ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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Non-smooth solutions in incompressible fluid dynamics

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March 2021

Abstract

This work is devoted to the study of the main models which describe the motion of incompressible fluids, namely the Navier-Stokes, together with their hypodissipative version, and the Euler equations. We will mainly focus on the analysis of non-smooth weak solutions to those equations. Most of the results have been obtained by using the *convex integration* techniques introduced by Camillo De Lellis and László Székelyhidi in the context of the Euler equations, which recently led to the proof of the Onsager's conjecture on the anomalous dissipation of the kinetic energy. With various refinements of those iterative schemes we prove ill-posedness of Leray-Hopf weak solutions of the hypodissipative Navier-Stokes equations, sharpness of the kinetic energy regularity for Euler, typicality results in the sense of Baire's category for both Euler and Navier-Stokes, estimate on the dimension of the singular set in time of non-conservative Hölder weak solutions of the Euler equations. Moreover, building on different techniques, we also address some regularizing effects of those equations in various classes of weak solutions with some fractional differentiability in terms of Hölder, Sobolev and Besov regularity. The latter make use of new abstract interpolation results for multilinear operators which we developed for our specific context but which may also have independent interests.

Keywords: Incompressible fluids, Euler equations, Navier-Stokes equations, weak solutions, Leray-Hopf solutions, Ill-posedness, convex integration, Baire category, non-conservative solutions, regularizing effects.

Contents

Chapter 1

Introduction

For several decades the equations which describe the motions of fluids have attracted the attention of many mathematicians for both their intrinsic mathematical beauty and their usefulness in several applications of practical nature. They describe the physics of many phenomena of scientific and engineering interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. These equations, in their full and simplified form, help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution and many other things.

Even if these models have been proposed almost 200 years ago, many of the main related mathematical questions still remain open. Indeed from 2000 the incompressible Navier-Stokes equations, which are unquestionably the most famous equations in this context, are the content of one of the *Millennium Prize Problems* stated by the *Clay Mathematics Institute*. It is hard to briefly explain where the difficulty of these problems comes from, but at a first empirical stage it can be related to the extremely chaotic and irregular motion of the fluid particles along the flow, which can be easily observed in several everyday phenomena. This complicated, and apparently disordered, behavior is what physicists named *Turbulence*, or more specifically a *turbulent flow*, which is experimentally known to be a consequence of the accumulation of energy at finer and finer scales that overcomes the damping coming from the viscosity of the fluid. That is why turbulence is usually observed in low viscous fluids, or analogously, in a high Reynolds number regime, being the Reynolds number proportional to the inverse of the viscosity. The complexity, together with the usefulness in real life applications, of such high Reynolds number flows has driven many physicists from the last century (Prandtl, Richardson, Taylor, Heisenberg, Kolmogorov, Onsager...) to formulate statistical theories that were able to predict their chaotic motion. The success of those theories in modeling the statistics of turbulent fluids has been astounding but to date rigorous results to formally validate many of those statistical predictions are still missing. The aim of the mathematical theory of fluid dynamics is to build the missing bridge between the physical theories based on experimental observations and the rigorous properties of solutions, if they exist. It should be then clear that a possible way to catch the essence of Turbulence is to consider non-smooth solutions of such equations, namely solutions with a very low regularity. For instance, solutions of the Navier-Stokes equations are not expected to be uniformly smooth when the viscosity parameter goes to 0, thus they can converge to distributional solutions of Euler with little regularity and which may anomalously dissipate formally conserved quantities such as the kinetic energy. This is indeed the goal of this thesis: the study of non-smooth solutions of the main incompressible fluid models considered by the scientific community, namely the Navier-Stokes, together with their corresponding fractional version, and Euler equations.

This work contains results that have been achieved by the present author during his PhD path and it is mostly focused on the analysis of the wild, somehow non-physical, weak solutions that naturally arise in the study of Turbulence. It ranges from basic regularization properties for various regularity classes, ill-posedness problems, conservation and/or dissipation of the main meaningful physical quantities, typicality results for weak solutions, structure of time singularities etc... Obviously, it is not an exhaustive reference for the wide available mathematical literature on the topic, for which we refer to the classical monographs [21, 48, 50] and references therein.

Most of the results contained in this thesis make use of the so called *convex integration* technique that has been introduced in the context of fluid dynamics by Camillo De Lellis and László Székelyhidi in the last 15 years. These new revolutionary ideas have reinvigorated the attention on many of the physical theories described above, leading to the proof of various results, remained open for decades, with a huge impact on the whole mathematical community active in the study of partial differential equations. After explaining in detail what are the equations considered in this work, together with their main properties and the related open questions, we will also give an historical overview of these techniques here in the introduction, which will end with a more detailed description about the content of each chapter of which this work is composed.

Consistently with all the results presented in the next chapters, we fix our spatial domain to be the $3-$dimensional torus $\mathbb{T}^3$. Clearly, many of the subsequent properties remain valid also for more general $d-$dimensional domains but we prefer to stick with this assumption for a greater clarity of exposure. There are two main reasons why this particular spatial domain has been chosen: at first it avoids all the technical difficulties coming from the boundary and, on the other side, it still models the most physically relevant $3-$dimensional case.

1.1 The Navier-Stokes equations

The Navier-Stokes equations are a set of partial differential equations which describe the motion of viscous fluids, named after the two physicists Claude-Louis Navier and George Gabriel Stokes

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \mu \Delta u = 0 \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = \overline{u}. \end{cases} \tag{1.1}$$

The vector field $u : \mathbb{T}^3 \times (0,T) \to \mathbb{R}^3$ represents the velocity of the fluid, the scalar function $p : \mathbb{T}^3 \times (0,T) \to \mathbb{R}$ the hydrodynamic pressure, $\overline{u}$ is a given divergence free initial datum and $\mu > 0$ is the kinematic viscosity of the fluid. When the time T of existence is infinite, we say that the solution is global in time. We will denote by u^i the $i-$th component of the vector field u. The symbol $u\otimes u$ denotes the 3×3 matrix whose components are $(u\otimes u)_{ij}=u^iu^j$ and consequently, by following the usual convention, its divergence is obtained by computing it column-wise. Moreover, by using the incompressibility constraint $\operatorname{div} u=\partial_i u^i=0$, the latter term can be rewritten as

$$(\operatorname{div}(u\otimes u))^i=\partial_j\left(u^ju^i\right)=u^j\partial_j u^i=(u\cdot\nabla)u^i,$$

where the usual convention of summing over repeated indexes has been used. Since it appears as a gradient, it is clear that the pressure is always determined up to a constant which can only depend on time. This is the reason why equations (1.1) are usually coupled with the constraint

$$\int_{\mathbb{T}^3} p(x,t)\,dx=0\quad\forall t\geq 0,$$

which guarantees the uniqueness of the pressure.

The Navier-Stokes equations mathematically express conservation of momentum and conservation of mass for Newtonian fluids. They arise from applying Isaac Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of the velocity) and a pressure term. The difference between them and the closely related Euler equations (discussed in the next section) is that the Navier-Stokes equations take viscosity into account while the Euler equations model inviscid flows. The presence of viscosity is due to the internal friction between particles and it is responsible for the kinetic energy dissipation of the fluid. Indeed by setting the kinetic energy e_u to be

$$e_u(t)=\frac{1}{2}\int_{\mathbb{T}^3}|u(x,t)|^2\,dx,$$

for a sufficiently smooth solution of (1.1), we have

$$e_u(t)+\mu\int_0^t\int_{\mathbb{T}^3}|\nabla u(x,\tau)|^2\,dxd\tau=e_{\overline{u}},\tag{1.2}$$

for every $t\in(0,\infty)$. The previous *global energy equality* is obtained by integrating in space-time its corresponding local version

$$\partial_t\frac{|u|^2}{2}+\operatorname{div}\left(\left(\frac{|u|^2}{2}+p\right)u\right)+\mu|\nabla u|^2=\mu\operatorname{div}\left((\nabla u)^T u\right)=\mu\Delta\left(\frac{|u|^2}{2}\right),$$

that is obtained by scalar multiply the first equation in (1.1) by u, together with the relations

$$u\cdot\operatorname{div}(u\otimes u)=u^i\partial_j(u^ju^i)=u^ju^i\partial_j u^i=u^j\partial_j\frac{|u|^2}{2}=\operatorname{div}\left(\frac{|u|^2}{2}u\right),$$
$$u\cdot\Delta u=u^i\partial^2_{jj}u^i=\partial_j\left(u^i\partial_j u^i\right)-\partial_j u^i\partial_j u^i=\operatorname{div}\left((\nabla u)^T u\right)-|\nabla u|^2.$$

The energy equality (1.2) specifies the dissipation rate of kinetic energy asserting that the latter is proportional to both the viscosity of the fluid and its averaged gradient: this is not only important from a physical point of view but it is also the key point in showing the existence of an appropriate notion of weak solutions, since it guarantees compactness properties to a sequence of solutions of a suitable regularized version of (1.1).

Equations (1.1) are of great interest in a purely mathematical sense. Despite their wide range of practical uses, it has not yet been proven whether global smooth solutions always exist in three dimensions. This is the so called Navier-Stokes existence and smoothness problem and is the content of one of the Millennium Prize Problems stated by the Clay Mathematics Institute. More precisely we have

Problem 1.1. *Let $\mu > 0$ be given. Is it true that for every $\overline{u} \in C^\infty(\mathbb{T}^3)$ there exist a couple $u, p \in C^\infty(\mathbb{T}^3 \times (0,\infty))$ solving* (1.1)*?*

As it usually happens in the study of partial differential equations, the difficulty of proving the existence of regular solutions leads to different notions of weak solutions. The most successful is surely the notion of *Leray weak solution*.

Definition 1.2. *Let $\mu > 0$ and $\overline{u} \in L^2(\mathbb{T}^3)$ such that* $\operatorname{div}\overline{u} = 0$. *A Leray weak solution of* (1.1) *is a divergence free vector field u such that $u \in L^\infty((0,\infty);L^2(\mathbb{T}^3)) \cap L^2((0,\infty);W^{1,2}(\mathbb{T}^3))$ and*

$$\int_0^\infty \int_{\mathbb{T}^3} (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi + \mu u \cdot \Delta \varphi)\, dxdt = -\int_{\mathbb{T}^3} \overline{u}(x)\varphi(x,0)\, dx, \tag{1.3}$$

for every test vector field $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,\infty))$ such that $\operatorname{div}\varphi = 0$. *Moreover the following energy inequality*

$$\frac{1}{2}\int_{\mathbb{T}^3} |u(x,t)|^2\, dx + \mu \int_0^t \int_{\mathbb{T}^3} |\nabla u(x,\tau)|^2 dxd\tau \le \frac{1}{2}\int_{\mathbb{T}^3} |\overline{u}(x)|^2\, dx \tag{1.4}$$

holds for almost every $t > 0$.

The previous definition (1.3) can be formally obtained by scalar multiplying the first equation in the system (1.1) by the smooth test vector field φ and integrating by parts. Note that, as a consequence of the solenoidal nature of φ, the pressure p does not appear in the weak formulation, but it can easily be recovered a posteriori as the unique zero average solution of

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes u).$$

The previous elliptic equation for the pressure can be formally obtained by computing the divergence of (1.1) and it comes from the fact that relation (1.3) implies that the vector field $\partial_t u + \operatorname{div}(u \otimes u) - \mu \Delta u$ is (weakly) irrotational. For analysis in which the initial datum $\overline{u}$ does not play any role, one can choose the test function $\varphi \in C_c^\infty((0,\infty) \times \mathbb{T}^3)$ which makes the term $\int_{\mathbb{T}^3} \overline{u}(x)\varphi(x,0)\, dx$ disappear.

In his seminal work [46] from 1934, the french mathematician Jean Leray proved the existence of the above described solutions. His proof, later refined by Hopf in [38], relies on a suitable regularization of the Navier-Stokes equations which preserves the energy properties described above, obtaining a sequence of smooth approximate solutions which, thanks to (1.2), are uniformly bounded in the energy space $L^2((0,\infty);W^{1,2}(\mathbb{T}^3))$. By compactness, the sequence (up to extracting a converging subsequence) converges to an actual solution of (1.1) and the energy inequality (1.4) is a consequence of the weak lower semicontinuity of the norms in reflexive Banach spaces. Moreover, by the same arguments introduced by Leray, it can be proved that these solutions are smooth outside a closed set of Hausdorff dimension $1/2$. This was first obseved by Scheffer in [57], which was also the starting point of the partial regularity theory for Navier-Stokes which culminated with the Caffarelli-Kohn-Niremberg result [11] in which, by introducing the notion of *suitable weak* solutions, the authors proved that the set of space-time singular point has zero $1-$dimensional parabolic Hausdorff measure. This is still the best partial regularity result available in the literature.

To date the uniqueness and regularity question of such solutions is still open and represents one of the most challenging problems in fluid dynamics. We remark that a small refinement of Leray's argument implies the validity of the following stronger version of (1.4)

$$\frac{1}{2}\int_{\mathbb{T}^3}|u(x,t)|^2\,dx+\mu\int_s^t\int_{\mathbb{T}^3}|\nabla u(x,\tau)|^2dxd\tau\leq\frac{1}{2}\int_{\mathbb{T}^3}|u(x,s)|^2\,dx, \tag{1.5}$$

for almost every $s\geq 0$ and every $t>s$. The latter energy inequality gives the additional information that the total energy of the system is a non-increasing function of time[1]. Weak solutions satisfying (1.5) are usually called *Leray-Hopf weak solutions*.

After the work of Leray several conditional uniqueness and smoothness results have been proved, culminating in the Prodi-Serrin cryterion [54, 58] (see also [35] for the limit case $L^\infty((0,T);L^3(\mathbb{R}^3))$.

Theorem 1.3. *Let u be a Leray weak solution of* (1.1). *If $u\in L^r((0,T);L^q(\mathbb{T}^3))$ for some $r\in[2,\infty)$, $q\in(3,\infty)$ such that $\frac{2}{r}+\frac{3}{q}\leq 1$, then u is smooth and unique.*

Clearly the short list of results we presented here does not cover all the huge mathematical literature available on the Navier-Stokes equations but it is enough for the purposes of this thesis. For a wider and more detailed discussion on the topic we refer to the monographs [36, 48, 60].

[1] It can be shown that every Leray solution is continuous in time with values in $L^2(\mathbb{T}^3)$ endowed with the weak topology, thus (1.5) implies that the kinetic energy, being well defined for every time t, is monotone in the classical sense.

1.2 The Euler equations

In fluid dynamics, the Euler equations are a set of quasilinear hyperbolic equations governing inviscid flows. They are named after the Swiss mathematician and physicist Leonhard Euler

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = \overline{u} \end{cases} \tag{1.6}$$

We observe that the previous system can also be formally obtained from (1.1) by letting $\mu \to 0$. By the absence of viscosity and external forces acting on the fluid, there is no physical reason why the kinetic energy can be dissipated. Indeed, with the same computations already done in the previous section, we get

$$\frac{d}{dt} e_u = 0, \tag{1.7}$$

for a sufficiently smooth solution u. As in the Navier-Stokes case, it is not known wether smooth solutions exist globally in time and this represents another big open question in this field.

The most used definition of weak solution of (1.6) is the following

Definition 1.4. *Let* $\overline{u} \in L^2(\mathbb{T}^3)$ *such that* $\operatorname{div} \overline{u} = 0$. *A divergence free vector field* $u \in L^\infty((0,T); L^2(\mathbb{T}^3))$ *is a weak solution of* (1.6) *if*

$$\int_0^T \int_{\mathbb{T}^3} (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi)\, dx dt = -\int_{\mathbb{T}^3} \overline{u}(x) \varphi(x,0)\, dx, \tag{1.8}$$

for every test vector field $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,\infty))$ *such that* $\operatorname{div} \varphi = 0$.

It is clear that in the above definition the assumption $u \in L^2(\mathbb{T}^3 \times (0,T))$ would suffice to make (1.8) work. The reason why the L^∞ in time assumption is used is due to the fact that only solutions whose kinetic energy is uniformly bounded in time are considered. This is important for the physical meaning of the model. Clearly, assuming $u \in L^\infty((0,T); L^2(\mathbb{T}^3))$ does not prohibit to the solution to increase its kinetic energy in time, which would be an extremely non-physical phenomenon. That is why a slightly stronger definition of weak solutions has been introduced.

Definition 1.5. *An admissible weak solution of* (1.6) *is a weak solution in the sense of the previous definition, such that moreover*

$$e_u(t) \leq e_u(0) \tag{1.9}$$

for almost every $t \geq 0$.

This notion of weak solution can be viewed as the analogous of Definition 1.2 and it plays both a physical and a mathamatical role. Indeed, on one hand it prohibits solutions to generate kinetic energy from nowhere and on the other side it is also important to get a suitable weak-strong uniqueness result

[3]. However, the lack of compactness for such equations due to the absence of viscosity does not allow to prove the existence of such solutions from a general initial datum $\bar{u} \in L^2(\mathbb{T}^3)$. The problem of existence of weak solutions, as well as the existence of global in time smooth solutions starting from smooth initial data, remains a formidable open question. We now focus on an issue that has aroused more interest in recent years: the energy conservation and Onsager's conjecture.

1.2.1 Non-smooth solutions and Turbulence

In the last decade, considerable attention has been devoted to the study of Hölder continuous weak solutions of (1.6), since they naturally arise in incompressible hydrodynamics models, starting from the celebrated prediction of Kolmogorov's Theory of Turbulence [44]. In this context, one of the most investigated problems is the Onsager's conjecture on the kinetic energy conservation for Hölder continuous weak solutions of (1.6). Indeed, in 1949, for solutions $u \in L^\infty((0,T);C^\beta(\mathbb{T}^3))$, Lars Onsager predicted that anomalous dissipation of the kinetic energy e_u may occur only in the low regularity regime $\beta < \frac{1}{3}$, while in the case $\beta > \frac{1}{3}$, some rigidity of the equation prohibits the existence of such wild non-conservative weak solutions. In recent years the Onsager's conjecture has been completely solved but the question on what happens in the critical case $\beta = \frac{1}{3}$ is still open. It is worth to mention that considering weak solutions $u \in C^\beta(\mathbb{T}^3 \times [0,T])$ is not more restrictive with respect to the ones considered by Onsager. Indeed in Chapter 2, it is shown that every $u \in L^\infty((0,T);C^\beta(\mathbb{T}^3))$ enjoys the same $\beta-$Hölder regularity in time by using simple mollification estimates. This property has been first observed in [39] where the proof is based on a Littlewood-Paley decomposition of the velocity.

The energy conservation question was first tackled by Eyink in [34] but the first proof for the whole range $\beta > \frac{1}{3}$ has been given in [22] by Constantin, E and Titi in the slightly more general Besov class $L^3\left((0,T);B^\beta_{3,\infty}(\mathbb{T}^3)\right)$. They noticed that the quadratic commutator obtained by regularizing in space equations (1.6), enjoys a corresponding improved (quadratic) estimate. We refer to Chapter 2 for a precise description of the technique, where the same convolution estimates are used to prove some of the results therein. In particular by using a suitable refinement of the Constantin, E and Titi proof we prove that

$$|e_u(t) - e_u(s)| \leq C|t-s|^{\frac{2\beta}{1-\beta}}, \tag{1.10}$$

whenever $u \in L^\infty((0,T);C^\beta(\mathbb{T}^3))$, for some $\beta \in (0,1)$. The previous regularity estimate on the kinetic energy of Hölder continuous weak solutions of Euler implies that even in the range in which it is not necessarily constant, the energy has however some constraint. Property (1.10) has been first observed in [39] for any $\beta \leq \frac{1}{3}$, thus the real novelty of the proof that we propose in Chapter 2, in addition to its simplicity, is that it works with no restrictions on β, allowing us to deduce both the Hölder regularity of the kinetic energy and the energy conservation for $\beta > \frac{1}{3}$, since in the latter case the Hölder exponent in (1.10) is bigger than 1.

The negative part of the Onsager conjecture has seen incredible developments in recent years. The first proof of solutions with a non-constant energy profile was given in [56] by Scheffer. He constructed L^2 weak solutions of Euler with compact support in time, thus strictly speaking they are not *dissipative*, as dissipative solutions are required to have non-increasing energy. The existence of dissipative weak solutions was first proven by Shnirelman in [59], but also in this case they were only in L^2. In the groundbreaking papers [26, 27] De Lellis and Székélyhidi Jr. made a significant progress towards the Onsager's conjecture providing the first construction of dissipative Hölder continuous weak solutions by introducing the so called *convex integration* in the context of fluid dynamics. After a series of advancements [4, 6, 40], the full range $\beta < \frac{1}{3}$ was eventually achieved by Philip Isett in [41]. The present thesis develops new aspects of these iterative techniques and leads to the new results of chapters 5, 6, 7 and 8. They all follow the Hölder-based convex integration of [7], with the exception of Chapter 8, which follows the L^p-based convex integration introduced in [8], in which Tristan Buckmaster and Vlad Vicol, by introducing some substantial new ideas which can be summarised under the word *intermittency*, proved the non-uniqueness of L^2 weak solution to Navier-Stokes. By pushing further intermittency, it has been proven in the recent paper [14] the sharpness of the Prodi-Serrin exponents of Theorem 1.3. Being quite technical, we postponed the detailed description of these techniques to the respective chapters.

If instead of looking at Hölder solutions one considers the Sobolev class H^β, it is know that the energy conservation happens if $\beta > \frac{5}{6}$. This is an easy consequence of the refinement of the Constantin, E and Titi proof given in [13], together with the embeddings $H^{\frac{5}{6}} \subset W^{\frac{1}{3},3} \subset B^{\frac{1}{3}}_{3,c(\mathbb{N})}$. This indicates that in the weaker Sobolev regularity, solutions could dissipate the kinetic energy even above the Onsager barrier $\frac{1}{3}$. The recent remarkable result [9] is the first one in this direction and it shows the existence of non-conservative weak solutions of Euler in the space $L^\infty((0,T);H^\beta(\mathbb{T}^3))$, with $\beta < \frac{1}{2}$. This result also represents the first example in which a spatial intermittent convex integration scheme is implemented to achieve an high regularity for the limit solution, since in the previous ones [5] and [8] only very small values β were allowed.

1.3 Fractional Navier-Stokes equations

The fractional Navier-Stokes equations have a long history. They have been considered by J. L. Lions in the sixties. The dissipative term of the classical Navier-Stokes equations is substituted by a (generally non local) operator $(-\Delta)^\gamma$, where γ might be an arbitrary positive real number.

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p + \mu(-\Delta)^\gamma u = 0 \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = \overline{u}. \end{cases} \tag{1.11}$$

There are different ways to rigorously define the operator $(-\Delta)^\gamma$. One of them is to define it as the symbol $|k|^{2\gamma}$ in the Fourier space. More precisely, in the periodic setting, we have

$$(-\Delta)^\gamma u = \sum_{k\in\mathbb{Z}^3} |k|^{2\gamma} u_k e^{2\pi i k\cdot x},$$

where u_k is the $k-$th Fourier coefficient of the vector field u in space. If $\gamma = 1$ they coincide with (1.1). When $\gamma < 1$ they are called *hypodissipative* Navier-Stokes equations, meaning that they dissipate less kinetic energy with respect to the full Laplacian $-\Delta$. If $\gamma > 1$ we have the *hyperdissipative* version. For the purposes of this work we restrict ourselves to the case $\gamma < 1$. As for the equations already discussed, their corresponding energy equality

$$\frac{1}{2}\int_{\mathbb{T}^3} |u(x,t)|^2\,dx + \mu\int_0^t\int_{\mathbb{T}^3}\left|(-\Delta)^{\gamma/2}u(x,\tau)\right|^2 dx d\tau = \frac{1}{2}\int_{\mathbb{T}^3}|\overline{u}(x)|^2\,dx$$

can be derived for smooth solutions by multiplying the first equation in (1.11) by u and integrating by parts. Also in this hypodissipative regime, the existence of smooth solutions is not known. The analogous notion of Leray weak solutions can be also given.

Definition 1.6. *Let* $\mu > 0$ *and* $\overline{u} \in L^2(\mathbb{T}^3)$ *such that* $\operatorname{div}\overline{u} = 0$. *A Leray weak solution of* (1.11) *is a divergence free vector field u such that* $u \in L^\infty((0,\infty);L^2(\mathbb{T}^3)) \cap L^2((0,\infty);W^{\gamma,2}(\mathbb{T}^3))$ *and*

$$\int_0^\infty\int_{\mathbb{T}^3} (u\cdot\partial_t\varphi + u\otimes u : \nabla\varphi - \mu u\cdot(-\Delta)^\gamma\varphi)\,dxdt = -\int_{\mathbb{T}^3}\overline{u}(x)\varphi(x,0)\,dx, \tag{1.12}$$

for every test vector field $\varphi \in C_c^\infty(\mathbb{T}^3\times[0,\infty))$ *such that* $\operatorname{div}\varphi = 0$. *Moreover the following energy inequality*

$$\frac{1}{2}\int_{\mathbb{T}^3} |u(x,t)|^2\,dx + \mu\int_0^t\int_{\mathbb{T}^3}\left|(-\Delta)^{\gamma/2}u(x,\tau)\right|^2 dx d\tau \le \frac{1}{2}\int_{\mathbb{T}^3}|\overline{u}(x)|^2\,dx \tag{1.13}$$

holds for almost every $t > 0$.

For any $\gamma > 0$, the same proof of Leray can be used to deduce the existence of solutions in the sense of Definition 1.6, since the presence of even minimal dissipation gives compactness. Unlike the full Navier-Stokes equations, it has recently been proved by Colombo, De Lellis and the present author, that this notion of solution is ill-posed if $\gamma < \frac{1}{3}$. The proof is based on adapting the Hölder convex integration to (1.11) by noticing that, if $\gamma < \frac{1}{2}$, the dissipative term can be incorporated in the iterative scheme as an error, since in this case, its differential order does not exceed 1, which is the leading order of differentiability of Euler. The more restrictive $\gamma < \frac{1}{3}$ comes from imposing that our solutions are of Leray type. This is the content of Chapter 5.

1.4 Outline and description of the thesis

We end the introduction with the organization of the thesis by describing the content of each chapter.

Chapter 2

This chapter corresponds to [17], a joint work with Maria Colombo. It contains various regularity properties for solutions $u \in L^\infty((0,T),C^\beta(\mathbb{T}^3))$ to both Euler and the hypodissipative Navier-Stokes equations. In particular we show that the same $\beta-$Hölder regularity transfers in time, while the pressure p enjoys the twice 2β regularity. The technique is based on a space regularization of the equation and the time regularity is achieved by linking the mollification parameter to the desired time scale. Moreover, we prove that a suitable Hölder regularity holds for the energies of both equations (1.6) and (1.11). The latter properties are proved in the slightly more general class of Besov spatial regularity, while keeping the L^∞ assumption in time. To prove the double regularity of the pressure we show some improved Schauder's estimate for the Laplace equations with a specific right hand side. In most of the proofs, the quadratic structure of the commutator introduced in [22], which comes from the mollification, plays an important role. In the Euler case all these properties have been first observed in [39], where however the proof is based on different techniques, while for the hypodissipative Navier-Stokes equations they are completely new. In addition to the great simplicity of our proofs, the real novelty with respect to [39] is the proof of the regularity (1.10) with no restriction on β, from which also the energy conservation follows in the rigid regime $\beta > \frac{1}{3}$.

Chapter 3

In this chapter we generalize all the regularising properties of the Euler equations from Chapter 2 to the wider class of Besov and Sobolev weak solutions. It builds on the work [18], jointly obtained with Maria Colombo and Luigi Forcella. As for the Hölder case, by assuming that the solution has a Besov or Sobolev regularity in space (together with some integrability in time), we show that the same fractional weak differentiability transfers in time and the pressure p is always double as regular as u. The strength of this work is that it builds on an abstract and robust technique: we prove some new abstract interpolation results for quite general multilinear operators, from which we deduce improved Calderón-Zygmund estimates for elliptic PDEs with a particular right hand side. In particular the double spatial regularity of the pressure directly follows. These improved general regularity estimates are also interesting by themselves, since they are in contrast with some common believes that better regularity estimates are a consequence of precise cancellations properties that can only be observed when the kernel which provides the solution is explicit.

Chapter 4

The aim of this chapter is to investigate the Helicity conservation for the incompressible Euler equations. It is based on the singled authored paper [28]. Helicity is an integral physical quantity that can be topologically interpreted as a measure of linkage and/or knottedness of vortex lines in the fluid flow. As for the most known kinetic energy, for sufficiently smooth solutions Helicity is conserved [12, 13]. Unlike the Onsager's conjecture, in this case the threshold of the fractional differentiability needed to imply conservation is $\frac{2}{3}$. Building again on the same mollification technique of Chapter 2, we prove that the Helicity of solutions $u \in L^\infty((0,T);W^{\beta,3}(\mathbb{T}^3))$, enjoys a suitable Hölder regularity. We also prove a new conservation result by treating the velocity and its curl as two independent functions, which in general is not implied by the ones already present in the literature. Even if Helicity had a very little attention in the mathematical community, it could be one of the key points in the better understanding of the convex integration techniques, since the regularity required would cross the actual barrier in which those iterative schemes work.

Chapter 5

Here we prove the ill-posedness of the Leray-Hopf weak solutions of the hypodissipative Navier-Stokes equations when the power of the fractional dissipation is $\gamma < \frac{1}{3}$. This chapter contains the work [29] and it represents an improvement with respect to [16] where Colombo, De Lellis and the author of this thesis proved the non-uniqueness up to $\gamma < \frac{1}{5}$. Moreover, we also show that dissipative weak solutions to Euler can be obtained as a vanishing viscosity limit of Leray-Hopf weak solutions of a suitable fractional Navier-Stokes system. The same problem for the full Navier-Stokes equations (1.1) represents one of the main open questions in the field. The non-uniqueness proof we propose in this chapter builds on the Hölder convex integration scheme proposed in [7]. To adapt their scheme in our dissipative setting we also prove some new stability estimates for linear non-local advection-diffusion equations, from which we deduce a local in time existence result of smooth solutions to (1.11).

Cahpter 6

This chapter deals with a conjecture of Philip Isett and Sung-Jin Oh [43] on the sharpness of the energy regularity (1.10). It is the content of [32], obtained in collaboration with Riccardo Tione. The conjecture asserts that solutions whose energy regularity (1.10) is sharp are residual in the space of all $\beta-$Hölder weak solutions to Euler. We give a (partial) positive answer to this conjecture by proving that the residuality holds if one considers a space of slightly more regular solutions. Our proof is based on a refinement of the Hölder-based convex integration scheme, by noticing that using such iterative techniques it is possible to achieve any energy profile whose regularity is arbitrarily close to the sharp one. We show how this implies an empty interior condition in the right space of solutions, from which we conclude the residuality property. We also explain why our proof does not imply the full conjecture in the natural space, which remains open. Our, even if partial, positive answer to their question implies that below the Onsager's threshold, oscillations of the energy are highly unstable under perturbations of solutions, which clearly confirms the well known fact that instability phenomena frequently appear in the turbulent regime.

Chapter 7

Following the work [30] obtained in collaboration with Silja Haffter, this chapter deals with the size of the singular set for the wild Hölder continuous weak solutions constructed via convex integration. For $\beta < \frac{1}{3}$, we consider $\beta-$Hölder weak solutions of the incompressible Euler equations that do not conserve the kinetic energy. We prove that for such solutions the closed and non-empty set of singular times $\mathscr{B}$ satisfies $\dim_{\mathscr{H}} \mathscr{B} \geq \frac{2\beta}{1-\beta}$. This lower bound on the Hausdorff dimension of the singular set in time is intrinsically linked to the Hölder regularity of the kinetic energy and we conjecture it to be sharp. As a first step in this direction, for every $\beta < \beta' < \frac{1}{3}$ we are able to construct, via a convex integration scheme, non-conservative C^{β} weak solutions of the incompressible Euler system such that $\dim_{\mathscr{H}} \mathscr{B} \leq \frac{1}{2} + \frac{1}{2}\frac{2\beta'}{1-\beta'}$. This result gives more and new insights on the topic but also leaves an interesting open question on how to fill the gap to reach the lower bound that seems to be sharp. The structure of the solutions that we construct in this chapter allows moreover to deduce a strong non-uniqueness result for weak solutions of the incompressible Euler equations emanating from every regular initial datum.

Chapter 8

This chapter contains a couple of typicality results for weak solutions of the Navier-Stokes equations. It follows a joint work with Maria Colombo and Massimo Sorella [19]. By adapting the same ideas of [32] to the L^p-based convex integration scheme introduced in [8], we prove that the Leray weak solutions, in the sense of Definition 1.2, are a nowhere dense set in the space of all solutions with finite kinetic energy and the solutions which are smooth in some open time interval are meager. The key idea to prove the respectively empty iterior conditions is to localize the convex integration construction in the (open) time interval in which the Leray solution is smooth. The latter property had already been observed by Leray himself in his seminal paper [46].

Some other works

While the above mentioned works are all under the same guiding thread of the analysis of non-smooth solutions to the main incompressible fluid models, we also mention that a couple of results [31,33] have been obtained in parallel to the main topic of the PhD project. They address a Calculus of Variations question posed by Denis Serre on the higher integrability of the determinant of divergence-free matrix fields. In [31] we proved the sharpness of his result relying again on the elegant Baire category theory as in the Euler equation from above. This subsequently generated [33] in which, in collaboration with Denis Serre, we showed how this sharp integrability is also linked to the upper semicontinuity of the corresponding functional in the spirit of the pioneering result of Fonseca and Müller, that however can not be applied in this context. Since they would be out of context in this thesis, we preferred not to add them to this manuscript.

Chapter 2

Regularity results for the Fractional Navier-Stokes equations in Hölder spaces

2.1 Introduction

In the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, we consider the fractional Navier-Stokes equations that we recall here

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p + \mu(-\Delta)^\alpha u = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } (0,T) \times \mathbb{T}^3 . \tag{2.1}$$

We are concerned with a class of distributional solutions of the previous system, which exhibit an Hölder spatial regularity. Hölder solutions are of particular importance for the Euler equations in the context of hydrodynamic turbulence, starting from a celebrated prediction of Kolmogorov's theory [44]: the velocity increments in turbulent flows should obey on average a universal scaling law corresponding to the Hölder exponent $\frac{1}{3}$

$$< |u(x+\delta x) - u(x)|^p >^{1/p} \leq C(p)|\delta x|^{1/3}.$$

In the following we exploit a regularizing property of the Euler equations, namely that weak solutions with spatial Hölder regularity $u \in L^\infty((0,T);C^\theta(\mathbb{T}^3))$ are in fact $\theta-$Hölder continuous also in time. Moreover, the associated pressure is almost $2\theta-$Hölder continuous in space-time and the corresponding kinetic energy profile is $\frac{2\theta}{1-\theta}-$Hölder continuous. This property can be observed in all the non conservative solutions constructed to validate the Onsager conjecture and it was first proved by Isett [39]. In his work, this regularization is obtained as a consequence of the regularity for advective derivatives, and involves refined and technical estimates on the Paley-Littlewood decomposition of the solution. Our proof is based on completely different ideas, involving a regularization of the equation as in [22] as well as their commutator estimate. The method is quite flexible and indeed we perform it not only for Euler, but also for the fractional Navier-Stokes system.

Theorem 2.1. *Let $\theta \in (0,1)$, $\alpha \in \left(0,\frac{1}{2}\right)$, $\mu \geq 0$, and let (u,p) be a weak solution of* (2.1) *such that $u \in L^\infty((0,T);C^\theta(\mathbb{T}^3))$. Then there exists $C_\mu > 0$, depending only on μ, such that*

$$\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)} \leq C_\mu\left(\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} + \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\right). \tag{2.2}$$

Moreover there exists $C_\theta > 0$ and for every $\varepsilon > 0$ small, a constant $C_{\theta,\mu,\varepsilon} > 0$ such that

(i) *if* $\theta \in \left(0,\frac{1}{2}\right)$

$$\|p\|_{L^\infty((0,T);C^{2\theta}(\mathbb{T}^3))} \leq C_\theta \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}, \tag{2.3}$$

$$\|p\|_{C^{2\theta-\varepsilon}([0,T]\times\mathbb{T}^3)} \leq C_{\theta,\mu,\varepsilon}\left(\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} + \|u\|^3_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\right); \tag{2.4}$$

(ii) *if* $\theta \in \left(\frac{1}{2},1\right)$

$$\|p\|_{L^\infty((0,T);C^{1,2\theta-1}(\mathbb{T}^3))} \leq C_\theta \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}, \tag{2.5}$$

$$\|p\|_{C^{1,2\theta-1-\varepsilon}([0,T]\times\mathbb{T}^3)} \leq C_{\theta,\mu,\varepsilon}\sum_{m=2}^{4} \|u\|^m_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}. \tag{2.6}$$

The improved $2\theta-$Hölder space regularity of the pressure in (2.3) and (2.5) was previously established in [20]. The assumption $\alpha < \frac{1}{2}$ is absolutely natural in this context: indeed, for α above this threshold any Hölder continuous solution to the $\alpha-$Navier-Stokes equation is in fact smooth by simple bootstrap arguments, based on the regularization of the “fractional heat equation” part of (2.1), considering the nonlinearity and the pressure as a right-hand side.

In the following result, we consider the kinetic and total energies of the system (2.1)

$$e_u(t) = \frac{1}{2}\int_{\mathbb{T}^3} |u|^2(x,t)\,dx, \qquad E_u(t) = e_u(t) + \mu\int_0^t\int_{\mathbb{T}^3}\left|(-\Delta)^{\alpha/2}u\right|^2(x,r)\,dxdr \tag{2.7}$$

which coincide for the Euler equations, namely when $\mu = 0$. We show that, instead of asking $u \in L^\infty((0,T);C^\theta(\mathbb{T}^3))$, a suitable spatial Besov regularity on the velocity u is enough to guarantee Hölder regularity of the energies. This is obviously due to their "integral" nature.

Theorem 2.2. *Let $\theta \in (0,1)$, $\mu \geq 0$, $\alpha \in (0,\frac{1}{2})$, with $\alpha < \theta$ if $\mu > 0$. Let $u \in L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))$ be a weak solution of* (2.1)*. Then if $\mu = 0$ (namely, for the Euler system) we have*

$$|e_u(t) - e_u(s)| \leq C_{u,\theta}|t-s|^{\frac{2\theta}{1-\theta}},$$

where $C_{u,\theta} = C_\theta\left([u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))} + [u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}\right)$; if $\mu > 0$ (namely, for the hypodissipative Navier-Stokes system) and $\theta \leq 1/3$ we have

$$|E_u(t) - E_u(s)| \leq C_{u,\theta,\alpha}|t-s|^{\frac{2(\theta-\alpha)}{1-3\theta+2(\theta-\alpha)}}, \tag{2.8}$$

where $C_{u,\theta,\alpha} = C_{\theta,\alpha}\left([u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))} + [u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}\right)$*; if* $\mu > 0$ *and* $\theta > 1/3$ *the energy* E_u *is constant in time.*

Note that the previous theorem implies the energy conservation for (2.1), in particular e_u and E_u are conserved, respectively, if $\theta > \frac{1}{3}$ and if $\theta > \max\{\frac{1}{3}, \alpha\}$, since both the Hölder exponents are greater than 1.

To prove the time regularities, we look at a regularized version of (2.1) in the spirit of the proof of conservation of the energy for $\theta > \frac{1}{3}$ by Constantin, E and Titi in [22]. To do that, let $\rho \in C_c^\infty(B_1(0))$ be a standard non negative kernel such that $\int_{B_1(0)} \rho(x)\,dx = 1$. For any $\delta > 0$ we define $\rho_\delta = \delta^{-3}\rho(\frac{x}{\delta})$ and we consider the mollifications (in space) of u and p

$$u_\delta(t,x) = (u * \rho_\delta)(t,x) = \int_{B_\delta(x)} u(t,y)\rho_\delta(x-y)\,dy, \qquad p_\delta(t,x) = (p * \rho_\delta)(t,x)\,.$$

Thus, mollifying equations (2.1) one gets

$$\partial_t u_\delta + \operatorname{div}(u_\delta \otimes u_\delta) + \nabla p_\delta + \mu(-\Delta)^\alpha u_\delta = \operatorname{div} R_\delta\,, \tag{2.9}$$

where $R_\delta = u_\delta \otimes u_\delta - (u \otimes u)_\delta$. It is easy to see that the energy identity for u_δ becomes

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^3} |u_\delta|^2\,dx + \mu\int_{\mathbb{T}^3} \left|(-\Delta)^{\alpha/2} u_\delta\right|^2 dx = \int_{\mathbb{T}^3} u_\delta \cdot \operatorname{div} R_\delta\,dx = -\int_{\mathbb{T}^3} R_\delta : \nabla u_\delta\,dx\,. \tag{2.10}$$

Then we estimate the variation of u, e_u and E_u between two times $s < t$ through u_δ, e_{u_δ} and E_{u_δ} respectively, and we optimize the choice of δ in terms of $|t-s|$.

Regarding the pressure, taking the divergence of the first equation in (2.1) p solves

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes u) \tag{2.11}$$

and the solution is unique up to the renormalization $\int_{\mathbb{T}^3} p(t,x)\,dx = 0$, for every $t \in [0,T]$. By Schauder estimates we infer

$$\|p\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} \leq C_\theta \|u \otimes u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} \leq C_\theta \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\,. \tag{2.12}$$

To improve the regularity as stated in (2.3), namely to show that p is not only $\theta-$Hölder continuous but actually $2\theta-$Hölder continuous, we exploit the quadratic structure of the right hand side $\operatorname{div}\operatorname{div}(u \otimes u)$, together with the solenoidal nature of the vector field u. The space regularity of the pressure in $\mathbb{T}^3$ is then a direct consequence of Proposition 2.3 and Lemma 2.4 below. Relying on different representation formulas for the pressure in bounded domains, similar estimates were previously deduced in [20]. Finally, the time regularity of the pressure is obtained by differentiating (2.11)

$$\Delta\partial_t p = \operatorname{div}\operatorname{div}\operatorname{div}(u \otimes u \otimes u) + \operatorname{div}\operatorname{div}\left(2\nabla p \otimes u + \mu(-\Delta)^\alpha u \otimes u + \mu u \otimes (-\Delta)^\alpha u\right),$$

and by exploiting again the structure of the right-hand side. In this case the presence of the fractional Laplacian in the right-hand side introduces a technical difficulty to the analysis.

2.2 Some improved Schauder estimates

To prove the space regularity of the pressure we exploit the explicit formulae for the potential theoretic solution of the Laplace equation in $\mathbb{R}^3$. To this end, we denote by $\Phi(x)=\frac{1}{4\pi|x|}$ the fundamental solution of the Laplace operator $-\Delta$, which enjoys the estimates $\left|D^k\Phi(x)\right|\le C(k)|x|^{-1-k}$ for all $k\in\mathbb{N}$. We recall that given $R\in C^\theta_c(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ compactly supported, the potential theoretic solution of $-\Delta p=\operatorname{div}\operatorname{div} R$ is the only solution $p\in C^\theta(\mathbb{R}^3)$ which vanishes at infinity and it is given by the formula (with the Einstein summation convention)

$$p(x)=\int_{B_{R_0}(x_0)}\partial^2_{ij}\Phi(x-y)\big(R^{ij}(y)-R^{ij}(x)\big)\,dy-R^{ij}(x)\int_{\partial B_{R_0}(x_0)}\partial_i\Phi(x-y)n_j(y)\,dy\,, \tag{2.13}$$

where $B_{R_0}(x_0)$ is any ball containing the support of R (and R is thought to be extended to 0 outside its support) and $n(y)$ is the normal to $B_{R_0}(x_0)$ at y. Notice that the first integrand is not singular around x thanks to the Hölder regularity of R. Given any parameter $\lambda=\theta,\mu,\varepsilon$ we will explicitly write C_λ to denote constants which depend only on λ.

Proposition 2.3. *Let $\beta,\gamma\in(0,1)$ and $v,w,z\in C^0(\mathbb{R}^3)$ be solenoidal vector fields compactly supported. If $p,q:\mathbb{R}^3\to\mathbb{R}$ are the potential theoretic solutions of*

$$-\Delta p=\operatorname{div}\operatorname{div}(v\otimes w)\qquad \textit{and}\qquad -\Delta q=\operatorname{div}\operatorname{div}\operatorname{div}(v\otimes w\otimes z)\,, \tag{2.14}$$

then there exist a constant $C_{\beta,\gamma}>0$ such that the following holds

(i) if $\beta+\gamma\in(0,1)$ then $\|p\|_{C^{\beta+\gamma}(\mathbb{R}^3)}\le C_{\beta,\gamma}\|v\|_{C^\beta(\mathbb{R}^3)}\|w\|_{C^\gamma(\mathbb{R}^3)}$,

(ii) if $\beta+\gamma\in(1,2)$ then $\|p\|_{C^{1,\beta+\gamma-1}(\mathbb{R}^3)}\le C_{\beta,\gamma}\|v\|_{C^\beta(\mathbb{R}^3)}\|w\|_{C^\gamma(\mathbb{R}^3)}$,

(iii) if $\beta+\gamma\in(1,2)$ then

$$\begin{aligned}\|q\|_{C^{\beta+\gamma-1}(\mathbb{R}^3)}\le{}& C_{\beta,\gamma}\|v\|_{C^0(\mathbb{R}^3)}\|w\|_{C^\beta(\mathbb{R}^3)}\|z\|_{C^\gamma(\mathbb{R}^3)}\\ &+C_{\beta,\gamma}\|v\|_{C^\beta(\mathbb{R}^3)}\big(\|w\|_{C^0(\mathbb{R}^3)}\|z\|_{C^\gamma(\mathbb{R}^3)}+\|w\|_{C^\gamma(\mathbb{R}^3)}\|z\|_{C^0(\mathbb{R}^3)}\big).\end{aligned} \tag{2.15}$$

Taking $\beta=\gamma=\theta$ in the previous proposition and $v=w=u$, one obtains that the potential theoretic solutions p and q of $-\Delta p=\operatorname{div}\operatorname{div}(u\otimes u)$ and $-\Delta q=\operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u)$ obey

$$\|p\|_{C^{2\theta}(\mathbb{R}^3)}\le C_\theta\|u\|^2_{C^\theta(\mathbb{R}^3)}\qquad \text{if } \theta\in\big(0,\tfrac12\big), \tag{2.16}$$

$$\|p\|_{C^{1,2\theta-1}(\mathbb{R}^3)}\le C_\theta\|u\|^2_{C^\theta(\mathbb{R}^3)},\quad \|q\|_{C^{2\theta-1}(\mathbb{R}^3)}\le C_\theta\|u\|_{C^0(\mathbb{R}^3)}\|u\|^2_{C^\theta(\mathbb{R}^3)}\quad \text{if } \theta\in\big(\tfrac12,1\big).$$

However, the more general nature of Proposition 2.3 will be useful to deal with Theorem 2.1 (*ii*); in this case, we will take advantage not only of the structure of the equation for p, ∇p and $\partial_t p$, but also of the (previously showed) regularity of u in time, and for this scope we will need Proposition 2.3 in its generality, including the non-symmetric nature of (2.15) with respect to v, w and z. We do not expect (2.16) to hold for $\theta = \frac{1}{2}$ due to the usual loss in Schauder estimates in integer Hölder spaces; for $\theta = \frac{1}{2}$ the estimate reads $\|p\|_{C^{1-\varepsilon}(\mathbb{R}^3)} \le C_\varepsilon \|u\|^2_{C^{\frac{1}{2}}(\mathbb{R}^3)}$.

Proof. (i) We will prove that

$$[p]_{C^{\beta+\gamma}(\mathbb{R}^3)} \le C\|v\|_{C^\beta(\mathbb{R}^3)}\|w\|_{C^\gamma(\mathbb{R}^3)}. \tag{2.17}$$

The estimate for $\|p\|_{C^0(\mathbb{R}^3)}$ (as well as the one for $[p]_{C^{\min\{\beta,\gamma\}}(\mathbb{R}^3)}$) follows from the standard Schauder estimates. For any $x_1, x_2 \in \mathbb{R}^3$, we define $\bar{x} = \frac{x_1+x_2}{2}$ and $\lambda = |x_1 - x_2|$. Since $\operatorname{div} v = \partial_i v^i = 0 = \operatorname{div} w = \partial_i w^i$, we observe that the equation for p can be rewritten as

$$\begin{aligned} -\Delta p = \partial_{ij}(v^i w^j) = \partial_i w^j \partial_j v^i &= \partial_i (w^j - w^j(x_2))\partial_j(v^i - v^i(x_1)) \\ &= \partial_{ij}\big((v^i - v^i(x_1))(w^j - w^j(x_2))\big) \\ &= \partial_{ij}\big((v^i - v^i(x_1))(w^j - w^j(x_2)) - v^i(x_1)w^j(x_2)\big). \end{aligned}$$

Since the function $(v^i - v^i(x_1))(w^j - w^j(x_2)) - v^i(x_1)w^j(x_2)$ is compactly supported in B_1, we conclude that the potential theoretic solution associated to it is the same as the potential theoretic solution associated to $v^i w^j$, namely by (2.13) applied with $B_{R_0}(x_0) = B_{R_0}(\bar{x})$ it is given by the representation formula

$$\begin{aligned} p(x) = \int_{B_{R_0}(\bar{x})} & \partial^2_{ij}\Phi(x-y)\big[\big(v^i(y) - v^i(x_1)\big)\big(w^j(y) - w^j(x_2)\big) \\ & - \big(v^i(x) - v^i(x_1)\big)\big(w^j(x) - w^j(x_2)\big)\big]\,dy \\ - \big[\big(v^i(x) - v^i(x_1)\big)&\big(w^j(x) - w^j(x_2)\big) - v^i(x_1)w^j(x_2)\big] \int_{\partial B_{R_0}(\bar{x})} \partial_i \Phi(x-y) n_j(y)\,dy\,, \end{aligned}$$

for every $x \in \mathbb{R}^3$. Through the isometry $y \to x_1 + x_2 - y$, using that $\partial_i \Phi$ is odd and $n(y) = -n(x_1 + x_2 - y)$, we observe that

$$\begin{aligned} \int_{\partial B_{R_0}(\bar{x})} \partial_i \Phi(x_1 - y) n_j(y)\,dy &= \int_{\partial B_{R_0}(\bar{x})} \partial_i \Phi(y - x_2) n_j(x_1 + x_2 - y)\,dy \\ &= \int_{\partial B_{R_0}(\bar{x})} \partial_i \Phi(x_2 - y) n_j(y)\,dy\,. \end{aligned}$$

Hence, we rewrite the incremental quotient as

$$\begin{aligned} & p(x_1) - p(x_2) \\ & \quad = \int_{B_{R_0}(\bar{x})} \big[\partial^2_{ij}\Phi(x_1 - y) - \partial^2_{ij}\Phi(x_2 - y)\big]\,\big(v^i(y) - v^i(x_1)\big)\big(w^j(y) - w^j(x_2)\big)\,dy\,. \end{aligned}$$

Splitting the contributions of $y \in B_\lambda(\bar{x})$ and $y \in B^c_\lambda(\bar{x})$,

$$
\begin{aligned}
&|p(x_1) - p(x_2)| \\
&\quad \le \int_{(B_{R_0} \setminus B_\lambda)(\bar{x})} \left|\partial^2_{ij}\Phi(x_1 - y) - \partial^2_{ij}\Phi(x_2 - y)\right| \left|v^i(y) - v^i(x_1)\right| \left|w^j(y) - w^j(x_2)\right| dy \\
&\quad + \int_{B_\lambda(\bar{x})} \left(\left|D^2\Phi(x_1 - y)\right| + \left|D^2\Phi(x_2 - y)\right|\right) \left|v^i(y) - v^i(x_1)\right|\left|w^j(y) - w^j(x_2)\right| dy \\
&\quad = I + II
\end{aligned} \tag{2.18}
$$

Using the decay of $\left|D^2\Phi\right|$ we estimate the second integral in the right-hand side of (2.18) with

$$
\begin{aligned}
II &\le [v]_{C^\beta}[w]_{C^\gamma} \int_{B_\lambda(\bar{x})} \left[\lambda^\gamma \left|D^2\Phi(x_1 - y)\right||x_1 - y|^\beta + \lambda^\beta \left|D^2\Phi(x_2 - y)\right||x_2 - y|^\gamma\right] dy \\
&\le C[v]_{C^\beta}[w]_{C^\gamma} \left(\int_{B_{2\lambda}(x_1)} \frac{\lambda^\gamma}{|x_1 - y|^{3-\beta}} dy + \int_{B_{2\lambda}(x_2)} \frac{\lambda^\beta}{|x_2 - y|^{3-\gamma}} dy \right) \\
&\le C\lambda^{\beta+\gamma}[v]_{C^\beta}[w]_{C^\gamma}.
\end{aligned}
$$

By the decay of $|D^3\Phi|$, in particular since for every point $\tilde{x} \in B_{\lambda/2}(\bar{x})$ and for every $y \in B^c_\lambda(\bar{x})$ we have $|\tilde{x} - y| \ge |\bar{x} - y| - |\tilde{x} - \bar{x}| \ge \frac{|\bar{x}-y|}{2}$ and $\left|D^3\Phi(\tilde{x} - y)\right| \le \frac{C}{|\tilde{x}-y|^4} \le \frac{C}{|\bar{x}-y|^4}$, we have

$$
\begin{aligned}
I &\le \lambda [v]_{C^\beta}[w]_{C^\gamma} \int_{B^c_\lambda(\bar{x})} \left(\int_0^1 \left|D^3\Phi(tx_1 + (1-t)x_2 - y)\right| dt \right) |x_1 - y|^\beta |x_2 - y|^\gamma dy \\
&\le C\lambda [v]_{C^\beta}[w]_{C^\gamma} \int_{B^c_\lambda(\bar{x})} \frac{|x_1 - y|^\beta |x_2 - y|^\gamma}{|\bar{x} - y|^4} dy \\
&\le C\lambda [v]_{C^\beta}[w]_{C^\gamma} \int_{B^c_\lambda(\bar{x})} \frac{1}{|\bar{x} - y|^{4-\beta-\gamma}} dy \le C\lambda^{\beta+\gamma}[v]_{C^\beta}[w]_{C^\gamma}.
\end{aligned}
$$

This concludes the proof of (i) (notice that in the last line we used that $\beta + \gamma < 1$).

(ii) If $\beta + \gamma \in \left(\frac{1}{2}, 1\right)$ we have that for every partial derivative ∂_k and for every given $x \in \mathbb{R}^3$

$$
-\Delta\partial_k p(y) = \partial^3_{ijk}(v^i(y)w^j(y)) = \partial^2_{ij}\partial_k\big((v^i(y) - v^i(x))(w^j(y) - w^j(x))\big).
$$

Since $\partial_k\big((v^i - v^i(x))(w^j - w^j(x))\big)$ is compactly supported we can use again the representation formula (2.13) getting

$$
\partial_k p(x) = \int_{B_{R_0}(x_0)} \partial^2_{ij}\Phi(x - y)\partial_k\big((v^i(y) - v^i(x))(w^j(y) - w^j(x))\big) dy
$$

Integrating by parts (this can be easily justified approximating u with smooth functions) and letting $R_0 \to \infty$ we obtain

$$\partial_k p(x) = \int_{\mathbb{R}^3} \partial^3_{ijk}\Phi(x-y)(v^i(y)-v^i(x))(w^j(y)-w^j(x))\,dy\,. \tag{2.19}$$

For every $x_1, x_2 \in \mathbb{R}^3$ we define $\bar{x} = \frac{x_1+x_2}{2}$, $\lambda = |x_1 - x_2|$ and we write

$$\begin{aligned}
\partial_k p(x_1) - \partial_k p(x_2) &= \int_{B_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_1-y)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_1))\,dy \\
&- \int_{B_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_2-y)(v^i(y)-v^i(x_2))(w^j(y)-w^j(x_2))\,dy \\
&+ \int_{B^c_\lambda(\bar{x})} \big(\partial^3_{ijk}\Phi(x_1-y) - \partial^3_{ijk}\Phi(x_2-y)\big)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_1))\,dy \\
&+ \int_{B^c_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_2-y)(v^i(y)-v^i(x_2))(w^j(x_2)-w^j(x_1))\,dy \\
&+ \int_{B^c_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_2-y)(v^i(x_2)-v^i(x_1))(w^j(y)-w^j(x_1))\,dy\,,
\end{aligned}$$

and, arguing as in the proof for $\beta+\gamma<1$, it is easy to see that each of the above integrals is estimated by $C\lambda^{\beta+\gamma-1}[v]_{C^\beta}[w]_{C^\gamma}$ from which the estimate on p in (ii) follows.

