Hölder norms). *Let $\gamma, \varepsilon > 0$ and $\beta \geq 0$ such that $2\gamma + \beta + \varepsilon \leq 1$, and let $f : \mathbb{T}^3 \to \mathbb{R}^3$. If $f \in \mathcal{C}^{0,2\gamma+\beta+\varepsilon}$, then $(-\Delta)^\gamma f \in \mathcal{C}^\beta$, moreover there exists a constant $C = C(\varepsilon)$ such that*

$$\|(-\Delta)^\gamma f\|_\beta \leq C(\varepsilon)[f]_{2\gamma+\beta+\varepsilon}. \tag{3.8.1}$$

Moreover, for every $\gamma \in (0,1)$, $\varepsilon > 0$ such that $0 < \gamma + \varepsilon \leq 1$, and f as above, there exists $C = C(\varepsilon) > 0$ such that

$$\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\gamma}{2}} f\right|^2 (x) \mathrm{d}x \leq C(\varepsilon)[f]^2_{\gamma+\varepsilon} \qquad \forall f \in \mathcal{C}^{\gamma+\varepsilon}(\mathbb{T}^3). \tag{3.8.2}$$

The last two bullets deserve a somewhat more detailed explanation.
The starting point of the proof in [15] is the following proposition ([15, Proposition 2.2]).

Proposition 3.8.1. *Let $E_1, E_2 > 1$. Assume $\mathcal{E}$ is a family of smooth functions on $[0,1]$ with the property that:*

(i) ${}^1/_2 \leq e(t) \leq 1$ *for every t and every $e \in \mathcal{E}$;*

(ii) *$e(0)$ is the same for every $e \in \mathcal{E}$;*

(iii) *$e'(0)$ is the same for every $e \in \mathcal{E}$;*

(iv) $\sup_{e\in\mathcal{E}} \|e\|_{\mathcal{C}^1} = E_1$*;*

(v) $\sup_{e\in\mathcal{E}} \|e\|_{\mathcal{C}^2} = E_2$.

Then for each $e \in \mathcal{E}$ it is possible to produce a corresponding pair (v_e, p_e) for which the following holds.

(a) *(v_e, p_e) solves the fractional Navier-Stokes equations* (2.1.3.1)*;*

(b) *Each v_e satisfies $\int |v_e|^2(t,x)\mathrm{d}x = e(t)$ for all $t \in [0,1]$;*

(c) *If $\beta = \theta + \varepsilon < {}^1/_5$ for some suitably small ε (depending only on α), then we have the explicit estimate*

$$\|v_e\|_{\mathcal{C}^\beta} \leq C(\theta,\varepsilon)\max\left\{E_1^{2\theta+3\varepsilon}, E_2^{\frac{2\theta+4\varepsilon}{3}}\right\};$$

(d) *The initial datum $v_\varepsilon(\cdot,0)$ is the same for every $e \in \mathcal{E}$.*

With this result in hand, the idea of the proof of **Theorem 2.4.2.4** in [15] is to choose energy profiles e where there is a quadratic relation between E_2 and E_1, and where e' "almost saturates" the bound on the $\mathcal{C}^1$ norm. More specifically, for a constant $K > 1$, we choose $E_1 = 2K + 2$ and $E_2 = CK^2$, and we require $e' \leq -2K + 2$. This gives us $\|v_e\|_{\mathcal{C}^\beta} \lesssim K^\gamma$ for some $\gamma < 1$. If we now look at the admissibility condition, we see that the dissipation term can be estimated by $CK^\gamma(t-s)$. Since the kinetic energy is ${}^e/_2$, we can estimate it from above by $(t-s)(1-K)$, so that the admissibility condition can be ensured by choosing K large enough so that $CK^\gamma < K - 1$.

In the paper [26], **Theorem 2.4.2.5** is proved. The strategy employed therein is a combinations of innovations similar to those described above with the strategy used in [8] for the Euler equations in the case of Hölder exponent $\beta < {}^1/_3$. This allows the author of [26] to reach the threshold $\theta < {}^1/_3$ for the exponent of the Laplacian. Note that such a strategy requires local existence and uniqueness results for solutions of fractional Navier-Stokes, as well as estimates for the norms of such solutions, which we have already seen in **Theorem 2.4.1.1** and **Theorem 2.4.1.2**, both of which were taken from [26, Section 3].

Chapter 4

Strategy towards the density theorem

4.1 Subsolutions and their existence

As seen in Chapter 3, the first step to setting up a convex integration scheme is to find a relaxation of the equations, and define a notion of approximate solutions, also called subsolutions. To this end, consider a pair (v, p) which solves the fractional Navier-Stokes equations (2.1.3.1). Consider an averaging process $\overline{}$ which is linear and commutes with derivatives (e.g. a mollification), and write $v = \overline{v} + w$, where $\overline{v}$ is the "mean flow", and w is the "fluctuation". If we take the average of the fractional Navier-Stokes equations, we get:

$$\begin{cases} \partial_t \overline{v} + \operatorname{div} \overline{v \otimes v} + \nabla \overline{p} + (-\Delta)^\theta \overline{v} = 0 \\ \operatorname{div} \overline{v} = 0 \end{cases}.$$

The second equation implies $\operatorname{div} w = 0$, since $\operatorname{div} w = \operatorname{div}(v - \overline{v}) = \operatorname{div} v - \operatorname{div} \overline{v} = 0 - 0$. The first equation can be rewritten as:

$$\partial_t \overline{v} + \operatorname{div}(\overline{v} \otimes \overline{v}) + \nabla \overline{p} + (-\Delta)^\theta \overline{v} = -\operatorname{div}(\overline{v \otimes v} - \overline{v} \otimes v) =: -\operatorname{div} \overline{R}.$$

We thus have that $(\overline{v}, \overline{p})$ "almost solves" the fractional Navier-Stokes equations, and $-\operatorname{div} \overline{R}$ is, in some sense, an error term. Noting that, by Jensen's inequality, $\overline{R}$ is positive semidefinite, we have a definition of subsolutions, which is very similar to the one used in [18, 19, 23].

Definition 4.1.1 (Subsolutions and strict subsolutions). *A* subsolution *for the fractional Navier-Stokes equations is a triple* $(v, p, R) : (0, T) \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R} \times \mathcal{S}^{3\times 3}_{\geq 0}$ *such that* $v \in L^2_{loc}$, $R \in L^1_{loc}$, *p is a distribution, the equations*

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\theta v = -\operatorname{div} R \\ \operatorname{div} v = 0 \end{cases} \tag{4.1.1}$$

hold in the sense of distributions in $(0, T) \times \mathbb{T}^3$, *and moreover* $R \geq 0$ *a.e., i.e. it is positive semidefinite a.e.. If* $R \in \mathcal{S}^{3\times 3}_+$ *a.e., then the subsolution is said to be* strict.

This system is known as *fractional Navier-Stokes-Reynolds system*, and is in perfect analogy to the *Euler-Reynolds system* (3.1.1) introduced in Chapter 3 for the Euler equations. The two notions of subsolution are also analogous to those introduced in Chapter 3. The following existence lemma holds for strict subsolutions.

Lemma 4.1.1 (Existence of strict subsolutions). *Let* $w \in L^2(\mathbb{T}^3)$ *with* $\operatorname{div} w = 0$. *For any* $\delta > 0$ *there exists a smooth strict subsolution* $(\tilde{v}, \tilde{p}, \tilde{R})$ *defined on* $[0,T)$ *and a time* $T_\eta \le T$ *such that*

$$\|\tilde{v}|_{t=0} - w\|_{L^2(\mathbb{T}^3)} \le \delta, \tag{4.1.2}$$

and

$$\frac{1}{2}\int_{\mathbb{T}^3} \big(|\tilde{v}|^2(t,x) + \operatorname{tr}\tilde{R}(t,x)\big)\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}\tilde{v}\right|^2(s,x)\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\int_{\mathbb{T}^3} |w|^2(x)\mathrm{d}x + \delta \qquad \forall t \in [0, T_\eta]. \tag{4.1.3}$$

Moreover, $T_\eta \to 0$ *as* $\eta \to 0$.

The following proof is inspired by that of [67, Lemma 6.8, p. 38].
Proof.
Fix $\rho \in C_c^\infty(B_1(0))$ a standard mollification kernel in space, and define:

$$\rho_\varepsilon(x) \coloneqq \varepsilon^{-3}\rho(x\varepsilon^{-1}).$$

To ensure the regularity of the initial datum, we consider the smoothed datum

$$w_0 \coloneqq w * \rho_{\eta_0},$$

where

$$\eta_0 \coloneqq \max\left\{\eta : \|w * \rho_\eta - w\|_{L^2} \le \frac{\delta}{3} \wedge \int_{\mathbb{T}^3} [|w_0|^2 - |w|^2](x)\mathrm{d}x \le \frac{2}{3}\delta\right\}. \tag{4.1.4}$$

By **Theorem 2.4.2.1**, there exists a solution $(\tilde{v}, \tilde{p})$ with initial datum w_0, where $\tilde{p}$ can be recovered uniquely once we impose $\int \tilde{p} = 0$.
We now fix a standard mollification kernel in time $\chi \in C_c^\infty((-1,0))$ and, with ρ_ε, ρ as defined above, we define

$$\chi_\varepsilon(t) \coloneqq \frac{1}{\varepsilon}\chi\left(\frac{t}{\varepsilon}\right)$$

$$v(t,x) \coloneqq \int_t^{t+\varepsilon} (\tilde{v} * \rho_\varepsilon)(s,x)\chi_\varepsilon(t-s)\mathrm{d}s,$$

$$p(t,x) \coloneqq \int_t^{t+\varepsilon} (\tilde{p} * \rho_\varepsilon)(s,x)\chi_\varepsilon(t-s)\mathrm{d}s,$$

$$R(t,x) \coloneqq \overline{\tilde{v} \otimes \tilde{v}} - v \otimes v,$$

where

$$\overline{f} = \int_{t}^{t+\varepsilon} (f * \rho_\varepsilon)(s,x)\,\chi_\varepsilon(t-s)\mathrm{d}s.$$

By construction and since $(\tilde{v}, \tilde{p})$ solves (2.1.3.1), (v, p, R) is a smooth solution of (4.1.1), i.e.

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\theta v = -\operatorname{div} R \\ \operatorname{div} v = 0 \end{cases}.$$

By using Jensen's inequality on $[t, t+\varepsilon] \times \mathbb{T}^3$ with the measure $\rho_\varepsilon(x-y)\,\chi_\varepsilon(t-s)\mathrm{d}x\mathrm{d}s$, we conclude that

$$R = \overline{\tilde{v} \otimes \tilde{v}} - v \otimes v \geq 0. \tag{4.1.5}$$

Coming to the initial datum, we have that

$$v|_{t=0} = \int_{0}^{\varepsilon} ((\tilde{v} - w_0) * \rho_\varepsilon)(s,x)\,\chi_\varepsilon(-s)\mathrm{d}s + w_0 * \rho_\varepsilon.$$

Taking the L^2 norm, we can easily obtain that

$$\|v|_{t=0} - w_0\|_{L^2(\mathbb{T}^3)} \leq \sup_{t\in[0,\varepsilon]} \|\tilde{v}(t,\cdot) - w_0\|_{L^2(\mathbb{T}^3)} + \|w_0 * \rho_\varepsilon - w_0\|_{L^2(\mathbb{T}^3)} =: \sup_{t\in[0,\varepsilon]} I_t + II_\varepsilon.$$

II_ε can be made as small as we desire by choosing ε small enough. Let ε_0 be the maximal parameter such that $II_\varepsilon < {}^{\delta}/_{3}$. As for $\sup I_t$, using $\tilde{v}_t(x) := \tilde{v}(t,x)$, we can obtain that

$$\begin{aligned} I_t^2 &= \int_{\mathbb{T}^3} |\tilde{v}_t - w_0|^2 \mathrm{d}x = \int_{\mathbb{T}^3} |\tilde{v}_t|^2 - |w_0|^2 \mathrm{d}x - 2\int_0^t \int_{\mathbb{T}^3} \langle \partial_t \tilde{v}, w_0\rangle \mathrm{d}x\mathrm{d}s \\ &\overset{*}{\leq} 2\int_0^t \int_{\mathbb{T}^3} -(\tilde{v}_s \otimes \tilde{v}_s) : \mathfrak{D} w_0 + \left\langle \tilde{v}_s, (-\Delta)^\theta w_0 \right\rangle \mathrm{d}x\mathrm{d}s \\ &\overset{\bullet}{\leq} 2\sqrt{C(1-2\theta)} \int_0^t \|\tilde{v}_t\|_{L^2} [w_0]_1 \mathrm{d}s + 2\int_0^t \|\tilde{v}_t\|_{L^2}^2 \|\mathfrak{D} w_0\|_{L^\infty} \mathrm{d}s \\ &\leq 2t \|D w_0\|_{L^\infty} \|w_0\|_{L^2} (\sqrt{C(1-2\theta)} + 2\|w_0\|_{L^2}) \leq K(w) t \|\mathfrak{D} w_0\|_{L^\infty}. \end{aligned}$$

In $*$, we used the fact that $\|\tilde{v}_t\|_{L^2}^2 \leq \|w_0\|_{L^2}^2$, i.e. (1.3.1), as well as the fact that $(\tilde{v}, \tilde{p})$ is a solution of (2.1.3.1) and the fact that $\operatorname{div} w_0 = 0$. In $\bullet$, we used **Theorem 3.8.1**, choosing $\varepsilon = 1 - 2\theta$. In the last step, we used that $\|w_0\|_{L^2} \leq \|w\|_{L^2}$. This becomes arbitrarily small if we choose t appropriately small which, since we are taking $t \leq \varepsilon$, reduces to choosing ε small enough. Since $D w_0 = D\rho_{\eta_0} * w = \eta_0^{-4} D\rho(\eta_0^{-1}\cdot) * w$, Hölder's inequality yields

$$\|D w_0\|_{L^\infty} \leq \eta_0^{-4} \|D\rho(\eta_0^{-1} x)\|_{L_x^2(B_{\eta_0})} \|w\|_{L^2} \leq \eta_0^{-4} \eta_0^2 \|D\rho\|_{C^0} \|w\|_{L^2} = C(w)\eta_0^{-2},$$

so that, to ensure $\sup_{[0,\varepsilon]} I_t \leq {}^{\delta}/_{3}$, we choose

$$\varepsilon \leq \frac{\delta \eta_0^2}{3C(w)K(w)} =: \tilde{\varepsilon}.$$

Choosing $\varepsilon := \min\{\varepsilon_0, \tilde{\varepsilon}\}$ thus yields

$$\|v|_{t=0} - w_0\|_{L^2} \leq \frac{2}{3}\delta \wedge \|w_0 - w\|_{L^2} \leq \frac{\delta}{3} \implies \|v|_{t=0} - w\|_{L^2} \leq \delta.$$

We have thus obtained (4.1.2). As for (4.1.3), we first notice that, by the definition of R, we have that

$$\int_{\mathbb{T}^3} |v|^2(t,x) + \operatorname{tr} R(t,x) \mathrm{d}x = \int_{\mathbb{T}^3} \overline{|\tilde{v}|^2}(t,x) \mathrm{d}x.$$

We have thus reduced (4.1.3) to the following inequality:

$$\frac{1}{2} \int_{\mathbb{T}^3} \overline{|\tilde{v}|^2}(t,x) + \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 (s,x) \mathrm{d}x \mathrm{d}s \leq \int_{\mathbb{T}^3} |w|^2(x) \mathrm{d}x + \delta.$$

Since $(\tilde{v}, \tilde{p})$ satisfies (1.3.1), we can see that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}^3} \overline{|\tilde{v}|^2}(t,x) + \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 (s,x) \mathrm{d}x \mathrm{d}s &\leq \frac{1}{2} \int_{\mathbb{T}^3} |w|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} |w_0|^2 - |w|^2 \mathrm{d}x \\ &\quad + \frac{1}{2} \int_{\mathbb{T}^3} [\overline{|\tilde{v}|^2} - |\tilde{v}|^2](t,x) \mathrm{d}x \\ &\quad + \int_0^t \int_{\mathbb{T}^3} \left[\left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}\right|^2 \right] (s,x) \mathrm{d}x \mathrm{d}s \\ &=: \frac{1}{2} \int_{\mathbb{T}^3} |w|^2 \mathrm{d}x + I + II + III. \end{aligned}$$

Our desired estimate (4.1.3) will then follow from

$$I \leq \frac{\delta}{3} \qquad II \leq \frac{\delta}{3} \qquad III \leq \frac{\delta}{3}. \tag{4.1.6}$$

The first of these relations follows from (4.1.4).
The second relation in (4.1.6) is the reason why (4.1.3) only holds for small times. Indeed, if we define $N(t) := \int |\tilde{v}|^2(t,x) \mathrm{d}x$, we can see that

$$II(t) = (\chi_\varepsilon * N)(t) - N(t).$$

To deduce our desired estimate, we would require this to be smaller than ${}^{\delta}/_{3}$ for all t, or at least for a.e. t, since deducing the estimate for a.e. t implies that it holds for all

t. However, this would mean $\chi_\varepsilon * N \to N$ uniformly a.e.. If N is not continuous, this is impossible since $\chi_\varepsilon N$ is smooth. The best that we can obtain for $\tilde{v}$ on $[0,T]$ is $\tilde{v} \in C^0_t L^2_{w,x}$, that is $t \mapsto \tilde{v}(t,\cdot)$ is continuous w.r.t. the weak L^2 topology. However, that is not enough. Indeed, with this continuity, the continuity of N would imply $\tilde{v} \in C^0_t L^2_x$, which is not known in general. However, since w_0, there exists T_η such that on $[0,T_\eta]$ there exists a smooth solution, for which N is indeed continuous. This implies the desired uniform convergence on $[0,T_\eta]$, and thus the desired estimate for $t \le T_\eta$.
Coming to the third relation, we first rewrite and bound the integral of the first integrand:

$$\int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 (s,x) \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{T}^3} \left| \int_s^{s+\varepsilon} \int_{\mathbb{T}^3} \rho_\varepsilon(x-y)(-\Delta)^{\frac{\theta}{2}} \tilde{v}(y,\tau)\, \chi_\varepsilon(t-\tau) \mathrm{d}y\mathrm{d}\tau \right|^2 \mathrm{d}x\mathrm{d}s$$
$$\le \int_0^t \int_s^{s+\varepsilon} \int_{\mathbb{T}^3} \rho_\varepsilon * \left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}\right|^2 (x,\tau)\, \chi_\varepsilon(t-\tau) \mathrm{d}x\mathrm{d}\tau\mathrm{d}s,$$

where we used Jensen's inequality in the second step. Therefore, the remaining term is estimated as:

$$III \le \int_0^t \int_{\mathbb{T}^3} \left[\overline{\left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}\right|^2} (s,x) - \left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}\right|^2 (s,x) \right] \mathrm{d}x\mathrm{d}s.$$

We now note that, since $\tilde{v} \in L^2_t H^\theta_x$, we have that $(-\Delta)^{\theta/2} \tilde{v} \in L^2_t L^2_x$, and thus $|(-\Delta)^{\theta/2} \tilde{v}|^2 \in L^1_t L^1_x$. Therefore, $\overline{|(-\Delta)^{\theta/2} \tilde{v}|^2} - |(-\Delta)^{\theta/2} \tilde{v}|^2 \to 0$ in $L^1_t L^1_x$, and the third relation of (4.1.6) reduces to an opportune choice of ε. Note that, in this case, the uniform convergence is not a problem since we are integrating in t.
Summing up, (v,p,R) is a smooth solution of (4.1.1), which satisfies (4.1.3) and (4.1.2), and $R \ge 0$ by (4.1.5). The proof of **Lemma 4.1.1** is thus complete. ◇

4.2 Two "stronger" kinds of subsolutions

Since we are setting up a convex integration scheme which aims to prove a density result, we must introduce strong and adapted subsolutions analogous to the notions seen in Chapter 3 (Section 3.7). However, the notions we will use are a bit different to those from that chapter. In particular, the notion of strong subsolution extends the ones of [19] and [18]. As in [18], the Reynolds stress is controlled by a power of the trace. However, the exponent γ will only act on the "reduced" trace $\rho\Omega^{-1}$, where $\Omega > 0$ is a constant whose role is explained in Section 4.3.

Definition 4.2.1 (Strong subsolutions). *A* strong subsolution with parameters $\gamma, \Omega > 0$ *is a subsolution* (v,p,R) *such that in addition* $\operatorname{tr} R$ *is a function of* t *only and, if*

$$\rho(t) := \frac{1}{3}(\operatorname{tr} R)(t) \qquad\qquad \varrho(t) := \frac{\rho(t)}{\Omega},$$

then

$$\left|\mathring{R}(t,x)\right| \le \Omega\, \varrho^{1+\gamma}(t) \qquad \forall (t,x). \tag{4.2.1}$$

Remark 4.2.1 (On strength and parameters). *In our schemes ϱ will be sufficiently small so that in particular $\varrho^\gamma \le r_0$, where r_0 is the geometric constant in [19, Definition 3.2], thus leading to the conclusion that* (4.2.1) *implies that our strong subsolutions are also strong in the sense of [19], provided $\Omega = O(1)$ (specifically $\Omega\varrho^\gamma \le r_0$). Note also that, if (v, p, R) is a strong subsolution for some parameters $\gamma, \Omega > 0$ with $\varrho < 1$, then it is also a strong subsolution for any $0 < \gamma' < \gamma$ with the same Ω.*

The last notion of subsolution has vanishing Reynolds stress at time $t = 0$ and the $\mathcal{C}^1$-norms blow up at certain rates as the Reynolds stress goes to zero. Such adapted subsolutions have been introduced in [19, 18]. The blow-up rate in the remainder of this thesis is analogous to the one of [18]. Differently from [18], the blow-up is controlled by the "reduced" trace ϱ rather than the "full" trace ρ, and the estimates include a power of Ω.

Definition 4.2.2 (Adapted subsolutions). *Given $\gamma, \Omega > 0, 0 < \beta < 1/3$, and ν satisfying*

$$\nu > \frac{1-3\beta}{2\beta}, \tag{4.2.2}$$

*we call a triple (v, p, R) a $\mathcal{C}^\beta$-*adapted subsolution *on* $[0, T]$ with parameters γ, Ω, ν *if $(v, p, R) \in \mathcal{C}^\infty((0, T] \times \mathbb{T}^3) \cap \mathcal{C}([0, T] \times \mathbb{T}^3)$ is a strong subsolution with parameters γ, Ω with initial datum*

$$v(0, \cdot) \in \mathcal{C}^\beta(\mathbb{T}^3) \quad \textit{and} \quad R(0, \cdot) \equiv 0, \tag{4.2.3}$$

and, setting $\rho(t) := 1/3 \operatorname{tr} R(t, x)$ and $\varrho := \rho\Omega^{-1}$, for all $t > 0$ we have that $\rho(t) > 0$ and there exist $\alpha \in (0, 1)$ and $C \ge 1$ such that

$$\|v\|_{1+\alpha} \le C\Omega^{\frac{1}{2}}\varrho^{-(1+\nu)} \tag{4.2.4}$$

$$|\partial_t \varrho| \le C\Omega^{\frac{1}{2}}\varrho^{-\nu}. \tag{4.2.5}$$

4.3 General strategy

The remainder of this thesis closely follows the convex integration strategy adopted by [18] in the Euler setting and described in Section 3.7.
Chapter 5 shows how to obtain a strong subsolution from a strict one.
Chapter 6 states and proves the two propositions that allow us to make each step in the two convex integration schemes, the gluing step of Sections 6.1-6.3, and the perturbation step of Section 6.4.
Chapter 7 states and proves two propositions that allow us to approximate one kind of subsolution (as defined in the previous sections) with another. More specifically, in Section 7.1 we approximate strict subsolutions with $\mathcal{C}^\beta$-adapted ones, whereas in Section 7.2 we approximate $\mathcal{C}^\beta$-adapted subsolutions with weak solutions. The latter of those results uses the parameters we will introduce in this section in (4.3.1).
Chapter 8 proves the main theorem starting from the results of Chapter 7.
In passing from one subsolution to the next, the $\mathcal{C}^0$ and $\mathcal{C}^1$ norms of the various subsolutions are estimated in terms of parameters (δ_q, λ_q), where $\delta_q^{1/2}$ is the amplitude (in space)

of $w_q \coloneqq v_q - v_{q-1}$, and λ_q is the oscillation frequency (in space) of w_q. The parameters, however, are partially different from those chosen in [18] and closer to the ones used in [19]. More precisely, we define

$$\lambda_q \coloneqq 2\pi \lceil a^{b^q} \rceil \qquad \delta_q \coloneqq \Lambda \zeta_q = \delta \lambda_1^{2\beta} \lambda_q^{-2\beta} \qquad \zeta_q \coloneqq \lambda_q^{-2\beta} \qquad \Lambda \coloneqq \delta \lambda_1^{2\beta}, \quad (4.3.1)$$

where

- $\lceil x \rceil$ denotes the ceiling of x, i.e. the smallest integer $n \geq x$;
- δ is a small parameter;
- $\beta \in (0, 1/3)$ and $b \in (1, 3/2)$ control the Hölder exponent of the scheme and are required to satisfy
$$1 < b < \frac{1-\beta}{2\beta}. \quad (4.3.2)$$
- $a \gg 1$ is sufficiently large to absorb various q-independent constants in the course of the proofs.

The parameter Λ, and thus the distinction between δ_q and ζ_q, were absent in [18]. They are added here to make sure $\delta_1 = \delta$, thus making (5.2.2) an a-independent estimate. Thus, in particular, we are allowed to bound Λ from below, since such a bound will be satisfied for a large enough, but not to bound it from above, which would cause δ to depend on a. With this choice of parameters, we must require the conditions

$$\Lambda \geq 1 \quad (4.3.3)$$

$$\frac{1}{3} > \beta > \theta + \varepsilon', \quad (4.3.4)$$

for some positive ε'. Condition (4.3.3) merely requires a to be sufficiently large.
The main convex integration step will consist in stating that, for a certain universal constant $M > 1$, some sufficiently small $\alpha, \gamma > 0$, and a sufficiently large $a \gg 1$, if (v_q, p_q, R_q) is a strong subsolution satisfying

$$\|\mathring{R}_q\|_0 \leq \Lambda \varrho_q^{1+\gamma} \quad (4.3.5)$$

$$\|v_q\|_{1+\alpha} \leq M \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \quad (4.3.6)$$

$$\frac{3}{4}\delta_{q+2} \leq \rho_q \leq \frac{7}{2}\delta_{q+1} \quad (4.3.7)$$

$$|\partial_t \rho_q| \leq \rho_q \delta_q^{\frac{1}{2}} \lambda_q \quad (4.3.8)$$

$$\|v_q\|_{\theta+\varepsilon} \leq M\left(1 + \sum_{i=0}^{q} \lambda_i^{\theta+\varepsilon-\beta}\right), \quad (4.3.9)$$

where $\rho_q := {}^1/_3 \operatorname{tr} R_q$, and $\varrho_q := \Lambda^{-1}\rho_q$, then there exists $(v_{q+1}, p_{q+1}, R_{q+1})$ a strong subsolution satisfying the conditions (4.3.5)-(4.3.9) with q replaced by $q+1$ as well as the following additional estimate

$$\left\|v_{q+1}-v_q\right\|_0+\lambda_{q+1}\left\|v_{q+1}-v_q\right\|_{H^{-1}}+\lambda_{q+1}^{-1-\alpha}\left\|v_{q+1}-v_q\right\|_{1+\alpha} \leq M\delta_{q+1}^{\frac{1}{2}}.$$

The proof consists of three steps:

1. A mollification step, moving from (v_q, p_q, R_q) to $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, R_{\ell_{q,i}})$, where the mollification parameter $\ell_{q,i}$ varies on suitably chosen subintervals, as required by the different orders of the upper and lower bounds on ρ_q in (4.3.7);

2. A gluing step, which goes from $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, R_{\ell_{q,i}})$ to $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$;

3. A perturbation step going from $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ to $(v_{q+1}, p_{q+1}, R_{q+1})$.

The change in condition (4.3.5) with respect to [18] was made in order to prevent the new definition of δ_q from causing bounds of the form $\Lambda^A \leq 1$, with $A > 0$, to appear in the proofs. Condition (4.3.9) was added in order to control the new trace terms.
Chapter 6 addresses all three steps. The gluing step was introduced in [40] to ensure $\mathring{\overline{R}}_q$ is supported in pairwise disjoint time intervals. This allows us to construct the perturbation as $w = \sum_i (w_{o,i} + w_{c,i})$, where the $w_{o,i}$ are Mikado flows with pairwise disjoint supports and $\operatorname{supp} w_{c,i} \subseteq \operatorname{supp} w_{o,i}$, thus preventing $w \otimes w$ from containing "mixed terms" $w_{o,i} \otimes w_{o,j}$ with $i \neq j$, which are harder to deal with.
Fixing $\alpha > 0, \gamma > 0$, we also define

$$\ell_q := \frac{\zeta_{q+2}^{\frac{1+\gamma}{2}}}{\zeta_q^{\frac{1}{2}}\lambda_q\lambda_{q+1}^{2\alpha}} = \frac{\delta_{q+2}^{\frac{1+\gamma}{2}}\Lambda^{-\frac{\gamma}{2}}}{\delta_q^{\frac{1}{2}}\lambda_q\lambda_{q+1}^{2\alpha}} \tag{4.3.10}$$

$$\tau_q := \frac{\ell_q^{4\alpha}}{\delta_q^{\frac{1}{2}}\lambda_q}. \tag{4.3.11}$$

Remark 4.3.1 (Homogeneity in Λ of ℓ_q, τ_q). *ℓ_q, as well as the $\ell_{q,i}$ defined in Section 6.3, are 0-homogeneous in Λ, whereas τ_q is ${}^1/_2$-homogeneous. The last property allows us to cancel the $\Lambda^{1/2}$ factors we will see appearing in the course of the proof.*

We also assume

$$\frac{\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q}{\lambda_{q+1}^{1-15\alpha-\beta\gamma}} \leq \delta_{q+2}, \tag{4.3.12}$$

which can be achieved if a is sufficiently large assuming $(15\alpha+\beta\gamma)b < (b-1)(1-\beta-2b\beta)$. Moreover, we assume

$$\lambda_{q+1}^{-1} \leq \ell_q \leq \lambda_q^{-1}. \tag{4.3.13}$$

The right inequality in (4.3.13) is evident from the definition. The left inequality can be reduced to $-b < \beta b^2(1+\gamma)+\beta-1-2b\alpha$, which can easily be verified for $\alpha = 0 = \gamma$, and thus also for α, γ sufficiently small. We will in fact need the following sharper bound:

$$\lambda_{q+1}^{1-\overline{N}} \leq \ell_q^{\overline{N}+1}, \tag{4.3.14}$$

which can be achieved by imposing the following condition:

$$\overline{N}[(b-1)(1-\beta(b+1))-\gamma\beta b^2-2\alpha b] > 1+b+(1+\gamma)\beta b^2+2\alpha b-\beta. \tag{4.3.15}$$

The above conditions can be obtained by choosing, in this order

- b, β as in (4.3.2), so that in particular $\beta(1+b) < 1$;
- $0 < \alpha, \gamma$ sufficiently small depending on b, β;
- $\overline{N} \in \mathbb{N}$ sufficiently large depending on b, β, α, γ so as to get (4.3.15).

One last notational remark: $A \lesssim B$ (resp. $A \gtrsim B$) will mean $A \leq C(b,\beta,\alpha,\gamma,M)B$ (resp. $A \geq C(b,\beta,\alpha,\gamma,M)B$), or $C(N,b,\beta,\alpha,\gamma,M)$ if norms depending on N are involved (e.g. $C^{N+1+\alpha}$-norms). $A \sim B$ will mean $A \lesssim B$ and $A \gtrsim B$. Note that C does not depend on $a \gg 1$.

Chapter 5

First approximation

As discussed in Section 4.3 above, the first step in the proof of the main result of this thesis is to approximate strict subsolutions with strong subsolutions. The strategy to that end, as in [19] and [18], proceeds in two steps:

- We first prove a proposition that gives a "perturbation strategy" for smooth solutions of (4.1.1);
- We then apply it to strict subsolutions and obtain, as a corollary, that they can be approximated by strong subsolutions with additional properties necessary to then start the first convex integration scheme (as is done in Section 7.1).

This chapter is devoted to those two steps.

5.1 The approximation proposition

The first of the above steps is provided by an analogue of [19, Proposition 3.1].

Proposition 5.1.1. *Let* $(\tilde{v},\tilde{p},\tilde{R})$ *be a smooth solution of* (4.1.1)*, and* $S \in \mathcal{C}^\infty([0,T]\times\mathbb{T}^3;\mathcal{S}^{3\times 3}_+)$ *be a smooth positive-definite matrix field. Fix* $\overline{\alpha}\in(0,1)$ *and* $\varepsilon > 0$*. Then for any* $\lambda > 1$ *there exists a smooth solution* $(\check{v},\check{p},\check{R})$ *of* (4.1.1) *with*

$$(\check{v},\check{p},\check{R}) = (\tilde{v},\tilde{p},\tilde{R}) \quad \textit{for } t \notin \operatorname{supp}\operatorname{tr} S, \tag{5.1.1}$$

$$\int_{\mathbb{T}^3} \left(|\check{v}|^2 + \operatorname{tr}\check{R}\right)(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^3} \left(|\tilde{v}|^2 + \operatorname{tr}\tilde{R}\right)(x,t)\,\mathrm{d}x \qquad \forall t \in [0,T], \tag{5.1.2}$$

and the following estimates hold

$$\|\check{v}-\tilde{v}\|_{H^{-1}} \le \frac{C}{\lambda} \tag{5.1.3}$$

$$\|\check{v}\|_k \le C\lambda^k \qquad k = 1,2 \tag{5.1.4}$$

$$\left\|\tilde{R}-\check{R}-S\right\|_N \le \frac{C}{\lambda^{1-2\theta-\overline{\alpha}-N}} \tag{5.1.5}$$

Moreover, $\operatorname{tr}(\tilde{R}(x,t) - \check{R}(x,t) - S(x,t)) =: \mu(t)$ *is a function of t only and satisfies*

$$|\mu'|(t) \le C\lambda^{\overline{\alpha}}. \tag{5.1.6}$$

The constant $C \ge 1$ above depends on $(\tilde{v}, \tilde{p}, \tilde{R}), S$ and $\overline{\alpha}$, but not on λ. Finally, defining

$$\check{\mathcal{J}}(t) := \int_0^t \int_{\mathbb{T}^3} \left(\left|(-\Delta)^{\frac{\theta}{2}} \check{v}(x,s)\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}(x,s)\right|^2 \right) \mathrm{d}x \mathrm{d}s,$$

we have that

$$\left|\partial_t \check{\mathcal{J}}(t)\right| \le C\lambda^{2(\theta+\varepsilon)}. \tag{5.1.7}$$

Proof.

Define the inverse flow of $\tilde{v}$, $\Phi : \mathbb{T}^3 \times [0,T] \to \mathbb{T}^3$, as the solution of

$$\begin{cases} \partial_t \Phi(x,t) + (\tilde{v} \cdot \nabla)\Phi(x,t) = 0 \\ \Phi(x,0) = x \qquad x \in \mathbb{T}^3 \end{cases},$$

and set

$$\overline{R}(x,t) = \mathfrak{D}\Phi(x,t) S(x,t) \mathfrak{D}^T \Phi(x,t).$$

Observe that $\overline{R}$ is defined on the compact set $\mathbb{T}^3 \times [0,T]$ and, being continuous, has a compact image $\mathcal{N}_0 := \overline{R}(\mathbb{T}^3 \times [0,T]) \subset \mathcal{S}_+^{3\times 3}$.