(iii) We note that for every choice of x_1, x_2, x_0 we can write $q = q^1 + q^2$, where

$$\begin{aligned}
-\Delta q^1 &= \partial^2_{ij}\partial_k\big((v^i - v^i(x_1))(w^j - w^j(x_2))(z^k - z^k(x_0))\big) \\
-\Delta q^2 &= \partial^3_{ijk}\big((v^i(x_1)w^j z^k\big) + \partial^3_{ijk}\big((v^i w^j(x_2) z^k\big) + \partial^3_{ijk}\big((v^i w^j z^k(x_0)\big)\,.
\end{aligned}$$

Since the right hand side of the Poisson equation for q^2 has exactly the same structure of the one for $\partial_k p$ (the only difference are the constants but they do not play any role and they can be estimated by their respective C^0 norm) in the previous computations, we can infer that q^2 enjoys the estimate (2.15). For q^1 we can use the same formula as in (2.19) choosing $x_0 = x_m$ when we have to evaluate $q^1(x_m)$. Thus for $m = 1,2$ we can write

$$q^1(x_m) = \int_{\mathbb{R}^3} \partial^3_{ijk}\Phi(x_m-y)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_2))(z^k(y)-z^k(x_m))\,dy$$

and again, letting $\lambda = |x_1 - x_2|$, $\bar{x} = \frac{x_1+x_2}{2}$ and splitting the contributions in $B_\lambda(\bar{x})$ and $B^c_\lambda(\bar{x})$ we write

$$
\begin{aligned}
&q^1(x_1) - q^1(x_2)\\
&= \int_{B_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_1-y)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_2))(z^k(y)-z^k(x_1))\,dy\\
&- \int_{B_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_2-y)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_2))(z^k(y)-z^k(x_2))\,dy\\
&+ \int_{B^c_\lambda(\bar{x})} \partial^3_{ijk}\Phi(x_1-y)(v^i(y)-v^i(x_1))(w^j(y)-w^j(x_2))(z^k(x_2)-z^k(x_1))\,dy\\
&+ \int_{B^c_\lambda(\bar{x})} \left(\partial^3_{ijk}\Phi(x_1-y)-\partial^3_{ijk}\Phi(x_2-y)\right)(v^i(y)-v^i(x_1))\\
&\qquad\qquad (w^j(y)-w^j(x_2))(z^k(y)-z^k(x_2))\,dy\,.
\end{aligned}
$$

We estimate each term in the same spirit as the previous computations to get

$$
\begin{aligned}
&|q^1(x_1)-q^1(x_2)|\\
&\le C\|v\|_{C^\beta}\|w\|_{C^0}\|z\|_{C^\gamma}\int_{B_\lambda(\bar{x})}\frac{1}{|x_1-y|^{4-\beta-\gamma}}\,dy + C\|v\|_{C^0}\|w\|_{C^\beta}\|z\|_{C^\gamma}\\
&\times\left(\int_{B_\lambda(\bar{x})}\frac{1}{|x_2-y|^{4-\beta-\gamma}}\,dy + \lambda^\gamma\int_{B^c_\lambda(\bar{x})}\frac{1}{|\bar{x}-y|^{4-\beta}}\,dy + \lambda\int_{B^c_\lambda(\bar{x})}\frac{1}{|\bar{x}-y|^{5-\beta-\gamma}}\,dy\right)\\
&\le C\lambda^{\beta+\gamma-1}\left(\|v\|_{C^0(\mathbb{R}^3)}\|w\|_{C^\beta(\mathbb{R}^3)}\|z\|_{C^\gamma(\mathbb{R}^3)} + \|v\|_{C^\beta(\mathbb{R}^3)}\|w\|_{C^0(\mathbb{R}^3)}\|z\|_{C^\gamma(\mathbb{R}^3)}\right),
\end{aligned}
$$

which concludes the proof of (ii). □

In order to adapt the previous proposition to periodic solutions in $\mathbb{R}^3$ (thus without any decay at infinity) we will use the following lemma.

Lemma 2.4. *Let $\theta \in (0,1)$. For any $u \in C^\theta(\mathbb{T}^3)$ such that $\operatorname{div} u = 0$, there exists a vector field $\tilde{u} \in C^\theta(\mathbb{R}^3)$ compactly supported in $B_{12}(0)$ and a positive constant $C_\theta > 0$ such that $\operatorname{div}\tilde{u} = 0$, $\tilde{u} \equiv u$ in $B_6(0)$ and*

$$\|\tilde{u}\|_{C^\theta(\mathbb{R}^3)} \le C_\theta \|u\|_{C^\theta(\mathbb{T}^3)}\,. \tag{2.20}$$

Proof. Assume for now that $\int_{\mathbb{T}^3} u = 0$. Since $\operatorname{div} u = 0$ on $\mathbb{T}^3$ then there exists a vector potential $A : \mathbb{T}^3 \to \mathbb{R}^3$ such that $u = \operatorname{curl} A$ and $-\Delta A = \operatorname{curl} u$. Moreover by Schauder estimates we have $\|A\|_{C^{1,\theta}(\mathbb{T}^3)} \le C_\theta\|u\|_{C^\theta(\mathbb{T}^3)}$. Now think A to be defined periodically to the whole space $\mathbb{R}^3$. Choose a smooth cut-off function $0 \le \varphi \le 1$ such that $\operatorname{supp}\varphi \subset B_{12}$, $\varphi \equiv 1$ on B_6 and $\|\varphi\|_{C^2} \le C$. Define $\tilde{u} = \operatorname{curl}\tilde{A}$ where $\tilde{A} = A\varphi_R$. Trivially $\operatorname{div}\tilde{u} = 0$ and we also have the following estimate

$$\|\tilde{u}\|_{C^\theta(\mathbb{R}^3)} \le \|\tilde{A}\|_{C^{1,\theta}(\mathbb{R}^3)} \le C\|\varphi\|_{C^{1,\theta}(\mathbb{R}^3)}\|A\|_{C^{1,\theta}(\mathbb{T}^3)} \le C_\theta\|u\|_{C^\theta(\mathbb{T}^3)}\,.$$

Moreover $\tilde{u}$ satisfies $\tilde{u} = \operatorname{curl}\tilde{A} = \operatorname{curl} A = u$ in $B_6(0)$. When the average of u is not zero, one can repeat the proof with the only difference that $u = \operatorname{curl} A + \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} u$ in this case. □

Note that the choice of $B_6(0)$ in the previous lemma is to ensure that the cube (and thus the torus) $[-\pi,\pi]^3 \subset B_6(0)$. Since we will work on functions u,p that solve (2.14) in $\mathbb{T}^3$, we can take the extension $\tilde{u}$ given by Lemma 2.4 and define $\tilde{p},\tilde{q}$ as

$$-\Delta\tilde{p} = \operatorname{div}\operatorname{div}(\tilde{v}\otimes\tilde{w}) \quad \text{in } \mathbb{R}^3.$$

$$-\Delta\tilde{q} = \operatorname{div}\operatorname{div}\operatorname{div}(\tilde{v}\otimes\tilde{w}\otimes\tilde{z}) \quad \text{in } \mathbb{R}^3.$$

Thus we can write $p = p-\tilde{p}+\tilde{p}$ and $q = q-\tilde{q}+\tilde{q}$, where $\tilde{p}$ and $\tilde{q}$ satisfy Proposition 2.3, while $p-\tilde{p}$ and $q-\tilde{q}$ are harmonic in $B_6(0)$. Thus we have the following

Corollary 2.5. *If $v,w,z\in C^0(\mathbb{T}^3)$, then Proposition 2.3 holds also for the (unique) zero-average solutions p and q of* (2.14) *in* $\mathbb{T}^3$.

2.3 Velocity and pressure regularity

Here we prove Theorem 2.1.

Time regularity of u

To prove (2.2), it is enough to show that u is $\theta-$Hölder in time, uniformly in space. For any $s,t\in[0,T]$ we estimate $v\mu$

$$|u(t,x)-u(s,x)| \le |u(t,x)-u_\delta(t,x)| + |u_\delta(t,x)-u_\delta(s,x)| + |u_\delta(s,x)-u(s,x)|. \tag{2.21}$$

Using (B.5) we get

$$|u(t,x)-u_\delta(t,x)| \le C\delta^\theta \|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3)))} \qquad \forall t\in[0,T],$$

thus we are only left with the second term in the right hand side of (2.21). Using the equation (2.9), the estimates (2.12) and (B.4), Theorem C.1, we have

$$\begin{aligned}
|u_\delta(t,x)-u_\delta(s,x)| &\le |t-s|\|\partial_t u_\delta\|_{L^\infty((0,T)\times\mathbb{T}^3)} \\
&\le |t-s|\left(\|\operatorname{div}(u\otimes u)_\delta\|_{L^\infty} + \|\nabla p_\delta\|_{L^\infty} + \mu\,\|(-\Delta)^\alpha u_\delta\|_{L^\infty}\right) \\
&\le C|t-s|\left(\delta^{\theta-1}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} + \mu\|u_\delta\|_{L^\infty((0,T);C^{2\alpha+\varepsilon}(\mathbb{T}^3))}\right),
\end{aligned}$$

Since $\alpha\in\left(0,\frac{1}{2}\right)$ we can choose ε such that $2\alpha+\varepsilon<1$, getting

$$|u_\delta(t,x)-u_\delta(s,x)| \le C|t-s|\delta^{\theta-1}\left(\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} + \mu\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\right),$$

Finally we choose $\delta = |t-s|$, from which the claim follows.

Space regularity for p, for $\theta \in (0,1)$

Estimates (2.3) and (2.5) follow from Corollary 2.5.

Time regularity for p, for $\theta < \frac{1}{2}$

For any $s,t \in [0,T]$, such that $|t-s| = \delta < 1$ we estimate via the triangular inequality and thanks to the space regularity of p and (B.5)

$$\begin{aligned} |p(t,x) - p(s,x)| &\leq 2 \sup_{t\in[0,T]} |p(t,x) - p_\delta(t,x)| + |p_\delta(t,x) - p_\delta(s,x)| \\ &\leq C\delta^{2\theta} \|p\|_{L^\infty((0,T);C^{2\theta}(\mathbb{T}^3))} + |p_\delta(t,x) - p_\delta(s,x)| \\ &\leq C\delta^{2\theta} \|u\|^2_{L^\infty((0,T);C^{\theta}(\mathbb{T}^3))} + |p_\delta(t,x) - p_\delta(s,x)| \,. \end{aligned} \tag{2.22}$$

To estimate the last term we consider the equation solved by p_δ

$$-\Delta p_\delta = \operatorname{div}\operatorname{div}((u\otimes u)_\delta) = \operatorname{div}\operatorname{div}(u_\delta \otimes u_\delta - R_\delta)$$

and hence the one for $p_\delta(t,\cdot) - p_\delta(s,\cdot)$

$$\begin{aligned} -\Delta(p_\delta(t,x) - p_\delta(s,x)) &= \operatorname{div}\operatorname{div}(R_\delta(s,x) - R_\delta(t,x) + u_\delta(t,x)\otimes u_\delta(t,x) - u_\delta(s,x)\otimes u_\delta(s,x)) \\ &= \operatorname{div}\operatorname{div}\left[R_\delta(s,x) - R_\delta(t,x) + \int_s^t \left(\frac{d}{dr}u_\delta(r,x)\otimes u_\delta(r,x) + u_\delta(r,x)\otimes \frac{d}{dr}u_\delta(r,x)\right) dr\right] \\ &= \operatorname{div}\operatorname{div}\left[R_\delta(s,x) - R_\delta(t,x) + \int_s^t \big((\operatorname{div}(u_\delta\otimes u_\delta) - \nabla p_\delta - \operatorname{div} R_\delta - \mu(-\Delta)^\alpha u_\delta)\otimes u_\delta \right. \\ &\quad \left. + u_\delta \otimes (\operatorname{div}(u_\delta\otimes u_\delta) - \nabla p_\delta - \operatorname{div} R_\delta - \mu(-\Delta)^\alpha u_\delta)\big)\, dr\right] . \end{aligned} \tag{2.23}$$

Defining the commutator

$$T^\alpha(u_\delta) = (-\Delta)^\alpha(u_\delta\otimes u_\delta) - (-\Delta)^\alpha u_\delta \otimes u_\delta - u_\delta \otimes (-\Delta)^\alpha u_\delta \,, \tag{2.24}$$

and denoting by $p^1_{s,t}$, $p^2_{s,t}$, $p^3_{s,t}, p^4_{s,t}, p^5_{s,t}$ the unique 0-average solutions of

$$-\Delta p^1_{s,t} = \operatorname{div}\operatorname{div}(R_\delta(s,x) - R_\delta(t,x)) \,,$$

$$\Delta p^2_{s,t} = \int_s^t \operatorname{div}\operatorname{div}((\operatorname{div} R_\delta + \nabla p_\delta)\otimes u_\delta + u_\delta \otimes (\operatorname{div} R_\delta + \nabla p_\delta))\, dr \,,$$

$$-\Delta p^3_{s,t} = \int_s^t \operatorname{div}\operatorname{div}(\operatorname{div}(u_\delta\otimes u_\delta)\otimes u_\delta + u_\delta \otimes \operatorname{div}(u_\delta\otimes u_\delta))\, dr \,,$$

$$-\Delta p^4_{s,t} = \mu \int_s^t \operatorname{div}\operatorname{div} T^\alpha(u_\delta)\, dr\,,$$

$$\Delta p^5_{s,t} = \mu \int_s^t \operatorname{div}\operatorname{div} (-\Delta)^\alpha (u_\delta \otimes u_\delta)\, dr\,,$$

we have that

$$p_\delta(t,x) - p_\delta(s,x) = p^1_{s,t} + p^2_{s,t} + p^3_{s,t} + p^4_{s,t} + p^5_{s,t}\,.$$

By Schauder estimates, estimating R_δ by (B.2) and p_δ by (B.4) and thanks to the space regularity of p proved above in (2.3), $p^1_{s,t}$ and $p^2_{s,t}$ enjoy the estimate

$$\begin{aligned}\|p^1_{s,t}\|_{L^\infty(\mathbb{T}^3)} &\le \|p^1_{s,t}\|_{C^\varepsilon(\mathbb{T}^3)} \le C\big(\|R_\delta(t,\cdot)\|_{C^\varepsilon(\mathbb{T}^3)} + \|R_\delta(s,\cdot)\|_{C^\varepsilon(\mathbb{T}^3)}\big)\\ &\le C\delta^{2\theta-\varepsilon}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))},\end{aligned}$$

$$\begin{aligned}\|p^2_{s,t}\|_{L^\infty(\mathbb{T}^3)} &\le \|p^2_{s,t}\|_{C^\varepsilon(\mathbb{T}^3)} \le C|t-s|\|(\operatorname{div} R_\delta + \nabla p_\delta)\otimes u_\delta\|_{L^\infty((0,T);C^\varepsilon(\mathbb{T}^3))}\\ &\le C|t-s|\Big(\|R_\delta\|_{L^\infty((0,T);C^{1,\varepsilon}(\mathbb{T}^3))} + \|p_\delta\|_{L^\infty((0,T);C^{1,\varepsilon}(\mathbb{T}^3))}\Big)\|u_\delta\|_{L^\infty((0,T);C^\varepsilon(\mathbb{T}^3))}\\ &\le C|t-s|\delta^{2\theta-\varepsilon-1}\|u\|^3_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\,.\end{aligned}$$

Note that $p^3_{s,t}$ is the integral in time of $\Delta^{-1}\operatorname{div}\operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta\otimes u_\delta)$, which from Corollary 2.5 and (B.3) satisfies

$$\begin{aligned}\|p^3_{s,t}\|_{L^\infty(\mathbb{T}^3)} \le \|p^3_{s,t}\|_{C^\varepsilon(\mathbb{T}^3)} &\le C|t-s|\|u_\delta\|_{L^\infty((0,T)\times\mathbb{T}^3)}\|u_\delta\|^2_{L^\infty((0,T);C^{\frac{1+\varepsilon}{2}}(\mathbb{T}^3))}\\ &\le C|t-s|\|u\|_{L^\infty((0,T)\times\mathbb{T}^3)}\big(\delta^{\theta-\frac{1+\varepsilon}{2}}\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\big)^2\\ &\le C|t-s|\delta^{2\theta-1-\varepsilon}\|u\|^3_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\,.\end{aligned}$$

Choosing ε such that $\frac{\varepsilon}{2} < \alpha$, by Schauder estimates and (C.4) we have

$$\begin{aligned}\|p^4_{s,t}\|_{L^\infty(\mathbb{T}^3)} &\le \|p^4_{s,t}\|_{C^\varepsilon(\mathbb{T}^3)}\\ &\le C|t-s|\|T^\alpha(u_\delta)\|_{L^\infty((0,T);C^\varepsilon(\mathbb{T}^3))}\\ &\le C|t-s|\|u_\delta\|^2_{L^\infty((0,T);C^{\alpha+\varepsilon/2}(\mathbb{T}^3))}\,.\end{aligned}$$

To estimate $p^5_{s,t}$ we note that every solution of $\Delta q = \operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta)$ enjoy the estimate (by Proposition 2.5) $\|q\|_{L^\infty((0,T);C^{2\alpha+\varepsilon}(\mathbb{T}^3))} \le C\|u_\delta\|^2_{L^\infty((0,T);C^{\alpha+\varepsilon/2}(\mathbb{T}^3))}$, and since $p^5_{s,t} = \int_s^t (-\Delta)^\alpha q\, dr$, by Theorem C.1 we infer

$$\|p^5_{s,t}\|_{L^\infty(\mathbb{T}^3)} \le C|t-s|\|q\|_{L^\infty((0,T);C^{2\alpha+\varepsilon}(\mathbb{T}^3))} \le C|t-s|\|u_\delta\|^2_{L^\infty((0,T);C^{\alpha+\varepsilon/2}(\mathbb{T}^3))}\,.$$

In the case $\alpha < \theta$, if ε is sufficiently small, we have (since $\delta < 1$)

$$\|u_\delta\|^2_{L^\infty((0,T);C^{\alpha+\varepsilon/2}(\mathbb{T}^3))} \le \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} \le \delta^{2\theta-1}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\,,$$

while, if $\alpha \geq \theta$, using (B.3) we have

$$\|u_\delta\|^2_{L^\infty((0,T);C^{\alpha+\varepsilon/2}(\mathbb{T}^3))} \leq \delta^{2(\theta-\alpha)-\varepsilon}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} \leq \delta^{2\theta-1}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

Thus we deduce

$$\|p^4_{s,t}\|_{L^\infty(\mathbb{T}^3)} + \|p^5_{s,t}\|_{L^\infty(\mathbb{T}^3)} \leq C|t-s|\delta^{2\theta-1}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

Since $\delta = |t-s|$ we conclude that $|p_\delta(t,x) - p_\delta(s,x)| \leq C|t-s|^{2\theta-\varepsilon}$, so that (2.4) holds true.

Time regularity for ∇p, for $\theta > \frac{1}{2}$

By the equation solved by p, for $0 < s < t$ we have

$$\begin{aligned}-\Delta(p(t)-p(s)) &= \partial^2_{ij}(u^i(t)u^j(t) - u^i(s)u^j(s)) \\ &= \partial^2_{ij}\big((u^i(t)-u^i(s))u^j(t) + u^i(s)(u^j(t)-u^j(s))\big)\end{aligned}$$

By Corollary 2.5 we can apply Proposition 2.3 (ii) with $\beta = 1-\theta+\varepsilon$ and $\gamma = \theta$ to obtain

$$\|\nabla p(t) - \nabla p(s)\|_{C^\varepsilon(\mathbb{T}^3)} \leq C\|u(t)-u(s)\|_{C^{1-\theta+\varepsilon}(\mathbb{T}^3)}\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

Interpolating the $C^{1-\theta+\varepsilon}$-norm between C^0 and C^θ (since $\theta > \frac{1}{2}$) and by the time regularity of u in (2.2), we have

$$\begin{aligned}\|u(t)-u(s)\|_{C^{1-\theta+\varepsilon}(\mathbb{T}^3)} &\leq \|u(t)-u(s)\|^{\frac{1-\theta+\varepsilon}{\theta}}_{C^\theta(\mathbb{T}^3)}\|u(t)-u(s)\|^{\frac{2\theta-1-\varepsilon}{\theta}}_{C^0(\mathbb{T}^3)} \\ &\leq \|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}|t-s|^{2\theta-1-\varepsilon} \\ &\leq \left(\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))} + \|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}\right)|t-s|^{2\theta-1-\varepsilon},\end{aligned}$$

which proves that for any $x \in \mathbb{T}^3$

$$\big|\nabla p(t,x) - \nabla p(s,x)\big| \leq C|t-s|^{2\theta-1-\varepsilon} \tag{2.25}$$

Space regularity for $\partial_t p$, for $\theta > \frac{1}{2}$

With the previous arguments, $p \in C^{0,1}([0,T]\times\mathbb{T}^3)$. Hence $\partial_t p \in L^\infty$ and we can look at the equation solved (in distributional sense) by it, obtained by differentiating in time (2.11)

$$-\Delta\partial_t p = \operatorname{div}\operatorname{div}\partial_t(u\otimes u).$$

Note that, defining $T^\alpha(u_\delta)$ as in (2.24), for every $\delta > 0$ we have

$$\begin{aligned}\partial_t(u_\delta\otimes u_\delta) &= \partial_t u_\delta \otimes u_\delta + u_\delta \otimes \partial_t u_\delta \\ &= -\operatorname{div}(u_\delta\otimes u_\delta\otimes u_\delta) + \operatorname{div} R_\delta \otimes u_\delta + u_\delta \otimes \operatorname{div} R_\delta \\ &\quad - \nabla p_\delta \otimes u_\delta - u_\delta \otimes \nabla p_\delta + \mu T^\alpha(u_\delta) - \mu(-\Delta)^\alpha(u_\delta\otimes u_\delta)\end{aligned}$$

and, since by (B.2) $\operatorname{div} R_\delta \to 0$ uniformly and by Proposition C.3, $T^\alpha(u_\delta) \to T^\alpha(u)$ uniformly, we have that $\partial_t p$ solves distributionally

$$\Delta \partial_t p = \operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) + \operatorname{div}\operatorname{div}\big(2\nabla p\otimes u - \mu T^\alpha(u) + \mu(-\Delta)^\alpha(u\otimes u)\big)\,. \tag{2.26}$$

Hence we can write $\partial_t p = q^1+q^2+q^3+q^4$, where

$$\Delta q^1 = \operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) \qquad \Delta q^2 = 2\operatorname{div}\operatorname{div}(\nabla p\otimes u)$$

$$\Delta q^3 = \mu\operatorname{div}\operatorname{div} T^\alpha(u) \qquad -\Delta q^4 = \mu\operatorname{div}\operatorname{div}(-\Delta)^\alpha(u\otimes u)$$

In turn by the estimate on $q=q^1$ from Corollary 2.5

$$\|q^1(t)\|_{C^{2\theta-1}(\mathbb{T}^3)} \le C\|u(t)\|^3_{C^\theta(\mathbb{T}^3)}$$

and by Schauder estimates, together with the regularity of p, we have

$$\begin{aligned}\|q^2(t)\|_{C^{2\theta-1}(\mathbb{T}^3)} &\le C\|\nabla p\otimes u\|_{C^{2\theta-1}(\mathbb{T}^3)}\\ &\le C\|\nabla p(t)\|_{C^{2\theta-1}(\mathbb{T}^3)}\|u(t)\|_{C^\theta(\mathbb{T}^3)} \le C\|u(t)\|^3_{C^\theta(\mathbb{T}^3)}\,.\end{aligned}$$

Applying (C.4) and (C.5) with $\beta = 2(\theta-\alpha)$ (choosing ε small enough such that $\theta < 1-\varepsilon$) and by Schauder estimates we deduce

$$\|q^3(t)\|_{C^{2\theta-1}(\mathbb{T}^3)} \le C\|T^\alpha(u)(t)\|_{C^{2\theta-1}(\mathbb{T}^3)} \le C\|u(t)\|^2_{C^\theta(\mathbb{T}^3)} \le C\|u(t)\|^2_{C^\theta(\mathbb{T}^3)}\,.$$

Notice that $q^4 = \mu(-\Delta)^\alpha p$, thus by (2.5) we have

$$\|q^4(t)\|_{C^{2\theta-1}(\mathbb{T}^3)} \le C\|p(t)\|_{C^{1,2\theta-1}(\mathbb{T}^3)} \le C\|u(t)\|^2_{C^\theta(\mathbb{T}^3)}\,.$$

Time regularity for $\partial_t p$, for $\theta > \frac{1}{2}$

For any $0<s<t$, by the equation for $\partial_t p$ in (2.26), we have that $\partial_t p(t,x) - \partial_t p(s,x) = p^1_{s,t} + p^2_{s,t} + p^3_{s,t}$ where $p^1_{s,t}, p^2_{s,t}$, and $p^3_{s,t}$ are the unique 0-average solutions in $\mathbb{T}^3$ of

$$\begin{aligned}\Delta p^1_{s,t} &= \operatorname{div}\operatorname{div}\operatorname{div}\big(u(t)\otimes u(t)\otimes u(t) - u(s)\otimes u(s)\otimes u(s)\big)\\ &= \operatorname{div}\operatorname{div}\operatorname{div}\big((u(t)-u(s))\otimes u(t)\otimes u(t) + u(s)\otimes(u(t)-u(s))\otimes u(t)\\ &\quad + u(s)\otimes u(s)\otimes(u(t)-u(s))\big)\end{aligned}$$

$$\Delta p^2_{s,t} = \operatorname{div}\operatorname{div}\big(2\nabla p(t)\otimes u(t) - 2\nabla p(s)\otimes u(s) - \mu T^\alpha(u(t)) + \mu T^\alpha(u(s))\big)$$

$$\Delta p^3_{s,t} = \mu(-\Delta)^\alpha \operatorname{div}\operatorname{div}\big(u(t)\otimes u(t) - u(s)\otimes u(s)\big)\,.$$

To estimate $p^1_{s,t}$, for any ε small we apply Corollary 2.5, with particular reference to Proposition 2.3 (iii), with $\beta = 1-\theta+\varepsilon$ and $\gamma = \theta$, in such a way that the factor $u(t)-u(s)$ gets each time only the $C^{1-\theta+\varepsilon}$ norm and not the C^θ norm. We obtain that

$$\|p^1_{s,t}\|_{L^\infty(\mathbb{T}^3)} \le \|p^1_{s,t}\|_{C^\varepsilon(\mathbb{T}^3)} \le C\|u(t)-u(s)\|_{C^{1-\theta+\varepsilon}(\mathbb{T}^3)}\|u\|^2_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

Next, we interpolate the $C^{1-\theta+\varepsilon}$ norm between C^0 and C^θ and finally we use the C^θ regularity in time of u to obtain that

$$\begin{aligned}\|u(t)-u(s)\|_{C^{1-\theta+\varepsilon}(\mathbb{T}^3)} &\le C\|u(t)-u(s)\|_{C^\theta(\mathbb{T}^3)}^{\frac{1-\theta+\varepsilon}{\theta}}\|u(t)-u(s)\|_{C^0(\mathbb{T}^3)}^{\frac{2\theta-1-\varepsilon}{\theta}}\\ &\le C\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}^{\frac{1-\theta+\varepsilon}{\theta}}|t-s|^{2\theta-1-\varepsilon}\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}^{\frac{2\theta-1-\varepsilon}{\theta}}\\ &\le C|t-s|^{2\theta-1-\varepsilon}\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}.\end{aligned}$$

Notice that

$$T^\alpha(u(t))-T^\alpha(u(s)) = T^\alpha\big(u(t)-u(s),u(t)\big)+T^\alpha\big(u(s),u(t)-u(s)\big).$$

Now if $2\alpha > \theta$ we use part (i) of Proposition C.3 with $k_1 = 2\alpha-\theta+\frac{\varepsilon}{2}$, $k_2 = \theta-\frac{\varepsilon}{2}$, $\beta=\varepsilon$ and we get

$$\|T^\alpha\big(u(t)-u(s),u(t)\big)\|_{C^\varepsilon(\mathbb{T}^3)} \le C\|u(t)-u(s)\|_{C^{2\alpha-\theta+\varepsilon}(\mathbb{T}^3)}\|u(t)\|_{C^\theta(\mathbb{T}^3)},$$

while if $2\alpha\le\theta$ we choose $k_1=\varepsilon$, $k_2 = 2\alpha-\varepsilon$, $\beta=\varepsilon$, getting

$$\begin{aligned}\|T^\alpha\big(u(t)-u(s),u(t)\big)\|_{C^\varepsilon(\mathbb{T}^3)} &\le C\|u(t)-u(s)\|_{C^{3\varepsilon/2}(\mathbb{T}^3)}\|u(t)\|_{C^{2\alpha-\varepsilon/2}(\mathbb{T}^3)}\\ &\le C\|u(t)-u(s)\|_{C^{2\varepsilon}(\mathbb{T}^3)}\|u(t)\|_{C^\theta(\mathbb{T}^3)}.\end{aligned}$$

Interpolating again between C^0 and C^θ and using the Hölder regularity of u in time, we obtain

$$\|T^\alpha\big(u(t)-u(s),u(t)\big)\|_{C^{\varepsilon/2}(\mathbb{T}^3)} \le C|t-s|^{2\theta-1-\varepsilon}\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

Summarizing we achieved

$$\|T^\alpha(u(t))-T^\alpha(u(s))\|_{C^{\varepsilon/2}(\mathbb{T}^3)} \le C|t-s|^{2\theta-1-\varepsilon}\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}\|u\|_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}. \tag{2.27}$$

Moreover by interpolation we estimate

$$\begin{aligned}\|\nabla p(t)\otimes u(t)-\nabla p(s)\otimes u(s)\|_{C^{\varepsilon/2}(\mathbb{T}^3)} &\le \|\nabla p(t)\otimes u(t)-\nabla p(s)\otimes u(s)\|_{C^0(\mathbb{T}^3)}^{1-\frac{\varepsilon}{2(2\theta-1-\varepsilon/2)}}\\ &\quad\|\nabla p(t)\otimes u(t)-\nabla p(s)\otimes u(s)\|_{C^{2\theta-1-\varepsilon/2}(\mathbb{T}^3)}^{\frac{\varepsilon}{2(2\theta-1-\varepsilon/2)}}\\ &\le C|t-s|^{2\theta-1-\varepsilon}\|\nabla p\otimes u\|_{C^{2\theta-1-\varepsilon/2}([0,T]\times\mathbb{T}^3)}\\ &\le C|t-s|^{2\theta-1-\varepsilon}\|\nabla p\|_{C^{2\theta-1-\varepsilon/2}([0,T]\times\mathbb{T}^3)}\|u\|_{C^\theta([0,T]\times\mathbb{T}^3)}.\end{aligned} \tag{2.28}$$

By Schauder estimates, together with (2.27) and (2.28), we conclude

$$\|p^2_{s,t}\|_{C^{\varepsilon/2}(\mathbb{T}^3)} \leq C|t-s|^{2\theta-1-\varepsilon}.$$

We note that $p^3_{s,t} = -\mu(-\Delta)^\alpha(p(t)-p(s))$, thus by Theorem C.1 and (2.25) we have

$$\|p^3_{s,t}\|_{C^{\varepsilon}(\mathbb{T}^3)} \leq C\|p(t)-p(s)\|_{C^1(\mathbb{T}^3)} \leq C|t-s|^{2\theta-1-\varepsilon},$$

From which we conclude

$$[\partial_t p(x)]_{C^{2\theta-1-\varepsilon}([0,T])} \leq C\|u\|^3_{L^\infty((0,T);C^\theta(\mathbb{T}^3))}.$$

2.4 Energy regularity

Here we prove Theorem 2.2.

The case $\mu = 0$ (Euler)

Let $s,t \in [0,T]$. We wish to find a proper estimate for $|e_u(t)-e_u(s)|$. To do this we split it in three terms as follows d

$$|e_u(t)-e_u(s)| \leq |e_u(t)-e_{u_\delta}(t)| + |e_{u_\delta}(t)-e_{u_\delta}(s)| + |e_{u_\delta}(s)-e_u(s)|, \tag{2.29}$$

for some parameter $\delta > 0$ that will be fixed at the end of the proof. Assume that $u \in L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))$. Using (B.1) and (B.8) with $r = \frac{3}{2}$ we can estimate

$$|e_u(t)-e_{u_\delta}(t)| \leq \frac{1}{2}\int_{\mathbb{T}^3} \left|(|u|^2)_\delta - |u_\delta|^2\right|(x,t)\,dx \leq C\delta^{2\theta}[u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}.$$

We are now left with the second therm in the right hand side of (2.29). By (2.10) we get

$$\begin{aligned}|e_{u_\delta}(t)-e_{u_\delta}(s)| &\leq |t-s|\left\|\frac{de_{u_\delta}}{dt}\right\|_{L^\infty(0,T)} \\ &\leq C|t-s|\|R_\delta\|_{L^\infty((0,T);L^{3/2}(\mathbb{T}^3))}\|\nabla u_\delta\|_{L^\infty((0,T);L^3(\mathbb{T}^3))},\end{aligned}$$

and using (B.7) and (B.8) we obtain

$$|e_{u_\delta}(t)-e_{u_\delta}(s)| \leq C|t-s|\delta^{3\theta-1}[u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}.$$

Thus we have achieved

$$|e_u(t)-e_u(s)| \leq C\left([u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))} + [u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}\right)\left(\delta^{2\theta} + |t-s|\delta^{3\theta-1}\right),$$

from which choosing $\delta = |t-s|^{\frac{1}{1-\theta}}$ we conclude

$$|e_u(t)-e_u(s)| \leq C\left([u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))} + [u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}\right)|t-s|^{\frac{2\theta}{1-\theta}}.$$

The case $\mu > 0$ (Hypodissipative Navier-Stokes)

We assume that $u \in L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))$ and we spilt

$$|E_u(t) - E_u(s)| \leq |E_u(t) - E_{u_\delta}(t)| + |E_{u_\delta}(t) - E_{u_\delta}(s)| + |E_{u_\delta}(s) - E_u(s)|, \tag{2.30}$$

Using (B.1) and (B.8) with $r = \frac{3}{2}$ we can estimate

$$|E_u(t) - E_{u_\delta}(t)| \leq \frac{1}{2}\int_{\mathbb{T}^3} \left|(|u|^2)_\delta - |u_\delta|^2\right| dx + \mu \int_0^t \int_{\mathbb{T}^3} \left|(|(-\Delta)^{\alpha/2} u|^2)_\delta - |(-\Delta)^{\alpha/2} u_\delta|^2\right| dxdr$$

Using (B.8) with $r = \frac{3}{2}$ and with $r = 1$ we have respectively

$$\frac{1}{2}\int_{\mathbb{T}^3} \left|(|u|^2)_\delta - |u_\delta|^2\right|(x,t)\, dx \leq C\delta^{2\theta}[u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))},$$

$$\int_0^t \int_{\mathbb{T}^3} \left|(|(-\Delta)^{\alpha/2} u|^2)_\delta - |(-\Delta)^{\alpha/2} u_\delta|^2\right|(x,r)\, dxdr \leq C\delta^{2(\theta-\alpha)}[(-\Delta)^{\alpha/2} u]^2_{L^\infty((0,T);B^{\theta-\alpha}_{2,\infty}(\mathbb{T}^3))}.$$

Since $\|(-\Delta)^{\alpha/2} u(t)\|_{W^{k,2}(\mathbb{T}^3)} \leq \|u(t)\|_{W^{\alpha+k,2}(\mathbb{T}^3)}$ for both $k = 0, 1$, by interpolation we also get

$$[(-\Delta)^{\alpha/2} u(t)]_{B^{\theta-\alpha}_{2,\infty}(\mathbb{T}^3)} \leq \|u(t)\|_{B^\theta_{2,\infty}(\mathbb{T}^3)}.$$

Thus we have achieved

$$|E_u(t) - E_{u_\delta}(t)| \leq C\delta^{2(\theta-\alpha)}[u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}.$$

Note that the second term in the right hand side of (2.30) is estimated by the same expression for the case $\mu = 0$, thus we get

$$|E_{u_\delta}(t) - E_{u_\delta}(s)| \leq |t-s| \left\|\frac{dE_{u_\delta}}{dt}\right\|_{L^\infty(0,T)} \leq C|t-s|\delta^{3\theta-1}[u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}.$$

Thus we have obtained

$$|E_u(t) - E_u(s)| \leq C\left([u]^2_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))} + [u]^3_{L^\infty((0,T);B^\theta_{3,\infty}(\mathbb{T}^3))}\right)\left(\delta^{2(\theta-\alpha)} + |t-s|\delta^{3\theta-1}\right).$$

Hence if $\theta < 1/3$ choosing $\delta = |t-s|^{\frac{1}{1-3\theta+2(\theta-\alpha)}}$ we conclude the validity of (2.8); if $\theta > 1/3$ we let $\delta \to 0$ in this inequality to deduce the conservation of energy.

Chapter 3

Regularity results for the Euler equations in Besov and Sobolev spaces

3.1 Introduction

In the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, we consider the incompressible Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \qquad \text{in } (0,T) \times \mathbb{T}^3. \tag{3.1}$$

In the last chapter we analyzed the case of Hölder continuous solutions. The following theorem provides a regularization property of the Euler equations, for solutions which enjoy some a priori Sobolev or Besov regularity in space. Roughly speaking, we prove that the pressure associated to any such solution enjoys double regularity in space with respect to u, and that both u and p enjoy a corresponding time regularity.

Theorem 3.1. *Let* (u,p) *be a distributional solution to* (3.1) *in* $(0,T) \times \mathbb{T}^3$*, for some* $T < \infty$*. For any* $\theta \in (0,1)$*,* $s \in [1,\infty]$*,* $r \in (1,\infty)$*, the following implications are true:*

(i) if $u \in L^{2s}((0,T); B^{\theta}_{2r,\infty}(\mathbb{T}^3))$*, then* $u \in B^{\theta}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$ *and* $p \in L^s((0,T); B^{2\theta}_{r,\infty}(\mathbb{T}^3))$*;*

(ii) if $u \in L^{3s}((0,T); B^{\theta}_{4r,\infty}(\mathbb{T}^3))$ *and* $\theta > 1/2$*, then* $p \in B^{2\theta-1-\beta}_{s,\infty}((0,T); B^{1+\beta}_{r,\infty}(\mathbb{T}^3))$ *for any* $\beta \in [0, 2\theta - 1)$*;*

(iii) if $u \in L^{3s}((0,T); B^{\theta}_{3r,\infty}(\mathbb{T}^3))$ *and if* $\theta \le 1/2$*, then* $p \in B^{2\theta-\varepsilon}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$*, for any* $\varepsilon > 0$*. Moreover in the case* $\theta > 1/2$ *we have* $p \in W^{1,s}((0,T); B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$*;*

(iv) if $u \in L^{6s}((0,T); B^{\theta}_{6r,\infty}(\mathbb{T}^3))$ *and* $\theta > 1/2$*, then* $\partial_t p \in B^{2\theta-1-\varepsilon}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$*, for any* $\varepsilon > 0$*.*

Then we obtain the following corollary on the Sobolev solutions by considering suitable embeddings between Sobolev and Besov spaces.

Corollary 3.2. *Let* (u,p) *be a distributional solution to* (3.1) *in* $(0,T)\times\mathbb{T}^3$*, for some* $T<\infty$*. For any* $\theta\in(0,1)$*,* $s\in[1,\infty]$*,* $r\in(1,\infty)$*, the following implications hold true:*

(*i*) *if* $u\in L^{2s}((0,T);W^{\theta,2r}(\mathbb{T}^3))$*, then* $u\in W^{\theta-\varepsilon,s}((0,T);L^r(\mathbb{T}^3))$ *and* $p\in L^s((0,T);W^{2\theta-\varepsilon,r}(\mathbb{T}^3))$;

(*ii*) *if* $\theta\le 1/2$ *and* $u\in L^{3s}((0,T);W^{\theta,3r}(\mathbb{T}^3))$*, or if* $\theta>1/2$ *and* $u\in L^{6s}((0,T);W^{\theta,6r}(\mathbb{T}^3))$*, then* $p\in W^{2\theta-\varepsilon,s}((0,T);L^r(\mathbb{T}^3))$.

When $s=r=\infty$, identifying $W^{\theta,\infty}$ with the corresponding Hölder space, the previous theorem corresponds formally to Theorem 2.1 from the previous chapter: roughly speaking, it says that if (u,p) is a distributional solution to (3.1), $\theta\in(0,1)$ and $u\in L^\infty((0,T);C^\theta(\mathbb{T}^3))$, then $u\in C^{\theta-\varepsilon}((0,T);L^\infty(\mathbb{T}^3))$, namely $u\in C^{\theta-\varepsilon}((0,T)\times\mathbb{T}^3)$ and $p\in C^{2\theta-\varepsilon}((0,T)\times\mathbb{T}^3)$.

Theorem 3.1 follows from two main ingredients: on one side, we obtain the time regularity by estimating, for any time increment h, some norm $||u(t+h)-u(t)||$ by comparison between u and the convolution of u with a mollification kernel at some scale δ, which is then chosen as an appropriate function of h. On the other side, to obtain the double regularity of the pressure we look at

$$-\Delta p=\operatorname{div}\operatorname{div}(u\otimes u),\tag{3.2}$$

which is the formal equation solved by p. We consider a bilinear operator which associates to two divergence-free vector fields (u,v) the solution to $-\Delta p=\operatorname{div}\operatorname{div}(u\otimes v)$ and we apply an abstract interpolation result for bilinear operators (see Theorem 3.8 below). While the arguments of Chapter 2 were based on the classical representation formulae for potential-theoretic solutions of the Poisson equation, in this chapter we employ real interpolation methods which seem to be new in the present context.

3.2 Abstract multilinear interpolation

In this section we provide some estimates for multilinear operators, by means of abstract real interpolation methods. They are the core of this chapter and the proof of Theorem 3.1 relies on them. We start by recalling some definitions and basic facts about interpolation spaces and we refer the reader to the classical monographs [2, 49, 61] for further details.

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two real Banach spaces. The couple (X,Y) is said to be an interpolation couple if both X and Y are continuously embedded in a topological Hausdorff vector space. For any interval $I\subseteq(0,\infty)$ we denote by $L^r_*(I)$ the Lebesgue space of r-summable functions with respect to the measure dt/t. Let use notice that in particular $L^\infty(I)=L^\infty_*(I)$. Moreover, we recall the definition of the K-function, by introducing the following notation.

Definition 3.3. *Given* $x\in X+Y$ *we define* $\Omega(x)=\{(a,b)\in X\times Y : a+b=x\}\subset X\times Y$*. For every* $x\in X+Y$ *and* $t>0$*, the K-function is defined by*

$$K(t,x,X,Y)=\inf_{\Omega(x)}\{\|a\|_X+t\|b\|_Y\}.\tag{3.3}$$

If no confusion can occur, we simply write $K(t,x)$ instead of $K(t,x,X,Y)$.

Definition 3.4. *Let $\theta \in (0,1)$ and $r \in [1,\infty]$. We set*

$$(X,Y)_{\theta,r} = \left\{ x \in X+Y \text{ s.t. } t \mapsto t^{-\theta} K(t,x) \in L^r_*(0,\infty) \right\}$$

endowed with the norm

$$\|x\|_{(X,Y)_{\theta,r}} = \|t^{-\theta} K(\cdot,x)\|_{L^r_*}.$$

For these spaces we have the following inclusions

$$X \cap Y \hookrightarrow (X,Y)_{\theta,r} \hookrightarrow (X,Y)_{\theta,s} \hookrightarrow X+Y, \tag{3.4}$$

for every $\theta \in (0,1)$ and $r,s \in [1,\infty]$ with $r \leq s$. Moreover if $\gamma > \theta$ we also have $(X,Y)_{\gamma,r} \hookrightarrow (X,Y)_{\theta,s}$, for every $r,s \in [1,\infty]$, provided $Y \hookrightarrow X$.

The following two remarks will be useful in the proof of Theorem 3.8.

Remark 3.5. *When $Y \hookrightarrow X$, the definition of K in* (3.3) *does not change if instead of* $\Omega(x)$ *we consider the set $\tilde{\Omega}(x) = \{(a,b) \in \Omega(x) \text{ s.t. } \|a\|_X \leq \|x\|_X\}$; in other words,*

$$K(t,x,X,Y) = \inf_{\Omega(x)} \{\|a\|_X + t\|b\|_Y\} = \inf_{\tilde{\Omega}(x)} \{\|a\|_X + t\|b\|_Y\}.$$

Indeed, since $Y \hookrightarrow X$, one can choose $a = x$ and $b = 0$ in (3.3)*, obtaining $K(t,x) \leq \|x\|_X$. On the other hand, we have that $\|a\|_X + t\|b\|_Y > \|x\|_X$ for all $(a,b) \in \tilde{\Omega}(x)^c$.*

Remark 3.6. *Consider again the case $Y \hookrightarrow X$. Since $a+b = x$, we have*

$$\|a\|_X + \|b\|_X \leq 2\|a\|_X + \|x\|_X \leq 3\|x\|_X, \qquad \forall\, (a,b) \in \tilde{\Omega}(x).$$

It is well known that $\left((X,Y)_{\theta,r}, \|\cdot\|_{(X,Y)_{\theta,r}}\right)$ is a Banach space. Furthermore, we recall that a linear operator T behaves nicely with respect to interpolation, i.e. if $T \in \mathscr{L}(X_1,Y_1) \cap \mathscr{L}(X_2,Y_2)$, then $T \in \mathscr{L}((X_1,X_2)_{\theta,r},(Y_1,Y_2)_{\theta,r})$ for any $\theta \in (0,1)$ and $r \in [1,\infty]$.

Instead of linear operators, our aim is to treat the case of multilinear operators, in particular bilinear and trilinear ones. It is worth mentioning that there exists a wide literature on Interpolation Theory for multilinear operators, see for example the works [2], [37], [47] and [51], but at the best of our knowledge the following results are new. We also emphasise that they are precisely designed for the applications to incompressible fluid models of the next section. In what follows, a conjugate pair (s,s') is a couple of reals satisfying the usual duality $\frac{1}{s} + \frac{1}{s'} = 1$.

Theorem 3.7. *Let (X_1,X_2) and (Y_1,Y_2) be two interpolation couples. Let T be a bilinear operator satisfying*

$$\|T(a_1,a_2)\|_{Y_1} \le C_0\|a_1\|_{X_1}\|a_2\|_{X_1}, \tag{3.5}$$
$$\|T(b_1,b_2)\|_{Y_2} \le C_0\|b_1\|_{X_2}\|b_2\|_{X_2}, \tag{3.6}$$

and

$$\|T(a,b)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}} + \|T(b,a)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}} \le C_0\|a\|_{X_1}\|b\|_{X_2}, \tag{3.7}$$

for some constant $C_0>0$ independent of $a,a_1,a_2\in X_1$ and $b,b_1,b_2\in X_2$, where we implicitly assume that T is well defined between the spaces involved in the previous estimates. Then, for any $\theta,\gamma\in(0,1)$, $r,s,s'\in[1,\infty]$ with s,s' a conjugate pair,

$$\|T(x_1,x_2)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}} \le C_0\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\|x_2\|_{(X_1,X_2)_{\theta,rs'}} \qquad \forall x_1\in(X_1,X_2)_{\gamma,rs},\, \forall x_2\in(X_1,X_2)_{\theta,rs'}.$$

In particular, for $\gamma=\theta$ and $s=s'=2$, we get

$$\|T(x,x)\|_{(Y_1,Y_2)_{\theta,r}} \le C_0\|x\|^2_{(X_1,X_2)_{\theta,2r}}, \quad \forall x\in(X_1,X_2)_{\theta,2r}.$$

Proof. Let $x_1\in(X_1,X_2)_{\gamma,sr}$ and $x_2\in(X_1,X_2)_{\theta,rs'}$. Then we can write $x_1=a_1+b_1$ and $x_2=a_2+b_2$ for some $a_1,a_2\in X_1$ and $b_1,b_2\in X_2$, by definition. Since T is bilinear we have

$$T(x_1,x_2)=T(a_1,a_2)+T(a_1,b_2)+T(b_1,a_2)+T(b_1,b_2).$$

From (3.7) we know that $T(a_1,b_2)\in(Y_1,Y_2)_{\frac{1}{2},\infty}$, hence for any $t,\varepsilon>0$ there exist $T_1\in Y_1$ and $T_2\in Y_2$ such that $T(a_1,b_2)=T_1+T_2$ and

$$\begin{aligned}\|T_1\|_{Y_1}+t\|T_2\|_{Y_2} &\le (1+\varepsilon)K(t,T(a_1,b_2),Y_1,Y_2)\\ &\le (1+\varepsilon)\sqrt{t}\|T(a_1,b_2)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}} \le (1+\varepsilon)C_0\sqrt{t}\|a_1\|_{X_1}\|b_2\|_{X_2}.\end{aligned} \tag{3.8}$$

Similarly, we can decompose $T(b_1,a_2)=U_1+U_2$ with $U_1\in Y_1$ and $U_2\in Y_2$ with estimate

$$\|U_1\|_{Y_1}+t\|U_2\|_{Y_2} \le (1+\varepsilon)C_0\sqrt{t}\|a_2\|_{X_1}\|b_1\|_{X_2}. \tag{3.9}$$

Therefore we can write $T(x_1,x_2)=V+W$, where

$$\begin{aligned}V&=T(a_1,a_2)+T_1+U_1\in Y_1,\\ W&=T(b_1,b_2)+T_2+U_2\in Y_2.\end{aligned}$$

Summing up (3.5)–(3.9) yields to

$$\begin{aligned}\|V\|_{Y_1}+t\|W\|_{Y_2} &\le (1+\varepsilon)C_0\left(\|a_1\|_{X_1}\|a_2\|_{X_1}+\sqrt{t}\left(\|a_1\|_{X_1}\|b_2\|_{X_2}+\|a_2\|_{X_1}\|b_1\|_{X_2}\right)+t\|b_1\|_{X_2}\|b_2\|_{X_2}\right)\\ &= (1+\varepsilon)C_0\left(\|a_1\|_{X_1}+\sqrt{t}\|b_1\|_{X_2}\right)\left(\|a_2\|_{X_1}+\sqrt{t}\|b_2\|_{X_2}\right),\end{aligned}$$

which in turn implies

$$K(t,T(x_1,x_2),Y_1,Y_2) \le (1+\varepsilon)C_0K(\sqrt{t},x_1,X_1,X_2)K(\sqrt{t},x_2,X_1,X_2). \tag{3.10}$$

Multiplying (3.10) by $t^{-(\gamma+\theta)/2}$ and by taking the $L^r_*(0,\infty)$-norm we get, by means of the Hölder inequality with conjugate exponents s and s',

$$\begin{aligned}\|T(x_1,x_2)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}} &= \|(\cdot)^{-(\theta+\gamma)/2}K(\cdot,T(x_1,x_2))\|_{L^r_*}\\ &\le (1+\varepsilon)C_0\left(\|(\cdot)^{-s\gamma/2}K^s(\sqrt{\cdot},x_1)\|^{1/s}_{L^r_*}\|(\cdot)^{-s'\theta/2}K^{s'}(\sqrt{\cdot},x_2)\|^{1/s'}_{L^r_*}\right)\\ &= (1+\varepsilon)C_0\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\|x_2\|_{(X_1,X_2)_{\theta,rs'}},\end{aligned}$$

and since the last inequality holds true for any $\varepsilon > 0$, we are done. □

Let us now focus on trilinear operators, for which a similar result as in Theorem 3.7 can be proved. In what follows, it will be useful to consider interpolation couples (X_1,X_2) such that $X_2 \hookrightarrow X_1$. For sake of clarity, we require that the trilinear operator in the statement is totally symmetric, i.e. $T(a_1,a_2,a_3) = T\left(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}\right)$ for every permutation σ, even though a suitable adaptation would work without this requirement.

Theorem 3.8. *Let $C_0 > 0$, (X_1,X_2) and (Y_1,Y_2) be two interpolation couples with $X_2 \hookrightarrow X_1$. Let T be a trilinear and symmetric operator satisfying the following conditions*

$$\|T(a_1,a_2,a_3)\|_{Y_1} \le C_0\|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}, \tag{3.11}$$

$$\|T(b_1,b_2,b_3)\|_{Y_2} \le C_0\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big), \tag{3.12}$$

and

$$\|T(a_1,b_2,b_3)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}} \le C_0\|a_1\|_{X_1}\Big(\|b_2\|_{X_2}\|b_3\|_{X_1} + \|b_2\|_{X_1}\|b_3\|_{X_2}\Big), \tag{3.13}$$

where we implicitly assume that T is well defined between the spaces involved in the previous estimates. Then for any $\gamma,\theta \in (0,1)$ and $r,s \in [1,\infty]$, for every x_1,x_2,x_3 we have

$$\begin{aligned}\|T(x_1,x_2,x_3)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}} \le 3C_0\Bigg(&\|x_1\|_{X_1}\|x_2\|_{(X_1,X_2)_{\gamma,rs}}\|x_3\|_{(X_1,X_2)_{\theta,rs'}}\\ &+\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\Big(\|x_2\|_{X_1}\|x_3\|_{(X_1,X_2)_{\theta,rs'}} + \|x_2\|_{(X_1,X_2)_{\theta,rs'}}\|x_3\|_{X_1}\Big)\Bigg).\end{aligned} \tag{3.14}$$

In particular, for $\gamma = \theta$ and $s = s' = 2$, we get

$$\|T(x,x,x)\|_{(Y_1,Y_2)_{\theta,r}} \le 3C_0\|x\|_{X_1}\|x\|^2_{(X_1,X_2)_{\theta,2r}}, \quad \forall x \in (X_1,X_2)_{\theta,2r}.$$

Proof. We assume without loss of generality that $\theta \geq \gamma$. Consider $x_1 \in (X_1,X_2)_{\gamma,rs}$ and $x_2,x_3 \in (X_1,X_2)_{\theta,rs'}$. For $k=1,2,3$ we write $x_k = a_k + b_k$ with $a_k \in X_1$ and $b_k \in X_2$; therefore we expand

$$T(x_1,x_2,x_3) = U + V + W$$

where

$$\begin{aligned} U &= T(a_1,a_2,a_3) + T(b_1,a_2,a_3) + T(a_1,b_2,a_3) + T(a_1,a_2,b_3), \\ V &= T(b_1,b_2,a_3) + T(b_1,a_2,b_3) + T(a_1,b_2,b_3), \\ W &= T(b_1,b_2,b_3). \end{aligned}$$

Since $X_2 \hookrightarrow X_1$ we have that $b_k \in X_1$ for any $k=1,2,3$, then by (3.11) we can control U as

$$\begin{aligned} \|U\|_{Y_1} \leq C_0 \Big(& \|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1} + \|b_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1} \\ & + \|a_1\|_{X_1}\|b_2\|_{X_1}\|a_3\|_{X_1} + \|a_1\|_{X_1}\|a_2\|_{X_1}\|b_3\|_{X_1} \Big). \end{aligned} \tag{3.15}$$

The symmetry of the operator T and (3.13) imply that every term defining V belongs to $(Y_1,Y_2)_{\frac{1}{2},\infty}$. Let us consider without loss of generality the term $T(b_1,b_2,a_3)$; as already done in Theorem 3.7, for any $t,\varepsilon > 0$ there exist $T_1 \in Y_1$ and $T_2 \in Y_2$ such that $T(b_1,b_2,a_3) = T_1 + T_2$ and

$$\begin{aligned} \|T_1\|_{Y_1} + t\|T_2\|_{Y_2} &\leq (1+\varepsilon)K(t,T(b_1,b_2,a_3),Y_1,Y_2) \leq (1+\varepsilon)\sqrt{t}\|T(b_1,b_2,a_3)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}} \\ &\leq (1+\varepsilon)C_0\sqrt{t}\|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\right). \end{aligned}$$

We point out that the elements T_1 and T_2 actually depend on a_3,b_1,b_2,ε and t as well. The same consideration for the other two terms defining V yields, for any $t,\varepsilon > 0$, to the existence of $V_1 \in Y_1$ and $V_2 \in Y_2$ such that $V = V_1 + V_2$ and

$$\begin{aligned} \|V_1\|_{Y_1} + t\|V_2\|_{Y_2} \leq (1+\varepsilon)C_0\sqrt{t}\Big(& \|a_1\|_{X_1}\left(\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_2\|_{X_2}\|b_3\|_{X_1}\right) \\ & + \|a_2\|_{X_1}\left(\|b_1\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_3\|_{X_1}\right) + \|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\right)\Big). \end{aligned} \tag{3.16}$$

By using (3.12) we also get

$$\|T(b_1,b_2,b_3)\|_{Y_2} \leq C_0\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big). \tag{3.17}$$

By combining (3.15), (3.16) and (3.17) we obtain, for any $t,\varepsilon > 0$, a decomposition of $T(x_1,x_2,x_3) =$

$(U+V_1)+(V_2+W)$, with $U+V_1 \in Y_1$ and $V_2+W \in Y_2$ such that

$$\begin{aligned}
\|U+V_1\|_{Y_1}+t\|V_2+W\|_{Y_2} \leq (1+\varepsilon)C_0\Big(&\|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}+\|b_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}\\
&+\|a_1\|_{X_1}\|b_2\|_{X_1}\|a_3\|_{X_1}+\|a_1\|_{X_1}\|a_2\|_{X_1}\|b_3\|_{X_1}+\sqrt{t}\Big(\|a_1\|_{X_1}\left(\|b_2\|_{X_1}\|b_3\|_{X_2}+\|b_2\|_{X_2}\|b_3\|_{X_1}\right)\\
&+\|a_2\|_{X_1}\left(\|b_1\|_{X_1}\|b_3\|_{X_2}+\|b_1\|_{X_2}\|b_3\|_{X_1}\right)+\|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2}+\|b_1\|_{X_2}\|b_2\|_{X_1}\right)\Big)\\
&+t\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2}+\|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2}+\|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big)\Big)\\
\leq (1+\varepsilon)C_0\Big(&\left(\|a_1\|_{X_1}+\|b_1\|_{X_1}\right)\left(\|a_2\|_{X_1}+\sqrt{t}\|b_2\|_{X_2}\right)\left(\|a_3\|_{X_1}+\sqrt{t}\|b_3\|_{X_2}\right)\\
&+\left(\|a_1\|_{X_1}+\sqrt{t}\|b_1\|_{X_2}\right)\left(\|a_2\|_{X_1}+\|b_2\|_{X_1}\right)\left(\|a_3\|_{X_1}+\sqrt{t}\|b_3\|_{X_2}\right)\\
&+\left(\|a_1\|_{X_1}+\sqrt{t}\|b_1\|_{X_2}\right)\left(\|a_2\|_{X_1}+\sqrt{t}\|b_2\|_{X_2}\right)\left(\|a_3\|_{X_1}+\|b_3\|_{X_1}\right)\Big)=R(t)
\end{aligned}$$

which clearly implies

$$K(t,T(x_1,x_2,x_3),Y_1,Y_2) \leq R(t). \tag{3.18}$$

Now, by using Theorem 3.5 and Theorem 3.6 and by taking the infima over all the sets $\tilde{\Omega}(x_k) = \{(a_k,b_k) \in \Omega(x_k) \text{ s.t. } \|a_k\|_{X_1} \leq \|x_k\|_{X_1}\}$ for $k=1,2,3$ in the right-hand side of (3.18), we achieve

$$\begin{aligned}
K(t,T(x_1,x_2,x_3),Y_1,Y_2) \leq 3(1+\varepsilon)C_0\big(&\|x_1\|_{X_1}K(\sqrt{t},x_2)K(\sqrt{t},x_3)\\
&+K(\sqrt{t},x_1)\left(\|x_2\|_{X_1}K(\sqrt{t},x_3)+\|x_3\|_{X_1}K(\sqrt{t},x_2)\right)\big).
\end{aligned}$$

Multiplying by $t^{-(\theta+\gamma)/2}$ the last inequality, taking the $L^r_*(0,\infty)$-norm and using the Hölder inequality with s,s' as conjugate pair, we obtain (3.14) by letting $\varepsilon \to 0$. □

We recall that interpolation theory also provides the following useful characterization of Besov spaces (see for instance [2, Theorem 6.2.4]).