By **Lemma C.1** there exists a smooth vector field $W : \mathcal{N}_0 \times \mathbb{T}^3 \to \mathbb{T}^3$ satisfying the differential equations (C.1) and the integral equations (C.2). Define

$$w_o(x,t) = \mathfrak{D}\Phi^{-1} W(\overline{R}, \lambda\Phi(x,t))$$
$$w_c(x,t) = \frac{1}{\lambda} \operatorname{curl}(\mathfrak{D}^T \Phi U(\overline{R}, \lambda\Phi(x,t))) - w_o,$$

where $U = U(R,\xi)$ is defined as in (C.7) and thus satisfies $\operatorname{curl}_\xi U = W$. Moreover, set

$$\check{v} := \tilde{v} + w_o + w_c \qquad \check{p} = \tilde{p} + \overline{p} \qquad \check{R} := \tilde{R} - S - \mathring{\mathcal{E}}^{(1)} - \mathcal{E}^{(2)},$$

where

$$\begin{aligned} \overline{p} &:= -\frac{1}{3}(w_c \cdot \check{v} + w_o \cdot w_c) \\ \mathring{\mathcal{E}}^{(1)} &:= \mathcal{R}(F) + (w_c \otimes \check{v} + w_o \otimes w_c + \overline{p}\mathrm{Id}) \\ F &:= \operatorname{div}(w_o \otimes w_o - S) + (\partial_t + \tilde{v} \cdot \nabla) w_o \\ &\quad + [(w_o + w_c) \cdot \nabla]\tilde{v} + \partial_t w_c + (-\Delta)^\theta (w_o + w_c) \\ \mathcal{E}^{(2)} &:= \frac{1}{3} \cdot \fint_{\mathbb{T}^3} \left(|\check{v}|^2 - |\tilde{v}|^2 - \operatorname{tr} S \right) \mathrm{d}x \cdot \mathrm{Id}, \end{aligned}$$

with $\mathcal{R}$ defined as in (C.11). By construction, the relation (5.1.2) holds, $\mathring{\mathcal{E}}^{(1)}$ is traceless, $\mathcal{E}^{(2)}$ is only t-dependent (and thus $\operatorname{div}\mathcal{E}^{(2)} = 0$), F is mean-free (and thus $\operatorname{div}\mathcal{R}(F) = F$), and $(\check{v}, \check{p}, \check{R})$ solves (4.1.1). To verify this last claim, we can see that

$$\begin{aligned} \operatorname{div}\mathring{\mathcal{E}}^{(1)} &= \operatorname{div}(\check{v} \otimes \check{v} - \tilde{v} \otimes \tilde{v} - S + \overline{p}\mathrm{Id}) + \partial_t(\check{v} - \tilde{v}) + (-\Delta)^\theta (w_o + w_c) \\ &= \partial_t \check{v} + \operatorname{div}(\check{v} \otimes \check{v} - S + \tilde{R}) + \nabla \check{p} + (-\Delta)^\theta \check{v}. \end{aligned}$$

We call $w \coloneqq w_o + w_c = \check{v} - \tilde{v}$. Recall that

$$\begin{aligned} W(R,\xi) &= \sum_{k\neq 0} a_k(R) A_k e^{ik\cdot\xi} \\ U(R,\xi) &= \sum_{k} a_k(R) \frac{ik\times A_k}{|k|^2} e^{ik\cdot\xi}, \end{aligned}$$

which are respectively (C.3) and (C.7), with the a_k satisfying (C.5). This allows us to decompose

$$w_o = \sum_{k\neq 0} \mathfrak{D}\Phi^{-1} a_k(\mathfrak{D}\Phi S \mathfrak{D}^T\Phi) A_k e^{ik\cdot\lambda\Phi} = \sum_{k\neq 0} b_k e^{ik\cdot\lambda\Phi} \tag{5.1.8}$$

$$w_c = \frac{i}{\lambda} \sum_{k\neq 0} \nabla(a_k(\mathfrak{D}\Phi S \mathfrak{D}^T\Phi)) \times \frac{\mathfrak{D}^T\Phi(k\times A_k)}{|k|^2} e^{ik\cdot\lambda\Phi} = \sum_{k\neq 0} \frac{c_k}{\lambda} e^{ik\cdot\lambda\Phi}. \tag{5.1.9}$$

We next obtain estimate (5.1.5). To that end, we decompose $\mathring{\mathcal{E}}^{(1)}$ as follows:

$$\begin{aligned} \mathring{\mathcal{E}}^{(1)} &= \mathcal{R}\operatorname{div}(w_o\otimes w_o - S) + \mathcal{R}[(\partial_t + v\cdot\nabla) w_o] \\ &\quad + \mathcal{R}(w\cdot\nabla v) + \mathcal{R}(\partial_t w_c) + (w_c \mathring{\otimes} \check{v} + w_o \mathring{\otimes} w_c) + \mathcal{R}((-\Delta)^\theta w) \\ &=: I + II + III + IV + IV + V + VI. \end{aligned}$$

Noting that V is the traceless part of $w_c\otimes\check{v} + w_o\otimes w_c$, and any estimate on this will imply the same estimate for V, we proceed to reproduce the arguments used in [19] estimate the first five terms, and then prove estimates for $\mathcal{E}^{(2)}$ and VI.

We start by estimating I. By (C.4) we have that

$$w_o\otimes w_o - S = \sum_k \mathfrak{D}\Phi^{-1} C_k(\overline{R}) \mathfrak{D}\Phi^{-T} e^{i\lambda k\cdot\Phi} =: \sum_k d_k e^{u\lambda k\cdot\Phi}.$$

Using (C.8), we see that

$$\operatorname{div}(d_k e^{i\lambda k\cdot\Phi} = \operatorname{div} d_k e^{i\lambda k\cdot\Phi},$$

so that

$$\operatorname{div}(w_o\otimes w_o - S) = \sum_k \operatorname{div}(d_k) e^{i\lambda k\cdot\Phi}.$$

We can thus estimate term I by (C.14):

$$\begin{aligned} \|I\|_0 &\leq \sum_k \left\| \mathcal{R}(\operatorname{div}(d_k) e^{i\lambda k\cdot\Phi}) \right\|_\alpha \lesssim \sum_k \left[\frac{\|\operatorname{div} d_k\|_0}{|k\lambda|^{1-\alpha}} + \frac{\|\operatorname{div} d_k\|_{N+\alpha} + \|\operatorname{div} d_k\|_0 \|\Phi\|_{N+\alpha}}{|k\lambda|^{N-\alpha}} \right] \\ &\lesssim \frac{1}{\lambda^{1-\alpha}} \sum_k \left[\frac{\|d_k\|_1}{|k|^{1-\alpha}} + \frac{\|d_k\|_{N+1+\alpha} + \|d_k\|_1 \|\Phi\|_{N+\alpha}}{|k\lambda|^{N-1} |k|^{1-\alpha}} \right]. \end{aligned}$$

Since the coefficients d_k are smooth, they will satisfy a bound of the form $\|d_k\| \lesssim |k|^{-5}$. This yields that $\operatorname{div}(w_o\otimes w_o - S)$ satisfies (5.1.5) in the case $N=0$. The other cases can be tackled by observing that any derivative will add a factor of λ to the estimate when differentiating the exponential.

As for the terms II, III, IV, since the coefficients b_k, c_k also satisfy an estimate of the form $\|b_k\| \lesssim |k|^{-5}$, a similar application of (C.14) to the one above will estimate them once we rewrite them as:

$$\begin{aligned} II &= \sum_k \mathcal{R}\left[(\partial_t + v\cdot\nabla)(b_k)e^{i\lambda k\cdot\Phi}\right] \\ III &= \sum_k \mathcal{R}\left[\left(b_k + \frac{c_k}{\lambda}\right)\cdot\nabla v e^{i\lambda k\cdot\Phi}\right] \\ IV &= \sum_k \mathcal{R}\left(\frac{\partial_t c_k}{\lambda}e^{i\lambda k\cdot\Phi}\right). \end{aligned}$$

Coming to V, we can rewrite it as:

$$V = \frac{1}{\lambda}\sum_k (c_k \otimes \check{v} + w_o \otimes c_k)e^{i\lambda k\cdot\Phi},$$

which easily yields that

$$\|V\|_0 \lesssim \frac{1}{\lambda}.$$

For all these terms, we once again note that any extra derivative on the exponential costs a factor of λ.

$\mathcal{E}^{(2)}$ is estimated in a similar fashion to how we estimate $\mathring{\mathcal{E}}^{(1)}$ below.

Finally, to estimate $\mathcal{R}((-\Delta)^\theta w)$, using the fact that $[\mathcal{R}, (-\Delta)^\theta] = 0$ and (3.8.1), we see that

$$\begin{aligned} \left\|\mathcal{R}((-\Delta)^\theta w)\right\|_0 \lesssim \|\mathcal{R}w\|_{2\theta+\overline{\alpha}} \lesssim \sum_k \Bigg(&\frac{\left\|b_k + \lambda^{-1}c_k\right\|_0}{|\lambda k|^{1-\overline{\alpha}-2\theta}} + \frac{\left\|b_k + \lambda^{-1}c_k\right\|_{N+2\theta+\overline{\alpha}}}{|\lambda k|^{N-2\theta-\overline{\alpha}}} \\ &+ \frac{\left\|b_k + \lambda^{-1}c_k\right\|_0 \|\Phi\|_{N+2\theta+\overline{\alpha}}}{|\lambda k|^{N-2\theta-\overline{\alpha}}}\Bigg) \\ \lesssim \lambda^{\overline{\alpha}+2\theta-1}\cdot \sum_k &\left(\frac{1}{|k|^{7-\overline{\alpha}-2\theta}} + \frac{1 + C_\Phi(N,\alpha,\theta)}{\lambda^{N-1}|k|^{N+6-2\theta-\overline{\alpha}}}\right), \end{aligned}$$

where we used (C.5) to get the extra $|k|^{-6}$ in each term, and the boundedness of Φ to get the $C_\Phi(N,\alpha,\theta)$.

Concerning (5.1.4), the smoothness of Φ, S combined with (C.5) gives us

$$\max\{\|c_k\|_N, \|b_k\|_N\} \lesssim |k|^{-m}, \tag{5.1.10}$$

for all integers $m > 0$, where the b_k and c_k are as in the decompositions of w_o, w_c above. This easily allows us to conclude that

$$\|w\|_N \lesssim \lambda^N,$$

since differentiating the exponential gives us a factor of λ for each derivative. We then note that

$$\|\check{v}\|_N \le \|\tilde{v}\|_N + \|w\|_N \lesssim 1 + \lambda^N \lesssim \lambda^N,$$

where the second step used the smoothness of $\tilde{v}$. For $N=1,2$ the above reduces to (5.1.4).

Next, we prove estimate (5.1.3). Let $f \in H^1$, and observe that:

$$\left|\int_{\mathbb{T}^3} (\check{v}-\tilde{v}) f \mathrm{d}x\right| = \left|\int_{\mathbb{T}^3} w f \mathrm{d}x\right| = \frac{1}{\lambda}\left|\int_{\mathbb{T}^3} \mathrm{curl}(\mathfrak{D}^T \Phi U(\overline{R},\lambda\Phi(x,t))) f \mathrm{d}x\right|$$
$$\lesssim \frac{1}{\lambda}\left\|\mathfrak{D}^T \Phi U(\overline{R},\lambda\Phi(x,t))\right\|_{L^2}\|\nabla f\|_{L^2} \lesssim \frac{1}{\lambda}\|f\|_{H^1},$$

since $\mathfrak{D}^T \Phi U(\overline{R},\lambda\Phi(x,t)) \in \mathcal{C}^\infty \subset L^2$. This yields (5.1.3).

To continue, we note that $\mu = \mathrm{tr}\,\mathcal{E}^{(2)} = \fint |\check{v}|^2 - |\tilde{v}|^2 - \mathrm{tr}\, S \mathrm{d}x$. $\tilde{v}$ and $\mathrm{tr}\, S$ are both smooth, so they are bounded. In order to estimate $\int |\check{v}(t)|^2$, note that the following energy identity for $\check{v}$ follows from (4.1.1):

$$\begin{aligned}\partial_t \frac{1}{2}|\check{v}|^2 + \mathrm{div}\left(\check{v}\left(\frac{|\check{v}|^2}{2}+\check{p}\right)\right) + \check{v}\cdot(-\Delta)^\theta \tilde{v} =& -\check{v}\cdot \mathrm{div}(\check{R}+\mathcal{R}((-\Delta)^\theta w)+\mathcal{E}^{(2)})\\ =& -\check{v}\cdot\mathrm{div}(\tilde{R}-S)\\ &+\check{v}\cdot\mathrm{div}(\mathring{\mathcal{E}}^{(1)}-\mathcal{R}((-\Delta)^\theta w)).\end{aligned}$$

Moreover, we already saw above that

$$\left\|\mathring{\mathcal{E}}^{(1)}-\mathcal{R}((-\Delta)^\theta w)\right\|_0 \lesssim \lambda^{\overline{\alpha}-1}.$$

These two bounds, by integrating in x and using (5.1.4), yield

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\fint \frac{1}{2}|\check{v}|^2\mathrm{d}x\right| \lesssim \fint |\check{v}|\left|\mathrm{div}(\tilde{R}-S)\right| + |\nabla\check{v}|\left|\mathring{\mathcal{E}}^{(1)}-\mathcal{R}(((-\Delta)^\theta w))\right|\mathrm{d}x \le C(1+\lambda^{\overline{\alpha}}).$$

Thus, the estimate (5.1.6) is proved.

The last thing left is to estimate $|\partial_t\check{\mathcal{J}}|$. Combining some simple calculations with (5.1.4), (3.8.1), (3.8.2), and (A.2), we obtain that

$$\begin{aligned}\left|\partial_t\check{\mathcal{J}}\right| =& \left|\int_{\mathbb{T}^3}\left|(-\Delta)^{\frac{\theta}{2}}\check{v}\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}}\tilde{v}\right|^2 \mathrm{d}x\mathrm{d}s\right| = \left|\int_{\mathbb{T}^3}[2(-\Delta)^{\frac{\theta}{2}}\tilde{v}+(-\Delta)^{\frac{\theta}{2}}w]\cdot(-\Delta)^{\frac{\theta}{2}}w\mathrm{d}x\mathrm{d}s\right|\\ \lesssim& \left|\int_{\mathbb{T}^3} 2(-\Delta)^{\frac{\theta}{2}}\tilde{v}\cdot(-\Delta)^{\frac{\theta}{2}}w\mathrm{d}x\mathrm{d}s\right| + [w]^2_{\theta+\varepsilon}\mathrm{d}s \lesssim \underbrace{\|\tilde{v}\|_{\theta+\varepsilon}\|w\|_{\theta+\varepsilon}}_{=:I} + \lambda^{2(\theta+\varepsilon)}.\end{aligned}$$

The velocity $\tilde{v}$ is bounded by smoothness, so $I \le K(\tilde{v},\theta,\varepsilon)\lambda^{\theta+\varepsilon}$. Since $\lambda > 1$, this yields (5.1.7), thus concluding the proof. ◇

5.2 From strict to strong subsolutions

The second of the two steps in this chapter is given by the following corollary, which is an adaptation to the hypodissipative Navier-Stokes case of [18, Corollary 4.1].

Corollary 5.2.1 (Strict to strong). *Let $(\tilde{v},\tilde{p},\tilde{R})$ be a smooth strict subsolution on $[0,T]$. There exist $\tilde{\delta},\gamma>0$ such that the following holds.*

For any $0<\delta<\tilde{\delta}$, $\alpha,\gamma>0$ and $0<\varepsilon<\beta-\theta$ sufficiently small, there exists a smooth strong subsolution $(\check{v},\check{p},\check{R})$ with $\check{R}(x,t)=\check{\rho}(t)\,\mathrm{Id}+\mathring{\check{R}}(x,t)$, and a "dissipative trace term" as isolated in ***Proposition 5.1.1****, i.e.*

$$\check{\mathcal{T}}(t):=\int_0^t\int_{\mathbb{T}^3}\left(\left|(-\Delta)^{\frac{\theta}{2}}\tilde{v}(x,s)\right|^2-\left|(-\Delta)^{\frac{\theta}{2}}\check{v}(x,s)\right|^2\right)\mathrm{d}x\mathrm{d}s,$$

such that, for all $t\in[0,T]$

$$\int_{\mathbb{T}^3}\left(|\tilde{v}(x,t)|^2+\operatorname{tr}\tilde{R}(x,t)\right)\mathrm{d}x=\int_{\mathbb{T}^3}\left(|\check{v}(x,t)|^2+\operatorname{tr}\check{R}(x,t)\right)\mathrm{d}x \tag{5.2.1}$$

$$\frac{3}{4}\delta\le\check{\rho}\le\frac{5}{4}\delta \tag{5.2.2}$$

$$\left|\mathring{\check{R}}\right|\le\Lambda\check{\varrho}^{1+\gamma} \tag{5.2.3}$$

$$\|\check{v}-\tilde{v}\|_{H^{-1}}\le\delta\lambda_0^{-1} \tag{5.2.4}$$

$$\|\check{v}\|_{1+\alpha}\le\delta_0\lambda_0^{1+\alpha} \tag{5.2.5}$$

$$|\partial_t\check{\rho}(t)|\le\delta\delta_0^{\frac{1}{2}}\lambda_0 \tag{5.2.6}$$

$$\left|\partial_t\check{\mathcal{T}}(t)\right|\le\Lambda^{\frac{1}{2}}\delta_0^{\frac{1}{2}}\lambda_0^{\theta+\varepsilon} \tag{5.2.7}$$

$$\|\check{v}\|_{\theta+\varepsilon}\le K(1+\delta_0^{\frac{1}{2}}\lambda_0^{\theta+\varepsilon}). \tag{5.2.8}$$

where $\check{\varrho}:=\Lambda^{-1}\check{\rho}$, the constant K depends on $(\tilde{v},\tilde{p},\tilde{R})$ and ε, the parameters $\delta_q,\lambda_q,\zeta_q,\Lambda$ are defined as in (4.3.1) *with sufficiently large a, and α is the small parameter from Section 4.3.*

The proof will proceed by first reducing all the claims in the corollary to a series of conditions on λ, and then, at the end, proving that all those conditions can be satisfied simultaneously. This is necessary because some of them are upper bounds on λ, and some are lower bounds.

Proof.

Let
$$\tilde{\delta}:=\frac{1}{2}\inf\left\{\tilde{R}(x,t)\xi\cdot\xi:|\xi|=1,x\in\mathbb{T}^3,t\in[0,T]\right\}.$$

Since $\tilde{R}$ is a smooth positive definite tensor on a compact set, $\tilde{\delta}>0$. Then $S:=\tilde{R}-\delta\,\mathrm{Id}$ is positive definite for any $\delta<\tilde{\delta}$. We may in addition assume without loss of generality that $\delta\le1$. We apply **Proposition 5.1.1** with $(\tilde{v},\tilde{p},\tilde{R}),S$, and $\overline{\alpha}\in(0,1),\varepsilon>0$ to be chosen below. This yields a smooth solution $(\check{v},\check{p},\check{R})$ of (4.1.1) with properties (5.1.2), (5.1.3)-(5.1.5), and (5.1.6). We first note that (5.2.1) coincides with (5.1.2). Next, we observe that $\check{R}-\tilde{R}+S=\check{R}-\delta\,\mathrm{Id}$, so that, since $\mu(t)=\operatorname{tr}(\check{R}-\tilde{R}+S)$ is a function of time only, the function

$$\check{\rho}=\frac{1}{3}\operatorname{tr}\check{R}=\frac{1}{3}\operatorname{tr}(\check{R}-\delta\,\mathrm{Id})+\delta=\frac{1}{3}\operatorname{tr}(\check{R}-\tilde{R}+S)+\delta \tag{5.2.9}$$

is independent of x.

Let us now prove (5.2.2). By the above and (5.1.5) for $N=0$, we have that

$$|\check{\rho}-\delta| = \frac{1}{3}\left|\mathrm{tr}(\check{R}-\tilde{R}+S)\right| \le \left\|\check{R}-\tilde{R}+S\right\|_0 \le C\lambda^{2\theta+\overline{\alpha}-1}, \tag{5.2.10}$$

We require now the following condition on λ:

$$C\lambda^{2\theta-1+\overline{\alpha}} \le \delta_0^{\frac{1}{2}}\lambda_0^{2\theta-1+\overline{\alpha}}. \tag{5.2.11}$$

Then we notice that, for γ sufficiently small and a sufficiently large, we have that

$$\delta_q^{\frac{1}{2}}\lambda_q^{2\theta+\overline{\alpha}-1} \le \frac{1}{4}\delta_{q+1}\zeta_{q+1}^{\gamma}. \tag{5.2.12}$$

Indeed, rewriting the above in terms of λ_q, it reads

$$\Lambda^{\frac{1}{2}}\lambda_q^{-\beta+2\theta+\overline{\alpha}-1} \le \Lambda\lambda_q^{-2b\beta(1+\gamma)}.$$

Since $\Lambda \ge 1$ by (4.3.3), this reduces to showing that

$$-\beta+2\theta+\overline{\alpha}-1 < -2b\beta(1+\gamma). \tag{5.2.13}$$

and taking a sufficiently large. In turn, (5.2.13) can be proved using that, by assumption, $\theta<\beta, 2b\beta<1-\beta$ (see (4.3.2)), and taking $\overline{\alpha},\gamma$ sufficiently small. Thus (5.2.12) is proved.

Now from (5.2.10), (5.2.11), and (5.2.12) for $q=0$, it follows that

$$|\check{\rho}-\delta| \le \frac{1}{4}\delta_1\zeta_1^{\gamma} = \frac{1}{4}\Lambda^{-\gamma}\delta^{1+\gamma} \le \frac{1}{4}\delta, \tag{5.2.14}$$

where in the last inequality we used the fact that $\delta<1<\Lambda$. We have thus proved (5.2.2).

From this estimate we can in turn deduce (5.2.3). Indeed, since $\mathring{R} = \mathring{\check{R}} - \mathring{\tilde{R}} + \mathring{S}$, by chaining the inequalities (5.1.5) for $N=0$, (5.2.11), and (5.2.12) for $q=0$, we analogously deduce that:

$$\left|\mathring{\check{R}}\right| \le \frac{1}{4}\delta_1\zeta_1^{\gamma} = \frac{1}{4}\Lambda^{-\gamma}\delta^{1+\gamma} \le \left(\frac{3}{4}\right)^{1+\gamma}\delta^{1+\gamma}\Lambda^{-\gamma} \le \check{\rho}^{1+\gamma}\Lambda^{-\gamma} \le \Lambda\check{\varrho}^{1+\gamma}.$$

The bound (5.2.4) follows from (5.1.3) together with the following condition on λ:

$$C\lambda^{-1} \le \delta\lambda_0^{-1}. \tag{5.2.15}$$

To obtain (5.2.5), we first use standard interpolation estimates together with (5.1.4) to obtain that

$$\|\check{v}\|_{1+\alpha} \le C_I\|\check{v}\|_1^{1-\alpha}\|\check{v}\|_2^{\alpha} \le C_I C\lambda^{1+\alpha}.$$

Therefore, (5.2.5) reduces to the following condition on λ:

$$CC_I\lambda^{1+\alpha} \le \delta_0^{\frac{1}{2}}\lambda_0^{1+\alpha}, \tag{5.2.16}$$

The estimate (5.2.6) follows from (5.1.6) and (5.2.9), giving

$$|\partial_t \check{\rho}| = \frac{1}{3}\big|\partial_t \operatorname{tr}(\check{R}-\tilde{R}+S)\big| \le \frac{C}{3}\lambda^{\overline{\alpha}}.$$

Therefore, (5.2.6) amounts to

$$\frac{C}{3}\lambda^{\overline{\alpha}} \le \delta\Lambda^{\frac{1}{2}}\lambda_0^{1-\beta}. \tag{5.2.17}$$

Since by (5.1.7) one has that $|\partial_t\check{\mathcal{T}}| \le C\lambda^{2(\theta+\varepsilon)}$, to obtain (5.2.7) we require

$$C\lambda^{2(\theta+\varepsilon)} \le \Lambda^{\frac{1}{2}}\delta_0^{\frac{1}{2}}\lambda_0^{\theta+\varepsilon}. \tag{5.2.18}$$

Finally, to obtain (5.2.8), we note that $\check{v}$ is smooth and thus bounded by a constant C_0, so that, by interpolation and (5.1.4), we have that

$$\|\check{v}\|_{\theta+\varepsilon} \le C_I\|\check{v}\|_0^{1-\theta-\varepsilon}\|\check{v}\|_1^{\theta+\varepsilon} \le C_I C_0^{1-\theta-\varepsilon}(C\lambda)^{\theta+\varepsilon}.$$

Therefore, we will require

$$C_I C_0^{1-\theta-\varepsilon}(C\lambda)^{\theta+\varepsilon} \le \delta_0^{\frac{1}{2}}\lambda_0^{\theta+\varepsilon}. \tag{5.2.19}$$

To conclude the proof of the corollary, we now show that, for suitable choices of $\delta,\gamma,\overline{\alpha}$, there exists a λ satisfying conditions (5.2.11), (5.2.15), (5.2.16), (5.2.17), (5.2.18), and (5.2.19).

In particular, for fixed constants $\overline{C},\overline{K}$ independent of the parameters $a,\delta,b,\overline{\alpha},\beta$, the following conditions must be satisfied by λ:

$$\lambda \ge \overline{K}\delta_0^{\frac{1}{2}(2\theta-1+\overline{\alpha})^{-1}}\lambda_0 \tag{5.2.20}$$

$$\lambda \ge \overline{K}\delta^{-1}\lambda_0 \tag{5.2.21}$$

$$\lambda \le \overline{C}\lambda_0\delta_0^{\frac{1}{2+2\alpha}} \tag{5.2.22}$$

$$\lambda \le \overline{C}\delta^{\frac{1}{\overline{\alpha}}}\Lambda^{\frac{1}{2\overline{\alpha}}}\lambda_0^{\frac{1-\beta}{\overline{\alpha}}} \tag{5.2.23}$$

$$\lambda \le \overline{C}\Lambda^{\frac{1}{4(\theta+\varepsilon)}}\delta_0^{\frac{1}{4(\theta+\varepsilon)}}\lambda_0^{\frac{1}{2}} \tag{5.2.24}$$

$$\lambda \le \overline{C}\delta_0^{\frac{1}{2}(\theta+\varepsilon)^{-1}}\lambda_0 \tag{5.2.25}$$

First we choose $\delta < 1$, and $\overline{\alpha},\gamma$ sufficiently small, and then show that, for a sufficiently large there exists a λ satisfying all the above inequalities.

First of all, notice that, since $\delta_0 = \delta\lambda_0^{2\beta(b-1)} \gg 1$ if δ is fixed and a is sufficiently large, then (5.2.21) implies (5.2.20), and (5.2.22) implies (5.2.25) independently of the choice of $\alpha > 0$, since $\theta+\varepsilon < \beta < \frac{1}{3} < 1+\alpha$.

Hence, we are left with showing that (5.2.21) is compatible with (5.2.22)-(5.2.24).

The compatibility of (5.2.21) and (5.2.22), independently of $\alpha > 0$, is straightforward, since $\delta_0 \gg 1$ when a is sufficiently large.

Inequality (5.2.23) does not contradict (5.2.21) provided we choose $\overline{\alpha}$ so small that $\frac{1-\beta}{\overline{\alpha}} > 1$, and then a sufficiently large.

The compatibility of (5.2.21) with (5.2.24) rewrites as

$$\lambda_0^{\frac{1}{2}} \leq \frac{\overline{C}\delta}{\overline{K}} \Lambda^{\frac{1}{4(\theta+\varepsilon)}} \delta_0^{\frac{1}{4(\theta+\varepsilon)}}$$

and, inserting the definitions of $\delta_0, \Lambda, \lambda_1$, as

$$\lambda_0^{\frac{1}{2}} \leq \frac{\overline{C}}{\overline{K}} \delta^{\frac{1}{2(\theta+\varepsilon)}+1} \lambda_0^{(2b-1)\left(\frac{\beta}{2(\theta+\varepsilon)}\right)}.$$

Hence the above reduces to showing that

$$\frac{1}{2} \leq \frac{\beta(2b-1)}{2(\theta+\varepsilon)},$$

which holds since $b > 1$ and $\theta + \varepsilon < \beta$.

The proof is thus complete. $\diamond$

Chapter 6

The iterative step

6.1 Partitioning the time interval

As stated in Section 4.3, the proof of the theorem continues with two convex integration schemes. The first one approximates strong subsolutions (and thus strict subsolution by the result of the previous chapter) with adapted subsolutions, whereas the second one approximates adapted subsolutions with weak solutions.

Each step of these iterations consists of a mollification step, a gluing step, and a perturbation step. The first two are performed together in **Proposition 6.3.1**, whereas the last one is carried out in **Proposition 6.4.1**. In order to perform the gluing step, we need a suitable partition of the time interval $[0,T]$ into intervals of length τ_q.

Definition 6.1.1 (Decomposing the time interval). *Let $0 \leq T_1 < T_2 \leq T$ such that $T_2 - T_1 > 4\tau_q$. We define sequences of intervals $\{I_i\}, \{J_i\}$ as follows. Let*

$$t_i := i\tau_q \qquad I_i := \left[t_i + \frac{1}{3}\tau_q, t_i + \frac{2}{3}\tau_q\right] \cap [0,T], \tag{6.1.1}$$

and let

$$\underline{n} := \begin{cases} \min\left\{i : t_i - \frac{2}{3}\tau_q \geq T_1\right\} & T_1 > 0 \\ 0 & T_1 = 0 \end{cases} \qquad \overline{n} := \max\left\{i : t_i + \frac{2}{3}\tau_q \leq T_2\right\}. \tag{6.1.2}$$

Moreover, define

$$\begin{aligned} J_i &:= \left(t_i - \frac{1}{3}\tau_q, t_i + \frac{1}{3}\tau_q\right) \cap [0,T] \qquad \underline{n} \leq i \leq \overline{n} \\ J_{\underline{n}-1} &:= \left[0, t_{\underline{n}} - \frac{2}{3}\tau_q\right) \qquad J_{\overline{n}+1} := \left(t_{\overline{n}} + \frac{2}{3}\tau_q, T\right]. \end{aligned} \tag{6.1.3}$$

These form a pairwise disjoint decomposition of $[0,T]$:

$$[0,T] = J_{\underline{n}-1} \cup I_{\underline{n}-1} \cup [J_{\underline{n}} \cup \ldots \cup J_{\overline{n}}] \cup I_{\overline{n}} \cup J_{\overline{n}+1}, \tag{6.1.4}$$

and

$$t_{\underline{n}} < T_1 + \frac{5}{3}\tau_q < T_2 - \frac{5}{3}\tau_q < t_{\overline{n}}. \tag{6.1.5}$$

Moreover, if $T_1 > 0$, $\underline{n} \geq 1$, otherwise we have both that $\underline{n} = 0$ and that $J_{\underline{n}-1} \cup I_{\underline{n}-1} = \varnothing$.

6.2 A technical lemma

We now prove a technical lemma, which will be used to simplify the proof of the gluing step in the next section. To state it, we need a couple of definitions.

Given a strong subsolution (v,p,R) with $\rho \coloneqq \frac{1}{3}\operatorname{tr}R$ and $\varrho \coloneqq \Lambda^{-1}\rho$, we will define:

$$(\rho_i,\, \varrho_i, \ell_{q,i}) \coloneqq \left(\rho(t_i),\, \varrho(t_i),\, \frac{\varrho_i^{\frac{1+\gamma}{2}}}{\zeta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_q^{-\alpha}}\right), \tag{6.2.1}$$

where α,γ are the parameters of Section 4.3. Assuming $a \gg 1$ is sufficiently large (as in (4.3.13), depending on α,γ,b) and $\rho \gtrsim \delta_{q+2}$, we may ensure that

$$\lambda_{q+1}^{-1} \le \ell_q \le \ell_i \le \lambda_q^{-1}. \tag{6.2.2}$$

Since we will always be working with $\theta < \beta$, recalling $\ell_q^{-1} \le \lambda_{q+1}$, and assuming $\varepsilon \le \alpha$, we observe that

$$\tau_q \ell_{q,i}^{-2\theta-\varepsilon} \le \tau_q \ell_q^{-2\theta-\varepsilon} \le \ell_q^{3\alpha}\Lambda^{-\frac{1}{2}}\left(\zeta_q^{\frac{1}{2}}\lambda_q\right)^{-1}\lambda_{q+1}^{2\theta} \le \ell_q^{3\alpha}\lambda_q^{\beta-1+2b\theta} < \ell_q^{3\alpha}, \tag{6.2.3}$$

for b sufficiently close to 1.

Lemma 6.2.1 (Material derivative estimates for subsolutions and potentials). *Let (v,p,R), (v',p',R') be two solutions of* (4.1.1)*, and let $z \coloneqq (-\Delta)^{-1}\operatorname{curl} v, z' \coloneqq (-\Delta)^{-1}\operatorname{curl} v$. Then the following estimates hold for every $N \in \mathbb{N}, \alpha \in (0,1)$:*

$$\begin{aligned}
\left\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(v-v')\right\|_{N+\alpha} &\lesssim \left\|v-v'\right\|_{N+\alpha}(\|v\|_{1+\alpha} + \left\|v'\right\|_{1+\alpha}) \\
&\quad + \left\|v-v'\right\|_{\alpha}(\|v\|_{N+1+\alpha} + \left\|v'\right\|_{N+1+\alpha}) \\
&\quad + \left\|R-R'\right\|_{N+1+\alpha}
\end{aligned} \tag{6.2.4}$$

$$\begin{aligned}
\left\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(z'-z)\right\|_{N+\alpha} &\lesssim \left\|z'-z\right\|_{N+\alpha}\|v\|_{1+\alpha} \\
&\quad + \left\|z'-z\right\|_{\alpha}\|v\|_{N+1+\alpha} \\
&\quad + \left\|R'-R\right\|_{N+\alpha}.
\end{aligned} \tag{6.2.5}$$

If in addition we assume that $v_{t_0} - v'_{t_0} = 0$ for some time t_0 and the following estimates hold:

$$\max\{\|v\|_{N+1+\alpha}, \left\|v'\right\|_{N+1+\alpha}\} \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{-N} \tag{6.2.6}$$

$$\max\{\|v\|_{\theta+\varepsilon}, \left\|v'\right\|_{\theta+\varepsilon}\} \lesssim \Lambda^{\frac{1}{2}} \tag{6.2.7}$$

$$\max\{\|R\|_{N+\alpha}, \left\|R'\right\|_{N+\alpha}\} \lesssim \Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-N-\alpha}, \tag{6.2.8}$$

then the following estimates hold on $[t_0, t_0 + \frac{4}{3}\tau_q]$*:*

$$\|v' - v\|_{N+\alpha} \lesssim \Lambda^{\frac{1}{2}} \varrho_i^{\frac{1+\gamma}{2}} \ell_{q,i}^{-N} \ell_q^{\alpha} \tag{6.2.9}$$

$$\begin{aligned}\|(-\Delta)^\theta(v' - v)\|_{N+\alpha} &\lesssim \Lambda^{\frac{1}{2}} \varrho_i^{\frac{1+\gamma}{2}} \ell_{q,i}^{-2\theta-\varepsilon-N} \ell_q^{\alpha} \\ &\lesssim \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-1-N-\alpha}\end{aligned} \tag{6.2.10}$$

$$\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(v' - v)\|_{N+\alpha} \lesssim \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-1-N+2\alpha} \tag{6.2.11}$$

$$\|z' - z\|_{N+\alpha} \lesssim \Lambda \tau_q \varrho_i^{1+\gamma} \ell_{q,i}^{-N} \ell_q^{-\alpha} \tag{6.2.12}$$

$$\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(z' - z)\|_{N+\alpha} \lesssim \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-N-\alpha} \tag{6.2.13}$$

$$\begin{aligned}\|(-\Delta)^\theta(z' - z)\|_{N+\alpha} &\lesssim \Lambda \tau_q \varrho_i^{1+\gamma} \ell_{q,i}^{-N-2\theta-\varepsilon} \ell_q^{-\alpha} \\ &\lesssim \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-N} \ell_q^{-\alpha}\end{aligned} \tag{6.2.14}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} |v'|^2 - |v|^2 \mathrm{d}x\right| \lesssim \Lambda \varrho_i^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha}. \tag{6.2.15}$$

Note that, since the proofs require the Schauder estimates of **Lemma A.4**, this result does not hold for $\alpha = 0$.

Proof.

First of all, we observe that:

$$\begin{aligned}(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(v - v') &= -\nabla p - \operatorname{div} R - [(v - v')\cdot\nabla]v' + \nabla p' + \operatorname{div} R' \\ &= -[(v - v')\cdot\nabla]v' - \nabla(p - p') - \operatorname{div}(R - R').\end{aligned}$$

The first and third term clearly satisfy (6.2.4). We thus focus on the pressure term. Taking divergence:

$$\Delta(p - p') = \operatorname{div}\{-(v\cdot\nabla)(v - v') - [(v - v')\cdot\nabla]v' - \operatorname{div}(R - R')\}.$$

We then note that:

$$\operatorname{div}[a\cdot\nabla b - b\cdot\nabla a] = a\cdot\nabla \operatorname{div} b - b\cdot\nabla \operatorname{div} a, \tag{6.2.16}$$

which, combined with Schauder estimates, yields:

$$\begin{aligned}\|\nabla(p - p')\|_{N+\alpha} \le \|p - p'\|_{N+1+\alpha} &\lesssim \|[(v - v')\cdot\nabla](v + v') + \operatorname{div}(R - R')\|_{N+\alpha} \\ &\lesssim \|v - v'\|_{N+\alpha}\|v + v'\|_{1+\alpha} + \|v - v'\|_{\alpha}\|v + v'\|_{N+1+\alpha} + \|R - R'\|_{N+1+\alpha}.\end{aligned}$$

This yields the desired estimate (6.2.4).

Coming then to the potentials, we observe that, since the velocities will be chosen so that $\int v = \int v' = 0$, and recalling $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$, we have $\operatorname{curl} z = v$ and $\operatorname{curl} z' = v'$, and moreover:

$$\begin{aligned}(v\cdot\nabla)(v' - v) &= \operatorname{curl}((v\cdot\nabla)(z' - z)) + \operatorname{div}(((z' - z)\times\nabla)v) \\ \\ [(v' - v)\cdot\nabla]v' &= \operatorname{div}(((z' - z)\times\nabla)v'^T),\end{aligned}$$

so, recalling the above transport equation:

$$(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(v - v') = -[(v-v')\cdot\nabla]v' - \nabla(p-p') - \operatorname{div}(R-R').$$

we see we can rewrite it as:

$$\begin{aligned}\operatorname{curl}[(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(z'-z)] = &-\operatorname{div}(((z'-z)\times\nabla)v + ((z'-z)\times\nabla)v'^T)\\ &-\nabla(p'-p) - \operatorname{div}(R'-R).\end{aligned}$$

Recall once more the identity $\operatorname{curl}^2 = \nabla\operatorname{div} - \Delta$, so that, taking curl, our transport equation becomes:

$$\begin{aligned}-\Delta[(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(z'-z)] = &-\nabla\operatorname{div}[(v\cdot\nabla)(z'-z)]\\ &-\operatorname{curl}\operatorname{div}(((z'-z)\times\nabla)v + ((z'-z)\times\nabla)v'^T)\\ &-\operatorname{curl}\operatorname{div}(R'-R).\end{aligned}$$

Recalling (6.2.16) and that $\operatorname{div} v = 0 = \operatorname{div}(z'-z)$, Schauder estimates then give us, for $N \geq 2$:

$$\begin{aligned}\left\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)(z'-z)\right\|_{N+\alpha} \lesssim &\left\|((z'-z)\cdot\nabla)v\right\|_{N+\alpha}\\ &+\left\|((z'-z)\times\nabla)v + ((z'-z)\times\nabla)v'^T\right\|_{N+\alpha}\\ &+\left\|R'-R\right\|_{N+\alpha}\\ \lesssim &\left\|z'-z\right\|_{N+\alpha}\|v\|_{1+\alpha} + \left\|z'-z\right\|_\alpha\|v\|_{N+1+\alpha}\\ &+\left\|z'-z\right\|_{N+\alpha}\left\|v'\right\|_{1+\alpha} + \left\|z'-z\right\|_\alpha\left\|v'\right\|_{N+1+\alpha}\\ &+\left\|R'-R\right\|_{N+\alpha},\end{aligned}$$

our second desired estimate. As for $N=0$ and $N=1$, we use $(-\Delta)^{-1}$ and the RHS will only exhibit order-0 operators, which are continuous $C^A \to C^A$ for all $A \in \mathbb{R}^+$.

We must now obtain (6.2.9)-(6.2.15). We first remark that (6.2.11) follows easily by plugging (6.2.6)-(6.2.8) and (6.2.9) into (6.2.4), while (6.2.13) follows by plugging (6.2.6)-(6.2.8) and (6.2.12) into (6.2.5). (6.2.12) can be obtained from (6.2.5) by the same argument we shall now use to deduce (6.2.9) from (6.2.4), so we will only prove (6.2.9).

To that end, we first set $N=0$. With this choice, plugging (6.2.6)-(6.2.8) into (6.2.4) gives

$$\left\|(\partial_t + v\cdot\nabla + (-\Delta)^\theta)v' - v\right\|_\alpha \lesssim \left\|v'-v\right\|_\alpha \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} + \Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-1-\alpha}.$$

We can then use **Proposition B.2** on $[t_0, t_0 + \frac{4}{3}\tau_q]$, since then $|t-t_0| \leq \frac{4}{3}\tau_q < [v_t'-v_t]_1^{-1}$, and thus obtain that

$$\left\|v_t'-v_t\right\|_\alpha \lesssim \int_{t_0}^{t} \left\|v_s'-v_s\right\|_\alpha \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} + \Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-1-\alpha}\,\mathrm{d}s.$$

By Grönwall's inequality, we then have

$$\left\|v_t'-v_t\right\|_\alpha \lesssim \Lambda\tau_q\varrho_i^{1+\gamma}\ell_{q,i}^{-1-\alpha}e^{\int_{t_0}^{t}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}} \lesssim \Lambda^{\frac{1}{2}}\varrho_i^{\frac{1+\gamma}{1}}\frac{\varrho_i^{\frac{1+\gamma}{2}}}{\zeta_q\lambda_q^{1+\alpha}\ell_q^{-1-\alpha}}\ell_q^{4\alpha-\alpha}\lambda_q^\alpha\ell_{q,i}^{-\alpha} \lesssim \Lambda^{\frac{1}{2}}\varrho_q^{\frac{1+\gamma}{2}}\ell_q^\alpha,$$

which is (6.2.9) in the case $N=0$.

We then continue by setting $N=1$. We plug (6.2.6)-(6.2.8) and the case $N=0$ of (6.2.9) into (6.2.4), thus obtaining

$$\begin{aligned}\left\|(\partial_t+v\cdot\nabla+(-\Delta)^\theta)(v-v')\right\|_{1+\alpha} &\lesssim \left\|v-v'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\\ &\quad+\Lambda^{\frac{1}{2}}\varrho_i^{\frac{1+\gamma}{2}}\ell_q^{\alpha}\cdot\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{-1}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha}\\ &\lesssim \left\|v-v'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha}.\end{aligned}$$

Applying **Proposition B.2**, we conclude

$$\begin{aligned}\left\|v-v'\right\|_{1+\alpha} &\lesssim \int_{t_0}^{t}\left\|v-v'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha}\mathrm{d}s\\ &\quad+\int_{t_0}^{t}(t-s)\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\left[\left\|v_s-v_s'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha}\right]\mathrm{d}s\\ &\lesssim \int_{t_0}^{t}\left\|v-v'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha}\mathrm{d}s,\end{aligned}$$

so that, by Grönwall

$$\left\|v-v'\right\|_{1+\alpha}\lesssim\Lambda\tau_q\varrho_i^{1+\gamma}\ell_{q,i}^{-2-\alpha},$$

which, again as before, gives us (6.2.9) for $N=1$.

With these two cases on our hands, we can tackle the general case. Plug (6.2.9) for $N=0$ and $N=1$, as well as (6.2.6)-(6.2.8), into (6.2.4) to get

$$\begin{aligned}\left\|(\partial_t+v\cdot\nabla+(-\Delta)^\theta)(v-v')\right\|_{N+\alpha} &\lesssim \left\|v-v'\right\|_{N+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\\ &\quad+\Lambda^{\frac{1}{2}}\varrho_i^{\frac{1+\gamma}{2}}\ell_q^{\alpha}\cdot\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{-N}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-N-\alpha}\\ &\lesssim \left\|v-v'\right\|_{N+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N-\alpha}.\end{aligned}$$

Applying, once again, **Proposition B.2**, we conclude that

$$\begin{aligned}\left\|v-v'\right\|_{N+\alpha} &\lesssim \int_{t_0}^{t}\left\|v-v'\right\|_{N+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-N-1-\alpha}\mathrm{d}s\\ &\quad+\int_{t_0}^{t}(t-s)\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\left[\left\|v_s-v_s'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-N-1-\alpha}\right]\mathrm{d}s\\ &\lesssim \int_{t_0}^{t}\left\|v-v'\right\|_{1+\alpha}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}+\Lambda\varrho_i^{1+\gamma}\ell_{q,i}^{-N-1-\alpha}\mathrm{d}s,\end{aligned}$$

and using the $N=1$ case of (6.2.9) and the fact that $(t-s)\|v_s - v'_s\|_{1+\alpha} \lesssim 1$:

$$\left\|v-v'\right\|_{N+\alpha} \lesssim \int_{t_0}^{t} \left\|v-v'\right\|_{N+\alpha} \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} + \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-N-1-\alpha} \mathrm{d}s,$$

which yields the other cases of (6.2.9) by Grönwall's inequality.

The first inequality in (6.2.10) is obtained by simply interpolating (6.2.9), since $\|(-\Delta)^\theta (v'-v)\|_{N+\alpha} \lesssim \|v'-v\|_{N+\alpha+2\theta+\varepsilon}$ by **Theorem 3.8.1**. To obtain the second inequality, note that

$$\begin{aligned}\Lambda^{\frac{1}{2}} \varrho_i^{\frac{1+\gamma}{2}} \ell_{q,i}^{-2\theta-\varepsilon-N} \ell_q^{\alpha} &= \Lambda \varrho_i^{1+\gamma} \tau_q \ell_q^{-4\alpha} \lambda_q^{-\alpha} \ell_q^{\alpha} \ell_{q,i}^{-2\theta-\varepsilon-N-1} \ell_q^{\alpha} \\ &\leq \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-N-1-\alpha} \cdot \ell_q^{-4\alpha} \lambda_q^{-\alpha} \ell_q^{\alpha} \ell_q^{3\alpha},\end{aligned}$$

where we used (6.2.3) in the last step. This yields (6.2.10). Estimate (6.2.14) is obtained similarly.

We are then left only with (6.2.15). To obtain it, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} \left|v'\right|^2 - |v|^2 \mathrm{d}x = \int_{\mathbb{T}^3} 2v' \partial_t v' - 2v \partial_t v \mathrm{d}x,$$

and using the fact that $(v,p,R),(v',p',R')$ are subsolutions and $\operatorname{div} v = 0 = \operatorname{div} v'$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} \left|v'\right|^2 - |v|^2 \mathrm{d}x = -2 \int_{\mathbb{T}^3} v'((-\Delta)^\theta v' + \operatorname{div} R') - v((-\Delta)^\theta v + \operatorname{div} R) \mathrm{d}x.$$

Integration by parts then gives us that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} \left|v'\right|^2 - |v|^2 \mathrm{d}x\right| &\leq 2\left|\int_{\mathbb{T}^3} \mathfrak{D}v : R \mathrm{d}x\right| + 2\left|\int_{\mathbb{T}^3} \mathfrak{D}v' R' \,\mathrm{d}x\right| + 2\left|\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}} v\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}} v'\right|^2 \mathrm{d}x\right| \\ &\lesssim \|v\|_{1+\alpha} \|R\|_\alpha + \left\|v'\right\|_{1+\alpha} \left\|R'\right\|_\alpha + \left\|v+v'\right\|_{\theta+\varepsilon} \left\|v-v'\right\|_{\theta+\varepsilon} \\ &\overset{*}{\lesssim} \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \cdot \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-\alpha} + \Lambda^{\frac{1}{2}} \cdot \Lambda \varrho_i^{1+\gamma} \ell_{q,i}^{-\theta-\varepsilon-1-\alpha} \\ &= \Lambda \varrho_i^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha} (F_1 + F_2),\end{aligned}$$

where we used (6.2.9) in $*$ and we write

$$\begin{aligned} F_1 &\coloneqq \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_{q,i}^{-\alpha} \tau_q \ell_q^{-2\alpha} \\ F_2 &\coloneqq \Lambda^{\frac{1}{2}} \tau_q^2 \ell_q^{-2\alpha} \ell_{q,i}^{-1-\theta} \ell_{q,i}^{-\varepsilon-\alpha}. \end{aligned}$$

It is easy to see that

$$F_1 = \ell_q^{4\alpha} \lambda_q^{\alpha} \ell_{q,i}^{-\alpha} \ell_q^{-2\alpha} \lesssim 1.$$

To prove that F_2 is also bounded above by a constant, we note that

$$F_2 = f(\alpha,\varepsilon,\gamma)\frac{\Lambda^{\frac{1}{2}}}{\delta_q\lambda_q^2}\frac{(\zeta_q^{\frac{1}{2}}\lambda_q)^{1+\theta}}{\varrho_{q,i}^{\frac{1+\theta}{2}}},$$

where $f(\alpha,\varepsilon,\gamma)\sim 1$ for $\alpha,\gamma,\varepsilon \ll 1$. We then observe that (6.3.2), together with the fact that $\theta < \frac{1}{3}, b > 1$, implies that

$$\frac{\Lambda^{\frac{1}{2}}}{\delta_q\lambda_q^2}\frac{(\zeta_q^{\frac{1}{2}}\lambda_q)^{1+\theta}}{\varrho_{q,i}^{\frac{1+\theta}{2}}} \le \varrho_{q,i}^{-\frac{2}{3}}(\zeta_q^{\frac{1}{2}}\lambda_q)^{-\frac{2}{3}} \lesssim \zeta_{q+2}^{-\frac{2}{3}}\lambda_q^{-\frac{2}{3}(1-\beta)} = \lambda_q^{\frac{4}{3}\beta b^2-\frac{2}{3}(1-\beta)} = \lambda_q^{-\frac{2}{3}(1-\beta-2b^2\beta)} \le 1.$$

We have thus proved that

$$F_2 \lesssim 1,$$

provided $\alpha,\gamma,\varepsilon \ll 1$ are sufficiently small and a is sufficiently large. The proof is thus complete. ◇

6.3 Gluing step

We can finally state and prove the proposition that contains the gluing step.

Proposition 6.3.1 (Gluing step). *Let b,β,α,γ and $(\delta_q,\lambda_q,\Lambda,\zeta_q,\ell_q,\tau_q)$ be as in Section 4.3, with*

$$\alpha b < \beta\gamma \tag{6.3.1}$$

$$b^2(1+\gamma) < \frac{1-\beta}{2\beta}, \tag{6.3.2}$$

Let $[T_1,T_2]\subset[0,T]$ with $|T_2-T_1| > 4\tau_q$. Let (v_q,p_q,R_q) be a strong subsolution on $[0,T]$ which on $[T_1,T_2]$ satisfies the estimates

$$\frac{3}{4}\delta_{q+2} \le \rho_q \le \frac{7}{2}\delta_{q+1} \tag{6.3.3}$$

$$\|\mathring{R}_q\|_0 \le \Lambda\varrho_q^{1+\gamma} \tag{6.3.4}$$

$$\|v_q\|_{1+\alpha} \le M\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} \tag{6.3.5}$$

$$\|v_q\|_{\theta+\varepsilon} \le M\left(1+\sum_{i=0}^{q}\delta_i^{\frac{1}{2}}\lambda_i^{\theta+\varepsilon}\right) \tag{6.3.6}$$

$$|\partial_t\rho_q| \lesssim \rho_q\delta_q^{\frac{1}{2}}\lambda_q \tag{6.3.7}$$

with some constant $M>0$, where

$$\rho_q := \frac{1}{3}\operatorname{tr}R_q \qquad\qquad \varrho_q := \frac{\rho_q}{\Lambda}.$$

Define $\rho_{q,i}$, $\varrho_{q,i},\ell_{q,i}$ as in (6.2.1)*, using (v_q,p_q,R_q) as the starting point.*

Then, provided $a \gg 1$ is sufficiently large, there exists $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ solution of (4.1.1) on $[0,T]$ such that

$$(\overline{v}_q, \overline{p}_q, \overline{R}_q) = (v_q, p_q, R_q) \qquad \text{on } [0,T] \setminus [T_1, T_2], \tag{6.3.8}$$

and on $[T_1, T_2]$ the following estimates hold:

$$\left\|\overline{v}_q - v_q\right\|_\alpha \lesssim \Lambda^{\frac{1}{2}} \overline{\varrho}_q^{\frac{1+\gamma}{2}} \ell_q^\alpha \tag{6.3.9}$$

$$\left\|\overline{v}_q\right\|_{1+\alpha} \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \tag{6.3.10}$$

$$\left\|\overline{v}_q\right\|_{\theta+\varepsilon} \le M\left(1 + \sum_{i=0}^{q+1} \delta_i^{\frac{1}{2}} \lambda_i^{\theta+\varepsilon}\right) \tag{6.3.11}$$

$$\left\|\mathring{\overline{R}}_q\right\|_0 \lesssim \Lambda \overline{\varrho}_q^{1+\gamma} \ell_q^{-2\alpha} \tag{6.3.12}$$

$$\frac{7}{8}\rho_q \le \overline{\rho}_q \le \frac{9}{8}\rho_q \tag{6.3.13}$$

$$\left|\partial_t \overline{\rho}_q\right| \lesssim \overline{\rho}_q \delta_q^{\frac{1}{2}} \lambda_q \tag{6.3.14}$$

$$\left|\rho_q - \overline{\rho}_q\right| = \frac{1}{3}\left|\int_{\mathbb{T}^3} \left(\left|v_q\right|^2 - \left|\overline{v}_q\right|^2\right) \mathrm{d}x\right| \lesssim \Lambda \overline{\varrho}_q^{1+\gamma} \lambda_q^{-\alpha} \ell_q^\alpha. \tag{6.3.15}$$

Moreover, on $[t_{\underline{n}}, t_{\overline{n}}]$ the following additional estimates hold for $t \in I_{i-1} \cup J_i \cup I_i$:

$$\left\|\overline{v}_q\right\|_{N+1+\alpha} \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_{q,i}^{-N} \tag{6.3.16}$$

$$\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \Lambda \overline{\varrho}_q^{1+\gamma} \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lambda_q^{-\alpha} \tag{6.3.17}$$

$$\left\|(\partial_t + \overline{v}_q \cdot \nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \Lambda \overline{\varrho}_q^{1+\gamma} \tau_q^{-1} \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lambda_q^{-\alpha}. \tag{6.3.18}$$

Regarding the support of the Reynolds stress, we have that

$$\mathring{\overline{R}}_q(\cdot, t) \equiv 0 \qquad \forall t \in \bigcup_{i=\underline{n}}^{\overline{n}} J_i. \tag{6.3.19}$$

In terms of energy, we have that

$$\int_{\mathbb{T}^3} \left(\left|\overline{v}_q\right|^2(x,t) + \operatorname{tr}\overline{R}_q(x,t)\right) \mathrm{d}x = \int_{\mathbb{T}^3} \left(\left|v_q\right|^2(x,t) + \operatorname{tr} R_q(x,t)\right) \mathrm{d}x, \tag{6.3.20}$$

and the function

$$\mathcal{T}_g := \int_{\mathbb{T}^3} \int_0^t \left(\left|(-\Delta)^{\frac{\theta}{2}} \overline{v}_q\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}} v_q\right|^2\right) \mathrm{d}s \mathrm{d}x,$$

satisfies

$$\left|\partial_t \mathcal{T}_g\right| \lesssim \Lambda^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\theta+\varepsilon}, \tag{6.3.21}$$

and therefore

$$\left|\mathcal{T}_g(t)\right| \lesssim t\Lambda\overline{\varrho}_q^{1+\gamma}\zeta_q^{\frac{1}{2}}\ell_{q,i}^{-N}\ell_q^{-6\alpha} \leq T\Lambda\overline{\varrho}_q^{1+\gamma}\zeta_q^{\frac{1}{2}}\ell_{q,i}^{-N}\ell_q^{-6\alpha}.$$

Finally, if $2\alpha < \beta\gamma$*, then*

$$\left\|\overline{v}_q - v_q\right\|_\alpha \lesssim \delta_{q+1}^{\frac{1}{2}}\ell_q^{(\frac{2}{b}+\frac{1}{2})\alpha}. \tag{6.3.22}$$

The proof closely follows the gluing procedure of [18, Section 6], which in turn draws heavily from [8, Sections 3-4]. Recall that the solution is left unchanged outside $[T_1, T_2]$ and the gluing only happens in that interval. More precisely, recalling the decomposition (6.1.4):

- The gluing procedure is carried out in the interval

$$J_{\underline{n}} \cup \ldots \cup J_{\overline{n}} = \left(t_{\underline{n}} - \frac{1}{3}\tau_q, t_{\overline{n}} + \frac{1}{3}\tau_q\right); \tag{6.3.23}$$

- The subsolution is left unchanged in $J_{\underline{n}-1} \cup J_{\overline{n}+1}$;
- The intervals $I_{\underline{n}-1}$ and $I_{\overline{n}}$ are used as cutoff regions between the glued and unglued subsolutions.

Recall also that, since the trace $\rho_q = {}^1/_3\,\mathrm{tr}R_q$ has different lower and upper bounds on $[T_1, T_2]$ (respectively of order δ_{q+2} and δ_{q+1}), mollification with different parameters $\ell_{q,i}$ depending on $\rho_q(t_i)$ on intervals of size τ_q around the points t_i is necessary.

Proof. (**Proposition 6.3.1**)

Step 1: Mollification

Let φ be a standard mollification kernel in space and define

$$\begin{aligned}
v_{\ell_{q,i}} &:= v_q * \varphi_{\ell_{q,i}}\\
p_{\ell_{q,i}} &:= p_q * \varphi_{\ell_{q,i}} + \frac{1}{3}(|v_q|^2 * \varphi_\ell - |v_{\ell_{q,i}}|^2)\\
\mathring{R}_{\ell_{q,i}} &:= \mathring{R}_q * \varphi_{\ell_{q,i}} + (v_q \mathring{\otimes} v_q) * \varphi_{\ell_{q,i}} - v_{\ell_{q,i}} \mathring{\otimes} v_{\ell_{q,i}}.
\end{aligned}$$

With this definition, (4.1.1) holds for the triple $(v_{\ell_{q,i}}, p_{\ell_{q,i}}.R_{\ell_{q,i}})$. Using the estimates (6.3.4) and (6.3.5), together with (A.5), we deduce that

$$\left\|v_{\ell_{q,i}} - v_q\right\|_\alpha \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i} = \Lambda^{\frac{1}{2}}\varrho_{q,i}^{\frac{1+\gamma}{2}}\ell_q^\alpha \tag{6.3.24}$$

$$\left\|v_{\ell_{q,i}}\right\|_{N+1+\alpha} \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{-N} \tag{6.3.25}$$

$$\left\|v_{\ell_{q,i}}\right\|_{\theta+\varepsilon} \lesssim \Lambda^{\frac{1}{2}} \tag{6.3.26}$$

$$\begin{aligned}
\left\|\mathring{R}_{\ell_{q,i}}\right\|_{N+\alpha} &\lesssim \Lambda\varrho_q^{1+\gamma}\ell_{q,i}^{-N-\alpha} + \delta_q\lambda_q^{2+2\alpha}\ell_{q,i}^{2-N-\alpha}\\
&\lesssim \Lambda\varrho_q^{1+\gamma}\ell_{q,i}^{-N-\alpha} + \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^\alpha
\end{aligned} \tag{6.3.27}$$

$$\left|\int_{\mathbb{T}^3} |v_q|^2 - |v_{\ell_{q,i}}|^2 \mathrm{d}x\right| \lesssim \delta_q\lambda_q^{2+2\alpha}\ell_{q,i}^2 = \Lambda\varrho_{q,i}^{1+\gamma}\ell_q^{2\alpha}. \tag{6.3.28}$$

To obtain (6.3.28), we also use the trivial identity $\int f * \varphi_\ell = \int f$ for $f = |v_q|^2$.

Step 2: Gluing procedure

Let $\{I_i\}_{\underline{n}\le i\le \overline{n}}$ be the sequence of intervals corresponding to $[T_1,T_2]$ according to **Definition 6.1.1** above. We now fix a partition of unity on $[0,T]$

$$\sum_{i=\underline{n}-1}^{\overline{n}+1} \chi_i \equiv 1$$

subordinate to the decomposition (6.1.4), i.e. $[0,T]=J_{\underline{n}-1}\cup I_{\underline{n}-1}\cup[J_{\underline{n}}\cup\ldots\cup J_{\overline{n}}]\cup I_{\overline{n}}\cup J_{\overline{n}+1}$. More precisely, for each $\underline{n}\le i\le\overline{n}$, the function $\chi_i\ge 0$ satisfies

$$\operatorname{supp}\chi_i\subset I_{i-1}\cup J_i\cup I_i, \qquad \chi_i\Big|_{J_i}\equiv 1 \qquad \left|\partial_t^N\chi_i\right|\lesssim \tau_q^{-N}\quad \forall N\ge 0,$$

whereas for $i=\overline{n}+1, i=\underline{n}-1$ we have

$$\operatorname{supp}\chi_{\overline{n}+1}\subset I_{\overline{n}}\cup J_{\overline{n}+1} \qquad \chi_{\overline{n}+1}\Big|_{J_{\overline{n}+1}}\equiv 1 \qquad \left|\partial_t^N\chi_{\overline{n}+1}\right|\lesssim \tau_q^{-N}\quad \forall N\ge 0,$$
$$\operatorname{supp}\chi_{\underline{n}-1}\subset J_{\underline{n}-1}\cup I_{\underline{n}-1} \qquad \chi_{\overline{n}+1}\Big|_{J_{\underline{n}-1}}\equiv 1 \qquad \left|\partial_t^N\chi_{\underline{n}-1}\right|\lesssim \tau_q^{-N}\quad \forall N\ge 0.$$

We define

$$\overline{v}_q := \sum_{i=\underline{n}-1}^{\overline{n}+1}\chi_i v_i, \qquad \overline{p}_q^{(1)} := \sum_{i=\underline{n}-1}^{\overline{n}+1}\chi_i p_i, \tag{6.3.29}$$

where (v_i,p_i) is defined as follows. For $\underline{n}\le i\le\overline{n}$ we define (v_i,p_i) as the solution of

$$\begin{cases}\partial_t v_i+\operatorname{div}(v_i\otimes v_i)+\nabla p_i+(-\Delta)^\theta v_i=0\\ \operatorname{div} v_i=0\\ v_i(\cdot,t_i)=v_{\ell_{q,i}}(\cdot,t_i)\end{cases}, \tag{6.3.30}$$

and set $(v_i,p_i)=(v_q,p_q)$ for $i=\underline{n}-1$ and $i=\overline{n}+1$. Thus, we note first of all that $\operatorname{div}\overline{v}_q=0$, and moreover

$$(\overline{v}_q,\overline{p}_q^{(1)})=(v_q,p_q), \qquad t\in[0,T]\setminus[T_1,T_2].$$

Next, we define $\overline{R}_q$. We have that $\chi_i+\chi_{i+1}=1$ for $t\in J_i\cup I_i\cup J_{i+1}$, and therefore

$$\begin{aligned}\partial_t\overline{v}_q+\operatorname{div}(\overline{v}_q\otimes\overline{v}_q)+\nabla\overline{p}_q^{(1)}+(-\Delta)^\theta\overline{v}_q&=\partial_t\chi_i\cdot(v_i-v_{i+1})\\&\quad-\chi_i\cdot(1-\chi_i)\operatorname{div}((v_i-v_{i+1})\otimes(v_i-v_{i+1}))\\&\quad-\operatorname{div}(\chi_iR_i+(1-\chi_i)R_{i+1})\end{aligned}$$

for all $\underline{n}-1\le i\le\overline{n}$, where we wrote $R_i=0$ for $\underline{n}\le i\le\overline{n}$ and $R_i=R_q$ otherwise. Thus, recalling the operator $\mathcal{R}$ from **Definition C.1**, set

$$\begin{aligned}\overline{\mathring{R}}_q^{(1)}&:=\begin{cases}-\partial_t\chi_i\mathcal{R}(v_i-v_{i+1})\\ \quad+\chi_i\cdot(1-\chi_i)(v_i-v_{i+1})\mathring{\otimes}(v_i-v_{i+1}) & t\in I_i\\ 0 & t\in J_i\cup J_{i+1}\end{cases}\\ \overline{\mathring{R}}_q^{(2)}&:=\sum_{i=\underline{n}-1}^{\overline{n}+1}\chi_i\mathring{R}_i=(\chi_{\underline{n}-1}+\chi_{\overline{n}+1})\mathring{R}_q,\end{aligned} \tag{6.3.31}$$

and

$$\overline{p}_q^{(2)} := \sum_{i=\underline{n}-1}^{\overline{n}+1} \chi_i \cdot (1-\chi_i) \left(|v_i - v_{i+1}|^2 - \fint_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \mathrm{d}x \right). \tag{6.3.32}$$

Finally, we define

$$\overline{R}_q = \overset{\circ}{\overline{R}}{}_q^{(1)} + \overset{\circ}{\overline{R}}{}_q^{(2)} + \overline{\rho}_q \mathrm{Id}, \qquad \overline{p}_q := \overline{p}_q^{(1)} + \overline{p}_q^{(2)}, \tag{6.3.33}$$

where

$$\overline{\rho}_q := \rho_q + \frac{1}{3} \fint_{\mathbb{T}^3} \left(|v_q|^2 - |\overline{v}_q|^2 \right) \mathrm{d}x. \tag{6.3.34}$$

Define also

$$\mathcal{T}_g(t) := \frac{1}{3} \fint_{\mathbb{T}^3} \left(\left| (-\Delta)^{\frac{\theta}{2}} v_q \right|^2 - \left| (-\Delta)^{\frac{\theta}{2}} \overline{v}_q \right|^2 \right) \mathrm{d}x.$$

By construction, we have that

$$\partial_t \overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla \overline{p}_q = -\operatorname{div} \overline{R}_q,$$

and (6.3.8) and (6.3.20) hold. Moreover

$$\overset{\circ}{\overline{R}}_q = 0 \qquad \forall t \in \bigcup_{i=\underline{n}}^{\overline{n}} J_i.$$

Step 3: Stability estimates on classical solutions

Throughout this step and the next, we will assume estimate (6.3.13), which will be proved in Step 5 below. This estimate will allow us to replace $\overline{\rho}_q$ with ρ_q and vice-versa whenever we need to do so in our estimates, since the two are of the same order.
Let us consider for the moment $\underline{n} \leq i \leq \overline{n}$. We recall the classical existence result for solutions of (6.3.30) found in **Theorem 2.4.1.2**, by which (v_i, p_i) in (6.3.29) above is defined at least on an interval of length $\sim \|v_{\ell_{q,i}}\|_{1+\alpha}^{-1}$. By (6.3.25) and (4.3.11), we have that

$$\left\| v_{\ell_{q,i}} \right\|_{1+\alpha} \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} = \tau_q^{-1} \ell_q^{4\alpha} \lambda_q^{\alpha} \leq \tau_q^{-1}.$$

Therefore, provided $a \gg 1$ is sufficiently large, v_i is defined on $I_{i-1} \cup J_i \cup I_i$, so that $\overline{v}_q$ in (6.3.29) is well-defined.

Next, we deduce from (6.3.7) that $|\partial_t \log \rho_q| \leq \delta_q^{1/2} \lambda_q = \tau_q^{-1} \ell_q^{4\alpha}$, so that, by assuming $a \gg 1$ is sufficiently large, we may ensure that

$$\rho_q(t_1) \leq 4 \rho_q(t_2) \qquad \forall t_1, t_2 \in I_{i-1} \cup J_i \cup I_i, \tag{6.3.35}$$

for any i. In particular $\rho_q \sim \rho_{q,i}$ and $\varrho_q \sim \varrho_{q,i}$ in $I_{i-1} \cup J_i \cup I_i$.

We then note that, thanks to (6.3.25), (6.3.26), (6.3.27), (6.3.35), and the Hölder estimates in **Theorem 2.4.1.2**, we can apply **Lemma 6.2.1** to $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, R_{\ell_{q,i}})$ and $(v_i, p_i, 0)$, thus obtaining (6.2.9), which reads

$$\left\| v_i - v_{\ell_{q,i}} \right\|_{N+\alpha} \lesssim \Lambda^{\frac{1}{2}} \varrho_{q,i}^{\frac{1+\gamma}{2}} \ell_{q,i}^{-N} \ell_q^{\alpha}. \tag{6.3.36}$$

The case $N=0$ of (6.3.36), together with (6.3.24), leads to

$$\left\|\overline{v}_q - v_q\right\|_\alpha \leq \sum_{j=i-1}^{i+1}(\left\|v_j - v_{\ell_{q,j}}\right\|_\alpha + \left\|v_{\ell_{q,j}} - v_q\right\|_\alpha) \lesssim \Lambda^{\frac{1}{2}} \varrho_{q,i}^{\frac{1+\gamma}{2}} \ell_q^\alpha. \tag{6.3.37}$$

By (6.3.13), this is equivalent to (6.3.9).

The case $N=1$ of (6.3.36) leads to

$$\left\|v_i - v_{\ell_{q,i}}\right\|_{1+\alpha} \lesssim \Lambda^{\frac{1}{2}} \varrho_{q,i}^{\frac{1+\gamma}{2}} \ell_{q,i}^{-1} \ell_q^\alpha = \Lambda^{\frac{1}{2}} \zeta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_q^{-\alpha} \ell_q^\alpha = \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha}.$$

Combining the above estimate with (6.3.5) outside the gluing region and in $I_{\underline{n}-1} \cup I_{\overline{n}}$, and with (6.3.25) in $J_{\underline{n}} \cup I_{\underline{n}} \cup \ldots \cup I_{\overline{n}-1} \cup J_{\overline{n}}$, we deduce that (6.3.10) is verified. More generally, as we did above for $N=0$, we deduce from (6.3.25) and (6.3.36) that

$$\left\|\overline{v}_q\right\|_{1+N+\alpha} \leq \sum_{j=i-1}^{i+1} \chi_j \Big(\left\|v_j - v_{\ell_{q,j}}\right\|_{1+N+\alpha} + \left\|v_{\ell_{q,j}}\right\|_{1+N+\alpha}\Big) \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_{q,i}^{-N} \quad \forall t \in \operatorname{supp} \chi_i.$$

We have used the fact that $\ell_{q,i} \sim \ell_{q,i+1} \sim \ell_{q,i-1}$. The above inequality coincides with (6.3.16).

We also remark the following simple interpolation of the $N=0$ and $N=1$ cases of (6.3.36), which will be used in Step 5 below.

$$\left\|v_i - v_{\ell_{q,i}}\right\|_{\theta+\varepsilon} \lesssim \left\|v_i - v_{\ell_{q,i}}\right\|_\alpha^{1-\theta-\varepsilon} \left\|v_i - v_{\ell_{q,i}}\right\|_{1+\alpha}^{\theta+\varepsilon} \lesssim \Lambda \tau_q \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{-\theta-\varepsilon-1-\alpha}. \tag{6.3.38}$$

Further in the proof, we will need estimates for $\|v_i - v_{i+1}\|_{N+\alpha}$ and $\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(v_i - v_{i+1})\|_{N+\alpha}$. Concerning the former, by applying the triangle inequality, we see that

$$\|v_i - v_{i+1}\|_{N+\alpha} \leq \left\|v_i - v_{\ell_{q,i}}\right\|_{N+\alpha} + \left\|v_{\ell_{q,i}} - v_{\ell_{q,i+1}}\right\|_{N+\alpha} + \left\|v_{\ell_{q,i+1}} - v_{i+1}\right\|_{N+\alpha}.$$

The first and third term are estimated by (6.3.36). Coming to the second one, we note that

$$\left\|v_{\ell_{q,i}} - v_{\ell_{q,i+1}}\right\|_\alpha \leq \left\|v_{\ell_{q,i}} - v_q\right\|_\alpha + \left\|v_q - v_{\ell_{q,i+1}}\right\|_\alpha.$$

Both of these terms obey the bound (6.3.39) by (6.3.24). For the cases $N \neq 0$, we instead note that

$$\left\|v_{\ell_{q,i}} - v_{\ell_{q,i+1}}\right\|_{N+\alpha} \leq \left\|v_{\ell_{q,i}}\right\|_{N+\alpha} + \left\|v_{\ell_{q,i+1}}\right\|_{N+\alpha},$$

and conclude, by (6.3.25), that both terms obey the bound (6.3.39). Thus, we conclude that

$$\|v_i - v_{i+1}\|_{N+\alpha} \lesssim \Lambda^{\frac{1}{2}} \varrho_{q,i}^{\frac{1+\gamma}{2}} \ell_{q,i}^{-N} \ell_q^\alpha. \tag{6.3.39}$$

As for the material derivative, we note that

$$\begin{aligned}
\left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(v_i - v_{i+1})\right\|_{N+\alpha} \leq{}& \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(v_i - v_{\ell_{q,i}})\right\|_{N+\alpha} \\
&+ \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(v_{\ell_{q,i}} - v_{\ell_{q,i+1}})\right\|_{N+\alpha} \\
&+ \left\|(v_{\ell_{q,i}} - v_{\ell_{q,i+1}}) \cdot \nabla(v_{\ell_{q,i+1}} - v_{i+1})\right\|_{N+\alpha} \\
&+ \left\|(\partial_t + v_{\ell_{q,i+1}} \cdot \nabla)(v_{\ell_{q,i+1}} - v_{i+1})\right\|_{N+\alpha} \\
=:{}& I + II + III + IV.
\end{aligned}$$

We start by estimating I. Combining the two estimates (6.2.11) and (6.2.10) from **Lemma 6.2.1**, we obtain that

$$
\begin{aligned}
I &\leq \left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla + (-\Delta)^\theta)(v_i - v_{\ell_{q,i}})\right\|_{N+\alpha} + \left\|(-\Delta)^\theta(v_i - v_{\ell_{q,i}})\right\|_{N+\alpha} \\
&\lesssim \lesssim \Lambda \varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-1-N-\alpha} \\
IV &\leq \left\|(\partial_t + v_{\ell_{q,i+1}}\cdot\nabla + (-\Delta)^\theta)(v_{i+1} - v_{\ell_{q,i+1}})\right\|_{N+\alpha} + \left\|(-\Delta)^\theta(v_{i+1} - v_{\ell_{q,i+1}})\right\|_{N+\alpha} \\
&\lesssim \Lambda \varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-1-N-\alpha},
\end{aligned}
$$

where we used the fact that $(v_{\ell_{q,i+1}}, p_{\ell_{q,i+1}}, R_{\ell_{q,i+1}})$ and $(v_{i+1}, p_{i+1}.0)$ also satisfy the assumptions of **Lemma 6.2.1**, since $\varrho_{q,i} \sim \varrho_{q,i+1}$. Since $v_{\ell_{q,i}} - v_{\ell_{q,i+1}}$ obeys the bound (6.3.39) as seen above, by using **Lemma 6.2.1**, we conclude that II also obeys the above bound. We now consider III, which can be estimated as follows:

$$
\begin{aligned}
III &= \left\|(v_{\ell_{q,i}} - v_{\ell_{q,i+1}})\cdot\nabla(v_{i+1} - v_{\ell_{q,i+1}})\right\|_{N+\alpha} \\
&\lesssim \Lambda^{\frac{1}{2}}\varrho_{q,i}^{\frac{1+\gamma}{2}}\ell_{q,i}^{\alpha}\cdot\Lambda^{\frac{1}{2}}\varrho_{q,i}^{\frac{1+\gamma}{2}}\ell_{q,i}^{\alpha}\cdot\ell_{q,i}^{-N-1} = \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{2\alpha-N-1},
\end{aligned}
$$

in particular satisfying the same bound as I. We thus conclude that

$$
\left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(v_i - v_{i+1})\right\|_{N+\alpha} \lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-1-N-\alpha} = \Lambda^{\frac{1}{2}}\varrho_{q,i}^{\frac{1+\gamma}{2}}\tau_q^{-1}\ell_q^{3\alpha}\lambda_q^{\alpha}\ell_{q,i}^{-N-\alpha}. \tag{6.3.40}
$$

Step 4: Estimates on the new Reynolds stress

As is done in [8, Section 3.3], we define the vector potentials

$$
z_i \coloneqq (-\Delta)^{-1}\operatorname{curl} v_i \qquad z_{\ell_{q,i}} \coloneqq (-\Delta)^{-1}\operatorname{curl} v_{\ell_{q,i}}, \qquad z_q \coloneqq (-\Delta)^{-1}\operatorname{curl} v_q
$$

and, by **Lemma 6.2.1**, obtain that

$$
\left\|z_i - z_{\ell_{q,i}}\right\|_{N+\alpha} \lesssim \Lambda\tau_q\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-\alpha}. \tag{6.3.41}
$$

$$
\left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla + (-\Delta)^\theta)(z_i - z_{\ell_{q,i}})\right\|_{N+\alpha} \lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-\alpha}. \tag{6.3.42}
$$

$$
\begin{aligned}
\left\|(-\Delta)^\theta(z_i - z_{\ell_{q,i}})\right\|_{N+\alpha} &\lesssim \left\|z_i - z_{\ell_{q,i}}\right\|_{N+\alpha+2\theta+\varepsilon} \lesssim \left\|v_i - v_{\ell_{q,i}}\right\|_{N+\alpha-1+2\theta+\varepsilon} \\
&\lesssim \Lambda\tau_q\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N-2\theta-\varepsilon}\ell_q^{-\alpha} \lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-\alpha}.
\end{aligned} \tag{6.3.43}
$$