Proposition 3.9. *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz open set. For any $\theta \in (0,1)$, $r,s \in [1,\infty]$ and $\sigma_1 \neq \sigma_2 \in \mathbb{Z}$,*

$$(W^{\sigma_1,r}(\Omega),W^{\sigma_2,r}(\Omega))_{\theta,s} = B^{(1-\theta)\sigma_1+\theta\sigma_2}_{r,s}(\Omega). \tag{3.19}$$

Moreover, the same holds if we restrict all spaces in (3.19) *to the linear subspace of divergence-free vector fields.*

Even if the previous proposition is also valid for spaces of negative order, for the sake of simplicity, we did not define, in Appendix A, Sobolev and Besov spaces of order less than or equal to 0. However, we will apply Proposition 3.9 only for the Besov spaces of strictly positive θ. The statement for divergence-free vector fields follows instead from the same proof as (3.19), since the construction in the interpolation is based on mollification at a suitable scale, and convolutions preserve the divergence-free structure of the vector fields.

3.3 Velocity and pressure regularity

The following result about elliptic equations follows by a direct application of Theorem 3.7 and Theorem 3.8 of the previous section. The reader can compare the following proposition with Proposition 2.3 obtained for Hölder spaces through estimates on a representation formula for p and q.

Proposition 3.10. *Let $\gamma, \theta \in (0,1)$ and $r \in (1,\infty)$. Let $u,w,z : \mathbb{T}^3 \to \mathbb{R}^3$ be divergence-free vector fields and let $p,q : \mathbb{T}^3 \to \mathbb{R}^3$ be the unique 0-average solutions of*

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes w), \tag{3.20}$$

$$-\Delta q = \operatorname{div}\operatorname{div}\operatorname{div}(u \otimes w \otimes z). \tag{3.21}$$

Then, for any $s \in [1,\infty]$, we have

$$\|p\|_{B^{\gamma+\theta}_{r,s}} \le C \|u\|_{B^{\gamma}_{2r,2s}} \|w\|_{B^{\theta}_{2r,2s}}. \tag{3.22}$$

Furthermore, if $\theta + \gamma > 1$

$$\|q\|_{B^{\gamma+\theta-1}_{r,s}} \le C\Big(\|u\|_{L^{3r}} \|w\|_{B^{\gamma}_{3r,2s}} \|z\|_{B^{\theta}_{3r,2s}} + \|u\|_{B^{\gamma}_{3r,2s}} \Big(\|w\|_{L^{3r}} \|z\|_{B^{\theta}_{3r,2s}} + \|w\|_{B^{\theta}_{3r,2s}} \|z\|_{L^{3r}} \Big)\Big). \tag{3.23}$$

Proof. We denote by $W^{1,r}_{\text{div}}$ the linear subspace of $W^{1,r}$ made by divergence-free vector fields (and similarly for $B^{\theta}_{r,s,\text{div}}$). Let $T(u,w)$ be the operator that for each couple (u,w) associate the unique 0-average solution of (3.20). By the Calderón-Zygmund theory, we have

$$\|T(u,w)\|_{L^r} \le C \|u\|_{L^{2r}} \|w\|_{L^{2r}}.$$

Moreover since $\operatorname{div} u = \operatorname{div} w = 0$ the right-hand side of (3.20) can be rewritten as

$$\operatorname{div}\operatorname{div}(u \otimes w) = \partial^2_{ij}(u^i w^j) = \partial_j (u^i \partial_i w^j) = \partial_j u^i \partial_i w^j,$$

thus we can use again Calderón-Zygmund to get

$$\|T(u,w)\|_{W^{1,r}} \le C \|u\|_{L^{2r}} \|w\|_{W^{1,2r}}$$

and

$$\|T(u,w)\|_{W^{2,r}} \le C \|u\|_{W^{1,2r}} \|w\|_{W^{1,2r}}.$$

Since, by Proposition 3.9, we have the embedding $W^{1,r}_{\text{div}} \hookrightarrow B^1_{r,\infty,\text{div}} = (L^r_{\text{div}}, W^{2,r}_{\text{div}})_{\frac{1}{2},\infty}$, we can apply Theorem 3.7 with $X_1 = L^{2r}_{\text{div}}$, $X_2 = W^{1,2r}_{\text{div}}$, $Y_1 = L^r_{\text{div}}$, $Y_2 = W^{2,r}_{\text{div}}$, hence obtaining (3.22). Note that it is important that all the spaces above consist of divergence-free vector fields.
The proof of (3.23) follows similarly as a consequence of Calderón-Zygmund and Theorem 3.8, with $X_1 = L^{3r}_{\text{div}}$, $X_2 = W^{1,3r}_{\text{div}}$, $Y_1 = W^{-1,r}_{\text{div}}$ and $Y_2 = W^{1,r}_{\text{div}}$ once one notices that the solenoidal nature of u,w,z implies that

$$\begin{aligned}\operatorname{div}\operatorname{div}\operatorname{div}(u \otimes w \otimes z) &= \partial^3_{ijk}(u^i w^j z^k) = \partial^2_{ij}(\partial_k u^i w^j z^k) + \partial^2_{ij}(u^i \partial_k w^j z^k) \\ &= \partial_j(\partial_k u^i \partial_i w^j z^k + \partial_k u^i w^j \partial_i z^k) + \partial_i(\partial_j u^i \partial_k w^j z^k + u^i \partial_k w^j \partial_j z^k). \quad \square\end{aligned}$$

Remark 3.11. *The regularity estimates for the pressure of the proposition above are also a generalization of previously known results contained in* [1, Lemmas 7.9, 7.10 and 7.14]*, where some Lipschitz regularity of the vector fields is assumed. Proposition 3.10 is however more general, both because it proves the double regularity of the pressure based only on the Besov regularity of the vector field and because it does not require boundedness or Lipschitz assumptions on the vector field, which are not satisfied for instance by the solutions built by convex integration methods.*

Moreover, the above double regularity results on the pressure do not depend on the specific structure given by the Laplacian but also apply to more general elliptic operators. Indeed the Calderón-Zygmund estimates in the extremal spaces L^r, $W^{1,r}$ and $W^{2,r}$ is enough to apply our abstract interpolation theorems.

We consider now a weak solution (u,p) of the incompressible Euler equations (3.1). The pressure p solves

$$-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u), \tag{3.24}$$

thus it can be uniquely determined if one imposes that $\int_{\mathbb{T}^3} p(t,x)\,dx = 0$, for any time $t\in(0,T)$. For every $\theta\in(0,1)$ and $r\in(1,\infty)$, a direct application of Calderón-Zygmund leads to

$$\|p(t)\|_{B^{\theta}_{r,\infty}} \le C\|u(t)\|^2_{B^{\theta}_{2r,\infty}}. \tag{3.25}$$

Since our solutions are just weak solutions, we will need to mollify (3.1) in order to justify some computations; moreover, we will tune the convolution parameter in terms of the time increment h. By regularizing (in space) the equations (3.1), one gets that the couple $(u_\delta,p_\delta)=(u*\varphi_\delta, p*\varphi_\delta)$ solves

$$\begin{cases} \partial_t u_\delta + \operatorname{div}(u_\delta\otimes u_\delta) + \nabla p_\delta = \operatorname{div} R_\delta \\ \operatorname{div} u_\delta = 0\,, \end{cases} \tag{3.26}$$

where $R_\delta = u_\delta\otimes u_\delta - (u\otimes u)_\delta$. We can now prove our main theorem.

Proof of Theorem 3.1. Let $h>0$ be a time increment. When it will help the readability we will also put in the constants C all the norms of u and p which are already known to be finite. We prove the theorem for $s<\infty$, since the case $s=\infty$ is a simple adaptation and it is easier using the identification $B^{\theta}_{\infty,\infty}=C^{\theta}$. In the following, given an interval I, the function $\chi_I(\cdot)$ will denote the usual characteristic function of the set I.

Proof of (i). Assume that $u\in L^{2s}((0,T);B^{\theta}_{2r,\infty}(\mathbb{T}^3))$, for some $s\in[1,\infty)$. We split

$$\|u(t+h)-u(t)\|_{L^r} \le \|u(t+h)-u_\delta(t+h)\|_{L^r} + \|u_\delta(t+h)-u_\delta(t)\|_{L^r} + \|u_\delta(t)-u(t)\|_{L^r}. \tag{3.27}$$

Using (B.6) we have $\|u_\delta(t)-u(t)\|_{L^r}\le C\delta^{\theta}\|u(t)\|_{B^{\theta}_{r,\infty}}$ for every $t\in(0,T)$, from which we deduce

$$\begin{aligned}\left(\int_0^{T-h}\|u(t+h)-u_\delta(t+h)\|^s_{L^r}\,dt\right)^{\frac1s} + \left(\int_0^{T-h}\|u(t)-u_\delta(t)\|^s_{L^r}\,dt\right)^{\frac1s} &\le C\delta^{\theta}\|u\|_{L^s(B^{\theta}_{r,\infty})}\\ &\le C\delta^{\theta}\|u\|_{L^{2s}(B^{\theta}_{2r,\infty})}.\end{aligned}$$

In the last inequality we used the fact that both the time and spatial domains are bounded. We are left with the second term in the right-hand side of (3.27). Since u_δ solves (3.26), using also (B.7) and (3.25) we get

$$\begin{aligned}\|u_\delta(t+h)-u_\delta(t)\|_{L^r} &\le \int_t^{t+h}\|\partial_t u_\delta(\tau)\|_{L^r}\,d\tau \le \int_t^{t+h}\Big(\|\mathrm{div}\,(u\otimes u)_\delta(\tau)\|_{L^r}+\|\nabla p_\delta(\tau)\|_{L^r}\Big)\,d\tau\\ &\le C\delta^{\theta-1}\int_t^{t+h}\big(\|u\otimes u(\tau)\|_{B^\theta_{r,\infty}}+\|p(\tau)\|_{B^\theta_{r,\infty}}\big)\,d\tau \le C\delta^{\theta-1}\int_t^{t+h}\|u(\tau)\|^2_{B^\theta_{2r,\infty}}\,d\tau.\end{aligned}$$

By the Hölder inequality with conjugate exponents s and $\frac{s}{s-1}$ we deduce

$$\|u_\delta(t+h)-u_\delta(t)\|^s_{L^r}\le C\delta^{(\theta-1)s}h^{s-1}\int_0^T\chi(\tau)_{(t,t+h)}\|u(\tau)\|^{2s}_{B^\theta_{2r,\infty}}\,d\tau,$$

from which, by integrating in time, we conclude

$$\begin{aligned}\int_0^{T-h}\|u_\delta(t+h)-u_\delta(t)\|^s_{L^r}\,dt &\le C\delta^{(\theta-1)s}h^{s-1}\int_0^{T-h}\int_0^T\chi(\tau)_{(t,t+h)}\|u(\tau)\|^{2s}_{B^\theta_{2r,\infty}}\,d\tau\,dt\\ &\le C\delta^{(\theta-1)s}h^s\|u\|^{2s}_{L^{2s}(B^\theta_{2r,\infty})},\end{aligned}$$

where in the last inequality we also used $\int_0^{T-h}\chi(t)_{(\tau-h,\tau)}\,dt\le h$. By choosing $\delta=h$, we achieve

$$\left(\int_0^{T-h}\|u(t+h)-u(t)\|^s_{L^r}\,dt\right)^{\frac{1}{s}}\le Ch^\theta\left(\|u\|_{L^{2s}(B^\theta_{2r,\infty})}+\|u\|^2_{L^{2s}(B^\theta_{2r,\infty})}\right),$$

from which, by taking the supremum all over $h\in(0,T)$, we conclude $u\in B^\theta_{s,\infty}((0,T);L^r(\mathbb{T}^3))$. Since p solves (3.24), we can use (3.22) with $u=w=u(t)$, $\gamma=\theta$, $s=\infty$, getting

$$\|p(t)\|_{B^{2\theta}_{r,\infty}}\le C\|u(t)\|^2_{B^\theta_{2r,\infty}}. \tag{3.28}$$

Taking the $L^s(0,T)$-norm, we deduce that $p\in L^s((0,T);B^{2\theta}_{r,\infty}(\mathbb{T}^3))$, namely that (i) holds.

Proof of (ii). Let $\theta>1/2$ and $\beta\in[0,2\theta-1)$. Note that

$$-\Delta(p(t+h)-p(t))=\mathrm{div}\,\mathrm{div}\Big(\big(u(t+h)-u(t)\big)\otimes u(t+h)+u(t)\otimes\big(u(t+h)-u(t)\big)\Big).$$

Thus, by using (3.22) with $\gamma=1-\theta+\beta$, $s=\infty$, we get

$$\|p(t+h)-p(t)\|_{B^{1+\beta}_{r,\infty}}\le C\|u(t+h)-u(t)\|_{B^{1-\theta+\beta}_{2r,\infty}}\Big(\|u(t+h)\|_{B^\theta_{2r,\infty}}+\|u(t)\|_{B^\theta_{2r,\infty}}\Big), \tag{3.29}$$

and taking the $L^s(0,T-h)$-norm in time, by also using the Hölder inequality, we achieve

$$\left(\int_0^{T-h}\|p(t+h)-p(t)\|^s_{B^{1+\beta}_{r,\infty}}\,dt\right)^{\frac{1}{s}} \le C\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^{1-\theta+\beta}_{2r,\infty}}\,dt\right)^{\frac{2}{3s}}\|u\|_{L^{3s}(B^\theta_{2r,\infty})}. \quad (3.30)$$

By the interpolation inequality (A.3), the Hölder inequality, and since $u\in B^\theta_{\frac{3s}{2},\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (i), we can estimate

$$\begin{aligned}\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^{1-\theta+\beta}_{2r,\infty}}\,dt &\le \int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}\frac{2\theta-1-\beta}{\theta}}_{L^{2r}}\|u(t+h)-u(t)\|^{\frac{3s}{2}\frac{1-\theta+\beta}{\theta}}_{B^\theta_{2r,\infty}}\,dt\\ &\le \left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{L^{2r}}\,dt\right)^{\frac{2\theta-1-\beta}{\theta}}\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^\theta_{2r,\infty}}\,dt\right)^{\frac{1-\theta+\beta}{\theta}}\\ &\le Ch^{\frac{3s}{2}(2\theta-1-\beta)}\|u\|^{\frac{3s}{2}\frac{2\theta-1-\beta}{\theta}}_{B^\theta_{\frac{3s}{2},\infty}(L^{2r})}\|u\|^{\frac{3s}{2}\frac{1-\theta+\beta}{\theta}}_{L^{\frac{3s}{2}}(B^\theta_{2r,\infty})}\le Ch^{\frac{3s}{2}(2\theta-1-\beta)}.\end{aligned}$$

By plugging this last estimate in (3.30), we conclude that $p\in B^{2\theta-1-\beta}_{s,\infty}((0,T);B^{1+\beta}_{r,\infty}(\mathbb{T}^3))$, since we get

$$\left(\int_0^{T-h}\|p(t+h)-p(t)\|^s_{B^{1+\beta}_{r,\infty}}\,dt\right)^{\frac{1}{s}}\le Ch^{2\theta-1-\beta}.$$

Proof of (iii). In order to prove the Besov regularity in time of the pressure, we split

$$\|p(t+h)-p(t)\|_{L^r}\le\|p(t+h)-p_\delta(t+h)\|_{L^r}+\|p_\delta(t+h)-p_\delta(t)\|_{L^r}+\|p_\delta(t)-p(t)\|_{L^r}. \quad (3.31)$$

Using (B.6) and (3.28), we have, for every $t\in(0,T)$,

$$\|p_\delta(t)-p(t)\|_{L^r}\le C\delta^{2\theta}\|p(t)\|_{B^{2\theta}_{r,\infty}}\le C\delta^{2\theta}\|u(t)\|^2_{B^\theta_{2r,\infty}}\le C\delta^{2\theta}\|u(t)\|^2_{B^\theta_{3r,\infty}},$$

from which we deduce

$$\int_0^{T-h}\|p(t+h)-p_\delta(t+h)\|^s_{L^r}\,dt+\int_0^{T-h}\|p(t)-p_\delta(t)\|^s_{L^r}\,dt\le C\delta^{2\theta s}\|u\|^{2s}_{L^{3s}(B^\theta_{3r,\infty})}.$$

It remains to prove the estimate for the middle term $\|p_\delta(t+h)-p_\delta(t)\|_{L^r}$ in the right-hand side of (3.31). Notice that $p_\delta(t+h)-p_\delta(t)$ solves

$$\begin{aligned}-\Delta(p_\delta(t+h)-p_\delta(t)) &= \operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+u_\delta(t+h)\otimes u_\delta(t+h)-u_\delta(t)\otimes u_\delta(t)\Big)\\ &=\operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+\int_t^{t+h}\Big(\frac{d}{d\tau}u_\delta(\tau,x)\otimes u_\delta(\tau,x)+u_\delta(\tau,x)\otimes\frac{d}{d\tau}u_\delta(\tau,x)\Big)\,d\tau\Big)\\ &=\operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+\int_t^{t+h}\Big((\operatorname{div}(u_\delta\otimes u_\delta)-\nabla p_\delta-\operatorname{div}R_\delta)\otimes u_\delta\\ &\quad+u_\delta\otimes(\operatorname{div}(u_\delta\otimes u_\delta)-\nabla p_\delta-\operatorname{div}R_\delta)\Big)\,d\tau\Big).\end{aligned}$$

Thus $p_\delta(t+h)-p_\delta(t)=q^1+q^2+q^3$, where q^1,q^2,q^3 are the unique 0-average solutions to

$$
\begin{aligned}
-\Delta q^1 &= \operatorname{div}\operatorname{div}\left(R_\delta(t,x)-R_\delta(t+h,x)\right),\\
\Delta q^2 &= 2\int_t^{t+h}\operatorname{div}\operatorname{div}\left((\operatorname{div}R_\delta+\nabla p_\delta)\otimes u_\delta\right)d\tau,\\
-\Delta q^3 &= \int_t^{t+h}\operatorname{div}\operatorname{div}\operatorname{div}\left(u_\delta\otimes u_\delta\otimes u_\delta\right)d\tau.
\end{aligned}
$$

By Calderón-Zygmund, (B.8) and (B.7) we have that

$$\|q^1(t)\|_{L^r}\le C\big(\|R_\delta(t+h)\|_{L^r}+\|R_\delta(t)\|_{L^r}\big)\le C\delta^{2\theta}\left(\|u(t+h)\|^2_{B^\theta_{3r,\infty}}+\|u(t)\|^2_{B^\theta_{3r,\infty}}\right),$$

and

$$\|q^2(t)\|_{L^r}\le C\int_t^{t+h}\left(\|\operatorname{div}R_\delta(\tau)\|_{L^{\frac{3r}{2}}}+\|\nabla p_\delta(\tau)\|_{L^{\frac{3r}{2}}}\right)\|u_\delta(\tau)\|_{L^{3r}}\,d\tau\le C\delta^{2\theta-1}\int_t^{t+h}\|u(\tau)\|^3_{B^\theta_{3r,\infty}}\,d\tau.$$

Hence, by taking the $L^s(0,T-h)$-norm, we deduce

$$\|q^1\|_{L^s(L^r)}\le C\delta^{2\theta}\|u\|^2_{L^{3s}(B^\theta_{3r,\infty})}. \tag{3.32}$$

and, similarly to above, by the Hölder inequality we have

$$
\begin{aligned}
\int_0^{T-h}\|q^2(t)\|^s_{L^r}\,dt &\le C\delta^{(2\theta-1)s}h^{s-1}\int_0^{T-h}\left(\int_0^T\chi_{(t,t+h)}(\tau)\|u(\tau)\|^{3s}_{B^\theta_{3r,\infty}}\,d\tau\right)dt\\
&\le C\delta^{(2\theta-1)s}h^s\|u\|^{3s}_{L^{3s}(B^\theta_{3r,\infty})}.
\end{aligned} \tag{3.33}
$$

For q^3 we can use, for any $\varepsilon>0$, (3.23) with $\theta=\gamma=(1+\varepsilon)/2$, $s=\infty$, $u=w=z=u_\delta(t)$, getting

$$\|q^3(t)\|_{L^r}\le\|q^3(t)\|_{B^\varepsilon_{r,\infty}}\le C\int_t^{t+h}\|u_\delta(\tau)\|_{L^{3r}}\|u_\delta(\tau)\|^2_{B^{\frac{1+\varepsilon}{2}}_{3r,\infty}}\,d\tau. \tag{3.34}$$

By (A.4) and the estimate (B.7), we have

$$\|u_\delta(t)\|_{B^{\frac{1+\varepsilon}{2}}_{3r,\infty}}\le\|u_\delta(t)\|^{\frac{1-\varepsilon}{2(1-\theta)}}_{B^\theta_{3r,\infty}}\|u_\delta(t)\|^{\frac{1+\varepsilon-2\theta}{2(1-\theta)}}_{W^{1,3r}}\le C\delta^{\theta-\frac{1+\varepsilon}{2}}\|u(t)\|_{B^\theta_{3r,\infty}}.$$

Plugging this last estimate in (3.34), we achieve

$$\|q^3(t)\|_{L^r}\le C\delta^{2\theta-1-\varepsilon}\int_t^{t+h}\|u(\tau)\|^3_{B^\theta_{3r,\infty}}\,d\tau,$$

from which we deduce

$$\|q^3\|_{L^s(L^r)} \le C\delta^{2\theta-1-\varepsilon} h \|u\|^3_{L^{3s}(B^\theta_{3r,\infty})}. \tag{3.35}$$

Choosing $\delta = h$, from (3.32), (3.33) and (3.35), we conclude

$$\left(\int_0^{T-h} \|p_\delta(t+h) - p_\delta(t)\|^s_{L^r}\, dt\right)^{\frac{1}{s}} \le Ch^{2\theta-\varepsilon}\left(\|u\|^2_{L^{3s}(B^\theta_{3r,\infty})} + \|u\|^3_{L^{3s}(B^\theta_{3r,\infty})}\right),$$

which implies that $p \in B^{2\theta-\varepsilon}_{s,\infty}((0,T);L^r(\mathbb{T}^3))$. If now $\theta > 1/2$, we have to prove that $p \in W^{1,s}((0,T);B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. It is enough to show that $\partial_t p \in L^s((0,T);B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. Indeed by point (i) of the Theorem 3.1 $p \in L^{\frac{3s}{2}}((0,T);B^{2\theta}_{\frac{3r}{2},\infty}(\mathbb{T}^3)) \hookrightarrow L^s((0,T);B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. Thus we can write, by using (3.38), $\partial_t p = q^1 + q^2$ where q^1, q^2 are the unique 0-average solutions of

$$\begin{aligned}\Delta q^1 &= \operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u),\\ \Delta q^2 &= 2\operatorname{div}\operatorname{div}(\nabla p\otimes u).\end{aligned}$$

Since, by (3.28),

$$\|\nabla p(t)\|_{B^{2\theta-1}_{\frac{3r}{2},\infty}} \le C\|u(t)\|^2_{B^\theta_{3r,\infty}},$$

by Calderón-Zygmund we get

$$\|q^2(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|(\nabla p\otimes u)(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|\nabla p(t)\|_{B^{2\theta-1}_{\frac{3r}{2},\infty}}\|u(t)\|_{B^\theta_{3r,\infty}} \le C\|u(t)\|^3_{B^\theta_{3r,\infty}}.$$

Moreover, by (3.23) with $\gamma = \theta$, $s = \infty$ and $u = w = z = u(t)$,

$$\|q^1(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|u(t)\|^3_{B^\theta_{3r,\infty}}.$$

Hence, by taking the $L^s(0,T)$-norm we obtain

$$\|\partial_t p\|_{L^s(B^{2\theta-1}_{r,\infty})} \le \|q^1\|_{L^s(B^{2\theta-1}_{r,\infty})} + \|q^2\|_{L^s(B^{2\theta-1}_{r,\infty})} \le C\|u\|^3_{L^{3s}(B^\theta_{3r,\infty})},$$

which concludes the proof of (iii).

Proof of (iv). By Lemma 3.12 we have that $\partial_t p$ solves (3.38). Therefore $\partial_t p(t+h) - \partial_t p(t) = q^1 + q^2$ where

$$\begin{aligned}\Delta q^1 &= \operatorname{div}\operatorname{div}\operatorname{div}(u(t+h)\otimes u(t+h)\otimes u(t+h) - u(t)\otimes u(t)\otimes u(t))\\ &= \operatorname{div}\operatorname{div}\operatorname{div}\big((u(t+h)-u(t))\otimes u(t+h)\otimes u(t+h) + u(t)\otimes(u(t+h)-u(t))\otimes u(t+h)\\ &\quad + u(t)\otimes u(t)\otimes(u(t+h)-u(t))\big),\end{aligned}$$

$$\Delta q^2 = 2\operatorname{div}\operatorname{div}\big(\nabla p(t+h)\otimes u(t+h) - \nabla p(t)\otimes u(t)\big).$$

To estimate q^1, for any small $\varepsilon > 0$, we apply (3.23) with $\gamma = 1-\theta+\varepsilon$ and $s=\infty$, in such a way that the factor $u(t+h)-u(t)$ gets only the $B^{1-\theta+\varepsilon}_{3r,\infty}$-norm and not the $B^{\theta}_{3r,\infty}$-norm. Thus we get

$$\|q^1(t)\|_{L^r} \le \|q^1(t)\|_{B^{\varepsilon}_{r,\infty}} \le C\|u(t+h)-u(t)\|_{B^{1-\theta+\varepsilon}_{3r,\infty}}\big(\|u(t+h)\|^2_{B^{\theta}_{3r,\infty}} + \|u(t)\|^2_{B^{\theta}_{3r,\infty}}\big).$$

Integrating in time on $(0,T-h)$ yields to

$$\int_0^{T-h} \|q^1(t)\|^s_{L^r}\,dt \le C\int_0^{T-h} \|u(t+h)-u(t)\|^s_{B^{1-\theta+\varepsilon}_{3r,\infty}}\big(\|u(t+h)\|^{2s}_{B^{\theta}_{3r,\infty}} + \|u(t)\|^{2s}_{B^{\theta}_{3r,\infty}}\big)\,dt$$

and by the Cauchy-Schwarz inequality we get

$$\int_0^{T-h} \|q^1(t)\|^s_{L^r}\,dt \le C\left(\int_0^{T-h} \|u(t+h)-u(t)\|^{2s}_{B^{1-\theta+\varepsilon}_{3r,\infty}}\,dt\right)^{\frac{1}{2}} \|u\|^{2s}_{L^{4s}(B^{\theta}_{3r,\infty})}.$$

Now, by (A.3) together with the Hölder inequality in time, we have

$$\begin{aligned}\int_0^{T-h} \|u(t+h)-u(t)\|^{2s}_{B^{1-\theta+\varepsilon}_{3r,\infty}}\,dt &\le \int_0^{T-h} \|u(t+h)-u(t)\|^{2s\frac{2\theta-1-\varepsilon}{\theta}}_{L^{3r}} \|u(t+h)-u(t)\|^{2s\frac{1-\theta+\varepsilon}{\theta}}_{B^{\theta}_{3r,\infty}}\,dt\\ &\le \left(\int_0^{T-h} \|u(t+h)-u(t)\|^{2s}_{L^{3r}}\,dt\right)^{\frac{2\theta-1-\varepsilon}{\theta}} \left(\int_0^{T-h} \|u(t+h)-u(t)\|^{2s}_{B^{\theta}_{3r,\infty}}\,dt\right)^{\frac{1-\theta+\varepsilon}{\theta}}\\ &\le Ch^{2s(2\theta-1-\varepsilon)}\|u\|^{2s\frac{2\theta-1-\varepsilon}{\theta}}_{B^{\theta}_{2s,\infty}(L^{3r})}\|u\|^{2s\frac{1-\theta+\varepsilon}{\theta}}_{L^{2s}(B^{\theta}_{3r,\infty})} \le Ch^{2s(2\theta-1-\varepsilon)},\end{aligned}$$

where in the last inequality we used $u\in B^{\theta}_{3s,\infty}((0,T);L^{3r}(\mathbb{T}^3)) \hookrightarrow B^{\theta}_{2s,\infty}((0,T);L^{3r}(\mathbb{T}^3))$, that comes from (i). Thus we conclude with

$$\int_0^{T-h} \|q^1(t)\|^s_{L^r}\,dt \le Ch^{s(2\theta-1-\varepsilon)}. \tag{3.36}$$

Similarly, we obtain

$$\begin{aligned}\int_0^{T-h} \|q^2(t)\|^s_{L^r}\,dt &\le C\int_0^{T-h} \|(\nabla p\otimes u)(t+h)-\nabla p\otimes u)(t)\|^s_{L^r}\,dt \le Ch^{s(2\theta-1-\varepsilon)}\|\nabla p\otimes u\|^s_{B^{2\theta-1-\varepsilon}_{s,\infty}(L^r)}\\ &\le Ch^{s(2\theta-1-\varepsilon)}\left(\|\nabla p\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\|u\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\right)^s\\ &\le Ch^{s(2\theta-1-\varepsilon)}\left(\|\nabla p\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\|u\|_{B^{\theta}_{2s,\infty}(L^{2r})}\right)^s \le Ch^{s(2\theta-1-\varepsilon)},\end{aligned} \tag{3.37}$$

where we used that $u\in B^{\theta}_{2s,\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (i), and $\nabla p\in B^{2\theta-1-\varepsilon}_{2s,\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (ii). Summing up (3.36) and (3.37) we obtain $\partial_t p\in B^{2\theta-1-\varepsilon}_{s,\infty}((0,T);L^r(\mathbb{T}^3))$, as desired. □

Lemma 3.12. *Let $u \in L^{3s}((0,T);B^\theta_{3r,\infty}(\mathbb{T}^3))$ for some $r,s \in [1,\infty]$ and $\theta \in (1/2,1)$. Then $\partial_t p$ solves*

$$\Delta\partial_t p = \operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) + 2\operatorname{div}\operatorname{div}(\nabla p\otimes u), \tag{3.38}$$

in the distributional sense.

Proof. For every $\delta > 0$, we denote by p^δ the unique 0-average solution of

$$-\Delta p^\delta = \operatorname{div}\operatorname{div}(u_\delta \otimes u_\delta).$$

Note that by Calderón-Zygmund, $p^\delta \to p$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$ as $\delta \to 0$. Thus $\partial_t p^\delta \to \partial_t p$ in distribution. Since $\partial_t u_\delta \in L^{\frac{3s}{2}}((0,T);C^\infty(\mathbb{T}^3))$ from (3.26), we can compute

$$\begin{aligned}\partial_t \operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta) = 2\operatorname{div}\operatorname{div}(\partial_t u_\delta\otimes u_\delta) &= -\operatorname{div}\operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta\otimes u_\delta)\\ &\quad - 2\operatorname{div}\operatorname{div}(\nabla p_\delta\otimes u_\delta) + 2\operatorname{div}\operatorname{div}(\operatorname{div}R_\delta\otimes u_\delta).\end{aligned}$$

Obviously $u_\delta \to u$ in $L^{3s}((0,T);L^{3r}(\mathbb{T}^3))$. By (B.8), since $\theta > 1/2$ we have that $\operatorname{div}R_\delta \to 0$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$. Moreover by (i) in Theorem 3.1 we also have $\nabla p_\delta \to \nabla p$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$. Thus we conclude that in the distributional sense

$$\partial_t \operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta) \to -\operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) - 2\operatorname{div}\operatorname{div}(\nabla p\otimes u). \qquad \square$$

Remark 3.13. *In the above proof, one can make explicit quantitative estimates on the quantities which appear in the statement of Theorem 3.1. For instance, as regards (i) we have*

$$\begin{aligned}\|u\|_{B^\theta_{s,\infty}(L^r)} &\le C\left(\|u\|_{L^s(B^\theta_{r,\infty})} + \|u\|^2_{L^{2s}(B^\theta_{2r,\infty})}\right),\\ \|p\|_{L^s(B^{2\theta}_{r,\infty})} &\le C\|u\|^2_{L^{2s}(B^\theta_{2r,\infty})},\end{aligned}$$

for a constant $C > 0$ depending only on r,s,θ.

Remark 3.14 (The case $r = 1$). *When $r = 1$, the statements (i) and (ii) of Theorem 3.1 on the pressure may not be true in general. On the positive side, if $u \in L^{3s}((0,T);W^{1,1}(\mathbb{T}^3))$, the compensated compactness methods* [15] *give that the pressure belongs to $L^{\frac{3s}{2}}((0,T);W^{2,1}(\mathbb{T}^3))$ (namely, the result with $r = 1$ and $\theta = 1$ would hold). On the other side, however, if $r = 1$ and $\theta = 0$, the lack of the Calderón-Zygmund theory gives us that a solution p to* (8.2) *is in general not more than in the weak-$L^1(\mathbb{T}^3)$ space. Trying to repeat the proof of the abstract interpolation result of Theorem 3.7, as we did in Proposition 3.10 for $r = 1$, this constitutes a problem because we would need to apply the interpolation result with $Y_1 = L^1_{\text{weak},\operatorname{div}}$, $Y_2 = W^{2,1}_{\operatorname{div}}$. Hence, Theorem 3.7 would only give us that $p(t) \in (L^1_{\text{weak}}(\mathbb{T}^3), W^{2,1}(\mathbb{T}^3))_{\theta,1}$ and it is unclear if such space would coincide with a suitable Besov-type space.*

Proof of Theorem 3.2. The proof is just a consequence of (i), (ii) and (iv) of Theorem 3.1 together with the embeddings $W^{\theta,r} \hookrightarrow B^\theta_{r,\infty} \hookrightarrow W^{\gamma,r}$, that hold true for any $r \in [1,\infty]$ and $\theta,\gamma \in (0,1)$ with $\theta > \gamma$. $\square$

Chapter 4

Helicity regularity and conservation for incompressible Euler

4.1 Introduction

In this chapter we consider again the incompressible Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0, \end{cases} \tag{4.1}$$

in the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Letting $\omega = \operatorname{curl} u$, by taking the curl of the first equation in (4.1) one also gets the evolution equation for the vorticity ω, which is

$$\partial_t \omega + \operatorname{curl}\operatorname{div}(u \otimes u) = \partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0\,. \tag{4.2}$$

Thanks to the peculiar structure (and its related cancellation properties) of the nonlinearity $\operatorname{div}(u \otimes u)$ one can prove that, at least for smooth solutions, we have conservation of the helicity $H = H(t)$ that is defined as

$$H(t) = \int_{T^3} u(x,t) \cdot \omega(x,t)\, dx.$$

The sharpest result in the literature on the helicity conservation has been proved in [13] assuming $u \in L^3((0,T); B^{2/3}_{3,c(\mathbb{N})}(\mathbb{T}^3))$. Note that the Sobolev spaces used in this work satisfy $W^{\theta,p} \hookrightarrow B^{\theta}_{p,c(\mathbb{N})}$, thus one has helicity conservation also for $u \in L^3((0,T); W^{\frac{2}{3},3}(\mathbb{T}^3))$. Here we propose a different approach which is to treat the velocity and the vorticity as two different functions. We prove the following

Theorem 4.1. *Let $0 < \theta, \alpha < 1$ and $1 \le p, q, r, \kappa \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{\kappa} = 1$. Suppose that u is a weak solution of (4.1) such that $u \in L^{2r}((0,T); W^{\theta,2p}(\mathbb{T}^3))$ and $\omega = \operatorname{curl} u \in L^{\kappa}((0,T); W^{\alpha,q}(\mathbb{T}^3))$. If $2\theta + \alpha \ge 1$ then H is constant.*

It is not clear how this result relates to [13] since for general exponents α and θ one does not have the embeddings in $B^{2/3}_{3,c(\mathbb{N})}$.

A similar result to Theorem 4.1 has already been proved in [12]. Indeed in [12] the author proved the helicity conservation assuming $\omega = \operatorname{curl} u \in C^0((0,T);L^{3/2}(\mathbb{T}^3)) \cap L^3((0,T);B^{\alpha}_{9/5,\infty}(\mathbb{T}^3))$ for every $\alpha > \frac{1}{3}$. Theorem 4.1 is then a generalization since it treats the velocity and the vorticity separately. Indeed a direct consequence of our theorem is that in order to prove the helicity conservation it suffices to assume $\omega \in L^3((0,T);W^{\alpha,q}(\mathbb{T}^3))$ for any $\alpha > 0$ and any $q > \frac{9}{4+3\alpha}$. We refer to Remark 4.6 for a precise discussion.

Since in our incompressible setting the velocity u is completely determined by its $\operatorname{curl} u$ (thanks to the existence of a potential) then there is a range in which Theorem 4.1 is just a consequence of the conservation proved in [13] for $u \in L^3((0,T);B^{2/3}_{3,c(\mathbb{N})}(\mathbb{T}^3))$, and also a range where the hypothesis on u in Theorem 4.1 is redundant. Thus an interesting case is when the regularity assumption on the $\operatorname{curl} u$ is as weak as possible (see Remark 4.5 for a more precise discussion). For this reason one can choose $p = \infty$ and $q = 1$ getting the following

Corollary 4.2. *Let $0 < \theta, \alpha < 1$ and $1 \le r, \kappa \le \infty$ such that $\frac{1}{r} + \frac{1}{\kappa} = 1$. If $u \in L^{2r}((0,T);C^{\theta}(\mathbb{T}^3))$ is a weak solution of (4.1) such that $\omega = \operatorname{curl} u \in L^{\kappa}((0,T);W^{\alpha,1}(\mathbb{T}^3))$, where $2\theta + \alpha \ge 1$, then the helicity is constant.*

Note that the hypothesis used in Corollary 4.2 in general do not imply $u \in L^3((0,T);B^{2/3}_{3,c(\mathbb{N})}(\mathbb{T}^3))$.

A natural question is to ask whether the helicity as some regularity also in the range in which it is not necessarily constant. To answer this question, instead of the time integrability L^r_t we assume uniformity, namely L^∞_t, showing the following

Theorem 4.3. *Let $0 < \theta, \alpha < 1$ and $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that u is a weak solution of (4.1) such that $u \in L^\infty((0,T);W^{\theta,2p}(\mathbb{T}^3))$ and $\omega \in L^\infty((0,T);W^{\alpha,q}(\mathbb{T}^3))$. Then there exist a constant $C > 0$ such that*

$$|H(t) - H(s)| \le C|t-s|^{\frac{\alpha+\theta}{1-\theta}}. \tag{4.3}$$

Theorem 4.4. *Let $\frac{1}{2} < \theta < 1$ and suppose that u is a weak solution of (4.1) such that $u \in L^\infty((0,T);W^{\theta,3}(\mathbb{T}^3))$. Then there exist a constant $C > 0$ such that*

$$|H(t) - H(s)| \le C|t-s|^{\frac{2\theta-1}{1-\theta}}. \tag{4.4}$$

We remark that the assumptions L^∞ in time is fundamental in order to get Hölder regularity of $H = H(t)$, but weaker assumptions as L^r would also imply suitable Sobolev regularity. However, we are not going to exploit such hypothesis. Moreover the assumption $\theta > \frac{1}{2}$ is necessary since $u \in W^{\frac{1}{2},2}(\mathbb{T}^3)$ is the minimal assumption to ensure that the Helicity is well defined.

The proofs of Theorem 4.3 and Theorem 4.4 make use of the same technique introduced in Chapter 2, since with this kind of equations one can easily prove Hölder regularity of energy and helicity by looking at the regularized versions of (4.1) and (4.2). Note that the previous theorems still give the helicity conservation if the two Hölder exponents in (4.3) and (4.4) are bigger than 1, which is $2\theta+\alpha>1$ and $\theta>\frac{2}{3}$ respectively. The reader might be confused about the critical hypothesis $2\theta+\alpha=1$ and $\theta=\frac{2}{3}$, which in Theorem 4.3 and Theorem 4.4 respectively just imply Lipschitz continuity of the helicity instead of conservation, but we remark that the borderline conservation is achieved in Theorem 4.1 and in [13] thanks to a limit procedure which is missing in Theorem 4.3.

Since our Corollary 4.2 shows the conservation of the helicity if $2\theta+\alpha\geq 1$, then choosing $\theta<\frac{1}{3}$ and the corresponding $\alpha=1-2\theta$, there might exist solutions such that $H=H(t)$ is constant but the energy is not. However we are not able to produce such solutions since in the current works on Hölder based convex integration techniques we do not have a strong control on the $\operatorname{curl} u$ in some Sobolev space as the one required here. In a similar direction and in view of the helicity conservation of [13], one could also aim to construct $\theta-$Hölder continuous solutions for some $\theta\in\left(\frac{1}{2},\frac{2}{3}\right)$, which do not conserve the helicity, but to date there is no Hölder based convex integration scheme which crosses the barrier $\theta=\frac{1}{3}$. We remark that in the Sobolev setting, this barrier $\frac{1}{3}$ has been crossed in the recent work [9] in which the authors constructed $L^\infty((0,T);H^{1/2-}(\mathbb{T}^3))$ weak solutions of Euler whose kinetic energy is not constant in time.

4.2 Helicity for smooth solutions

Before proving Theorem 4.1 we start considering the helicity for a smooth solution u of (4.1). By smoothness we can directly compute the first derivative of $H=H(t)$, using equations (4.1) and (4.2), getting

$$\begin{aligned}\frac{d}{dt}H(t)&=\int_{\mathbb{T}^3}(\partial_t u\cdot\omega+u\cdot\partial_t\omega)\,dx\\&=-\int_{\mathbb{T}^3}\big((u\cdot\nabla)u+\nabla p\big)\cdot\omega\,dx-\int_{\mathbb{T}^3}\big((u\cdot\nabla)\omega-(\omega\cdot\nabla)u\big)\cdot u\,dx\\&=-\int_{\mathbb{T}^3}\operatorname{div}\left(p\,\omega+u(u\cdot\omega)-\frac{|u|^2}{2}\,\omega\right)dx=0\,,\end{aligned}$$

where we used the following relations

$$\begin{aligned}\omega\cdot(u\cdot\nabla)u+u\cdot(u\cdot\nabla)\omega&=\operatorname{div}\big(u(u\cdot\omega)\big)\\u\cdot(\omega\cdot\nabla)u&=\frac{1}{2}\operatorname{div}\big(|u|^2\omega\big)\\\omega\cdot\nabla p&=\operatorname{div}(p\,\omega)\,.\end{aligned}$$

Thus in the smooth setting, the previous computations easily show that the helicity is constant.

In order to deal with weak solutions (and so with low regularity) we have to mollify the equation (4.1) getting an evolution equation for the smooth quantities (u_δ, p_δ), with an "error" forcing term which is due to the non-linearity. The crucial observation in [22] is that this error has a particular commutator structure and thus satisfies better estimates than u_δ. Since we also have to deal with the vorticity ω, we will mollify both equations (4.1) and (4.2) and we will see that the commutators have exactly the same structure.

Before proving our main results we start with two remarks about the hypothesis of Theorem 4.1.

Remark 4.5. *Theorem 4.1 is stated for any couple of exponents $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The reader may wonder if the assumption on u could be redundant, since it could be a consequence of the one on the* curl *u. Indeed in our incompressible setting we have $u = \operatorname{curl}\left((-\Delta)^{-1}\operatorname{curl} u\right)$. In particular, if $\operatorname{curl} u \in W^{\alpha,q}(\mathbb{T}^3)$, $1 < q < \infty$, by Calderón-Zygmund we get $u \in W^{1+\alpha,q}(\mathbb{T}^3)$ and by Sobolev embeddings we have that $W^{1+\alpha,q} \hookrightarrow W^{\theta,\frac{2q}{1-q}}$ if*

$$q > \frac{9}{5 + 2(\alpha - \theta)}. \tag{4.5}$$

In the case $q = 1$ we have $u \in W^{1+\alpha-\varepsilon,1}(\mathbb{T}^3)$ for any $\varepsilon > 0$, but this is obviously not enough to guarantee any Hölder regularity on u.

Remark 4.6. *For any $\alpha > 0$, we assume $\omega \in W^{\alpha,q}(\mathbb{T}^3)$ and we choose $\theta = \frac{1-\alpha}{2}$, so that the helicity is preserved. Then by* (4.5) *we have that $u \in W^{\frac{1-\alpha}{2},\frac{2q}{1-q}}(\mathbb{T}^3)$ if*

$$q > \frac{9}{4 + 3\alpha}.$$

Thus we have that the assumption $\alpha > \frac{1}{3}$ in [12] *is not necessary if one assume more on the integrability exponent q.*

We highlight that if $u \in W^{\frac{1}{2},2}(\mathbb{T}^3)$ then the helicity is the action of the distribution $\operatorname{curl} u \in W^{-\frac{1}{2},2}(\mathbb{T}^3)$ on the velocity u and it can be represented as

$$H(t) = \int_{\mathbb{T}^3} (-\Delta)^{\frac{1}{4}} u \cdot (-\Delta)^{-\frac{1}{4}} \operatorname{curl} u \, dx. \tag{4.6}$$

Note that by Cauchy-Schwarz and Calderón-Zygmund we have

$$|H(t)| \leq \left\| (-\Delta)^{\frac{1}{4}} u(t) \right\|_{L^2(\mathbb{T}^3)} \left\| (-\Delta)^{-\frac{1}{4}} \operatorname{curl} u(t) \right\|_{L^2(\mathbb{T}^3)} \leq C \|u(t)\|^2_{W^{\frac{1}{2},2}(\mathbb{T}^3)}.$$

4.3 Helicity conservation for weak solutions

We first mollify equations (4.1) and (4.2), getting

$$\partial_t u_\delta + \operatorname{div}(u_\delta \otimes u_\delta) + \nabla p_\delta = \operatorname{div} R_\delta\,, \tag{4.7}$$

$$\partial_t \omega_\delta + (u_\delta \cdot \nabla)\omega_\delta - (\omega_\delta \cdot \nabla) u_\delta = \operatorname{curl}\operatorname{div} R_\delta\,, \tag{4.8}$$

where $R_\delta = u_\delta \otimes u_\delta - (u \otimes u)_\delta$. Now we consider the helicity H_δ related to the smooth vector fields u_δ, ω_δ, namely the function

$$H_\delta(t) = \int_{\mathbb{T}^3} u_\delta(x,t) \cdot \omega_\delta(x,t)\, dx\,. \tag{4.9}$$

By the regularity of u and ω it is clear that for almost every $t \geq 0$, $H_\delta(t) \to H(t)$ as $\delta \to 0$. We can now compute the time derivative of H_δ. Using (4.7) and (4.8) as in Section 4.2, we get

$$\begin{aligned}\frac{d}{dt} H_\delta(t) = &- \int_{\mathbb{T}^3} \operatorname{div}\Big(p_\delta\, \omega_\delta + u_\delta (u_\delta \cdot \omega_\delta) - \frac{|u_\delta|^2}{2}\, \omega_\delta\Big)\, dx \\ &+ \int_{\mathbb{T}^3} \omega_\delta \cdot \operatorname{div} R_\delta\, dx + \int_{\mathbb{T}^3} u_\delta \cdot \operatorname{curl}\operatorname{div} R_\delta\, dx \\ &= -2 \int_{\mathbb{T}^3} \nabla \omega_\delta : R_\delta\, dx\,, \end{aligned} \tag{4.10}$$

where in the last equality we integrated by parts. Thus we have that

$$\big|H_\delta(t) - H_\delta(0)\big| \leq 2 \int_0^t \int_{\mathbb{T}^3} |\nabla \omega_\delta|(x,s) |R_\delta|(x,s)\, dx ds \leq 2 \int_0^t \|\nabla \omega_\delta(s)\|_{L^q(\mathbb{T}^3)} \|R_\delta(s)\|_{L^p(\mathbb{T}^3)}\, ds\,, \tag{4.11}$$

and by Proposition B.3 we conclude that

$$\big|H_\delta(t) - H_\delta(0)\big| \leq C \delta^{2\theta + \alpha - 1} \int_0^t [\omega(s)]_{W^{\alpha,q}(\mathbb{T}^3 \llcorner B_\delta)} [u(s)]^2_{W^{\theta,2p}(\mathbb{T}^3 \llcorner B_\delta)}\, ds\,.$$

The unusual notation $W^{\alpha,q}(\mathbb{T}^3 \llcorner B_\delta)$ denotes the sobolev norm restricted to the ball B_δ, see Appendix A. Note that in the previous estimate we used two conjugate exponents $1 < p, q < \infty$. The case in which one of them is equal to 1 (or equivalently ∞) is analogous. Finally, using Hölder inequality with exponents r, κ we achieve

$$\big|H_\delta(t) - H_\delta(0)\big| \leq C \delta^{2\theta + \alpha - 1} [\omega]_{L^\kappa((0,T); W^{\alpha,q}(\mathbb{T}^3 \llcorner B_\delta))} [u]^2_{L^{2r}((0,T); W^{\theta,2p}(\mathbb{T}^3 \llcorner B_\delta))}\,,$$

thus the claim follows by letting $\delta \to 0$.

4.4 Helicity regularity for weak solutions

Proof of Theorem 4.3

Now we will see how the L^∞ in time assumption leads to some Hölder continuity of the helicity, even without the assumption $2\theta+\alpha \geq 1$.

We define $H_\delta(t)$ as in (4.9). For any couple of times s,t we estimate

$$\left|H(t)-H(s)\right| \leq \left|H(t)-H_\delta(t)\right| + \left|H_\delta(t)-H_\delta(s)\right| + \left|H_\delta(s)-H(s)\right|. \tag{4.12}$$

By the L^∞_t assumption, both the first and the third term can be estimated by using (B.11) with $r=1$ and $p=q=2$ as follows

$$\left|H(t)-H_\delta(t)\right| + \left|H_\delta(s)-H(s)\right| \leq C\delta^{\theta+\alpha}\|u\|_{L^\infty((0,T);W^{\theta,p}(\mathbb{T}^3))}\|\omega\|_{L^\infty((0,T);W^{\alpha,q}(\mathbb{T}^3))},$$

where, in order to apply (B.11), we also used the the property $H(t)=\int_{\mathbb{T}^3} u\cdot\omega = \int_{\mathbb{T}^3}(u\cdot\omega)_\delta$. We are left with the second summand in the right hand side of (4.12). We have that

$$\left|H_\delta(t)-H_\delta(s)\right| \leq |t-s|\left\|\frac{d}{dt}H_\delta\right\|_{L^\infty(0,T)},$$

and by (4.10), together with Proposition B.3, we get

$$\left|H_\delta(t)-H_\delta(s)\right| \leq C|t-s|\delta^{2\theta+\alpha-1}\|u\|^2_{L^\infty((0,T);W^{\theta,2p}(\mathbb{T}^3))}\|\omega\|_{L^\infty((0,T);W^{\alpha,q}(\mathbb{T}^3))}.$$

Combining the previous estimates with (4.12) we achieved

$$\left|H(t)-H(s)\right| \leq C(\delta^{\theta+\alpha}+|t-s|\delta^{2\theta+\alpha-1}),$$

for some constant $C>0$, which depends on both u,ω. Finally, by choosing $\delta=|t-s|^{\frac{1}{1-\theta}}$ we can conclude

$$\left|H(t)-H(s)\right| \leq C|t-s|^{\frac{\alpha+\theta}{1-\theta}}.$$

Proof of Theorem 4.4

The proof runs in the same way as the one for Theorem 4.3. By equation (4.10) and using (B.10) and (B.11) we have

$$\begin{aligned}\left|H_\delta(t)-H_\delta(s)\right| &\leq |t-s|\left\|\frac{d}{dt}H_\delta\right\|_{L^\infty(0,T)} \leq 2|t-s|\|\nabla\mathrm{curl}\,u_\delta\|_{L^\infty((0,T);L^3(\mathbb{T}^3))}\|R_\delta\|_{L^\infty((0,T);L^{3/2}(\mathbb{T}^3))}\\ &\leq C|t-s|\delta^{3\theta-2}[u]^3_{L^\infty((0,T);W^{\theta,3}(\mathbb{T}^3))}.\end{aligned}$$

Since, for every $\delta > 0$,

$$H(t) = \int_{\mathbb{T}^3} (-\Delta)^{1/4} u \cdot (-\Delta)^{-1/4} \operatorname{curl} u \, dx = \int_{\mathbb{T}^3} \big((-\Delta)^{1/4} u \cdot (-\Delta)^{-1/4} \operatorname{curl} u\big)_\delta \, dx,$$

by applying (B.11) with $r = 1$, we deduce that for every $t \geq 0$

$$\begin{aligned} \big|H(t) - H_\delta(t)\big| &\leq C\delta^{2\theta-1} [(-\Delta)^{1/4} u]_{L^\infty((0,T);W^{\theta-1/2,2}(\mathbb{T}^3))} [(-\Delta)^{-1/4} \operatorname{curl} u]_{L^\infty((0,T);W^{\theta-1/2,2}(\mathbb{T}^3))} \\ &\leq C\delta^{2\theta-1} [u]^2_{L^\infty((0,T);W^{\theta,2}(\mathbb{T}^3))}, \end{aligned}$$

where in the last inequality we also used Calderón-Zygmund estimates. Thus we have

$$\begin{aligned} \big|H(t) - H(s)\big| &\leq \big|H(t) - H_\delta(t)\big| + \big|H_\delta(t) - H_\delta(s)\big| + \big|H_\delta(s) - H(s)\big| \\ &\leq C\Big(\delta^{2\theta-1} [u]^2_{L^\infty((0,T);W^{\theta,2}(\mathbb{T}^3))} + |t-s|\delta^{3\theta-2} [u]^3_{L^\infty((0,T);W^{\theta,2}(\mathbb{T}^3))}\Big), \end{aligned}$$

from which we can conclude by choosing $\delta = |t-s|^{\frac{1}{1-\theta}}$.

Chapter 5

Nonuniqueness of Leray-Hopf weak solutions to the hypodissipative Navier-Stokes

5.1 Introduction

In this chapter we consider the Cauchy problem for the incompressible fractional Navier-Stokes equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\gamma v = 0 \\ \operatorname{div} v = 0 \\ v(\cdot, 0) = \overline{v}, \end{cases} \tag{5.1}$$

in the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3 \setminus \mathbb{Z}^3$.

We are interested in Leray-Hopf weak solutions of (5.1), namely weak solutions $v \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}^+, H^\gamma(\mathbb{T}^3))$ satisfying the global energy inequality

$$\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\, dx + \int_s^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\gamma/2} v\right|^2 (x,\tau)\, dx\, d\tau \le \frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,s)\, dx, \qquad \forall\, 0 \le s < t. \tag{5.2}$$

As for the Navier-Stokes equations (i.e. the case $\gamma = 1$), it is known that such solutions exist and it is also known that, if the power γ of the Laplacian is suitably small, then these solutions are not unique. Indeed in [16] the authors proved the ill-posedness in the case $\gamma < 1/5$. The question about uniqueness is still open for a general power γ. In this chapter we partially answer this question, proving the non-uniqueness of such solutions in the range $0 < \gamma < 1/3$. More precisely the main result is the following

Theorem 5.1. *Let $\gamma < 1/3$. Then there are initial data $\overline{v} \in L^2(\mathbb{T}^3)$ with $\operatorname{div} \overline{v} = 0$ for which there exist infinitely many Leray solutions v of (5.1) in $[0, +\infty) \times \mathbb{T}^3$. More precicely, if $\gamma < \beta < 1/3$, there are initial data $\overline{v} \in C^\beta(\mathbb{T}^3)$ with $\operatorname{div} \overline{v} = 0$ and a positive time T such that*

(*i*) *there are infinitely many Leray-Hopf solutions v of (5.1) and moreover $v \in C^\beta(\mathbb{T}^3 \times [0, T])$;*

(ii) *such solutions strictly dissipate the total energy in* $[0,T]$*, i.e. the function (of time only)*

$$E_{tot}(t) = \frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx + \int_0^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\gamma/2} v\right|^2 (x,\tau)\,dx\,d\tau \tag{5.3}$$

is strictly decreasing in $[0,T]$.

The proof of Theorem 5.1 is achieved by using the "convex integration methods" introduced by C. De Lellis and L. Székelyhidi for the incompressible Euler equations, in particular the costruction used in [7], where the authors, thanks to the new ideas introduced by Daneri and Székelyhidi in [25] and P. Isett in [41], proved the existence of $C^{1/3-}_{x,t}$ solutions of Euler equations with prescribed kinetic energy. This methods can be also used to prove the ill-posedness for the distributional solutions of the Navier-Stokes equations (i.e. $\gamma = 1$). Indeed, recently, in [10] T. Buckmaster and V. Vicol proved the existence of infinitely many weak solutions of the Navier-Stokes equations with bounded kinetic energy. The solutions constructed in [8] do not even have finite energy dissipation in the sense of $E_{tot} < \infty$, thus they are not of Leray-Hopf type.

In order to use the argument proposed in [7], we have to construct exact solutions of (5.1) in small time intervals. The corresponding stability estimates of such solutions, with respect to the initial data, are also needed. To this aim we prove new stability estimates for classical solutions of non-local advection-diffusion equations.

Following [16] we will see that if the exponent γ is not too large (in particular $\gamma < 1/3$), then the methods used in [7] to produce Hölder continuous solutions to the Euler equations with prescribed kinetic energy can be adapted to equations (5.1). Then we will be able to produce (different) solutions with different kinetic energy profile, let all of them start from the same initial data and keep under control the dissipative part in the definition of E_{tot} (see (5.3)).