By the triangle inequality, (3.8.1), (6.3.43), and (6.3.42), we thus conclude that

$$
\left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(z_i - z_{\ell_{q,i}})\right\|_{N+\alpha} \lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-\alpha}. \tag{6.3.44}
$$

Both (6.3.41) and (6.3.44) are valid in $I_{i-1}\cup J_i\cup I_i$ for any $\underline{n} \leq i \leq \overline{n}$.
The sequel of this proof will require estimates on $z_i - z_{i+1}$ and $(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(z_i - z_{i+1})$. We first use a triangle inequality to obtain

$$
\|z_i - z_{i+1}\|_{N+\alpha} \leq \left\|z_i - z_{\ell_{q,i}}\right\|_{N+\alpha} + \left\|z_{\ell_{q,i}} - z_{\ell_{q,i+1}}\right\|_{N+\alpha} + \left\|z_{\ell_{q,i+1}} - z_{i+1}\right\|_{N+\alpha}. \tag{6.3.45}
$$

The first and third term are estimated by (6.3.41), so we only have to estimate the second one. We note that $z_{\ell_{q,i}} = z_q * \varphi_{\ell_{q,i}}$ and $z_{\ell_{q,i+1}} = z_q * \varphi_{\ell_{q,i+1}}$, so that, using (A.5) and Schauder estimates (**Lemma A.4**), we get

$$\left\| z_{\ell_{q,i}} - z_{\ell_{q,i+1}} \right\|_{N+\alpha} \lesssim \|z_q\|_{2+\alpha} (\ell_{q,i}^{2-N} + \ell_{q,i+1}^{2-N}) \lesssim \Lambda \tau_q \varrho_{q,i}^{1+\gamma} \ell_q^{-2\alpha} \lambda_q^{-\alpha} \ell_{q,i}^{-N}. \tag{6.3.46}$$

The final estimate for $z_i - z_{i+1}$, combining (6.3.45), (6.3.46), and (6.3.41), is thus

$$\begin{aligned} \|z_i - z_{i+1}\|_{N+\alpha} &\leq \|z_i - z_{\ell_{q,i}}\|_{N+\alpha} + \left\| z_{\ell_{q,i}} - z_{\ell_{q,i+1}} \right\|_{N+\alpha} + \left\| z_{\ell_{q,i+1}} - z_{i+1} \right\|_{N+\alpha} \\ &\lesssim \Lambda \varrho_{q,i}^{1+\gamma} \tau_q \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lambda_q^{-\alpha}. \end{aligned} \tag{6.3.47}$$

As for the material derivatives, we must estimate

$$\begin{aligned} \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(z_i - z_{i+1})\right\|_{N+\alpha} &\leq \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(z_i - z_{\ell_{q,i}})\right\|_{N+\alpha} \\ &\quad + \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla)(z_{\ell_{q,i}} - z_{\ell_{q,i+1}})\right\|_{N+\alpha} \\ &\quad + \left\|(v_{\ell_{q,i}} - v_{\ell_{q,i+1}}) \cdot \nabla(z_{\ell_{q,i+1}} - z_{i+1})\right\|_{N+\alpha} \\ &\quad + \left\|(\partial_t + v_{\ell_{q,i+1}} \cdot \nabla)(z_{\ell_{q,i+1}} - z_{i+1})\right\|_{N+\alpha} \\ &=: I + II + III + IV. \end{aligned}$$

The terms I and IV are estimated by (6.3.44). To estimate II, we apply **Lemma 6.2.1** to $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, R_{\ell_{q,i}})$ and $(v_{\ell_{q,i+1}}, p_{\ell_{q,i+1}}, R_{\ell_{q,i+1}})$ and use (6.3.25) and (6.3.46), obtaining that

$$\left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla + (-\Delta)^\theta)(z_{\ell_{q,i}} - z_{\ell_{q,i+1}})\right\|_{N+\alpha} \lesssim \Lambda \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{-N-\alpha} (\ell_{q,i}^{\alpha} \ell_q^{2\alpha} + 1).$$

We then note that, by interpolation

$$\left\|(-\Delta)^\theta (z_{\ell_{q,i}} - z_{\ell_{q,i+1}})\right\|_{N+\alpha} \lesssim \Lambda \tau_q \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{-N-2\theta-\varepsilon} \ell_q^{-2\alpha} \lambda_q^{-\alpha} \lesssim \Lambda \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lambda_q^{-\alpha}.$$

The above two bounds combine to yield

$$\begin{aligned} II &\leq \left\|(\partial_t + v_{\ell_{q,i}} \cdot \nabla + (-\Delta)^\theta)(z_{\ell_{q,i}} - \ell_{q,i+1})\right\|_{N+\alpha} + \left\|(-\Delta)^\theta (z_{\ell_{q,i}} - z_{\ell_{q,i+1}})\right\|_{N+\alpha} \\ &\lesssim \Lambda \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lambda_q^{-\alpha}. \end{aligned} \tag{6.3.48}$$

Coming to III, we estimate it by combining (6.3.41) with (6.3.39):

$$\begin{aligned} \left\|(v_{\ell_{q,i}} - v_{\ell_{q,i+1}}) \cdot \nabla(z_{i+1} - z_{\ell_{q,i+1}})\right\|_{N+\alpha} &\leq \left\| v_{\ell_{q,i}} - v_{\ell_{q,i+1}} \right\|_{N+\alpha} \left\| z_{i+1} - z_{\ell_{q,i+1}} \right\|_{1+\alpha} \\ &\quad + \left\| v_{\ell_{q,i}} - v_{\ell_{q,i+1}} \right\|_{\alpha} \left\| z_{i+1} - z_{\ell_{q,i+1}} \right\|_{N+1+\alpha} \\ &\lesssim \Lambda^{\frac{1}{2}} \varrho_{q,i}^{\frac{1+\gamma}{2}} \ell_{q,i}^{\alpha} \cdot \Lambda \tau_q \varrho_{q,i}^{1+\gamma} \ell_q^{-\alpha} \cdot \ell_{q,i}^{-N-1} \\ &\leq \Lambda \varrho_{q,i}^{1+\gamma} \ell_{q,i}^{\alpha-N}. \end{aligned} \tag{6.3.49}$$

Combining (6.3.44), (6.3.48), and (6.3.49), we thus obtain that

$$\left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(z_i - z_{i+1})\right\|_{N+\alpha} \lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-2\alpha}\lambda_q^{-\alpha}. \tag{6.3.50}$$

Recalling the expression for $\mathring{\overline{R}}_q$ in (6.3.33) and the fact that $\mathcal{R}\,\mathrm{curl}: C^\alpha \to C^\alpha$ is bounded as a consequence of Schauder's estimates (**Lemma A.4**), using (6.3.35), (6.3.47), and (6.3.39), we obtain that

$$\begin{aligned}
\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} &\lesssim \tau_q^{-1}\|z_i - z_{i+1}\|_{N+\alpha} + \|v_i - v_{i+1}\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha \\
&\lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-2\alpha}\lambda_q^{-\alpha} + \Lambda^{\frac12}\varrho_{q,i}^{\frac{1+\gamma}{2}}\ell_{q,i}^{-N+\alpha}\cdot\Lambda^{\frac12}\varrho_{q,i}^{\frac{1+\gamma}{2}}\ell_{q,i}^{\alpha} \\
&\lesssim \Lambda\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-\alpha}(\ell_q^{-\alpha}\lambda_q^{-\alpha} + \ell_{q,i}^{3\alpha}).
\end{aligned}$$

This, together with (6.3.13), gives us (6.3.17).
As for (6.3.18), reasoning as in the proof of [8, Proposition 4.3], we note that

$$\begin{aligned}
\left\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim\; &\tau_q^{-2}\|z_i - z_{i+1}\|_{N+\alpha} + \tau_q^{-1}\left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(z_i - z_{i+1})\right\|_{N+\alpha} \\
&+ \tau_q^{-1}\left\|v_{\ell_{q,i}}\right\|_{1+\alpha}\|z_i - z_{i+1}\|_{N+\alpha} + \tau_q^{-1}\left\|v_{\ell_{q,i}}\right\|_{N+1+\alpha}\|z_i - z_{i+1}\|_\alpha \\
&+ \tau_q^{-1}\|v_i - v_{i+1}\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha \\
&+ \left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(v_i - v_{i+1})\right\|_{N+\alpha}\|v_i - v_{i+1}\|_\alpha \\
&+ \left\|(\partial_t + v_{\ell_{q,i}}\cdot\nabla)(v_i - v_{i+1})\right\|_{\alpha}\|v_i - v_{i+1}\|_{N+\alpha} \\
&+ \left\|(v_{\ell_{q,i}} - \overline{v}_q)\cdot\nabla\mathring{\overline{R}}_q\right\|_{N+\alpha}.
\end{aligned}$$

Combining the above bound on $\mathring{\overline{R}}_q$ with (6.3.47), (6.3.50), (6.3.25), (6.3.39), (6.3.40), and the bound (6.3.37) applied to $v_{\ell_{q,i}} - \overline{v}_q$, we obtain that

$$\left\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \Lambda\tau_q^{-1}\varrho_{q,i}^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-2\alpha}\lambda_q^{-\alpha}, \tag{6.3.51}$$

which yields (6.3.18) once combined with (6.3.13).

Step 5: $\overline{\rho}_q$, $\mathcal{T}_g$, and (6.3.22)

Next, we estimate $\overline{\rho}_q$, recalling its definition in (6.3.34). We wish to estimate $\overline{\rho}_q - \rho_q$. We note that

$$\left|\overline{\rho}_q - \rho_q\right| = \frac13\left|\int_{\mathbb{T}^3}|\overline{v}_q|^2 - |v_q|^2\mathrm{d}x\right| \le \left|\int_{\mathbb{T}^3}|\overline{v}_q|^2 - |v_{\ell_{q,i}}|^2\mathrm{d}x\right| + \left|\int_{\mathbb{T}^3}|v_{\ell_{q,i}}|^2 - |v_q|^2\mathrm{d}x\right|.$$

The second term above is already estimated by (6.3.28), so we proceed to estimate the first term.

As in [8, Proposition 4.4], one has that

$$\left|\overline{v}_q\right|^2-\left|v_{\ell_{q,i}}\right|^2=\chi_i(|v_i|^2-|v_{\ell_{q,i}}|^2)+(1-\chi_i)(|v_{i+1}|^2-|v_{\ell_{q,i+1}}|^2)$$
$$+(1-\chi_i)(|v_{\ell_{q,i+1}}|^2-|v_{\ell_{q,i}}|^2)-\chi_i(1-\chi_i)|v_i-v_{i+1}|^2.$$

Therefore

$$\begin{aligned}\left|\int_{\mathbb{T}^3}\left|\overline{v}_q\right|^2-\left|v_{\ell_{q,i}}\right|^2\mathrm{d}x\right| &\leq \left|\int_{\mathbb{T}^3}|v_i|^2-\left|v_{\ell_{q,i}}\right|^2\mathrm{d}x\right|+\left|\int_{\mathbb{T}^3}|v_{i+1}|^2-\left|v_{\ell_{q,i+1}}\right|^2\mathrm{d}x\right| \\ &+\left|\int_{\mathbb{T}^3}\left|v_{\ell_{q,i}}\right|^2-\left|v_{\ell_{q,i+1}}\right|^2\mathrm{d}x\right|+\left|\int_{\mathbb{T}^3}|v_i-v_{i+1}|^2\mathrm{d}x\right|. \end{aligned} \tag{6.3.52}$$

We start by estimating the fourth term as follows by using (6.3.39):

$$\left|\int_{\mathbb{T}^3}|v_i-v_{i+1}|^2\mathrm{d}x\right|\leq\|v_i-v_{i+1}\|_\alpha^2\lesssim\Lambda\varrho_{q,i}^{1+\gamma}\ell_q^{2\alpha}. \tag{6.3.53}$$

We then proceed to estimate the third term in (6.3.52) by using the triangle inequality, (6.3.28), and the fact $\varrho_{q,i}\sim\varrho_{q,i+1}$:

$$\begin{aligned}\left|\int_{\mathbb{T}^3}\left(\left|v_{\ell_{q,i}}\right|^2-\left|v_{\ell_{q,i+1}}\right|^2\right)\mathrm{d}x\right| &\leq\left|\int_{\mathbb{T}^3}\left(\left|v_{\ell_{q,i}}\right|^2-\left|v_q\right|^2\right)\mathrm{d}x\right|+\left|\int_{\mathbb{T}^3}\left(\left|v_q\right|^2-\left|v_{\ell_{q,i+1}}\right|^2\right)\mathrm{d}x\right| \\ &\lesssim\Lambda\varrho_{q,i}^{1+\gamma}\ell_q^{2\alpha}. \end{aligned} \tag{6.3.54}$$

The first and second terms in (6.3.52) are estimated in similar ways, so we only estimate the former. To that end, we proceed in a way similar to [26, Proposition 5.5]. We start by applying **Lemma 6.2.1** to $(v_i,p_i,0)$ and $(v_{\ell_{q,i}},p_{\ell_{q,i}},\mathring{R}_{\ell_{q,i}})$ to obtain that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|v_i|^2-\left|v_{\ell_{q,i}}\right|^2\mathrm{d}x\right|\lesssim\Lambda\varrho_{q,i}^{1+\gamma}\tau_q^{-1}\ell_q^{2\alpha}. \tag{6.3.55}$$

Combining (6.3.52)-(6.3.55), we conclude that

$$\left|\int_{\mathbb{T}^3}\left|\overline{v}_q\right|^2-\left|v_{\ell_{q,i}}\right|^2\mathrm{d}x\right|\lesssim\Lambda\varrho_{q,i}^{1+\gamma}\ell_q^{2\alpha}. \tag{6.3.56}$$

Estimates (6.3.56) and (6.3.28) imply

$$\left|\int_{\mathbb{T}^3}\left|\overline{v}_q\right|^2-\left|v_q\right|^2\mathrm{d}x\right|\lesssim\Lambda\varrho_{q,i}^{1+\gamma}\ell_q^{2\alpha}.$$

This proves in particular that $\overline{\rho}_q \sim \rho_q$ and (6.3.13), as well as (6.3.15).
Similarly, applying **Lemma 6.2.1** to (v_q, p_q, R_q) and $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, \mathring{R}_{\ell_{q,i}})$ first, and to $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ and $(v_{\ell_{q,i}}, p_{\ell_{q,i}}, R_{\ell_{q,i}})$ afterwards, we also deduce that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} \left|v_{\ell_{q,i}}\right|^2 - \left|v_q\right|^2 \mathrm{d}x\right| \lesssim \Lambda \varrho_{q,i}^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha} \tag{6.3.57}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} \left|\overline{v}_q\right|^2 - \left|v_{\ell_{q,i}}\right|^2 \mathrm{d}x\right| \lesssim \Lambda \varrho_{q,i}^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha}. \tag{6.3.58}$$

Combining (6.3.57) and (6.3.58), we get

$$\left|\partial_t \overline{\rho}_q - \partial_t \rho_q\right| \lesssim \Lambda \varrho_{q,i}^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha}. \tag{6.3.59}$$

To prove (6.3.14) note that

$$\Lambda \varrho_{q,i}^{1+\gamma} \tau_q^{-1} \ell_q^{2\alpha} \lesssim \rho_q \delta_q^{\frac{1}{2}} \lambda_q \cdot \lambda_{q+1}^{2\alpha - 2\beta\gamma} \lesssim \rho_q \delta_q^{\frac{1}{2}} \lambda_q,$$

where we used the definitions of τ_q, ζ_{q+1}, (6.3.3), the relations $\ell_q^{-1} \leq \lambda_{q+1}$ and $\Lambda \varrho_{q,i} = \rho_{q,i} \sim \rho_q$, and the fact that $\alpha < \alpha b < \beta\gamma$, which follows from (6.3.1) since $b > 1$. Therefore, since we showed above that $\overline{\rho}_q \sim \rho_q$, we have (6.3.14).
It remains to estimate $\|\mathring{\overline{R}}_q\|_0$ on $[T_1, T_2]$ in order to verify (6.3.12) for the Reynolds stress. We already obtained (6.3.17) on $J_{\underline{n}} \cup \ldots \cup J_{\overline{n}}$ (recall the decomposition (6.1.4)). Moreover, on $J_{\underline{n}-1} \cup J_{\overline{n}+1}$ the subsolution remains unchanged, so there is nothing to prove. We are then left with the task of proving (6.3.12) on the cut-off regions $I_{\underline{n}-1}$ and $I_{\overline{n}}$.
To do so, we need to estimate $\|v_i - v_q\|_\alpha$ and $\|z_i - z_q\|_\alpha$. For the former, we combine estimates of $v_i - v_{\ell_{q,i}}$ and of $v_{\ell_{q,i}} - v_q$. For the latter, we only need to estimate $\|z_{\ell_{q,i}} - z_q\|_\alpha$, since we already handled $z_i - z_{\ell_{q,i}}$ above. One has that, by (A.5), **Lemma A.4** (Schauder estimates), and (6.3.5)

$$\left\|z_{\ell_{q,i}} - z_q\right\|_\alpha \lesssim \left\|z_q\right\|_{2+\alpha} \ell_{q,i}^2 \lesssim \left\|\operatorname{curl} v_q\right\|_\alpha \ell_{q,i}^2 \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_{q,i}^2 = \Lambda \tau_q \varrho_{q,i}^{1+\gamma} \ell_q^{-2\alpha} \lambda_q^{-\alpha},$$

which gives us (6.3.12) as desired.
We then have to verify (6.3.11) and (6.3.21). To that end, we observe that

$$\left\|\overline{v}_q\right\|_{\theta+\varepsilon} \leq \left\|\overline{v}_q - v_q\right\|_{\theta+\varepsilon} + \left\|v_q\right\|_{\theta+\varepsilon}.$$

The second term is estimated by (6.3.6). As for the first one, we note that

$$\left\|\overline{v}_q - v_q\right\|_\alpha \lesssim \Lambda^{\frac{1}{2}} \overline{\varrho}_q^{\frac{1+\gamma}{2}} \ell_q^\alpha \lesssim \delta_{q+1}^{\frac{1}{2}} \zeta_{q+1}^{\frac{\gamma}{2}} \ell_q^\alpha \leq \delta_{q+1}^{\frac{1}{2}} \ell_q^\alpha,$$

where the first step is due to (6.3.37). We can thus estimate $\|\overline{v}_q - v_q\|_{\theta+\varepsilon}$ by interpolation:

$$\begin{aligned}\left\|\overline{v}_q - v_q\right\|_{\theta+\varepsilon} &\lesssim \left\|\overline{v}_q - v_q\right\|_\alpha^{1-\theta-\varepsilon} \left\|\overline{v}_q - v_q\right\|_{1+\alpha}^{\theta+\varepsilon} \lesssim (\delta_{q+1}^{\frac{1}{2}} \ell_q^\alpha)^{1-\theta-\varepsilon} \cdot \left(\delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha}\right)^{\theta+\varepsilon} \\ &\leq \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\theta+\varepsilon} \cdot \ell_q^{\alpha(1-2\theta-2\varepsilon)}.\end{aligned}$$

Lastly

$$\left|\partial_t \mathcal{J}_g\right| = \left|\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}\overline{v}_q\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}}v_q\right|^2 \mathrm{d}x\right| \lesssim \left\|\overline{v}_q + v_q\right\|_{\theta+\varepsilon}\left\|\overline{v}_q - v_q\right\|_{\theta+\varepsilon} \lesssim \Lambda^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{\theta+\varepsilon},$$

thus proving (6.3.21).
We conclude the proof by obtaining (6.3.22). To that end, we first note that (6.3.13) and (6.3.3) combine to give us $\overline{\varrho}_q \lesssim \zeta_{q+1}$. Combining this with (6.3.9), we conclude that

$$\left\|\overline{v}_q - v_q\right\|_{\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}}\zeta_{q+1}^{\frac{\gamma}{2}}\ell_q^{\alpha},$$

thus reducing (6.3.22) to

$$\lambda_{q+1}^{-\beta\gamma}\ell_q^{(\frac{1}{2}-\frac{2}{b})\alpha} \lesssim 1.$$

Since $\ell_q^{-1} \leq \lambda_{q+1}$, the above follows from

$$-\beta\gamma + \frac{2}{b}\alpha - \frac{1}{2}\alpha < 0 \iff \alpha < \frac{2b\beta\gamma}{4-b}.$$

We recall that we wish to obtain (6.3.22) only under the assumption $2\alpha < \beta\gamma$. This means the above relation follows from

$$\frac{2b}{4-b} > \frac{1}{2} \iff 5b > 4,$$

which in turn follows from $b > 1$. The proof is thus complete. ◇

Remark 6.3.1 (Multi-gluing). *Proposition 6.3.1 can easily be extended to a pairwise disjoint union of intervals* $[T_1^{(i)}, T_2^{(i)}] \subset [0,T]$ *with* $T_2^{(i)} - T_1^{(i)} \geq 4\tau_q$ *and* $T_2^{(i)} < T_1^{(i+1)}$.

6.4 Perturbation step

Proposition 6.4.1 (Main Perturbation Step). *Let* $b, \beta, \alpha, \gamma, (\delta_q, \lambda_q, \Lambda, \zeta_q, \ell_q, \tau_q)$ *be as in Section 4.3 with*

$$\alpha < \beta\gamma. \tag{6.4.1}$$

Let $[T_1, T_2] \subset [0,T]$ *and let* t_i, I_i, J_i *be as in* (6.1.1). *Let* (v, p, R) *be a smooth strong subsolution on* $[T_1, T_2]$ *satisfying*

$$\|R\|_1 \lesssim \Lambda\varrho^{1+\gamma}\ell_q^{-2\alpha}\ell_{q,i}^{-1} \tag{6.4.2}$$

$$\delta_{q+2} \lesssim \rho \lesssim \delta_{q+1} \tag{6.4.3}$$

$$\|v\|_0 \leq C_P \tag{6.4.4}$$

on $K_i \coloneqq [(i-1+\frac{1}{3})\tau_q, (i+\frac{2}{3})\tau_q]$ *for* $\underline{n}-1 \le i \le \overline{n}+1$*, where the* $\ell_{q,i}$ *are defined as in* (6.2.1)*, and* C_P *is a geometric constant. Further, let* $\psi : [0,T] \to [0,1]$ *be a cutoff function and* $S_\psi \in C^\infty(\mathbb{T}^3 \times [T_1,T_2]; \mathcal{S}^{3\times 3})$ *be a smooth matrix field with*

$$S_\psi(x,t) = \sigma_\psi(t)\,\mathrm{Id} + \mathring{S}_\psi(x,t) = \Lambda\varsigma_\psi(t)\,\mathrm{Id} + \mathring{S}_\psi(x,t), \tag{6.4.5}$$

where $S_\psi = \psi^2 S$*,* $\mathring{S}_\psi = \psi^2 \mathring{S}$ *is traceless,* $\sigma_\psi = \psi^2\sigma$*, and* $(\varsigma, \varsigma_\psi) \coloneqq \Lambda^{-1}(\sigma, \sigma_\psi)$*. Suppose* ψ *satisfies*

$$\left|\psi'\right| \lesssim \delta_q^{\frac{1}{2}}\lambda_q, \tag{6.4.6}$$

and σ *satisfies*

$$0 \le \sigma(t) \le 4\delta_{q+1} \tag{6.4.7}$$

$$\sigma|_{K_i} \lesssim \rho(t_i) \tag{6.4.8}$$

$$|\partial_t\sigma| \lesssim \sigma\delta_q^{\frac{1}{2}}\lambda_q, \tag{6.4.9}$$

Moreover, assume that for any $N \ge 0, t \in I_{i-1} \cup J_i \cup I_i, \underline{n} \le i \le \overline{n}$

$$\left\|\mathring{S}\right\|_{N+\alpha} \lesssim \Lambda\varsigma^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-2\alpha} \tag{6.4.10}$$

$$\|v\|_{N+1+\alpha} \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{-N} \tag{6.4.11}$$

$$\|v\|_{\theta+\varepsilon} \le M\left(1 + \sum_{i=0}^{q+1}\Lambda^{\frac{1}{2}}\lambda_i^{\theta+\varepsilon-\beta}\right) \tag{6.4.12}$$

$$\left\|(\partial_t + v\cdot\nabla)\mathring{S}\right\|_{N+\alpha} \lesssim \Lambda\varsigma^{1+\gamma}\ell_{q,i}^{-N}\ell_q^{-6\alpha}\delta_q^{\frac{1}{2}}\lambda_q, \tag{6.4.13}$$

Finally, assume that

$$\operatorname{supp}\mathring{S}_\psi \subseteq \mathbb{T}^3 \times \bigcup_i I_i. \tag{6.4.14}$$

Then, provided $a \gg 1$ *is sufficiently large (depending on the implicit constants in* (6.4.9)*,* (6.4.10)*,* (6.4.11)*, and* (6.4.13)*), there exist smooth* $(\tilde{v}, \tilde{p}) \in C^\infty(\mathbb{T}^3 \times [T_1,T_2]; \mathbb{R}^3 \times \mathbb{R})$ *and a smooth matrix field* $\mathcal{E} \in C^\infty(\mathbb{T}^3 \times [T_1,T_2]; \mathcal{S}^{3\times 3})$ *with* $\operatorname{supp}\mathcal{E} \subset \operatorname{supp} S_\psi$ *such that, setting* $\tilde{R} \coloneqq R - S_\psi - \mathcal{E}$*, the triple* $(\tilde{v}, \tilde{p}, \tilde{R})$ *is a strong subsolution with*

$$\int_{\mathbb{T}^3} |\tilde{v}|^2 + \operatorname{tr}\tilde{R}\,\mathrm{d}x = \int_{\mathbb{R}^3} |v|^2 + \operatorname{tr}R\,\mathrm{d}x \qquad \forall t. \tag{6.4.15}$$

Moreover, we have the estimates

$$\|\tilde{v} - v\|_{H^{-1}} \le \frac{M}{2}\delta_{q+1}^{\frac{1}{2}}\ell_{q,i}^{-1}\lambda_{q+1}^{-1} \tag{6.4.16}$$

$$\|\tilde{v} - v\|_0 \le \frac{M}{2}\delta_{q+1}^{\frac{1}{2}} \tag{6.4.17}$$

$$\|\tilde{v} - v\|_{1+\alpha} \le \frac{M}{2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1+\alpha} \tag{6.4.18}$$

$$\|\tilde{v} - v\|_{\theta+\varepsilon} \le M\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{\theta+\varepsilon}, \tag{6.4.19}$$

and the error $\mathcal{E}$ satisfies the estimates

$$\|\mathcal{E}\|_0 \leq \delta_{q+2}\lambda_{q+1}^{-6\alpha} \tag{6.4.20}$$

$$|\partial_t \operatorname{tr}\mathcal{E}| \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1-6\alpha}. \tag{6.4.21}$$

Finally, setting

$$\begin{aligned}\mathcal{T}_p(t) &:= \frac{1}{3}\int_0^t\int_{\mathbb{T}^3}\left(\left|(-\Delta)^{\frac{\theta}{2}}\tilde{v}\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}}v\right|^2\right)\mathrm{d}x\mathrm{d}s \\ &= \frac{1}{3}\int_0^t\int_{\mathbb{T}^3}\left((-\Delta)^{\frac{\theta}{2}}(\tilde{v}+v)\cdot(-\Delta)^{\frac{\theta}{2}}(\tilde{v}-v)\right)\mathrm{d}x\mathrm{d}s,\end{aligned}$$

we have that

$$\left|\partial_t\mathcal{T}_p\right| \lesssim \Lambda\lambda_q^{\theta+\varepsilon-\beta}, \tag{6.4.22}$$

for any $\varepsilon > 0$. Thus, $\mathcal{T}_p\,\mathrm{Id}$ satisfies (6.4.21) *for all $t\in[0,T]$, and* (6.4.20) *only for small times.*

The proof extends [18, Section 7], which is a localization of the argument carried out in [8, Section 5]. The difference between [18] and [8] is that the latter absorbs the whole R with the perturbation flow, whereas the former, as well as the proof below, aims to only absorb S.
Proof.

Step 1: Squiggling Stripes and the Stress Tensors $\tilde{S}_i$

As in [8, Lemma 5.3], we choose a family of smooth non-negative $\eta_i = \eta_i(x,t)$ with the following properties:

$$\eta_i \in \mathcal{C}^\infty(\mathbb{T}^3\times[T_1,T_2];[0,1]) \tag{6.4.23}$$

$$\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_j = \varnothing \qquad \forall i\neq j \tag{6.4.24}$$

$$\mathbb{T}^3\times I_i \subset \{(x.t) : \eta_i(x,t) = 1\} \tag{6.4.25}$$

$$\begin{aligned}\operatorname{supp}\eta_i &\subseteq \mathbb{T}^3\times(J_i\cup I_i\cup J_{i+1}) \\ &= \mathbb{T}^3\times\left\{\left(t_i - \frac{1}{3}\tau_q, t_i+\frac{1}{3}\tau_q\right)\cap[0,T]\right\}\end{aligned} \tag{6.4.26}$$

$$\left\|\partial_t^N\eta_i\right\|_m \leq C(N,m)\tau_q^{-N} \qquad N,m\geq 0, \tag{6.4.27}$$

$$\exists c_0 > 0 : \quad f_\eta(t) \geq c_0 \quad \forall t\in[0,T] \tag{6.4.28}$$

where the c_0 in (6.4.28) is a geometric constant and we write

$$f_\eta(t) := \sum_i \int_{\mathbb{T}^3}\eta_i^2(x,t)\mathrm{d}x.$$

Define

$$\sigma_i(x,t) := |\mathbb{T}^3| \frac{\eta_i^2(x,t)}{\sum_j \int \eta_j^2(y,t)\mathrm{d}y} \sigma_\psi(t),$$

so that $\sum_i \int_{\mathbb{T}^3} \sigma_i \mathrm{d}x = |\mathbb{T}^3|\sigma_\psi = \int_{\mathbb{T}^3} \sigma_\psi \mathrm{d}x$. Using the inverse flow Φ_i starting at time t_i

$$\begin{cases} (\partial_t + v\cdot\nabla)\Phi_i = 0 \\ \Phi_i(x,t_i) = x \end{cases},$$

set

$$S_i := \sigma_i \,\mathrm{Id} + \eta_i^2 \mathring{S}_\psi$$

$$\tilde{S}_i := \frac{\mathfrak{D}\Phi_i S_i \mathfrak{D}^T\Phi_i}{\sigma_i} = \mathfrak{D}\Phi_i \left(\mathrm{Id} + \frac{\sum_j \int \eta_j^2}{|\mathbb{T}^3|\sigma} \mathring{S} \right) \mathfrak{D}^T\Phi_i.$$

One can check from (6.4.28)-(6.4.27), (6.4.7), and (6.4.8) that

$$\|\sigma_i\|_0 \leq 4|\mathbb{T}^3| c_0^{-1}\delta_{q+1} \tag{6.4.29}$$

$$\|\sigma_i\|_N \lesssim \rho_i := \rho(t_i) \lesssim \delta_{q+1}, \tag{6.4.30}$$

and moreover, since by (6.4.14) $\operatorname{supp}\mathring{S}_\psi \subseteq \left\{\sum_i \eta_i^2 = 1\right\}$

$$\frac{1}{3}\mathrm{tr}\sum_i \int_{\mathbb{T}^3} S_i \mathrm{d}x = \frac{1}{3}\mathrm{tr}\, S_\psi. \qquad \sum_i \mathring{S}_i = \mathring{S}_\psi \tag{6.4.31}$$

We next claim that for all (x,t)

$$\tilde{S}_i(x,t) \in B_{\frac{1}{2}}(\mathrm{Id}) \subset \mathcal{S}_+^{3\times 3}, \tag{6.4.32}$$

where $B_{\frac{1}{2}}(\mathrm{Id})$ is the ball of radius $\frac{1}{2}$ centered at the identity Id in $\mathcal{S}^{3\times 3}$. Indeed, by the classical estimates on transport equations reported in **Proposition B.1**

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \lesssim \tau_q \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} = \ell_q^{4\alpha}\lambda_q^\alpha \leq \ell_q^{3\alpha} \tag{6.4.33}$$

for $t \in J_i \cup I_i \cup J_{i+1}$, since this is an interval of length $|J_i \cup I_i \cup J_{i+1}| \sim \tau_q$. Using (6.4.7), (6.4.10) and (4.3.13), we also have that, for any $N \geq 0$

$$\left\| \frac{\eta_i^2 \mathring{S}_\psi}{\sigma_i} \right\|_N \lesssim \left\| \frac{\mathring{S}}{\sigma} \right\|_N \lesssim \varsigma^\gamma \ell_{q,i}^{-N} \ell_q^{-2\alpha} \lesssim \zeta_{q+1}^\gamma \lambda_{q+1}^{2\alpha} \ell_{q,i}^{-N} = \lambda_{q+1}^{2\alpha - 2\beta\gamma} \ell_{q,i}^{-N}. \tag{6.4.34}$$

Then, using the decomposition

$$\tilde{S}_i - \mathrm{Id} = \mathfrak{D}\Phi_i \frac{\eta_i^2 \mathring{S}_\psi}{\sigma_i} \mathfrak{D}\Phi_i^T + \mathfrak{D}\Phi_i(\mathfrak{D}\Phi_i^T - \mathrm{Id}) + \mathfrak{D}\Phi_i - \mathrm{Id},$$

we deduce from (6.4.33)-(6.4.34) that

$$\left|\tilde{S}_i - \mathrm{Id}\right| \lesssim (1+\ell_q^{3\alpha})\lambda_{q+1}^{2\alpha-2\beta\gamma}(1+\ell_q^{3\alpha}) + 2\ell_q^{3\alpha} \leq \frac{1}{2},$$

provided $a \gg 1$ is sufficiently large, since we assumed $\alpha < \beta\gamma$ in (6.4.1). This verifies (6.4.32).

Step 2: The perturbation w.

Now we can define the perturbation term as

$$w_o := \sum_i \sqrt{\sigma_i}(\mathfrak{D}\Phi_i)^{-1} W(\tilde{S}_i, \lambda_{q+1}\Phi_i) = \sum_i w_{oi},$$

where W are the Mikado flows on the compact set $B_{\frac{1}{2}}(\mathrm{Id})$ as defined in **Lemma C.1**. Notice that the supports of the w_{oi} are disjoint and, using the Fourier series representation of the Mikado flows

$$w_{oi} := \sum_{k\neq 0} (\mathfrak{D}\Phi_i)^{-1} b_{i,k} A_k e^{i\lambda_{q+1}k\cdot\Phi_i}, \tag{6.4.35}$$

where we write

$$b_{i,k}(x,t) := \sqrt{\sigma_i(x,t)}\, a_k(\tilde{S}_i(x,t)).$$

We define w_c so that $w := w_o + w_c$ is divergence-free:

$$w_c := \frac{i}{\lambda_{q+1}} \cdot \sum_{i,k\neq 0} \mathfrak{D}(b_{i,k}) \times \frac{\mathfrak{D}\Phi_i^T \cdot (k\times A_k)}{|k|^2} e^{i\lambda_{q+1}k\cdot\Phi_i} = \sum_{i,k\neq 0} \frac{c_{i,k}}{\lambda_{q+1}} e^{i\lambda_{q+1}k\cdot\Phi_i},$$

where we write

$$c_{i,k} = \mathfrak{D}(b_{i,k}) \times \frac{i\mathfrak{D}\Phi_i^T \cdot (k\times A_k)}{|k|^2}. \tag{6.4.36}$$

Define then

$$\begin{aligned} w &:= w_o + w_c \\ \tilde{v} &:= v + w \\ \tilde{p} &:= p - \sum_i \sigma_i \\ \mathcal{E}(x,t) &:= \mathring{\mathcal{E}}^{(1)}(x,t) + \mathcal{E}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \mathring{\mathcal{E}}^{(1)} &:= \mathcal{R}(\partial_t \tilde{v} + \mathrm{div}(\tilde{v}\otimes\tilde{v}) + \nabla\tilde{p} + (-\Delta)^\theta \tilde{v} + \mathrm{div}(R - S_\psi)) \\ &= \mathcal{R}(\partial_t w + \mathrm{div}(w\otimes v + v\otimes w + w\otimes w) + \nabla(\tilde{p} - p) - \mathrm{div}\, S_\psi), \end{aligned} \tag{6.4.37}$$

with $\mathcal{R}$ being the anti-divergence operator defined in **Definition C.1**, and

$$\mathcal{E}^{(2)}(t) := \frac{\mathrm{Id}}{3} \int_{\mathbb{T}^3} \left(|\tilde{v}|^2 - |v|^2 - \mathrm{tr}\, S_\psi\right) \mathrm{d}x. \tag{6.4.38}$$

Equations (6.4.15) and (4.1.1) follow by construction.