For the reader convenience we recall here the fractional Navier-Stokes equations with some viscosity $\mu > 0$

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + \mu(-\Delta)^{\gamma} v = 0 \\ \operatorname{div} v = 0\,. \end{cases} \tag{5.4}$$

Using the main iterative proposition (Proposition 5.9) we are able to show the existence of dissipative solutions of Euler which can be obtained as a vanishing viscosity limit of solutions of (5.4). The main idea is taken from [8] where the authors proved that Hölder continuous solutions of Euler arise as a strong limit in $C^0([0,T];L^2(\mathbb{T}^3))$ as $\mu \to 0$, of weak solutions of the classical Navier-Stokes equations. Again by the restriction $\gamma < 1/3$, we are able to produce a sequence Leray-Hopf weak solutions of (5.4) converging to a dissipative solution of Euler, as $\mu \to 0$. More precicely we prove the following

Theorem 5.2. *Let* $\beta' < 1/3$*. There exist dissipative solutions* $v \in C^{\beta'}([0,T]\times\mathbb{T}^3)$ *of Euler such that, if* $0 < \gamma < \beta'$*, there exists a sequence* $\mu_n \to 0$ *and a sequence* $v^{(\mu_n)}$ *of Leray-Hopf weak solutions of* (5.4) *such that* $v^{(\mu_n)} \to v$ *strongly in* $C^0([0,T];C^{\beta''}(\mathbb{T}^3))$ *for every* $\beta'' < \beta'$.

Also in this case, if we only want to require that the sequence $v^{(\mu_n)}$ is just a sequence of weak solutions of (5.4), bounded in $L^\infty((0,T);L^2(\mathbb{T}^3))$, we could also prove that for any $\gamma < 1/2$ there exists a sequence of solutions of (5.4) converging to any Hölder solution of Euler, as $\mu \to 0$, but in order to be consistent with the arguments of this work, we will not enter in these details.

5.2 Proof of the nonuniqueness

In order to show Theorem 5.1 we will prove a slightly more general result about (5.1). Indeed, using the inductive scheme proposed in [7], we are able to prove the following

Theorem 5.3. *Let* $e : [0,1] \to \mathbb{R}^+$ *with the following properties*

(i) $1/2 \le e(t) \le 1$, $\forall t \in [0,1]$;

(ii) $\sup_t |e'(t)| \le K$, *for some* $K > 1$.

Then for all $\gamma < \beta < 1/3$ *there exists a couple* (v,p), *solving*

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\gamma v = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{5.5}$$

in the sense of distributions, such that $v \in C^\beta(\mathbb{T}^3 \times [0,1])$ *and*

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x,t)\,dx, \tag{5.6}$$

$$\|v\|_\beta \le C_\beta K^{4/9}, \tag{5.7}$$

where C_β *is a constant depending only on* β. *Moreover, given any two energy profiles* e_1 *and* e_2 *such that* $e_1(0) = e_2(0)$, *then the two corresponding solutions* $v^{(1)}$ *and* $v^{(2)}$ *start from the same initial data, i.e.* $v^{(1)}(\cdot,0) \equiv v^{(2)}(\cdot,0)$.

We end this section by proving Theorem 5.1, then the rest of the paper will be devoted to the proof of Theorem 5.3.

Proof of Theorem 5.1. Elementary arguments produce for every $K > 1$ an infinite set $\mathscr{E}_K$ of smooth functions $e : [0,1] \to \mathbb{R}$ with the following properties:

(i) $1/2 \le e(t) \le 1$, $\forall t \in [0,1]$;

(ii) $\|e\|_{C^1([0,1])} \le 2K + 2$;

(iii) $e(0) = 1$;

(iv) $e'(t) \le -2K + 2$, $\forall t \in [0, \frac{1}{4K}]$;

(v) for any pair of distinct elements of $\mathscr{E}_K$ there is a sequence of times converging to 0 where they take different values.

For each $e \in \mathscr{E}_K$, we now use Theorem 5.3 to produce infinitely many weak solutions satisfying

(a) $e(t) = \frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx$;

(b) $v \in C^\beta(\mathbb{T}^3 \times [0,1]), \quad \forall \beta < 1/3$;

(c) $v(\cdot,0) = \overline{v}$, for some $\overline{v} \in C^\beta(\mathbb{T}^3)$;

(d) $\|v\|_\beta \le C_\beta K^{4/9}$.

Let $T = 1/4K$. We have to show that all these solutions strictly dissipate the total energy, which is equivalent to

$$\frac{1}{2}\big(e(s) - e(t)\big) > \int_s^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\gamma/2} v\right|^2 (x,\tau)\,dx\,d\tau, \quad \forall 0 \le s < t \le T\,. \tag{5.8}$$

By our assumptions on the functions $e(t)$ and using Corollary C.2 we have

$$\frac{1}{2}\big(e(s) - e(t)\big) \ge (K-1)(t-s), \quad \forall 0 \le s < t \le T;$$
$$\int_s^t \int_{\mathbb{T}^3} \left|(-\Delta)^{\gamma/2} v\right|^2 (x,\tau)\,dx\,d\tau \le (t-s) C_\varepsilon \|v\|^2_{\gamma+\varepsilon}.$$

Chosing ε so that $\gamma + \varepsilon = \beta$, we see that (5.8) holds if the constant K satisfies

$$K - 1 > C_{\beta,\gamma} K^{8/9}, \tag{5.9}$$

where $C_{\beta,\gamma}$ depends only on γ and β, but not on K. It is clear that there exists a K (big enough) such that (5.9) is satisfied. Thus we have proved the existence of infinitely many Leray-Hopf solutions in the interval $[0,T]$ satisfying (a) and (b) of Theorem 5.1. Finally, each solutions can be prolonged to Leray-Hopf solutions for every $t \ge 0$, thus the proof is concluded. □

5.3 Local smooth solutions

Maximum principle and stability estimates

We begin by stating a maximum principle result for a non-local operator. The proof is standard, since, as for the local case (i.e. using the Laplacian), we have that $(-\Delta)^\gamma u(x_0) \ge 0$ whenever x_0 is a global maximum point of u (see for instance the integral representation formula (C.1)).

Theorem 5.4. *Define $Q_T = \mathbb{T}^3 \times (0,T]$. Let L be the pseudo-differential operator defined as $Lu = (v \cdot \nabla)u + \mu(-\Delta)^\gamma u$, where $u : \mathbb{T}^3 \times [0,T] \to \mathbb{R}$, $v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ is a given vector field and $\mu > 0$, $0 < \gamma \le 1$. The following holds:*

(i) *if* $u_t + Lu \le 0$ *in* Q_T, *then* $\max_{\overline{Q}_T} u = \max_{\mathbb{T}^3\times\{0\}} u$;

(ii) *if* $u_t + Lu \ge 0$ *in* Q_T, *then* $\min_{\overline{Q}_T} u = \min_{\mathbb{T}^3\times\{0\}} u$.

In Using Theorem 5.4 we can prove a stability estimate for a general class of non-local parabolic equations. Indeed we have

Proposition 5.5. *Let* $u : \mathbb{T}^3 \times [t_0, T] \to \mathbb{R}^3$ *be a solution of the Cauchy problem*

$$\begin{cases} u_t + Lu = f \\ u(\cdot, t_0) = u_0 \,. \end{cases} \tag{5.10}$$

Then for any $t \in [t_0, T]$ *we have*

$$\|u(t)\|_0 \le \|u_0\|_0 + \int_{t_0}^t \|f(s)\|_0 \, ds \,, \tag{5.11}$$

$$[u(t)]_1 \le [u_0]_1 e^{(t-t_0)[v]_1} + \int_{t_0}^t e^{(t-s)[v]_1} [f(s)]_1 \, ds \,, \tag{5.12}$$

and, more generally, for any $N \ge 2$ *there exists a constant* $C = C_N$ *so that*

$$\begin{aligned} [u(t)]_N \le\ & ([u_0]_N + C(t-t_0)[v]_N [u_0]_1)\, e^{C(t-t_0)[v]_1} \\ & + \int_{t_0}^t e^{(t-s)[v]_1} \left([f(s)]_N + C(t-s)[v]_N [f(s)]_1\right) ds \,. \end{aligned} \tag{5.13}$$

Proof. We may assume that u and f are two scalar functions, indeed we can work on each component of equation (5.10). Note also that Theorem 5.4 is invariant under the time shifting $t \mapsto t + t_0$.

Defining

$$w = u - \int_{t_0}^t \|f(s)\|_0 \, ds \,,$$

we have

$$\begin{cases} w_t + Lw = f - \|f(t)\|_0 \le 0 \\ w(\cdot, t_0) = u_0 \,. \end{cases}$$

Thus, by Theorem 5.4, we have

$$u(x,t) \le \|u_0\|_0 + \int_{t_0}^t \|f(s)\|_0 \, ds. \tag{5.14}$$

Applying the same argument to the function $\tilde{w} = u + \int_{t_0}^t \|f(s)\|_0 \, ds$, we get the bound from below, showing (5.11).

Next, differentiate (5.10) in the x variable to obtain

$$(Du)_t + LDu = Df - DvDu \,.$$

Applying (5.11) to Du yields

$$[u(t)]_1 \leq [u_0]_1 + \int_{t_0}^{t} \big([f(s)]_1 + [v]_1[u(s)]_1\big)\, ds\,,$$

and by Grönwall's inequality we get (5.12). Now, differentiating (5.10) N times yields

$$(D^N u)_t + LD^N u = D^N f + \sum_{k=0}^{N-1} c_{k,N} D^{k+1} u D^{N-k} v\,. \tag{5.15}$$

Using again (5.11) we can estiamte

$$[u(t)]_N \leq [u_0]_N + \int_{t_0}^{t} \big([f(s)]_N + C([v]_N[u(s)]_1 + [v]_1[u(s)]_N)\big)\, ds\,,$$

and plugging the estimate (5.12), we get

$$\begin{aligned}[u(t)]_N \leq [u_0]_N + C(t-t_0)[v]_N[u_0]_1 e^{(t-t_0)[v]_1} + \int_{t_0}^{t} \Big([f(s)]_N \\ + C[v]_N \int_{t_0}^{s} e^{(s-r)[v]_1}[f(r)]_1\, dr + C[v]_1[u(s)]_N\Big)\, ds\,,\end{aligned}$$

and Grönwall's inequality finally leads to (5.13). □

Using Proposition 5.5 we also get the following

Proposition 5.6. *Assume* $0 \leq (t-t_0)[v]_1 \leq 1$. *Then, any solution u of* (5.10) *satisfies*

$$\|u(t)\|_\alpha \leq e^\alpha \left(\|u_0\|_\alpha + \int_{t_0}^{t} \|f(\cdot,\tau)\|_\alpha\, d\tau\right)\,, \tag{5.16}$$

for all $0 \leq \alpha \leq 1$*, and, more generally, for any* $N \geq 1$ *and* $0 \leq \alpha < 1$

$$[u(t)]_{N+\alpha} \lesssim [u_0]_{N+\alpha} + (t-t_0)[v]_{N+\alpha}[u_0]_1 + \int_{t_0}^{t} \Big([f(\tau)]_{N+\alpha} + (t-\tau)[v]_{N+\alpha}[f(\tau)]_1\Big)\, d\tau\,, \tag{5.17}$$

where the implicit constant depends only on N and α*.*

Proof. For any $\alpha \in [0,1]$, let

$$w(x,t;h) = \frac{\delta_h u(x,t)}{|h|^\alpha} = \frac{u(x+h,t) - u(x,t)}{|h|^\alpha}\,.$$

We have that this new function w satisfies (see equation (4.13) in [23])

$$(\partial_t + \mu(-\Delta)^\gamma + v\cdot\nabla_x + \delta_h v\cdot\nabla_h)\, w = \alpha \frac{\delta_h v}{|h|}\cdot\frac{h}{|h|} w + \frac{\delta_h f}{|h|^\alpha}\,,$$

Thus by the maximum principle[1] (5.11) and since $\sup_{h,x}|w(x,t;h)| = [u(t)]_\alpha$, we get

$$[u(t)]_\alpha \le [u_0]_\alpha + \int_{t_0}^{t} \big(\alpha[v(s)]_1[u(s)]_\alpha + [f(s)]_\alpha\big)\,ds\,,$$

from which, by Grönwall's inequality, (5.16) follows.

To get the higher order bounds (5.17) just differentiate the equation N times as in (5.15) and apply the previous argument with

$$w(x,t;h) = \frac{\delta_h D^N u(x,t)}{|h|^\alpha} = \frac{D^N u(x+h,t) - D^N u(x,t)}{|h|^\alpha}\,,$$

then (5.17) is again a consequence of (5.11) and Grönwall's inequality. □

Local smooth solutions to fractional Navier-Stokes

We want to consider exact (smooth) solutions to the fractional Navier-Stokes equations

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p + \mu(-\Delta)^\gamma v = 0 \\ \operatorname{div} v = 0 \\ v(\cdot,0) = u_0, \end{cases} \tag{5.18}$$

in the periodic setting $\mathbb{T}^3 \times [0,T]$, where $\gamma \in (0,1)$ and $\mu \ge 0$. We define the space

$$V^m = \{v \in H^m(\mathbb{T}^3) \,:\, \text{div } v = 0\}\,.$$

We start with the following

Theorem 5.7. *For any $m \ge 3$ there exists a constant $c_m = c(m)$ such that the following holds. Given any initial condition $u_0 \in V^m$ and $T_m = c_m\|u_0\|_V^{-1}$ there exists a unique solution $v \in C([0,T_m],V^m) \cap C^1([0,T_m],V^{m-2})$. Moreover we have the estimate*

$$\|v(t)\|_{V^m} \le \|u_0\|_{V^m} e^{c_m \int_0^t \|\nabla v(s)\|_0\,ds} \qquad \forall t \in [0,T_m]\,. \tag{5.19}$$

For a proof of Theorem 5.7 we refer to [50] (Theorem 3.4 in Chapter 3). Notice that that theorem is stated for the classical Navier-Stokes equations. The proof uses the so called "energy method" and it can be easily adapted to any power γ of the Laplacian in the equations (5.18).

We now want to prove that there exists a maximal time of existence (independent on m) of such solution. In particular, if the initial datum is smooth, we get the local existence of a smooth solution of (5.18). We also prove some stability estimates of such solution in Hölder spaces, since they will play a crucial role in the iterative construction.

[1] Here the maximum principle is applied in both the variables x, h.

Proposition 5.8. *For any $\mu \geq 0$ and any $0 < \alpha < 1$ there exists a constant $c = c(\alpha) > 0$ with the following property. Given any initial data $u_0 \in C^\infty$, and $T \leq c\|u_0\|_{1+\alpha}^{-1}$, there exists a unique solution $v : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ of* (5.18). *Moreover, v obeys the bounds*

$$\|v\|_{N+\alpha} \lesssim \|u_0\|_{N+\alpha} \,. \tag{5.20}$$

for all $N \geq 1$, where the implicit constant depends on N and $\alpha > 0$.

Proof of Proposition 5.8. We first show that all solutions given by Theorem 5.7 exist in the interval $[0,T]$, for any $T \lesssim \|u_0\|_{1+\alpha}^{-1}$. Fix any $\alpha \in (0,1)$ and let T^* be the maximal time such that

$$T^* \sup_{0 \leq t \leq T^*} [v(t)]_1 \leq 1 \,.$$

Suppose $T^* < c\|u_0\|_{1+\alpha}^{-1}$, for some constant $c = c(\alpha)$ to be fixed later (we will see that this contraddicts the assumption on the maximality of T^*, in particular $T^* \geq c\|u_0\|_{1+\alpha}^{-1}$). Using Schauder estimate on $-\Delta p = \nabla v^T : \nabla v$ we have

$$\|p(t)\|_{2+\alpha} \lesssim \|v(t)\|_{1+\alpha}^2 \,,$$

thus, differentiating the equation in the x variable we get

$$\|(\partial_t + v \cdot \nabla + \mu(-\Delta)^\gamma) Dv\|_\alpha \lesssim \|v(t)\|_{1+\alpha}^2 \,.$$

By Proposition 5.6, for any $0 \leq t \leq T^*$, we have

$$\|v(t)\|_{1+\alpha} \lesssim \|u_0\|_{1+\alpha} + \int_0^t \|v(s)\|_{1+\alpha}^2 \, ds \,.$$

Finally, using Grönwall's inequality we get the estimate

$$\|v(t)\|_{1+\alpha} \lesssim \|u_0\|_{1+\alpha} < \frac{1}{T^*} \quad \forall t \in [0,T^*] \,,$$

where in the last inequality we have choosen the constant $c = c(\alpha)$ to get it "strict". Obviously, this contraddicts the hypothesis on the maximality of T^*, and also gives the a priori estimate (5.20) for $N = 1$, which together with (5.19), gives the existence of a smooth solution in the interval $[0,T]$, for any $T \leq c\|u_0\|_{1+\alpha}^{-1}$.

We are left with the higher-order bounds (5.20) for $N \geq 2$. For any multi-index θ with $|\theta| = N$ we have

$$\partial_t \partial^\theta v + v \cdot \nabla \partial^\theta v + \mu(-\Delta)^\gamma \partial^\theta v + [\partial^\theta, v \cdot \nabla] v + \nabla \partial^\theta p = 0.$$

Using again Schauder estimates for the pressure we obtain

$$\|\nabla \partial^\theta p\|_\alpha \lesssim \|\mathrm{tr}\,(\nabla v \nabla v)\|_{N-1+\alpha} \lesssim \|v\|_{1+\alpha} \|v\|_{N+\alpha}.$$

Therefore

$$\|(\partial_t + v \cdot \nabla + \mu(-\Delta)^\gamma) \partial^\theta v\|_\alpha \lesssim \|v\|_{1+\alpha} \|v\|_{N+\alpha},$$

and (5.20) follows by applying (5.16) and Grönwall's inequality. □

5.4 The main theorems as a consequence of an inductive proposition

As already outlined, the main construction is taken from [7], thus we are not going to prove all technical details about the mechanism of the convex integration scheme. However all the proofs of the propositions involving the structure of the Navier-Stokes equations (different from the Euler ones), are completely self contained.

5.4.1 Inductive proposition

First of all, we impose for the moment that

$$\sup_{t\in[0,1]} |e'(t)| \le 1 \tag{5.21}$$

(we will see later that this can be done provided that we impose some conditions on the parameters appearing in the iteration).

Let then $q \geq 0$ be a natural number. At a given step q we assume to have a triple $(v_q, p_q, \mathring{R}_q)$ to the fractional Navier-Stokes Reynolds system, namely such that

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q + \mu(-\Delta)^\gamma v_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0\,, \end{cases} \tag{5.22}$$

to which we add the constraints

$$\operatorname{tr} \mathring{R}_q = 0\,, \tag{5.23}$$

$$\int_{\mathbb{T}^3} p_q(x,t)\, dx = 0\,. \tag{5.24}$$

In (5.22) the viscosity μ is just some small constant (in particular $\mu < 1$) depending on some parameters of the inductive construction. In what follows we will see that this coefficient comes from a "technical rescaling" on the equations (5.1).

The size of the approximate solution v_q and the error $\mathring{R}_q$ will be measured by a frequency λ_q and an amplitude δ_q, which are given by

$$\lambda_q = 2\pi \lceil a^{(b^q)} \rceil \tag{5.25}$$

$$\delta_q = \lambda_q^{-2\beta} \tag{5.26}$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$, $a > 1$ is a large parameter, $b > 1$ is close to 1 and $0 < \beta < 1/3$ is the exponent of Theorem 5.3. The parameters a and b are then related to β.

We proceed by induction, assuming the estimates

$$\|\mathring{R}_q\|_0 \le \delta_{q+1}\lambda_q^{-3\alpha} \tag{5.27}$$

$$\|v_q\|_1 \le M\delta_q^{1/2}\lambda_q \tag{5.28}$$

$$\|v_q\|_0 \le 1-\delta_q^{1/2} \tag{5.29}$$

$$\delta_{q+1}\lambda_q^{-\alpha} \le e(t) - \int_{\mathbb{T}^3} |v_q|^2\,dx \le \delta_{q+1} \tag{5.30}$$

where $0<\alpha<1$ is a small parameter to be chosen suitably (which will depend upon β), and M is a universal constant.

Proposition 5.9. *There exists a universal constant M with the following property. Let $0<\beta<1/3$, $0<\gamma<1/3$ and*

$$1 < b < \min\left\{\frac{1-\beta}{2\beta}, \frac{1-\beta}{2\gamma}, \frac{4}{3}\right\}. \tag{5.31}$$

Then there exists an α_0 depending only on β and b, such that for any $0<\alpha<\alpha_0$ there exists an a_0 depending on β, b, α and M, such that for any $a \ge a_0$ the following holds: given a strictly positive function $e:[0,T]\to\mathbb{R}^+$ satisfying (5.21), and a triple $(v_q, p_q, \mathring{R}_q)$ solving (5.22)-(5.24) and satisfying the estimates (5.27)–(5.30), then there exists a solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ to (5.22)-(5.24) satisfying (5.27)–(5.30) with q replaced by $q+1$. Moreover, we have

$$\|v_{q+1}-v_q\|_0 + \frac{1}{\lambda_{q+1}}\|v_{q+1}-v_q\|_1 \le M\delta_{q+1}^{1/2}. \tag{5.32}$$

Furthermore, $v_{q+1}(\cdot,0)$ depends only on $e(0)$ and $v_q(\cdot,0)$.

The proof of Proposition 5.9 is summarized in the Sections 5.5.1, 5.5.2 and 5.5.3. We show next that this proposition immediately implies Theorem 5.3.

5.4.2 Prescribing the energy

Here we prove Theorem 5.3. First of all, we fix any Hölder exponent $\beta<1/3$ and also the parameters b and α, the first satisfying (5.31) and the second smaller than the threshold given in Proposition 5.9. Next we show that, without loss of generality, we may further assume the energy profile satisfies

$$\inf_t e(t) \ge \delta_1\lambda_0^{-\alpha}, \qquad \sup_t e(t) \le \delta_1, \quad \text{and} \quad \sup_t e'(t) \le 1, \tag{5.33}$$

provided the parameter a is chosen sufficiently large. To see this, we first make the following transformations

$$\tilde{v}(x,t) = \mu\, v(x,\mu t) \qquad \tilde{p}(x,t) = \mu^2 p(x,\mu t). \tag{5.34}$$

Thus if we choose

$$\mu = \delta_1^{1/2},$$

the stated problem reduces to finding a solution $(\tilde{v}, \tilde{p})$ of

$$\begin{cases} \partial_t \tilde{v} + \tilde{v}\cdot\nabla\tilde{v} + \nabla\tilde{p} + \mu(-\Delta)^\gamma \tilde{v} = 0 \\ \operatorname{div}\tilde{v} = 0 \end{cases} \tag{5.35}$$

with the energy profile given by

$$\tilde{e}(t) = \mu^2 e(\mu t),$$

for which we have (using our assumptions on the function $e(t)$)

$$\inf_t \tilde{e}(t) \geq \delta_1 \inf_t e(t) \geq \frac{\delta_1}{2}, \qquad \sup_t \tilde{e}(t) \leq \delta_1, \qquad \text{and} \qquad \sup_t \tilde{e}'(t) \leq \delta_1^{3/2} K.$$

If a is chosen sufficiently large, in particualar $a \geq a_0 K^{1/3\beta}$, then we can ensure

$$\sup_t \tilde{e}'(t) \leq \delta_1^{3/2} K \leq 1, \qquad \text{and} \qquad \frac{1}{2} \geq \lambda_0^{-\alpha}.$$

Now we apply Proposition 5.9 iteratively with $(v_0, R_0, p_0) = (0,0,0)$. Indeed the pair (v_0, R_0) trivially satisfies (5.27)–(5.29), whereas the estimate (5.30) and (5.21) follows as a consequence of (5.33). Notice that by (5.32) v_q converges uniformly to some continuous $\tilde{v}$. Moreover, we recall that the pressure is determined by

$$\Delta p_q = \operatorname{div}\operatorname{div}(-v_q \otimes v_q + \mathring{R}_q) \tag{5.36}$$

and (5.24) and thus p_q is also converging to some pressure $\tilde{p}$ (for the moment only in L^r for every $r < \infty$). Since $\mathring{R}_q \to 0$ uniformly, the pair $(\tilde{v}, \tilde{p})$ solves equations (5.35). Observe that using (5.32) we also infer

$$\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{\beta'} \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_0^{1-\beta'} \|v_{q+1} - v_q\|_1^{\beta'} \lesssim \sum_{q=0}^{\infty} \delta_{q+1}^{\frac{1-\beta'}{2}} \left(\delta_{q+1}^{1/2} \lambda_q\right)^{\beta'} \lesssim \sum_{q=0}^{\infty} \lambda_q^{\beta'-\beta}$$

and hence that v_q is uniformly bounded in $C^0([0,1]; C^{\beta'}(\mathbb{T}^3))$ for all $\beta' < \beta$. Using the last inequality and the definitions of the parameter λ_q we also have that if a is chosen sufficiently large, then

$$\|\tilde{v}\|_{\beta'} \leq 1, \qquad \forall \beta' < \beta.$$

Since $\delta_{q+1} \to 0$ as $q \to \infty$, from (5.30) we have

$$\int_{\mathbb{T}^3} |\tilde{v}|^2 \, dx = \tilde{e}(t),$$

If now we use the transformation

$$v(x,t) = \mu^{-1}\, \tilde{v}(x, \mu^{-1} t) \quad \text{and} \quad p(x,t) = \mu^{-2} \tilde{p}(x, \mu^{-1} t),$$

then it is clear that the pair (v, p) solves (5.5) and it satisfies (5.6) and (5.7). The time regularity is a consequence of (2.2) from Chapter 2.

5.4.3 Dissipative Euler solutions as vanishing viscosity limit

Here we prove Theorem 5.2. Let $v \in C^{\beta'}(\mathbb{T}^3 \times [0,T])$ be a dissipative solution of Euler, with the kinetic energy profile satisfying the assumptions $(i)-(v)$ in the proof of Theorem 5.1 (note that the proof of the existence of such solution is given in [7]). Using the rescaling (5.34), with $\mu = (2\|v\|_0)^{-1}$, we can assume that $\|v\|_0 \le 1/2$.

We fix two positive kernels (Friedrichs mollifiers) φ and ψ, respectively in space and time. Let $\delta_n = a^{-b^{n+2}}$ and $\mu_n = \delta_n^{1+\beta'}$. Since v solves Euler, the smooth function $v_n = (v * \varphi_{\delta_n}) * \psi_{\delta_n}$ solves the following Navier-Stokes Reynolds equations

$$\partial_t v_n + \operatorname{div}(v_n \otimes v_n) + \nabla p_n + \mu_n(-\Delta)^\gamma v_n = \operatorname{div} \mathring{R}_n ,$$

with

$$\mathring{R}_n = v_n \mathring{\otimes} v_n - (v \mathring{\otimes} v)_n + \mu_n \mathscr{R}(-\Delta)^\gamma v_n ,$$

where $f \mathring{\otimes} g$ is the traceless part of the matrix $f \otimes g$ and $\mathscr{R}$ is the operator defined in (D.1). We also define the energy as

$$e_n(t) = \int_{\mathbb{T}^3} |v_n|^2 \, dx + \delta_{n+1} \lambda_n^{-\alpha} . \tag{5.37}$$

Using standard mollification estimates and (B.2) we have

$$\|v_n\|_1 \lesssim \delta_n^{\beta'-1} ,$$
$$\|\mathring{R}_n\|_0 \lesssim \delta_n^{2\beta'} + \mu_n [v_n]_1 \lesssim \delta_n^{2\beta'} .$$

Thus, if we chose $\gamma < \beta < \beta'$ and the parameter a large enough, we can guarantee that (5.27)-(5.30) hold for $q = n$, provided that b is sufficiently near 1 and α is small. We can now apply Proposition 5.9 (inductively for $q \ge n$) in order to obtain a solution $v^{(\mu_n)}$ of (5.4), and since $\gamma < \beta$ (as already done in the proof of Theorem 5.1) we can guarantee that $v^{(\mu_n)}$ is indeed a Leray-Hopf weak solution. Moreover by (5.32) we have

$$\|v^{(\mu_n)} - v_n\|_{\beta''} \le \sum_{q \ge n} \|v_{q+1} - v_q\|_{\beta''} \lesssim \sum_{q \ge n} a^{(\beta''-\beta)b^{q+1}} .$$

Thus, provided that the parameter a is chosen even larger, we can ensure that

$$\|v^{(\mu_n)} - v\|_{\beta''} \le \|v^{(\mu_n)} - v_n\|_{\beta''} + \|v_n - v\|_{\beta''} \le \frac{1}{n} , \qquad \forall \beta'' < \beta ,$$

and this concludes the proof of the theorem. We also remark that $e_n(t) \to \int_{\mathbb{T}^3} |v|^2 \, dx$ as $n \to +\infty$.

5.5 The convex integration scheme and proof of the iterative proposition

The rest of this chapter is devoted to the proof of Proposition 5.9. To simplify several estimates we will assume that α is small enough so to have

$$\lambda_q^{3\alpha} \leq \left(\frac{\delta_q}{\delta_{q+1}}\right)^{3/2} \leq \frac{\lambda_{q+1}}{\lambda_q}\,, \tag{5.38}$$

in which we also need that a is big enough to nullify any constant from the ratio $\lambda_q/a^{(b^q)}$, which can be easily bounded as

$$2\pi \leq \frac{\lambda_q}{a^{b^q}} \leq 4\pi\,. \tag{5.39}$$

Following the construction of [7] we subdivide the proof in three stages, in each of which we modify v_q: mollification, gluing and perturbation.

5.5.1 Mollification step

The first stage is mollification: we mollify v_q (in space) at length scale

$$\ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+3\alpha/2}}\,. \tag{5.40}$$

Fix a standard mollification kernel ψ, we define

$$\begin{aligned} v_\ell &= v_q * \psi_\ell \\ \mathring{R}_\ell &= \mathring{R}_q * \psi_\ell - (v_q \mathring{\otimes} v_q) * \psi_\ell + v_\ell \mathring{\otimes} v_\ell\,. \end{aligned}$$

These functions obey the equation

$$\begin{cases} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell + \mu(-\Delta)^\gamma v_\ell = \operatorname{div} \mathring{R}_\ell \\ \operatorname{div} v_\ell = 0\,, \end{cases} \tag{5.41}$$

in view of (5.22). Observe, again choosing α sufficiently small and a sufficiently large we can assume

$$\lambda_q^{-3/2} \leq \ell \leq \lambda_q^{-1}\,, \tag{5.42}$$

which will be used in order to simplify several estimates. From standard mollification estimates we obtain the following bounds (we refer to [7] for a detailed proof).

Proposition 5.10.

$$\|v_\ell - v_q\|_0 \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\alpha}, \tag{5.43}$$

$$\|v_\ell\|_{N+1} \lesssim \delta_q^{1/2}\lambda_q \ell^{-N} \qquad \forall N \geq 0, \tag{5.44}$$

$$\|\mathring{R}_\ell\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \geq 0. \tag{5.45}$$

$$\left| \int_{\mathbb{T}^3} |v_q|^2 - |v_\ell|^2 \, dx \right| \lesssim \delta_{q+1}\ell^\alpha. \tag{5.46}$$

5.5.2 Gluing step

In the second stage we glue together exact solutions to the fractional Navier-Stokes equations in order to produce a new $\overline{v}_q$, close to v_q, whose associated Reynolds stress error has support in pairwise disjoint temporal regions of length τ_q in time, where

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}. \tag{5.47}$$

Note that we have the CFL-like condition

$$2\tau_q \|v_\ell\|_{1+\alpha} \overset{(5.44)}{\lesssim} \tau_q \delta_q^{1/2}\lambda_q \ell^{-\alpha} \lesssim \ell^\alpha \ll 1 \tag{5.48}$$

as long as a is sufficiently large. More precisely, we aim to construct a new triple $(\overline{v}_q, \overline{\mathring{R}}_q, \overline{p}_q)$ solving the Navier-Stokes Reynolds equation (5.22) such that the temporal support of $\overline{\mathring{R}}_q$ is contained in pairwise disjoint intervals I_i of length $\sim \tau_q$ and such that the gaps between neighbouring intervals is also of length $\sim \tau_q$.

For each i, let $t_i = i\tau_q$, and consider smooth solutions of the fractional Navier-Stokes equations

$$\begin{cases} \partial_t v_i + \operatorname{div}(v_i \otimes v_i) + \nabla p_i + \mu(-\Delta)^\gamma v_i = 0 \\ \operatorname{div} v_i = 0 \\ v_i(\cdot, t_i) = v_\ell(\cdot, t_i). \end{cases} \tag{5.49}$$

defined over their own maximal interval of existence. An immediate consequence of (5.44), (5.47) and Proposition 5.8 is the following

Corollary 5.11. *If a is sufficiently large, for $0 \leq (t - t_i) \leq 2\tau_q$, we have*

$$\|v_i\|_{N+\alpha} \lesssim \delta_q^{1/2}\lambda_q \ell^{1-N-\alpha} \lesssim \tau_q^{-1}\ell^{1-N+\alpha} \qquad \text{for any } N \geq 1. \tag{5.50}$$

We will now show that for $0 \leq (t - t_i) \leq 2\tau_q$, v_i is close to v_ℓ and by the identity

$$v_i - v_{i+1} = (v_i - v_\ell) - (v_{i+1} - v_\ell),$$

the vector field v_i is also close to v_{i+1}.

Proposition 5.12. *For* $0 \leq (t-t_i) \leq 2\tau_q$, $N \geq 0$ *and* $0 < \mu < 1$ *we have*

$$\|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N-1+\alpha}, \tag{5.51}$$

$$\|\nabla(p_\ell - p_i)\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N-1+\alpha}, \tag{5.52}$$

$$\|L_{t,\ell,\gamma}(v_i - v_\ell)\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N-1+\alpha}, \tag{5.53}$$

$$\|D_{t,\ell}(v_i - v_\ell)\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N-1+\alpha}, \tag{5.54}$$

where we write

$$D_{t,\ell} = \partial_t + v_\ell \cdot \nabla \qquad L_{t,\ell,\gamma} = D_{t,\ell} + \mu(-\Delta)^\gamma. \tag{5.55}$$

Proof. Let us first consider (5.51) with $N = 0$. From (5.41) and (5.49) we have

$$L_{t,\ell,\gamma}(v_\ell - v_i) = (v_i - v_\ell) \cdot \nabla v_i - \nabla(p_\ell - p_i) + \operatorname{div} \mathring{R}_\ell. \tag{5.56}$$

In particular, using

$$\Delta(p_\ell - p_i) = \operatorname{div}\big(\nabla v_\ell (v_i - v_\ell)\big) + \operatorname{div}\big(\nabla v_i (v_i - v_\ell)\big) + \operatorname{div}\operatorname{div} \mathring{R}_\ell, \tag{5.57}$$

estimates (5.45) and (5.50), and Proposition F.1 (recall that $\partial_i \partial_j (-\Delta)^{-1}$ is given by $1/3\,\delta_{ij}$ + a Calderón-Zygmund operator), we conclude

$$\|\nabla(p_\ell - p_i)(\cdot,t)\|_\alpha \leq \delta_q^{1/2} \lambda_q \ell^{-\alpha} \|v_i - v_\ell\|_\alpha + \delta_{q+1} \ell^{-1+\alpha}.$$

Thus, using (5.45) and the definition of τ_q, we have

$$\|L_{t,\ell,\gamma}(v_\ell - v_i)\|_\alpha \lesssim \delta_{q+1} \ell^{-1+\alpha} + \tau_q^{-1} \|v_\ell - v_i\|_\alpha \tag{5.58}$$

By applying (5.16) we obtain

$$\|(v_\ell - v_i)(\cdot,t)\|_\alpha \lesssim |t - t_i| \delta_{q+1} \ell^{-1+\alpha} + \int_{t_i}^t \tau_q^{-1} \|(v_\ell - v_i)(\cdot,s)\|_\alpha \, ds.$$

Applying Grönwall's inequality and using the assumption $0 \leq (t - t_i) \leq 2\tau_q$ we obtain

$$\|v_i - v_\ell\|_\alpha \lesssim \tau_q \delta_{q+1} \ell^{-1+\alpha}, \tag{5.59}$$

i.e. (5.51) for the case $N = 0$. Then as a consequence of (5.58) we obtain (5.53) for $N = 0$.

Next, consider the case $N \geq 1$ and let θ be a multiindex with $|\theta| = N$. Commuting the derivative ∂^θ with the material derivative $\partial_t + v_\ell \cdot \nabla$ we have

$$\begin{aligned}
\|L_{t,\ell,\gamma}\partial^\theta(v_\ell - v_i)\|_\alpha &\lesssim \|\partial^\theta L_{t,\ell,\gamma}(v_\ell - v_i)\|_\alpha + \|[v_\ell \cdot \nabla, \partial^\theta](v_\ell - v_i)\|_\alpha \\
&\lesssim \|\partial^\theta L_{t,\ell,\gamma}(v_\ell - v_i)\|_\alpha + \|v_\ell\|_{N+\alpha} \|v_\ell - v_i\|_{1+\alpha} + \|v_\ell\|_{1+\alpha} \|v_\ell - v_i\|_{N+\alpha} \\
&\lesssim \|\partial^\theta L_{t,\ell,\gamma}(v_\ell - v_i)\|_\alpha + \|v_\ell\|_{N+1+\alpha} \|v_\ell - v_i\|_\alpha + \|v_\ell\|_{1+\alpha} \|v_\ell - v_i\|_{N+\alpha},
\end{aligned}$$

On the other hand differentiating (5.56) leads to

$$\begin{aligned}\|\partial^\theta L_{t,\ell,\gamma}(v_\ell - v_i)\|_\alpha &\lesssim \|v_\ell - v_i\|_{N+\alpha}\|v_i\|_{1+\alpha} + \|v_\ell - v_i\|_\alpha \|v_i\|_{N+1+\alpha} + \|p_\ell - p_i\|_{N+1+\alpha} + \|\mathring{R}_\ell\|_{N+1+\alpha} \\ &\lesssim \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha} + \delta_{q+1}\ell^{-N-1+\alpha} + \|\nabla(p_\ell - p_i)\|_{N+\alpha}\,, \end{aligned} \tag{5.60}$$

where we have used (5.59). Furthermore, from (5.57) we also obtain, using Corollary 5.11 and (5.59)

$$\begin{aligned}\|\nabla(p_\ell - p_i)\|_{N+\alpha} &\lesssim (\|v_\ell\|_{N+1+\alpha} + \|v_i\|_{N+1+\alpha})\|v_\ell - v_i\|_\alpha \\ &\quad + (\|v_\ell\|_{1+\alpha} + \|v_i\|_{1+\alpha})\|v_\ell - v_i\|_{N+\alpha} + \|\mathring{R}_\ell\|_{N+1+\alpha} \\ &\lesssim \delta_{q+1}\ell^{-N-1+\alpha} + \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha}\,. \end{aligned} \tag{5.61}$$

Summarizing, for any multiindex θ with $|\theta| = N$ we obtain

$$\|L_{t,\ell,\gamma}\partial^\theta(v_\ell - v_i)\|_\alpha \lesssim \delta_{q+1}\ell^{-N-1+\alpha} + \tau_q^{-1}\|v_\ell - v_i\|_{N+\alpha}.$$

Therefore, invoking once more (5.16) we deduce

$$\|(v_\ell - v_i)(\cdot,t)\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N-1+\alpha} + \int_{t_i}^t \tau_q^{-1}\|(v_\ell - v_i)(\cdot,s)\|_{N+\alpha}\,ds,$$

and hence, using Grönwall's inequality and the assumption $0 \le (t - t_i) \le 2\tau_q$ we obtain (5.51). From (5.61) and (5.60) we then also conclude (5.52) and (5.53). We are only left with (5.54). By Theorem C.1 and estimate (5.51) we have

$$\mu\|(-\Delta)^\gamma(v_\ell - v_i)\|_{N+\alpha} \lesssim \|v_\ell - v_i\|_{N+2\gamma+2\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-N-1-2\gamma-\alpha}\,.$$

If a is chosen sufficiently large we can ensure $\ell^{-1} \le \lambda_{q+1}$ and, using (5.31), we get

$$\tau_q\ell^{-2\gamma-2\alpha} \le \frac{\lambda_{q+1}^{2\gamma}}{\delta_q^{1/2}\lambda_q} \le 1$$

from which we deduce

$$\mu\|(-\Delta)^\gamma(v_\ell - v_i)\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N-1+\alpha}\,. \tag{5.62}$$

Finally, combining (5.53), (5.62) and triangular inequality, we get (5.54) □

Define the vector potentials to the solutions v_i as

$$z_i = \mathscr{B}v_i = (-\Delta)^{-1}\operatorname{curl} v_i, \tag{5.63}$$

where $\mathscr{B}$ is the Biot-Savart operator, so that

$$\operatorname{div} z_i = 0 \qquad \text{and} \qquad \operatorname{curl} z_i = v_i - \int_{\mathbb{T}^3} v_i\,. \tag{5.64}$$

Our aim is to obtain estimates for the differences $z_i - z_{i+1}$.

Proposition 5.13. *For* $0 \le (t-t_i) \le 2\tau_q$*, we have that*

$$\|z_i - z_{i+1}\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+\alpha}\,, \tag{5.65}$$

$$\|D_{t,\ell}(z_i - z_{i+1})\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha}\,. \tag{5.66}$$

Proof. Set $\tilde{z}_i = \mathscr{B}(v_i - v_\ell)$ and observe that $z_i - z_{i+1} = \tilde{z}_i - \tilde{z}_{i+1}$. Hence, it suffices to estimate $\tilde{z}_i$ in place of $z_i - z_{i+1}$.

The estimate on $\|\nabla \tilde{z}_i\|_{N-1+\alpha}$ for $N \ge 1$ follows directly from (5.51) and the fact that $\nabla \mathscr{B}$ is a bounded operator on Hölder spaces:

$$\|\nabla \tilde{z}_i\|_{N-1+\alpha} \le \|\nabla \mathscr{B}(v_i - v_\ell)\|_{N-1+\alpha} \|v_i - v_\ell\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell^{-N+\alpha}\,. \tag{5.67}$$

Next, observe that

$$\partial_t (v_i - v_\ell) + v_\ell \cdot \nabla (v_i - v_\ell) + (v_i - v_\ell) \cdot \nabla v_i + \nabla (p_i - p_\ell) + \mu (-\Delta)^\gamma (v_i - v_\ell) + \operatorname{div} \mathring{R}_\ell = 0. \tag{5.68}$$

Since $v_i - v_\ell = \operatorname{curl} \tilde{z}_i$ with $\operatorname{div} \tilde{z}_i = 0$, we have[2]

$$\begin{aligned} v_\ell \cdot \nabla (v_i - v_\ell) &= \operatorname{curl}\big((v_\ell \cdot \nabla) \tilde{z}_i\big) + \operatorname{div}\big((\tilde{z}_i \times \nabla) v_\ell\big) \\ ((v_i - v_\ell) \cdot \nabla) v_i &= \operatorname{div}\big((\tilde{z}_i \times \nabla) v_i^T\big), \end{aligned}$$

so that we can write (5.68) as

$$\operatorname{curl}\left(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i + \mu(-\Delta)^\gamma \tilde{z}_i\right) = -\operatorname{div}\big((\tilde{z}_i \times \nabla) v_\ell + (\tilde{z}_i \times \nabla) v_i^T\big) - \nabla (p_i - p_\ell) - \operatorname{div} \mathring{R}_\ell. \tag{5.69}$$

Taking the curl of (5.69) the pressure term drops out. Using in addition that $\operatorname{div} \tilde{z}_i = \operatorname{div}(v_i - v_\ell) = 0$ and the identity $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div}$, we then arrive at

$$-\Delta\big(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i + \mu(-\Delta)^\gamma \tilde{z}_i\big) = F,$$

where

$$F = -\nabla \operatorname{div}\left((\tilde{z}_i \cdot \nabla) v_\ell\right) - \operatorname{curl}\operatorname{div}\big((\tilde{z}_i \times \nabla) v_\ell + (\tilde{z}_i \times \nabla) v_i^T\big) - \operatorname{curl}\operatorname{div} \mathring{R}_\ell.^3$$

Consequently,

$$\begin{aligned} \|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i + \mu(-\Delta)^\gamma \tilde{z}_i\|_{N+\alpha} &\lesssim (\|v_i\|_{N+1+\alpha} + \|v_\ell\|_{N+1+\alpha}) \|\tilde{z}_i\|_\alpha \\ &\quad + (\|v_i\|_{1+\alpha} + \|v_\ell\|_{1+\alpha}) \|\tilde{z}_i\|_{N+\alpha} + \|\mathring{R}_\ell\|_{N+\alpha} \\ &\lesssim \tau_q^{-1} \|\tilde{z}_i\|_{N+\alpha} + \tau_q^{-1} \ell^{-N} \|\tilde{z}_i\|_\alpha + \delta_{q+1} \ell^{-N+\alpha}. \end{aligned} \tag{5.70}$$

[2] Here we use the notation $[(z \times \nabla) v]^{ij} = \varepsilon_{ikl} z^k \partial_l v^j$ for vector fields z, v.

[3] In deriving the latter equality we have used the identity $\nabla \operatorname{div}((v_\ell \cdot \nabla)\tilde{z}_i) = \nabla \operatorname{div}((\tilde{z}_i \cdot \nabla) v_\ell)$, which follows easily from the fact that both v_ℓ and $\tilde{z}_i$ are divergence free.

Setting $N = 0$ and using (5.16) and Grönwall's inequality we obtain

$$\|\tilde{z}_i\|_\alpha \lesssim \tau_q \delta_{q+1} \ell^\alpha ,$$

which together with (5.67) gives (5.65). Using (5.65) into (5.70) we get

$$\|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i + \mu(-\Delta)^\gamma \tilde{z}_i\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha} .$$

Thus we conclude

$$\begin{aligned}\|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i\|_{N+\alpha} &\lesssim \delta_{q+1}\ell^{-N+\alpha} + \|(-\Delta)^\gamma \tilde{z}_i\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} + \|\tilde{z}_i\|_{N+2\gamma+2\alpha} \\ &\lesssim \delta_{q+1}\ell^{-N+\alpha}\left(1 + \tau_q \ell^{-2\gamma-2\alpha}\right) \le \delta_{q+1}\ell^{-N+\alpha} .\end{aligned}$$

□

Proceding as in [7], we now glue the solutions v_i together in order to construct $\overline{v}_q$. Let

$$\begin{aligned}&t_i = i\tau_q, \qquad I_i = \left[t_{i+1} + \tfrac{1}{3}\tau_q, t_{i+1} + \tfrac{2}{3}\tau_q\right] \cap [0,T] ,\\ &J_0 = \left[0, t_1 + \tfrac{1}{3}\tau_q\right) , \qquad J_i = \left(t_{i+1} - \tfrac{1}{3}\tau_q, t_{i+1} + \tfrac{1}{3}\tau_q\right) \cap [0,T] \quad i \ge 1 .\end{aligned}$$

Note that $\{I_i, J_i\}_i$ is a decomposition of $[0,T]$ into pairwise disjoint intervals. Note also that these definitions of J_i, I_i are slightly different from the one used in [7]. The reason is that our stability estimates for smooth solutions of the fractional Navier-Stokes equations hold for $0 \le t - t_i \le \tau_q$ as opposed to $|t - t_i| \le \tau_q$ in [7].

We define a partition of unity $\{\chi_i\}_i$ in time with the following properties:

- The cut-offs form a partition of unity
$$\sum_i \chi_i \equiv 1. \tag{5.71}$$
- $\operatorname{supp}\chi_i \cap \operatorname{supp}\chi_{i+2} = \emptyset$ and moreover
$$\begin{aligned}\operatorname{supp}\chi_0 &\subset \left[0, t_1 + \tfrac{2}{3}\tau_q\right) ,\\ \operatorname{supp}\chi_i &\subset I_{i-1} \cup J_i \cup I_i ,\\ \chi_i(t) &= 1 \quad \text{ for } t \in J_i .\end{aligned} \tag{5.72}$$
- For any i and N we have
$$\|\partial_t^N \chi_i\|_0 \lesssim \tau_q^{-N} . \tag{5.73}$$

We define

$$\begin{aligned}\overline{v}_q &= \sum_i \chi_i v_i ,\\ \overline{p}_q^{(1)} &= \sum_i \chi_i p_i .\end{aligned}$$

Observe that $\operatorname{div}\overline{v}_q = 0$. Furthermore, if $t \in I_i$, then $\chi_i + \chi_{i+1} = 1$ and $\chi_j = 0$ for $j \neq i, i+1$, therefore on I_i:

$$\begin{aligned}\overline{v}_q &= \chi_i v_i + (1-\chi_i) v_{i+1}\\ \overline{p}_q^{(1)} &= \chi_i p_i + (1-\chi_i) p_{i+1}\end{aligned}$$

and

$$\partial_t \overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla \overline{p}_q^{(1)} + \mu(-\Delta)^\gamma \overline{v}_q = \partial_t \chi_i (v_i - v_{i+1}) - \chi_i(1-\chi_i)\operatorname{div}\left((v_i - v_{i+1}) \otimes (v_i - v_{i+1})\right).$$

On the other hand, if $t \in J_i$ then $\chi_i = 1$ and $\chi_j(\tilde{t}) = 0$ for all $j \neq i$ for all $\tilde{t}$ sufficiently close to t (since J_i is open). Then for all $t \in J_i$ we have

$$\overline{v}_q = v_i, \quad \overline{p}_q^{(1)} = p_i,$$

and, from (5.49),

$$\partial_t \overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla \overline{p}_q^{(1)} + (-\Delta)^\gamma \overline{v}_q = 0.$$

In order to define the new Reynolds tensor, we recall the operator $\mathscr{R}$ from (D.1). Thus we define

$$\begin{aligned}\mathring{\overline{R}}_q &= \partial_t \chi_i \mathscr{R}(v_i - v_{i+1}) - \chi_i(1-\chi_i)(v_i - v_{i+1}) \mathring{\otimes} (v_i - v_{i+1})\\ \overline{p}_q^{(2)} &= -\chi_i(1-\chi_i)\left(|v_i - v_{i+1}|^2 - \int_{\mathbb{T}^3} |v_i - v_{i+1}|^2\, dx\right),\end{aligned}$$

for $t \in I_i$ and $\mathring{\overline{R}}_q = 0$, $\overline{p}_q^{(2)} = 0$ for $t \notin \bigcup_i I_i$. Furthermore, we set

$$\overline{p}_q = \overline{p}_q^{(1)} + \overline{p}_q^{(2)}$$

It follows from the preceding discussion and Proposition D.1 that

- $\mathring{\overline{R}}_q$ is a smooth symmetric and traceless 2-tensor;
- For all $(x,t) \in \mathbb{T}^3 \times [0,T]$

$$\begin{cases} \partial_t \overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla \overline{p}_q + \mu(-\Delta)^\gamma \overline{v}_q = \operatorname{div} \mathring{\overline{R}}_q,\\ \operatorname{div} \overline{v}_q = 0;\end{cases}$$

- $\operatorname{supp} \mathring{\overline{R}}_q \subset \mathbb{T}^3 \times \bigcup_i I_i$.

Next, we estimate the various Hölder norms of $\overline{v}_q$ and $\mathring{\overline{R}}_q$.

Proposition 5.14. *The velocity field* $\overline{v}_q$ *and the new Reynolds stress tensor* $\mathring{\overline{R}}_q$ *satisfy the following estimates*

$$\|\overline{v}_q - v_\ell\|_\alpha \lesssim \delta_{q+1}^{1/2}\ell^\alpha \tag{5.74}$$

$$\|\overline{v}_q - v_\ell\|_{N+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-N+\alpha} \tag{5.75}$$

$$\|\overline{v}_q\|_{1+N} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N} \tag{5.76}$$

$$\|\mathring{\overline{R}}_q\|_{N+\alpha} \lesssim \delta_{q+1}\ell^{-N+\alpha} \tag{5.77}$$

$$\|(\partial_t + \overline{v}_q\cdot\nabla)\mathring{\overline{R}}_q\|_{N+\alpha} \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N-\alpha} \tag{5.78}$$

for all $N \geq 0$*. Moreover the difference of the energies of* $\overline{v}_q$ *and* v_ℓ *satisfies*

$$\left|\int_{\mathbb{T}^3} |\overline{v}_q|^2 - |v_\ell|^2 dx\right| \lesssim \delta_{q+1}\ell^\alpha\,. \tag{5.79}$$

Proof. The estimates (5.74)–(5.78) are consequence of Propositions 5.12 and 5.63 (the proof can be found in [7]). However we prove explicitly (5.79) since it involves the structure of the dissipative term.

Observe that for $t \in I_i$

$$\begin{aligned}\overline{v}_q\otimes\overline{v}_q &= (\chi_i v_i + (1-\chi_i)v_{i+1})\otimes(\chi_i v_i + (1-\chi_i)v_{i+1})\\ &= \chi_i v_i\otimes v_i + (1-\chi_i)v_{i+1}\otimes v_{i+1} - \chi_i(1-\chi_i)(v_i - v_{i+1})\otimes(v_i - v_{i+1}),\end{aligned}$$

so that, taking the trace:

$$|\overline{v}_q|^2 - |v_\ell|^2 = \chi_i(|v_i|^2 - |v_\ell|^2) + (1-\chi_i)(|v_{i+1}|^2 - |v_\ell|^2) - \chi_i(1-\chi_i)|v_i - v_{i+1}|^2$$

Next, recall that v_i and v_ℓ are smooth solutions of (5.49) and (5.41) respectively, therefore

$$\left|\frac{d}{dt}\int_{\mathbb{T}^3} |v_i|^2 - |v_\ell|^2\,dx\right| = 2\left|\int_{\mathbb{T}^3}\nabla v_\ell : \mathring{R}_\ell\,dx\right| + 2\mu\left|\int_{\mathbb{T}^3}\left(\left|(-\Delta)^{\gamma/2}v_i\right|^2 - \left|(-\Delta)^{\gamma/2}v_\ell\right|^2\right)dx\right|.$$

Using (5.45) and (5.50), we estimate

$$\left|\int_{\mathbb{T}^3}\nabla v_\ell : \mathring{R}_\ell\,dx\right| \lesssim \|\nabla v_\ell\|_0\|\mathring{R}_\ell\|_0 \lesssim \delta_q^{1/2}\lambda_q\delta_{q+1} \lesssim \tau_q^{-1}\delta_{q+1}\ell^\alpha\,.$$

Moreover, since $\|v_q\|_\gamma \leq 1$ for every $\gamma < \beta$ (as already exploited in the proof of Proposition 5.9), by (5.51), Theorem C.1 and Cauchy-Schwarz inequality we have, for all $t \in I_i$

$$\left|\int_{\mathbb{T}^3}\left(\left|(-\Delta)^{\gamma/2}v_i\right|^2 - \left|(-\Delta)^{\gamma/2}v_\ell\right|^2\right)dx\right| \lesssim \|v_i - v_\ell\|_{\gamma+\alpha} \lesssim \tau_q\delta_{q+1}\ell^{-1-\gamma} \lesssim \tau_q^{-1}\delta_{q+1}\ell^\alpha\,,$$

where in the last inequality (remember the restriction $\gamma < 1/3$) we have used

$$\ell^{-1-\gamma} \leq \ell^{-4/3} \overset{(5.40)}{=} \frac{\left(\delta_q^{1/2}\lambda_q\right)^{4/3}}{\delta_{q+1}^{2/3}} \lambda_q^{2\alpha} \overset{(5.47)}{=} \tau_q^{-2}\ell^{\alpha}\ell^{5\alpha/3}\lambda_q^{2\alpha}\left(\tau_q\delta_{q+1}^{-1}\right)^{2/3} \overset{(5.42)}{\leq} \tau_q^{-2}\ell^{\alpha}.$$

Moreover, $v_i = v_\ell$ for $t = t_i$. Therefore, after integrating in time we deduce

$$\left|\int_{\mathbb{T}^3} |v_i|^2 - |v_\ell|^2 \, dx\right| \lesssim \delta_{q+1}\ell^{\alpha},$$

for all $t \in I_i$. Furthermore, using (5.51) and $\delta_{q+1}^{1/2}\tau_q\ell^{-1} = \ell^{2\alpha}\lambda_q^{3\alpha/2} \overset{(5.42)}{\leq} \lambda_q^{-\alpha/2} \leq 1$

$$\int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx \lesssim \|v_i - v_{i+1}\|_\alpha^2 \lesssim \tau_q^2 \delta_{q+1}^2 \ell^{-2+2\alpha} \lesssim \delta_{q+1}\ell^{2\alpha},$$

in I_i. Therefore

$$\left|\int |\bar{v}_q|^2 - |v_\ell|^2 dx\right| \lesssim \delta_{q+1}\ell^{\alpha},$$

which concludes the proof.

□

5.5.3 Perturbation

We will now outline the construction of the perturbation w_{q+1}, where

$$v_{q+1} = w_{q+1} + \bar{v}_q .$$

The perturbation w_{q+1} is highly oscillatory and will be based on the Mikado flows introduced in [25]. Their main properties can be found in Appendix E.

First of all note that as a corollary of (5.30), (5.46) and (5.79), by choosing a sufficiently large we can ensure that

$$\frac{\delta_{q+1}}{2\lambda_q^{\alpha}} \leq e(t) - \int_{\mathbb{T}^3} |\bar{v}_q|^2 \, dx \leq 2\delta_{q+1} . \tag{5.80}$$

Starting with the solution $(\bar{v}_q, \bar{p}_q, \mathring{\bar{R}}_q)$, we then produce a new solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ of the Navier-Stokes Reynolds system (5.22) with estimates

$$\|v_{q+1} - \bar{v}_q\|_0 + \lambda_{q+1}^{-1}\|v_{q+1} - \bar{v}_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2} \tag{5.81}$$

$$\|\mathring{R}_{q+1}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}} . \tag{5.82}$$

$$\left|e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx - \frac{\delta_{q+2}}{2}\right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}} , \tag{5.83}$$

cf. Propositions 5.18, 5.19 and 5.20. Then Proposition 5.9 is just a consequence of estimates (5.81)-(5.83), Proposition 5.10 and Proposition 5.14 (again, a detailed proof can be found in [7]).