Step 3: Estimates on the perturbation

The estimates on $\tilde{v}$ follow similarly to the ones for v_{q+1} in [8, Sections 5-6]. Obtaining those requires estimates on the coefficients $b_{i,k}, c_{i,k}$, which in turn require estimates of $\tilde{S}_i$ and estimates of $\mathfrak{D}\Phi_i$. The latter read as follows:

$$\|\mathfrak{D}\Phi_i - \mathrm{Id}\|_N + \left\|(\mathfrak{D}\Phi_i)^{-1} - \mathrm{Id}\right\|_N \lesssim \ell_q^{3\alpha}\ell_{q,i}^{-N} \tag{6.4.39}$$

$$\|\mathfrak{D}\Phi_i\|_N + \left\|(\mathfrak{D}\Phi_i)^{-1}\right\|_N \lesssim \ell_q^{3\alpha\mathbb{1}_{N\neq 0}}\ell_{q,i}^{-N} \tag{6.4.40}$$

$$\|(\partial_t + v\cdot\nabla)\mathfrak{D}\Phi_i\|_N \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_q^{3\alpha\mathbb{1}_{N\neq 0}}\ell_{q,i}^{-N}. \tag{6.4.41}$$

To obtain these, we first observe that Φ_i is a diffeomorphism, which implies both $\mathfrak{D}\Phi_i$ and $(\mathfrak{D}\Phi_i)^{-1}$ are bounded, thus yielding the $N=0$ case of (6.4.40). To obtain (6.4.39), we start by combining (6.4.33) with the $N=0$ case of (6.4.40), thus obtaining that

$$\left\|(\mathfrak{D}\Phi_i)^{-1} - \mathrm{Id}\right\|_0 \leq \left\|(\mathfrak{D}\Phi_i)^{-1}\right\|_0 \|\mathrm{Id} - \mathfrak{D}\Phi_i\|_0 \lesssim \ell_q^{3\alpha}.$$

This yields (6.4.39) for $N=0$. For $N\geq 1$, we note that

$$\|\mathfrak{D}\Phi_i - \mathrm{Id}\|_N + \left\|(\mathfrak{D}\Phi_i)^{-1} - \mathrm{Id}\right\|_N \lesssim \|\mathfrak{D}\Phi_i - \mathrm{Id}\|_0 + \left\|(\mathfrak{D}\Phi_i)^{-1} - \mathrm{Id}\right\|_0 + \left\|\mathfrak{D}^2\Phi_i\right\|_{N-1} + \left\|\mathfrak{D}(\mathfrak{D}\Phi_i)^{-1}\right\|_{N-1}.$$

The other cases of (6.4.39) follow by combining its $N=0$ case with the $N\geq 1$ cases of (6.4.40).
The estimates for $\|\mathfrak{D}\Phi_i\|_N$ for $N\geq 1$ follow from **Proposition B.1**. By combining (C.9) with **Lemma A.1**, we obtain that

$$\|(\partial_t + v\cdot\nabla)\mathfrak{D}\Phi_i\|_N \lesssim \|\mathfrak{D}\Phi_i\|_N\|\nabla v\|_0 + \|\nabla v\|_N\|\mathfrak{D}\Phi_i\|_0.$$

Estimates (6.4.40) and (6.4.11) then yield (6.4.41).
To complete the proof of (6.4.40), we are left with estimating $\|(\mathfrak{D}\Phi_i)^{-1}\|_N$. We note that $\mathfrak{D}^N(\Phi_i\circ\Phi_i^{-1}) = 0$ for $N\geq 1$. We then use the Leibniz rule and the chain rule to write

$$\begin{aligned}
\mathfrak{D}^2(\Phi_i\circ\Phi_i^{-1}) &= \mathfrak{D}((\mathfrak{D}\Phi_i\circ\Phi_i^{-1})\mathfrak{D}\Phi_i^{-1}) = (\mathfrak{D}^2\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})^2 + (\mathfrak{D}\Phi_i\circ\Phi_i^{-1})\mathfrak{D}^2\Phi_i^{-1}\\
\mathfrak{D}^3(\Phi_i\circ\Phi_i^{-1}) &= \mathfrak{D}(\mathfrak{D}^2(\Phi_i\circ\Phi_i^{-1}))\\
&= (\mathfrak{D}^3\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})^3 + 3(\mathfrak{D}^2\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})(\mathfrak{D}^2\Phi_i^{-1}) + (\mathfrak{D}\Phi_i\circ\Phi_i^{-1})\mathfrak{D}^3\Phi_i^{-1}\\
\mathfrak{D}^4(\Phi_i\circ\Phi_i^{-1}) &= \mathfrak{D}(\mathfrak{D}^3(\Phi_i\circ\Phi_i^{-1}))\\
&= (\mathfrak{D}^4\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})^4 + 6(\mathfrak{D}^3\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})^2\mathfrak{D}^2\Phi_i^{-1} + 3(\mathfrak{D}^2\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}^2\Phi_i^{-1})^2\\
&\quad + 4(\mathfrak{D}^2\Phi_i\circ\Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})(\mathfrak{D}^3\Phi_i^{-1}) + (\mathfrak{D}\Phi_i\circ\Phi_i^{-1})\mathfrak{D}^4\Phi_i^{-1}.
\end{aligned}$$

From these, we can see that

$$\begin{aligned}
\left\|\mathfrak{D}^2\Phi_i^{-1}\right\|_0 &\leq \left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|\mathfrak{D}^2\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^2 \lesssim \ell_q^{3\alpha}\ell_{q,i}^{-1}\\
\left\|\mathfrak{D}^3\Phi_i^{-1}\right\|_0 &\leq \left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left(\left\|\mathfrak{D}^3\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^3 + 3\left\|\mathfrak{D}^2\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|\mathfrak{D}^2\Phi_i^{-1}\right\|_0\right) \lesssim \ell_q^{3\alpha}\ell_{q,i}^{-2}\\
\left\|\mathfrak{D}^4\Phi_i^{-1}\right\|_0 &\leq \left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left(\left\|\mathfrak{D}^4\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^4 + 6\left\|\mathfrak{D}^3\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^2\left\|\mathfrak{D}^2\Phi_i^{-1}\right\|_0\right.\\
&\quad \left. + 3\left\|\mathfrak{D}^2\Phi_i\right\|_0\left\|\mathfrak{D}^2\Phi_i^{-1}\right\|_0^2 + 4\left\|\mathfrak{D}^2\Phi_i\right\|_0\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|\mathfrak{D}^3\Phi_i^{-1}\right\|_0\right) \lesssim \ell_q^{3\alpha}\ell_{q,i}^{-3}.
\end{aligned}$$

These two examples show us that

$$\mathfrak{D}^M(\Phi_i \circ \Phi_i^{-1}) = (\mathfrak{D}\Phi_i \circ \Phi_i^{-1})\mathfrak{D}^M\Phi_i^{-1} + \text{other terms},$$

where the other terms are of the form $(\mathfrak{D}^{M-(m-1)n}\Phi_i \circ \Phi_i^{-1})(\mathfrak{D}\Phi_i^{-1})^{\ell}(\mathfrak{D}^m\Phi_i^{-1})^n$, where $m < M$. If we assume (6.4.40) for $N < M-1$, we see that such terms are estimated as $\ell_{q,i}^{-(k-1)}\ell_{q,i}^{-(m-1)n} = \ell_{q,i}^{-(1-M)}$, thus so is $\mathfrak{D}^M\Phi_i^{-1} = \mathfrak{D}(\mathfrak{D}^{M-1}\Phi_i^{-1})$, which proves (6.4.40) for $N = M-1$. Thus, by induction, the estimate (6.4.40) is proved.
The following estimates then follow precisely as in [8, Propositions 5.7 and 5.9]:

$$\left\|\tilde{S}_i\right\|_N \lesssim \ell_{q,i}^{-N} \tag{6.4.42}$$

$$\left\|b_{i,k}\right\|_N \lesssim \rho_i^{\frac{1}{2}}|k|^{-6}\ell_{q,i}^{-N} \tag{6.4.43}$$

$$\left\|c_{i,k}\right\|_N \lesssim \rho_i^{\frac{1}{2}}|k|^{-6}\ell_{q,i}^{-N-1} \tag{6.4.44}$$

$$\left\|D_t\tilde{S}_i\right\|_N \lesssim \tau_q^{-1}\ell_{q,i}^{-N} \tag{6.4.45}$$

$$\left\|D_t b_{i,k}\right\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\tau_q^{-1}\ell_{q,i}^{-N}|k|^{-6} \tag{6.4.46}$$

$$\left\|D_t c_{i,k}\right\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\tau_q^{-1}\ell_{q,i}^{-N-1}|k|^{-6}. \tag{6.4.47}$$

To obtain (6.4.42), observe first that, by its definition, we have that

$$\tilde{S}_i = \mathfrak{D}\Phi_i\mathfrak{D}^T\Phi_i + \mathfrak{D}\Phi_i\frac{\sum_j \int_{\mathbb{T}^3}\eta_j^2 \mathrm{d}x}{|\mathbb{T}^3|\sigma}\mathring{S}\mathfrak{D}^T\Phi_i,$$

and therefore

$$\left\|\tilde{S}_i\right\|_N \lesssim \|\mathfrak{D}\Phi_i\|_N + \|\mathfrak{D}\Phi_i\|_N\left\|\frac{\mathring{S}}{\sigma}\right\|_0 + \left\|\frac{\mathring{S}}{\sigma}\right\|_N,$$

where we used that $\|\mathfrak{D}\Phi_i\|_0 \lesssim 1$. By (6.4.40), the first term above obeys (6.4.42). To estimate the remaining two terms, we use (6.4.10) and (6.4.7) to obtain that

$$\left\|\frac{\mathring{S}}{\sigma}\right\|_N \lesssim \frac{\left\|\mathring{S}\right\|_{N+\alpha}}{\sigma} \lesssim \varsigma^{\gamma}\ell_{q,i}^{-N}\ell_q^{-2\alpha} \lesssim \ell_{q,i}^{-N}\lambda_{q+1}^{2\alpha-2\beta\gamma}. \tag{6.4.48}$$

Estimate (6.4.42) then follows from (6.4.40) and the assumption (6.4.1), i.e. that $\alpha < \beta\gamma$.
The proof of (6.4.45) follows a similar strategy, making use of the relations (6.4.40), (6.4.41), (6.4.48), (6.4.27), and (6.4.9).
To prove (6.4.43) and (6.4.46), we first prove some estimates on $\sqrt{\sigma_i}$. Firstly, we note that, thanks to (6.4.28) and (6.4.8)

$$\|\sqrt{\sigma_i}\|_N \leq \frac{|\mathbb{T}^3|^{\frac{1}{2}}}{\sqrt{C_0}}\sqrt{\sigma_\psi}\|\eta_i\|_N \leq \frac{|\mathbb{T}^3|^{\frac{1}{2}}\psi}{c_0^{\frac{1}{2}}}\rho_i^{\frac{1}{2}} \lesssim \rho_i^{\frac{1}{2}}. \tag{6.4.49}$$

As for the material derivative, similar computations, combined with a straightforward decomposition of $D_t\sqrt{\sigma_i}$, (6.4.27), (6.4.11), (6.4.8), (6.4.28), the fact $\psi \leq 1$, (6.4.6), and (6.4.9), yield the following bound:

$$\|D_t\sqrt{\sigma_i}\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\tau_q^{-1}\ell_{q,i}^{-N}. \tag{6.4.50}$$

We will also use the following bound on the derivatives of f_η, obtained by making use of (6.4.27):

$$\|f_\eta\|_N = \left\|\sum_j \int_{\mathbb{T}^3} 2\eta_j\partial_t\eta_j \mathrm{d}y\right\|_N \lesssim \sum_j \int_{\mathbb{T}^3} 2\left(\|\eta_j\|_N\|\partial_t\eta_j\|_0 + \|\eta_j\|_0\|\partial_t\eta_j\|_N\right)\mathrm{d}y \leq K\tau_q^{-1}, \tag{6.4.51}$$

where $K > 0$ is a constant.

To prove (6.4.43) and (6.4.46), we note that

$$\begin{aligned}
\|b_{i,k}\|_N &\lesssim \|\sqrt{\sigma_i}\|_N\|a_k(\tilde{S}_i)\|_0 + \|\sqrt{\sigma_i}\|_0\|a_k(\tilde{S}_i)\|_N \\
\|D_t b_{i,k}\| &\lesssim \|D_t\sqrt{\sigma_i}\|_N\|a_k(\tilde{S}_i)\|_0 + \|D_t\sqrt{\sigma_i}\|_n\|a_k(\tilde{S}_i)\|_N + \|\sqrt{\sigma_i}\|_N\|D_t[a_k(\tilde{S}_i)]\|_0 + \|\sqrt{\sigma_i}\|_0\|D_t[a_k(\tilde{S}_i)]\|_N.
\end{aligned}$$

The bounds (6.4.43) and (6.4.46) then readily follow by combining (6.4.49), (6.4.50), and the following applications of (A.4):

$$\begin{aligned}
\|a_k(\tilde{S}_i)\|_N &\lesssim \|\mathfrak{D}a_k\|_0\|\mathfrak{D}\tilde{S}_i\|_{N-1} + \|\mathfrak{D}a_k\|_{N-1}\|\mathfrak{D}\tilde{S}_i\|_0^N \lesssim \|a_k\|_N(\|\tilde{S}_i\|_N + \|\tilde{S}_i\|_1^N), \\
\|D_t(a_k(\tilde{S}_i))\|_N &\leq \|(\mathfrak{D}a_k)(\tilde{S}_i)\|_N\|D_t\tilde{S}_i\|_0 + \|(\mathfrak{D}a_k)(\tilde{S}_i)\|_0\|D_t\tilde{S}_i\|_N \\
&\lesssim (\|a_k\|_{N+1}\|\tilde{S}_i\|_1^N + \|a_k\|_2\|\tilde{S}_i\|_N)\|D_t\tilde{S}_i\|_0 + \|a_k\|_1\|D_t\tilde{S}_i\|_N.
\end{aligned}$$

To prove (6.4.44), we note that, by Leibniz rule

$$\|c_{i,k}\|_N \lesssim \sum_{i=0}^N \|b_{i,k}\|_{i+1}\|\mathfrak{D}^T\Phi_i\|_{N-i},$$

from which (6.4.44) follows by (6.4.43) and (6.4.40). Coming finally to (6.4.47), we start by noting that

$$D_t\nabla(b_{i,k}) = \nabla D_t(b_{i,k}) + [v\cdot\nabla,\nabla](b_{i,k}).$$

This means that

$$\|D_t c_{i,k}\|_N \lesssim \sum_{i=0}^N \left(\|\nabla D_t(b_{i,k})\|_i + \|[v\cdot\nabla,\nabla](b_{i,k})\|_i\right)\|\mathfrak{D}^T\Phi_i\|_{N-i} + \sum_{i=0}^N \|\mathfrak{D}b_{i,k}\|_i\|D_t\mathfrak{D}^T\Phi_i\|_{N-i}.$$

Since, by the estimates (6.4.41), (6.4.40), (6.4.43), (6.4.46), and (6.4.11) on the factors here involved, we see that this scales like $\ell_{q,i}^{-N}$, we will only need to prove the case $N = 0$. In that case, we obtain that

$$\|D_t c_{i,k}\|_0 \lesssim \|D_t(b_{i,k})\|_1\|\mathfrak{D}\Phi_i\|_0 + \|[v\cdot\nabla,\nabla](b_{i,k})\|_0\|\mathfrak{D}\Phi_i\|_0 + \|b_{i,k}\|_1\|D_t\mathfrak{D}\Phi_i\|_0 =: I + II + III.$$

We now note that

$$\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} = \tau_q^{-1}\ell_q^{3\alpha} \leq \tau_q^{-1}. \tag{6.4.52}$$

By (6.4.46), (6.4.40), (6.4.43), (6.4.41), and (6.4.52), we see that I, II are estimated as desired. We are then left with proving that II also satisfies this bound. To this end, we rewrite the commutator as

$$[v\cdot\nabla,\nabla](b_{i,k}) = \sum_{j\ell}(v_j\partial_j(\partial_\ell b_{i,k}) - \partial_\ell(v_j\partial_j b_{i,k}))e_\ell = -\sum_{j\ell}\partial_\ell v_j\partial_j b_{i,k}e_\ell = \nabla v\nabla b_{i,k}.$$

It then follows from (6.4.40), (6.4.43), and (6.4.52) that II satisfies the same bound as I and III, thus proving (6.4.47).
In turn, the estimates on $\tilde{v}$ in (6.4.17)-(6.4.18) follow from the ones just given precisely as in [8, Corollary 5.8, pp. 23-24]. Indeed, once we note that

$$\left\|\nabla(e^{i\lambda_{q+1}k\cdot\Phi_i})\right\|_0 \leq \lambda_{q+1}|k|\|\mathfrak{D}\Phi_i\|_0 \leq 2\lambda_{q+1}|k|, \tag{6.4.53}$$

we can deduce from the estimates above that

$$\begin{aligned}\|w_{o,i}\|_N \lesssim \sum_{i,k}\left\|\mathfrak{D}\Phi_i^{-1}\right\|_N\left\|b_{i,k}\right\|_0\left\|e^{i\lambda_{q+1}k\cdot\Phi_i}\right\|_0 + \sum_{i,k}\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|b_{i,k}\right\|_N\left\|e^{i\lambda_{q+1}k\cdot\Phi_i}\right\|_0 \\ + \sum_{i,k}\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|b_{i,k}\right\|_0\left\|e^{i\lambda_{q+1}k\cdot\Phi_i}\right\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^N.\end{aligned}$$

The $w_{o,i}$ have pairwise disjoint supports, so the sum over i always consists of a single term, which yields that the desired estimates hold for w_o. The estimates on $c_{i,k}\lambda_{q+1}^{-1}$ are always better than those on $b_{i,k}$, meaning that any estimate that holds for w_o holds for w_c as well. Thus, (6.4.17) follows directly, and (6.4.18) and (6.4.19) follow by interpolation.
Coming to (6.4.16), the fact that w_c satisfies this bound can be easily deduced from (6.4.44), which tells us that $\lambda_{q+1}^{-1}\|c_{i,k}\|_0 \lesssim \delta_{q+1}^{1/2}|k|^{-6}\ell_{q,i}^{-1}\lambda_{q+1}^{-1}$. To estimate w_o, we use a procedure similar to the one employed in Section 5 to prove (5.1.3), replacing (5.1.10) with (6.4.43).

Step 4: Estimates on the new Reynolds term $\mathring{\mathcal{E}}^{(1)}$.

The aim of this section is to prove $\mathring{\mathcal{E}}^{(1)}$ satisfies (6.4.20), namely

$$\left\|\mathring{\mathcal{E}}^{(1)}\right\|_0 \leq \delta_{q+2}\lambda_{q+1}^{-6\alpha}.$$

Drawing from [8], we decompose $\mathring{\mathcal{E}}^{(1)}$ as

$$\begin{aligned}\mathring{\mathcal{E}}^{(1)} &= \mathcal{R}\left(\partial_t w + \operatorname{div}(v\otimes w + w\otimes v + w\otimes w) - \sum_i\nabla\sigma_i + (-\Delta)^\theta w - \operatorname{div} S_\psi\right)\\ &= \mathcal{R}\left(\partial_t w + w\cdot\nabla v + v\cdot\nabla w + \operatorname{div}(w\otimes w) - \sum_i\nabla\sigma_i + (-\Delta)^\theta w - \operatorname{div} S_\psi\right)\\ &= \underbrace{\mathcal{R}(w\cdot\nabla v)}_{\text{Nash error}=:\mathcal{E}_N} + \underbrace{\mathcal{R}((\partial_t + v\cdot\nabla)w)}_{\text{Transport error}=:\mathcal{E}_T} + \underbrace{\mathcal{R}\left[\operatorname{div}(w\otimes w - \mathring{S}_\psi) - \sum_i\nabla\sigma_i\right]}_{\text{Oscillation error}=:\mathcal{E}_O} + \underbrace{\mathcal{R}((-\Delta)^\theta w)}_{\text{Dissipation error}=:\mathcal{E}_D}.\end{aligned}$$

We then note that, since the $w_{o,i}$ have disjoint supports and $w_o = \sum_i w_{o,i}$, by (6.4.31), we have that

$$\operatorname{div}(w \otimes w - \mathring{S}_\psi) - \sum_i \nabla \sigma_i = \operatorname{div}(w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c) + \sum_i \left[\operatorname{div}\left(w_{o,i} \otimes w_{o,i} - \eta_i^2 \mathring{S}_\psi - \frac{1}{3} \operatorname{Id} \operatorname{tr} S_i \right) \right]. \tag{6.4.54}$$

We now rewrite the first three terms using the definition of w, (6.4.54) (to rewrite $\mathcal{E}_O$), and the fact that $D_t e^{i\lambda_{q+1} k \cdot \Phi_i} = 0$ (to rewrite $\mathcal{E}_T$):

$$\begin{aligned}
\mathcal{E}_N &= \sum_{i,k} \mathcal{R}\big((\mathfrak{D}\Phi_i^{-1} b_{i,k} A_k + \lambda_{q+1}^{-1} c_{i,k}) \cdot \nabla v e^{i\lambda_{q+1} k \cdot \Phi_i}\big) \\
\mathcal{E}_T &= \sum_{i,k} \mathcal{R}\big((D_t \mathfrak{D}\Phi_i^{-1} b_{i,k} A_k + \mathfrak{D}\Phi_i^{-1} D_t b_{i,k} A_k + \lambda_{q+1}^{-1} D_t c_{i,k}) e^{i\lambda_{q+1} k \cdot \Phi_i}\big) \\
\mathcal{E}_O &= \mathcal{R} \operatorname{div}(w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c) + \sum_i \mathcal{R} \operatorname{div}(w_{o,i} \otimes w_{o,i} - S_i) \\
&= \mathcal{R} \operatorname{div}(w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c) + \sum_{i,k} \mathcal{R}\big(\operatorname{div}(\sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}^T \Phi_i^{-1}) e^{i\lambda_{q+1} k \cdot \Phi_i}\big),
\end{aligned}$$

where the C_k are as defined in **Lemma C.1**. We now note that the leading order terms are

$$\begin{aligned}
\mathcal{E}_N^{(L)} &\coloneqq \sum_{i,k} \mathcal{R}\big((\mathfrak{D}\Phi_i^{-1} b_{i,k} A_k) \cdot \nabla v e^{i\lambda_{q+1} k \cdot \Phi_i}\big) \\
\mathcal{E}_T^{(L)} &\coloneqq \sum_{i,k} \mathcal{R}\big((D_t \mathfrak{D}\Phi_i^{-1} b_{i,k} + \mathfrak{D}\Phi_i^{-1} D_t b_{i,k}) A_k e^{i\lambda_{q+1} k \cdot \Phi_i}\big) \\
\mathcal{E}_O^{(L)} &\coloneqq \mathcal{R} \operatorname{div}(w_o \otimes w_c + w_c \otimes w_o) + \sum_{i,k} \mathcal{R}\big(\operatorname{div}(\sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}^T \Phi_i^{-1}) e^{i\lambda_{q+1} k \cdot \Phi_i}\big) \eqqcolon \mathcal{E}_O^{(L,1)} + \mathcal{E}_O^{(L,2)}.
\end{aligned}$$

We start by estimating $\mathcal{E}_O^{(L,1)}$. Since $\mathcal{R} \operatorname{div}$ is Calderón-Zygmund, we have that

$$\|\mathcal{R} \operatorname{div}(w_o \otimes w_c + w_c \otimes w_o)\|_\alpha \lesssim \|w_o\|_\alpha \|w_c\|_0 + \|w_o\|_0 \|w_c\|_\alpha.$$

From (6.4.43), (6.4.40), and (6.4.44), we can conclude that

$$\|w_o\|_N \lesssim \rho_i^{\frac{1}{2}} \lambda_{q+1}^N \qquad \|w_c\|_N \lesssim \rho_i^{\frac{1}{2}} \lambda_{q+1}^{N-1} \ell_{q,i}^{-1}.$$

By interpolation, this lets us conclude that

$$\left\| \mathcal{E}_O^{(L,1)} \right\|_\alpha \lesssim \rho_i \ell_{q,i}^{-1} \lambda_{q+1}^{\alpha-1}.$$

To estimate the other leading terms, we start by using **Lemma C.3** on all three:

$$\left\|\mathcal{E}_N^{(L)}\right\|_\alpha \lesssim \sum_{i,k}\left(\frac{\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_0}{|k\lambda_{q+1}|^{1-\alpha}}+\frac{\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_{N+\alpha}+\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_0\|\Phi_i\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}\right)$$

$$\left\|\mathcal{E}_T^{(L)}\right\|_\alpha \lesssim \sum_{i,k}\frac{\left\|D_t\mathfrak{D}\Phi_i^{-1}b_{i,k}+\mathfrak{D}\Phi_i^{-1}D_tb_{i,k}\right\|_0}{|k\lambda_{q+1}|^{1-\alpha}}$$
$$+\sum_{i,k}\frac{\left\|D_t\mathfrak{D}\Phi_i^{-1}b_{i,k}+\mathfrak{D}\Phi_i^{-1}D_tb_{i,k}\right\|_{N+\alpha}+\left\|D_t\mathfrak{D}\Phi_i^{-1}b_{i,k}+\mathfrak{D}\Phi_i^{-1}D_tb_{i,k}\right\|_0\|\Phi_i\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}$$

$$\left\|\mathcal{E}_O^{(L,2)}\right\|_\alpha \lesssim \sum_{i,k}\frac{\left\|\operatorname{div}(\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\mathfrak{D}^T\Phi_i^{-1})\right\|_0}{|k\lambda_{q+1}|^{1-\alpha}}$$
$$+\sum_{i,k}\frac{\left\|\operatorname{div}(\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\mathfrak{D}^T\Phi_i^{-1})\right\|_{N+\alpha}+\left\|\operatorname{div}(\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\mathfrak{D}^T\Phi_i^{-1})\right\|_0\|\Phi_i\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}.$$

To estimate $\mathcal{E}_N^{(L)}$, we combine (6.4.40), (6.4.43), and (6.4.11) with a Leibniz inequality:

$$\sum_{i,k}\left(\frac{\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_0}{|k\lambda_{q+1}|^{1-\alpha}}+\frac{\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_{N+\alpha}+\left\|\mathfrak{D}\Phi_i^{-1}b_{i,k}\cdot\nabla v\right\|_0\|\Phi_i\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}\right)$$
$$\lesssim \sum_{i,k}\frac{\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|b_{i,k}\right\|_0\|\nabla v\|_0}{|k\lambda_{q+1}|^{1-\alpha}}+\sum_{i,k}\frac{\left\|\mathfrak{D}\Phi_i^{-1}\right\|_0\left\|b_{i,k}\right\|_0\|\nabla v\|_0\|\Phi_i\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}+\sum_{i,k}\frac{\left\|\mathfrak{D}\Phi_i^{-1}\right\|_\alpha\left\|b_{i,k}\right\|_\alpha\|\nabla v\|_{N+\alpha}}{|k\lambda_{q+1}|^{N-\alpha}}$$
$$+\sum_{i,k}\frac{\left\|\mathfrak{D}\Phi_i^{-1}\right\|_{N+\alpha}\left\|b_{i,k}\right\|_\alpha\|\nabla v\|_\alpha+\left\|\mathfrak{D}\Phi_i^{-1}\right\|_\alpha\left\|b_{i,k}\right\|_{N+\alpha}\|\nabla v\|_\alpha}{|k\lambda_{q+1}|^{N-\alpha}}$$
$$\lesssim \frac{\rho_i^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}}{\lambda_{q+1}^{1-\alpha}}+\frac{\ell_{q,i}^{-N-3\alpha}\rho_i^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}}{\lambda_{q+1}^{N-\alpha}}.$$

The above holds for any N. If we choose N to be the $\overline{N}$ from Section 4.3, by (6.4.3) and (4.3.14), we conclude that

$$\left\|\mathcal{E}_N^{(L)}\right\|_\alpha \lesssim \frac{\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q}{\lambda_{q+1}^{1-2\alpha}}.$$

Notice how the leading order term here is the one that does not depend on N thanks to the ℓ_q in (4.3.14). This is also true of $\mathcal{E}_T^{(L)}$ and $\mathcal{E}_O^{(L,2)}$.
$\mathcal{E}_T^{(L)}$ is estimated in a similar manner, using (6.4.40), (6.4.41), (6.4.43), (6.4.46), and (4.3.14).
As for $\mathcal{E}_O^{(L,1)}$, we first ensure that adding a derivative, whichever factor it lands on, costs at most $\ell_{q,i}^{-1}$. This ensures that the leading term is the first one, because of that gain of ℓ_q mentioned above. We then estimate the leading term.
By (6.4.40) and (6.4.43), differentiating $b_{i,k}$ or $\mathfrak{D}\Phi_i^{-1}$ costs $\ell_{q,i}^{-1}$, and by (6.4.30), differentiating σ_i does not cost anything, so we are left with showing that $C_k(\tilde{S}_i)$ scales like $\ell_{q,i}^{-N}$.

Thanks to (A.4), (C.6), and (6.4.42), we have that

$$\left\|C_k(\tilde{S}_i)\right\|_N \lesssim \|C_k\|_1 \left\|\mathfrak{D}\tilde{S}_i\right\|_{N-1} + \|\nabla C_k\|_{N-1} \left\|\tilde{S}_i\right\|_1^N \lesssim |k|^{-6} \ell_{q,i}^{-N}. \tag{6.4.55}$$

We then use (6.4.7), (6.4.40), and the above estimate on $C_k(\tilde{S}_i)$ to estimate the leading term:

$$\begin{aligned}\sum_{i,k} \frac{\left\|\operatorname{div}(\sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}^T \Phi_i^{-1})\right\|_0}{\left|k\lambda_{q+1}\right|^{1-\alpha}} & \\ \lesssim \sum_{i,k} & \frac{\|\sigma_i\|_1 \left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^2 \left\|C_k(\tilde{S}_i)\right\|_0 + \|\sigma_i\|_0 \left\|\mathfrak{D}\Phi_i^{-1}\right\|_0^2 \left\|C_k(\tilde{S}_i)\right\|_1}{|k\lambda_{q+1}|^{1-\alpha}} \\ & + \sum_{i,k} \frac{2\|\sigma_i\|_0 \left\|\mathfrak{D}\Phi_i^{-1}\right\|_1 \left\|C_k(\tilde{S}_i)\right\|_0 \left\|\mathfrak{D}^T \Phi_i^{-1}\right\|_0}{|k\lambda_{q+1}|^{1-\alpha}} \lesssim \rho_i \ell_{q,i}^{-1} \lambda_{q+1}^{-1}.\end{aligned}$$

We thus obtain that

$$\begin{aligned}\|\mathcal{E}_N\|_\alpha &\lesssim \frac{\delta_q^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}^{1-2\alpha}} \\ \|\mathcal{E}_T\|_\alpha &\lesssim \frac{\delta_q^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}^{1-5\alpha}} \\ \|\mathcal{E}_O\|_\alpha &\lesssim \frac{\rho_i}{\ell_{q,i} \lambda_{q+1}^{1-\alpha}}.\end{aligned}$$

The relation (4.3.12) easily yields that the above terms satisfy (6.4.20) for $a \gg 1$ sufficiently large, since

$$\begin{aligned}\frac{\delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}^{1-3\alpha}} &\le \frac{\delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}^{1-5\alpha}} \le \delta_{q+2} \lambda_{q+1}^{-6\alpha} \\ \frac{\rho_i}{\ell_{q,i} \lambda_{q+1}^{1-\alpha}} &\le \frac{\Lambda \zeta_{q+1}^{\frac{1-\gamma}{2}} \zeta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_q^{-\alpha}}{\lambda_{q+1}^{1-\alpha}} \lesssim \frac{\delta_q^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_q}{\lambda_{q+1}^{1-3\alpha-\beta\gamma}} \lesssim \delta_{q+2} \lambda_{q+1}^{-6\alpha}.\end{aligned} \tag{6.4.56}$$

Coming to $\mathcal{E}_D$, which is not present in [8], we estimate it as follows:

$$\left\|\mathcal{R}((-\Delta)^\theta w)\right\|_0 \lesssim \|\mathcal{R}w\|_{2\theta+\varepsilon} \lesssim \|\mathcal{R}w\|_0^{1-2\theta-\varepsilon} \|\mathcal{R}w\|_1^{2\theta+\varepsilon}. \tag{6.4.57}$$

At this point, we use **Lemma C.3** to obtain that

$$\begin{aligned}\|\mathcal{R}w_o\|_0 \lesssim \|\mathcal{R}w_o\|_\alpha &\lesssim \sum_{i,k} \left(\frac{\left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_0}{|k|^{1-\alpha}} + \frac{\left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_{N+\alpha} + \left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_0 \|\Phi_i\|_{N+\alpha}}{|k|^{N-\alpha}} \right) \\ &\lesssim \delta_{q+1}^{\frac{1}{2}} \cdot \sum_{k\neq 0} \left(\frac{1}{\lambda_{q+1}^{1-\alpha} |k|^{7-\alpha}} + \frac{\ell_q^{-N-\alpha}}{\lambda_{q+1}^{N-\alpha} |k|^{N-\alpha+7}} \right) \lesssim \frac{\delta_{q+1}^{\frac{1}{2}}}{\lambda_{q+1}^{1-\alpha}},\end{aligned} \tag{6.4.58}$$

the last step being due to (4.3.14). We also note that

$$\|\mathcal{R}w_o\|_1 = \max_i \|\mathcal{R}\partial_i w_o\|_0.$$

Proceeding on the $\mathcal{R}\partial_i w_o$ as we did on $\mathcal{R}w_o$ then yields

$$\|\mathcal{R}w_o\|_1 \lesssim \max_i \|\mathcal{R}\partial_t w_o\|_\alpha \lesssim \sum_{i,k} \left(\frac{\left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_1}{|k|^{1-\alpha}} + \frac{\left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_{N+1+\alpha} + \left\|\mathfrak{D}\Phi_i^{-1} b_{i,k}\right\|_1 \|\Phi_i\|_{N+\alpha}}{|k|^{N-\alpha}} \right)$$
$$\lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\alpha}. \tag{6.4.59}$$

Such estimates analogously also hold for $\mathcal{R}w_c$, and thus for $\mathcal{R}w$. Thus, by (6.4.57)-(6.4.59)

$$\left\|\mathcal{R}(-\Delta)^\theta w\right\|_0 \lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\alpha} \lambda_{q+1}^{2\theta+\varepsilon-1}.$$

In particular, for $a \gg 1$ large enough, (6.4.20) is satisfied if

$$2\theta + \varepsilon - 1 + \alpha - \beta < -2b\beta - 6\alpha \iff 7\alpha + \varepsilon < 1 + \beta - 2\theta - 2b\beta. \tag{6.4.60}$$

Since $\theta < \beta$, and $2b\beta < 1 - \beta$ by (4.3.2), we have that $1 + \beta - 2\theta - 2b\beta > 0$. Thus, (6.4.60) above holds for α, ε sufficiently small.

Step 5: Estimates on the new Reynolds term $\mathcal{E}^{(2)}$.

Now we turn to $\mathcal{E}^{(2)}$. Consider the decomposition

$$\left|\mathcal{E}^{(2)}\right| = \frac{1}{3}\left| \fint_{\mathbb{T}^3} |\tilde{v}|^2 - |v|^2 - \operatorname{tr} S_\psi \right|$$
$$\leq \frac{1}{3}\left| \fint_{\mathbb{T}^3} |w_o|^2 - \operatorname{tr} S_\psi \right| + \frac{1}{3}\left| \fint_{\mathbb{T}^3} 2w \cdot v \right| + \frac{1}{3}\left| \fint_{\mathbb{T}^3} 2w_c \cdot w_o + |w_c|^2 \right|, \tag{6.4.61}$$

and proceed as in [8, Proposition 6.2]. In the case of the first term, we will bound the whole tensor, and therefore the trace. For the other terms, only the trace will be estimated. Concerning the first term in (6.4.61), thanks to (6.4.31), $\sum \int \sigma_i = \int \sigma_\psi$, so that two cancellations occur:

$$\int w_o \otimes w_o - S_\psi \, \mathrm{d}x = \sum_{i,k\neq 0} \int \sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}\Phi_i^{-T} e^{i\lambda_{q+1} k\cdot\Phi_i} \mathrm{d}x + \int \left(\sum_i \sigma_i - \sigma_\psi \right) \mathrm{Id}\, \mathrm{d}x$$
$$= \sum_{i,k\neq 0} \int Z_{i,k} e^{i\lambda_{q+1} k\cdot\Phi_i} \mathrm{d}x, \tag{6.4.62}$$

where we write $Z_{i,k} \coloneqq \sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}\Phi_i^{-T}$. Using (C.13), (C.6), and (4.3.14), we obtain that

$$\left| \int_{\mathbb{T}^3} \sum_{i,k\neq 0} Z_{i,k} e^{i\lambda_{q+1} k\cdot\Phi_i} \right| \lesssim \sum_{i,k\neq 0} \frac{\|Z_{i,k}\|_{\overline{N}} + \|Z_{i,k}\|_0 \|\Phi_i\|_{\overline{N}}}{|\lambda_{q+1} k|^{\overline{N}}} \lesssim \sum_{k\neq 0} \frac{\delta_{q+1} \ell_q^{-\overline{N}}}{\lambda_{q+1}^{\overline{N}} |k|^{\overline{N}}} \lesssim \frac{\delta_{q+1} \ell_q}{\lambda_{q+1}}. \tag{6.4.63}$$

The second inequality above is easily justified by using (6.4.30), (6.4.39), and (6.4.55) to estimate $Z_{i,k}$ as follows:

$$\begin{aligned}\|Z_{i,k}\|_N &\lesssim \|\sigma_i\|_N\|\mathfrak{D}\Phi_i^{-1}\|_0^2\|C_k(\tilde{S}_i)\|_0 + \|\sigma_i\|_0\|\mathfrak{D}\Phi_i^{-1}\|_0^2\|C_k(\tilde{S}_i)\|_N \\ &\quad + 2\|\sigma_i\|_0\|\mathfrak{D}\Phi_i^{-1}\|_N\|C_k(\tilde{S}_i)\|\|\mathfrak{D}\Phi_i^{-1}\|_0 \\ &\lesssim |k|^{-6}\delta_{q+1}\ell_q^{-N}.\end{aligned}$$

To estimate the second term in (6.4.61), observe that

$$w\cdot v = \sum_{i,k}((\mathfrak{D}\Phi_i)^{-1}b_{i,k} + \lambda_{q+1}^{-1}c_{i,k})\cdot v e^{i\lambda_{q+1}k\cdot\Phi_i},$$

so that, combining **Lemma C.3**, (6.4.43), (6.4.44), (6.4.11), (6.4.39), and (4.3.14), we obtain that

$$\begin{aligned}\left|\fint 2w\cdot v\mathrm{d}x\right| &\lesssim \sum_{i,k}\frac{\left\|((\mathfrak{D}\Phi_i)^{-1}b_{i,k}+\lambda_{q+1}^{-1}c_{i,k})\cdot v\right\|_{\overline{N}} + \left\|((\mathfrak{D}\Phi_i)^{-1}b_{i,k}+\lambda_{q+1}^{-1}c_{i,k})\cdot v\right\|_0\|\mathfrak{D}\Phi_i\|_{\overline{N}}}{\lambda_{q+1}^{\overline{N}}|k|^{\overline{N}}} \\ &\lesssim \delta_{q+1}^{\frac12}\cdot\delta_q^{\frac12}\lambda_q^{1+\alpha}\cdot\ell_q\lambda_{q+1}^{-1} \lesssim \delta_{q+1}^{\frac12}\delta_q^{\frac12}\lambda_q^{\alpha}\lambda_{q+1}^{-1}.\end{aligned} \tag{6.4.64}$$

Concerning the third term in (6.4.61), note that the estimates on w_c are always no coarser than those for w_o, so if we estimate $\int w_o\cdot w_c$ well, the whole term is estimated well. To this end, we observe that

$$\begin{aligned}\left|\int w_o\cdot w_c\mathrm{d}x\right| &\le \sum_i\sum_{0\neq k\neq l}\left|\int(\mathfrak{D}\Phi_i)^{-1}b_{i,k}\lambda_{q+1}^{-1}c_{i,l-k}e^{i\lambda l\cdot\Phi}\mathrm{d}x\right| \\ &\lesssim \sum_i\sum_{0\neq k\neq l\neq 0}\frac{\left\|(\mathfrak{D}\Phi_i)^{-1}b_{i,k}\lambda_{q+1}^{-1}c_{i,l-k}\right\|_{\overline{N}} + \left\|(\mathfrak{D}\Phi_i)^{-1}b_{i,k}\lambda_{q+1}^{-1}c_{i,l-k}\right\|_0\|\mathfrak{D}\Phi_i\|_{\overline{N}}}{\lambda_{q+1}^{\overline{N}}|l|^{\overline{N}}} \\ &\quad + \sum_i\sum_{k\neq 0}\left|\int(\mathfrak{D}\Phi_i)^{-1}b_{i,k}\lambda_{q+1}^{-1}c_{i,-k}\mathrm{d}x\right| =: I + II,\end{aligned}$$

where we used **Lemma C.3** in the case $l\neq 0$, as well as the fact that the $w_{o,i}$ and $w_{c,i}$ have disjoint support so we do not have products of the form $b_{i,k}c_{j,l-k}$ for $i\neq j$. The term I is easily estimated as $\delta_{q+1}\ell_q^{-\overline{N}}\lambda_{q+1}^{-\overline{N}} \lesssim \delta_{q+1}\ell_q\lambda_{q+1}^{-1}$, so that I satisfies the same estimate as the second term in (6.4.61). As for II, (6.4.43) and (6.4.44) easily yield $II \lesssim \rho_i\ell_{q,i}^{-1}\lambda_{q+1}^{-1}$. Therefore

$$\left|\int w_o\cdot w_c\mathrm{d}x\right| \lesssim \frac{\delta_{q+1}\ell_q}{\lambda_{q+1}} + \frac{\rho_i\ell_q^{-1}}{\lambda_{q+1}}. \tag{6.4.65}$$

Combining (6.4.61)-(6.4.65) with the fact that $\int|w_c|^2$ also satisfies (6.4.65), we arrive at

$$\left|\mathcal{E}^{(2)}\right| \lesssim \frac{\delta_{q+1}\ell_q}{\lambda_{q+1}} + \frac{\delta_{q+1}^{\frac12}\delta_q^{\frac12}\lambda_q^{1+\alpha}\ell_q}{\lambda_{q+1}} + \frac{\rho_i\ell_{q,i}^{-1}}{\lambda_{q+1}}.$$

By (4.3.12), we thus conclude that, for $a\gg 1$ sufficiently large, $\mathcal{E}^{(2)}$ satisfies (6.4.20). Combining with the fact (obtained in the previous step) that $\mathring{\mathcal{E}}^{(1)}$ satisfies (6.4.20), we thus conclude that (6.4.20) holds.