Recall that $\mathring{\overline{R}}_q$ is supported in the set $\mathbb{T}^3 \times \bigcup_i I_i$, whereas, from (5.72) it follows that $[0,T] \setminus \bigcup_i I_i = \bigcup_i J_i$, where the open intervals J_i have length $|J_i| = \frac{2}{3}\tau_q$ each, except for the first J_0 and last one, which might be shortened by the intersection with $[0,T]$, more precisely

$$J_i = \left(t_{i+1} - \tfrac{1}{3}\tau_q, t_{i+1} + \tfrac{1}{3}\tau_q\right) \cap [0,T].$$

We start by defining smooth non-negative cut-off functions $\eta_i = \eta_i(x,t)$ with the following properties

- (i) $\eta_i \in C^\infty(\mathbb{T}^3 \times [0,T])$ with $0 \le \eta_i(x,t) \le 1$ for all (x,t);
- (ii) $\operatorname{supp} \eta_i \cap \operatorname{supp} \eta_j = \emptyset$ for $i \neq j$;
- (iii) $\mathbb{T}^3 \times I_i \subset \{(x,t) : \eta_i(x,t) = 1\}$;
- (iv) $\operatorname{supp} \eta_i \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1}$;
- (v) There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0,T]$

$$\sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\,dx \ge c_0.$$

The next lemma is taken from [7].

Lemma 5.15. *There exists cut-off functions $\{\eta_i\}_i$ with the properties (i)-(v) above and such that for any i and $n,m \ge 0$*

$$\|\partial_t^n \eta_i\|_m \le C(n,m)\tau_q^{-n}$$

where $C(n,m)$ are geometric constants depending only upon m and n.

Define

$$\rho_q(t) = \frac{1}{3}\left(e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2\,dx\right)$$

and

$$\rho_{q,i}(x,t) = \frac{\eta_i^2(x,t)}{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy}\rho_q(t)$$

Define the backward flows Φ_i for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{cases} (\partial_t + \overline{v}_q \cdot \nabla)\Phi_i = 0 \\ \Phi_i(x,t_i) = x. \end{cases}$$

Define

$$R_{q,i} = \rho_{q,i}\mathrm{Id} - \eta_i^2 \mathring{\bar{R}}_q$$

and

$$\tilde{R}_{q,i} = \frac{\nabla\Phi_i R_{q,i} (\nabla\Phi_i)^T}{\rho_{q,i}}. \tag{5.84}$$

We note that, because of properties (ii)-(iv) of η_i,

- $\operatorname{supp} R_{q,i} \subset \operatorname{supp} \eta_i$;
- on $\operatorname{supp} \mathring{\bar{R}}_q$ we have $\sum_i \eta_i^2 = 1$;
- $\operatorname{supp} \tilde{R}_{q,i} \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1}$;
- $\operatorname{supp} \tilde{R}_{q,i} \cap \operatorname{supp} \tilde{R}_{q,j} = \emptyset$ for all $i \neq j$.

Lemma 5.16. *For $a \gg 1$ sufficiently large we have*

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \leq \frac{1}{2} \qquad \text{for } t \in \operatorname{supp}(\eta_i). \tag{5.85}$$

Furthermore, for any $N \geq 0$

$$\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq |\rho_q(t)| \leq \delta_{q+1} \quad \text{for all } t, \tag{5.86}$$

$$\|\rho_{q,i}\|_0 \leq \frac{\delta_{q+1}}{c_0}, \tag{5.87}$$

$$\|\rho_{q,i}\|_N \lesssim \delta_{q+1}, \tag{5.88}$$

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q, \tag{5.89}$$

$$\|\partial_t \rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}. \tag{5.90}$$

Moreover, for all (x,t)

$$\tilde{R}_{q,i}(x,t) \in B_{1/2}(\mathrm{Id}) \subset \mathscr{S}_+^{3\times 3},$$

where $B_{1/2}(\mathrm{Id})$ denotes the metric ball of radius $1/2$ around the identity Id in the space $\mathscr{S}^{3\times 3}$.

Proof of Lemma 5.16. For the estimates (5.86)-(5.88) we refer to [7]. Note that by the definition of the cut-off functions η_i

$$c_0 \leq \sum_i \int_{\mathbb{T}^3} \eta_i^2(y,t)\,dy \leq 2. \tag{5.91}$$

To prove (5.89) and (5.90) we first note that

$$\left|\frac{d}{dt}\int |\bar{v}_q(x,t)|^2\,dx\right| \leq 2\left|\int \nabla\bar{v}_q \cdot \mathring{\bar{R}}_q\,dx\right| + 2\mu \int \left|(-\Delta)^{\gamma/2}\bar{v}_q\right|^2 dx \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q \ell^\alpha$$

Thus[4]

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q$$

Then, since $\|\partial_t \eta_j\|_N \lesssim \tau_q^{-1}$ and $\delta_q^{1/2}\lambda_q \leq \tau_q^{-1}$, using (5.91), the estimate (5.90) follows. □

The constant M

The principal term of the perturbation can be written as

$$w_o = \sum_i \left(\rho_{q,i}(x,t)\right)^{1/2} (\nabla\Phi_i)^{-1} W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i) = \sum_i w_{o,i}\,, \tag{5.92}$$

where Lemma E.1 is applied with $\mathscr{N} = \overline{B}_{1/2}(\mathrm{Id})$, namely the closed ball (in the space of symmetric 3×3 matrices) of radius $1/2$ centered at the identity matrix. From Lemma 5.16 it follows that $W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i)$ is well defined. Using the Fourier series representation of the Mikado flows (E.3) we can write

$$w_{o,i} = \sum_{k\neq 0} (\nabla\Phi_i)^{-1} b_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}\,,$$

where

$$b_{i,k}(x,t) = \left(\rho_{q,i}(x,t)\right)^{1/2} a_k(\tilde{R}_{q,i}(x,t)).$$

The following is a crucial point of our construction, which ensures that the constant M of Proposition 5.9 is geometric and in particular independent of all the parameters of the construction.

Lemma 5.17. *There is a geometric constant $\bar{M}$ such that*

$$\|b_{i,k}\|_0 \leq \frac{\bar{M}}{|k|^4}\delta_{q+1}^{1/2}\,. \tag{5.93}$$

The previous lemma follows from the definition of the Mikado flows given in Appendix E and we refer to [7] for a more precise discussion.

We are finally ready to define the constant M of Proposition 5.9: from Lemma 5.17 it follows trivially that the constant is indeed geometric and hence independent of all the parameters entering in the statement of Proposition 5.9.

We can now define the geometric constant M as

$$M = 64\bar{M} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|k|^4}\,, \tag{5.94}$$

[4]Note that $\|\partial_t e\|_0 \leq 1 \leq \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q$ since $\delta_{q+1}\delta_q^{1/2}\lambda_q = \lambda_{q+1}^{-2\beta}\lambda_q^{1-\beta} \geq a^{b^q(1-\beta-2\beta b)} \geq 1$. Recall that $b < \frac{1-\beta}{2\beta}$.

where $\bar{M}$ is the constant of Lemma 5.17. We also define

$$w_c = \frac{-i}{\lambda_{q+1}} \sum_{i,k\neq 0} \left[\operatorname{curl} \left(\left(\rho_{q,i}\right)^{1/2} \frac{\nabla\Phi_i^T (k \times a_k(\tilde{R}_{q,i}))}{|k|^2} \right) \right] e^{i\lambda_{q+1} k\cdot\Phi_i} =: \sum_{i,k\neq 0} c_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}.$$

Then by direct computations one can check that

$$w_{q+1} = w_o + w_c = \frac{-1}{\lambda_{q+1}} \operatorname{curl} \left(\sum_{i,k\neq 0} (\nabla\Phi_i)^T \left(\frac{ik \times b_{k,i}}{|k|^2} \right) e^{i\lambda_{q+1}k\cdot\Phi_i} \right), \tag{5.95}$$

thus the perturbation w_{q+1} is divergence free. Note that the dependence of $w_{q+1}(\cdot,0)$ on the function $e(t)$ is only trough the value $e(0)$.

5.5.4 The final Reynolds stress and conclusions

Upon letting

$$\overline{R}_q = \sum_i R_{q,i},$$

the new Reynolds stress will be split in two main component: the Euler error $\mathring{R}^E_{q+1}$ and the dissipative error $\mathring{R}^D_{q+1}$, i.e.

$$\mathring{R}_{q+1} = \mathring{R}^E_{q+1} + \mathring{R}^D_{q+1}, \tag{5.96}$$

where

$$\begin{aligned} \mathring{R}^E_{q+1} &= \mathscr{R}\left(w_{q+1}\cdot\nabla\bar{v}_q + \partial_t w_{q+1} + \bar{v}_q\cdot\nabla w_{q+1} + \operatorname{div}\left(-\overline{R}_q + (w_{q+1}\otimes w_{q+1})\right)\right) \\ \mathring{R}^D_{q+1} &= \mu\mathscr{R}\left((-\Delta)^\gamma w_{q+1}\right). \end{aligned}$$

Notice that all three terms in (5.96) are of the form $\mathscr{R}f$, where f has always zero mean. Notice also that the definition of $\mathring{R}^E_{q+1}$ is the same as in [7] and that due to the dissipative term $(-\Delta)^\gamma$ we have to put also $\mathring{R}^D_{q+1}$ in the definition of the new Reynolds stress in order to ensure that the system (5.22) is satisfied at the step $q+1$. Indeed, with this definition one may verify that

$$\begin{cases} \partial_t v_{q+1} + \operatorname{div}(v_{q+1}\otimes v_{q+1}) + \nabla p_{q+1} + \mu(-\Delta)^\gamma v_{q+1} = \operatorname{div}(\mathring{R}_{q+1}), \\ \operatorname{div} v_{q+1} = 0, \end{cases}$$

where the new pressure is defined by

$$p_{q+1}(x,t) = \bar{p}_q(x,t) - \sum_i \rho_{q,i}(x,t) + \rho_q(t). \tag{5.97}$$

We now state a proposition taken from [7].

Proposition 5.18. *For $t \in \tilde{I}_i$ and any $N \geq 0$*

$$\|(\nabla\Phi_i)^{-1}\|_N + \|\nabla\Phi_i\|_N \lesssim \ell^{-N}, \tag{5.98}$$

$$\|\tilde{R}_{q,i}\|_N \lesssim \ell^{-N}, \tag{5.99}$$

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}|k|^{-6}\ell^{-N}, \tag{5.100}$$

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1}|k|^{-6}\ell^{-N-1}. \tag{5.101}$$

Moreover assuming a is sufficiently large, the perturbations w_o, w_c and w_q satisfy the following estimates

$$\|w_o\|_0 + \frac{1}{\lambda_{q+1}}\|w_o\|_1 \leq \frac{M}{4}\delta_{q+1}^{1/2} \tag{5.102}$$

$$\|w_c\|_0 + \frac{1}{\lambda_{q+1}}\|w_c\|_1 \lesssim \delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{-1} \tag{5.103}$$

$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}}\|w_{q+1}\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2} \tag{5.104}$$

where the constant M depends solely on the constant c_0 in (5.91). *In particular, we obtain* (5.81).

We are now ready to complete the proof of Proposition 5.9 by proving the remaining estimates (5.82) and (5.83). The estimate (5.83) is a consequence of Proposition 5.18 and Lemma 5.16 and does not involve the different structure of the Navier-Stokes equations with respect to the Euler ones, thus for the proof of the next proposition we refer to [7].

Proposition 5.19. *The energy of v_{q+1} satisfies the following estimate:*

$$\left|e(t) - \int_{\mathbb{T}^3}|v_{q+1}|^2\,dx - \frac{\delta_{q+2}}{2}\right| \lesssim \frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

For the inductive estimate on $\mathring{R}_{q+1}$ we have the following

Proposition 5.20. *The Reynolds stress error $\mathring{R}_{q+1}$ defined in* (5.96) *satisfies the estimate*

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}. \tag{5.105}$$

In particular, (5.82) *holds.*

Proof. For the first term in the definition of the new Reynolds stress tensor we have

$$\|\mathring{R}_{q+1}^E\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}.$$

We are not going to give the proof of the last estimate because, as already explained, it can be found in [7, Proposition 6.1]. To estimate $\mathring{R}^D_{q+1}$ we first note that $\mu < 1$ and the two operators $\mathscr{R}$ and $(-\Delta)^\gamma$ commute, therefore we can first estimate $\|\mathscr{R}w_{q+1}\|_0$ and $\|\mathscr{R}w_{q+1}\|_1$ from which, using Theorem C.1 and interpolation in Hölder spaces, we conclude

$$\|\mathring{R}^D_{q+1}\|_0 \lesssim \|\mathscr{R}w_{q+1}\|_{\gamma+\alpha} \lesssim \|\mathscr{R}w_{q+1}\|_0^{1-2\gamma-\alpha}\|\mathscr{R}w_{q+1}\|_1^{2\gamma+\alpha}.$$

By the definition of the new perturbations we have

$$w_c = \sum_{i,k\neq 0} c_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}$$
$$w_o = \sum_{i,k\neq 0} L_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i},$$

where $L_{i,k} = (\nabla\Phi_i)^{-1}b_{i,k}$. Using Proposition 5.18 we have

$$\|L_{i,k}\|_N \leq \|(\nabla\Phi_i)^{-1}\|_N\|b_{i,k}\|_0 + \|(\nabla\Phi_i)^{-1}\|_0\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}|k|^{-6}\ell^{-N}. \tag{5.106}$$

Using Proposition D.2 and (5.106) we estimate

$$\begin{aligned}\|\mathscr{R}w_o\|_0 \leq \|\mathscr{R}w_o\|_\alpha &\lesssim \sum_{i,k\neq 0} \frac{\|L_{i,k}\|_0}{\lambda_{q+1}^{1-\alpha}|k|^{1-\alpha}} + \frac{\|L_{i,k}\|_{N+\alpha} + \|L_{i,k}\|_0\|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}|k|^{N-\alpha}} \\ &\lesssim \delta_{q+1}^{1/2}\sum_{k\neq 0}\frac{1}{\lambda_{q+1}^{1-\alpha}|k|^{7-\alpha}} + \frac{\ell^{-N-\alpha}}{\lambda_{q+1}^{N-\alpha}|k|^{N-\alpha+7}} \lesssim \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-\alpha}},\end{aligned}$$

where in the last inequality we have chosen N big enough. It is not difficult to see that we also have

$$\|\mathscr{R}w_o\|_1 \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{\alpha},$$

since each time we take a space derivative the largest contribution is given by differentiating the fast exponential, thereby worsening the estimate by a factor λ_{q+1}. Thus by interpolation we conclude

$$\|(-\Delta)^\gamma\mathscr{R}w_o\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-\alpha}}\lambda_{q+1}^{2\gamma}. \tag{5.107}$$

Now we observe that the estimate on the coefficients $c_{i,k}$ are better then those for the $L_{i,k}$'s, so that we also bound

$$\|(-\Delta)^\gamma\mathscr{R}w_c\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-\alpha}}\lambda_{q+1}^{2\gamma}. \tag{5.108}$$

Finally, combining (5.107), (5.108) and the restriction $\gamma < 1/3$ we get

$$\|\mathring{R}^D_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-\alpha}}\lambda_{q+1}^{2\gamma} \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}.$$

□

Chapter 6

Sharp energy regularity and typicality results for the Euler equations

6.1 Introduction

In this chapter we consider the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \qquad \text{in } \mathbb{T}^3 \times [0,T]. \tag{6.1}$$

We know that for smooth solutions one has energy conservation, namely

$$\frac{d}{dt} e_v(t) = \frac{d}{dt} \int_{\mathbb{T}^3} |v|^2(x,t)\, dx = 0, \qquad \forall t \in [0,T].$$

For weak solutions $v \in L^\infty((0,T); C^\theta(\mathbb{T}^3))$ it is known, and was previously conjectured by Lars Onsager, that the threshold for the energy conservation is $\theta = 1/3$.

As observed in [39], and also proved in Chapter 2, given any solution $v \in L^\infty((0,T); C^\theta(\mathbb{T}^3))$, it can be shown that the associated kinetic energy e_v satisfies

$$|e_v(t) - e_v(s)| \leq C\, |t-s|^{\frac{2\theta}{1-\theta}} \qquad \forall t,s \in [0,T], \tag{6.2}$$

which in particular implies the conservation if $\theta > 1/3$, but also shows a peculiar Hölder regularity of the energy. Throughout this chapter, we will sometimes use the shorter notation

$$\theta^* = \frac{2\theta}{1-\theta}.$$

P. Isett and S.-J. Oh conjectured in [43, Conjecture 1] that this exponent is optimal in the following sense

Conjecture. For any $\theta < \frac{1}{3}$, there exists a solution to (6.1) in the class $v \in C^\theta(\mathbb{R}\times\mathbb{T}^n)$ whose energy profile $e(t)$ fails to have any regularity above the exponent $\frac{2\theta}{1-\theta}$, in the sense that $e_v(t) \notin W^{\frac{2\theta}{1-\theta}+\varepsilon,p}(I)$, for every $\varepsilon > 0$, $p \geq 1$ and every open time interval $I \subset \mathbb{R}$. Furthermore, the set of all such solutions v with the above property is residual (in the sense of category) within the space of all $C^\theta(\mathbb{R}\times\mathbb{T}^n)$ weak solutions, endowed with the topology from the C^θ norm.

In this chapter we solve this conjecture in a slightly smaller space than C^θ. This is due to some technical reasons and we postpone the discussion about this choice at the end of the introduction. The first of our main results is the following

Theorem 6.1. *Fix $\gamma > 0$ and $\theta \in (0,1/3)$ such that $\frac{2\theta}{1-\theta}+\gamma < 1$. For every strictly positive $e \in C^{\frac{2\theta}{1-\theta}+\gamma}([0,T])$, there exists a vector field $v \in C^\theta(\mathbb{T}^3\times[0,T])$ that solves* (6.1) *in the distributional sense and such that*

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x,t)\,dx, \qquad \forall t \in [0,T].$$

The proof of this result follows closely the one of [7]. In particular, our Theorem 6.1 states the same conclusion of [7, Theorem 1.1], except for the fact that we are dropping the hypothesis on the smoothness of the function e. We remark that such sharpness of the energy regularity was first proven in [42, 43] for any $\theta \in (0,1/5)$. Here we extend the result to the whole range $(0,1/3)$, even though it must be noted that in [42, 43] the energy profile is allowed to vanish, while in the scheme of [7], and thus in ours, this is not. A small refinement of Theorem 6.1, coupled with a suitable h-principle, also yields that weak solutions $v \in C^\theta(\mathbb{T}^3\times[0,T])$ belonging to a proper, yet quite large, subset of the space of all weak solutions have typically a kinetic energy e_v which is not more regular than $C^{\frac{2\theta}{1-\theta}}([0,T])$. To state it in a more precise way we set

$$X_\theta = \overline{\left\{ v \in \bigcup_{\theta'>\theta} C^{\theta'}(\mathbb{T}^3\times[0,T]) \,:\, v \text{ weakly solves (6.1)} \right\}}^{\|\cdot\|_{C^\theta_{x,t}}}, \tag{6.3}$$

endowed with the distance

$$d(u,v) = \|u-v\|_{C^\theta_{x,t}}. \tag{6.4}$$

It is clear that (X_θ, d) is a complete metric space. We also define

$$W^{\theta^*} = \bigcup_{I\subset[0,T]} \bigcup_{p\geq 1} \bigcup_{\varepsilon>0} W^{\theta^*+\varepsilon,p}(I)$$

and

$$Y_\theta = \left\{ v \in X_\theta \,:\, e_v \in C^{\theta^*}([0,T]) \setminus W^{\theta^*} \right\}. \tag{6.5}$$

We prove the following

Theorem 6.2. *For any $\theta \in (0,1/3)$, the set Y_θ is residual in X_θ.*

Given a metric space (X, dist), a subset $Y \subset X$ is said to be residual if its complement Y^c is contained in a countable union of closed sets with empty interior. The set Y^c is then called meager. Baire's Theorem asserts that a complete metric space is not meager. Therefore, the previous Theorem yields some immediate corollaries. First, it implies that the kinetic energy of the typical solution in X_θ is not of bounded variation, thus not monotonic, in any open subset of $[0,T]$. Thus, Theorem 6.2 shows a very irregular behaviour of the energy of solutions, in sharp contrast with the conservation of the energy in the case $\theta > 1/3$. We refer the reader to [42,43] for further discussions. A second immediate corollary of Theorem 6.2 is that, for every $\theta \in (0,1/3)$, there exists a weak solution v of Euler such that $e_v \in C^{\theta^*}([0,T])$ but $e_v \notin C^{\theta^*+\gamma}([0,T])$, for any $\gamma > 0$. Let us note in passing that this also yields a weak $C^\theta(\mathbb{T}^3 \times [0,T])$ solution of (6.1) that is not in $C^{\theta+\gamma}(\mathbb{T}^3 \times [0,T])$, for any γ. Indeed, from (6.2) it is clear that Y_θ can not contain solutions v that are more Hölder regular than $C^\theta(\mathbb{T}^3 \times [0,T])$. While the residuality property implies that the kinetic energy of many $C^\theta(\mathbb{T}^3 \times [0,T])$ solutions enjoys the sharp regularity (6.2), it must be noted that X_θ might not contain all the $C^\theta(\mathbb{T}^3 \times [0,T])$ solutions of Euler, since in general not all the $C^\theta(\mathbb{T}^3 \times [0,T])$ functions can be obtained as limit of more regular ones. In particular it is not clear if the same statement is true if one considers as a complete metric space in Theorem 6.2 all the $C^\theta(\mathbb{T}^3 \times [0,T])$ solutions of (6.1), endowed with the same distance $\mathrm{dist}(u,v) = \|u-v\|_{C^\theta_{x,t}}$. This would solve [43, Conjecture 1] completely. We refer the reader to Section 6.5 for a more detailed discussion on this problem.

Another natural question is about the topological properties of the smooth solutions in this setting. To this end we define S to be the set of all smooth solutions of (6.1), and similarly as before we also set

$$C_\theta = \left\{ v \in C^\theta(\mathbb{T}^3 \times [0,T]) \,:\, v \text{ weakly solves } (6.1) \right\},$$

together with the natural distance (6.4). Note at first that, as a corollary of Theorem 6.2, one already gets that $S \subset Y_\theta^c$ which obviously implies that S is a meager set in X_θ. However, in this case, a stronger result can be proved

Theorem 6.3. *For any $\theta \in (0,1/3)$, the set S of all smooth solutions of (6.1) is nowhere dense in C_θ.*

We recall that in a complete metric space, a nowhere dense set is a set whose closure has empty interior. Thus Theorem 6.3 is stronger with respect to the corollary that Theorem 6.2 would give from two points of view. Firstly, every nowhere dense set is also meager. Secondly, the corresponding topological property it is proved in a larger, and also more natural, space C_θ.

6.2 The main inductive proposition

We will follow the construction given in [7] dropping the hypothesis of the smoothness of the energy. In this chapter we will use the same shorter notation $\|f\|_\theta$ introduced in Appendix A to denote Hölder norms.

Let $q \geq 0$ be a natural number. At a given step q we assume to have a smooth triple $(v_q, p_q, \mathring{R}_q)$ solving the Euler-Reynolds system, namely such that

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0, \end{cases} \tag{6.6}$$

to which we add the constraints

$$\operatorname{tr} \mathring{R}_q = 0, \tag{6.7}$$

$$\int_{\mathbb{T}^3} p_q(x,t)\, dx = 0. \tag{6.8}$$

To measure the size of the approximate solution v_q and the error $\mathring{R}_q$, we use a frequency λ_q and an amplitude δ_q, defined through these relations:

$$\lambda_q = 2\pi \lceil a^{(b^q)} \rceil, \tag{6.9}$$

$$\delta_q = \lambda_q^{-2\beta}, \tag{6.10}$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$, $a > 1$ is a large parameter, $b > 1$ is close to 1 and $0 < \beta < 1/3$. The parameters a and b will depend on β and on other quantities. We proceed by induction, assuming the estimates

$$\|\mathring{R}_q\|_0 \leq \delta_{q+1} \lambda_q^{-3\alpha} \tag{6.11}$$

$$\|v_q\|_1 \leq M \delta_q^{1/2} \lambda_q \tag{6.12}$$

$$\|v_q\|_0 \leq 1 - \delta_q^{1/2} \tag{6.13}$$

$$\delta_{q+1} \lambda_q^{-\alpha} \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2\, dx \leq \delta_{q+1} \tag{6.14}$$

where $0 < \alpha < 1$ is a small parameter to be chosen suitably, in dependence of β and other quantities, and M is a universal constant.

We now state the main inductive proposition

Proposition 6.4. *There exists a universal constant M with the following property. Let $0 < \beta < \eta < 1/3$, $E > 0$, and*

$$1 < b < \sqrt{\frac{\eta^*}{\beta^*}}. \tag{6.15}$$

Then there exists an α_0 depending on β, η and b, such that for any $0 < \alpha < \alpha_0$ there exists an a_0 depending on β, b, α, η, E and M, such that for any $a \geq a_0$ the following holds: given a triple

$(v_q, p_q, \mathring{R}_q)$ *solving* (6.6)-(6.8) *and satisfying the estimates* (6.11)–(6.14) *for some strictly positive* $e \in C^{\eta^*}([0,T])$ *with*

$$\|e\|_{\eta^*} \le E,$$

there exists a solution $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ *to* (6.6)-(6.8) *satisfying* (6.11)–(6.14) *for the same function* e *with* q *replaced by* $q+1$. *Moreover, we have*

$$\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}} \|v_{q+1} - v_q\|_1 \le M \delta_{q+1}^{1/2}. \tag{6.16}$$

The reader may notice that there are four main differences with respect to [7, Proposition 2.1]. First of all the statement is fomulated in a slightly different way than in [7, Proposition 2.1], in order to highlight the fact that the parameter a_0 is uniform once one has chosen the $C^{\eta^*}([0,T])$ norm of e. Moreover, we drop the smoothness hypothesis on the function e, we allow the parameter a_0 to depend on E and finally we suppose in (6.15) a different relation between the parameters b and β. Notice that our relation (6.15) is more restrictive than the one used in [7], indeed we have

$$1 < b < \sqrt{\frac{\eta^*}{\beta^*}} < \sqrt{\frac{1}{\beta^*}} = \sqrt{\frac{1-\beta}{2\beta}} < \frac{1-\beta}{2\beta}. \tag{6.17}$$

6.3 Proof of the main theorems

In this section we prove our two main theorems. As in [7], the proof of Theorem 6.1 is a direct consequence of Proposition 6.4 and we are going to prove it for the reader's convenience. Theorem 6.2 will still be an application of the iterative proposition. Indeed, through a h-principle comparable to [7, Theorem 1.3], we will be able to write the set Y_θ^c as a countable union of closed set with empty interior.

6.3.1 Solutions with a non-smooth energy

Here we prove Theorem 6.1. First of all, fix γ, θ and e as in the statement of the theorem. In order to apply Proposition 6.4 we choose $\eta \in (0, 1/3)$ to be the only solution of $\eta^* = \theta^* + \gamma$ and β such that $\theta < \beta < \eta$. Consequently we also fix the parameters b and α appearing in the statement of Proposition 6.4, the first satisfying (6.15) and the second lower than the threshold α_0. Recall the invariance of the Euler equations under the rescaling

$$v(x,t) \mapsto v_\Gamma(x,t) = \Gamma v(x, \Gamma t) \quad \text{and} \quad p(x,t) \mapsto p_\Gamma(x,t) = \Gamma^2 p(x, \Gamma t), \tag{6.18}$$

for any $\Gamma > 0$. Thus, with an the appropriate rescaling, we can further assume that the energy profile satisfies

$$\delta_1 \lambda_0^{-\alpha} \le \inf_t e(t) \le \sup_t e(t) \le \delta_1.$$

Then we can apply inductively Proposition 6.4 starting with the triple $(v_0, p_0, \mathring{R}_0) = (0,0,0)$. Indeed v_0 and $\mathring{R}_0$ trivially satisfy estimates (6.11)-(6.13) and by the rescaling on the energy we also get (6.14) for $q=0$. By (6.16) we have

$$\sum_{q=0}^{\infty} \|v_{q+1}-v_q\|_\theta \lesssim \sum_{q=0}^{\infty} \|v_{q+1}-v_q\|_0^{1-\theta} \|v_{q+1}-v_q\|_1^\theta \lesssim \sum_{q=0}^{\infty} \delta_{q+1}^{1/2} \lambda_{q+1}^{\theta} \lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{\theta-\beta} < \infty \tag{6.19}$$

and hence v_q converges in $C^0([0,T];C^\theta(\mathbb{T}^3))$ to a function v. Moreover, by [17, Theorem 1.1], we have that $v \in C^\theta(\mathbb{T}^3 \times [0,T])$. By taking the divergence of the first equation in (6.6), we get that p_q is the unique 0-average solution of

$$-\Delta p_q = \operatorname{div}\operatorname{div}(v_q \otimes v_q - \mathring{R}_q)$$

and since $v_q \otimes v_q - \mathring{R}_q \to v \otimes v$ uniformly, p_q is also converging to some function p in $L^r(\mathbb{T}^3 \times [0,T])$, for any $r < \infty$. Hence it is clear that the limit couple (v,p) solves (6.1) in the distributional sense. Finally, by (6.14), as $q \to \infty$, we also get

$$e(t) = \int_{\mathbb{T}^3} |v|^2(x,t)\,dx \quad \forall t \in [0,T],$$

which concludes the proof of the theorem.

6.3.2 Residuality of wild solutions

Here we prove Theorem 6.2. We want to show that Y_θ^c is meager in X_θ. Let us enumerate the intervals with rational endpoints inside $[0,T]$, $(I_r)_{r\in\mathbb{N}}$, and let $(q_s)_s$ be a countable and dense subset of $[1,+\infty)$. By (6.5) we can write

$$Y_\theta^c = \bigcup_{m,n,r,s\in\mathbb{N}} C_{m,n,r,s},$$

where

$$C_{m,n,r,s} = \left\{ v \in X_\theta \, : \, \|e_v\|_{W^{\theta^*+\frac{1}{m},q_s}(I_r)} \leq n \right\}.$$

It is easily seen that $C_{m,n,r,s}$ are closed subsets of X_θ. Suppose by contradiction that there exist $\overline{m}, \overline{n}, \overline{r}, \overline{s}$ such that $\mathscr{C} = C_{\overline{m},\overline{n},\overline{r},\overline{s}}$ has a nonempty interior. Thus there exists $\varepsilon > 0$ and $u_0 \in \mathscr{C}$ such that

$$B_\varepsilon(u_0) = \{v \in X_\theta : \|v - u_0\|_{C^\theta_{x,t}} \leq \varepsilon\} \subset \mathscr{C}. \tag{6.20}$$

By the definition of X_θ, we can find a solution of (6.1), $u \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$, $\theta' > \theta$, such that $\|u - u_0\|_{C^\theta_{x,t}} \leq \frac{\varepsilon}{3}$. Moreover, (6.20) implies that

$$B_{\frac{\varepsilon}{2}}(u) \subset \mathscr{C}. \tag{6.21}$$

From now on, we assume that

$$\theta^* < (\theta')^* < \theta^* + \frac{1}{2\bar{m}}. \tag{6.22}$$

This can be done simply by choosing a possibly smaller θ' and exploiting the embedding $C^\alpha(\mathbb{T}^3 \times [0,T]) \subset C^\beta(\mathbb{T}^3 \times [0,T])$, for any $\beta \leq \alpha$. Now fix parameters $\theta'', \beta, \eta > 0$ such that $\theta < \theta' < \theta'' < \beta < \eta$ and for which $\eta^* < \theta^* + \frac{1}{2\bar{m}}$. This can be done in view of (6.22). Fix moreover a function (of time only) $f \in C^{\eta^*}([0,T]) \setminus W^{\eta^*}$, such that $1/2 \leq f \leq 1$ and set

$$e(t) = \int_{\mathbb{T}^3} |u|^2 \, dx + \frac{\rho}{2} f(t), \tag{6.23}$$

for some small parameter $\rho > 0$. These choices imply that the energy $e = e(t)$ satisfies

$$e \notin W^{\theta^* + \frac{1}{\bar{m}}, q_{\bar{s}}}(I_{\bar{r}}). \tag{6.24}$$

Now we claim that, if ρ is chosen sufficiently small, depending on $\theta, \theta', \theta'', \beta, \eta$ and $\bar{m}$, then there exists a solution of (6.1) $v \in C^{\theta''}(\mathbb{T}^3 \times [0,T])$ such that

$$\|u - v\|_{C^\theta_{x,t}} \leq \frac{\varepsilon}{3}, \tag{6.25}$$

$$e_v(t) = e(t), \quad \forall t \in [0,T]. \tag{6.26}$$

It is clear that the claim implies a contradiction with (6.21). Indeed, since $\theta'' > \theta$, we have $v \in X_\theta$. Therefore, by (6.21) and (6.25), we get $e_v \in W^{\theta^* + \frac{1}{\bar{m}}, q_{\bar{s}}}(I_{\bar{r}})$, but this is in contradiction with (6.26) and (6.24). This would conclude the proof of the present theorem, hence we are only left with the proof of the claim.

To prove the claim, we want to apply Proposition 6.4. First, as in the proof of Theorem 6.1, we use the rescaling (6.18) on u with $\Gamma = \min\{(2\|u\|_0)^{-1}, 1\}$ to obtain a new solution $\tilde{u} \in C^{\theta'}(\mathbb{T}^3 \times [0, T/\Gamma])$. If $\|u\|_0 = 0$, we work with the convention that $\Gamma = 1$. For every map $w \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$, we denote with $\tilde{w}$ the map obtained through the rescaling (6.18) with Γ defined above. Notice that there exist constants $c_1(\|u\|_0), c_2(\|u\|_0) > 0$ such that

$$c_1 \|\tilde{w}_1 - \tilde{w}_2\|_{C^{\theta'}_{x,t}} \leq \|w_1 - w_2\|_{C^{\theta'}_{x,t}} \leq c_2 \|\tilde{w}_1 - \tilde{w}_2\|_{C^{\theta'}_{x,t}}, \quad \forall w_1, w_2 \in C^{\theta'}(\mathbb{T}^3 \times [0,T]), \tag{6.27}$$

and that

$$e_{\tilde{w}}(t) = \Gamma^2 e_w(\Gamma t), \quad \forall t \in [0, T/\Gamma], \forall w \in C^{\theta'}(\mathbb{T}^3 \times [0,T]). \tag{6.28}$$

Therefore, we also define

$$\tilde{e}(t) = \Gamma^2 e(\Gamma t), \quad \forall t \in [0, T/\Gamma]. \tag{6.29}$$

Moreover, Proposition 6.4 requires a smooth starting triple. For this reason we consider a space-time mollification of $\tilde{u}$, $u_\delta = (\tilde{u} * \varphi_\delta) * \psi_\delta$, where φ_δ and ψ_δ are standard mollifiers in space and time respectively and $\delta > 0$ is a parameter that will be fixed later on. Of course, u_δ is smooth and solves the following Euler-Reynolds system

$$\partial_t u_\delta + \operatorname{div}(u_\delta \otimes u_\delta) + \nabla p_\delta = \operatorname{div} \mathring{R}_\delta,$$

where $\mathring{R}_\delta = u_\delta \mathring{\otimes} u_\delta - (\tilde{u} \mathring{\otimes} \tilde{u})_\delta$ and the trace part of the commutator $u_\delta \otimes u_\delta - (\tilde{u} \otimes \tilde{u})_\delta$ is inside the pressure p_δ.

We now want to take $(u_\delta, p_\delta, R_\delta)$ as a starting point for the iterative scheme given by Proposition 6.4. In order to do so, we need to guarantee estimates (6.11), (6.12), (6.13) and to find $\rho > 0$ for which also (6.14) is satisfied with $q = 0$. Recall the definition of λ_q and δ_q of (6.10) and (6.9). We make the following choice of the parameters

$$\delta = \left(\delta_1 \lambda_0^{-4\alpha}\right)^{\frac{1}{2\theta'}} \text{ and } \rho = \frac{\delta_1}{\Gamma^2}.$$

Notice that with this choice, obviously both δ and ρ depend on the parameters appearing in Proposition 6.4. In particular the energy profile depends on a, but this will not be a problem since we will bound $\|e\|_{\eta^*}$ independently of a, see also remark 6.5 for a more thorough explanation. Finally, we will use another parameter $\sigma > 0$ to measure the (small) distance between u_δ and the solution given by Proposition 6.4. We start with (6.13). Using (B.5) and the rescaling, we get

$$\|u_\delta\|_0 \le \|u_\delta - \tilde{u}\|_0 + \|\tilde{u}\|_0 \le C\delta^{\theta'} + \frac{1}{2} \le C\lambda_0^{-2\alpha}\delta_1^{1/2} + \frac{1}{2},$$

where $C = C(\|u\|_{C^{\theta'}_{x,t}}) > 0$.It is clear that we can find a sufficiently large a such that

$$C\lambda_0^{-2\alpha}\delta_1^{1/2} + \frac{1}{2} \le 1 - \delta_1^{1/2}.$$

Therefore, (6.13) is fulfilled. Let us now show (6.11) and (6.12). First, by (B.2), we have

$$\|\mathring{R}_\delta\|_0 \lesssim \delta^{2\theta'} = \delta_1 \lambda_0^{-4\alpha},$$

so that again if $\alpha > 0$ is fixed, then (6.11) holds for $q = 0$ if a is large enough. Moreover, through (B.4),

$$\|u_\delta\|_1 \lesssim \delta^{\theta'-1} = (\delta_1 \lambda_0^{-4\alpha})^{\frac{\theta'-1}{2\theta'}},$$

and using the definition of δ_q and λ_q, one verifies that (6.12) holds if a is large enough and $b > 1$ is chosen in such a way that

$$b < \frac{(\theta')^*}{\beta^*} - \frac{2\alpha}{\beta}. \tag{6.30}$$

But since $\beta < \theta'$, if α is sufficiently small (depending on b, β and θ') there exists $b > 1$ sufficiently close to 1 such that (6.30) holds. We are left with the estimate on the energy (6.14). By using (B.2), we estimate

$$\begin{aligned}\tilde{e}(t) - \int_{\mathbb{T}^3} |u_\delta|^2\,dx = \int_{\mathbb{T}^3} |\tilde{u}|^2\,dx + \frac{\delta_1}{2} f(\Gamma t) - \int_{\mathbb{T}^3} |u_\delta|^2\,dx = \int_{\mathbb{T}^3} \left((|\tilde{u}|^2)_\delta - |u_\delta|^2 \right)\,dx + \frac{\delta_1}{2} f(\Gamma t)\\ \leq C\delta^{2\theta'} + \frac{\delta_1}{2} \leq C\delta_1 \lambda_0^{-4\alpha} + \frac{\delta_1}{2},\end{aligned}$$

where the second equality is true in view of the fact that the mollification preserves the mean of every periodic function. If a is large enough,

$$C\delta_1 \lambda_0^{-4\alpha} + \frac{\delta_1}{2} \leq \delta_1,$$

hence the upper bound of (6.14) holds. Similarly we have

$$\int_{\mathbb{T}^3} \left((|\tilde{u}|^2)_\delta - |u_\delta|^2 \right)\,dx + \frac{\delta_1}{2} f(\Gamma t) \geq -C\delta^{2\theta'} + \frac{\delta_1}{4} = -C\delta_1 \lambda_0^{-4\alpha} + \frac{\delta_1}{2} \geq \delta_1 \lambda_0^{-\alpha},$$

where, to guarantee the last inequality, we took again the parameter a large enough. Now we observe that, since $\delta_1 \leq 1$ for any choice of the parameters,

$$\|\tilde{e}\|_{\eta^*} \lesssim \|e_u\|_{\eta^*} + \|f\|_{\eta^*},$$

hence independently of a, there exists a constant $E > 0$ such that

$$\|\tilde{e}\|_{\eta^*} \leq E, \quad \forall a \in (0, +\infty).$$

Therefore we are in place to apply Proposition 6.4 to get a solution $\tilde{v} \in C^{\theta''}(\mathbb{T}^3 \times [0, T/\Gamma])$ of (6.1), for any $\theta < \theta'' < \beta$. Moreover

$$e_{\tilde{v}}(t) = \int_{\mathbb{T}^3} |\tilde{v}|^2\,dx = \tilde{e}(t) \tag{6.31}$$

and, as already done in (6.19), we have the estimate

$$\|\tilde{v} - u_\delta\|_\theta \lesssim \sum_{q \geq 1} \lambda_q^{\theta - \beta} < \sigma, \tag{6.32}$$

provided a is chosen sufficiently large. Of course the choice of a depends on σ, that will be fixed at the end of the proof. By the triangular inequality we also get

$$\|\tilde{v} - \tilde{u}\|_\theta \leq \|\tilde{v} - u_\delta\|_\theta + \|u_\delta - \tilde{u}\|_\theta \lesssim \sigma, \tag{6.33}$$

having once again estimated

$$\|u_\delta - \tilde{u}\|_\theta \lesssim \delta^{\theta' - \theta} = (\delta_1 \lambda_0^{-4\alpha})^{\frac{\theta' - \theta}{2\theta'}} \leq \sigma,$$

the last estimate again being true if a is chosen large enough, depending on σ. Notice that this is possible since $\theta' > \theta$. By Proposition G.2, we also get

$$\|\tilde{v} - \tilde{u}\|_{C^{\theta}_{x,t}} \lesssim \sigma. \tag{6.34}$$

In order to finish the proof of the claim, we scale back the map $\tilde{v}$ and the energy $\tilde{e}$ through the rescaling (6.18), with $1/\Gamma$ instead of Γ. We define $v(x,t) = \Gamma^{-1}\tilde{v}(x,\Gamma^{-1}t)$. Now (6.34) and (6.27) yield

$$\|v - u\|_{C^{\theta}_{x,t}} \lesssim \sigma.$$

We fix $\sigma > 0$ in such a way that

$$\|v - u\|_{C^{\theta}_{x,t}} \leq \frac{\varepsilon}{3},$$

and this gives us (6.25). Moreover, as $\tilde{v} \in C^{\theta''}(\mathbb{T}^3 \times [0,T/\Gamma])$ was a solution of (6.1), then also $v \in C^{\theta''}(\mathbb{T}^3 \times [0,T])$ is a weak solution of (6.1). The last thing to check for the proof of the claim is (6.26). By (6.31), we have

$$e_{\tilde{v}}(t) = \tilde{e}(t).$$

Using (6.28) and (6.29), we can write

$$\Gamma^2 e_v(\Gamma t) = e_{\tilde{v}}(t) = \tilde{e}(t) = \Gamma^2 e(\Gamma t), \quad \forall t \in [0,T/\Gamma],$$

so that

$$e_v(t) = e(t), \quad \forall t \in [0,T],$$

thus proving (6.26) and hence concluding the proof of the claim.

Remark 6.5. *Since the choice in the previous proof of the energy profile depends on a, we wish to clarify in this remark the dependences of the parameters appearing in the proof of the claim. First, we fixed parameters* $0 < \beta < \theta' < 1/3$, *and we chose* $b > 1$ *in such a way that at the same time* (6.30) *and*

$$b < \sqrt{\frac{\theta'^*}{\beta^*}}$$

hold. By choosing $\alpha \in (0,\alpha_1)$, *where* α_1 *is small enough, this can be guaranteed. Note that in this way* α_1 *only depends on* β, θ' *and b, as stated in Proposition 6.4. Therefore, we can always consider* $\alpha_1 \leq \alpha_0$, *where* α_0 *is the number appearing in Proposition 6.4. Next, we have proved that there exists* a_1 *large enough such that for* $a \geq a_1$, *we can guarantee estimates* (6.11), (6.12), (6.13) *and* (6.14) *for* $q = 0$, *for any function e of the form* (6.23). *This* a_1 *only depends on* $\beta, b, \alpha, \theta'$ *and u. Moreover, in the last steps it is required to take a large enough so that inequality* (6.32) *holds. This yields therefore a number* $a_2 \geq a_1$ *that depends on* ε, $E = \|e_u\|_{\eta^*} + \|f\|_{\eta^*}$ *and the universal constant C of Proposition G.2. Therefore* a_2 *now depends only on* $\beta, b, \alpha, \theta'$ *and E, since* u, ε *and C are fixed from the start of the proof of the claim. We can therefore take any* $a_2 \geq a_0$, *where* a_0 *is the parameter appearing in Proposition 6.4. Hence we take* $\alpha = \frac{\alpha_2}{2}$, $a = 2a_2$. *These choices define uniquely e as in* (6.23) *and let us prove the claim.*

We end this section with the proof of Theorem 6.3. Since it follows closely the one of Theorem 6.2, we avoid to give all the technical details already given in the previous proof

6.3.3 Smooth solutions are nowhere dense

Here we prove Theorem 6.3. We want to prove that $\overline{S}$ has empty interior, which is equivalent to say that for any $u_0 \in \overline{S}$ and for any $\varepsilon > 0$, there exists a $v \in C_\theta$ such that

$$v \notin \overline{S}, \tag{6.35}$$
$$d(u_0, v) < \varepsilon. \tag{6.36}$$

The closure and the distance appearing in the lines above are all referred to the C^θ topology. Since $u_0 \in \overline{S}$, there exists a smooth solution u of (6.1) such that

$$d(u_0, u) < \frac{\varepsilon}{2}. \tag{6.37}$$

In particular u is a smooth subsolution whose associated Reynolds stress is zero, and by applying the same rescaling (6.18) with $\Gamma = \min\{(2\|u\|_0)^{-1}, 1\}$ we can guarantee that the rescaled solution $\tilde{u}$, satisfies (6.12) and (6.13) by choosing the parameter a large enough. Since $\tilde{u}$ a smooth solution, its kinetic energy is constant, denoted by $E_{\tilde{u}}$. Moreover, by choosing a non constant and smooth function $1/2 \leq f \leq 1$, with the choice

$$e(t) = E_{\tilde{u}} + \frac{\delta_1}{2} f(t)$$

also condition (6.14) is satified. As in the proof of Theorem 6.2, we can now apply Proposition 6.4 in order to get a solution $\tilde{v} \in C^\theta(\mathbb{T}^3 \times [0, T/\Gamma])$, such that $e_{\tilde{v}} \equiv e$. Moreover, by choosing the parameter a large enough, we can also ensure that $\tilde{v}$ is C^θ close to $\tilde{u}$. By rescaling these maps back, we thus get a solution $v \in C_\theta$ with a non constant energy profile, such that

$$d(v, u) < \frac{\varepsilon}{2}. \tag{6.38}$$

From (6.37) and (6.38) we obviously deduce (6.36). Moreover, the fact that the kinetic energy e_v is not constant implies that v cannot be obtained as a uniform limit of smooth solutions, showing also (6.35).

6.4 Proof of the inductive proposition

The proof of the main iterative proposition given in [7] is subdivided in three steps

1. mollification: $(v_q, \mathring{R}_q) \mapsto (v_\ell, \mathring{R}_\ell)$;
2. gluing : $(v_\ell, \mathring{R}_\ell) \mapsto (\overline{v}_q, \mathring{\overline{R}}_q)$;
3. perturbation: $(\overline{v}_q, \mathring{\overline{R}}_q) \mapsto (v_{q+1}, \mathring{R}_{q+1})$.

In the proof of [7, Proposition 2.1], the energy function e only appears in the perturbation step and both the mollification and the gluing steps are independent on its choice. Thus, also in our case, given the triple $(v_q, p_q, \mathring{R}_q)$ there will exists a new triple $(\overline{v}_q, \overline{p}_q, \mathring{\overline{R}}_q)$ solving the Euler Reynolds system such that the temporal support of $\mathring{\overline{R}}_q$ is contained in pairwise disjoint intervals I_i of length comparable to

$$\tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2}\lambda_q}.$$

More precisely, for any $n \in \mathbb{Z}$ let

$$t_n = \tau_q n, \qquad I_n = \left[t_n + \frac{1}{3}\tau_q, t_n + \frac{2}{3}\tau_q\right] \cap [0,T], \qquad J_n = \left[t_n - \frac{1}{3}\tau_q, t_n + \frac{1}{3}\tau_q\right] \cap [0,T].$$

We have

$$\operatorname{supp} \mathring{\overline{R}}_q \subset \bigcup_{n\in\mathbb{Z}} I_n \times \mathbb{T}^3.$$

Moreover the following estimates hold

$$\|v_q - \overline{v}_q\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\alpha} \tag{6.39}$$

$$\|\overline{v}_q\|_{1+N} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N} \tag{6.40}$$

$$\left\|\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha} \tag{6.41}$$

$$\left\|\partial_t \mathring{\overline{R}}_q + (\overline{v}_q \cdot \nabla)\mathring{\overline{R}}_q\right\|_{N+\alpha} \lesssim \delta_{q+1} \delta_q^{1/2} \lambda_q \ell^{-N-\alpha} \tag{6.42}$$

$$\left|\int_{\mathbb{T}^3} |\overline{v}_q|^2 - |v_\ell|^2 \, dx\right| \lesssim \delta_{q+1} \ell^{\alpha}, \tag{6.43}$$

for any $N \geq 0$, where the small parameter ℓ is defined as

$$\ell = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2} \lambda_q^{1+3\alpha/2}}$$

and it comes from the mollification step. We observe that by choosing α sufficiently small and a sufficiently large we can assume

$$\lambda_q^{-3/2} \leq \ell \leq \lambda_q^{-1}. \tag{6.44}$$

We also state another inequality we will need in the following, that is a consequence of (B.2),(6.14), and (6.43)

$$\frac{\delta_{q+1}}{2\lambda_q^{\alpha}} \leq e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \leq 2\delta_{q+1}. \tag{6.45}$$

Thus we can pass to the perturbation step. The aim is to find a triple $(v_{q+1}, p_{q+1}, \mathring{R}_q)$ which solves (6.6) with the estimates

$$\|v_{q+1}-\overline{v}_q\|_0+\lambda_{q+1}^{-1}\|v_{q+1}-\overline{v}_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2} \tag{6.46}$$

$$\|\mathring{R}_{q+1}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}} \tag{6.47}$$

$$\left| e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx-\frac{\delta_{q+2}}{2}\right| \leq C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}+\frac{\delta_{q+2}}{4}. \tag{6.48}$$

Note that estimates (6.46) and (6.47) are the same stated in [7], while (6.48) is slightly different due to the term $\delta_{q+2}/4$. This does not affect the iteration and Proposition 6.4 is still a direct consequence of estimates (6.46)-(6.48). However, since estimate (6.48) is different than the one used in [7], we give a complete proof of Proposition 6.4.

Proof of Proposition 6.4

By using (6.39) and (6.46) we estimate

$$\|v_{q+1}-v_q\|_0 \leq \|v_{q+1}-\overline{v}_q\|_0+\|\overline{v}_q-v_q\|_0 \leq \frac{M}{2}\delta_{q+1}^{1/2}+C\delta_{q+1}^{1/2}\lambda_q^{-\alpha},$$

where the constant C depends only on α, β and M. Thus if a is chosen sufficiently large we can guarantee

$$\|v_{q+1}-v_q\|_0 \leq M\delta_{q+1}^{1/2}. \tag{6.49}$$

Similarly, by using (6.12), (6.40) and (6.46), we have

$$\|v_{q+1}-v_q\|_1 \leq \|v_{q+1}-\overline{v}_q\|_1+\|\overline{v}_q\|_1+\|v_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2}\lambda_{q+1}+(C+M)\,\delta_q^{1/2}\lambda_q.$$

Again, if a is chosen sufficiently large, we can ensure

$$\|v_{q+1}-v_q\|_1 \leq M\delta_{q+1}^{1/2}\lambda_{q+1},$$

which, together with (6.49), gives (6.16). By (6.12), (6.13) and (6.16) we get

$$\|v_{q+1}\|_0 \leq \|v_{q+1}-v_q\|_0+\|v_q\|_0 \leq \frac{M}{2}\delta_{q+1}^{1/2}+1-\delta_q^{1/2} \leq 1-\delta_{q+1}^{1/2},$$
$$\|v_{q+1}\|_1 \leq \|v_{q+1}-v_q\|_1+\|v_q\|_1 \leq \frac{M}{2}\delta_{q+1}^{1/2}\lambda_{q+1}+M\delta_q^{1/2}\lambda_q \leq M\delta_{q+1}^{1/2}\lambda_{q+1}$$

where we also chose the parameter a sufficiently large to guarantee the last inequalities of the previous estimates. In particular this shows that v_{q+1} obeys (6.12) and (6.13) in which q is replaced by $q+1$. Estimate (6.11) for $\mathring{R}_{q+1}$ is a direct consequence of (6.47) and the parameters inequality

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} \leq \frac{\delta_{q+2}}{\lambda_{q+1}^{8\alpha}}. \tag{6.50}$$

Indeed, by taking the logarithms, the last inequality holds by choosing a sufficiently large if

$$-\beta-\beta b+1-b+2b^2\beta+8b\alpha<0,$$

but this is true since $b<\frac{1-\beta}{2\beta}$ (see (6.17)) and α is chosen sufficiently small. We are only left with estimate (6.14) for v_{q+1}. By (6.48) and (6.50) we have

$$e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx \leq \frac{\delta_{q+2}}{2}+C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}+\frac{\delta_{q+2}}{4} \leq \frac{3}{4}\delta_{q+2}+C\frac{\delta_{q+2}}{\lambda_{q+1}^{6\alpha}},$$

thus, for a sufficiently large a, we get

$$e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx \leq \delta_{q+2}. \tag{6.51}$$

Finally, again by (6.48) we have

$$e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx \geq \frac{\delta_{q+2}}{2}-C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}-\frac{\delta_{q+2}}{4} \geq \left(\frac{1}{4}-\frac{C}{\lambda_{q+1}^{6\alpha}}\right)\delta_{q+2},$$

and, since for a sufficiently large a we can ensure that

$$\frac{1}{4}-\frac{C}{\lambda_{q+1}^{6\alpha}} \geq \frac{1}{\lambda_{q+1}^{\alpha}},$$

we end up with

$$e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx \geq \delta_{q+2}\lambda_{q+1}^{-\alpha},$$

which together with (6.51) gives (6.14) and concludes the proof of the proposition.

6.4.1 Perturbation

We will now outline the construction of the perturbation w_{q+1}, where

$$v_{q+1}=w_{q+1}+\bar{v}_q\,.$$

The perturbation w_{q+1} is highly oscillatory and will be based on the Mikado flows from Appendix E.

We define the smooth non-negative cut-off functions $\eta_i=\eta_i(x,t)$ with the following properties

(i) $\eta_i \in C^\infty(\mathbb{T}^3 \times [0,T])$ with $0 \le \eta_i(x,t) \le 1$ for all (x,t);

(ii) $\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_j = \emptyset$ for $i \neq j$;

(iii) $\mathbb{T}^3 \times I_i \subset \{(x,t) : \eta_i(x,t) = 1\}$;

(iv) $\operatorname{supp}\eta_i \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1}$;

(v) There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0,T]$

$$\sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\,dx \ge c_0. \tag{6.52}$$

The next lemma is taken from [7].

Lemma 6.6. *There exists cut-off functions $\{\eta_i\}_i$ with the properties (i)-(v) above and such that for any i and $n,m \ge 0$*

$$\|\partial_t^n \eta_i\|_m \le C(n,m)\tau_q^{-n}$$

where $C(n,m)$ are geometric constants depending only upon m and n.

Analogously to [7], we will now define the perturbations that are necessary to show (6.46)-(6.48). Since the energy profile is not smooth, we will need to mollify it. To do so we will henceforth consider e to be extended on the whole $\mathbb{R}$ as $e(t) = e(0)$ for all $t < 0$ and $e(t) = e(T)$ for all $t > T$, in such a way that the extension is still in $C^{\eta^*}(\mathbb{R})$. With this convention we define

$$e_q(t) = (e * \psi_{\varepsilon_q})(t),$$

where ψ_{ε_q} is a standard mollifier and

$$\varepsilon_q = \left(\frac{\delta_{q+2}}{4E}\right)^{\frac{1}{\eta^*}}. \tag{6.53}$$

Define also

$$\rho_q(t) = \frac{1}{3}\left(e_q(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\overline{v}_q|^2\,dx\right)$$

and

$$\rho_{q,i}(x,t) = \frac{\eta_i^2(x,t)}{\sum_j \int_{\mathbb{T}^3} \eta_j^2(y,t)\,dy}\rho_q(t)$$

Define the backward flows Φ_i for the velocity field $\overline{v}_q$ as the solution of the transport equation

$$\begin{cases} (\partial_t + \overline{v}_q \cdot \nabla)\Phi_i = 0 \\ \Phi_i(x,t_i) = x. \end{cases}$$

Define

$$R_{q,i} = \rho_{q,i}\mathrm{Id} - \eta_i^2 \mathring{\bar{R}}_q$$

and

$$\tilde{R}_{q,i} = \frac{\nabla\Phi_i R_{q,i} (\nabla\Phi_i)^T}{\rho_{q,i}}. \tag{6.54}$$

We note that, because of properties (ii)-(iv) of η_i,

- $\operatorname{supp} R_{q,i} \subset \operatorname{supp} \eta_i$;
- on $\operatorname{supp} \mathring{\bar{R}}_q$ we have $\sum_i \eta_i^2 = 1$;
- $\operatorname{supp} \tilde{R}_{q,i} \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1}$;
- $\operatorname{supp} \tilde{R}_{q,i} \cap \operatorname{supp} \tilde{R}_{q,j} = \emptyset$ for all $i \neq j$.

Lemma 6.7. *For $a \gg 1$ sufficiently large we have*

$$\|\nabla\Phi_i - \mathrm{Id}\|_0 \leq \frac{1}{2} \qquad \text{for } t \in \operatorname{supp}(\eta_i). \tag{6.55}$$

Furthermore, for any $N \geq 0$

$$\frac{\delta_{q+1}}{8\lambda_q^\alpha} \leq |\rho_q(t)| \leq \delta_{q+1} \quad \text{for all } t\,, \tag{6.56}$$

$$\|\rho_{q,i}\|_0 \leq \frac{\delta_{q+1}}{c_0}\,, \tag{6.57}$$

$$\|\rho_{q,i}\|_N \lesssim \delta_{q+1}\,, \tag{6.58}$$

$$\|\partial_t \rho_q\|_0 \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\,, \tag{6.59}$$

$$\|\partial_t \rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}\,. \tag{6.60}$$

Moreover, for all (x,t)

$$\tilde{R}_{q,i}(x,t) \in B_{1/2}(\mathrm{Id}) \subset \mathscr{S}_+^{3\times 3}\,,$$

where $B_{1/2}(\mathrm{Id})$ denotes the metric ball of radius $1/2$ around the identity Id *in the space $\mathscr{S}^{3\times 3}$.*

Proof. We write

$$\rho_q(t) = \frac{1}{3}\left(e_q(t) - \int_{\mathbb{T}^3} |\bar{v}_q|^2\,dx - \frac{\delta_{q+2}}{2}\right) = \frac{1}{3}\left(e_q(t) - e(t) + e(t) - \int_{\mathbb{T}^3} |\bar{v}_q|^2\,dx - \frac{\delta_{q+2}}{2}\right),$$

thus by (6.45) we get

$$\frac{1}{3}\left(\frac{\delta_{q+1}}{2\lambda_q^\alpha} - \frac{\delta_{q+2}}{2} - |e_q(t) - e(t)|\right) \leq |\rho_q(t)| \leq \frac{1}{3}\left(|e_q(t) - e(t)| + 2\delta_{q+1} + \frac{\delta_{q+2}}{2}\right). \tag{6.61}$$

By using (B.5) and the fact that $[e]_{\eta^*} \le E$, we also get

$$|e_q(t) - e(t)| \le [e]_{\eta^*} \varepsilon_q^{\eta^*} \le \delta_{q+2}$$

and, by plugging it into (6.61), we achieve

$$\frac{\delta_{q+1}}{6\lambda_q^{\alpha}} - \frac{\delta_{q+2}}{2} \le |\rho_q(t)| \le \frac{2}{3}\delta_{q+1} + \frac{\delta_{q+2}}{2}.$$

It is easy to show that by choosing a sufficiently large we can guarantee (6.56). Note that by definition of the cut-off function η_i

$$c_0 \le \sum_i \int_{\mathbb{T}^3} \eta_i^2(x,t)\,dx \le 2 \tag{6.62}$$

and hence we obtain (6.57). Since $|\nabla^N \eta_j| \lesssim 1$, the bound (6.58) also follows. For the bound (6.55) and the fact that $\tilde{R}_{q,i}(x,t) \in B_{1/2}(\mathrm{Id}) \subset \mathscr{S}_+^{3\times 3}$ we refer to [7, Lemma 5.4]. To prove (6.59), we first use (6.40), (6.41) to estimate

$$\left|\frac{d}{dt}\int_{\mathbb{T}^3} |\bar{v}_q|^2\,dx\right| = 2\left|\int_{\mathbb{T}^3} \nabla \bar{v}_q \cdot \mathring{\bar{R}}_q\,dx\right| \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q.$$

Moreover, by (B.4), we have

$$|\partial_t e_q| \le [e]_{\eta^*} \varepsilon_q^{\eta^*-1} \le C\delta_{q+2}^{1-1/\eta^*},$$

where the constant C depends on η and E. Thus (6.59) is implied by the following parameters inequality

$$C\delta_{q+2}^{1-1/\eta^*} \le \delta_{q+1}\delta_q^{1/2}\lambda_q. \tag{6.63}$$

Using the definition of the parameters δ_q and λ_q it can be checked that the last inequality holds if one chose a big enough (depending on b, β, η and E) provided that

$$\left(\frac{1}{\eta^*} - 1\right) b^2 + b - \frac{1}{\beta^*} < 0.$$

Since b satisfies (6.15) we have

$$\left(\frac{1}{\eta^*} - 1\right) b^2 + b - \frac{1}{\beta^*} < \left(\frac{1}{\eta^*} - 1\right)\frac{\eta^*}{\beta^*} + \frac{\eta^*}{\beta^*} - \frac{1}{\beta^*} = 0,$$

thus (6.63) holds. Finally, since $\|\partial_t \eta_j\|_N \lesssim \tau_q^{-1}$ and $\tau_q^{-1} \ge \delta_q^{1/2}\lambda_q$, using (6.62), also the estimate (6.60) follows. □

The constant M

The principal term of the perturbation can be written as

$$w_o = \sum_i \left(\rho_{q,i}(x,t)\right)^{1/2} (\nabla\Phi_i)^{-1} W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i) = \sum_i w_{o,i}\,, \tag{6.64}$$

where Lemma E.1 is applied with $\mathcal{N} = \overline{B}_{1/2}(\mathrm{Id})$, namely the closed ball (in the space of symmetric 3×3 matrices) of radius $1/2$ centered at the identity matrix.

From Lemma 6.7 it follows that $W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i)$ is well defined. Using the Fourier series representation of the Mikado flows (E.3) we can write

$$w_{o,i} = \sum_{k\neq 0} (\nabla\Phi_i)^{-1} b_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}\,,$$

where

$$b_{i,k}(x,t) = \left(\rho_{q,i}(x,t)\right)^{1/2} a_k(\tilde{R}_{q,i}(x,t)).$$

By the definition of $w_{o,i}$ and (E.2) we compute

$$\begin{aligned} w_{o,i}\otimes w_{o,i} &= \rho_{q,i}\nabla\Phi_i^{-1}(W\otimes W)(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i)\nabla\Phi_i^{-T} \\ &= \rho_{q,i}\nabla\Phi_i^{-1}\tilde{R}_{q,i}\nabla\Phi_i^{-T} + \sum_{k\neq 0}\rho_{q,i}\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i} \\ &= R_{q,i} + \sum_{k\neq 0}\rho_{q,i}\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}. \end{aligned} \tag{6.65}$$

The following is a crucial point of the construction, which ensures that the constant M of Proposition 6.4 is geometric and in particular independent of all the parameters of the construction. It follows from the definition of the Mikado flows given in Appendix E and we refer to [7] for a detailed proof.

Lemma 6.8. *There is a geometric constant $\bar{M}$ such that*

$$\|b_{i,k}\|_0 \leq \frac{\bar{M}}{|k|^4}\delta_{q+1}^{1/2}\,. \tag{6.66}$$

We are finally ready to define the constant M of Proposition 6.4: from Lemma 6.8 it follows trivially that the constant is indeed geometric and hence independent of all the parameters of the statement of Proposition 6.4. We can now define the geometric constant M as

$$M = 64\bar{M}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\frac{1}{|k|^4}\,, \tag{6.67}$$

where $\bar{M}$ is the constant of Lemma 6.8. We also define

$$w_c = \frac{-i}{\lambda_{q+1}}\sum_{i,k\neq 0}\left[\mathrm{curl}\left(\left(\rho_{q,i}\right)^{1/2}\frac{\nabla\Phi_i^T(k\times a_k(\tilde{R}_{q,i}))}{|k|^2}\right)\right]e^{i\lambda_{q+1}k\cdot\Phi_i} =: \sum_{i,k\neq 0} c_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\,.$$

Then by direct computations one can check that

$$w_{q+1} = w_o + w_c = \frac{-1}{\lambda_{q+1}} \operatorname{curl} \left(\sum_{i,k \neq 0} (\nabla \Phi_i)^T \left(\frac{ik \times b_{k,i}}{|k|^2} \right) e^{i\lambda_{q+1} k \cdot \Phi_i} \right), \tag{6.68}$$

thus the perturbation w_{q+1} is divergence free.

6.4.2 The final Reynolds stress and conclusions

Upon letting

$$\overline{R}_q = \sum_i R_{q,i},$$

we define the new Reynolds stress as follows

$$\mathring{R}_{q+1} = \mathscr{R} \left(w_{q+1} \cdot \nabla \overline{v}_q + \partial_t w_{q+1} + \overline{v}_q \cdot \nabla w_{q+1} + \operatorname{div} \left(-\overline{R}_q + w_{q+1} \otimes w_{q+1} \right) \right), \tag{6.69}$$

where the operator $\mathscr{R}$ is the one defined in (D.1). With this definition one may verify that

$$\begin{cases} \partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} = \operatorname{div}(\mathring{R}_{q+1}) \\ \operatorname{div} v_{q+1} = 0, \end{cases}$$

where the new pressure is defined by

$$p_{q+1}(x,t) = \bar{p}_q(x,t) - \sum_i \rho_{q,i}(x,t) + \rho_q(t). \tag{6.70}$$

The following proposition is taken from [7].

Proposition 6.9. *For $t \in I_i \cup J_i \cup J_{i+1}$ and any $N \geq 0$*

$$\|(\nabla \Phi_i)^{-1}\|_N + \|\nabla \Phi_i\|_N \lesssim \ell^{-N}, \tag{6.71}$$

$$\|\tilde{R}_{q,i}\|_N \lesssim \ell^{-N}, \tag{6.72}$$

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} |k|^{-6} \ell^{-N}, \tag{6.73}$$

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} |k|^{-6} \ell^{-N-1}. \tag{6.74}$$

Moreover assuming a is sufficiently large, the perturbations w_o, w_c and w_q satisfy the following estimates

$$\|w_o\|_0 + \frac{1}{\lambda_{q+1}} \|w_o\|_1 \leq \frac{M}{4} \delta_{q+1}^{1/2} \tag{6.75}$$

$$\|w_c\|_0 + \frac{1}{\lambda_{q+1}} \|w_c\|_1 \lesssim \delta_{q+1}^{1/2} \ell^{-1} \lambda_{q+1}^{-1} \tag{6.76}$$

$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}} \|w_{q+1}\|_1 \leq \frac{M}{2} \delta_{q+1}^{1/2} \tag{6.77}$$

where the constant M depends solely on the constant c_0 in (6.52). *In particular, we obtain* (6.46).

We are now ready to complete the proof of Proposition 6.4 by proving the remaining estimates (6.48) and (6.47). We start with the energy increment

Proposition 6.10. *The energy of v_{q+1} satisfies the following estimate*

$$\left|e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx-\frac{\delta_{q+2}}{2}\right|\le C\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}+\frac{\delta_{q+2}}{4}.$$

In particular, (6.48) *holds.*

Proof. By definition we have $v_{q+1}=\overline{v}_q+w_{q+1}=\overline{v}_q+w_o+w_c$, thus we have

$$\begin{aligned}\left|e(t)-\int_{\mathbb{T}^3}|v_{q+1}|^2\,dx-\frac{\delta_{q+2}}{2}\right|&\le\left|e(t)-\int_{\mathbb{T}^3}|w_o|^2\,dx-\frac{\delta_{q+2}}{2}-\int_{\mathbb{T}^3}|\overline{v}_q|^2\,dx\right|\\&\quad+\left|\int_{\mathbb{T}^3}|w_c|^2\,dx+2\int_{\mathbb{T}^3}w_o\cdot w_c\,dx+2\int_{\mathbb{T}^3}w_{q+1}\cdot\overline{v}_q\,dx\right|.\end{aligned}\tag{6.78}$$

The estimate on the second term in the right hand side of (6.78) is just a a consequence of (6.40) and Proposition 7.8 and for a complete we refer to [7, Proposition 6.2], in which it is proved that

$$\left|\int_{\mathbb{T}^3}|w_c|^2\,dx+2\int_{\mathbb{T}^3}w_o\cdot w_c\,dx+2\int_{\mathbb{T}^3}w_{q+1}\cdot\overline{v}_q\,dx\right|\lesssim\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

Now recall that from (6.65) and the definition of $R_{q,i}$ we have

$$\begin{aligned}\int_{\mathbb{T}^3}|w_o|^2\,dx&=\sum_i\int_{\mathbb{T}^3}\operatorname{tr}R_{q,i}\,dx+\int_{\mathbb{T}^3}\sum_{i,k\neq0}\rho_{q,i}\nabla\Phi_i^{-1}\operatorname{tr}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx\\&=3\sum_i\int_{\mathbb{T}^3}\rho_{q,i}\,dx+\int_{\mathbb{T}^3}\sum_{i,k\neq0}\rho_{q,i}\nabla\Phi_i^{-1}\operatorname{tr}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx\\&=3\rho_q(t)+\int_{\mathbb{T}^3}\sum_{i,k\neq0}\rho_{q,i}\nabla\Phi_i^{-1}\operatorname{tr}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx\\&=e_q(t)-\frac{\delta_{q+2}}{2}-\int_{\mathbb{T}^3}|\overline{v}_q|^2\,dx+\int_{\mathbb{T}^3}\sum_{i,k\neq0}\rho_{q,i}\nabla\Phi_i^{-1}\operatorname{tr}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx.\end{aligned}$$

As a consequence of (E.6), Lemma 6.7 and Proposition 7.8 we have

$$\left|\int_{\mathbb{T}^3}\sum_{i,k\neq0}\rho_{q,i}\nabla\Phi_i^{-1}\operatorname{tr}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\,dx\right|\lesssim\frac{\delta_q^{1/2}\delta_{q+1}^{1/2}\lambda_q^{1+2\alpha}}{\lambda_{q+1}}.$$

For a detailed proof of the previous estimate we again refer to [7, Proposition 6.2]. Thus we are only left with estimating $|e(t)-e_q(t)|$, but from (B.5), the definition of ε_q in (6.53) and the fact that $[e]_{C^{\eta^*}}\le E$, we get

$$|e(t)-e_q(t)|\le[e]_{\eta^*}\varepsilon_q^{\eta^*}\le\frac{\delta_{q+2}}{4},$$

which concludes the proof of the proposition. □

For the inductive estimate on $\mathring{R}_{q+1}$ we refer to [7, Proposition 6.1]

Proposition 6.11. *The Reynolds stress error* $\mathring{R}_{q+1}$ *defined in* (6.69) *satisfies the estimate*

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-4\alpha}}. \tag{6.79}$$

In particular, (6.47) *holds.*

6.5 The gap for the full conjecture

In this section, we wish to comment on why we need to introduce the space X_θ (see (6.3)), since clearly the most natural choice for X_θ would have simply been the space of all $C^\theta(\mathbb{T}^3 \times [0,T])$ or $c^\theta(\mathbb{T}^3 \times [0,T])$ solutions of Euler equation. Here c^θ denotes the space of little Hölder continuous functions, namely the closure of smooth functions in the C^θ norm. We believe that such a discussion highlights some interesting features of the convex integration scheme.

The introduction of X_θ is related to the proof of Theorem 6.2 and to intrinsic properties of the iterative scheme of [7]. The proof of Theorem 6.2 uses the following strategy, that is quite standard in arguments involving Baire Theorem. As a first step, we rewrite Y_θ^c as union of closed sets $C_{m,n,r,s}$. The parameters m,n,s quantify an improvement in the regularity of elements of $C_{m,n,r,s}$. Secondly, one needs to prove that $C_{m,n,r,s}$ has empty interior. Equivalently, every element $u_0 \in C_{m,n,r,s}$ must be approximated in the $C^\theta(\mathbb{T}^3 \times [0,T])$ norm with elements $u \in X_\theta \setminus C_{m,n,r,s}$. This is where the convex integration scheme comes into play. The iterative procedure of [7] tells us, roughly speaking, that given a smooth subsolution $\bar{u}$ and a positive and smooth (or $C^{\theta^*+\gamma}([0,T])$, as proved in the present work) energy profile e, one can find an arbitrarily close solution u such that $e = e_u$, provided some initial estimates are verified. In order to obtain the desired "less regular" approximating sequence, it seems therefore rather natural to try to apply this result to the subsolution obtained by mollifying u_0, and choose an energy profile $e \in C^{\theta^*+1/2m}([0,T]) \setminus W^{\theta^*+1/2m}$.

Since one wishes to approximate a $C^\theta(\mathbb{T}^3 \times [0,T])$ solution with a sequence of smooth functions in the $C^\theta(\mathbb{T}^3 \times [0,T])$ topology, the first natural restriction is to take the complete metric space in which to apply the Baire argument to be a closed subset of $c^\theta(\mathbb{T}^3 \times [0,T])$. Once one can guarantee the fact that the mollifications of u_0 are close in the right topology to u_0, the next step is to use the convex integration scheme on a close enough space-time mollification of u_0, let us call it u_δ, $\delta > 0$ being the parameter of mollification. Let us moreover denote with R_δ the Reynold stress tensor of u_δ, i.e.

$$R_\delta = u_\delta \otimes u_\delta - (u_0 \otimes u_0)_\delta.$$

In order to apply the scheme, one needs to guarantee step 0 of the inductive estimates, i.e. (6.11),(6.12), (6.13), (6.14). We will now show that, by choosing any $\theta < \beta$ in order to have the $C^\theta(\mathbb{T}^3 \times [0,T])$

closeness of the resulting solution to u_δ (and therefore to u_0), (6.11) and (6.12) become impossible to guarantee using the estimates of Proposition B.1. Through these estimates, one wishes to find $\delta > 0$ and $\alpha > 0$ for which

$$\|\mathring{R}_\delta\|_0 \lesssim \delta^{2\theta} \leq \delta_1 \lambda_0^{-3\alpha} \text{ and } \|u_\delta\|_1 \lesssim \delta^{\theta-1} \leq M\delta_0^{1/2}\lambda_0.$$

These relations are anyway incompatible for any $\delta, \alpha > 0$ if

$$\delta_q = \lambda_q^{-2\beta} = a^{-2\beta b^q} \tag{6.80}$$

for $a, b > 1$. To see this, notice that a solution δ would need to satisfy also

$$\delta^{2\theta} \lesssim \delta_1 = \lambda_1^{-2\beta} \tag{6.81}$$

Moreover, the estimate on the C^1 norm can be rewritten as

$$\delta_0^{-\frac{1}{2(1-\theta)}} \lambda_0^{-\frac{1}{1-\theta}} \lesssim \delta. \tag{6.82}$$

Combining (6.80), (6.81) and (6.82), one obtains

$$a^{-\frac{1-\beta}{1-\theta}} \lesssim a^{-b\frac{\beta}{\theta}},$$

hence that the function $a \mapsto a^{b\frac{\beta}{\theta} - \frac{1-\beta}{1-\theta}}$ is bounded. Since for every $b > 1$, one has $b\frac{\beta}{\theta} - \frac{1-\beta}{1-\theta} > 0$ because of the inequality $\theta < \beta$, we find that a can not be taken freely in an open unbounded interval $(a_0, +\infty)$, hence Proposition 6.4 can not possibly be true in this setting. Nonetheless, as it is clearly stated in [7], we could have found many $C^\beta(\mathbb{T}^3 \times [0,T])$ solutions of (6.1) $C^\beta(\mathbb{T}^3 \times [0,T])$ close to u_δ, for $\beta < \theta$. This is obviously not sufficient for Theorem 6.2. On the other hand, if the starting point u_0 can be approximated in the $C^\theta(\mathbb{T}^3 \times [0,T])$ topology by more regular solutions, for instance in $C^{\theta'}(\mathbb{T}^3 \times [0,T])$, $\theta < \theta'$, then by the previous discussion it becomes clear that we can now start the scheme from these more regular points obtaining the desired estimates in $C^\theta(\mathbb{T}^3 \times [0,T])$. This is exactly the reason for introducing the space X_θ.

We conclude this discussion by noting that, even though it could not contain all the $C^\theta(\mathbb{T}^3 \times [0,T])$ solutions of (6.1), X_θ contains many elements. Indeed, by [7], for every smooth and positive energy profile e and for every $\theta < \theta' < 1/3$, we find a weak solution $u \in C^{\theta'}(\mathbb{T}^3 \times [0,T])$ of (6.1) with $e = e_u$. Since $\theta' > \theta$, $u \in X_\theta$.

Chapter 7

Dimension of the singular set of times of dissipative Hölder solutions to Euler

7.1 Introduction

This chapter concerns again the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0, \end{cases} \tag{7.1}$$

in the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3 \setminus \mathbb{Z}^3$.

A natural next question to ask is how irregular those wild solutions arising from the convex integration schemes described in the previous chapters are, or, more precisely, how small their non-empty singular set can be. In the following, we will only consider the singular set in time, that is the smallest closed set $\mathscr{B} \subseteq [0,T]$ such that $v \in C^\infty\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$. This question has recently been investigated in [5] in the context of the Navier-Stokes equations, where the existence of wild $C^0\left([0,T];L^2\left(\mathbb{T}^3\right)\right)$ weak solutions whose singular set in time has Hausdorff dimension strictly less than 1 has been established. Moreover, in the recent work [14], it has been shown that it is possible to construct non-conservative wild solutions of both Euler and Navier-Stokes equations whose singular set of times has arbitrarily small Hausdorff dimension if one requires only some low L^p integrability in time, where $1 \leq p < 2$. Specifically, in the context of the Euler equations, these solutions belong to $L^{3/2-}\left([0,T];C^{1/3}\left(\mathbb{T}^3\right)\right) \cap L^1\left([0,T];C^{1-}\left(\mathbb{T}^3\right)\right)$ and do not possess a uniform in time regularity.

The question on the size of the singular set in time of wild $C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ weak solutions of Euler as been raised in [14] and has not yet been investigated. In this chapter, we address this issue by studying the structure of the non-conservative weak solutions of Euler constructed in [41] and [7]. We first prove that the singular set in time of such solutions cannot be arbitrarily small. More precisely, we have the following

Theorem 7.1. *Let* $0 < \beta < \frac{1}{3}$ *and* $v \in C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ *be a non-conservative weak solution of* (7.1). *If* $\mathscr{B} \subseteq [0,T]$ *is a closed set such that* $v \in C^\infty\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$, *then* $\mathscr{H}^{\frac{2\beta}{1-\beta}}(\mathscr{B}) > 0$. *In particular, we have*

$$\dim_{\mathscr{H}}(\mathscr{B}) \geq \frac{2\beta}{1-\beta}.$$

The previous result is intrinsically related to the Hölder continuity of kinetic energy of the corresponding class of solutions. Indeed, a remarkable property of $\beta-$Hölder continuous weak solutions of Euler is that the corresponding kinetic energy e_v enjoys the peculiar Hölder regularity (1.10) that we recall here for the reader's convenience

$$|e(t) - e(s)| \leq C|t-s|^{\frac{2\beta}{1-\beta}}. \tag{7.2}$$

Since e_v is constant on $\mathscr{B}^c$, but not on $[0,T]$, Theorem 7.1 quantifies how big $\mathscr{B}$ has to be in order to allow the energy e_v to grow, in a $C^{2\beta/(1-\beta)}$ fashion, between its different values. In this way, Theorem 7.1 is a consequence of a general property of non-constant Hölder continuous functions that increase only on a set of given Hausdorff dimension (see Lemma 7.4 below).

Motivated by the sharpness of the energy regularity proved in Chapter 6 and its connection with the size of the set of singular times of non-conservative solutions, we make the following

Conjecture 7.2. *For every* $\beta < \frac{1}{3}$, *there exists a non-conservative weak solution* $v \in C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ *of* (7.1) *and a closed set* $\mathscr{B} \subset [0,T]$, *such that* $v \in C^\infty\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$ *and* $\dim_{\mathscr{H}}(\mathscr{B}) = \frac{2\beta}{1-\beta}$.

Observe that according to Theorem 7.1 such a solution necessarily satisifes $\mathscr{H}^{\frac{2\beta}{1-\beta}}(\mathscr{B}) > 0$. In this note, we make a first step towards the conjecture. More precisely, using the convex integration scheme of [7] together with the time localization introduced in [5], we prove the following

Theorem 7.3. *Let* $0 \leq \beta < \beta' < \frac{1}{3}$ *and let* $v_1, v_2 \in C^\infty\left(\mathbb{T}^3 \times [0,T]\right)$ *be two smooth solutions of* (7.1) *such that* $\int_{\mathbb{T}^3} v_1(x,t)\,dx = \int_{\mathbb{T}^3} v_2(x,t)\,dx$, *for all* $t \in [0,T]$. *There exists* $v \in C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ *which weakly solves* (7.1) *such that the following holds*

(i) $v\big|_{[0,T/3]} \equiv v_1$ *and* $v\big|_{[2T/3,T]} \equiv v_2$;

(ii) there exists a closed set $\mathscr{B} \subset [0,T]$ *such that* $v \in C^\infty\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$ *and* $\dim_{\mathscr{H}}(\mathscr{B}) \leq \frac{1}{2} + \frac{1}{2}\frac{2\beta'}{1-\beta'}$.

The previous result is in the spirit of the result [5, Theorem 1.1] for the Navier-Stokes equations: as the former, it gives on one hand a strong non-uniqueness result for the Cauchy problem of the Euler equations. Indeed, for any smooth initial datum $\bar{v} \in C^\infty\left(\mathbb{T}^3\right)$ one can choose $v_1 \in C^\infty\left(\mathbb{T}^3 \times [0,T]\right)$ as the smooth solution such that $v_1(0,\cdot) = \bar{v}$, where $T > 0$ is its maximal time of existence, and as v_2 any stationary smooth solution which differs from $\bar{v}$. This clearly shows that for every $\beta < \frac{1}{3}$, $C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ weak solutions are non-unique for every smooth initial datum. We remark that, in view of the weak-strong uniqueness result from [3], our solutions can not be admissible, in the sense that they do not verify $e_v(t) \le e_v(0)$ for every $t \in [0,T]$. For a non-uniqueness result on such solutions we refer to [24, 25], in which the L^2-density of wild initial data has been recently established up to the $\frac{1}{3}-$Onsager's critical threshold. On the other hand, Theorem 7.3 builds solutions that are smooth outside a compact set of quantifiable Hausdorff dimension. The hypothesis on the spatial averages of the two smooth solution is just a standard compatibility condition, since every continuous solution of (7.1) preserves its mean on the torus.

The loss given by the gap $\beta' > \beta$ is typical of such iterative schemes as already observed in [32, Theorem 1.1], while the gap between the Hausdorff dimension achieved in (ii) and the one of Conjecture 7.2 is an outcome of the implementation of the time localization of [5] in the scheme of [7], that we believe could be improved. We postpone the technical discussion of this issue to Section 7.2.8.

7.2 Outline of the proof and main iterative scheme

In order to construct Hölder continuous solutions of Euler we will base our construction on the convex integration scheme proposed in [7]. However, there will be two main differences: at first, since the main goal of our Theorem 7.3 is to ensure that the constructed solution is smooth in a large set of times, we need to introduce a time localization of the glued Reynolds stress as well as of the perturbation. This will be done by adapting the idea that has been introduced in [5] in the context of L^p-based convex integration for the incompressible Navier-Stokes equations. Second, since our purpose in not to prescribe a given energy profile $e = e(t)$, we will avoid all the technicalities coming from the energy iterations. We remark that, even if an energy profile will not be prescribed, the failure of energy conservation will still be a consequence of Theorem 7.3, since we can glue two solutions v_1 and v_2 whose kinetic energy differs. We begin with the simple proof of Theorem 7.1 and then we move on to the description of the main iteration.

7.2.1 Proof of Theorem 7.1

The following lemma asserts that a $\theta-$Hölder continuous function, defined on a $1-$dimensional domain, cannot increase *only* on a null set of the $\theta-$dimensional Hausdorff measure. Since we could not find a reference for it, we give a detailed proof.

Lemma 7.4. *Let $e \in C^\theta([0,T])$ for some $\theta \in (0,1)$ and let $\mathscr{B} \subset [0,T]$ be a closed set such that $\mathscr{H}^\theta(\mathscr{B}) = 0$. If $\frac{d}{dt}e = 0$ on $\mathscr{B}^c$, then $e(t) = e(0)$ for all $t \in [0,T]$.*

Proof. Since $\mathscr{H}^\theta(\mathscr{B}) = 0$, for every $\varepsilon > 0$, there exists a family of open balls $\{B_{r_i}(t_i)\}_i$, such that $\mathscr{B} \subset \bigcup_i B_{r_i}(t_i)$ and

$$\sum_i r_i^\theta < \varepsilon. \tag{7.3}$$

Moreover, since $\frac{d}{dt}e\Big|_{\mathscr{B}^c} = 0$, then the function e can not increase (nor decrease) on $\left(\bigcup_i B_{r_i}(t_i)\right)^c$. This implies that $\forall t \geq 0$, we have

$$|e(t) - e(0)| \leq \sum_i |e(t_i - r_i) - e(t_i + r_i)|,$$

which together with the θ-Hölder continuity of e and (7.3), allows us to conclude

$$|e(t) - e(0)| \leq C \sum_i r_i^{\theta} < C\varepsilon.$$

The claim then follows since $\varepsilon > 0$ was arbitrary. □

To prove Theorem 7.1, just notice that the kinetic energy e_v of any solution $v \in C^{\beta}\left(\mathbb{T}^3 \times [0,T]\right) \cap C^{\infty}\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$ always satisfies (7.2), and moreover, by the standard energy conservation for smooth solutions, we also get

$$\frac{d}{dt}e\Big|_{\mathscr{B}^c} = 0.$$

Then Lemma 7.4, together with the assumption that v is non-conservative, implies $\mathscr{H}^{\frac{2\beta}{1-\beta}}(\mathscr{B}) > 0$ and hence in particular the desired lower bound $\dim_{\mathscr{H}} \mathscr{B} \geq \frac{2\beta}{1-\beta}$.

7.2.2 Inductive proposition

For any index $q \in \mathbb{N}$ we will construct a smooth solution (v_q, R_q) of the Euler Reynolds system on $\mathbb{T}^3 \times [0,T]$

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} R_q \\ \operatorname{div} v_q = 0, \end{cases} \tag{7.4}$$

where R_q is a symmetric matrix. The pressure p_q will consequently be the unique zero average solution of

$$-\Delta p_q = \operatorname{div}\operatorname{div}(v_q \otimes v_q - R_q). \tag{7.5}$$

For any integer q, we define a frequency parameter λ_q and an amplitude parameter δ_q by

$$\lambda_q = 2\pi \lceil a^{b^q} \rceil,$$
$$\delta_q = \lambda_q^{-2\beta},$$

where $0 < \beta < \frac{1}{3}$ is the regularity exponent of Theorem 7.3, $b > 1$ is a number that is close to 1 and $a \gg 1$ is a large enough parameter that will be chosen at the end (depending on all the other parameters). We also introduce the parameter

$$\gamma \in (0, (b-1)(1-\beta)), \tag{7.6}$$

which will be the key parameter to measure the smallness of the singular set of the solution v that we construct, as well as the parameter $\alpha > 0$, which will be chosen sufficiently small (depending on β, b and γ), together with the universal geometric constant $M > 0$ that will be defined later in the construction.

At step q, we will assume the following iterative estimates on the couple (v_q, R_q)

$$\|R_q\|_0 \le \delta_{q+1}\lambda_q^{-\gamma-3\alpha}, \tag{7.7}$$

$$\|v_q\|_1 \le M\delta_q^{1/2}\lambda_q, \tag{7.8}$$

$$\|v_q\|_0 \le 1-\delta_q^{1/2}. \tag{7.9}$$

Here the Hölder norms will always only measure the spatial regularity; in other words, we take the supremum in time of the corresponding spatial Hölder norm (see Appendix A).

We will also inductively assume that the vector field v_q is an exact solution of (7.1) for a large set of times, or analogously that the support of the Reynolds stress R_q is contained in a finite union of thiny time intervals. To this aim, we follow the scheme of time localizations introduced in [5] and, for $q \ge 1$, we introduce the following two parameters

$$\theta_q = \frac{1}{\delta_{q-1}^{1/2}\lambda_{q-1}^{1+3\alpha}},$$

$$\tau_q = \lambda_{q-1}^{-\gamma}\theta_q.$$

For the special case $q = 0$, we set $\tau_0 = T/15$, while for θ_0 we don't need to assign any value. Observe that for every $q \ge 1$, we have as a consequence on the bounds on γ in (7.6) that

$$\theta_{q+1} \ll \tau_q \ll \theta_q \ll 1. \tag{7.10}$$

In order to ensure (ii) in Theorem 7.3, we split the time interval $[0,T]$ at step $q \ge 0$ into a closed good set $\mathscr{G}_q$ and an open bad set $\mathscr{B}_q$ such that $[0,T] = \mathscr{G}_q \cup \mathscr{B}_q$ and $\mathscr{G}_q \cap \mathscr{B}_q = \emptyset$. The Reynolds stress will be supported strictly inside the bad set and hence, on the good set v_q will be a smooth solution of (7.1). More precisely, we will inductively construct the sets $\mathscr{G}_q$ and $\mathscr{B}_q$ with the following properties:

(i) $\mathscr{G}_0 := [0, T/3] \cup [2T/3, T]$,

(ii) $\mathscr{G}_{q-1} \subset \mathscr{G}_q$ for all $q \ge 1$,

(iii) $\mathscr{B}_q$ is a finite union of disjoint open intervals of length $5\tau_q$,

(iv) the size of $\mathscr{B}_q$ is shrinking in q according to the rate

$$|\mathscr{B}_q| \le 10\frac{\tau_q}{\theta_q}|\mathscr{B}_{q-1}| \quad \forall q \ge 1, \tag{7.11}$$

(v) if $t \in \mathscr{G}_{q'}$ for some $q' < q$, then $v_q(t) = v_{q'}(t)$,

(vi) defining the "real" bad set $\widehat{\mathscr{B}}_q := \{t \in [0,T] : \operatorname{dist}(t, \mathscr{G}_q) > \tau_q\}$, we have that the Reynolds stress R_q is supported inside $\widehat{\mathscr{B}}_q$, or in other words

$$R_q(t) \equiv 0 \quad \text{for all } t \in \widehat{\mathscr{B}}_q^c, \tag{7.12}$$

(vii) on the complement of the real bad set, v_q (that from (vi) is a smooth solution of Euler) satisfies the better estimate

$$\|v_q(t)\|_{N+1} \lesssim \delta_{q-1}^{1/2} \lambda_{q-1} \ell_{q-1}^{-N} \quad \text{for all } t \in \widehat{\mathscr{B}}_q^c, \tag{7.13}$$

for all $q \geq 1$, where ℓ_{q-1} is the mollification parameter, as introduced in (7.27). Here, the symbol $\lesssim$ means that the constant in the inequality is allowed to depend on N, but not on any of the parameters and, in particular, not on q.

The following iterative proposition is the cornerstone of the proof of Theorem 7.3.

Proposition 7.5 (Iterative Proposition). *There exists a universal constant $M > 0$ such that the following holds. Fix $0 < \beta < \frac{1}{3}$, $1 < b < \frac{1-\beta}{2\beta}$ and*

$$0 < \gamma < \frac{(b-1)(1-\beta-2\beta b)}{b+1}. \tag{7.14}$$

Then, there exists $\alpha_0 = \alpha_0(\beta, b, \gamma) > 0$ such that for every $0 < \alpha < \alpha_0$, there exists $a_0 = a_0(\beta, b, \gamma, \alpha, M)$ such that for every $a \geq a_0$ the following holds.

Given a smooth couple (v_q, R_q) solving (7.4) *on $\mathbb{T}^3 \times [0,T]$ with the estimates* (7.7)–(7.9) *and a set $\mathscr{B}_q \subset [0,T]$ satisfying the properties* (i)–(vii) *above, there exists a smooth solution (v_{q+1}, R_{q+1}) to* (7.4) *on $\mathbb{T}^3 \times [0,T]$ and a set $\mathscr{B}_{q+1} \subset [0,T]$ satisfying both the estimates* (7.7)–(7.9) *and the properties* (i)–(vii) *with q replaced by $q+1$. Moreover, we have*

$$\|v_{q+1} - v_q\|_0 + \lambda_{q+1}^{-1} \|v_{q+1} - v_q\|_1 \leq M \delta_{q+1}^{1/2}. \tag{7.15}$$

The proof of the main inductive proposition will occupy almost all of the remaining chapter; we give a sketch of the different steps in the proof in the sections 7.2.5 and 7.2.6. Before doing so, we show how the size of the singular set in time is linked to the choice of the parameters and how the iterative proposition implies Theorem 7.3.

7.2.3 Size of the singular set in time

From (7.12), it follows that v_q is a smooth solution to Euler on $\mathbb{T}^3 \times \mathscr{G}_q$. Moreover, the estimate (7.15), together with the fact that $R_q \to 0$ uniformly from (7.7), will ensure that v_q converges strongly in $C^0\left(\mathbb{T}^3 \times [0,T]\right)$ to a weak solution v of (7.1) (see proof of Theorem 7.3). Property (v) guarantees that $v = v_q$ on $\mathscr{G}_q$ and hence the limit solution will be smooth in $\mathbb{T}^3 \times \mathscr{G}_q$. Since this holds for every

$q \geq 0$, we deduce that there exists a closed set $\mathscr{B} \subset [0,T]$, of zero Lebesgue measure, such that $v \in C^\infty\left(\mathbb{T}^3 \times \mathscr{B}^c\right)$ and moreover,

$$\mathscr{B} \subset \bigcap_{q\geq 0} \mathscr{B}_q \,. \tag{7.16}$$

The shrinking rate (7.11) allows us to estimate the Hausdorff (in fact the box-counting) dimension of the right-hand side. Indeed, using also the definition of the parameters τ_q, θ_q and λ_q, we have

$$|\mathscr{B}_q| \leq |\mathscr{B}_0| \prod_{q'=1}^{q} 10\frac{\tau_{q'}}{\theta_{q'}} = \frac{10^q T}{3} \prod_{q'=1}^{q} \lambda_{q'-1}^{-\gamma} \leq (40\pi)^q T a^{-\gamma\left(\frac{b^q-1}{b-1}\right)} \,. \tag{7.17}$$

Since by (iii) every $\mathscr{B}_q$ is made of disjoint intervals of length $5\tau_q$, this implies that for every $q \geq 0$, the set $\mathscr{B}_q$ (and hence $\mathscr{B}$) is covered by at most

$$(40\pi)^q T a^{-\gamma\left(\frac{b^q-1}{b-1}\right)} (5\tau_q)^{-1} \tag{7.18}$$

of such intervals. Since $\tau_q \to 0$ as $q \to \infty$, this shows that the box-counting dimension (and hence the Hausdorff dimension) of $\mathscr{B}$ is bounded by

$$\begin{aligned} \dim_b(\mathscr{B}) &\leq \lim_{q\to\infty} -\frac{\log\left((40\pi)^q T a^{-\gamma\left(\frac{b^q-1}{b-1}\right)}(5\tau_q)^{-1}\right)}{\log(5\tau_q)} \\ &\leq 1 + \lim_{q\to\infty} \frac{\gamma(b^q-1)\log a}{(b-1)\log\tau_q} \\ &= 1 - \frac{\gamma b}{(b-1)(1-\beta+3\alpha+\gamma)} \,, \end{aligned} \tag{7.19}$$

where in the last equality we used that by definition $\tau_q = \lambda_{q-1}^{-(1-\beta+\gamma+3\alpha)}$. Observe that for γ in the range (7.6), this dimension estimate makes sense for α small enough, that is $\dim_b(\mathscr{B}) \in (0,1)$.

7.2.4 Proof of Theorem 7.3

We fix $0 < \beta < \beta' < 1/3$ and we define the auxiliary parameter

$$\beta'' := \frac{\beta+\beta'}{2} \,.$$

Let $1 < b < \frac{1-\beta''}{2\beta''}$ and let $\gamma \in \left(0, \frac{(b-1)(1-\beta''-2\beta'' b)}{b+1}\right)$ yet to be chosen. We will apply Proposition 7.5 with the parameters (β'', b, γ) and we therefore fix admissible parameters $\alpha \in (0,\alpha_0)$ and $a \geq a_0$, where α_0 and a_0 are given by the proposition.

Let $v_1, v_2 \in C^\infty\left(\mathbb{T}^3 \times [0,T]\right)$ be two smooth solutions of (7.1) with the same spatial average. We construct the desired gluing v with an inductive procedure. To this aim, let $\eta : [0,T] \to [0,1]$ be a smooth cutoff function such that $\eta \equiv 1$ on $[0, {}^{2T}\!/_5]$ and $\eta \equiv 0$ on $[{}^{3T}\!/_5, T]$. Consequently, we define the starting velocity v_0 as

$$v_0(x,t) := \eta(t)v_1(x,t) + (1-\eta(t))v_2(x,t).$$

Recalling the inverse divergence operator (D.1) we define

$$R_0 = \partial_t \eta \mathscr{R}(v_1 - v_2) - \eta(1-\eta)(v_1 - v_2) \otimes (v_1 - v_2).$$

Note that, the first term in the definition of R_0 is well defined since $\int_{\mathbb{T}^3}(v_1 - v_2)\,dx = 0$ by assumption.

The smooth couple (v_0, R_0) solves (7.4) however, it does not verify the bounds (7.7)–(7.9) at $q = 0$. To bypass this problem, we exploit the invariance of the Euler equations under the rescaling

$$(v_0, R_0) \to \left(v_0^\varepsilon(x,t) = \varepsilon v_0(x, \varepsilon t), R_0^\varepsilon(x,t) = \varepsilon^2 R_0(x, \varepsilon t)\right). \tag{7.20}$$

Observe that $(v_0^\varepsilon, R_0^\varepsilon)$ is a smooth solution of (7.4) on $\mathbb{T}^3 \times \left[0, \varepsilon^{-1}T\right]$, with the properties that

$$v_0^\varepsilon\big|_{[0,\varepsilon^{-1}T/3]} \equiv v_1^\varepsilon \quad \text{and} \quad v_0^\varepsilon\big|_{\left[\varepsilon^{-1}2T/3, \varepsilon^{-1}T\right]} \equiv v_2^\varepsilon, \tag{7.21}$$
$$\|R_0^\varepsilon\|_0 = \varepsilon^2 \|R_0\|_0, \quad \|v_0^\varepsilon\|_0 = \varepsilon\|v_0\|_0 \quad \text{and } \|v_0^\varepsilon\|_1 = \varepsilon\|v_0\|_1.$$

This allows to choose ε small enough, depending on all the previous parameters and additionally also on T, v_1 and v_2, in order to satisfy (7.7)–(7.9) at $q = 0$. To be precise, we choose

$$\varepsilon = \min\left\{ \left(\frac{\delta_1 \lambda_0^{-(\gamma+3\alpha)}}{\|R_0\|_0}\right)^{1/2}, \frac{M\delta_0^{1/2}\lambda_0}{\|v_0\|_0}, \frac{1-\delta_0^{1/2}}{\|v_0\|_1} \right\}. \tag{7.22}$$

With this choice of ε, $(v_0^\varepsilon, R_0^\varepsilon)$ satisfies the estimates (7.7)–(7.9) as well as the properties (i)–(vii) for $q = 0$ (where for (vii) the constant in the inequality (7.13) depends on $\varepsilon, \|v_1\|_N, \|v_2\|_N$). We then apply inductively Proposition 7.5. We start from $q = 0$ with the couple $(v_0^\varepsilon, R_0^\varepsilon)$ solving (7.4) on $\mathbb{T}^3 \times \left[0, \varepsilon^{-1}T\right]$ and the bad set $\mathscr{B}_0 = (\varepsilon^{-1}T/3, \varepsilon^{-1}2T/3)$. In this way, we construct a sequence of smooth solutions $\left\{(v_q^\varepsilon, R_q^\varepsilon)\right\}_{q\geq 0}$ to (7.4) on $\mathbb{T}^3 \times \left[0, \varepsilon^{-1}T\right]$, with estimates (7.7)–(7.9) and (7.15), and with the corresponding bad set $\mathscr{B}_q$ obeying (i)–(vii). The bound (7.15) implies, together with the interpolation estimate (A.2), that

$$\sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_\beta \lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_1^\beta \|v_{q+1} - v_q\|_0^{1-\beta} \lesssim \sum_{q=0}^{\infty} \lambda_{q+1}^{\beta - \beta''} \lesssim 1.$$

Hence there a exists a strong limit

$$w = \lim_{q\to\infty} v_q^\varepsilon \in C^0\left(\left[0, \varepsilon^{-1}T\right], C^\beta\left(\mathbb{T}^3\right)\right). \tag{7.23}$$

By (7.7) we have that $R_q^\varepsilon \to 0$ uniformly, which implies that the limit w solves (7.1). From (2.2), we also recover the regularity in time and we deduce that in fact $w \in C^\beta\left(\mathbb{T}^3 \times \left[0, \varepsilon^{-1}T\right]\right)$.

Combining the properties (i), (ii) and (v) of the bad set and recalling the structure of v_0^ε from (7.21), we deduce that

$$w\big|_{[0,\varepsilon^{-1}T/3]} \equiv v_1^\varepsilon \quad \text{and} \quad w\big|_{\left[\varepsilon^{-1}2T/3,\varepsilon^{-1}T\right]} \equiv v_2^\varepsilon. \tag{7.24}$$

Moreover, as proven in Section 7.2.3, there exists a closed set $\mathscr{C} \subset \bigcap_{q\geq 0}\mathscr{B}_q \subset \left[0,\varepsilon^{-1}T\right]$ such that $w \in C^\infty\left(\mathbb{T}^3 \times \mathscr{C}^c\right)$ and

$$\dim_b(\mathscr{C}) \leq 1 - \frac{\gamma\beta''}{(b-1)(1-\beta''+3\alpha+\gamma)}.$$

We now come to the choice of the parameters b,γ and α. It is easy to observe that the infimum of the above dimension bound is reached in the limit as $\alpha \downarrow 0$, $\gamma \uparrow \frac{(b-1)(1-\beta''-2\beta''b)}{b+1}$ and $b \downarrow 1$. More precisely, we have

$$\inf_{b\in\left(1,\frac{1-\beta''}{2\beta''}\right)} \left\{ \inf_{\gamma\in\left(0,\frac{(b-1)(1-\beta''-2b\beta''))}{b+1}\right)} \left\{ \inf_{\alpha\in(0,\alpha_0)} \left\{ 1 - \frac{\gamma\beta''}{(b-1)(1-\beta''+3\alpha+\gamma)} \right\}\right\}\right\} = \frac{1}{2} + \frac{1}{2}\frac{2\beta''}{1-\beta''}. \tag{7.25}$$

Since by choice of $\beta'' > \beta'$, the right-hand side of (7.25) is strictly smaller than the desired dimension bound $\frac{1}{2}+\frac{1}{2}\frac{2\beta'}{1-\beta'}$, we can first choose b sufficiently close to 1 (depending on β''), then γ sufficiently close to $\frac{(1-b)(1-\beta''-2\beta''b)}{b+1}$ (depending on β'' and b) and finally α sufficiently close to 0, such that

$$\dim_b(\mathscr{C}) \leq 1 - \frac{\gamma\beta''}{(b-1)(1-\beta''+3\alpha+\gamma)} \leq \frac{1}{2} + \frac{1}{2}\frac{2\beta'}{1-\beta'}.$$

Finally, we rescale back and set $v(x,t) = \varepsilon^{-1}w\left(x,\varepsilon^{-1}t\right)$ to obtain a weak solution $v \in C^\beta\left(\mathbb{T}^3 \times [0,T]\right)$ which is a gluing of v_1 and v_2 (in the sense of Theorem 7.3) and which is smooth in $\mathbb{T}^3 \times \mathscr{B}^c$, where

$$\mathscr{B} = \left\{\varepsilon^{-1}t : t \in \mathscr{C}\right\}. \tag{7.26}$$

By scale-invariance, $\mathscr{B}$ obeys the same desired Hausdorff dimension bound as $\mathscr{C}$.

□

7.2.5 Gluing and localization step

As a first step in the proof of Proposition 7.5, we construct from the couple (v_q,R_q) and the set $\mathscr{B}_q$ a new couple $\left(\overline{v}_q,\overline{R}_q\right)$ solving (7.4) as well as a set $\mathscr{B}_{q+1}$ satisfying (i)–(vii), with q replaced by $q+1$. Whereas $\overline{v}_q$ will enjoy roughly the same estimates as v_q, the new Reynolds stress $\overline{R}_q$ will already be localized (in time) in a subset of $\widehat{\mathscr{B}}_{q+1}$, that is in disjoint intervals of length τ_{q+1}. The price of this localization in time will be worsened estimates on $\overline{R}_q$ with respect to R_q, proportional to shrinking rate (7.11).

Following the construction of [41], $\overline{v}_q$ will be a gluing of exact solutions of the Euler equations. In order to produce those solutions, we first mollify v_q in space at length scale ℓ, as it is typical in convex integration schemes for the Euler equations to avoid the loss of derivative problem. To this end, let φ be standard radial mollification kernel in space which we rescale with some parameter ℓ_q, that is $\varphi_{\ell_q}(x) = \ell_q^{-3}\varphi\left(\frac{x}{\ell_q}\right)$. For any $q \geq 1$, we choose the mollification parameter to be

$$\ell_q = \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}\lambda_q^{1+\gamma/2+3\alpha/2}}. \tag{7.27}$$

Observe that in view of (7.6), ℓ_q enjoys, for α small enough[1], the elementary bounds

$$\lambda_q^{-3/2} < \ell_q < \lambda_q^{-1}. \tag{7.28}$$

In what follows, we will usually drop the subscript q unless there is ambiguity about the step. We define the mollified functions

$$\begin{aligned} v_\ell &= v_q * \varphi_\ell, \\ R_\ell &= R_q * \varphi_\ell + v_\ell \otimes v_\ell - (v_q \otimes v_q) * \varphi_\ell, \\ p_\ell &= p_q * \varphi_\ell. \end{aligned}$$

In view of (7.4), we get that (v_ℓ, R_ℓ) is a smooth solution to the Euler-Reynolds system

$$\begin{cases} \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell = \operatorname{div} R_\ell \\ \operatorname{div} v_\ell = 0. \end{cases}$$

The choice of ℓ guarantees that both contributions in R_ℓ, the mollification of R_q and the commutator, are of equal size. In particular, R_ℓ will be of the size of R_q. More precisely we have the following

Proposition 7.6. *For any $N \geq 0$ we have*

$$\|v_\ell - v_q\|_0 \lesssim \delta_q^{1/2}\lambda_q \ell, \tag{7.29}$$

$$\|v_\ell\|_{N+1} \lesssim \delta_q^{1/2}\lambda_q \ell^{-N}, \tag{7.30}$$

$$\|R_\ell\|_{N+\alpha} \lesssim \delta_{q+1}\lambda_q^{-\gamma-3\alpha}\ell^{-N-\alpha}. \tag{7.31}$$

Here (and in what follows) the symbol $\lesssim$ means that the constant in the inequality may depend on the number of derivatives N, but not on any of the parameters of Proposition 7.5, neither on the step q.

[1] The upper bound $\ell_q \leq \lambda_q^{-1}$ holds for every $\alpha > 0$, while for the lower bound $\ell_q \geq \lambda_q^{-3/2}$ it suffices to require $\alpha < \beta$.

Proof. The estimates (7.29) and (7.30) follow from standard mollification estimates and (7.8). Indeed, we have

$$\|v_\ell - v_q\|_0 \lesssim \ell \|v_q\|_1 \lesssim \delta_q^{1/2}\lambda_q \ell\,,$$
$$\|v_\ell\|_{N+1} \lesssim \ell^{-N}\|v_q\|_1 \lesssim \delta_q^{1/2}\lambda_q \ell^{-N} \quad \forall N \geq 0\,.$$

Finally, by using (B.2), (7.7)–(7.8) and the choice of ℓ in (7.27) we get

$$\|R_\ell\|_{N+\alpha} \lesssim \ell^{-N-\alpha}\|R_q\|_0 + \ell^{2-N-\alpha}\|v_q\|_1^2 \lesssim \ell^{-N-\alpha}\left(\delta_{q+1}\lambda_q^{-\gamma-3\alpha} + \ell^2\delta_q\lambda_q^2\right) \lesssim \delta_{q+1}\lambda_q^{-\gamma-3\alpha}\ell^{-N-\alpha}\,.$$

□

To localize the glued Reynolds stress in time, we use the strategy of [5]. By the inductive hypothesis (vi), the real bad set $\widehat{\mathscr{B}}_q$, where the Reynolds stress R_q is supported, is a finite union of disjoint intervals of length $3\tau_q$. We will split each of these intervals in subintervals $[t_i, t_{i+1}]$ of length $t_{i+1} - t_i = \theta_{q+1}$ and we will build smooth solutions v_i of the Euler system with initial datum $v_i(t_i) = v_\ell(t_i)$. By the choice of θ_{q+1}, we have for large enough a that

$$2\theta_{q+1}\|v_\ell(t_i)\|_{1+\alpha} \ll 1, \tag{7.32}$$

which guarantees that v_i will exist for times $|t - t_i| \leq 2\theta_{q+1}$ (see Proposition 5.8). This will allow us to define $\overline{v}_q$ as the following gluing of smooth solutions

$$\overline{v}_q := \sum_i \eta_g^i v_i + (1-\eta_g)v_q\,, \tag{7.33}$$

where $\eta_g = \sum_i \eta_g^i$ is a smooth temporal cutoff between $\widehat{\mathscr{B}}_q$ and $\mathscr{B}_q$. The cutoffs η_g^i will be supported in an interval of length $< 2\theta_{q+1}$ around t_i and will be steep: $\partial_t \eta_g^i$ will be supported in two tiny (compared to their support) intervals I_i and I_{i+1} of length τ_{q+1} (see Section 7.3). By construction, $\overline{v}_q$ will solve an Euler Reynolds system with a Reynolds stress $\overline{R}_q$ which is localized where $\partial_t \eta_g$ is non-zero, that is in $\bigcup_i I_i$. Up to enlarging every I_i in length by $2\tau_{q+1}$ on either side, those intervals will form the new bad set $\mathscr{B}_{q+1}$.

Proposition 7.7. *Given a couple (v_q, R_q) solving* (7.4) *on $\mathbb{T}^3 \times [0,T]$ together with a set $\mathscr{B}_q \subset [0,T]$ satisfying the hypothesis of Proposition 7.5, there exists a smooth solution $\left(\overline{v}_q, \overline{R}_q\right)$ to* (7.4) *on $\mathbb{T}^3 \times [0,T]$ and an open set $\mathscr{B}_{q+1} \subset [0,T]$ satisfying the properties* (i)–(iv) *listed in Section 7.2.2 with q replaced by $q+1$, such that additionally*

$$\overline{v}_q(t) = v_q(t) \qquad \text{for all } t \in \mathscr{G}_q, \tag{7.34}$$
$$\overline{R}_q(t) = 0 \qquad \text{for all } t \in [0,T] \text{ such that } \operatorname{dist}\left(t, \mathscr{G}_{q+1}\right) \leq 2\tau_{q+1}. \tag{7.35}$$

Moreover, we have the estimates

$$\|\overline{v}_q - v_q\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\gamma/2-3\alpha/2}, \tag{7.36}$$

$$\|\overline{v}_q - v_\ell\|_{N+1} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N} \qquad \forall N \geq 0, \tag{7.37}$$

$$\|\overline{v}_q\|_{N+1} \lesssim \delta_q^{1/2} \lambda_q \ell^{-N} \qquad \forall N \geq 0, \tag{7.38}$$

$$\|\overline{R}_q\|_{N+\alpha} \lesssim \delta_{q+1} \ell^{-N+\alpha} \qquad \forall N \geq 0, \tag{7.39}$$

$$\|(\partial_t + \overline{v}_q \cdot \nabla)\overline{R}_q\|_{N+\alpha} \lesssim \delta_{q+1} \delta_q^{1/2} \lambda_q^{1+\gamma} \ell^{-N-2\alpha} \qquad \forall N \geq 0. \tag{7.40}$$

Observe that we are not explicitly requiring the properties (v)–(vii) on the new bad set $\mathscr{B}_{q+1}$; the latter will be however an easy consequence of the stronger properties (7.34), (7.35) and (7.38). The proof of the previous proposition will be given in Section 7.3.

7.2.6 Perturbation step

Although the gluing step allows to localize the Reynolds stress $\overline{R}_q$ already in much smaller intervals of time, we did not improve the size of the Reynolds stress yet. In fact, the estimates have been even worsened by the factor λ_q^γ, which can be view as the main reason why the Hausdorff dimension achieved in Theorem 7.3 is strictly bigger than the optimal one given in Conjecture 7.2. The precise discussion of this issue is postponed to Section 7.2.8 below.

In order reduce the size of the Reynolds stress, we will produce from $(\overline{v}_q, \overline{R}_q)$ and $\mathscr{B}_{q+1}$, given by Proposition 7.7, a new solution (v_{q+1}, R_{q+1}) to (7.4) with the Reynolds stress R_{q+1} still supported in $\widehat{\mathscr{B}}_{q+1}$ and verifying all the desired estimates. This will be done by adding a highly oscillatory perturbation w_{q+1} to $\overline{v}_q$. Indeed, this is the key ingredient of all convex integration schemes building on [26] and, as in [7, 41], the building blocks for the perturbation w_{q+1} are again the Mikado flows from Appendix E. In the presentation of the perturbation step, we will follow closely [7].

At difference from [7], we will need to localize the perturbation w_{q+1} in time to have support within $\widehat{\mathscr{B}}_{q+1}$. This will be achieved by means of steep temporal cutoffs, similar to the ones from the gluing step, and will be responsible for worsened estimates on R_{q+1} with respect to [7].

Proposition 7.8. *Let $(\overline{v}_q, \overline{R}_q)$ and the bad set $\mathscr{B}_{q+1} \subset [0,T]$ be as in Proposition 7.7. There exists a new smooth couple (v_{q+1}, R_{q+1}) which solves* (7.4) *in $\mathbb{T}^3 \times [0,T]$, and such that all the properties* (i)–(vii) *listed in Section 7.2.2 hold with q replaced by $q+1$. Moreover, we have the estimates*

$$\|v_{q+1} - \overline{v}_q\|_0 + \lambda_{q+1}^{-1} \|v_{q+1} - \overline{v}_q\|_1 \leq \frac{M}{2} \delta_{q+1}^{1/2}, \tag{7.41}$$

$$\|R_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}}, \tag{7.42}$$

where $M > 0$ is a universal geometric constant.

The proof of the previous proposition is the core of the convex integration scheme and will occupy most of this work. Being quite technical, we postpone it to Section 7.4.

7.2.7 Proof of Proposition 7.5

We prove how the main iterative Proposition 7.5 is a consequence of the two previous steps and we postpone their respective proofs in Sections 7.3 and 7.4 below.

We start by noticing that given a couple (v_q, R_q) solving (7.4) on $\mathbb{T}^3 \times [0,T]$ together with a set $\mathscr{B}_q \subset [0,T]$ satisfying the hypothesis of Proposition 7.5, Proposition 7.8 directly gives the smooth couple (v_{q+1}, R_{q+1}) solving (7.4) on $\mathbb{T}^3 \times [0,T]$, together with the bad set $\mathscr{B}_{q+1}$ (and consequently its complement $\mathscr{G}_{q+1}$), satisfying properties (i)–(vii). Thus we are left to check (7.15) and that estimates (7.7)–(7.9) hold with q replaced by $q+1$.