Step 6: Estimates on $\partial_t \operatorname{tr} \mathcal{E}$

Observe that $\mathring{\mathcal{E}}^{(1)}$ is traceless, whereas $\mathcal{E}^{(2)}$ is a function of t only. In order to estimate the time derivative of $\mathcal{E}^{(2)}$, observe that, since v is solenoidal, for every $F = F(x,t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} F = \int_{\mathbb{T}^3} D_t F,$$

where $D_t = \partial_t + v \cdot \nabla$. Therefore, using again the decomposition in (6.4.61), we have that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} |\tilde{v}|^2 - |v|^2 - \operatorname{tr} S_\psi \right| \le \left| \int_{\mathbb{T}^3} \operatorname{tr} \left[D_t \left(\sum_{i,k \neq 0} \sigma_i \mathfrak{D}\Phi_i^{-1} C_k(\tilde{S}_i) \mathfrak{D}\Phi_i^{-T} e^{i\lambda_{q+1} k \cdot \Phi_i} \right) \right] \right| \\ + \left| \int_{\mathbb{T}^3} D_t (2 w_c \cdot w_o + |w_c|^2) \right| + \left| \int_{\mathbb{T}^3} D_t (2 v \cdot w) \right|. \tag{6.4.66}$$

Let us first estimate $\|D_t w_o\|_0$. Recall from (C.10) that $D_t (\mathfrak{D}\Phi_i)^{-1} = \mathfrak{D}v (\mathfrak{D}\Phi_i)^{-1}$, which, combined with the fact that $D_t e^{i\lambda_{q+1} k \cdot \Phi_i} = 0$, yields

$$\begin{aligned} D_t w_o = & \sum_{i,k \neq 0} D_t \big(\sqrt{\sigma_i} a_k(\tilde{S}_i) \big) \mathfrak{D}\Phi_i^{-1} A_k e^{i\lambda_{q+1} k \cdot \Phi_i} \\ & + \sum_{i,k \neq 0} \sqrt{\sigma_i} a_k(\tilde{S}_i) \mathfrak{D}v \mathfrak{D}\Phi_i^{-1} A_k e^{i\lambda_{q+1} k \cdot \Phi_i} \\ = & \sum_{i,k \neq 0} \mathfrak{D}\Phi_i^{-1} D_t b_{i,k} e^{i\lambda_{q+1} k \cdot \Phi_i} + \sum_{i,k \neq 0} \mathfrak{D}v \mathfrak{D}\Phi_i^{-1} b_{i,k} e^{i\lambda_{q+1} k \cdot \Phi_i}. \end{aligned}$$

First notice that, by using (6.4.11), (6.4.39), and (6.4.43), we obtain that

$$\left\| \mathfrak{D}v \mathfrak{D}\Phi_i^{-1} b_{i,k} \right\|_0 \lesssim \frac{\delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha}}{|k|^6}.$$

As for the coefficients $\mathfrak{D}\Phi_i^{-1} D_t b_{i,k}$, combining (6.4.39) and (6.4.46) gives

$$\left\| \mathfrak{D}\Phi_i^{-1} D_t b_{i,k} \right\|_0 \lesssim \tau_q^{-1} \delta_{q+1}^{\frac{1}{2}} |k|^{-6} = \frac{\delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q \ell_q^{-4\alpha}}{|k|^6}.$$

Therefore

$$\|D_t w_o\|_0 \lesssim \delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q \ell_q^{-4\alpha}.$$

Observing that

$$D_t w_c = \sum_{i,k} \lambda_{q+1}^{-1} D_t c_{i,k} e^{i\lambda_{q+1} k \cdot \Phi_i},$$

which follows from $D_t e^{i\lambda_{q+1} k \cdot \Phi_i} = 0$ seen above, (6.4.47) implies

$$\|D_t w_c\|_0 \lesssim \delta_{q+1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q \ell_q^{-1-4\alpha} \lambda_{q+1}^{-1}.$$

Combining with $\|w_o\|_0+\|w_c\|_0 \lesssim \delta_{q+1}^{\frac{1}{2}}$ and using (4.3.12)-(4.3.13), we obtain that

$$
\begin{aligned}
\left|\int_{\mathbb{T}^3} D_t(w_o\otimes w_c+w_c\otimes w_o+w_c\otimes w_c)\right| &\lesssim \|D_t w_o\|_0\|w_c\|_0+\|w_o\|_0\|D_t w_c\|_0+\|D_t w_c\|_0\|w_c\|_0\\
&\lesssim \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\ell_q^{-4\alpha}=\delta_{q+1}^{\frac{1}{2}}(\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q)\ell_q^{-4\alpha}\\
&\lesssim \delta_{q+1}^{\frac{1}{2}}(\delta_{q+2}\lambda_{q+1}^{1-10}\alpha)\lambda_{q+1}^{4\alpha}=\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1-6\alpha}.
\end{aligned}
$$

The second term of (6.4.66) is thus estimated. We then similarly decompose the first term in (6.4.66) as

$$
\begin{aligned}
D_t\left[\sum_{i,k\neq 0}\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\nabla\Phi_i^{-1}e^{i\lambda_{q+1}k\cdot\Phi_i}\right] &= \sum_{i,k\neq 0} D_t\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\nabla\Phi_i^{-1}e^{i\lambda_{q+1}k\cdot\Phi_i}\\
&\quad+\sum_{i,k\neq 0}\sigma_i\mathfrak{D}v\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\nabla\Phi_i^{-1}e^{i\lambda_{q+1}k\cdot\Phi_i}\\
&\quad+\sum_{i,k\neq 0}\sigma_i\mathfrak{D}\Phi_i^{-1}D_t[C_k(\tilde{S}_i)]\nabla\Phi_i^{-1}e^{i\lambda_{q+1}k\cdot\Phi_i}\\
&\quad+\sum_{i,k\neq 0}\sigma_i\mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\nabla\Phi_i^{-1}\nabla v e^{i\lambda_{q+1}k\cdot\Phi_i}.
\end{aligned}
$$

In order to estimate this, we still need to estimate $D_t[C_k(\tilde{S}_i)]$ and $D_t\sigma_i$. To obtain the former, we first use (A.4):

$$
\begin{aligned}
\left\|D_t(C_k(\tilde{S}_i))\right\|_N &\leq \left\|(\mathfrak{D}C_k)(\tilde{S}_i)\right\|_N\left\|D_t\tilde{S}_i\right\|_0+\left\|(\mathfrak{D}C_k)(\tilde{S}_i)\right\|_0\left\|D_t\tilde{S}_i\right\|_N\\
&\lesssim (\|C_k\|_{N+1}\left\|\tilde{S}_i\right\|_1^N+\|C_k\|_2\left\|\tilde{S}_i\right\|_N)\left\|D_t\tilde{S}_i\right\|_0+\|C_k\|_1\left\|D_t\tilde{S}_i\right\|_N,
\end{aligned}
$$

We then use (C.6), (6.4.42), and (6.4.45) to conclude that

$$
\left\|D_t(C_k(\tilde{S}_i))\right\|_N \lesssim |k|^{-6}\tau_q^{-1}\ell_{q,i}^{-N}. \tag{6.4.67}
$$

Coming to $D_t\sigma_i$, we claim that

$$
\|D_t\sigma_i\|_N \lesssim \delta_{q+1}\tau_q^{-1}\ell_{q,i}^{-N}. \tag{6.4.68}
$$

To obtain (6.4.68), we set

$$
\begin{aligned}
h(t) &:= \sum_j\int\eta_j^2(x,t)\mathrm{d}x\\
D_t\sigma_i &= \frac{|\mathbb{T}^3|\psi^2\sigma}{h}2\eta_i D_t\eta_i+|\mathbb{T}^3|\eta_i^2\partial_t\left(\frac{\psi^2\sigma}{h}\right)=: I+II.
\end{aligned}
$$

We first estimate the term I. Recalling (6.4.27), (6.4.28), $\psi\leq 1$, and (6.4.7), we conclude that

$$
\|I(\cdot,t)\|_N \lesssim \frac{|\mathbb{T}^3|\psi^2\sigma}{h}(\|\eta_i\|_N\|D_t\eta_i\|_0+\|\eta_i\|_0\|D_t\eta_i\|_N)\lesssim \delta_{q+1}\tau_q^{-1}\ell_{q,i}^{-N}.
$$

As for the second term, we already see that, since the only factor depending on x is η_i^2 which, by (6.4.27), satisfies $\|\eta_i^2\|_N \lesssim 1$ for all N, the estimates for II will only depend on N via an a-independent constant, thus making it sufficient to estimate $\partial_t(\psi^2\sigma h^{-1})$ in C^0. To that end, we rewrite it as

$$\partial_t\left(\frac{\psi^2\sigma}{h}\right) = \frac{2\psi\psi'\sigma}{h} + \frac{\psi^2\partial_t\sigma}{h} - \frac{\psi^2\sigma h'}{h^2} =: T_1 + T_2 + T_3.$$

To estimate T_1, we recall (6.4.7), (6.4.6), and (6.4.28):

$$\|T_1\|_0 \le \frac{2\delta_q^{\frac{1}{2}}\lambda_q \cdot 4\delta_{q+1}}{c_0} \lesssim \tau_q^{-1}\delta_{q+1}.$$

Coming to T_2, by (6.4.7), (6.4.9), $\psi \le 1$, and (6.4.28), we obtain that

$$\|T_2\|_0 \le \frac{4C\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q}{c_0} \lesssim \delta_{q+1}\tau_q^{-1},$$

where C is the implicit constant in (6.4.9). Finally, to estimate T_3, we use (6.4.7), $\psi \le 1$, and (6.4.51):

$$\|T_3\|_0 \le \frac{4K\delta_{q+1}\tau_q^{-1}}{c_0^2} \lesssim \delta_{q+1}\tau_q^{-1},$$

where K is the implicit constant in (6.4.51). The estimate (6.4.68) is thus proved. By (6.4.68), (6.4.40), (6.4.55), (6.4.30), (6.4.11), (6.4.67), and (4.3.12), we conclude that

$$\begin{aligned}\left\| D_t\left[\sum_{i,k\neq 0}\sigma_i \mathfrak{D}\Phi_i^{-1}C_k(\tilde{S}_i)\nabla\Phi_i^{-1}e^{i\lambda_{q+1}k\cdot\Phi_i}\right]\right\|_0 &\lesssim \delta_{q+1}\tau_q^{-1} = \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\ell_q^{-4\alpha} \\ &\lesssim \frac{\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q}{\lambda_{q+1}^{1-10\alpha}}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1-10\alpha+4\alpha} \lesssim \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1-6\alpha}.\end{aligned}$$

Finally, to estimate the term involving $D_t(w\cdot v)$, we first note that

$$\int D_t(v\cdot w) = \int D_t v\cdot w + \int v\cdot D_t w = -\int(\nabla p + (-\Delta)^\theta v + \operatorname{div} R)\cdot w + \int v\cdot D_t w, \quad (6.4.69)$$

using (4.1.1) in the last step. To estimate the second term of (6.4.69), we write

$$v\cdot D_t w = \sum_{i,k\neq 0} h_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i},$$

where

$$h_{i,k} := v\cdot D_t[(\mathfrak{D}\Phi_i)^{-1}b_{i,k} + \lambda_{q+1}^{-1}c_{i,k}] = v\cdot[D_t(\mathfrak{D}\Phi_i)^{-1}b_{i,k} + (\mathfrak{D}\Phi_i)^{-1}D_t b_{i,k} + \lambda_{q+1}^{-1}D_t c_{i,k}].$$

By **Lemma A.1**, we obtain that

$$\begin{aligned}\|h_{i,k}\|_N \lesssim \|v\|_N&\left(\|D_t\mathfrak{D}\Phi_i^{-1}\|_0\|b_{i,k}\|_0 + \|\mathfrak{D}\Phi_i^{-1}\|_0\|D_t b_{i,k}\|_0 + \frac{1}{\lambda_{q+1}}\|D_t c_{i,k}\|_0\right) \\ &+ \|v\|_0\Big(\|D_t\mathfrak{D}\Phi_i^{-1}\|_N\|b_{i,k}\|_0 + \|D_t\mathfrak{D}\Phi_i^{-1}\|_0\|b_{i,k}\|_N \\ &\qquad + \|\mathfrak{D}\Phi_i\|_N\|D_t b_{i,ki}\|_0 + \|\mathfrak{D}\Phi_i^{-1}\|_0\|D_t b_{i,k}\|_N + \frac{1}{\lambda_{q+1}}\|D_t c_{i,k}\|_N\Big).\end{aligned}$$

Thus, using (6.4.4), (6.4.11) (in the form $\|v\|_{N+\alpha} \lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_{q,i}^{1-N} \lesssim \tau_q^{-1}\ell_q^{-N}$), (6.4.43)-(6.4.44), (6.4.46)-(6.4.47), and (6.4.40)-(6.4.41), we conclude that

$$\left\|h_{i,k}\right\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\tau_q^{-1}\ell_q^{-N} = \delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q\ell_q^{-4\alpha-N}.$$

With **Lemma C.3**, the above estimate yields that $\int v\cdot D_t w$ satisfies (6.4.21).
To deal with the first term of (6.4.69), we first note that, since $\operatorname{div} w = 0$, $\int \nabla p\cdot w = 0$. The term $\int \operatorname{div} R\cdot w$ can be estimated as follows:

$$\left|\int \operatorname{div} R\cdot w\mathrm{d}x\right| \le \|R\|_1\|w\|_0 \lesssim \Lambda\varrho^{1+\gamma}\ell_q^{-2\alpha}\ell_{q,i}^{-1}\cdot\delta_{q+1}^{\frac{1}{2}} \lesssim \Lambda^{\frac{1}{2}}\zeta_{q+1}^{\frac{1+\gamma}{2}}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_q^{-3\alpha}\cdot\delta_{q+1}^{\frac{1}{2}},$$

where we used (6.4.2) and (6.4.3). To conclude that the first term in (6.4.69) satisfies (6.4.21), we would require

$$\zeta_{q+1}^{\frac{1+\gamma}{2}}\zeta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\ell_q^{-3\alpha} \lesssim \zeta_{q+2}\lambda_{q+1}^{1-6\alpha}. \tag{6.4.70}$$

For α,γ sufficiently small, this follows from

$$-b\beta-\beta+1 < -2b^2\beta+b \iff 1-\beta-2b\beta < b(1-\beta-2b\beta) \iff 1-\beta-2b\beta > 0,$$

which in turn follows from (4.3.2).

Step 7: $\mathcal{T}_p$ and its derivative

The time derivative $\partial_t\mathcal{T}_p$ is readily estimated as

$$\begin{aligned}\left|\partial_t\mathcal{T}_p\right| &= \left|\int_{\mathbb{T}^3}(-\Delta)^{\frac{\theta}{2}}(2v+w)\cdot(-\Delta)^{\frac{\theta}{2}}w\mathrm{d}x\right| \\ &\le \left|\int_{\mathbb{T}^3}2(-\Delta)^{\frac{\theta}{2}}v\cdot(-\Delta)^{\frac{\theta}{2}}w\mathrm{d}x\right| + \int_{\mathbb{T}^3}\left|(-\Delta)^{\frac{\theta}{2}}w\right|^2\mathrm{d}x \\ &\lesssim (2\|v\|_{\theta+\varepsilon}\|w\|_0^{1-\theta-\varepsilon}\|w\|_1^{\theta+\varepsilon} + \|w\|_0^{2-2\theta-2\varepsilon}\|w\|_1^{2\theta+2\varepsilon}).\end{aligned}$$

By (6.4.12), we have that $\|v\|_{\theta+\varepsilon} \lesssim \Lambda^{\frac{1}{2}}$. As for w, we have that $\|w\|_N \lesssim \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^N$ (cfr. Step 3 above). Thus, recalling that $\theta+\varepsilon<\beta$

$$\left|\partial_t\mathcal{T}_p\right| \lesssim 2\Lambda^{\frac{1}{2}}\cdot\Lambda^{\frac{1}{2}}\lambda_{q+1}^{\theta+\varepsilon-\beta} + \Lambda\lambda_{q+1}^{2\theta+2\varepsilon-2\beta} \lesssim \Lambda\lambda_{q+1}^{\theta+\varepsilon-\beta}.$$

Since this is exactly (6.4.22), the proposition is proved. ◇

Remark 6.4.1 (The fractional dissipation term). *Note that* (6.4.22) *is stronger than* (6.4.21)*, since*

$$\Lambda\lambda_{q+1}^{\theta+\varepsilon-\beta} = \Lambda^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{\theta+\varepsilon} \lesssim \delta_{q+1}^{\frac{1}{2}}\delta_{q+2}\lambda_{q+1}^{1-6\alpha}.$$

Indeed, this inequality follows from $\theta+\varepsilon-\beta<1-6\alpha-\beta-2b\beta$ *which, for* α,ε *sufficiently small, follows from* (4.3.2) *and the fact* $\theta<\beta$.
However, $\mathcal{T}_p$ *is only estimated as follows:*

$$|\mathcal{T}_p(t)|\lesssim t\Lambda\lambda_{q+1}^{\theta+\varepsilon-\beta}.$$

To ensure that this satisfies (6.4.20) *for any* $q\geq 0$*, we would require*

$$0<\theta+\varepsilon-\beta<-2b^2\beta-3b\alpha \iff 3b\alpha<\beta-\theta-\varepsilon-2b^2\beta.$$

Seen as the above right-hand side is, in general, negative, we cannot require it. Thus, in general, $\mathcal{T}_p$ *only satisfies* (6.4.20) *if the* q *in the statement is sufficiently large, which is why we separated* $\mathcal{T}_p$ *from the other Reynolds terms.*
However, for $t\lesssim\Lambda^{-1}\lambda_{q+1}^{\beta-\varepsilon-\theta}$*, we can contrast the growth of* $\Lambda\lambda_{q+1}^{\theta+\varepsilon-\beta}$ *with the smallness of the time, meaning that* $\mathcal{T}_p$ *only satisfies* (6.4.20) *for a short period of time, or if* q *is sufficiently large.*

Remark 6.4.2 **(**$\mathcal{C}^0$ **estimate on the Reynolds stress).** *The requirement* (6.4.2) *is only used to obtain* (6.4.21)*, meaning we only need it on* $\operatorname{supp} S$*, since* $S=0 \implies \mathcal{E}=0$.

Chapter 7

Final approximations

7.1 From strict to adapted subsolutions: perturbing near $t = 0$

The aim of this section is to prove the following proposition, which provides the first convex integration scheme.

Proposition 7.1.1 (From strict to adapted subsolutions). *Let $(\tilde{v}, \tilde{p}, \tilde{R})$ be a smooth strict subsolution on $[0,T]$. Then, for any $\theta < \hat{\beta} < {}^1/_3$, $\nu > \frac{1-3\hat{\beta}}{2\hat{\beta}}$, and $\delta, \sigma > 0$, there exist $\gamma, \Omega > 0$ and a $C^{\hat{\beta}}$-adapted subsolution $(\hat{v}, \hat{p}, \hat{R})$ with parameters γ, Ω, ν such that $\hat{\rho} \leq \frac{5}{4}\delta$ and, for all $t \in [0,T]$*

$$\int_{\mathbb{T}^3} \left(|\hat{v}|^2 + \operatorname{tr}\hat{R}\right) \mathrm{d}x = \int_{\mathbb{T}^3} \left(|\tilde{v}|^2 + \operatorname{tr}\tilde{R}\right) \mathrm{d}x \tag{7.1.1}$$

$$\|\tilde{v} - \hat{v}_{C^0}\| \lesssim 1 + \delta^{\frac{1}{2}} \tag{7.1.2}$$

$$\|\tilde{v} - \hat{v}\|_{H^{-1}} < \sigma \tag{7.1.3}$$

Moreover, if we define

$$\hat{\mathcal{T}}(t) := \int_0^t \int_{\mathbb{T}^3} \left(\left|(-\Delta)^{\frac{\theta}{2}} \tilde{v}\right|^2 - \left|(-\Delta)^{\frac{\theta}{2}} \hat{v}\right|^2 \right) \mathrm{d}x\mathrm{d}s, \tag{7.1.4}$$

we have the bound

$$\left|\partial_t \hat{\mathcal{T}}\right| \lesssim \sum_q \Lambda \lambda_q^{\theta+\varepsilon-\beta}. \tag{7.1.5}$$

The $q = 0$ term of this sum is the largest, and is $\delta\lambda_1^{2\beta}\lambda_0^{\theta+\varepsilon-\beta}$, which is a-increasing. We now note that $a \to \infty$ for $\delta \to 0$, since we required $\delta\lambda_1^{2\beta} =: \Lambda \geq 1$ in (4.3.3), and thus need $\lambda_1^{2\beta} \to \infty$ for $\delta \to 0$. Therefore, for any $\eta > 0$, it can only be ensured that

$$\left|\hat{\mathcal{T}}\right|(t) \leq \eta \qquad t \in [0, \hat{T}(\eta, \delta, a)],$$

where $\hat{T}(\eta, \delta, a) \sim \eta\delta^{-1}\lambda_1^{-2\beta}\lambda_0^{\beta-\theta-\varepsilon} \to 0$ if $a \to \infty$ or $\eta \to 0$.

The proof closely follows the arguments of [18, Section 8]. Each stage contains a localized gluing step performed using **Proposition 6.3.1**, and a perturbation step performed using **Proposition 6.4.1**.

Proof. (***Proposition 7.1.1***)

Step 1: Setting the parameters of the scheme

Let $(\tilde{v},\tilde{p},\tilde{R})$ be a smooth strict subsolution and let $0<\hat{\beta}<\beta<\frac{1}{3}, \nu>0$. Choose $b>1$ according to (4.3.2), furthermore let $\tilde{\varepsilon}>0$ such that:

$$b(1+\tilde{\varepsilon})<\frac{1-\beta}{2\beta}. \tag{7.1.6}$$

Then, let $\tilde{\delta},\tilde{\gamma}>0$ be the constants given by **Corollary 5.2.1**, and choose $0<\alpha<1$ and $0<\gamma<\hat{\gamma}<\tilde{\gamma}$ so that:

- The inequalities (4.3.12), (4.3.13) are satisfied by both the pairs (α,γ) and $(\alpha,\hat{\gamma})$;
- The other conditions in Sections 6.3 and 6.4, namely (6.3.1)-(6.3.2) (and consequently (6.4.1)), (6.4.60), and (6.4.70), are satisfied by both the pairs (α,γ) and $(\alpha,\hat{\gamma})$;
- Condition (4.3.14) can hold for both pairs (α,γ) and $(\alpha,\hat{\gamma})$; since $\hat{\gamma}>\gamma$, relation (4.3.15) reduces this to:

$$(b-1)(1-\beta(b+1))-\hat{\gamma}\beta b^2-2\alpha b>0; \tag{7.1.7}$$

- The following conditions holds:

$$\nu>\frac{1-3\beta+\alpha}{2\beta} \tag{7.1.8}$$

$$\frac{\alpha}{\beta}<b\hat{\gamma}<\frac{3\alpha}{2\beta}, \qquad 0<b\gamma<\hat{\gamma}-\frac{\alpha}{\beta}, \qquad 3\alpha>2b\beta\gamma. \tag{7.1.9}$$

Having fixed $b,\beta,\alpha,\gamma,\hat{\gamma}$, we may choose $\overline{N}\in\mathbb{N}$ so that (4.3.14) is also valid. For $a\gg 1$ sufficiently large (to be determined) we then define (λ_q,δ_q) as in (4.3.1). Thus we are in the setting of Section 4.3.

Step 2: From strict to strong subsolution

We apply **Corollary 5.2.1** to obtain from $(\tilde{v},\tilde{p},\tilde{R})$ a strong subsolution (v_0,p_0,R_0) with $\delta=\delta_1$ such that the properties from (5.2.2) to (5.2.6) hold. By (5.2.2)-(5.2.6), (v_0,p_0,R_0) satisfies

$$\frac{3}{4}\delta_1\le\rho_0\le\frac{5}{4}\delta_1 \tag{7.1.10}$$

$$\left\|\mathring{R}_0(t)\right\|_0\le\Lambda\varrho_0^{1+\hat{\gamma}} \tag{7.1.11}$$

$$\|v_0\|_{H^{-1}}\le\lambda_0^{-1} \tag{7.1.12}$$

$$\|v_0\|_{1+\alpha}\le\delta_0^{\frac{1}{2}}\lambda_0^{1+\alpha} \tag{7.1.13}$$

$$|\partial_t\rho_0|\le\delta_1\delta_0^{\frac{1}{2}}\lambda_0. \tag{7.1.14}$$

Step 3: Inductive construction of (v_q, p_q, R_q)

Starting from (v_0, p_0, R_0), we show how to inductively construct a sequence $\{(v_q, p_q, R_q)\}_{q\in\mathbb{N}}$ of smooth strong subsolutions with:

$$R_q(x,t) = \rho_q(t)\,\mathrm{Id} + \mathring{R}_q(x,t)$$

which satisfy the following properties:

(a_q) For all $t \in [0,T]$

$$\int_{\mathbb{T}^3} \big(|v_q|^2 + \operatorname{tr} R_q\big)\mathrm{d}x = \int_{\mathbb{T}^3} \big(|v_0|^2 + \operatorname{tr} R_0\big)\mathrm{d}x;$$

(b_q) For all $t \in [0,T]$

$$\big\|\mathring{R}_q(t)\big\|_0 \le \Lambda\,\varrho_q^{1+\gamma};$$

(c_q) If $2^{-j}T < t \le 2^{-j+1}T$ for some $j = 1, \dots, q$, then

$$\frac{3}{8}\delta_{j+1} \le \rho_q \le 4\delta_j;$$

(d_q) For all $t \le 2^{-q}T$:

$$\big\|\mathring{R}_q(t)\big\|_0 \le \Lambda\,\varrho_q^{1+\hat{\gamma}}, \qquad \frac{3}{4}\delta_{q+1} \le \rho_q \le \frac{5}{4}\delta_{q+1};$$

(e_q) If $2^{-j}T < t \le 2^{-j+1}T$ for some $j = 1, \dots, q$, then

$$\begin{aligned} \big\|v_q\big\|_{1+\alpha} &\le M\delta_j^{\frac{1}{2}}\lambda_j^{1+\alpha} \\ \big|\partial_t \rho_q\big| &\lesssim \delta_{j+1}\delta_j^{\frac{1}{2}}\lambda_j, \end{aligned}$$

whereas if $t \le 2^{-q}T$

$$\begin{aligned} \big\|v_q\big\|_{1+\alpha} &\le M\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} \\ \big|\partial_t \rho_q\big| &\lesssim \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q. \end{aligned}$$

(f_q) For all $t \in [0,T]$ and $q \ge 1$:

$$\big\|v_q - v_{q-1}\big\|_{H^{-1}} \le M\delta_q^{\frac{1}{2}}(\zeta_q^{\frac{\gamma}{2}}\ell_{q-1}^{\alpha} + \ell_{q-1}^{-1}\lambda_q^{-1}) \qquad \big\|v_q - v_{q-1}\big\|_0 \le M\delta_q^{\frac{1}{2}}.$$

(g_q) $\|v_q\|_{\theta+\varepsilon} \le M\Big(1 + \Lambda^{\frac{1}{2}}\sum_{i=0}^{q}\lambda_i^{\theta+\varepsilon-\beta}\Big).$

Thanks to our choice of parameters in Step 1 above, (v_0, p_0, R_0) satisfies (7.1.10)-(7.1.14), and thus the inductive assumptions (a_0)-(g_0) (the last condition can be deduced from (5.2.8)).
Suppose then (v_q, p_q, R_q) is a smooth strong subsolution which satisfies (a_q)-(g_q). The construction of $(v_{q+1}, p_{q+1}, R_{q+1})$ consists of two steps: first a localized gluing step performed using **Proposition 6.3.1** to get from (v_q, p_q, R_q) to a smooth strong subsolution $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$, then a localized perturbation step done using **Proposition 6.4.1** to get $(v_{q+1}, p_{q+1}, R_{q+1})$ from $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$.
We apply **Proposition 6.3.1** with

$$[T_1, T_2] = [0, 2^{-q}T].$$

Then $T_2 - T_1 \geq 4\tau_q$, if $a \gg 1$ is sufficiently large. Moreover, by (d_q), (e_q), and (g_q), (v_q, p_q, R_q) fulfils the requirements of **Proposition 6.3.1** on $[T_1, T_2]$ with parameters $\alpha, \hat{\gamma} > 0$.
Then, by **Proposition 6.3.1**, we obtain a smooth strong subsolution $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ on $[0, T]$ such that $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ is equal to (v_q, p_q, R_q) on $[2^{-q}T, T]$, and on $[0, 2^{-q}T]$ satisfies

$$\begin{aligned}
\left\|\overline{v}_q - v_q\right\|_\alpha &\lesssim \Lambda^{\frac{1}{2}} \overline{\varrho}_q^{\frac{1+\hat{\gamma}}{2}} \ell_q^\alpha \\
\left\|\overline{v}_q\right\|_{1+\alpha} &\lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \\
\left\|\overline{v}_q\right\|_{\theta+\varepsilon} &\lesssim 1 + \sum_{i=0}^{q+1} \delta_i^{\frac{1}{2}} \lambda_i^{\theta+\varepsilon} \\
\left\|\mathring{\overline{R}}_q\right\|_0 &\leq \Lambda \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-2\alpha} \\
\frac{5}{8}\delta_{q+1} &\leq \overline{\rho}_q \leq \frac{3}{2}\delta_{q+1} \\
\left|\partial_t \overline{\rho}_q\right| &\lesssim \delta_{q+1} \delta_q^{\frac{1}{2}} \lambda_q.
\end{aligned} \tag{7.1.15}$$

Moreover, on $[0, t_{\overline{n}}]$ one has that

$$\begin{aligned}
\left\|\overline{v}_q\right\|_{N+1+\alpha} &\lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_q^{-N} \\
\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} &\lesssim \Lambda \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-N-2\alpha} \\
\left\|(\partial_t + \overline{v}_q \cdot \nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} &\lesssim \Lambda \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-N-6\alpha} \delta_q^{\frac{1}{2}} \lambda_q.
\end{aligned} \tag{7.1.16}$$

and

$$\mathring{\overline{R}}_q \equiv 0 \qquad t \in \bigcup_{i=0}^{\overline{n}} J_i. \tag{7.1.17}$$

Recalling **Definition 6.1.1** and (6.1.5) observe that

$$\left[0, \frac{3}{4} 2^{-q} T\right] \subset [0, t_{\overline{n}}], \tag{7.1.18}$$

provided $a \gg 1$ is chosen sufficiently large (e.g. so that $\frac{5}{3}\tau_q < \frac{1}{4}2^{-q}T$). Then, choose a cut-off function $\psi_q \in \mathcal{C}_C^\infty([0,\frac{3}{4}2^{-q}T];[0,1])$ such that

$$\psi_q(t) = \begin{cases} 1 & t \le 2^{-(q+1)}T \\ 0 & t > \frac{3}{4}2^{-q}T \end{cases} \tag{7.1.19}$$

and such that $|\psi_q'(t)| \lesssim 2^q$. By choosing $a \gg 1$ sufficiently large, we may assume that

$$|\psi_q'(t)| \le \frac{1}{2}\delta_q^{\frac{1}{2}}\lambda_q \tag{7.1.20}$$

for all q. Then, set

$$S_\psi \coloneqq \psi_q^2(\overline{R}_q - \delta_{q+2}\,\mathrm{Id}) = \psi_q^2 S.$$

Using (7.1.20), (7.1.9), (7.1.15)-(7.1.18), and the easy observation that $\overline{\rho}_q \lesssim \overline{\rho}_q - \delta_{q+2}$, we see that S_ψ and $(\overline{v}_q,\overline{p}_q,\overline{R}_q)$ satisfy the assumptions of **Proposition 6.4.1** on the interval $[0,t_{\overline{n}}]$ with parameters $\alpha,\hat{\gamma} > 0$. We have that

$$\sigma_\psi = \psi_q^2(\overline{\rho}_q - \delta_{q+2}) = \psi_q^2\sigma.$$

Recalling **Remark 6.4.2**, since $\operatorname{supp} S_\psi \subseteq [t_{\underline{n}},t_{\overline{n}}]$ where (6.3.17) holds, we can apply **Proposition 6.4.1**, thus obtaining a new subsolution $(v_{q+1},p_{q+1},\overline{R}_q - S_\psi - \mathcal{E}_{q+1})$ with

$$\begin{aligned} \|v_{q+1}-\overline{v}_q\|_0 + \ell_q\lambda_{q+1}\|v_{q+1}-\overline{v}_q\|_{H^{-1}} & \\ + \lambda_{q+1}^{-1-\alpha}\|v_{q+1}-\overline{v}_q\|_{1+\alpha} & \\ + \lambda_{q+1}^{-\theta-\varepsilon}\|v_{q+1}-\overline{v}_q\|_{\theta+\varepsilon} & \le M\delta_{q+1}^{\frac{1}{2}} \\ \int_{\mathbb{T}^3}|v_{q+1}|^2 - \operatorname{tr} S - \operatorname{tr}\mathcal{E}_{q+1} = \int_{\mathbb{T}^3}|\overline{v}_q|^2 \qquad & t\in[0,T], \end{aligned}$$

and such that the estimates (6.4.20) and (6.4.21) hold for $\mathcal{E}_{q+1}$. Let

$$R_{q+1} \coloneqq \overline{R}_q - S_\psi - \mathcal{E}_{q+1}.$$

We claim that $(v_{q+1},p_{q+1},R_{q+1})$ is a smooth strong subsolution satisfying (a_{q+1})-(g_{q+1}). Notice that (a_{q+1}) is satisfied by construction. Since $(v_{q+1},p_{q+1},R_{q+1}) = (v_q,p_q,R_q)$ for $t \ge 2^{-q}T$, we may restrict t to $[0,2^{-q}T]$ in the following, so that in particular (7.1.15) holds.