Estimate (7.15) is a consequence of (7.36), (7.38), (7.41) and the inductive assumption (7.8) on v_q. Indeed, we have

$$\|v_{q+1} - v_q\|_0 \le \|v_{q+1} - \bar{v}_q\|_0 + \|\bar{v}_q - v_q\|_0 \le \delta_{q+1}^{1/2} \left(\frac{M}{2} + C\lambda_q^{-\gamma/2} \right) \le M\delta_{q+1}^{1/2}, \tag{7.43}$$

where the last inequality holds if a is large enough. Similarly, we get a large enough (independently of q) that

$$\|v_{q+1} - v_q\|_1 \le \|v_{q+1} - \bar{v}_q\|_1 + \|\bar{v}_q - v_q\|_1 \le \delta_{q+1}^{1/2}\lambda_{q+1} \left(\frac{M}{2} + C\frac{\delta_q^{1/2}\lambda_q}{\delta_{q+1}^{1/2}\lambda_{q+1}} \right) \le \frac{2}{3}M\delta_{q+1}^{1/2}\lambda_{q+1}, \tag{7.44}$$

which together with (7.43), proves (7.15).

By using (7.8) and (7.44), we obtain

$$\|v_{q+1}\|_1 \le \|v_{q+1} - v_q\|_1 + \|v_q\|_1 \le \frac{2}{3}M\delta_{q+1}^{1/2}\lambda_{q+1} + M\delta_q^{1/2}\lambda_q \le M\delta_{q+1}^{1/2}\lambda_{q+1},$$

which ensures the validity of (7.8) at step $q+1$. Moreover,

$$\|v_{q+1}\|_0 \le \|v_{q+1} - v_q\|_0 + \|v_q\|_0 \le M\delta_{q+1}^{1/2} + 1 - \delta_q^{1/2} \le 1 - \delta_{q+1}^{1/2},$$

where again we assumed that a is large enough in order to guarantee the last inequality. Thus also (7.9) holds at step $q+1$. Finally, the proof of the last estimate (7.7) is a consequence of the following relation

$$\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}} \le \delta_{q+2}\lambda_{q+1}^{-\gamma-3\alpha}. \tag{7.45}$$

By using the parameters definitions, it is clear that (7.45) holds if

$$-\beta b - \beta + 1 + \gamma - b + 5\alpha b < -2\beta b^2 - \gamma b - 3\alpha b. \tag{7.46}$$

We notice that if the previous inequality holds for $\alpha = 0$, then (being an inequality between polynomials) there will be an $\alpha_0 = \alpha_0(\gamma, \beta, b) > 0$ such that (7.46) still holds for all $0 < \alpha < \alpha_0$. But if we set $\alpha = 0$, we obtain

$$(b+1)\gamma < -2\beta b^2 + (\beta+1)b + \beta - 1 = (b-1)(-2\beta b + 1 - \beta),$$

which holds by our choice of γ in (7.14). This concludes the proof of Proposition 7.5.

7.2.8 Gap with the conjectured exponent

By our inductive assumption (iv) on the shrinking rate of the bad set $\mathscr{B}_q$, it is clear that bigger is γ, the smaller is the dimension of the final singular set. By looking at (7.19), one may verify that the sharp Hausdorff dimension of Conjecture 7.2 would be achieved if γ could reach the threshold $\overline{\gamma}(b,\beta) \simeq (b-1)(1-\beta-2\beta b)$. More precisely, we would need that this upper bound $\overline{\gamma}$ satisfies

$$\lim_{b\to 1} \frac{\overline{\gamma}(b,\beta)}{b-1} = 1-3\beta. \tag{7.47}$$

In our case however, the restriction (7.14) on γ implies that the maximal $\gamma_{\max}(b,\beta)$ we can choose is only half of the sharp one from (7.47), or in other words, our upper bound on γ satisfies

$$\lim_{b\to 1} \frac{\gamma_{\max}(b,\beta)}{b-1} = \frac{1-3\beta}{2}.$$

With that being said, we will now try to explain where the restriction (7.14) comes from. To do that, we give an heuristic version of the inductive scheme.

Given the two parameters δ_q and λ_q as in Section 7.2.2, the aim is to find a perturbation w_{q+1} of size $\delta_{q+1}^{1/2}$, oscillating at frequency λ_{q+1}, that verifies (7.15), together with a new error R_{q+1} that is localized in intervals of length $\sim \tau_{q+1}$. By looking at the oscillation error in (7.99), we deduce that $\|w_{q+1}\|_0 \simeq \|\overline{R}_q\|_0^{1/2}$, since without that, it would be impossible to ensure that R_{q+1} is considerably smaller than R_q. This implies that

$$\|\overline{R}_q\|_0 \simeq \delta_{q+1}. \tag{7.48}$$

The stress tensor $\overline{R}_q$ is obtained by using the gluing technique introduced by Philip Isett in [41] and consequently, the corresponding glued velocity $\overline{v}_q$ has to be an exact solution of Euler in time intervals of length $\theta_{q+1} \simeq (\delta_q^{1/2}\lambda_q)^{-1}$. The only difference is that we need to shrink the temporal support of $\overline{R}_q$ to intervals of length $\sim \tau_{q+1} = \lambda_q^{-\gamma}\theta_{q+1} \ll \theta_{q+1}$ (see Section 7.3 for the detailed construction). This asymmetry between the two sizes implies $\|\overline{R}_q\|_0 \simeq \|R_q\|_0\lambda_q^{\gamma}$ (see estimates (7.7) and (7.39)), which together with (7.48), forces

$$\|R_q\|_0 \simeq \delta_{q+1}\lambda_q^{-\gamma} \tag{7.49}$$

to be the right inductive assumption at step q. The perturbation w_{q+1} can now cancel the error $\overline{R}_q$, but we still need to force $\left|\operatorname{supp}_t R_{q+1}\right| \simeq \tau_{q+1}$. By looking at the definition of the new Reynolds stress in (7.99), the easiest way is to localize the perturbation w_{q+1} in such intervals by means of steep temporal cut-offs; that is by setting

$$w_{q+1} := \eta_p \tilde{w}_{q+1},$$

where $\tilde{w}_{q+1}$ is a combination of highly oscillatory (at frequency λ_{q+1}) Mikado flows and η_p is a time cut-off such that $\left|\operatorname{supp}_t \eta_p\right| \simeq \tau_{q+1}$. This of course implies that $\|\partial_t \eta_p\|_0 \lesssim \tau_{q+1}^{-1}$ (see Lemma 7.14). In this way, we are inserting in R_{q+1} (in particular in the transport error R_{transp}) a term that looks like

$\partial_t \eta_p \mathscr{R} \tilde{w}_{q+1}$, which from Proposition D.2 satisfies

$$\left\|\partial_t \eta_p \mathscr{R} \tilde{w}_{q+1}\right\|_0 \lesssim \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}} \tau_{q+1}^{-1} \simeq \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q^{1+\gamma}}{\lambda_{q+1}}. \tag{7.50}$$

Finally, to close the inductive estimate, we need to check that the bound in (7.50) is below the one inductively assumed in (7.49), at step $q+1$, which is

$$\frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q^{1+\gamma}}{\lambda_{q+1}} \leq \delta_{q+2} \lambda_{q+1}^{-\gamma}. \tag{7.51}$$

As already done in (7.45), the previous relation gives the bound (7.14) on γ.

To end the discussion, we believe that a possible way to prove Conjecture 7.2, could be to find a way to construct $\overline{R}_q$ such that there is no λ_q^γ loss in size with respect to R_q, or more explicitly

$$\|\overline{R}_q\|_0 \simeq \|R_q\|_0 \quad \text{and} \quad \left|\operatorname{supp}_t \overline{R}_q\right| \simeq \tau_{q+1}.$$

In this way, the inductive hypothesis on the Reynolds stress becomes $\|R_q\|_0 \lesssim \delta_{q+1}$, and consequently, the $\lambda_{q+1}^{-\gamma}$ disappears from the right hand side of (7.51), allowing γ to reach the sharp threshold $\overline{\gamma}(b,\beta) \simeq (b-1)(1-\beta-2\beta b)$.

7.3 Gluing and localization step: Proof of Proposition 7.7

In this section, we prove Proposition 7.7. To make our construction compatible with the choice of all parameters, we choose a large enough (depending on all the parameters β, b, γ, α, but not on q) such that[2]

$$5\tau_{q+1} < \theta_{q+1} \quad \text{and} \quad 2\theta_{q+1} < \tau_q \,. \tag{7.52}$$

We now fix a couple (v_q, R_q) solving (7.4) on $\mathbb{T}^3 \times [0,T]$ together with a set $\mathscr{B}_q \subset [0,T]$ satisfying the hypothesis of Proposition 7.5. By assumption (vi) on $\mathscr{B}_q$, we can write $\mathscr{B}_q$ as a disjoint union of finitely many open intervals J_q of length $5\tau_q$ and consequently, by setting $\widehat{J}_q := \left\{t \in J_q : \operatorname{dist}(t, \mathscr{G}_q) > \tau_q\right\}$, we can write $\widehat{\mathscr{B}}_q$ as disjoint union of intervals $\widehat{J}_q$ of length $3\tau_q$. Observe that by (7.52), we have $\operatorname{dist}(\widehat{J}_q, \mathscr{G}_q) = \tau_q > 2\theta_{q+1}$.

For every such interval J_q, we will first construct a smooth solution $\left(\overline{v}_q, \overline{R}_q\right)$ to (7.4) on $\mathbb{T}^3 \times J_q$ and equidistributed intervals $\{I_i\}_{i=0}^{n+1} \subset J_q$ with $n := \left\lceil \frac{3\tau_q}{\theta_{q+1}} \right\rceil$ such that

[2]Indeed, in order to guarantee the first inequality, it suffices to require $5\lambda_q^{-\gamma} < 1$ which is enforced if a is such that $10\pi a^{-\gamma} < 1$. To ensure the second inequality, we observe that by (7.6), we have for α small enough

$$2\frac{\theta_{q+1}}{\tau_q} \lesssim a^{b^{q-1}(\gamma-(b-1)(1+3\alpha-\beta))} \lesssim a^{-((b-1)(1+3\alpha-\beta)-\gamma)/b} \ll 1\,.$$

(a) $\operatorname{dist}(I_i, I_{i+1}) = \theta_{q+1}$, $|I_i| = \tau_{q+1}$, and all the I_i lie in a $2\theta_{q+1}$ neighbourhood of $\widehat{J}_q$, that is

$$\bigcup_{i=0}^{n+1} I_i \subset \left\{ t \in J_q : \operatorname{dist}\left(t, \widehat{J}_q\right) < 2\theta_{q+1} \right\},$$

(b) $\operatorname{supp} \overline{R}_q \subset \mathbb{T}^3 \times \bigcup_{i=0}^{n+1} I_i$,

(c) $\overline{v}_q(t) = v_q(t) \quad \forall t \in \left\{ t \in J_q : \operatorname{dist}\left(t, \widehat{J}_q\right) \geq 2\theta_{q+1} \right\}$,

(d) $\left(\overline{v}_q, \overline{R}_q\right)$ satisfies the estimates (7.36)–(7.40) when restricted to times $t \in J_q$.

Properties (b) and (c) allows to extend the different $\left(\overline{v}_q, \overline{R}_q\right)$ (coming from different intervals J_q) to a smooth couple $\left(\overline{v}_q, \overline{R}_q\right)$ solving (7.4) on $\mathbb{T}^3 \times [0,T]$ by setting

$$\overline{v}_q(t) := v_q(t) \text{ and } \overline{R}_q(t) := 0 \quad \forall t \in \mathscr{G}_q = [0,T] \setminus \mathscr{B}_q .$$

By construction, $\overline{v}_q$ satisfies (7.34). The new bad set $\mathscr{B}_{q+1}$ is obtained by enlarging the intervals I_i by $2\tau_{q+1}$ on either side;

$$\mathscr{B}_{q+1} := \left\{ t \in [0,T] : t \in J_q \text{ for one of the disjoint intervals of } \mathscr{B}_q \text{ and } \operatorname{dist}\left(t, \bigcup_{i=0}^{n+1} I_i\right) < 2\tau_{q+1} \right\}. \tag{7.53}$$

Using (7.52) and (a), it is easy to see that $\mathscr{B}_{q+1}$ is made of disjoint intervals of length $5\tau_{q+1}$ and $\mathscr{B}_{q+1} \subset \bigcup J_q = \mathscr{B}_q$. In particular, the new bad set satisfies the properties (i)–(iii), and it remains to verify (iv). By construction

$$|\mathscr{B}_{q+1}| = 5\tau_{q+1} \frac{|\mathscr{B}_q|}{5\tau_q} \left(\left\lceil \frac{3\tau_q}{\theta_{q+1}} \right\rceil + 2 \right) \leq 10 |\mathscr{B}_q| \frac{\tau_{q+1}}{\theta_{q+1}},$$

where in the last inequality, we are assuming α small and a large enough. Thus also (iv) holds true. Finally, with this definition of $\mathscr{B}_{q+1}$, the property (7.35) is an immediate consequence of (b), and the estimates (7.36)–(7.40) are a consequence of (d). Indeed, for times $t \in \mathscr{B}_q$ the estimates hold by (d). For $t \in \mathscr{G}_q = [0,T] \setminus \mathscr{B}_q$ we have $\overline{v}_q(t) = v_q(t)$ and $\overline{R}_q(t) \equiv 0$ which makes estimates (7.36), (7.39) and (7.40) trivial. Estimates (7.37) and (7.38) hold then by triangular inequality, (7.30) and assumption (vii) (see also the remarks preceeding (7.76)).

For the rest of this section, we thus fix one of the intervals J_q and the corresponding $\widehat{J}_q$.

7.3.1 Construction of $(\overline{v}_q, \overline{R}_q)$

We start by picking the equidistributed times $t_0 < t_1 < \cdots < t_n$ by setting t_0 to be the left endpoint of the interval $\widehat{J}_q$ and by setting inductively $t_{i+1} := t_i + \theta_{q+1}$ until reaching

$$n := \left\lceil \frac{|\widehat{J}_q|}{\theta_{q+1}} \right\rceil = \left\lceil \frac{3\tau_q}{\theta_{q+1}} \right\rceil. \tag{7.54}$$

In other words, t_n is the right endpoint of $\widehat{J_q}$ in case θ_{q+1} happens to be a multiple of τ_q, otherwise it is the first time falling thereafter. This procedure is compatible with the choice of parameters by (7.52). For each $i = 0, \ldots, n$, we now consider the smooth solutions (v_i, p_i) of the Euler equations with initial datum $v_\ell(t_i)$ defined on their maximal time of existence, that is

$$\begin{cases} \partial_t v_i + \operatorname{div}(v_i \otimes v_i) + \nabla p_i = 0 \\ \operatorname{div} v_i = 0 \\ v_i(\cdot, t_i) = v_\ell(t_i), \end{cases} \tag{7.55}$$

where v_ℓ is the spatial mollification of v_q at length scale $\ell = \ell_q$ defined in (7.27). From Proposition 5.8, (7.32) and (7.30), it follows that each v_i exists for times $|t - t_i| \leq 2\theta_{q+1}$ and enjoys the estimate

$$\|v_i(t)\|_{N+\alpha} \lesssim \|v_\ell(t_i)\|_{N+\alpha} \lesssim \delta_q^{1/2} \lambda_q \ell^{1-N-\alpha}, \quad \text{for } |t - t_i| \leq 2\theta_{q+1} \text{ and } N \geq 1. \tag{7.56}$$

We now glue the exact solutions v_i by means of steep cutoffs η_g^i centered in t_i which are constructed in the following

Lemma 7.9. *Let $\widehat{J_q}$ be one of the disjoint intervals $\widehat{B}_q$ is made of and let $t_0 < t_1 < \cdots < t_n$ be the equidistributed points picked before. There exists a family of cutoff $\{\eta_g^i\}_{i=0}^n \in C_c^\infty((0,T))$ such that*

(a) $\eta_g(t) := \sum_{i=0}^n \eta_g^i(t) = 1 \quad \forall t \in \widehat{J_q}$,

(b) η_g^i *is supported in an interval of size* $< 2\theta_{q+1}$ *centered at* t_i. *More precisely,*

$$\operatorname{supp} \eta_g^i \subset \left(t_i - \frac{\theta_{q+1} + \tau_{q+1}}{2}, t_i + \frac{\theta_{q+1} + \tau_{q+1}}{2} \right),$$

(c) $0 \leq \eta_g^i \leq 1$ *and*

$$\eta_g^i(t) = 1 \quad \forall t \in \left[t_i - \frac{\theta_{q+1} - \tau_{q+1}}{2}, t_i + \frac{\theta_{q+1} - \tau_{q+1}}{2} \right],$$

(d) $\|\partial_t^N \eta_g^i\|_0 \lesssim \tau_{q+1}^{-N} \quad \forall N \geq 0$.

We omit the poof of the Lemma since it is standard. Observe that by construction, $\partial_t \eta_g$ is supported strictly inside $\bigcup_{i=0}^{n+1} I_i$, with

$$I_i := \left(t_i - \frac{\theta_{q+1} + \tau_{q+1}}{2}, t_i - \frac{\theta_{q+1} + \tau_{q+1}}{2} \right),$$

and that from (a) and (b), we have

$$\eta_g^{i-1}(t) = 1 - \eta_g^i(t) \quad \forall t \in I_i \quad \text{and} \quad i = 1, \ldots, n.$$

Since $\operatorname{supp}\eta_g^i \subset \{t : |t-t_i| \le 2\theta_{q+1}\}$, the following gluing of exact solutions is well-defined

$$\overline{v}_q(x,t) := \sum_{i=0}^{n} \eta_g^i(t) v_i(x,t) + (1-\eta_g(t)) v_q(x,t) \quad \text{for } t \in J_q \tag{7.57}$$

$$\overline{p}_q(x,t) := \sum_{i=0}^{n} \eta_g^i(t) p_i(x,t) + (1-\eta_g(t)) p_q(x,t) \quad \text{for } t \in J_q\,. \tag{7.58}$$

It follows that $\overline{v}_q$ is smooth and is an exact solution to Euler outside $\bigcup_{i=0}^{n+1} I_i$; more precisely

$$\partial_t \overline{v}_q + \operatorname{div}(\overline{v}_q \otimes \overline{v}_q) + \nabla \overline{p}_q$$
$$= \begin{cases} 0 \quad \text{in } J_q \setminus \bigcup_{i=0}^{n+1} I_i\,, \\ \partial_t \eta_g^i (v_i - v_{i-1}) - \eta_g^i (1-\eta_g^i) \operatorname{div}((v_i - v_{i-1}) \otimes (v_i - v_{i-1})) & \text{in } I_i \text{ for } i \in \{1,\dots,n\}, \\ \partial_t \eta_g^0 (v_0 - v_q) - \eta_g^0 (1-\eta_g^0) \operatorname{div}((v_0 - v_q) \otimes (v_0 - v_q)) & \text{in } I_0, \\ \partial_t \eta_g^n (v_n - v_q) - \eta_g^n (1-\eta_g^n) \operatorname{div}((v_n - v_q) \otimes (v_n - v_q)) & \text{in } I_{n+1}. \end{cases}$$

Recall from (D.1) the inverse divergence operator $\mathscr{R}$ acting on vector fields with zero average. Since all the v_i and v_q have all the same average, we can define the new localized Reynolds stress by

$$\overline{R}_q := \begin{cases} 0 \quad \text{in in } J_q \setminus \bigcup_{i=0}^{n+1} I_i\,, \\ \partial_t \eta_g^i \mathscr{R}(v_i - v_{i-1}) - \eta_g^i (1-\eta_g^i) ((v_i - v_{i-1}) \otimes (v_i - v_{i-1})) & \text{in } I_i \text{ for } i \in \{1,\dots,n\}\,, \\ \partial_t \eta_g^0 \mathscr{R}(v_0 - v_q) - \eta_g^0 (1-\eta_g^0) ((v_0 - v_q) \otimes (v_0 - v_q)) & \text{in } I_0, \\ \partial_t \eta_g^n \mathscr{R}(v_n - v_q) - \eta_g^n (1-\eta_g^n) ((v_n - v_q) \otimes (v_n - v_q)) & \text{in } I_{n+1}. \end{cases} \tag{7.59}$$

With this definition, the smooth couple $\left(\overline{v}_q, \overline{R}_q\right)$ solves (7.4) on $\mathbb{T}^3 \times J_q$ and has already the desired localization property

$$\operatorname{supp} \overline{R}_q \subset \mathbb{T}^3 \times \bigcup_{i=0}^{n+1} I_i \quad \text{with } |I_i| = \tau_{q+1} \text{ and } n := \left\lceil \frac{3\tau_q}{\theta_{q+1}} \right\rceil. \tag{7.60}$$

7.3.2 Stability estimates on $v_i - v_\ell$ and improved bounds on $v_\ell - v_q$ on $\widehat{\mathscr{B}}_q^c$

We first establish stability estimates on two adjacent exact smooth solutions of Euler, v_i and v_{i+1}. Since $(v_i - v_{i+1}) = (v_i - v_\ell) + (v_\ell - v_{i+1})$, it suffices to estimate $v_i - v_\ell$. The proof of the following proposition follows closely [7] with some minor changes. Here (and in what follows), we denote the material derivative by

$$D_{t,\ell} := \partial_t + (v_\ell \cdot \nabla) \tag{7.61}$$

Proposition 7.10. *For $|t-t_i| \le 2\theta_{q+1}$ we have the estimates*

$$\|(v_i - v_\ell)(t)\|_{N+\alpha} \lesssim \tau_{q+1}\delta_{q+1}\ell^{-N-1+\alpha} \qquad \forall N \ge 0\,, \tag{7.62}$$

$$\|\nabla(p_i - p_\ell)(t)\|_{N+\alpha} \lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha} \qquad \forall N \ge 0\,, \tag{7.63}$$

$$\|D_{t,\ell}(v_i - v_\ell)(t)\|_{N+\alpha} \lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha} \qquad \forall N \ge 0\,. \tag{7.64}$$

Proof of Proposition 7.10. Observe that $(v_\ell - v_i)$ is divergence-free and it solves

$$\partial_t(v_\ell - v_i) + (v_\ell\cdot\nabla)(v_\ell - v_i) + \nabla(p_\ell - p_i) = ((v_i - v_\ell)\cdot\nabla)v_i + \operatorname{div} R_\ell\,, \tag{7.65}$$

so that, by taking the divergence, we find the following equation for the pressure term

$$\Delta(p_\ell - p_i) = -\operatorname{div}(\nabla v_\ell(v_\ell - v_i)) + \operatorname{div}(\nabla v_i(v_i - v_\ell)) + \operatorname{div}\operatorname{div} R_\ell\,. \tag{7.66}$$

By Schauder estimates we get

$$\begin{aligned}\|\nabla(p_\ell - p_i)\|_\alpha &\lesssim (\|\nabla v_\ell\|_\alpha + \|\nabla v_i\|_\alpha)\,\|v_\ell - v_i\|_\alpha + \|\operatorname{div} R_\ell\|_\alpha \\ &\lesssim \delta_q^{1/2}\lambda_q\ell^{-\alpha}\|v_\ell - v_i\|_\alpha + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-1-\alpha}\,,\end{aligned} \tag{7.67}$$

where we used (7.30) and (7.56) (together with A.2) as well as (7.31) in the last inequality. Hence

$$\|D_{t,\ell}(v_\ell - v_i)(t)\|_\alpha \lesssim \delta_q^{1/2}\lambda_q\ell^{-\alpha}\|(v_\ell - v_i)(t)\|_\alpha + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-1-\alpha}, \quad \text{if } |t-t_i| \le 2\theta_{q+1}\,. \tag{7.68}$$

Since $(v_\ell - v_i)(t_i) = 0$, we deduce from Proposition G.1 that

$$\|(v_\ell - v_i)(t)\|_\alpha \lesssim \int_{t_i}^t \left(\delta_q^{1/2}\lambda_q\ell^{-\alpha}\|(v_\ell - v_i)(\tau)\|_\alpha + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-1-\alpha}\right)d\tau\,.$$

Using that $\theta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-\alpha} = \lambda_q^{-3\alpha}\ell^{-\alpha} \le \ell^\alpha \le 1$ by (7.28), we deduce from Grönwall's inequality that

$$\|(v_\ell - v_i)(t)\|_\alpha \lesssim \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-1-\alpha}\theta_{q+1} \lesssim \tau_{q+1}\delta_{q+1}\ell^{-1+\alpha}, \quad \text{if } |t-t_i| \le 2\theta_{q+1}\,. \tag{7.69}$$

Inserting this estimate in (7.67) and (7.68), we have obtained the claimed estimates for $N=0$.

For $N \ge 1$, we fix a spatial derivative ∂^θ of order $|\theta| = N$. We differentiate (7.65) and estimate, using the interpolation inequality for the Hölder norm of products (A.1) on the nonlinear term,

$$\begin{aligned}\|\partial^\theta D_{t,\ell}(v_\ell - v_i)\|_\alpha &\lesssim \|\nabla(p_\ell - p_i)\|_{N+\alpha} + \|v_\ell - v_i\|_{N+\alpha}\|\nabla v_i\|_\alpha + \|v_\ell - v_i\|_\alpha\|\nabla v_i\|_{N+\alpha} + \|\operatorname{div} R_\ell\|_{N+\alpha} \\ &\lesssim \|\nabla(p_\ell - p_i)\|_{N+\alpha} + \delta_q^{1/2}\lambda_q\ell^{-\alpha}\|v_i - v_\ell\|_{N+\alpha} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-1-\alpha}\,,\end{aligned} \tag{7.70}$$

where the second inequality is a consequence of (7.56), (7.69) and (7.31). Reusing the equation (7.66) for the pressure term and Proposition F.1, we have, arguing as before, that

$$\begin{aligned}&\|\nabla(p_\ell - p_i)\|_{N+\alpha} \\ &\lesssim (\|\nabla v_\ell\|_{N+\alpha} + \|\nabla v_i\|_{N+\alpha})\|v_\ell - v_i\|_\alpha + (\|\nabla v_\ell\|_\alpha + \|\nabla v_i\|_\alpha)\|v_\ell - v_i\|_{N+\alpha} + \|\operatorname{div} R_\ell\|_{N+\alpha} \\ &\lesssim \delta_q^{1/2}\lambda_q\ell^{-\alpha}\|v_i - v_\ell\|_{N+\alpha} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-1-\alpha}\,.\end{aligned} \tag{7.71}$$

We now write $\partial^\theta D_{t,\ell}(v_\ell - v_i) = D_{t,\ell}\partial^\theta(v_\ell - v_i) + [\partial^\theta, (v_\ell \cdot \nabla)](v_\ell - v_i)$ and observe, using the Leibniz rule, that the commutator $[\partial^\theta, (v_\ell \cdot \nabla)](v_\ell - v_i)$ involves only spatial derivatives of order at most N of $v_\ell - v_i$. Using again (A.1) to estimate all the nonlinear terms appearing in the commutator, (A.2) and Young, we have

$$
\begin{aligned}
&\|[\partial^\theta, (v_\ell \cdot \nabla)](v_\ell - v_i)\|_\alpha \\
&\lesssim \sum_{k=1}^{N} \|v_\ell\|_{k+\alpha}\|v_\ell - v_i\|_{N+1-k+\alpha} \lesssim \|v_\ell\|_{1+\alpha}\|v_\ell - v_i\|_{N+\alpha} + \|v_\ell\|_{N+1+\alpha}\|v_\ell - v_i\|_\alpha \\
&\lesssim \delta_q^{1/2}\lambda_q \ell^{-\alpha}\|v_\ell - v_i\|_{N+\alpha} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-1-\alpha},
\end{aligned}
$$

where the last inequality uses again (7.30) and (7.69). Collecting terms, we obtain

$$
\|D_{t,\ell}\partial^\theta(v_\ell - v_i)(t)\|_\alpha \lesssim \delta_q^{1/2}\lambda_q \ell^{-\alpha}\|(v_\ell - v_i)(t)\|_{N+\alpha} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-1-\alpha}, \quad \text{if } |t - t_i| \le 2\theta_{q+1}. \tag{7.72}
$$

Reusing Proposition G.1 together with the fact that $\partial^\theta(v_\ell - v_i)(t_i) = 0$, we have

$$
\|(v_\ell - v_i)(t)\|_{N+\alpha} \lesssim \int_{t_i}^{t} \left(\delta_q^{1/2}\lambda_q \ell^{-\alpha}\|(v_\ell - v_i)(\tau)\|_{N+\alpha} + \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha}\right) d\tau, \quad \text{if } |t - t_i| \le 2\theta_{q+1},
$$

where we also used again that $\lambda_q^{-3\alpha} \le \ell^{2\alpha}$ by (7.28). Closing a Grönwall exactly as before, we deduce (7.62). Inserting this estimate in (7.71) and (7.70), we conclude (7.63) and (7.64) as well. □

At difference from [7], $\overline{v}_q$ is not purely a gluing of exact solutions v_i to Euler from an initial datum $v_\ell(t_i)$. Instead, we glue the first exact solution v_0 to v_q in the interval I_0 and the last exact solution v_n to v_q in the interval I_{n+1}. This is necessary in order to guarantee that $\overline{v}_q(t) = v_q(t)$ outside the new bad set and hence the crucial property (v). In addition to Proposition 7.10, we thus need improved estimates (with respect to Proposition 7.6) on $v_q - v_\ell$ in $I_0 \cup I_{n+1}$. Since $v_q(t) \ne v_\ell(t)$, such estimates can no longer rely on stability estimates via closing a suitable Grönwall inequality, but solely on mollification estimates and the better estimates (7.13) of v_q on $\widehat{\mathscr{B}}_q^c$, which the inductive assumption (vii) guarantees.

Proposition 7.11. *For $t \in \widehat{\mathscr{B}}_q^c$, we have the estimates*

$$
\begin{aligned}
\|(v_q - v_\ell)(t)\|_{N+\alpha} &\lesssim \tau_{q+1}\delta_{q+1}\ell^{-N-1+\alpha} && \forall N \ge 0, && (7.73)\\
\|\nabla(p_q - p_\ell)(t)\|_{N+\alpha} &\lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha} && \forall N \ge 0, && (7.74)\\
\|D_{t,\ell}(v_q - v_\ell)(t)\|_{N+\alpha} &\lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha} && \forall N \ge 0. && (7.75)
\end{aligned}
$$

where $D_{t,\ell}$ is the material derivative defined in (7.61).

Proof. We recall from assumption (vii), that v_q satisfies the better estimates (7.13) on $\widehat{\mathscr{B}}_q^c$. Since $\ell_{q-1}^{-N} \le \ell_q^{-N}$ by the definition, we have in particular (with $\ell = \ell_q$ as before)

$$
\|v_q(t)\|_{N+1} \le \delta_{q-1}^{1/2}\lambda_{q-1}\ell^{-N} \quad \forall t \in \widehat{\mathscr{B}}_q^c \text{ and } \forall N \ge 0. \tag{7.76}
$$

We deduce from standard mollification estimates, as in the proof of Proposition 7.6, that for $t \in \widehat{\mathscr{B}}_q^c$

$$\|(v_q - v_\ell)(t)\|_{N+\alpha} \lesssim \ell^{1-N-\alpha}\delta_{q-1}^{1/2}\lambda_{q-1} = \tau_{q+1}\delta_{q+1}\ell^{-N-1+\alpha}\ell^{-2\alpha}\frac{\delta_{q-1}^{1/2}\lambda_{q-1}}{\delta_q^{1/2}\lambda_q} \lesssim \tau_{q+1}\delta_{q+1}\ell^{-N-1+\alpha} \tag{7.77}$$

where in the last inequality, we used (7.28) together with

$$\delta_{q-1}^{1/2}\lambda_{q-1}\big(\delta_q^{1/2}\lambda_q\big)^{-1} \le \lambda_q^{-6\alpha} \le \lambda_q^{-3\alpha}, \tag{7.78}$$

which holds true if we require that α is chosen sufficiently small in order to satisfy

$$6\alpha b \le (b-1)(1-\beta).$$

Observe that $(v_\ell - v_q)$ is divergence-free and that, since $R_q \equiv 0$ on $\mathbb{T}^3 \times \widehat{\mathscr{B}}_q^c$ by assumption (vi), v_q is an exact solution of Euler on $\mathbb{T}^3 \times \widehat{\mathscr{B}}_q^c$. Consequently, $v_\ell - v_q$ satisfies (7.65) (with v_i replaced by v_q) and $p_\ell - p_q$ satisfies (7.66) (with p_i replaced by p_q) on $\mathbb{T}^3 \times \widehat{\mathscr{B}}_q^c$. By Proposition F.1, we deduce, using (A.1), (7.77), (7.73) and (7.31), that for $t \in \widehat{\mathscr{B}}_q^c$

$$\begin{aligned}
&\|\nabla(p_q - p_\ell)(t)\|_{N+\alpha} \\
&\lesssim \big(\|\nabla v_\ell\|_{N+\alpha} + \|\nabla v_q\|_{N+\alpha}\big)\|v_\ell - v_q\|_\alpha + \big(\|\nabla v_\ell\|_\alpha + \|\nabla v_q\|_\alpha\big)\|v_\ell - v_q\|_{N+\alpha} + \|\operatorname{div} R_\ell\|_{N+\alpha} \\
&\lesssim \delta_q^{1/2}\lambda_q\tau_{q+1}\delta_{q+1}\ell^{-N-1} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-1-\alpha} \\
&\lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1+\alpha},
\end{aligned}$$

where we used that $\delta_{q-1}^{1/2}\lambda_{q-1} \le \delta_q^{1/2}\lambda_q$ and (7.28) in the last two inequalities. As for the material derivative, we have, using the equation for $v_\ell - v_q$ as in the proof of Proposition 7.10, by (7.74), (7.77), (7.73) and (7.31)

$$\begin{aligned}
\|D_{t,\ell}(v_q - v_\ell)\|_{N+\alpha} &\lesssim \|\nabla(p_\ell - p_q)\|_{N+\alpha} + \|\nabla v_q\|_\alpha\|v_\ell - v_q\|_{N+\alpha} + \|\nabla v_q\|_{N+\alpha}\|v_\ell - v_q\|_\alpha + \|\operatorname{div} R_\ell\|_{N+\alpha} \\
&\lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N-1-\alpha} + \tau_{q+1}\delta_{q+1}\ell^{-N-1}\delta_{q-1}^{1/2}\lambda_{q-1} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-\alpha}
\end{aligned}$$

which gives (7.75) by observing $\delta_{q-1}^{1/2}\lambda_{q-1} \le \delta_q^{1/2}\lambda_q$ and using (7.28). □

7.3.3 Proof of the estimates (7.36)–(7.38) on $\overline{v}_q$

We show how the estimates (7.36)–(7.38), when restricted to J_q, are an immediate consequence of Proposition 7.10 and 7.11. By construction

$$\overline{v}_q - v_q = \sum_{i=0}^n \eta_g^i(v_i - v_q) = \sum_{i=0}^n \eta_g^i(v_i - v_\ell) - \sum_{i=0}^n \eta_g^i(v_q - v_\ell).$$

Since $\operatorname{supp}\eta_g^i \subset \{t \in J_q : |t-t_i| \le 2\theta_{q+1}\}$, we can use (7.62) and (7.29) to estimate

$$\|\overline{v}_q - v_q\|_0 \le \sup_{i=0,\dots,n} \|\eta_g^i(v_i - v_\ell)\|_\alpha + \|v_\ell - v_q\|_0 \lesssim \tau_{q+1}\delta_{q+1}\ell^{-1+\alpha} + \delta_q^{1/2}\lambda_q\ell \lesssim \delta_{q+1}^{1/2}\lambda_q^{-\gamma/2-3\alpha/2},$$

which proves (7.36). To prove (7.37), we write

$$\overline{v}_q - v_\ell = \sum_{i=0}^n \eta_g^i(v_i - v_\ell) + (1-\eta_g)(v_q - v_\ell).$$

Since $\operatorname{supp}(1-\eta_g) \subset \widehat{\mathscr{B}}_q^c$ by construction, we can use (7.73) together with (7.62) to estimate

$$\|\overline{v}_q - v_\ell\|_0 \le \sup_{i=0,\dots,n} \|\eta_g^i(v_i - v_\ell)\|_\alpha + \|(1-\eta_g)(v_q - v_\ell)\|_\alpha \lesssim \tau_{q+1}\delta_{q+1}\ell^{-1+\alpha} \lesssim \delta_q^{1/2}\lambda_q\ell \tag{7.79}$$

and

$$\begin{aligned}\|\overline{v}_q - v_\ell\|_{N+1} &\le \sup_{i=0,\dots,n} \|\eta_g^i(v_i - v_\ell)\|_{N+1+\alpha} + \|(1-\eta_g)(v_q - v_\ell)\|_{N+1+\alpha} \\ &\lesssim \tau_{q+1}\delta_{q+1}\ell^{-N-2+\alpha} \lesssim \delta_q^{1/2}\lambda_q\ell^{-N} \quad \forall N \ge 0,\end{aligned}$$

proving (7.37). Finally, (7.38) follows immediately combining the former estimate with (7.30).

7.3.4 Estimates on the vector potentials

To improve the estimates on the Reynolds stress $\overline{R}_q$, it is useful to consider the vector potentials associated to v_i, v_ℓ and v_q defined by

$$\begin{aligned} z_i &= \mathscr{Z} v_i := (-\Delta)^{-1}\operatorname{curl} v_i \quad i = 0,\dots,n, \\ z_\ell &:= \mathscr{Z} v_\ell, \\ z_q &:= \mathscr{Z} v_q, \end{aligned}$$

where $\mathscr{Z}$ is the Bio-Savart operator. By construction, $\operatorname{div} z_i = \operatorname{div} z_\ell = \operatorname{div} v_q = 0$ and

$$\operatorname{curl} z_i = (-\Delta)^{-1}\operatorname{curl}\operatorname{curl} v_i = v_i \quad \operatorname{curl} z_\ell = v_\ell \quad \operatorname{curl} z_q = v_q,$$

since v_i, v_ℓ and v_q are divergence free. Thus, we view $z_i - z_\ell$ (and $z_q - v_\ell$) as potential of first order of $v_i - v_\ell$ (and $v_q - v_\ell$) and as such, we expect the stability estimates $z_i - z_\ell$ (and $z_q - z_\ell$) to improve by a factor of ℓ. We make this heuristic rigorous in the following

Proposition 7.12. *For* $|t - t_i| \le 2\theta_{q+1}$

$$\|(z_i - z_\ell)(t)\|_{N+\alpha} \lesssim \tau_{q+1}\delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \ge 0 \tag{7.80}$$

$$\|D_{t,\ell}(z_i - z_\ell)(t)\|_{N+\alpha} \lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \ge 0, \tag{7.81}$$

where $D_{t,\ell}$ *denotes the material derivative as defined in* (7.61).

The proof of Proposition 7.12 follows closely [7]. The next proposition on the other hand, should be seen as the analogue of Proposition 7.11 and exploits crucially that v_q is an exact solution of Euler on $\mathbb{T}^3 \times \widehat{\mathscr{B}}_q^c$ with better estimates.

Proposition 7.13. *For $t \in \widehat{\mathscr{B}}_q^c$ we have*

$$\|(z_q - z_\ell)(t)\|_{N+\alpha} \lesssim \tau_{q+1}\delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \geq 0, \tag{7.82}$$
$$\|D_{t,\ell}(z_q - z_\ell)(t)\|_{N+\alpha} \lesssim \lambda_q^{-\gamma}\delta_{q+1}\ell^{-N+\alpha} \qquad \forall N \geq 0\,. \tag{7.83}$$

where $D_{t,\ell}$ denotes the material derivative as defined in (7.61).

Proof of Proposition 7.12. We set $\tilde{z}_i := z_\ell - z_i$. Recall that $(v_\ell - v_i)$ solves (7.65), so that

$$\partial_t \operatorname{curl} \tilde{z}_i + (v_\ell \cdot \nabla)\operatorname{curl} \tilde{z}_i = -\nabla(p_\ell - p_i) - (\operatorname{curl} \tilde{z}_i \cdot \nabla)v_i + \operatorname{div} R_\ell\,. \tag{7.84}$$

We rewrite, using $\operatorname{div} \tilde{z}_i = \operatorname{div} v_\ell = 0$,

$$[(v_\ell \cdot \nabla)\operatorname{curl} \tilde{z}_i]^j = \partial_k\left(v_\ell^k [\operatorname{curl} \tilde{z}_i]^j\right) = [\operatorname{curl}((v_\ell \cdot \nabla)\tilde{z}_i)]^j + \partial_k\left([\tilde{z}_i \times \nabla v_\ell^k]^j\right),$$
$$[(\operatorname{curl} \tilde{z}_i \cdot \nabla)v_i]^j = \partial_k\left([\operatorname{curl} \tilde{z}_i]^k v_i^j\right) = \operatorname{div}\operatorname{curl}\left(\tilde{z}_i v_i^j\right) + \partial_k\left([\tilde{z}_i \times \nabla v_i^j]^k\right) = \partial_k\left([\tilde{z}_i \times \nabla v_i^j]^k\right),$$

where we used the convention to sum over repeated indices. Setting

$$[(z \times \nabla)v]^{jk} = [z \times \nabla v^k]^j = \varepsilon_{jlm} z^l \partial_m v^k\,,$$

where ε_{jlm} denotes the Levi-Civita symbol, we obtain that

$$\operatorname{curl}(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i) = -\operatorname{div}\left((\tilde{z}_i \times \nabla)v_\ell + [(\tilde{z}_i \times \nabla)v_i]^T\right) - \nabla(p_\ell - p_i) + \operatorname{div} R_\ell\,.$$

Taking the curl of the above equation and recalling that $(-\Delta) = \operatorname{curl}\operatorname{curl} + \nabla\operatorname{div}$, we find

$$(-\Delta)(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla)\tilde{z}_i) = F\,,$$

where

$$F = -\nabla\operatorname{div}((\tilde{z}_i \cdot \nabla)v_\ell) - \operatorname{curl}\operatorname{div}\left((\tilde{z}_i \times \nabla)v_\ell + [(\tilde{z}_i \times \nabla)v_i]^T\right) + \operatorname{curl}\operatorname{div} R_\ell\,. \tag{7.85}$$

We deduce from Proposition F.1 and (A.1) that

$$\begin{aligned}
\|D_{t,\ell}\tilde{z}_i\|_{N+\alpha} &\lesssim \|(\tilde{z}_i \cdot \nabla)v_\ell\|_{N+\alpha} + \|(\tilde{z}_i \times \nabla)v_\ell\|_{N+\alpha} + \|[(\tilde{z}_i \times \nabla)v_i]^T\|_{N+\alpha} + \|R_\ell\|_{N+\alpha} \\
&\leq (\|v_i\|_{N+1+\alpha} + \|v_\ell\|_{N+1+\alpha})\|\tilde{z}_i\|_\alpha + (\|v_i\|_{1+\alpha} + \|v_\ell\|_{1+\alpha})\|\tilde{z}_i\|_{N+\alpha} + \|R_\ell\|_{N+\alpha} \\
&\lesssim \delta_q^{1/2}\lambda_q \ell^{-N-\alpha}\|\tilde{z}_i\|_\alpha + \delta_q^{1/2}\lambda_q \ell^{-\alpha}\|\tilde{z}_i\|_{N+\alpha} + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-\alpha}\,, \qquad (7.86)
\end{aligned}$$

where the last inequality is a consequence of (7.30), (7.56) and (7.31). In particular for $N=0$, we deduce from Proposition G.1 that, since $\tilde{z}_i(t_i)=0$,

$$\|\tilde{z}_i(t)\|_\alpha \lesssim \int_{t_i}^t \left(\delta_q^{1/2}\lambda_q\ell^{-\alpha}\|\tilde{z}_i(\tau)\|_\alpha + \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-\alpha}\right)d\tau, \quad \text{if } |t-t_i|\leq 2\theta_{q+1}\,.$$

Since $\delta_q^{1/2}\lambda_q\ell^\alpha\theta_{q+1}\leq \ell^\alpha\leq 1$ and that $\lambda_q^{-3\alpha}\ell^{-\alpha}\leq\ell^\alpha$ by (7.28), by Grönwall's inequality we get

$$\|\tilde{z}_i(t)\|_\alpha \lesssim \theta_{q+1}\lambda_q^{-\gamma}\delta_{q+1}\ell^\alpha = \tau_{q+1}\delta_{q+1}\ell^\alpha, \quad \text{if } |t-t_i|\leq 2\theta_{q+1}\,.$$

Inserting this bound back in (7.86), we also obtain (7.81) for $N=0$. As for the higher derivatives, we simply observe that the operator $\nabla\mathscr{Z}$ is bounded on Hölder spaces by Proposition F.1 and hence for $N\geq 1$, we deduce from (7.62)

$$\|\tilde{z}_i\|_{N+\alpha} = \|\nabla\tilde{z}_i\|_{N-1+\alpha} = \|\nabla\mathscr{Z}(v_i-v_\ell)\|_{N-1-\alpha}\lesssim \tau_{q+1}\delta_{q+1}\ell^{-N+\alpha}\,.$$

The estimate (7.81) for $N\geq 1$ is obtained by writing $\partial^\theta D_{t,\ell}\tilde{z}_i = D_{t,\ell}\partial^\theta\tilde{z}_i + [\partial^\theta,(v_\ell\cdot\nabla)]\tilde{z}_i$ for a derivative ∂^θ with $|\theta|=N$, estimating separately the resulting commutator as in the proof of Proposition 7.10. □

Proof of Proposition 7.13. Observe that the operator $\mathscr{Z}$ commutes with convolution, hence $z_\ell = z_q * \varphi_\ell$. Moreover, as a consequence of assumption (vii) and the fact that $\ell_q\leq\ell_{q-1}$, we have on $\widehat{\mathscr{B}}_q^c$ (for α small enough) the better estimates (7.76)–(7.77). Since $(-\Delta)z_q = \operatorname{curl} v_q$, standard estimates for the Laplace equation give $\|z_q\|_2\leq\|z_q\|_{2+\alpha}\lesssim\|v_q\|_{1+\alpha}$, which together with standard mollification estimates implies

$$\|(z_q-z_\ell)(t)\|_\alpha \lesssim \ell^{2-\alpha}\|z_q(t)\|_2 \lesssim \ell^{2-\alpha}\delta_{q-1}^{1/2}\lambda_{q-1}\ell^{-\alpha} = \tau_{q+1}\delta_{q+1}\ell^{-2\alpha}\frac{\delta_{q-1}^{1/2}\lambda_{q-1}}{\delta_q^{1/2}\lambda_q}\lesssim \tau_{q+1}\delta_{q+1}\ell^\alpha \quad t\in\widehat{\mathscr{B}}_q^c,$$

where in the last inequality we reused (7.78). As for derivatives of higher order, we recall that $\nabla\mathscr{Z}$ is bounded on Hölder spaces by Proposition F.1. We then estimate using (7.77) for $t\in\widehat{\mathscr{B}}_q^c$

$$\|(z_\ell-z_q)(t)\|_{N+1+\alpha} = \|\nabla\mathscr{Z}(v_\ell-v_q)(t)\|_{N+\alpha}\lesssim\|(v_\ell-v_q)(t)\|_{N+\alpha}\lesssim\tau_{q+1}\delta_{q+1}\ell^{-N-1+\alpha}\,,$$

where the last inequality uses again α small enough as in (7.78). As for the material derivative, we observe that by assumption (vi), v_q is a smooth solution of Euler on $\mathbb{T}^3\times\widehat{\mathscr{B}}_q^c$. Hence we can argue as in the proof of Proposition 7.12 to obtain, for $t\in\widehat{\mathscr{B}}_q^c$

$$\begin{aligned}
&\|D_{t,\ell}(z_\ell-z_q)\|_{N+\alpha}\\
&\lesssim \left(\|v_\ell\|_{N+1+\alpha}+\|v_q\|_{N+1+\alpha}\right)\|z_\ell-z_q\|_\alpha + \left(\|v_\ell\|_{1+\alpha}+\|v_q\|_{1+\alpha}\right)\|z_\ell-z_q\|_{N+\alpha}+\|R_\ell\|_{N+\alpha}\\
&\lesssim \lambda_q^{-\gamma-3\alpha}\delta_{q+1}\ell^{-N-\alpha},
\end{aligned}$$

where the last inequality follows from combining the estimates (7.76), (7.30), (7.82) and (7.31). We conclude (7.83) recalling that $\lambda_q^{-3\alpha}\leq\ell^{2\alpha}$ by (7.28). □

7.3.5 Proof of the estimates (7.39)–(7.40) on $\overline{R}_q$

Since $\operatorname{supp}\overline{R}_q \subset \mathbb{T}^3 \times \bigcup_{i=0}^{n+1} I_i$ by construction, it suffices to prove both estimates on every interval I_i. As in [7], we will repeatedly use that $\mathscr{R}\mathrm{curl}$ is a bounded operator on Hölder spaces by Proposition F.1 and thereby we improve the estimates on terms of the form $\mathscr{R}(v_i - v_{i-1}) = \mathscr{R}\mathrm{curl}\,(z_i - z_{i-1})$ by passing to the vector potentials.

Consider now first the case $i \in \{1,\dots,n\}$. Recall from (7.59)

$$\overline{R}_q = \partial_t \eta_g^i \mathscr{R}(v_i - v_{i-1}) - \eta_g^i(1-\eta_g^i)((v_i - v_{i-1}) \otimes (v_i - v_{i-1})) \quad \text{ on } I_i\,.$$

Recall from Lemma 7.9 that $\|\partial_t^N \eta_g^i\|_0 \lesssim \tau_{q+1}^{-N}$ and that $\operatorname{supp}\eta_g^i \subset \{t : |t-t_i| \le 2\theta_{q+1}\}$. Therefore, we bound, using also (A.1), (7.62) and (7.80),

$$\begin{aligned}\|\overline{R}_q(t)\|_{N+\alpha} &\lesssim \tau_{q+1}^{-1}\|\mathscr{R}\mathrm{curl}\,(z_i - z_{i-1})\|_{N+\alpha} + \|v_i - v_{i-1}\|_{N+\alpha}\|v_i - v_{i-1}\|_\alpha \\ &\lesssim \tau_{q+1}^{-1}\|z_i - z_{i-1}\|_{N+\alpha} + \tau_{q+1}^2\delta_{q+1}^2\ell^{-N-2+2\alpha} \\ &\lesssim \delta_{q+1}\ell^{-N+\alpha}\,, \end{aligned} \tag{7.87}$$

which gives (7.39) on I_i. As for estimate (7.40), we begin by writing

$$\|(\partial_t + \overline{v}_q\cdot\nabla)\overline{R}_q\|_{N+\alpha} \le \|(v_\ell - \overline{v}_q)\cdot\nabla\overline{R}_q\|_{N+\alpha} + \|D_{t,\ell}\overline{R}_q\|_{N+\alpha} \tag{7.88}$$

where $D_{t,\ell}$ is defined in (7.61). Using (7.36), (7.79) and (7.87), we estimate the first term on I_i

$$\|(v_\ell - \overline{v}_q)\cdot\nabla\overline{R}_q\|_{N+\alpha} \lesssim \|v_\ell - \overline{v}_q\|_{N+\alpha}\|\nabla\overline{R}_q\|_0 + \|v_\ell - \overline{v}_q\|_0\|\nabla\overline{R}_q\|_{N+\alpha} \lesssim \delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N}\,, \tag{7.89}$$

which is better than the desired bound in (7.40). We are left to bound $\|D_{t,\ell}\overline{R}_q\|_{N+\alpha}$ on I_i. We compute (always on I_i)

$$\begin{aligned} D_{t,\ell}\overline{R}_q &= \partial_{tt}\eta_g^i\mathscr{R}(v_i - v_{i-1}) + \partial_t\eta_g^i\Big(\partial_t\mathscr{R}(v_i - v_{i-1}) + (v_\ell\cdot\nabla)\mathscr{R}(v_i - v_{i-1})\Big) \\ &\quad - \partial_t\big(\eta_g^i(1-\eta_g^i)\big)\big((v_i - v_{i-1})\otimes(v_i - v_{i-1})\big) \\ &\quad - \eta_g^i(1-\eta_g^i)\Big(\big(D_{t,\ell}(v_i - v_{i-1})\big)\otimes(v_i - v_{i-1}) + (v_i - v_{i-1})\otimes\big(D_{t,\ell}(v_i - v_{i-1})\big)\Big)\,. \end{aligned}$$

We rewrite

$$\partial_t\eta_g^i\Big(\partial_t\mathscr{R}(v_i - v_{i-1}) + (v_\ell\cdot\nabla)\mathscr{R}(v_i - v_{i-1})\Big) = \partial_t\eta_g^i\Big((\mathscr{R}\mathrm{curl})D_{t,\ell}(z_i - z_{i-1}) + [(v_\ell\cdot\nabla),\mathscr{R}\mathrm{curl}](z_i - z_{i-1})\Big)\,,$$

where $[(v_\ell\cdot\nabla),\mathscr{R}\mathrm{curl}]$ denotes the commutator involving the singular integral operator $\mathscr{R}\mathrm{curl}$. From Proposition F.2, (7.30) and (7.80), we have

$$\begin{aligned}\|[(v_\ell\cdot\nabla),\mathscr{R}\mathrm{curl}](z_i - z_{i-1})\|_{N+\alpha} &\lesssim \|v_\ell\|_{1+\alpha}\|z_i - z_{i-1}\|_{N+\alpha} + \|v_\ell\|_{N+1+\alpha}\|z_i - z_{i-1}\|_\alpha \\ &\lesssim \tau_{q+1}\delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N}\,. \end{aligned} \tag{7.90}$$

We can thus estimate (always on I_i) using (A.1) on products together with (7.80), (7.81), (7.90), (7.62) and (7.64)

$$\begin{aligned}\|D_{t,\ell}\overline{R}_q\|_{N+\alpha} &\lesssim \tau_{q+1}^{-2}\|z_i-z_{i-1}\|_{N+\alpha}+\tau_{q+1}^{-1}\big(\|D_{t,\ell}(z_i-z_{i-1})\|_{N+\alpha}+\|[(v_\ell\cdot\nabla),\mathscr{R}\mathrm{curl}](z_i-z_{i-1})\|_{N+\alpha}\big)\\&\quad+\tau_{q+1}^{-1}\|v_i-v_{i-1}\|_{N+\alpha}\|v_i-v_{i-1}\|_\alpha\\&\quad+\|D_{t,\ell}(v_i-v_{i-1})\|_{N+\alpha}\|v_i-v_{i-1}\|_\alpha+\|D_{t,\ell}(v_i-v_{i-1})\|_\alpha\|v_i-v_{i-1}\|_{N+\alpha}\\&\lesssim \tau_{q+1}^{-1}\delta_{q+1}\ell^{-N+\alpha}+\delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N}+\tau_{q+1}\delta_{q+1}^2\ell^{-N-2+2\alpha}\\&\lesssim \tau_{q+1}^{-1}\delta_{q+1}\ell^{-N+\alpha},\end{aligned}$$

where we used in the last inequality that $\tau_{q+1}\delta_q^{1/2}\lambda_q\le\lambda_q^{-3\alpha}\le\ell^\alpha$ by (7.28) and $\left(\tau_{q+1}\ell^{-1}\right)^2\le 1$. This proves (7.40) on I_i recalling (7.28).

Finally, let us prove the estimates (7.39)–(7.40) on I_0 and I_{n+1}. Recall from (7.59)

$$\overline{R}_q=\partial_t\eta_g^0\mathscr{R}(v_0-v_q)-\eta_g^0(1-\eta_g^0)((v_0-v_q)\otimes(v_0-v_q))\qquad\text{on } I_0.$$

Arguing as before and writing $z_0-z_q=(z_0-z_\ell)+(z_\ell-z_q)$ and $v_0-v_q=v_0-v_\ell+v_\ell-v_q$, we have for $t\in I_0\subset\widehat{\mathscr{B}}_q^c$ using (7.80), (7.82), (7.62) and (7.73)

$$\begin{aligned}\|\overline{R}_q(t)\|_{N+\alpha}&\lesssim\tau_{q+1}^{-1}\|(z_0-z_q)(t)\|_{N+\alpha}+\|(v_0-v_q)(t)\|_{N+\alpha}\|(v_0-v_q)(t)\|_\alpha\\&\lesssim\delta_{q+1}\ell^{-N+\alpha}+\tau_{q+1}^2\delta_{q+1}^2\ell^{-N-2+2\alpha}\\&\lesssim\delta_{q+1}\ell^{-N+\alpha}.\end{aligned}$$

As for estimate (7.40), we argue as in (7.88) and (7.89) to reduce ourselves to bound $\|D_{t,\ell}\overline{R}_q\|_{N+\alpha}$. Proceeding as before, we obtain for $t\in I_0\subset\widehat{\mathscr{B}}_q^c$ that

$$\begin{aligned}\|D_{t,\ell}\overline{R}_q(t)\|_{N+\alpha}&\lesssim\tau_{q+1}^{-2}\|z_0-z_q\|_{N+\alpha}+\tau_{q+1}^{-1}\big(\|D_{t,\ell}(z_0-z_q)\|_{N+\alpha}+\|[(v_\ell\cdot\nabla),\mathscr{R}\mathrm{curl}](z_0-z_q)\|_{N+\alpha}\big)\\&\quad+\tau_{q+1}^{-1}\|v_0-v_q\|_{N+\alpha}\|v_0-v_q\|_\alpha\\&\quad+\|D_{t,\ell}(v_0-v_q)\|_{N+\alpha}\|v_0-v_q\|_\alpha+\|D_{t,\ell}(v_0-v_q)\|_\alpha\|v_0-v_q\|_{N+\alpha}\\&\lesssim\tau_{q+1}^{-1}\delta_{q+1}\ell^{-N+\alpha}+\delta_{q+1}\delta_q^{1/2}\lambda_q\ell^{-N}+\tau_{q+1}\delta_{q+1}^2\ell^{-N-2+2\alpha},\end{aligned}$$

where we used Proposition F.2 to estimate the commutator as well as the estimates (7.82), (7.83), (7.30), (7.73) and (7.75). We conclude (7.40) on I_0. The estimates on I_{n+1} follow in the same way up to exchanging the role (v_0,η_g^0) with (v_n,η_g^n). This concludes the proof of Proposition 7.7.