Let us now prove (b_{q+1}). On the one hand

$$\begin{aligned} \|\mathring{R}_{q+1}\|_0 &= \left\|(1-\psi_q^2)\mathring{\overline{R}}_q - \mathring{\mathcal{E}}_{q+1}\right\|_0 \\ &\le (1-\psi_q^2)\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha} + \delta_{q+2}\lambda_{q+1}^{-3\alpha}\mathbb{1}_{\{\psi_q>0\}}, \end{aligned} \tag{7.1.21}$$

on the other hand

$$\begin{aligned} \rho_{q+1} &= (1-\psi_q^2)\Lambda\overline{\varrho}_q + \psi_q^2\delta_{q+2} + \frac{1}{3}\operatorname{tr}\mathcal{E}_{q+1} \\ &\ge (1-\psi_q^2)\overline{\rho}_q + \psi_q^2\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-3\alpha}\mathbb{1}_{\{\psi_q>0\}}. \end{aligned} \tag{7.1.22}$$

The proof of (b_{q+1}) thus reduces to assessing whether there exists a suitable γ such that

$$(1-\psi_q^2)\Lambda^{-\hat{\gamma}}\overline{\rho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}+\delta_{q+2}\lambda_{q+1}^{-3\alpha}\mathbb{1}_{\{\psi_q>0\}}\le\Lambda^{-\gamma}[(1-\psi_q^2)\overline{\rho}_q+\psi_q^2\delta_{q+2}-\delta_{q+2}\lambda_{q+1}^{-3\alpha}]^{1+\gamma}. \tag{7.1.23}$$

To this end set

$$\begin{aligned} F(s) &:= (1-s)\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}+\delta_{q+2}\lambda_{q+1}^{-3\alpha} \\ G(s) &:= (1-s)\overline{\rho}_q+s\delta_{q+2}-\delta_{q+2}\lambda_{q+1}^{-3\alpha} \\ H(s) &:= \Lambda^{-\gamma}G^{1+\gamma}(s)-F(s), \end{aligned}$$

and observe that (7.1.23) is equivalent to $H(\psi_q^2)\ge 0$ if $\psi_q>0$, and follows from this inequality also in case $\psi_q=0$. In particular, (7.1.23) follows from:

(i) $H(0)\ge 0$ and $H(1)\ge 0$;

(ii) $H'(0)\le 0$ and $H'(1)\le 0$.

(iii) $H''(s)\ge 0$.

We note next that, since $2b\beta\hat{\gamma}<3\alpha$

$$\delta_{q+2}\lambda_{q+1}^{-3\alpha}\lesssim\Lambda\overline{\varrho}_q^{1+\hat{\gamma}},$$

so that we have the estimates

$$F(0)\lesssim\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}, \qquad\qquad G(0)\gtrsim\overline{\rho}_q.$$

It is also clear that $G(s)\le\overline{\rho}_q$.
It is then easy to check that the requirement $H(0)\ge 0$, i.e. $F(0)\le\Lambda^{-\gamma}G^{1+\gamma}(0)$, amounts to $\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}\lesssim\Lambda\overline{\varrho}_q^{1+\gamma}$, i.e. $\overline{\varrho}_q^{\hat{\gamma}-\gamma}\ell_q^{-2\alpha}\lesssim 1$. Hence, since $\ell_q^{-1}\le\lambda_{q+1}$ by (4.3.13) and $\overline{\varrho}_q\gtrsim\zeta_{q+1}$ by (d_{q+1}), $H(0)\ge 0$ follows from

$$\hat{\gamma}-\frac{\alpha}{\beta}>\gamma, \tag{7.1.24}$$

provided $a\gg 1$ is sufficiently large to absorb geometric constants. The relation (7.1.24) follows from (7.1.9) since $b>1$.
The next requirement, $H(1)\ge 0$, i.e. $\lambda_{q+1}^{-3\alpha}\lesssim\zeta_{q+2}^{\gamma}(1-\lambda_{q+1}^{-3\alpha})^{1+\gamma}$, requires $3\alpha>2b\beta\gamma$ as found in (7.1.9), since $1-\lambda_{q+1}^{-3\alpha}\ge\frac{1}{2}$ for a sufficiently large.
The following condition, $H'(0)\le 0$, can be rewritten as

$$-\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}\ge(1+\gamma)(\overline{\varrho}_q-\zeta_{q+2}\lambda_{q+1}^{-3\alpha})^{\gamma}(\delta_{q+2}-\overline{\rho}_q)\Longleftrightarrow\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}\lesssim(\overline{\varrho}_q-\zeta_{q+2}\lambda_{q+1}^{-3\alpha})^{\gamma}(\overline{\rho}_q-\delta_{q+2}).$$

Noting that $\overline{\rho}_q\gtrsim\delta_{q+1}\gg\delta_{q+2}$ by (d_{q+1}), and therefore $\overline{\rho}_q-\delta_{q+2}\ge\frac{1}{2}\overline{\rho}_q$ for a sufficiently large, the above reduces to

$$\overline{\varrho}_q^{\hat{\gamma}-\gamma}\ell_q^{-2\alpha}\lesssim 1\Longleftarrow\lambda_{q+1}^{2\alpha-2\beta\hat{\gamma}+2\beta\gamma}=\zeta_{q+1}^{\hat{\gamma}-\gamma}\lambda_{q+1}^{2\alpha}\lesssim 1,$$

which follows from condition (7.1.24) deduced above.
We then need the condition $H'(1) \leq 0$, which can be rewritten as

$$-\Lambda \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-2\alpha} \geq (1+\gamma)\zeta_{q+2}^{\gamma}(1-\lambda_{q+1}^{-3\alpha})^{\gamma}(\delta_{q+2}-\overline{\rho}_q),$$

which similarly follows from (7.1.24).
The last condition, $H'' \geq 0$, follows from the fact that $F'' \equiv 0$ and $G'' \equiv 0$, and thus $H'' = \Lambda^{-\gamma}(1+\gamma)\gamma G^{\gamma-1}G'^2$ is positive.
Thus, our choice of $\alpha, \gamma, \hat{\gamma}$ in (7.1.9) guarantees that (7.1.23) holds, which yields (b_{q+1}).
Consider now (c_{q+1}), where we only need to consider the case $j = q+1$, i.e. the estimate on $[2^{-q-1}T, 2^{-q}T]$. Using (7.1.22), the fact that $\overline{\rho}_q \geq \delta_{q+2}$ for a large enough, and (7.1.15), we see that

$$\delta_{q+2}(1-\lambda_{q+1}^{-3\alpha}) \leq \rho_{q+1}(t) \leq \overline{\rho}_q(t) + \delta_{q+2}\lambda_{q+1}^{-3\alpha} \leq \frac{3}{2}\delta_{q+1} + \delta_{q+2}\lambda_{q+1}^{-3\alpha}.$$

Therefore (c_{q+1}) holds, provided $a \gg 1$ is sufficiently large.

Remark 7.1.1 (The reason for different-order bounds). *This is the reason why the Gluing step in Section 6.3 required different-order bounds on ρ_q. Indeed, if we tried to require $\rho_q \geq \frac{3}{4}\delta_{q+1}$ in that proposition, we would need to obtain it here, meaning we would need $\delta_{q+1}(1-\lambda_{q+1}^{-3\alpha})$ in the above chain. This would change the definition of S_ψ in a similar way, which may not be positive definite since $\overline{\rho}_q \not\geq \delta_{q+1}$ everywhere.*

Similarly, concerning (d_{q+1}), observe that for $t \leq 2^{-(q+1)}T$ we have that $\psi_q(t) = 1$, so that

$$\delta_{q+2}(1-\lambda_{q+1}^{-3\alpha}) \leq \rho_{q+1} \leq \delta_{q+2}(1+\lambda_{q+1}^{-3\alpha}).$$

Moreover, using (7.1.21) and the fact that $\psi_q = 1$ for $t \leq 2^{-(q+1)}T$

$$\left\|\mathring{R}_{q+1}\right\|_0 \leq \delta_{q+2}\lambda_{q+1}^{-3\alpha} \leq \Lambda\left(\frac{3}{4}\zeta_{q+2}\right)^{1+\hat{\gamma}},$$

where we used the fact that $2b\beta\hat{\gamma} < 3\alpha$ and chose $a \gg 1$ sufficiently large. Therefore (d_{q+1}), i.e. $\|\mathring{R}_{q+1}\|_0 \lesssim \Lambda \varrho_{q+1}^{1+\hat{\gamma}}$ and $\frac{3}{4}\delta_{q+2} \leq \rho_{q+1} \leq \frac{5}{4}\delta_{q+2}$, holds.
Concerning (e_{q+1}), it is once more enough to restrict to $t \leq 2^{-q}T$, i.e. the case $j = q+1$. From (7.1.15) and (6.4.18) we deduce that

$$\begin{aligned}
\left\|v_{q+1}\right\|_{1+\alpha} &\leq \left\|v_{q+1}-\overline{v}_q\right\|_{1+\alpha} + \left\|\overline{v}_q\right\|_{1+\alpha} \\
&\leq \frac{M}{2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1+\alpha} + C\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha} \\
&\leq M\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{1+\alpha},
\end{aligned}$$

where C is the implicit constant in (7.1.15), which can be absorbed by choosing $a \gg 1$ sufficiently large. The estimate on $|\partial_t \rho_{q+1}|$ similarly follows from the trace estimate of (7.1.15) and (6.4.21). (e_{q+1}) is thus proved.
(f_{q+1}) follows from (7.1.15), (6.4.17), and (6.4.16).
Finally, (g_{q+1}) easily follows from (6.4.19) and (7.1.15).

Step 4: Convergence to an adapted subsolution

We have thus obtained a sequence (v_q, p_q, R_q) satisfying (a_q)-(g_q).
From (f_q) it follows that (v_q, p_q) is a Cauchy sequence in C^0. Indeed, it is clear for $\{v_q\}$, and concerning $\{p_q\}$ we may use (4.1.1) to write

$$\Delta(p_{q+1}-p_q) = -\operatorname{div}\operatorname{div}(\mathring{R}_{q+1}-\mathring{R}_q+(v_{q+1}-v_q)\otimes v_q+v_{q+1}\otimes(v_{q+1}-v_q)),$$

and apply Schauder estimates (**Lemma A.4**). Similarly, $\{R_q\}$ also converges in C^0. Indeed, from the definition and using (6.3.12), (6.3.4), (6.4.10), (6.4.20), and (b_q), we have that

$$\begin{aligned}\left\|R_{q+1}-R_q\right\|_0 &= \left\|\overline{R}_q-R_q-S_\psi-\mathcal{E}_{q+1}\right\|_0\\ &\le \left\|\overline{R}_q\right\|_0+\left\|R_q\right\|_0+\left\|S_\psi\right\|_0+\left\|\mathcal{E}_{q+1}\right\|_0\\ &\lesssim \delta_{q+1}.\end{aligned}$$

For all $t>0$ there exists $q(t)\in\mathbb{N}$ such that

$$(v_q, p_q, R_q)(\cdot,t) = (v_{q(t)}, p_{q(t)}, R_{q(t)})(\cdot,t) \qquad \forall q\ge q(t),$$

thus (v_q, p_q, R_q) converges uniformly to a strong subsolution $(\hat{v},\hat{p},\hat{R})$ satisfying

$$\left\|\hat{R}\right\|_0 \le \Lambda\hat{\varrho}^{1+\gamma},$$

and, using (5.2.1) and (a_q)

$$\int_{\mathbb{T}^3}\left(|\hat{v}|^2+\operatorname{tr}\hat{R}\right)\mathrm{d}x = \int_{\mathbb{T}^3}\left(|\tilde{v}|^2+\operatorname{tr}\tilde{R}\right)\mathrm{d}x \qquad \forall t\in[0,T].$$

Furthermore, using (5.2.4) and (f_q)

$$\begin{aligned}\|\hat{v}-\tilde{v}\|_{H^{-1}} &\le \|v_0-\tilde{v}\|_{H^{-1}}+\|v_0-\hat{v}\|_{H^{-1}}\\ &\lesssim \delta_1\lambda_0^{-1}+\sum_{q=0}^{\infty}\left\|v_{q+1}-v_q\right\|_{H^{-1}}\\ &\lesssim \delta^{\frac{1}{2}}\zeta_q^{\frac{\gamma}{2}}\ell_q^{\alpha},\end{aligned}$$

leading to (7.1.3) for a sufficiently large. Using (f_q) and the fact that $\tilde{v}, \check{v}$ are smooth and thus bounded in C^0, (7.1.2) is proved similarly:

$$\begin{aligned}\|\hat{v}-\tilde{v}\|_{C^0} &\le \|v_0-\tilde{v}\|_{C^0}+\|v_0-\hat{v}\|_{C^0}\\ &\lesssim 1+\sum_{q=0}^{\infty}\left\|v_{q+1}-v_q\right\|_0\\ &\lesssim 1+\delta_1^{\frac{1}{2}}.\end{aligned}$$

Concerning the initial datum, from (e_q) and (f_q) we obtain by interpolation that $\hat{v}(\cdot,0)\in C^{\hat{\beta}}$, and from (d_q) we obtain that $\hat{R}(\cdot,0)=0$.

Finally, we verify conditions (4.2.4), and (4.2.5) for being a $\mathcal{C}^{\hat{\beta}}$-adapted subsolution. Let $t > 0$. Then there exists $q \in \mathbb{N}$ such that $t \in [2^{-q}T, 2^{-q+1}T]$. By (c_q) and (e_q)

$$\frac{3}{8}\delta_{q+1} \le \hat{\rho} \le 4\delta_q$$
$$\|\hat{v}\|_{1+\alpha} \le M\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}.$$

Therefore $\hat{\rho}^{-1} \ge \frac{1}{4}\delta_q^{-1}$, and hence, using (4.3.1) and (7.1.8), we deduce that

$$\|\hat{v}\|_{1+\alpha} \le \Lambda^{\frac{1}{2}}\hat{\varrho}^{-(1+\nu)},$$

for $a \gg 1$ sufficiently large. Similarly, using (e_q) and (7.1.8), we deduce that

$$|\partial_t\hat{\rho}| \lesssim \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q = \Lambda^{\frac{3}{2}}\lambda_q^{1-\beta}\lambda_{q+1}^{-2\beta} \sim \Lambda^{\frac{3}{2}}\lambda_q^{1-\beta-2b\beta} = \Lambda^{\frac{3}{2}}\zeta_q^{-\frac{1}{2\beta}(1-\beta-2b\beta)} \le \Lambda^{\frac{3}{2}}\zeta_q^{1-\frac{1-\beta}{2\beta}} \lesssim \Lambda^{\frac{3}{2}}\hat{\varrho}^{-\nu}.$$

Finally, a word about the term

$$\hat{\mathcal{T}} := \sum(\mathcal{T}_g^{(q)} + \mathcal{T}_d^{(q)}),$$

where $\mathcal{T}_g^{(q)}$ and $\mathcal{T}_d^{(q)}$ are the extra trace terms from the qth gluing and perturbation steps. We have that $|\partial_t\mathcal{T}_g^{(q)}| + |\partial_t\mathcal{T}_d^{(q)}| \lesssim \Lambda\lambda_q^{\theta+\varepsilon-\beta}$, thus proving (7.1.5). However, adding $\hat{\mathcal{T}}$ into $\hat{R}$ could compromise the adaptedness of $(\hat{v}, \hat{p}, \hat{R})$ by rendering (4.2.4)-(4.2.5) invalid, which is why we keep it separated and deal with it in the final argument. The estimate (7.1.5) implies that

$$\left|\hat{\mathcal{T}}(t)\right| \lesssim \sum t\Lambda\lambda_q^{\theta+\varepsilon-\beta}.$$

To be able to make it as small as we desire, we must contrast the a-growth of the $q = 0$ and $q = 1$ terms of this sum. This is easily achieved by requiring $t \le \Lambda^{-1}\lambda_0^{\beta-\theta-\varepsilon-\iota}$ for ι arbitrarily small. In any case, calling t_s the maximal time where $\hat{\mathcal{T}}$ can be estimated with small quantities, we have that

$$\lim_{a\to\infty} t_s = 0,$$

since we need $t_s\delta_0^{1/2}\lambda_0^{\theta+\varepsilon}$ to be small. ◇

7.2 Perturbing away from $t = 0$ to obtain weak solutions

The aim of this section is to prove the following proposition, which provides the second convex integration scheme.

Proposition 7.2.1 (From adapted subsolutions to weak solutions). *Let $\theta < \beta < \hat{\beta} < \frac{1}{3}$, $\gamma > 0$, and $\nu > 0$ with*

$$\frac{1-3\hat{\beta}}{2\hat{\beta}} < \nu < \frac{1-3\beta}{2\beta}. \tag{7.2.1}$$

The following holds for all $\delta < 1$.

If $(\hat{v},\hat{p},\hat{R})$ *is a* $\mathcal{C}^{\hat{\beta}}$*-adapted subsolution with parameters* γ,Ω,ν *and* $\hat{\rho}\leq\frac{5}{2}\delta$*, then, for all* $\sigma>0$*, there exists a* $\mathcal{C}^{\beta}$ *weak solution* v *of* (2.1.3.1) *with initial datum*

$$v(\cdot,0)=\hat{v}(\cdot,0) \tag{7.2.2}$$

and such that, for all $t\in[0,T]$

$$\int_{\mathbb{T}^3}|v|^2\mathrm{d}x=\int_{\mathbb{T}^3}\left(|\hat{v}|^2+\operatorname{tr}\hat{R}\right)\mathrm{d}x \tag{7.2.3}$$

$$\|v-\hat{v}\|_{C^0}\lesssim\delta^{\frac{1}{2}} \tag{7.2.4}$$

$$\|v-\hat{v}\|_{H^{-1}}<\sigma \tag{7.2.5}$$

Moreover, if we define

$$\mathcal{T}(t):=\int_0^t\int_{\mathbb{T}^3}\left(\left|(-\Delta)^{\frac{\theta}{2}}\hat{v}\right|^2-\left|(-\Delta)^{\frac{\theta}{2}}v\right|^2\right)\mathrm{d}x\mathrm{d}s, \tag{7.2.6}$$

once again we have the bound

$$|\partial_t\mathcal{T}|\lesssim\sum\Lambda\lambda_q^{\theta+\varepsilon-\beta}, \tag{7.2.7}$$

so that, like in the previous proposition, for any $\eta>0$*, it can be ensured that*

$$|\mathcal{T}(t)|\leq\eta\qquad\forall t\in[0,T(\eta,\delta,a)],$$

where $T(\eta,\delta,a)\sim\eta\delta^{-1}\lambda_1^{-2\beta}\lambda_0^{\beta-\theta-\varepsilon}\to 0$ *if* $\eta\to 0$ *or* $a\to\infty$*.*
Finally, consider the family of strong subsolutions $(\hat{v},\hat{p},\hat{R}+{}^{e}/_{3}\operatorname{Id})$*, where* $e:[0,T]\to\mathbb{R}$ *satisfies the following conditions:*

$$e(t)\leq\frac{5}{2}\delta-\hat{\rho}(t) \tag{7.2.8}$$

$$|\partial_t e|\leq\sqrt{\delta_0}\lambda_0 e. \tag{7.2.9}$$

$$e\geq 0, \tag{7.2.10}$$

$$e(0)=0. \tag{7.2.11}$$

This family can be used to yield infinitely many distinct weak solutions with the same initial data as (v,p)*.*

The proof closely follows the arguments of [18, Section 9]. We now start from an adapted subsolution and, by a convex integration scheme, build a sequence of strong subsolutions which converge to a solution of the fractional Navier-Stokes equation. As in **Proposition 7.1.1**, the convex integration scheme needs the localized gluing and perturbation arguments of **Proposition 6.3.1** (in the form of **Remark 6.3.1**) and **Proposition 6.4.1**. However, the choice of the cut-off functions will be, as in [19], dictated by the shape of the trace part of the Reynolds stress, and not fixed a priori as in **Proposition 7.1.1**. Before we start the proof, a remark needs to be made about starting the chain of **Proposition 6.3.1** and **Proposition 6.4.1** with worse estimates.

Remark 7.2.1 (Worse starting estimate). *In* **Proposition 6.3.1**, *if we replace* (6.3.4) *with*

$$\|\mathring{R}\|_0 \le \Lambda \varrho_q^{1+\gamma} \ell_q^{-\frac{2}{b}\alpha},$$

as we will need to do below, the estimates (6.3.27), (6.3.36), (6.3.37), (6.3.9), (6.3.10), (6.3.16), (6.3.12), (6.3.17), (6.3.15), (6.3.51), *and* (6.3.59) *will be worsened by a factor* $\ell_q^{-2/b\alpha}$. *In fact, we can gain a factor* ℓ_q^{α} *in* (6.3.12) *and* (6.3.17), *and a factor* $\ell_q^{\alpha}\lambda_q^{\alpha}$ *in* (6.3.15) *and* (6.3.51). *To keep the inductive estimates on the velocity gap* $\|v_{q+1}-v_q\|_0$ *and* $\|v_{q+1}-v_q\|_{H^{-1}}$, *the velocity* $\|v_{q+1}\|_0$, *and the derivative of the trace* $|\partial_t \rho_q|$, *we will need*

$$\begin{aligned} \Lambda^{\frac{1}{2}} \varrho_q^{\frac{1+\gamma}{2}} \ell_q^{(1-\frac{2}{b})\alpha} &\lesssim \delta_{q+1}^{\frac{1}{2}} \\ \delta_q^{\frac{1}{2}} \lambda_q \ell_q^{-\frac{2}{b}\alpha} &\lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \\ \varrho_{q,i}^{\gamma} \ell_q^{-(2+\frac{2}{b})\alpha} &\lesssim 1, \end{aligned} \tag{7.2.12}$$

all of which can easily be deduced by assuming $2\alpha < \beta\gamma$ *and* $\alpha < 2/9$. *The former assumption also yields* (6.3.22), *which will allow us to bound the* H^{-1} *norm of* $v_{q+1}-v_q$ *sufficiently tightly. If we then start the perturbation step of* **Proposition 6.4.1** *from estimates that we can obtain from the modified output estimates mentioned above, we can get the same output estimates from* **Proposition 6.4.1**.

Proof. (**Proposition 7.2.1**)

Step 1: Setting the parameters in the scheme

Let $(\hat{v},\hat{p},\hat{R})$ be a $C^{\hat{\beta}}$-adapted subsolution on $[0,T]$, with $\Omega = \Lambda$, satisfying the "strong" condition $|\mathring{\hat{R}}| \le \Lambda \hat{\varrho}^{1+\gamma}$ for some $\gamma > 0$ and (4.2.4) and (4.2.5) for some $\alpha, \nu > 0$ as in **Definition 4.2.2** of adapted subsolution, with

$$\frac{1-\hat{\beta}}{2\hat{\beta}} < 1+\nu < \frac{1-\beta}{2\beta}.$$

Fix $b > 0$ so that

$$b^2(1+\nu) < \frac{1-\beta}{2\beta}, \qquad 2\beta(b^2-1) < 1. \tag{7.2.13}$$

Observe that both the strongness condition (4.2.1) and the adaptedness conditions (4.2.4)-(4.2.5) remain valid for any $\hat{\gamma} < \gamma$ and $\alpha' \le \alpha$ (cfr. **Remark 4.2.1**). Then, we may assume that $\alpha, \hat{\gamma} > 0$ are sufficiently small, so that $(\hat{v},\hat{p},\hat{R})$ satisfies (4.2.1) for some $\hat{\gamma} > 0$ and (4.2.4)-(4.2.5) for some $\alpha, \nu > 0$, and furthermore choose γ so that

$$2\alpha < \beta\hat{\gamma} < \beta\gamma < 3\alpha \qquad b\beta\hat{\gamma} < 3\alpha. \tag{7.2.14}$$

For the reasons discussed in **Remark 7.2.1** above, and for another technical reason we will see below, we require

$$2\alpha < \hat{\beta}\gamma < 3\alpha. \tag{7.2.15}$$

Finally, having fixed $b, \hat{\beta}, \beta, \alpha, \gamma, \hat{\gamma}$, we may choose $\overline{N} \in \mathbb{N}$ so that (4.3.14) holds. For $a \gg 1$ sufficiently large (to be determined) we then define (λ_q, δ_q) as in (4.3.1) (using β). Thus, we are in the setting of Section 4.3.

Step 2: Conditions on (v_0, p_0, R_0) and the inductive construction of (v_q, p_q, R_q)

Differently from [18, Section 9], we can take $(v_0, p_0, R_0) = (\hat{v}, \hat{p}, \hat{R})$, since we are assuming $\hat{\rho} \leq \frac{5}{4}\delta_1 = \frac{5}{4}\delta$, which is a-independent. We do have some estimates to verify for (v_0, p_0, R_0), namely that, wherever $\rho_0 \geq \delta_{q+2}$

$$\begin{aligned} \|v_0\|_{1+\alpha} &\leq \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \\ |\partial_t \rho_0| &\leq \rho_0 \delta_q^{\frac{1}{2}} \lambda_q. \end{aligned} \tag{7.2.16}$$

Indeed, where $\rho_0 \geq \delta_{q+2}$, (7.2.13) easily yields

$$\begin{aligned} \Lambda^{\frac{1}{2}} \varrho_0^{-(1+\nu)} &\lesssim \Lambda^{\frac{1}{2}} \lambda_q^{2\beta b^2(1+\nu)} \leq \delta_q^{\frac{1}{2}} \lambda_q \\ \Lambda^{\frac{3}{2}} \varrho_0^{-\nu} &\lesssim \Lambda^{\frac{3}{2}} \lambda_q^{2\beta b^2 \nu} \leq \delta_{q+2} \delta_q^{\frac{1}{2}} \lambda_q, \end{aligned}$$

provided $a \gg 1$ is sufficiently large. These two relations, combined with (4.2.4) and (4.2.5), yield (7.2.16).
Start from (v_0, p_0, R_0), we will inductively construct a sequence (v_q, p_q, R_q) of smooth strong subsolutions for $q = 1, 2, \ldots$, with

$$R_q(x,t) = \rho_q(t)\,\mathrm{Id} + \mathring{R}_q(x,t),$$

satisfying the following properties:

(A_q) For all $t \in [0, T]$

$$\int_{\mathbb{T}^3} \big(|v_q|^2 + \operatorname{tr} R_q\big)\,\mathrm{d}x = \int_{\mathbb{T}^3} \big(|v_0|^2 + \operatorname{tr} R_0\big)\,\mathrm{d}x; \tag{7.2.17}$$

(B_q) For all $t \in [0, T]$

$$\rho_q \leq \frac{5}{2}\delta_{q+1}; \tag{7.2.18}$$

(C_q) For all $t \in [0, T]$

$$\big\|\mathring{R}_q\big\|_0 \leq \begin{cases} \Lambda \varrho_q^{1+\hat{\gamma}} \ell_q^{-\frac{2}{b}\alpha} & \rho_q \geq 2\delta_{q+2} \\ \Lambda \varrho_q^{1+\hat{\gamma}} & \frac{3}{2}\delta_{q+2} \leq \rho_q \leq 2\delta_{q+2} \\ \Lambda \varrho_q^{1+\gamma} & \rho_q \leq \frac{3}{2}\delta_{q+2} \end{cases}; \tag{7.2.19}$$

(D_q) If $\rho_q \geq \delta_{j+2}$ for some $j \geq q$, then

$$\big\|v_q\big\|_{1+\alpha} \leq M \delta_j^{\frac{1}{2}} \lambda_j^{1+\alpha} \tag{7.2.20}$$

$$\big|\partial_t \rho_q\big| \leq \rho_q \delta_j^{\frac{1}{2}} \lambda_j; \tag{7.2.21}$$

(E_q) For all $t \in [0,T]$ and $q \geq 1$

$$\left\|v_q - v_{q-1}\right\|_{H^{-1}} \lesssim (\zeta_q^{\frac{\gamma}{2}} \ell_q^{\frac{\alpha}{2}} + \delta_q^{\frac{1}{2}} \lambda_q^{-1}) \qquad \left\|v_q - v_{q-1}\right\|_0 \lesssim \delta_q^{\frac{1}{2}}. \tag{7.2.22}$$

(F_q) $\|v_q\|_{\theta+\varepsilon} \leq M\left(1 + \Lambda^{\frac{1}{2}} \sum_{i=0}^{q} \lambda_i^{\theta+\varepsilon-\beta}\right)$.

Thanks to our choice of parameters in Step 1 above, (v_0, p_0, R_0) satisfies (7.2.16), and therefore our inductive assumptions (A_0)-(F_0).
Suppose now (v_q, p_q, R_q) satisfies (A_q)-(F_q) above. Let

$$J_q := \left\{t \in [0,T] : \rho_q(t) > \frac{3}{2}\delta_{q+2}\right\}, \qquad K_q := \left\{t \in [0,T] : \rho_q(t) \geq 2\delta_{q+2}\right\}.$$

Being (relatively) open in $[0,T]$, J_q is a disjoint, possibly countable, union of (relatively) open intervals $(T_1^{(i)}, T_2^{(i)})$. Let

$$\mathcal{I}_q := \left\{i : (T_1^{(i)}, T_2^{(i)}) \cap K_q \neq \varnothing\right\},$$

and let $t_0 \in (T_1^{(i)}, T_2^{(i)}) \cap K_q$ for some $i \in \mathcal{I}_q$. Since K_q is compact, we may assume that the open interval $(T_1^{(i)}, t_0)$ is contained in $J_q \setminus K_q$. Using (7.2.21), we then have that

$$\frac{3}{2}\delta_{q+2} = \rho_q(T_1^{(i)}) \geq \rho_q(t_0) - |T_1^{(i)} - t_0| \sup_{J_q} |\partial_t \rho_q| \geq 2\delta_{q+2} - 2\delta_{q+2}\delta_q^{\frac{1}{2}}\lambda_q |T_1^{(i)} - t_0|,$$

hence

$$|T_1 - t_0| \geq \frac{1}{4}(\delta_q^{\frac{1}{2}}\lambda_q)^{-1} = \frac{\ell_q^{-4\alpha}}{4}\tau_q > 4\tau_q, \tag{7.2.23}$$

provided $a \gg 1$ is chosen sufficiently large. A similar estimate holds for $T_2^{(i)}$. Therefore $T_2^{(i)} - T_1^{(i)} > 4\tau_q$ for any $i \in \mathcal{I}_q$, so that $\mathcal{I}_q$ is a finite index set.
Next, we apply **Proposition 6.3.1** (in the form of **Remark 6.3.1**), keeping **Remark 7.2.1** in mind, to (v_q, p_q, R_q) on this disjoint union of intervals $\bigcup_{\mathcal{I}_q} J_{q,i}$. Since $\rho_q > \frac{3}{2}\delta_{q+2}$, from (A_q)-(F_q) and (7.2.13)-(7.2.14) we see that the assumptions of **Proposition 6.3.1** on (v_q, p_q, R_q) hold with parameter $\hat{\gamma}$. Then we obtain $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ such that, on J_q

$$\left\|\overline{v}_q(t) - v_q(t)\right\|_\alpha \lesssim \delta_{q+1}^{\frac{1}{2}} \ell_q^{(\frac{1}{2}+\frac{2}{b}-\frac{2}{b}\mathbb{1}_{K_q})\alpha} \lesssim \delta_{q+1}^{\frac{1}{2}} \ell_q^{\frac{\alpha}{2}} \qquad \text{(From (6.3.22))}$$

$$\left\|\overline{v}_q\right\|_{1+\alpha} \lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_q^{-\frac{2}{b}\alpha\mathbb{1}_{K_q}} \lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{1+\alpha} \qquad \text{(From (6.3.10))}$$

$$\left\|\mathring{\overline{R}}_q\right\|_0 \leq \overline{\rho}_q^{1+\hat{\gamma}} \ell_q^{-2\alpha+(1-\frac{2}{b})\alpha\mathbb{1}_{K_q}} \qquad \text{(From (6.3.12))}$$

$$\frac{7}{8}\rho_q \leq \Lambda\overline{\varrho}_q \leq \frac{9}{8}\rho_q \qquad \text{(From (6.3.13))}$$

$$\left|\partial_t \overline{\rho}_q\right| \lesssim \overline{\rho}_q \delta_q^{\frac{1}{2}} \lambda_q. \qquad \text{(From (6.3.14))}$$

Moreover, recalling (6.1.5), for any $i \in \mathscr{I}_q$ we have the following additional estimates valid for $t \in [T_1^{(i)} + 2\tau_q, T_2^{(i)} - 2\tau_q] \cap J_q$:

$$\begin{aligned} \left\|\overline{v}_q\right\|_{N+1+\alpha} &\lesssim \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \ell_q^{-N-\frac{2}{b}\alpha \mathbb{1}_{K_q}} \\ \left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} &\lesssim \overline{\Lambda} \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-N-2\alpha+(1-\frac{2}{b})\alpha \mathbb{1}_{K_q}} \\ \left\|(\partial_t + \overline{v}_q \cdot \nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} &\lesssim \Lambda \overline{\varrho}_q^{1+\hat{\gamma}} \ell_q^{-N-6\alpha+(1-\frac{2}{b})\alpha \mathbb{1}_{K_q}} \delta_q^{\frac{1}{2}} \lambda_q, \end{aligned} \tag{7.2.24}$$

and

$$\operatorname{supp} \mathring{\overline{R}}_q \subset \mathbb{T}^3 \times \bigcup_i I_i, \tag{7.2.25}$$

where $\{I_i\}_i$ are the intervals defined in (6.1.1).
Let us choose a cut-off function $\psi_q \in C_c^\infty(J_q; [0,1])$ such that

$$\operatorname{supp} \psi_q \subset \bigcup_{i \in \mathscr{I}_q} \left(T_1^{(i)} + 2\tau_q, T_2^{(i)} - 2\tau_q\right) \tag{7.2.26}$$

$$K_q \subset \{\psi_q = 1\} \tag{7.2.27}$$

$$\left|\psi_q'\right| \lesssim \delta_q^{\frac{1}{2}} \lambda_q. \tag{7.2.28}$$

Such a choice is made possible by (7.2.23). We then want to apply **Proposition 6.4.1** (using **Remark 7.2.1** above where $\rho_q \geq 2\delta_{q+2}$) to $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ with

$$S_\psi := \psi_q^2(\overline{R}_q - \delta_{q+2} \operatorname{Id}),$$

hence $\sigma_\psi = \psi_q^2(\overline{\rho}_q - \delta_{q+2})$. Using (7.2.28), (7.2.14), (7.2.13), (7.2.24), (7.2.25), (6.3.13)-(6.3.14), and (A_q)-(F_q), we see that S and $(\overline{v}_q, \overline{p}_q, \overline{R}_q)$ satisfy the required assumptions on the interval $[T_1^{(i)} + 2\tau_q, T_2^{(i)} - 2\tau_q]$ with parameters $\alpha, \hat{\gamma} > 0$. In particular, (6.4.2) (or its worsened form discussed in **Remark 7.2.1**) follows from (7.2.24), since we only need it on $\operatorname{supp} \psi_q \subseteq [T_i^{(1)} + 2\tau_q, T_i^{(2)} - 2\tau_q]$.
Proposition 6.4.1 gives then a new subsolution $(v_{q+1}, p_{q+1}, \overline{R}_q - S_\psi - \mathscr{E}_{q+1})$ with

$$\begin{aligned} &\left\|v_{q+1} - \overline{v}_q\right\|_0 + \left\|v_{q+1} - \overline{v}_q\right\|_{H^{-1}} \lambda_{q+1} \\ &\quad + \lambda_{q+1}^{-1-\alpha} \left\|v_{q+1} - \overline{v}_q\right\|_{1+\alpha} \\ &\quad + \lambda_{q+1}^{-\theta-\varepsilon} \left\|v_{q+1} - \overline{v}_q\right\|_{\theta+\varepsilon} \leq M \delta_{q+1}^{\frac{1}{2}} \qquad \text{(From (6.4.17) and (6.4.18))} \\ &\int_{\mathbb{T}^3} \left|v_{q+1}\right|^2 - S_\psi - \mathscr{E}_{q+1} = \int_{\mathbb{T}^3} \left|\overline{v}_q\right|^2 \qquad t \in [0,T]. \qquad \text{(From (6.4.15))} \end{aligned}$$

and such that $\mathscr{E}_{q+1}$ satisfies (6.4.20)-(6.4.21). Let

$$R_{q+1} := \overline{R}_q - S_\psi - \mathscr{E}_{q+1},$$

We claim that $(v_{q+1}, p_{q+1}, R_{q+1})$ is a smooth strong subsolution satisfying (A_{q+1})-(F_{q+1}). Notice that (A_{q+1}) is satisfied by construction. By definition of S_ψ, one has that

$$\rho_{q+1} = \overline{\rho}_q(1-\psi_q^2) + \psi_q^2\delta_{q+2} - \frac{1}{3}\operatorname{tr}\mathcal{E}_{q+1}$$
$$\mathring{R}_{q+1} = \mathring{\overline{R}}_q(1-\psi_q^2) - \mathring{\mathcal{E}}_{q+1}.$$

For $t \in K_q$, condition (B_{q+1}) follows easily from (6.4.20) and the fact that $K_q \subset \{\psi_q = 1\}$. For $t \notin J_q$, we have that

$$\rho_{q+1} = \rho_q \leq \frac{3}{2}\delta_{q+2} < \frac{5}{2}\delta_{q+2}.$$

For $t \in J_q \setminus K_q$, we have that $\overline{\rho}_q \leq \frac{9}{8}\rho_q \leq \frac{9}{8}\cdot 2\delta_{q+2} = \frac{9}{4}\delta_{q+2}$, which means

$$\rho_{q+1} \leq \frac{9}{4}\delta_{q+2}\left(1 - \frac{4}{9}\psi_q^2 + \frac{5}{9}\lambda_{q+1}^{-6\alpha}\right) \leq \frac{5}{4}\delta_{q+2}\left(\frac{9}{5} + \lambda_1^{-6\alpha}\right),$$

and if $\lambda_1^{-6\alpha} \leq \frac{1}{5}$, which is a matter of choosing a large enough, we have (B_{q+1}).
Note that, by the construction of ρ_{q+1}, we have that $J_q \subseteq K_{q+1}$, since on the whole of J_q we have that $\rho_{q+1} \sim \delta_{q+2} \gg \delta_{q+3}$. This is the reason why we required $\hat{\gamma} < \gamma$ and used the larger γ outside of J_q in (C_q): to make sure (C_{q+1}) was automatically verified outside J_q. This is in stark contrast to what happened in Section 7.1, where the perturbation regions $P_q \coloneqq [0, 2^{-q}T]$ satisfied the opposite inclusion $P_{q+1} \subseteq P_q$, and where we consequently required $\hat{\gamma} > \gamma$ to ensure the weaker "strongness condition" (b_{q+1}) in $P_q \setminus P_{q+1}$, while the stronger (d_{q+1}) only held in P_{q+1}, where $\psi_q = 1$.
By the above paragraph, in verifying conditions (C_{q+1})-(D_{q+1}), it suffices to restrict to the case when $\rho_{q+1} \geq 2\delta_{q+3}$ and $j = q+1$, respectively.
The argument showing (C_{q+1}) for $t \in K_{q+1}$ is similar to the proof of (b_{q+1}) in Step 3 of **Proposition 7.1.1** above. On the one hand

$$\begin{aligned}\|\mathring{R}_{q+1}\|_0 &= \left\|(1-\psi_q^2)\mathring{\overline{R}}_q - \mathring{\mathcal{E}}_{q+1}\right\|_0 \\ &\leq (1-\psi_q^2)\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha+(1-\frac{2}{b})\alpha\mathbb{1}_{\psi_q=1}} + \delta_{q+2}\lambda_{q+1}^{-6\alpha},\end{aligned}$$

on the other hand

$$\begin{aligned}\rho_{q+1} &= (1-\psi_q^2)\Lambda\overline{\varrho}_q + \psi_q^2\delta_{q+2} + \frac{1}{3}\operatorname{tr}\mathcal{E}_{q+1} \\ &\geq (1-\psi_q^2)\overline{\rho}_q + \psi_q^2\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha}.\end{aligned}$$

So where $\psi_q = 1$ we have the condition since, for a large enough, we can guarantee

$$\delta_{q+2}\lambda_{q+1}^{-6\alpha} \lesssim \delta_{q+2}\zeta_{q+2}^{\hat{\gamma}}(1-\lambda_{q+1}^{-6\alpha})^{1+\hat{\gamma}} \iff \lambda_{q+1}^{-6\alpha} \lesssim \zeta_{q+2}^{\hat{\gamma}}(1-\lambda_{q+1}^{-6\alpha})^{1+\hat{\gamma}}, \tag{7.2.29}$$

since $6\alpha > 2b\beta\hat{\gamma}$ is required in (7.2.14). If $\psi_q \neq 1$, however, we need

$$(1-\psi_q^2)\Lambda^{-\hat{\gamma}}\overline{\rho}_q^{1+\tilde{\gamma}}\ell_q^{-2\alpha} + \delta_{q+2}\lambda_{q+1}^{-6\alpha}\mathbb{1}_{\{\psi_q>0\}} \leq \Lambda^{-\hat{\gamma}}[(1-\psi_q^2)\overline{\rho}_q + \psi_q^2\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha}]^{1+\hat{\gamma}}\ell_{q+1}^{-\frac{2}{b}\alpha}. \tag{7.2.30}$$

To this end set

$$F(s) \coloneqq (1-s)\Lambda\overline{\varrho}_q^{1+\tilde{\gamma}}\ell_q^{-2\alpha} + \delta_{q+2}\lambda_{q+1}^{-6\alpha}$$
$$G(s) \coloneqq (1-s)\overline{\rho}_q + s\delta_{q+2} - \delta_{q+2}\lambda_{q+1}^{-6\alpha} = \overline{\rho}_q + s(\delta_{q+2} - \overline{\rho}_q) - \delta_{q+2}\lambda_{q+1}^{-6\alpha}$$
$$H(s) \coloneqq \Lambda^{-\tilde{\gamma}}G^{1+\tilde{\gamma}}(s)\ell_{q+1}^{-\frac{2}{b}\alpha} - F(s),$$

and, just like in **Proposition 7.1.1**, deduce that $H(\psi_q^2) \geq 0$ by proving that:

(i) $H(0) \geq 0$ and $H(1) \geq 0$;

(ii) $H'(0) \geq 0$ and $H'(1) \geq 0$.

(iii) $H''(s) \geq 0$.

To this end, we first obtain the estimates

$$\delta_{q+2}\lambda_{q+1}^{-6\alpha} \lesssim \Lambda\overline{\varrho}_q^{1+\hat{\gamma}}, \qquad F(0) \lesssim \Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha}, \qquad G(0) \gtrsim \overline{\rho}_q, \qquad G(s) \leq \overline{\rho}_q.$$

The first one follows from (7.2.29), (6.3.13), and the fact we are working for $t \in J_q$.. The second one follows from the first one. The fourth one is obvious, since $\overline{\rho}_q \geq \frac{7}{8}\frac{3}{2}\delta_{q+2}\frac{21}{16}\delta_{q+2} > \delta_{q+2}$. For the third one, we reduce it to $\delta_{q+2}\lambda_{q+1}^{-6\alpha} \lesssim \overline{\rho}_q$, and then it follows from the first estimate, since $\Lambda\overline{\varrho}_q^{1+\hat{\gamma}} \leq \overline{\rho}_q$. We then prove (i)-(v) as follows.

- It is easy to check that the two parts of (i) amount to

$$\Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha} \lesssim \Lambda\overline{\varrho}_q^{1+\hat{\gamma}}\ell_{q+1}^{-\frac{2}{b}\alpha}, \qquad \delta_{q+2}\lambda_{q+1}^{-6\alpha} \leq \Lambda^{-\hat{\gamma}}[\delta_{q+2}(1-\lambda_{q+1}^{-6\alpha})]^{1+\hat{\gamma}}\ell_{q+1}^{-\frac{2}{b}\alpha};$$

  the first one follows from $\ell_q \sim \ell_{q+1}^{1/b}$; the second one follows from (7.2.14) and the following relations, which hold for a sufficiently large:

$$1-\lambda_{q+1}^{-6\alpha} \geq \frac{1}{2} \iff \lambda_{q+1}^{-6\alpha} \leq \frac{1}{2}, \qquad \lambda_{q+1}^{-6\alpha} \leq \zeta_{q+2}^{\hat{\gamma}}\ell_{q+1}^{-\frac{2}{b}\alpha}2^{-1-\hat{\gamma}};$$

- The requirements (ii) can be rewritten as

$$\overline{\rho}_q^{1+\hat{\gamma}}\ell_q^{-2\alpha} \geq (1+\hat{\gamma})(\overline{\rho}_q - \delta_{q+2})\ell_{q+1}^{-\frac{2}{b}\alpha}\max\{[\overline{\rho}_q - \delta_{q+2}\lambda_{q+1}^{-6\alpha}]^{\hat{\gamma}}, \delta_{q+2}^{\hat{\gamma}}(1-\lambda_{q+1}^{-6\alpha})^{\hat{\gamma}}\},$$

  which easily follows for sufficiently small $\hat{\gamma}$ and sufficiently large a, since $\ell_q^{-2\alpha} \sim \ell_{q+1}^{-\frac{2}{b}\alpha}$;

- Note that $G'' = 0$ because G is linear in s, and the same is true of F'', meaning that (iii) is simply

$$0 \leq \Lambda^{-\hat{\gamma}}\hat{\gamma}(1+\hat{\gamma})G^{\hat{\gamma}-1}(s)G'^2(s)\ell_{q+1}^{-\frac{2}{b}\alpha}$$
$$= \Lambda^{-\hat{\gamma}}\hat{\gamma}(1+\hat{\gamma})[(1-s)\overline{\rho}_q + \delta_{q+2}(1-\lambda_{q+1}^{-6\alpha})]^{\hat{\gamma}-1}(\overline{\rho}_q - \delta_{q+2})^2\ell_{q+1}^{-\frac{2}{b}\alpha},$$

  which is obvious, since all those factors are positive.

We have thus obtained (C_{q+1}).
The velocity estimate in (D_{q+1}) for $j = q+1$ follows from (6.4.18) and (6.3.16). The trace estimate in (D_{q+1}) follows from (6.4.21) and (6.3.14). Finally, (E_{q+1}) follows precisely as (f_{q+1}) in the proof of **Proposition 7.1.1** in Section 7.1 above, and (F_{q+1}) is obtained just like (g_{q+1}). Keep in mind **Remark 7.2.1** above for all of these.
Thus, the inductive step is proved.
Finally, the convergence of $\{v_q\}$ to a solution of the hypodissipative Navier-Stokes equations as in the statement of **Proposition 7.2.1** (i.e. the one we are proving) follows easily from the sequence of estimates in (A_q)-(F_q), analogously to Step 4 of **Proposition 7.1.1** proved in Section 7.1 above.
The Navier-Stokes term $\mathcal{T}$ will be handled in the same way as $\hat{\mathcal{T}}$ was dealt with in **Proposition 7.1.1**, giving us once more that the maximal time t_s of "smallness" of $\mathcal{T}$ must satisfy

$$\lim_{a\to\infty} t_s = 0.$$

Step 3: From one to infinitely many

To obtain infinitely many solutions, we change the generalized kinetic energy of the initial subsolution as described in the statement of **Proposition 7.2.1**, i.e. by adding a trace term to the Reynolds stress. If we can iterate as described above, since the iteration preserves this energy, we will have infinitely many solutions, which have the same initial datum (since the scheme never perturbs at $t = 0$), which have different kinetic energies for some time t, implying they do not coincide. Naturally, we will need the perturbation e from the statement to satisfy $e(0) = 0$, otherwise it is not possible for the scheme to preserve the generalized kinetic energy without perturbing at $t = 0$, which the scheme does not do. This is the reason for requiring (7.2.11).
The main idea of this step is to change the kinetic energy by replacing $(\hat{v}, \hat{p}, \hat{R})$ with

$$(v_0', p_0', R_0') := (\hat{v}, \hat{p}, \hat{R} + {}^{e}/_{3}\,\mathrm{Id}),$$

as described in the statement of **Proposition 7.2.1**. While this clearly retains condition (4.2.3), since the initial datum is not changed, it does not necessarily preserve conditions (4.2.4), and (4.2.5). Looking at the details of the iteration scheme, however, we realize that those conditions are only needed to obtain the conditions (D_0). If we then show that the conditions $(A_0) - (F_0)$ (and thus also (D_0)) are maintained with such a perturbation, we need not worry about losing (4.2.4) and (4.2.5).
Conditions about the velocity are clearly preserved, and (A_0') and (E_0') are vacuous, so all we need is

$$\rho_0' \leq \frac{5}{2}\delta \tag{B_0'}$$

$$\left\|\mathring{R}_0'\right\|_0 \leq \begin{cases} \Lambda \varrho_0'^{1+\hat{\gamma}} \ell_q^{-\frac{2}{b}\alpha} & \rho_0' \geq 2\delta_2 \\ \Lambda \varrho_0'^{1+\hat{\gamma}} & \frac{3}{2}\delta_2 \leq \rho_0' \leq 2\delta_2 \\ \Lambda \varrho_0'^{1+\gamma} & \rho_0' \leq \frac{3}{2}\delta_2 \end{cases} \tag{C_0'}$$

$$\left|\partial_t \rho_0'\right| \leq \rho_0' \delta_0^{\frac{1}{2}} \lambda_0. \tag{D_0'.2, i.e. (7.2.21)}$$

Concerning (B_0'), the proposition assumes $\hat{\rho} \leq {}^5/_2\delta$, so that the condition is preserved by requiring (7.2.8). Since $\hat{\rho}(0) = 0$, e has the possibility to vary in a neighborhood of $t = 0$ without becoming negative.
$(D_0'.2)$ boils down to the following condition on e:

$$\begin{cases} |\partial_t e| \leq e\sqrt{\delta_0}\lambda_0 & e > 0 \\ |\partial_t e| + |e|\sqrt{\delta_0}\lambda_0 \leq \hat{\rho}\sqrt{\delta_0}\lambda_0 - |\partial_t \hat{\rho}| & \text{otherwise} \end{cases}.$$

To keep things simple, we require (7.2.9) and (7.2.10).
Coming to (C_0'), we first assume $e > 0$, which immediately yields, by the properties of $(\hat{v}, \hat{p}, \hat{R})$, that

$$\|\mathring{R}_0\|_0 \leq \begin{cases} \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\hat{\gamma}}\ell_q^{-\frac{2}{b}\alpha} & \hat{\rho} \geq 2\delta_2 \\ \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\hat{\gamma}} & \frac{3}{2}\delta_2 \leq \hat{\rho} \leq 2\delta_2 \\ \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\gamma} & \hat{\rho} \leq \frac{3}{2}\delta_2 \end{cases}.$$

Our goal is to obtain that

$$\|\mathring{R}_0\|_0 \leq \begin{cases} \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\hat{\gamma}}\ell_q^{-\frac{2}{b}\alpha} & \hat{\rho} + e \geq 2\delta_2 \\ \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\hat{\gamma}} & \frac{3}{2}\delta_2 \leq \hat{\rho} + e \leq 2\delta_2 \\ \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\gamma} & \hat{\rho} + e \leq \frac{3}{2}\delta_2 \end{cases}.$$

- We first note that $\hat{\rho} + e \leq {}^3/_2\delta_2 \implies \hat{\rho} \leq {}^3/_2\delta_2$, so in this case we have the desired estimate for $\|\mathring{R}_0\|_0$;
- If $\hat{\rho} \leq {}^3/_2\delta_2$ but ${}^3/_2\delta_2 \leq \hat{\rho} + e \leq 2\delta_2$, since $\hat{\gamma} < \gamma$ and $\|\mathring{R}_0\|_0 \leq \Lambda(\hat{\varrho} + \Lambda^{-1}e)^{1+\gamma}$, we have the desired estimate;
- If $\hat{\rho} \leq {}^3/_2\delta_2$ but $\hat{\rho} + e \geq 2\delta_2$, the desired estimate is even looser than in the previous item;
- If ${}^3/_2\delta_2 \leq \hat{\rho} \leq 2\delta_2$, then either $\hat{\rho} + e$ also satisfies this bound, in which case we have the desired estimate, or $\hat{\rho} + e \geq 2\delta_2$, in which case the desired estimate is looser;
- Finally, if $\hat{\rho} \geq 2\delta_2$, then so is $\hat{\rho} + e$, meaning again we have the desired estimate.

Thus, we need no additional conditions to obtain (C_0'). Summing up, the conditions we must impose on e are precisely (7.2.8)-(7.2.10). The proof is complete. $\diamond$

Chapter 8

Proof of the density theorem

Proof. (**Theorem 1.3.1**)

We choose $\eta > 0, \theta < \beta < \hat{\beta}, w \in L^2$ with $\operatorname{div} w = 0$. Using the above result, we obtain a smooth strict subsolution $(\tilde{v}', \tilde{p}', \tilde{R}')$ on $[0,T]$ such that (4.1.2)-(4.1.3) hold for some $\delta > 0$ which we will fix later. We now note that adding a smoothly time-dependent non-negative multiple of the identity to $\tilde{R}'$ does not change the fact that $(\tilde{v}', \tilde{p}', \tilde{R}')$ is a smooth strict subsolution. We may thus substitute our strict subsolution with

$$(\tilde{v}, \tilde{p}, \tilde{R}) := \left(\tilde{v}', \tilde{p}', \tilde{R}' + \frac{2}{3|\mathbb{T}^3|} e_K(t) \operatorname{Id} \right),$$

where K is a constant to be specified later in this proof, $0 \le e_K(t) \le ({}^{\delta}/_{2} - Kt)^+$, and $e_K(0) = {}^{\delta}/_{2}$. Combining the choice of e_K with (4.1.3), we obtain the following relations for ${}^{\delta}/_{2} - Kt > 0$ and $t \le \tilde{T}_\delta$ with $\tilde{T}_\delta$ given by **Lemma 4.1.1**:

$$\frac{1}{2} \int_{\mathbb{T}^3} \left(|\tilde{v}|^2(x,0) + \operatorname{tr} \tilde{R}(x,0) \right) \mathrm{d}x = \frac{1}{2} \int_{\mathbb{T}^3} |w|^2(x) \mathrm{d}x + \frac{3}{2}\delta \tag{8.0.1}$$

$$\frac{1}{2} \int_{\mathbb{T}^3} \left(|\tilde{v}|^2(x,t) + \operatorname{tr} \tilde{R}(x,t) \right) \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \left| (-\Delta)^{\frac{\theta}{2}} \tilde{v} \right|^2 (x,s) \mathrm{d}x \mathrm{d}s \le \frac{1}{2} \int_{\mathbb{T}^3} |w|^2(x) \mathrm{d}x + \frac{3}{2}\delta - Kt. \tag{8.0.2}$$

Indeed, passing from $(\tilde{v}', \tilde{p}', \tilde{R}')$ to $(\tilde{v}, \tilde{p}, \tilde{R})$ adds a term e_K to the left-hand side, since $\operatorname{tr} \tilde{R} = \operatorname{tr} \tilde{R}' + 2|\mathbb{T}^3|^{-1} e_K$. Now let $\tilde{v}_0$ be the initial datum of $\tilde{v}$, and note that

$$\begin{aligned} \int_{\mathbb{T}^3} \operatorname{tr} \tilde{R}(x,0) \mathrm{d}x &= \|w\|_{L^2}^2 - \|\tilde{v}_0\|_{L^2}^2 + 3\delta \le \|w - \tilde{v}_0\|_{L^2} (\|w\|_{L^2} + \|\tilde{v}_0\|_{L^2}) + 3\delta \\ &\le \delta(2\|w\|_{L^2} + \delta) + 3\delta \le C(w)\delta. \end{aligned} \tag{8.0.3}$$

Using **Proposition 7.1.1** and **Proposition 7.2.1**, we can produce a $C^{\hat{\beta}}$-adapted subsolution $(\hat{v}, \hat{p}, \hat{R})$ and a C^{β} weak solution (v, p), satisfying the integral equalities (7.1.1) and (7.2.3) and the H^{-1} estimates (7.1.3) and (7.2.5), and the functions $\hat{\mathcal{T}}, \mathcal{T}$ of (7.1.4) and (7.2.6).

Recall that we have that

$$\int_{\mathbb{T}^3} (|\tilde{v}|^2 + \operatorname{tr}\tilde{R})(x,t)\mathrm{d}x = \int_{\mathbb{T}^3} (|\hat{v}|^2 + \operatorname{tr}\hat{R})(x,t)\mathrm{d}x = \int_{\mathbb{T}^3} |v(x,t)|^2\mathrm{d}x, \tag{8.0.4}$$

and thus

$$\|v(t)\|_2^2 - \|\tilde{v}(t)\|_2^2 = \int \operatorname{tr}\tilde{R}(x,t)\mathrm{d}x. \tag{8.0.5}$$

Call v_0 the initial datum of v and of $\hat{v}$, and note that, by (7.1.3), (8.0.5), and (8.0.3), we have that

$$\|v_0 - \tilde{v}_0\|_2^2 = \|v_0\|_2^2 - \|\tilde{v}_0\|_2^2 - 2\cdot\int_{\mathbb{T}^3} \tilde{v}_0\cdot(v_0-\tilde{v}_0)\mathrm{d}x \le C(w)\delta + 2\sigma\|\tilde{v}_0\|_{H^1}.$$

Thus, we first choose δ sufficiently small so that $C(w)\delta < \frac{\eta^2}{2}$ and obtain $\tilde{v}$, then we fix $\sigma < \frac{\eta^2}{4\|\tilde{v}_0\|_{H^1}}$ and obtain $\hat{v}$, and finally we conclude that

$$\|v_0 - \tilde{v}_0\|_2^2 \le \eta^2 \implies \|\tilde{v}_0 - v_0\|_{L^2} \le \eta.$$

As for the admissibility condition, choosing K so that $|\partial_t(\mathcal{T}+\hat{\mathcal{T}})| \le K-1$, as is made possible by (7.1.5) and (7.2.7), we have that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}v\right|^2(x,s)\mathrm{d}s\mathrm{d}x \\
&\overset{(8.0.4)}{=} \frac{1}{2}\int_{\mathbb{T}^3} \Big(|\tilde{v}(x,t)|^2 + \operatorname{tr}\tilde{R}(x,t)\Big)\mathrm{d}x - (\hat{\mathcal{T}}+\mathcal{T})(t) + \int_0^t\int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}\tilde{v}\right|^2(x,s)\mathrm{d}s\mathrm{d}x \\
&\overset{(8.0.2)}{\le} \int_{\mathbb{T}^3} \frac{1}{2}|w|^2(x)\mathrm{d}x + \frac{3}{2}\delta - t \overset{(8.0.1)}{\le} \frac{1}{2}\int_{\mathbb{T}^3} \big(|\tilde{v}_0|^2 + \operatorname{tr}\tilde{R}(x,0)\big)\mathrm{d}x \overset{(8.0.4)}{=} \frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,0)\mathrm{d}x,
\end{aligned}$$

where the second-last inequality is strict for all $t \neq 0$ where (8.0.2) is valid. This yields the energy inequality for $t < T_\eta \coloneqq \min\{\tilde{T}_\delta, T'_\delta\}$, where $\tilde{T}_\delta$ is given by **Lemma 4.1.1**, and $T'_\delta \coloneqq {}^{\delta}/_{2K}$. Since we can only estimate $|\partial_t\mathcal{T} + \partial_t\hat{\mathcal{T}}|$ with a quantity which is potentially unbounded as $\delta \to 0$ (as seen in (7.1.5) and (7.2.7)), and $C(w)\delta < {}^{\eta^2}/_2$ implies $\delta \to 0$ as $\eta \to 0$, we conclude that both T'_δ and $\tilde{T}_\delta$ tend to zero as $\eta \to 0$. Thus, our time T_η of guaranteed admissibility satisfies

$$\lim_{\eta\to 0} T_\eta = 0.$$

So far, we have only obtained one solution for each η. Suppose that, from $(\tilde{v},\tilde{p},\tilde{R})$, we produced the adapted subsolution $(\hat{v},\hat{p},\hat{R})$, and from there the solution (v,p). As noted in **Proposition 7.2.1**, considering

$$(\hat{v}',\hat{p}',\hat{R}') \coloneqq \left(\hat{v},\hat{p},\hat{R} + \frac{e}{3}\mathrm{Id}\right)$$

with e satisfying a suitable set of conditions, we can obtain more weak solutions and ensure these solutions are admissible up to T_η. The required conditions are (7.2.8)-(7.2.11) from **Proposition 7.2.1** plus the following one, which ensures the admissibility of the new solutions:

$$\frac{1}{2}|\mathbb{T}^3|e(t) \leq \frac{1}{2}\int_{\mathbb{T}^3} \left(|\hat{v}(x,0)|^2 + \mathrm{tr}\hat{R}(x,0) - |\hat{v}(x,t)|^2 - \mathrm{tr}\hat{R}(x,t)\right)\mathrm{d}x$$
$$- \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\frac{\theta}{2}}\hat{v}(x,s)\right|^2 \mathrm{d}x\mathrm{d}s - \hat{\mathcal{J}} - Kt,$$

K being the same constant used to find (v,p). Since the right-hand side of the above inequality is strictly positive for all $t \neq 0$ where (v,p) is admissible, this condition is compatible with requiring that $e \geq 0$ as done above.
This completes the proof. ◇

Appendix A

Classical calculus inequalities

We begin this appendix with a few estimates concerning the Hölder norms we introduced in the notational section of Chapter 1. First of all, we have the following classical estimates.

Lemma A.1. *For* $0 \le s \le r$ *and* $f, g : \mathbb{T}^3 \to \mathbb{R}^d$

$$[fg]_r \le C(r)([f]_r \|g\|_0 + \|f\|_0 [g]_r) \tag{A.1}$$

$$[f]_s \le C(r,s) \|f\|_0^{1-\frac{s}{r}} [f]_r^{\frac{s}{r}}. \tag{A.2}$$

We also have the following estimates on the norms of compositions.

Lemma A.2. *For* $f : \mathbb{T}^3 \to S \subseteq \mathbb{R}^d$ *and* $\Psi : S \to \mathbb{R}$*:*

$$[\Psi \circ f]_m \le K(d,m)([\Psi]_1 \|\mathfrak{D} f\|_{m-1} + \|\nabla\Psi\|_{m-1} \|f\|_0^{m-1} \|f\|_m) \tag{A.3}$$

$$[\Psi \circ f]_m \le K(d,m)([\Psi]_1 \|\mathfrak{D} f\|_{m-1} + \|\nabla\Psi\|_{m-1} [f]_1^m). \tag{A.4}$$

We now recall the definition of convolution:

$$(f * g)(x) := \int_{\mathbb{T}^3} f(x-y) g(y) \mathrm{d}y.$$

We note that convolution is commutative, and that $D(f*g) = f*Dg = Df*g$. Moreover, we have the following estimates for the Hölder norms of convolutions.

Lemma A.3. *For all* $s, r \ge 0$*:*

$$\begin{aligned} \|f * \varphi_\ell\|_{r+s} &\le C(r,s) \ell^{-s} \|f\|_r \\ \|f - f * \varphi_\ell\|_r &\le C(r,s) \ell^1 \|f\|_{r+1} \\ \|f - f * \varphi_\ell\|_r &\le C(r,s) \ell^2 \|f\|_{r+2} \\ \|(fg) * \varphi_\ell - (f * \varphi_\ell)(g * \varphi_\ell)\|_r &\le C(r,s) \ell^{2-r} \|f\|_1 \|g\|_1, \end{aligned} \tag{A.5}$$

where φ *is a standard mollification kernel, i.e.* $\varphi \in C_c^\infty(B_1; [0,1])$ *and* $\int \varphi = 1$*, and* $\varphi_\ell := {}^1/_{\ell^3} \varphi({}^\cdot/_\ell)$.

The above lemmas can then be applied to the time slices of time-dependent vector fields, e.g. the velocities of subsolutions, with the notation $\|f(t,\cdot)\|_{C^r}, [f(t,\cdot)]_{C^r}$ for the (semi)norms of the slices. By taking supremum norms in time, the above inequalities can be formulated with $C^0_t C^r_x$ norms.

To conclude this appendix, we recall classical Schauder estimates (see e.g. the book [34]), which will be used in several places in this thesis.

Lemma A.4 (Schauder estimates). *For any $\alpha \in (0,1)$ and any $m \in \mathbb{N}$, there exists a constant $C(\alpha, m)$ with the following properties. If $\varphi, \psi : \mathbb{T}^3 \to \mathbb{R}$ are the unique solutions of*

$$\begin{cases} \Delta\varphi = f \\ \fint \varphi = 0 \end{cases} \qquad \begin{cases} \Delta\psi = \operatorname{div} F \\ \fint \psi = 0 \end{cases},$$

then

$$\|\varphi\|_{m+2+\alpha} \leq C(m,\alpha)\|f\|_{m+\alpha} \qquad \|\psi\|_{m+1+\alpha} \leq C(m,\alpha)\|F\|_{m+\alpha}.$$

Appendix B

Estimates on the transport and transport-diffusion equations

Associated to the Euler equations (2.1.1.1) is the transport equation, which describes how a quantity evolves with the flow of the fluid:

$$(\partial_t + v \cdot \nabla) f = g.$$

In particular, for $g = 0$, we say f *is transported along the flow*, meaning it is constant along particle trajectories. A priori estimates on solutions of this transport equations can be useful in proving results about the Euler equations, since the former becomes the first of the latter for $f = v$ and $g = -\nabla p$. Indeed, we will see in Chapter 6 that such estimates are a useful tool for the proof of the density theorems in Chapter 1. The estimates we will use are found in [8, Proposition B.1], which states the following.

Proposition B.1 (Estimates on the transport equation). *Assume* $|t - t_0| \|v_t\|_1 \leq 1$. *Then, any solution* f *of*

$$\begin{cases} (\partial_t + v \cdot \nabla) f = g \\ f_{t_0} = h \end{cases} \tag{B.1}$$

satisfies

$$\|f_t\|_0 \leq \|h\|_0 + \int_{t_0}^{t} \|g_\tau\| \mathrm{d}s$$

$$\|f_t\|_\alpha \leq e^\alpha \left(\|h\|_\alpha + \int_{t_0}^{t} \|g_s\|_\alpha \mathrm{d}s \right)$$

for all $0 \leq \alpha \leq 1$ *and, more generally, for any* $N \geq 1$ *and* $0 \leq \alpha < 1$

$$[f_t]_{N+\alpha} \lesssim [h]_{N+\alpha} + |t| [v_t]_{N+\alpha} [h]_1 + \int_{t_0}^{t} \big([g_s]_{N+\alpha} + (t-s)[v_s]_{N+\alpha} [g_s]_1 \big) \mathrm{d}s.$$

Define Φ to be the inverse of the flux X of v starting at time t_0 as the identity (i.e. ${}^{d}/_{dt} X = v(t,X)$ and $X(t_0,x) = x$). Under the same assumptions as above we have that

$$
\begin{aligned}
\|\nabla\Phi_t - \mathrm{Id}\|_0 &\lesssim |t|[v_t]_1 \\
[\Phi_t]_N &\lesssim |t|[v_t]_N \quad \forall N \geq 2.
\end{aligned}
$$

In the case of the (fractional) Navier-Stokes equations, the analogue of the transport equation seen above is the transport-diffusion equation, which adds the laplacian term to the transport equation:

$$(\partial_t + v\cdot\nabla + (-\Delta)^{\theta})f = g.$$

Once more, it reduces to the (fractional) Navier-Stokes equations once $f = v, g = -\nabla p$, so having a priori estimates on its solutions can be useful to establish properties of the (fractional) Navier-Stokes system. We thus close this appendix with the estimates on the transport-diffusion equation found in [26, Proposition 3.3]

Proposition B.2 (Estimates on the transport-diffusion equation). *Assume* $0 \leq (t-t_0)[v_t]_1 \leq 1, \nu > 0, 0 < \theta \leq 1$. *Then, any solution of*

$$
\begin{cases}
(\partial_t + v\cdot\nabla + \nu(-\Delta)^{\theta})u = f & \text{in } (t_0,T)\times\mathbb{T}^3 \\
u_{t_0} = g & \text{in } \mathbb{T}^3
\end{cases}
$$

satisfies

$$\|u_t\|_\alpha \leq e^{\alpha}\left(\|g\|_\alpha + \int_{t_0}^{t} \|f_s\|_\alpha \mathrm{d}s\right)$$

for all $0 \leq \alpha \leq 1$ *and, more generally, for any* $N \geq 1$ *and* $0 \leq \alpha < 1$

$$[u_t]_{N+\alpha} \lesssim [g]_{N+\alpha} + (t-t_0)[v_t]_{N+\alpha}[g]_1 + \int_{t_0}^{t} \big([f_s]_{N+\alpha} + (t-s)[v_s]_{N+\alpha}[f_s]_1\big)\mathrm{d}s,$$

where the implicit constants depends only on N, α.
In both of these lemmas, we have used the notation $f_t(x) \coloneqq f(t,x)$ *for the time-slices of functions defined on* $[0,T]\times\mathbb{T}^3$*, as introduced in the notational section of Chapter 1.*

Appendix C

Mikado flows and antidivergence

In this appendix, we collect some results regarding Mikado flows and a "stationary phase lemma" which is used at several points in the proof of the density results. The proofs of the results in this appendix can be found in [19].

Lemma C.1 (Mikado flows). *For any compact subset $\mathcal{N} \subset\subset \mathcal{S}^{3\times 3}_+$ there exists a smooth vector field $W : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3$ such that, for every $R \in \mathcal{N}$*

$$\begin{cases} \operatorname{div}_\xi (W(R,\xi) \otimes W(R,\xi)) = 0 \\ \operatorname{div}_\xi W(R,\xi) = 0 \end{cases}, \tag{C.1}$$

and

$$\begin{cases} \fint_{\mathbb{T}^3} W(R,\xi)\mathrm{d}\xi = 0 \\ \fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi)\mathrm{d}\xi = R \end{cases}. \tag{C.2}$$

Using Fourier series in ξ and the above integral and differential relations, we obtain that

$$W(R,\xi) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} a_k(R) A_k e^{ik\cdot\xi} \tag{C.3}$$

$$W(R,\xi) \otimes W(R,\xi) = R + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} C_k(R) e^{ik\cdot\xi}, \tag{C.4}$$

where the coefficients $a_k, C_k \in C^\infty$, the A_k satisfy $A_k \cdot k = 0, |A_k| = 1$, the C_k satisfy $C_k k = 0$, and moreover

$$\sup_{R \in \mathcal{N}} \left|\mathfrak{D}^N_R a_k(R)\right| = \|a_k\|_{C^N(\mathcal{N})} \le \frac{C(\mathcal{N},N,m)}{|k|^m} \tag{C.5}$$

$$\sup_{R \in \mathcal{N}} \left|\mathfrak{D}^N_R C_k(R)\right| = \|C_k\|_{C^N(\mathcal{N})} \le \frac{C(\mathcal{N},N,m)}{|k|^m}. \tag{C.6}$$

A fact used in Section 5.1 is that, if we set

$$U(R,\xi) := \sum_k a_k(R) \frac{ik \times A_k}{|k|^2} e^{ik\cdot\xi}, \tag{C.7}$$

then we have that $\operatorname{curl}_\xi U = W$. Indeed

$$\begin{aligned}
\operatorname{curl}_\xi U(R,\xi) &= \sum_{k\ell mn} \varepsilon_{\ell mn}\partial_m \left(a_k(R)\frac{ik\times A_k}{|k|^2}e^{ik\cdot\xi}\right)_n e_\ell \\
&= \sum_{k\ell mnpq} \varepsilon_{\ell mn}\varepsilon_{npq} a_k(R) ik_p (A_k)_q |k|^{-2}\cdot ik_m e^{ik\cdot\xi} e_\ell \\
&= -\sum_{k\ell mpq} (\delta_{\ell p}\delta_{mq} - \delta_{\ell q}\delta_{mp}) a_k(R) k_p |k|^{-2} (A_k)_q k_m e^{ik\cdot\xi} e_\ell \\
&= \sum_k a_k(R) A_k e^{ik\cdot\xi} - \sum_{kpq} a_k(R) k_p k_q |k|^{-2} (A_k)_q e^{ik\cdot\xi} e_p \\
&= W(R,\xi) - \sum_k a_k(R) \underbrace{(k\cdot A_k)}_{0} \frac{k}{|k|^2} e^{ik\cdot\xi}.
\end{aligned}$$

Continuing, we recall some elementary calculations for the reader's convenience. With the definitions we gave for $\nabla, \mathfrak{D}$, setting $D_t^{(v)} \coloneqq \partial_t + v\cdot\nabla$, we have that

$$\nabla e^{ik\cdot\Phi} = i\nabla\Phi\cdot k e^{ik\cdot\Phi} = i e^{k\cdot\Phi} k\cdot\mathfrak{D}\Phi \tag{C.8}$$

$$D_t^{(v)}(\mathfrak{D}\Phi) = \mathfrak{D}(D_t^{(v)}\Phi) - \mathfrak{D}\Phi\cdot\mathfrak{D}v. \tag{C.9}$$

Observing that $\mathfrak{D}\Phi\mathfrak{D}\Phi^{-1} = \mathrm{Id}$ and thus $0 = D_t^{(v)}(\mathfrak{D}\Phi\mathfrak{D}\Phi^{-1}) = D_t^{(v)}(\mathfrak{D}\Phi)\cdot\mathfrak{D}\Phi^{-1} + \mathfrak{D}\Phi\cdot D_t^{(v)}(\mathfrak{D}\Phi^{-1})$, we can see that

$$D_t^{(v)}\mathfrak{D}\Phi^{-1} = -\mathfrak{D}\Phi^{-1}D_t^{(v)}(\mathfrak{D}\Phi)\mathfrak{D}\Phi^{-1} = (\nabla\Phi^{-1}\nabla v)^T - \mathfrak{D}\Phi^{-1}\cdot\mathfrak{D}D_t^{(v)}\Phi\cdot\mathfrak{D}\Phi^{-1}. \tag{C.10}$$

We now introduce a certain "anti-divergence operator" which was used to obtain the new Reynolds stress R in the various approximation results.

Definition C.1 (Anti-divergence $\mathcal{R}$). *Define the operator $\diamond$ so that*

$$\begin{cases} \Delta\diamond v = v - \fint_{\mathbb{T}^3} v\,\mathrm{d}x \\ \int_{\mathbb{T}^3} \diamond v = 0 \end{cases},$$

and then define

$$\mathcal{R}v \coloneqq \frac{1}{4}(\mathfrak{D}\mathcal{P}\diamond v + (\mathfrak{D}\mathcal{P}\diamond v)^T) + \frac{3}{4}(\mathfrak{D}\diamond v + (\mathfrak{D}\diamond v)^T) - \frac{1}{2}(\operatorname{div}\diamond v)\,\mathrm{Id}, \tag{C.11}$$

$\mathcal{P}$ being the Leray projection onto divergence-free fields with zero average.

This operator satisfies the following properties.

Lemma C.2 (Divergence and $\mathcal{R}$). *For any C^∞ vector field v, $\mathcal{R}v \in \mathcal{S}_0^{3\times 3}$ is symmetric and trace-free, and moreover*

$$\operatorname{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v\,\mathrm{d}x. \tag{C.12}$$

Moreover, we have the following statement, which we was used numerous times throughout the thesis.

Lemma C.3 (Stationary Phase Lemma). *Let* $\alpha \in (0,1), N \geq 1$. *Let* $a \in \mathcal{C}^\infty(\mathbb{T}^3), \Phi \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ *be smooth functions and assume*

$$\frac{1}{K} \leq |\mathfrak{D}\Phi| \leq K \qquad \textit{on } \mathbb{T}^3.$$

Then

$$\left| \int_{\mathbb{T}^3} a(x) e^{ik\cdot\Phi} \mathrm{d}x \right| \leq C(K,N) \frac{\|a\|_N + \|a\|_0 \|\Phi\|_N}{|k|^N}, \tag{C.13}$$

and for the operator $\mathcal{R}$ *of* (C.11) *above we have that*

$$\left\| \mathcal{R}(a(x)e^{ik\cdot\Phi}) \right\|_\alpha \leq C(\alpha,K,N) \left(\frac{\|a\|_0}{|k|^{1-\alpha}} + \frac{\|a\|_{N+\alpha} + \|a\|_0 \|\Phi\|_{N+\alpha}}{|k|^{N-\alpha}} \right). \tag{C.14}$$

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