7.4 Perturbation

In this section we will construct the perturbation w_{q+1} and consequently define

$$v_{q+1}:=\overline{v}_q+w_{q+1},\tag{7.91}$$

where $(\bar{v}_q, \overline{R}_q)$ is a smooth solution of (7.4) as given by Proposition 7.7. Following the construction of [7], the perturbation will be highly oscillatory and it will be based on the Mikado flows. As for the gluing step, also here there will be some changes with respect to [7]. For instance, the fact that we are not interested in prescribing an energy profile, allows us to simplify the choice of the amplitude of the perturbation. For this reason, we will give a complete proof of all the estimates.

Let now $\mathscr{B}_{q+1} \subset [0,T]$ be the bad set belonging to $(\bar{v}_q, \overline{R}_q)$ (see Proposition 7.7). Note in particular that by Proposition 7.7, $\mathscr{B}_{q+1}$ already satisfies the size properties (i)–(iv) at step $q+1$ and we will leave the bad set $\mathscr{B}_{q+1}$ unchanged. Thus, to prove Proposition 7.8, we are left only to check the two estimates (7.41) and (7.42) as well as the properties (v)–(vii) (with q replaced by $q+1$). Since by Proposition 7.7, the couple $(\bar{v}_q, \overline{R}_q)$ already satisfies the more restrictive properties (7.34), (7.35) and (7.38), the properties (iv)–(vii) can be achieved by ensuring that the temporal support of w_{q+1} is contained in a τ_{q+1} neighbourhood of the time support of $\overline{R}_q$. In particular, this will ensure that $\operatorname{supp} R_{q+1} \subset \mathbb{T}^3 \times \{t \in \mathscr{B}_{q+1} : \operatorname{dist}(t, \mathscr{G}_{q+1}) > \tau_{q+1}\}$, or in other words, that the new Reynolds stress R_{q+1} is localized in the new real bad set $\widehat{\mathscr{B}}_{q+1}$ which is made of disjoint intervals of length $3\tau_{q+1}$.

A crucial relation that will allow us to close the estimates on R_{q+1} will be

$$\ell^{-1} \ll \lambda_{q+1}, \tag{7.92}$$

that is a consequence of $\gamma + 3\alpha < 2(b-1)(1-\beta)$. By our bound on γ in (7.14) (actually (7.6) would suffice here), the latter holds if α is sufficiently small.

7.4.1 The stress tensor $\tilde{R}_{q,i}$

Recall that $\overline{R}_q$ is supported in the set $\mathbb{T}^3 \times I$, where I is the union of disjoint intervals of length τ_{q+1}. Thus we can write

$$I = \bigcup_i I_i, \text{ where } |I_i| = \tau_{q+1}.$$

The following lemma gives the family of cutoffs that will allow us to localize the perturbation (and thus the new Raynolds stress) in the new real bad set $\widehat{\mathscr{B}}_{q+1}$.

Lemma 7.14. *There exist smooth cutoff functions $\{\eta_p^i\}_i$ such that $\eta_p^i\big|_{I_i} \equiv 1$, $\operatorname{supp} \eta_p^i \cap \operatorname{supp} \eta_p^j = \emptyset$ if $i \neq j$, $\operatorname{supp} \eta_p^i \subset \{t \in I_i : \operatorname{dist}(t, I_i) < \tau_{q+1}\}$. Moreover, for any i and $N \geq 0$ we have*

$$\|\partial_t^N \eta_p^i\|_0 \lesssim \tau_{q+1}^{-N}. \tag{7.93}$$

Let s_i be the middle point of I_i. Define the flows Φ_i associated to the velocity field $\bar{v}_q$ as the solution of

$$\begin{cases} (\partial_t + \bar{v}_q \cdot \nabla)\Phi_i = 0 \\ \Phi_i(x, s_i) = x. \end{cases}$$

Define also

$$\tilde{R}_{q,i} := \frac{\nabla\Phi_i \left(\delta_{q+1}\mathrm{Id} - \overline{R}_q\right) \nabla\Phi_i^T}{\delta_{q+1}}. \tag{7.94}$$

We have the following

Lemma 7.15. *For $a \gg 1$ sufficiently large, we have*

$$\|\nabla\Phi_i(t) - \mathrm{Id}\|_0 \leq \frac{1}{2}, \quad \forall t \in \operatorname{supp} \eta_p^i. \tag{7.95}$$

Moreover, for all $(x,t) \in \mathbb{T}^3 \times \operatorname{supp} \eta_p^i$

$$\tilde{R}_{q,i}(x,t) \in B_{\frac{1}{2}}(\mathrm{Id}) \subset \mathscr{S}_+^{3\times 3},$$

where $B_{\frac{1}{2}}(\mathrm{Id})$ denotes the ball of radius $\frac{1}{2}$ around the identity, in the space of positive definite matrices.

Proof. By applying (7.38) and (G.4) we obtain

$$\|\nabla\Phi_i(t) - \mathrm{Id}\|_0 \lesssim |t - s_i| \|\bar{v}_q\|_1 \lesssim \tau_{q+1}\delta_q^{1/2}\lambda_q \leq \lambda_q^{-\gamma}. \tag{7.96}$$

Furthermore, by definition we have

$$\begin{aligned}
\tilde{R}_{q,i} - \mathrm{Id} &= -\nabla\Phi_i \frac{\overline{R}_q}{\delta_{q+1}} \nabla\Phi_i^T + \nabla\Phi_i\nabla\Phi_i^T - \mathrm{Id} \\
&= -\nabla\Phi_i \frac{\overline{R}_q}{\delta_{q+1}} \nabla\Phi_i^T + (\nabla\Phi_i - \mathrm{Id})\nabla\Phi_i^T + (\nabla\Phi_i - \mathrm{Id})^T,
\end{aligned}$$

from which, by using (7.39) and (7.96), we obtain for $t \in \operatorname{supp} \eta_p^i$

$$\|\tilde{R}_{q,i} - \mathrm{Id}\|_0 \lesssim \frac{\|\overline{R}_q\|_0}{\delta_{q+1}} + \|\nabla\Phi_i - \mathrm{Id}\|_0 \lesssim \ell^\alpha + \lambda^{-\gamma}.$$

By choosing $a \gg 1$ large enough, we conclude $\tilde{R}_{q,i}(x,t) \in B_{\frac{1}{2}}(\mathrm{Id})$ for every $(x,t) \in \mathbb{T}^3 \times \operatorname{supp} \eta_p^i$. $\square$

7.4.2 The perturbation, the constant M and the properties (v) and (vii)

We define the principal part the the perturbation as

$$w_o := \sum_i \eta_p^i \delta_{q+1}^{1/2} \nabla\Phi_i^{-1} W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i) = \sum_i w_{o,i},$$

where Lemma E.1 is applied with $\mathscr{N} = \overline{B}_{\frac{1}{2}}(\mathrm{Id})$. Notice that from Lemma 7.15 it follows that $W(\tilde{R}_{q,i}, \lambda_{q+1}\Phi_i)$ is well defined. Using the Fourier series representation (E.3) we obtain

$$w_{o,i} = \sum_{k\neq 0} \eta_p^i \delta_{q+1}^{1/2} a_k(\tilde{R}_{q,i}) \nabla\Phi_i^{-1} A_k e^{i\lambda_{q+1} k\cdot\Phi_i}.$$

The choice of w_o is motivated by the fact that the vector fields $U_{i,k} = \nabla\Phi_i^{-1} A_k e^{i\lambda_{q+1} k\cdot\Phi_i}$ solve

$$(\partial_t + \overline{v}_q \cdot \nabla) U_{i,k} = \nabla \overline{v}_q^T U_{i,k} \,. \tag{7.97}$$

In particular, since $\operatorname{div} U_{i,k}(x, s_i) = 0$ for all $x \in \mathbb{T}^3$, $U_{i,k}$ remains divergence free.

For notational convenience we set

$$b_{i,k}(x,t) := \eta_p^i(t)\delta_{q+1}^{1/2} a_k(\tilde{R}_{q,i}) A_k,$$

so that we may write

$$w_{o,i} = \sum_{k\neq 0} \nabla\Phi_i^{-1} b_{i,k} e^{i\lambda_{q+1} k\cdot\Phi_i}.$$

The following lemma ensures that the constant M from Proposition 7.5 is geometric and, in particular, does not depend on all the parameters entering in the scheme.

Lemma 7.16. *There exists a geometric constant* $\overline{C} > 0$ *such that*

$$\|b_{i,k}\|_0 \leq \frac{\overline{C}}{|k|^5} \delta_{q+1}^{1/2}.$$

Proof. Apply (E.4) with $N = 0$, $m = 5$ and $\mathscr{N} = \overline{B}_{\frac{1}{2}}(\mathrm{Id})$. □

We are now ready to define the geometric constant M of Proposition 7.5.

Definition 7.17. *The constant M is defined as*

$$M = 64\overline{C} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|k|^4},$$

where $\overline{C}$ *is the constant of Lemma 7.16.*

To ensure that w_{q+1} is divergence free we will add a corrector term w_c to w_o. More precisely, in view of (E.7), we define

$$w_c := \frac{-i}{\lambda_{q+1}} \sum_{i,k\neq 0} \eta_p^i \delta_{q+1}^{1/2} \nabla a_k(\tilde{R}_{q,i}) \times \frac{\nabla\Phi_i^T (k \times A_k)}{|k|^2} e^{i\lambda_{q+1} k\cdot\Phi_i} = \sum_{i,k\neq 0} c_{i,k} e^{i\lambda_{q+1} k\cdot\Phi_i},$$

where

$$c_{i,k} := \frac{-i}{\lambda_{q+1}} \eta_p^i \delta_{q+1}^{1/2} \nabla a_k(\tilde{R}_{q,i}) \times \frac{\nabla\Phi_i^T (k \times A_k)}{|k|^2}.$$

By using (E.7), one can check that

$$w_{q+1} := w_o + w_c = \frac{-1}{\lambda_{q+1}} \operatorname{curl} \left(\sum_{i,k\neq 0} \nabla\Phi_i^T \frac{ik \times b_{i,k}}{|k|^2} e^{i\lambda_{q+1} k\cdot\Phi_i} \right),$$

from which we deduce $\operatorname{div} w_{q+1} = 0$. Finally, note that thanks to the cutoffs η_p^i from Lemma 7.14, we also get

$$\operatorname{supp} w_{q+1} \subset \mathbb{T}^3 \times \left\{ t \in \mathscr{B}_{q+1} : \operatorname{dist}(t, \mathscr{G}_{q+1}) > \tau_{q+1} \right\} = \widehat{\mathscr{B}}_{q+1} . \tag{7.98}$$

Recalling (7.34) and the inductive assumption (v) on v_q, this guarantees the property (v) at step $q+1$. Moreover, by (7.38) and (7.98) we also get (vii) at step $q+1$, since for $N \geq 0$

$$\|v_{q+1}(t)\|_{N+1} \leq \|\overline{v}_q(t) + w_{q+1}(t)\|_{N+1} = \|\overline{v}_q(t)\|_{N+1} \leq \delta_q^{1/2} \lambda_q \ell_q^{-N}, \quad \forall t \in \widehat{\mathscr{B}}_{q+1}^c .$$

7.4.3 The final Reynolds stress and property (vi)

We define the new Reynolds stress as

$$\begin{aligned} R_{q+1} &:= \mathscr{R}(w_{q+1} \cdot \nabla \overline{v}_q) + \mathscr{R}(\partial_t w_{q+1} + \overline{v}_q \cdot \nabla w_{q+1}) + \mathscr{R} \operatorname{div} (\overline{R}_q + w_{q+1} \otimes w_{q+1}) \\ &= R_{nash} + R_{transp} + R_{osc} . \end{aligned} \tag{7.99}$$

Notice that in all the three terms of the previous formula, the operator $\mathscr{R}$ is always applied to a divergence of a curl (thus to zero average vector fields). Moreover, by (7.98) and (7.35), we directly get

$$\operatorname{supp} R_{q+1} \subset \mathbb{T}^3 \times \widehat{\mathscr{B}}_{q+1},$$

which proves property (vi) at step $q+1$. With this definition, one may check that

$$\begin{cases} \partial_t v_{q+1} + \operatorname{div} (v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} = \operatorname{div} R_{q+1} \\ \operatorname{div} v_{q+1} = 0, \end{cases}$$

where the new pressure is defined as $p_{q+1} := \overline{p}_q$.

7.4.4 Estimates on the perturbation

We start by estimating all the terms entering in the definition of w_{q+1}.

Proposition 7.18. *For all $t \in \operatorname{supp} \eta_p^i$ and every $N \geq 0$, we have*

$$\|(\nabla \Phi_i)^{-1}\|_N + \|\nabla \Phi_i\|_N \lesssim \ell^{-N} , \tag{7.100}$$

$$\|\tilde{R}_{q,i}\|_N \lesssim \ell^{-N} , \tag{7.101}$$

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} |k|^{-6} \ell^{-N} , \tag{7.102}$$

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} |k|^{-6} \ell^{-N-1} . \tag{7.103}$$

Proof. Let $t \in \operatorname{supp} \eta_p^i$. From (G.4), (G.5), (7.38) and (7.95), we obtain

$$\|\nabla \Phi_i\|_N \lesssim \|\nabla \Phi_i\|_0 + [\nabla \Phi_i]_N \lesssim 1 + \|\nabla \Phi_i - \mathrm{Id}\|_0 + [\nabla \Phi_i]_N \lesssim 1 + \tau_{q+1} \|\nabla \overline{v}_q\|_N \lesssim \ell^{-N}.$$

Moreover, by also using (7.39) we get

$$\|\tilde{R}_{q,i}\|_N \lesssim \|\nabla\Phi_i\|_N\|\nabla\Phi_i\|_0 + \|\nabla\Phi_i\|_0^2 \left\|\frac{\overline{R}_q}{\delta_{q+1}}\right\|_N \lesssim \ell^{-N} + \ell^{-N+\alpha} \lesssim \ell^{-N},$$

which proves (7.101). Finally, by the two previous estimates we also deduce

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\|\tilde{R}_{q,i}\|_N|k|^{-6} \lesssim \delta_{q+1}^{1/2}\ell^{-N}|k|^{-6},$$

where the constant in the inequality only depends on N (see (E.4)). Similarly,

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1}|k|^{-6}\left(\|\tilde{R}_{q,i}\|_{N+1} + \|\nabla\Phi_i\|_N\right) \lesssim \delta_{q+1}^{1/2}\lambda_{q+1}^{-1}|k|^{-6}\ell^{-N-1}.$$

□

Corollary 7.19. *If $a \gg 1$ is sufficiently large, the perturbation satisfies the estimates*

$$\|w_o\|_0 + \frac{1}{\lambda_{q+1}}\|w_o\|_1 \le \frac{M}{4}\delta_{q+1}^{1/2}, \tag{7.104}$$

$$\|w_c\|_0 + \frac{1}{\lambda_{q+1}}\|w_c\|_1 \lesssim \delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{-1}, \tag{7.105}$$

$$\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}}\|w_{q+1}\|_1 \le \frac{M}{2}\delta_{q+1}^{1/2}. \tag{7.106}$$

In particular, (7.41) *holds.*

Proof. From (7.95), we deduce that $\|\nabla\Phi_i\|_0 \le 2$ on $\operatorname{supp}\eta_p^i$. Thus, since η_p^i have disjoint supports, from Lemma 7.16 we get

$$\|w_o\|_0 \le 2\delta_{q+1}^{1/2}\overline{C}\sum_{k\neq 0}\frac{1}{|k|^5} \le \frac{M}{32}\delta_{q+1}^{1/2}. \tag{7.107}$$

To estimate $\|w_o\|_1$, we first observe that

$$\left\|\nabla\left(e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\right\|_0 \le \lambda_{q+1}|k|\|\nabla\Phi_i\|_0 \le 2\lambda_{q+1}|k|.$$

Compute now

$$\nabla w_{o,i} = \sum_{k\neq 0}\nabla\Phi_i^{-1}b_{i,k}\nabla\left(e^{i\lambda_{q+1}k\cdot\Phi_i}\right) + \sum_{k\neq 0}\nabla\left(\nabla\Phi_i^{-1}b_{i,k}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}.$$

In particular, from Lemma 7.16 and Proposition 7.18 we infer

$$\|\nabla w_o\|_0 \le 4\delta_{q+1}^{1/2}\lambda_{q+1}\overline{C}\sum_{k\neq 0}\frac{1}{|k|^4} + C\delta_{q+1}^{1/2}\ell^{-1}\sum_{k\neq 0}\frac{1}{|k|^5} \le \frac{M}{16}\delta_{q+1}^{1/2}\lambda_{q+1} + C\delta_{q+1}^{1/2}\ell^{-1},$$

for some constant C which also depends on M. Thanks to the parameter inequality $\ell^{-1} \ll \lambda_{q+1}$ from (7.92), by choosing $a \gg 1$ sufficiently large, we get

$$\|\nabla w_o\|_0 \le \frac{M}{8}\delta_{q+1}^{1/2}\lambda_{q+1},$$

which, together with (7.107), gives (7.104). As a consequence of (7.103), we also obtain (7.105). Finally, estimate (7.106) follows by putting together (7.104) and (7.105) and using again $\ell^{-1} \ll \lambda_{q+1}$. □

We denote by $D_{t,q} := \partial_t + \overline{v}_q \cdot \nabla$ the advective derivative with respect to $\overline{v}_q$. We have

Proposition 7.20. *For $t \in \operatorname{supp}\eta_p^i$ and every $N \ge 0$ we have*

$$\|D_{t,q}\nabla\Phi_i\|_N \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}, \tag{7.108}$$

$$\|D_{t,q}\tilde{R}_{q,i}\|_N \lesssim \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-2\alpha}, \tag{7.109}$$

$$\|D_{t,q}c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\lambda_{q+1}^{-1}\ell^{-N-1-3\alpha}|k|^{-6}, \tag{7.110}$$

$$\|D_{t,q}b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-3\alpha}|k|^{-6}. \tag{7.111}$$

Proof. Observe that $D_{t,q}\nabla\Phi_i = -\nabla\Phi_i D\overline{v}_q$. Thus, from (7.38) and (7.100) we get

$$\|D_{t,q}\nabla\Phi_i\|_N \lesssim \|\nabla\Phi_i\|_0\|\overline{v}_q\|_{N+1} + \|\nabla\Phi_i\|_N\|\overline{v}_q\|_1 \lesssim \delta_q^{1/2}\lambda_q\ell^{-N}.$$

Differentiating (7.94) yields

$$D_{t,q}\tilde{R}_{q,i} = D_{t,q}\nabla\Phi_i\left(\mathrm{Id} - \frac{\overline{R}_q}{\delta_{q+1}}\right)\nabla\Phi_i^T - \nabla\Phi_i\frac{D_{t,q}\overline{R}_q}{\delta_{q+1}}\nabla\Phi_i^T + \nabla\Phi_i\left(\mathrm{Id} - \frac{\overline{R}_q}{\delta_{q+1}}\right)D_{t,q}\nabla\Phi_i^T.$$

Then, by (7.39), (7.40), (7.100) and (7.108) we get

$$\begin{aligned}\left\|D_{t,q}\tilde{R}_{q,i}\right\|_N &\lesssim \|D_{t,q}\nabla\Phi_i\|_N + \left\|D_{t,q}\nabla\Phi_i\right\|_0\left(\frac{\|\overline{R}_q\|_N}{\delta_{q+1}} + \left\|\nabla\Phi_i^T\right\|_N\right) \\ &\quad + \|\nabla\Phi_i\|_N\frac{\left\|D_{t,q}\overline{R}_q\right\|_0}{\delta_{q+1}} + \frac{\left\|D_{t,q}\overline{R}_q\right\|_N}{\delta_{q+1}} \\ &\lesssim \delta_q^{1/2}\lambda_q\ell^{-N} + \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-2\alpha} \lesssim \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-2\alpha},\end{aligned}$$

which gives (7.109). Compute now

$$\begin{aligned}D_{t,q}c_{i,k} = &-i\frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}}\left(\partial_t\eta_p^i\nabla\left(a_k\left(\tilde{R}_{q,i}\right)\right) + \eta_p^i D_{t,q}\nabla\left(a_k\left(\tilde{R}_{q,i}\right)\right)\right) \times \frac{\nabla\Phi_i^T(k\times A_k)}{|k|^2} \\ &-i\frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}}\eta_p^i\nabla\left(a_k\left(\tilde{R}_{q,i}\right)\right) \times \frac{D_{t,q}\nabla\Phi_i^T(k\times A_k)}{|k|^2}.\end{aligned}$$

Writing $D_{t,q}\nabla\tilde{R}_{q,i} = \nabla(D_{t,q}\tilde{R}_{q,i}) - \nabla\bar{v}_q\nabla\tilde{R}_{q,i}$, we have by Proposition 7.18, (7.93), the previous two estimates (7.108) –(7.109) and (7.38) that

$$
\begin{aligned}
\frac{|k|^6\lambda_{q+1}}{\delta_{q+1}^{1/2}}\|D_{t,q}c_{i,k}\|_N &\lesssim \|\partial_t\eta_p^i\|_0\|\tilde{R}_{q,i}\|_{N+1} + \|D_{t,q}\nabla\tilde{R}_{q,i}\|_N \\
&\quad + \left(\|\partial_t\eta_p^i\|_0\|\tilde{R}_{q,i}\|_1 + \|D_{t,q}\nabla\tilde{R}_{q,i}\|_0\right)\|\nabla\Phi_i^T\|_N \\
&\quad + \|\tilde{R}_{q,i}\|_{N+1}\|D_{t,q}\nabla\Phi_i^T\|_0 + \|\tilde{R}_{q,i}\|_1\|D_{t,q}\nabla\Phi_i^T\|_N \\
&\lesssim \tau_{q+1}^{-1}\ell^{-N-1} + \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-1-2\alpha} \\
&\quad + \left(\tau_{q+1}^{-1}\ell^{-1} + \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-1-2\alpha}\right)\ell^{-N} + \ell^{-N-1}\delta_q^{1/2}\lambda_q \\
&\lesssim \tau_{q+1}^{-1}\ell^{-N-1} + \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-1-2\alpha} \\
&= \delta_q^{1/2}\lambda_q^{1+\gamma}\left(\lambda_q^{3\alpha} + \ell^{-2\alpha}\right)\ell^{-N-1} \lesssim \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-1-3\alpha},
\end{aligned}
$$

where in the last inequality we also used that $\lambda_q^{3\alpha} \leq \ell^{-3\alpha}$ from (7.28).

With similar computations we also get (7.111). Indeed

$$
\begin{aligned}
\|D_{t,q}b_{i,k}\|_N &\lesssim \frac{\delta_{q+1}^{1/2}}{|k|^6}\left(\|\partial_t\eta_p^i\|_0\|\tilde{R}_{q,i}\|_N + \|D_{t,q}\tilde{R}_{q,i}\|_N\right) \\
&\lesssim \frac{\delta_{q+1}^{1/2}}{|k|^6}\left(\tau_{q+1}^{-1}\ell^{-N} + \delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-2\alpha}\right) \lesssim \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-N-3\alpha}|k|^{-6}.
\end{aligned}
$$

□

7.4.5 Estimate on the new Reynolds stress

In this final section, we prove our last estimate (7.42) in order to conclude the proof of Proposition 7.8.

Proposition 7.21. *The Reynolds stresses R_{nash}, R_{osc} and R_{transp} defined in* (7.99) *satisfy*

$$\|R_{nash}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-2\alpha}}, \tag{7.112}$$

$$\|R_{osc}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma/2}}{\lambda_{q+1}^{1-4\alpha}}, \tag{7.113}$$

$$\|R_{transp}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}}, \tag{7.114}$$

for $\alpha > 0$ sufficiently small and $a \gg 1$ large enough. In particular, (7.42) *holds.*

Proof of (7.112). We rewrite the term R_{nash} as

$$\mathscr{R}(w_{q+1}\cdot\nabla v_q)=\sum_{i,k\neq 0}\mathscr{R}\left(\left(\nabla\Phi_i^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}+c_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\cdot\nabla\bar{v}_q\right).$$

Using Proposition D.2 and Proposition 7.18, we estimate

$$\begin{aligned}&\left\|\mathscr{R}\left(\nabla\Phi_i^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\cdot\nabla\bar{v}_q\right)\right\|_\alpha\\ &\lesssim\frac{\|\nabla\Phi_i^{-1}b_{i,k}\cdot\nabla\bar{v}_q\|_0}{\lambda_{q+1}^{1-\alpha}}+\frac{\|\nabla\Phi_i^{-1}b_{i,k}\cdot\nabla\bar{v}_q\|_{N+\alpha}+\|\nabla\Phi_i^{-1}b_{i,k}\cdot\nabla\bar{v}_q\|_0\|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}\\ &\lesssim\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}|k|^6}+\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{N-\alpha}\ell^{N+\alpha}|k|^6}\lesssim\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-2\alpha}|k|^6}\left(1+\frac{\lambda_{q+1}}{(\lambda_{q+1}\ell)^N}\right),\end{aligned}$$

where in the last inequality we also used that $\ell^{-\alpha}\leq\lambda_{q+1}^{\alpha}$ by (7.92). We claim that, by choosing $N\gg 1$ sufficiently large (depending on β,γ,α,b), then the following holds

$$\frac{\lambda_{q+1}}{(\lambda_{q+1}\ell)^N}\leq 1, \tag{7.115}$$

for $a\gg 1$ sufficiently large. Indeed we have

$$\frac{\lambda_{q+1}}{(\lambda_{q+1}\ell)^N}\lesssim a^{b-N(b+\beta-1-\gamma-3\alpha-\beta b)}.$$

Thus, there exists N large enough such that (7.115) holds, if $b+\beta-1-\gamma-3\alpha-\beta b>0$, which is equivalent to

$$\gamma+3\alpha<b-1-\beta(b-1)=(b-1)(1-\beta). \tag{7.116}$$

Since the right hand side in (7.116) is strictly larger than the upper bound on γ in (7.14), we conclude that (7.116) (and so (7.115)) holds if $\alpha>0$ is sufficiently small. Hence we achieved

$$\left\|\mathscr{R}\left(\nabla\Phi_i^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\cdot\nabla\bar{v}_q\right)\right\|_\alpha\lesssim\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-2\alpha}|k|^6}.$$

Due to the fact that the estimates on the coefficients $c_{i,k}$ from Proposition 7.18 are better than the ones on the $b_{i,k}$ by (7.92), we also get that

$$\left\|\mathscr{R}\left(c_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\cdot\nabla\bar{v}_q\right)\right\|_\alpha\lesssim\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-2\alpha}|k|^6}.$$

Finally, summing over all the frequencies $k\neq 0$, we conclude the desired estimate (7.112). □

Proof of (7.113). We write the oscillation error as follows

$$\begin{aligned}R_{osc} &= \mathscr{R}\mathrm{div}\,(\overline{R}_q + w_{q+1}\otimes w_{q+1})\\ &= \mathscr{R}\mathrm{div}\,(\overline{R}_q - \delta_{q+1}\mathrm{Id} + w_o\otimes w_o) + \mathscr{R}\mathrm{div}\,(w_o\otimes w_c + w_c\otimes w_o + w_c\otimes w_c)\\ &=: \mathscr{O}_1 + \mathscr{O}_2.\end{aligned}$$

Since, by Schauder estimates, the operator $\mathscr{R}\mathrm{div} : C^\alpha(\mathbb{T}^3)\to C^\alpha(\mathbb{T}^3)$ is bounded, we deduce by using (7.104), (7.105) and (7.92)

$$\begin{aligned}\|\mathscr{O}_2\|_\alpha &\lesssim \|w_o\|_\alpha\|w_c\|_\alpha + \|w_c\|_\alpha^2 \lesssim \frac{\delta_{q+1}\lambda_{q+1}^{2\alpha}}{\lambda_{q+1}\ell} + \frac{\delta_{q+1}\lambda_{q+1}^{2\alpha}}{(\lambda_{q+1}\ell)^2}\\ &\lesssim \frac{\delta_{q+1}\lambda_{q+1}^{2\alpha}}{\lambda_{q+1}\ell} = \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma/2+3\alpha/2}}{\lambda_{q+1}^{1-2\alpha}} \le \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma/2}}{\lambda_{q+1}^{1-4\alpha}}.\end{aligned}$$

Thus, to conclude the desired estimate on R_{osc} we are only left with $\mathscr{O}_1$. Since by Lemma 7.14 the supports of η_p^i are disjoint and $\sum_i \left(\eta_p^i\right)^2 \equiv 1$ on the supp$\overline{R}_q$, we have

$$\mathscr{O}_1 = \mathscr{R}\mathrm{div}\sum_i \left(\left(\eta_p^i\right)^2\left(\overline{R}_q - \delta_{q+1}\mathrm{Id}\right) + w_{o,i}\otimes w_{o,i}\right).$$

Using (E.5), we can write

$$\begin{aligned}w_{o,i}\otimes w_{o,i} &= \delta_{q+1}\left(\eta_p^i\right)^2\nabla\Phi_i^{-1}(W\otimes W)(\tilde{R}_{q,i},\lambda_{q+1}\Phi_i)\nabla\Phi_i^{-T}\\ &= \delta_{q+1}\left(\eta_p^i\right)^2\nabla\Phi_i^{-1}\tilde{R}_{q,i}\nabla\Phi_i^{-T} + \sum_{k\neq 0}\delta_{q+1}\left(\eta_p^i\right)^2\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\\ &= \left(\eta_p^i\right)^2(\delta_{q+1}\mathrm{Id} - \overline{R}_q) + \sum_{k\neq 0}\delta_{q+1}\left(\eta_p^i\right)^2\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i},\end{aligned}$$

from which we deduce

$$\mathscr{O}_1 = \mathscr{R}\mathrm{div}\left(\sum_{i,k\neq 0}\delta_{q+1}\left(\eta_p^i\right)^2\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}e^{i\lambda_{q+1}k\cdot\Phi_i}\right).$$

Also, recalling (E.6)

$$\nabla\Phi_i^{-1}C_k\nabla\Phi_i^{-T}\nabla\Phi_i^T k = 0,$$

and consequently

$$\mathscr{O}_1 = \mathscr{R}\left(\sum_{i,k\neq 0}\delta_{q+1}\left(\eta_p^i\right)^2\mathrm{div}\,\left(\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\right).$$

Thus, again by Proposition D.2 and Proposition 7.18, we estimate on $\operatorname{supp}\eta_p^i$

$$\begin{aligned}
&\left\|\mathscr{R}\left(\operatorname{div}\left(\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}\right)e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\right\|_\alpha \lesssim \frac{\left\|\operatorname{div}\left(\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}\right)\right\|_0}{\lambda_{q+1}^{1-\alpha}}\\
&+\frac{\left\|\operatorname{div}\left(\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}\right)\right\|_{N+\alpha}+\left\|\operatorname{div}\left(\nabla\Phi_i^{-1}C_k(\tilde{R}_{q,i})\nabla\Phi_i^{-T}\right)\right\|_0\|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}\\
&\lesssim \frac{1}{\ell^{1+\alpha}\lambda_{q+1}^{1-\alpha}|k|^6}+\frac{1}{\lambda_{q+1}^{N-\alpha}\ell^{N+1+\alpha}|k|^6}\lesssim\frac{1}{\ell\lambda_{q+1}^{1-2\alpha}|k|^6},
\end{aligned}$$

where we have again chosen $N \gg 1$ large enough to get the desired estimate, together with $\ell^{-\alpha} \leq \lambda_{q+1}^{\alpha}$ (see (7.92)), for the last inequality. By summing over $k \neq 0$ we conclude, reusing (7.92), that

$$\|\mathscr{O}_1\|_\alpha \lesssim \frac{\delta_{q+1}}{\ell\lambda_{q+1}^{1-2\alpha}} \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma/2}}{\lambda_{q+1}^{1-4\alpha}}.$$

□

Proof of (7.114). We start by splitting the transport error into two parts

$$\mathscr{R}\left(\left(\partial_t+\bar{v}_q\cdot\nabla\right)w_{q+1}\right)=\mathscr{R}\left(\left(\partial_t+\bar{v}_q\cdot\nabla\right)w_o\right)+\mathscr{R}\left(\left(\partial_t+\bar{v}_q\cdot\nabla\right)w_c\right)=:\mathscr{T}_1+\mathscr{T}_2.$$

We start with $\mathscr{T}_1$. Applying (7.97) yields

$$\left(\partial_t+\bar{v}_q\cdot\nabla\right)w_o=\sum_{i,k\neq 0}\left(\nabla\bar{v}_q\right)^T\nabla\Phi_i^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}+\sum_{i,k\neq 0}\nabla\Phi_i^{-1}D_{t,q}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}. \tag{7.117}$$

We now apply Proposition D.2, together with Proposition 7.18, to obtain

$$\begin{aligned}
&\|\mathscr{R}\left(\left(\nabla\bar{v}_q\right)^T\nabla\Phi_i^{-1}b_{i,k}e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\|_\alpha\\
&\lesssim \frac{\left\|\left(\nabla\bar{v}_q\right)^T\nabla\Phi_i^{-1}b_{i,k}\right\|_0}{\lambda_{q+1}^{1-\alpha}}+\frac{\left\|\left(\nabla\bar{v}_q\right)^T\nabla\Phi_i^{-1}b_{i,k}\right\|_{N+\alpha}+\left\|\left(\nabla\bar{v}_q\right)^T\nabla\Phi_i^{-1}b_{i,k}\right\|_0\|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}\\
&\lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-\alpha}|k|^6}+\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{N-\alpha}\ell^{N+\alpha}|k|^6}\lesssim\frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}^{1-2\alpha}|k|^6},
\end{aligned} \tag{7.118}$$

where in the last inequality we have again chosen $\alpha > 0$ sufficiently small and $N \gg 1$ large enough.

The estimate on the second term in (7.117) follows by Proposition D.2 and Proposition 7.20. Indeed we have

$$\begin{aligned}
&\left\| \mathscr{R}\left(\nabla\Phi_i^{-1} D_{t,q} b_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\right\|_\alpha \\
&\lesssim \frac{\left\|\nabla\Phi_i^{-1} D_{t,q} b_{i,k}\right\|_0}{\lambda_{q+1}^{1-\alpha}} + \frac{\left\|\nabla\Phi_i^{-1} D_{t,q} b_{i,k}\right\|_{N+\alpha} + \left\|\nabla\Phi_i^{-1} D_{t,q} b_{i,k}\right\|_0 \|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}} \\
&\lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-3\alpha}}{\lambda_{q+1}^{1-\alpha}|k|^6} + \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-3\alpha} + \delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{1-3\alpha}}{\lambda_{q+1}^{N-\alpha}\ell^{N+\alpha}|k|^6} \\
&\lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}\ell^{-4\alpha}}{\lambda_{q+1}^{1-\alpha}|k|^6} \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}|k|^6},
\end{aligned} \tag{7.119}$$

where we also used $\ell^{-4\alpha} \leq \lambda_{q+1}^{4\alpha}$. Putting together (7.118) and (7.119), summing over all the frequencies $k \neq 0$, we conclude

$$\|\mathscr{T}_1\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}}. \tag{7.120}$$

To estimate $\mathscr{T}_2$ we observe that since $(\partial_t + \bar{v}_q\cdot\nabla)\Phi_i = 0$ by choice of Φ_i, we have

$$\left(\partial_t + \bar{v}_q\cdot\nabla\right) w_c = \sum_{i,k\neq 0}\left(D_{t,q}c_{i,k}\right) e^{i\lambda_{q+1}k\cdot\Phi_i}.$$

Then, applying once again Proposition D.2, we get

$$\left\|\mathscr{R}\left(\left(D_{t,q}c_{i,k}\right) e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\right\|_\alpha \lesssim \frac{\left\|D_{t,q}c_{i,k}\right\|_0}{\lambda_{q+1}^{1-\alpha}} + \frac{\left\|D_{t,q}c_{i,k}\right\|_{N+\alpha} + \left\|D_{t,q}c_{i,k}\right\|_0\|\Phi_i\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}.$$

Since from Proposition 7.20 the estimates on $D_{t,q}c_{i,k}$ are better than the ones for $D_{t,q}b_{i,k}$ (recall that $\ell^{-1} \leq \lambda_{q+1}$) we obtain, as for the estimate (7.119), that

$$\left\|\mathscr{R}\left(\left(D_{t,q}c_{i,k}\right) e^{i\lambda_{q+1}k\cdot\Phi_i}\right)\right\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}|k|^6},$$

from which, by summing over $k \neq 0$, we deduce

$$\|\mathscr{T}_2\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q^{1+\gamma}}{\lambda_{q+1}^{1-5\alpha}}. \tag{7.121}$$

Estimates (7.120) and (7.121) imply the validity of (7.114) and this concludes the proof of Proposition 7.21. □

Chapter 8

Typical wild solutions to the Navier-Stokes equations

8.1 Introduction

In this chapter we investigate some typicality questions for the Navier-Stokes equations in terms of Baire category. We recall the equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \Delta v = 0 \\ \operatorname{div} v = 0 \end{cases} \qquad \text{in } \mathbb{T}^3 \times [0,T] \tag{8.1}$$

We define the following complete metric space

$$\mathscr{D} := \left\{ v \in L^\infty((0,T); L^2(\mathbb{T}^3)) : v \textit{ is a distributional solution of } (8.1) \right\},$$

endowed with the metric $d_{\mathscr{D}}(u,v) := \|u - v\|_{L^\infty_t(L^2_x)}$, and its subsets

$$\mathscr{L} := \{ v \in \mathscr{D} : v \textit{ is a Leray–Hopf solution of } (8.1) \}$$
$$\mathscr{S} := \left\{ v \in \mathscr{D} : v \in C^\infty(\mathbb{T}^3 \times I) \textit{ for some open interval } I \subset (0,T) \right\}.$$

We refer to Chapter 1 for the definitions of Leray-Hopf solutions. Our main result is the following

Theorem 8.1. *The set $\mathscr{L}$ is nowhere dense in $\mathscr{D}$ while the set $\mathscr{S}$ is meagre in $\mathscr{D}$.*

We recall that $\mathscr{L}$ is nowhere dense in $\mathscr{D}$ if and only if the closure of $\mathscr{L}$ has empty interior. In particular, $\mathscr{L}$ is meagre in $\mathscr{D}$.

Theorem 8.1 is in the same spirit of the typicality results proved in Chapter 6. Unlike the latter, its proof makes use of the L^p-based convex integration scheme proposed in [5, 10] that we recall in the next sections.

8.2 The iterative proposition and proof of the main theorem

The proof of Theorem 8.1 is based on an iterative proposition, typical of convex integration schemes and analogous to [10, Section 7] and [5, Section 2]; in analogy with the latter, also here we use intermittent jets (see Section 3 below) as the fundamental building blocks. At difference to the previously cited works, we need to keep track of the kinetic energy in some intervals of time along the iteration in such a way to be able to prescribe it in the limit, and we also need to make sure with a simple use of time cutoffs that the support of the perturbation is localized in a converging sequence of enlarging sets. On the contrary, we don't use the cutoffs to obtain a small set of singular times for our limit, as was done in [5].

In turn the proof of Theorem 8.1 follows from the iterative proposition by proving the usual empty interior condition. To show that the subset $\mathscr{L}$ is nowhere dense in the metric space $\mathscr{D}$, we prove that for every $v \in \overline{\mathscr{L}}$ there are arbitrarily close elements which belong to $\mathscr{D} \setminus \overline{\mathscr{L}}$. Note that our set $\mathscr{L}$ is closed, so it is enough to consider $\mathscr{L}$ itself instead of its closure. In Step 1 of the proof we reduce to such statement, where we choose elements in $\mathscr{D} \setminus \mathscr{L}$ by imposing locally increasing kinetic energy.

We recall that a distributional solution of the system (8.1) is a vector field $v \in L^2(\mathbb{T}^3 \times (0,T);\mathbb{R}^3)$ such that

$$\int_0^T \int_{\mathbb{T}^3} (v \cdot \partial_t \varphi + v \otimes v : \nabla \varphi + v \cdot \Delta \varphi)\, dxdt = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (0,T);\mathbb{R}^3)$ such that $\operatorname{div} \varphi = 0$. The pressure does not appear in the distributional formulation because it can be recovered as the unique 0-average solution of

$$-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v). \tag{8.2}$$

A Leray-Hopf weak solution of the system (8.1) is a vector field $v \in L^2((0,T);H^1(\mathbb{T}^3)) \cap L^\infty((0,T);L^2(\mathbb{T}^3))$ and for a.e. $s \geq 0$ and for all $t \in [s,T]$ the following inequality holds

$$\int_{\mathbb{T}^3} \frac{|v(x,t)|^2}{2} dx + \int_s^t \int_{\mathbb{T}^3} |\nabla v(x,\tau)|^2 dx d\tau \leq \int_{\mathbb{T}^3} \frac{|v(x,s)|^2}{2} dx. \tag{8.3}$$

We also recall the following two lemmas that are respectively Lemma 3.7 and Lemma B.1 in [8].

Lemma 8.2. *Fix integers $N, \sigma \geq 1$ and let $\zeta > 1$ such that*

$$\frac{2\pi\sqrt{3}\zeta}{\sigma} \leq \frac{1}{3} \quad \text{and} \quad \zeta^4 \frac{(2\pi\sqrt{3}\zeta)^N}{\sigma^N} \leq 1. \tag{8.4}$$

Let $p \in \{1,2\}$ and let $f, g \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$. Suppose that there exists a constant $C_f > 0$ such that

$$\|\nabla^j f\|_{L^p} \leq C_f \zeta^j,$$

holds for all $0 \leq j \leq N+4$. Then we have that

$$\|f g_\sigma\|_{L^p} \leq C_0 C_f \|g_\sigma\|_{L^p},$$

where C_0 is a universal constant.

Lemma 8.3. *Fix $\kappa \geq 1$, $p \in (1,2]$, and a sufficiently large $L \in \mathbb{N}$. Let $a \in C^L(\mathbb{T}^3)$ be such that there exists $1 \leq \lambda \leq \kappa$, $C_a > 0$ with*

$$\|D^j a\|_{L^\infty} \leq C_a \lambda^j,$$

for all $0 \leq j \leq L$. Assume furthermore that $\int_{\mathbb{T}^3} a(x)\mathbb{P}_{\geq\kappa} f(x)dx = 0$. Then we have

$$\||\nabla|^{-1}(a\mathbb{P}_{\geq\kappa}(f))\|_{L^p} \lesssim C_a \left(1 + \frac{\lambda^L}{\kappa^{L-2}}\right) \frac{\|f\|_{L^p}}{\kappa}$$

for any $f \in L^p(\mathbb{T}^3)$, where the implicit constant depends on p and L.

Lemma 8.4. *Let $g : \mathbb{T}^3 \to \mathbb{R}$ such that*

$$\fint_{\mathbb{T}^3} g(x)dx = 0,$$

and let $g_\sigma : \mathbb{T}^3 \to \mathbb{R}$: $g_\sigma(x) := g(\sigma x)$. Let $f : \mathbb{T}^3 \to \mathbb{R}$ such that

$$\|\nabla f\|_{C^0} \leq C_f \zeta,$$

then we have

$$\left| \int_{\mathbb{T}^3} g_\sigma(x) f(x) dx \right| \lesssim \frac{C_f \zeta}{\sigma} \|g_\sigma\|_{L^1(\mathbb{T}^3)},$$

where $\lesssim$ means up to a universal constant.

The Navier–Stokes–Reynolds system

In this section, for every integer $q \geq 0$ we will highlight the construction of a solution $(v_q, p_q, \mathring{R}_q)$ to the Navier-Stokes-Reynolds system

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q - \Delta v_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0 \end{cases} \tag{8.5}$$

where the Reynolds stress $\mathring{R}_q$ is assumed to be a trace-free symmetric matrix valued function. Indeed for any matrix A we will use the notation $\mathring{A}$ to denote the traceless property.

Parameters

Define the frequency parameter $\lambda_q \to +\infty$ and the amplitudes parameter $\delta_q \to 0^+$ by

$$\lambda_q = 2\pi a^{(b^q)},$$
$$\delta_q = \lambda_q^{-2\beta}.$$

The sufficiently large (universal) parameter b is free, and so is the sufficiently small parameter $\beta = \beta(b)$. The parameter a is chosen to be a sufficiently large multiple of the geometric constant n_*. Moreover, we fix another parameter useful to prescribe a precise kinetic energy

$$\varepsilon_1 := \left(\frac{\varepsilon}{\sup_{\xi\in\Lambda} \|\gamma_\xi\|_{C^0} |\Lambda| C_0 4(2\pi)^3} \right)^2, \tag{8.6}$$

where $\sup_{\xi\in\Lambda} \|\gamma_\xi\|_{C^0}, |\Lambda|, C_0$ are all universal constants independent on q, more precisely: γ_ξ are functions defined in Lemma 8.6, Λ is the finite set defined in Lemma 8.6, C_0 is the constant given by Lemma 8.2, ε is a free constant that will be used in the proof of Theorem 8.1.

Moreover, we will use the intermittent jets (defined in Section 8.3) to define the new velocity increment at step $q+1$.

8.2.1 Inductive estimates and iterative proposition

We define new "slow" parameters, for all $q \geq 0$

$$s_q := \left(\frac{s}{2}\right)^{q+1}, \tag{8.7}$$

$$S_q := \sum_{i=0}^{q} s_i, \tag{8.8}$$

for some fixed parameter $s > 0$. By choosing $a_0(s)$ sufficiently large, we will guarantee that

$$s_{q+1}^{-1} \ll \lambda_q,$$

indeed s_q^{-1} is a slow parameter compared to λ_q. Moreover we define the local time interval, for some small number $s > 0$, for all $q \geq 0$

$$I_q := (t_0 - S_q, t_0 + S_q), \tag{8.9}$$

for some $t_0 \in (0,1)$ and $s = s(t_0) > 0$ sufficiently small such that

$$B_{2s}(t_0) := (t_0 - 2s, t_0 + 2s) \subset [0,1].$$

Observe that $I_q \subset B_{2s}(t_0)$ for all $q \geq 0$.

In the following, if not specified differently, every space norm is taken with respect to the sup in time localized in the interval $B_{2s}(t_0)$, i.e. for example: if $v \in L^\infty_t L^p_x$, we denote $\|v\|_{L^p}$ the quantity $\sup_{t\in B_{2s}(t_0)} \|v(t,\cdot)\|_{L^p_x}$. We use $\lesssim$ as an inequality that holds up to a constant independent on q.

For $q \geq 0$, we want to guarantee

$$\|v_q\|_{L^2} \leq 2\|v_0\|_{L^2} - \frac{\varepsilon}{4\pi}\delta_q^{1/2}, \tag{8.10a}$$

$$\|\mathring{R}_q\|_{L^1} \leq \lambda_q^{-3\zeta}\delta_{q+1}, \tag{8.10b}$$

$$\|v_q\|_{C^1_{x,t}(\mathbb{T}^3\times B_{2s}(t_0))} \leq \lambda_q^4, \tag{8.10c}$$

and moreover[1]

$$\frac{\delta_{q+1}}{\delta_1\lambda_q^{\zeta/2}} \leq e(t) - \int_{\mathbb{T}^3}|v_q(x,t)|^2dx \leq \frac{\delta_{q+1}\varepsilon_1}{\delta_1}, \text{ for all } t\in I_0, \tag{8.11a}$$

$$\mathrm{Supp}_T(\mathring{R}_q) \subset I_q, \tag{8.11b}$$

$$\mathrm{Supp}_T(v_q - v_{q-1}) \subset I_q, \text{ for all } q \geq 1, \tag{8.11c}$$

which are new with respect to the convex integration scheme proposed by Buckmaster and Vicol in [10, Section 7].

Proposition 8.5 (Iterative Proposition)**.** *Let $e : [0,T] \to (0,\infty)$ be a strictly positive smooth function. For every $\varepsilon, s > 0$ and $t_0 \in (0,T)$ there exist $b > 1$, $\beta(b) > 0$, $\zeta > 0$, $a_0 = a_0(\beta, b, \zeta, e, \varepsilon, s)$ such that for any $a \geq a_0$ which is a multiple of the geometric constant n_* of Lemma 8.6, the following holds. Let $(v_q, p_q, \mathring{R}_q)$ be a smooth triple solving the Navier-Stokes-Reynolds system* (8.5) *in $\mathbb{T}^3 \times B_{2s}(t_0)$ satisfying the inductive estimates* (8.10)-(8.11).

Then there exists a second smooth triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ which solves the Navier-Stokes-Reynolds system in $\mathbb{T}^3 \times B_{2s}(t_0)$ (8.5)*, satisfies the estimates* (8.10) *and* (8.11) *at level $q+1$. In addition, we have that*

$$\|v_{q+1} - v_q\|_{L^\infty(B_{2s}(t_0);L^2(\mathbb{T}^3))} \leq \frac{\varepsilon}{\delta_1^{1/2}4\pi}\delta_{q+1}^{1/2}. \tag{8.12}$$

8.2.2 Proof of Theorem 8.1

Step 1. Let $v \in L^\infty((0,T);L^2(\mathbb{T}^3))$ be a distributional solution of (8.1)*, such that $v \in C^\infty(\mathbb{T}^3 \times I)$, for some open interval $I \subset (0,T)$. Then, we prove the following claim: for every $\varepsilon > 0$, there exists a distributional solution $v_\varepsilon \in L^\infty((0,T);L^2(\mathbb{T}^3))$ of* (8.1) *such that*

$$\|v_\varepsilon - v\|_{L^\infty((0,T);(L^2(\mathbb{T}^3))} < \varepsilon \tag{8.13}$$

and the kinetic energy of v_ε is strictly increasing in a sub-interval of $(0,T)$.

Let $t_0 \in I$ and choose $s > 0$ such that $\overline{B}_{2s}(t_0) \subset I$. Let $g \in C^\infty([0,T];[\frac{\varepsilon_1}{2},\varepsilon_1])$ be such that

[1] Here $\mathrm{Supp}_T(u)$ denotes the closure of $\{t \in (0,1) : \exists x \in \mathbb{T}^3 \;\; u(x,t) \neq 0\}$.

$$g'(t_0) > \sup_{t\in(0,1)} \left| \frac{d}{dt} \int_{\mathbb{T}^3} |v(x,t)|^2 dx \right|,$$

and consider the kinetic energy (increasing in a neighbourhood of t_0)

$$e(t) := \int_{\mathbb{T}^3} |v(x,t)|^2 dx + g(t). \tag{8.14}$$

Since the function v is smooth in $\mathbb{T}^3 \times I$ we consider the smooth solution p, with zero average, in $\mathbb{T}^3 \times I$ of (8.2), and define the starting triple $(v_0, p_0, R_0) := (v, p, 0)$.

Clearly $(v, p, 0)$ satisfies the estimates (8.10) and (8.11) at step $q = 0$, up to enlarge ${a_0}^2$, thus we can apply Proposition 8.5 starting from the triple (v_0, p_0, R_0). Hence, we get a sequence $\{v_q\}_{q\in\mathbb{N}}$ that satisfies (8.10), (8.11) and moreover, from (8.12) we get

$$\sum_{q\geq 0} \|v_{q+1} - v_q\|_{L^2} \leq \frac{\varepsilon}{\delta_1^{1/2} 4\pi} \sum_{q\geq 0} \delta_{q+1}^{1/2} \leq \frac{\varepsilon}{\delta_1^{1/2} 4\pi} \sum_{q\geq 0} (a^{-\beta b})^{q+1} \leq \frac{\varepsilon}{2(1-a^{-\beta b})} < \varepsilon \tag{8.15}$$

where the last holds if a_0 is sufficiently large in order to have $a^{-\beta b} < 1/2$. Hence, there exists the limit $\tilde{v}_\varepsilon := \lim_{q\to\infty} v_q$, in $L^\infty(B_{2s}(t_0); L^2(\mathbb{T}^3))$ such that $\|\tilde{v}_\varepsilon - v\|_{L^\infty(B_{2s}(t_0); L^2(\mathbb{T}^3))} < \varepsilon$ and it is a distributional solution of the Navier-Stokes equations in $B_{2s}(t_0) \times \mathbb{T}^3$, because by (8.10b) we have that $\lim_{q\to\infty} \mathring{R}_q = 0$ in $L^\infty(B_{2s}(t_0); L^1(\mathbb{T}^3))$. One can verify that the vector field

$$v_\varepsilon = \begin{cases} \tilde{v}_\varepsilon & \textit{in } B_{2s}(t_0) \\ v & \textit{in } [0,T] \setminus B_{2s}(t_0), \end{cases}$$

still solves (8.1) in $[0,T] \times \mathbb{T}^3$ and satisfies (8.13). Moreover the kinetic energy of v_ε is increasing in a neighbourhood of t_0 thanks to (8.11a) and (8.14).

Step 2. We conclude the proof of Theorem 8.1.

Let v_0 be a distributional solution which is smooth in a subinterval of times and $\varepsilon > 0$; for instance, any Leray solution can be taken as v_0 since they are smooth outside a closed set of $\mathscr{H}^{1/2}$ measure 0. We apply the Step 1 and get a distributional solution of Navier-Stokes $v_\varepsilon \in L^\infty((0,T); L^2(\mathbb{T}^3))$ such that $\|v_\varepsilon - v_1\|_{L^\infty((0,T); L^2(\mathbb{T}^3))} < \varepsilon$ with increasing kinetic energy in a sub-interval of $[0,T]$ and therefore such that $v_\varepsilon \in \mathscr{D} \setminus \mathscr{L}$.

Since $\mathscr{L}$ is closed with respect to $L^\infty L^2$ convergence, we deduce that the interior of $\overline{\mathscr{L}}$ which coincides with the interior of $\mathscr{L}$, is empty.

To show that $\mathscr{S}$ is a meagre set in $\mathscr{D}$, we rewrite it as

$$\mathscr{S} \subset \bigcup_{s\in\mathscr{Q}^+} \bigcup_{t\in(0,1)\cap\mathscr{Q}} \{v \in \mathscr{D} : v \in C^\infty((t-s, t+s) \times \mathbb{T}^3)\},$$

and we notice that from Step 1 the right-hand side is a countable union of nowhere dense sets, hence it is meagre.

[2] To be precise we considered $v_{-1} = v_0$.

8.3 Intermittent jets

In this section we recall from [10] the definition and the main properties of intermittent jets we will use in the convex integration scheme.

A geometric lemma.

We start with a geometric lemma. A proof of the following version, which is essentially due to De Lellis and Székelyhidi Jr., can be found in [5, Lemma 4.1]. This lemma allows us to reconstruct any symmetric 3×3 stress tensor R in a neighbourhood of the identity as a linear combination of a particular basis.

Lemma 8.6. *Denote by $\overline{B^{sym}_{1/2}}(Id)$ the closed ball of radius $1/2$ around the identity matrix in the space of symmetric 3×3 matrices. There exists a finite set $\Lambda \subset \mathbb{S}^2 \cap \mathcal{Q}^3$ such that there exist C^∞ functions $\gamma_\xi : B^{sym}_{1/2}(Id) \to \mathbb{R}$ which obey*

$$R = \sum_{\xi \in \Lambda} \gamma_\xi^2(R) \xi \otimes \xi,$$

for every symmetric matrix R satisfying $|R - Id| \leq 1/2$. Moreover for each $\xi \in \Lambda$, let use define $A_\xi \in \mathbb{S}^2 \cap \mathcal{Q}^3$ to be an orthogonal vector to ξ. Then for each $\xi \in \Lambda$ we have that $\{\xi, A_\xi, \xi \times A_\xi\} \subset \mathbb{S}^2 \cap \mathcal{Q}^3$ form an orthonormal basis for $\mathbb{R}^3$. Furthermore, since we will periodize functions, let n_* be the l.c.m. of the denominators of the rational numbers ξ, A_ξ and $\xi \times A_\xi$, such that

$$\{n_* \xi, n_* A_\xi, n_* \xi \times A_\xi\} \subset \mathbb{Z}^3.$$

Vector fields

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with support contained in a ball of radius 1. We normalize Φ such that $\phi = -\Delta\Phi$ obeys

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \phi^2(x_1, x_2) dx_1 dx_2 = 1. \tag{8.16}$$

We remark that by definition ϕ has zero average. Define $\psi : \mathbb{R} \to \mathbb{R}$ to be a smooth, zero average function with support in the ball of radius 1 satisfying

$$\fint_{\mathbb{T}} \psi^2(x_3) dx_3 = \frac{1}{2\pi} \int_{\mathbb{R}} \psi^2(x_3) dx_3 = 1.$$

We define the parameters $r_\perp$, $r_{||}$ and μ as follows

$$r_\perp := r_{\perp,q+1} := \lambda_{q+1}^{-6/7}(2\pi)^{-1/7}, \tag{8.17a}$$

$$r_{||} := r_{||,q+1} := \lambda_{q+1}^{-4/7}, \tag{8.17b}$$

$$\mu := \mu_{q+1} := \lambda_{q+1}^{9/7}(2\pi)^{1/7}. \tag{8.17c}$$

We define $\phi_{r_\perp}, \Phi_{r_\perp}$,and $\psi_{r_{||}}$ to be the rescaled cut-off functions

$$\overline{\phi_{r_\perp}}(x_1,x_2) := \frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp},\frac{x_2}{r_\perp}\right),$$
$$\overline{\Phi_{r_\perp}}(x_1,x_2) := \frac{1}{r_\perp}\Phi\left(\frac{x_1}{r_\perp},\frac{x_2}{r_\perp}\right),$$
$$\overline{\psi_{r_{||}}}(x_3) := \left(\frac{1}{r_{||}}\right)^{1/2}\psi\left(\frac{x_3}{r_{||}}\right).$$

With this rescaling we have $\overline{\phi_{r_\perp}} = -r_\perp^2\Delta\overline{\Phi_{r_\perp}}$. Moreover the functions $\overline{\phi_{r_\perp}}$ and $\overline{\Phi_{r_\perp}}$ are supported in the ball of radius $r_\perp$ in $\mathbb{R}^2$, $\psi_{r_{||}}$ is supported in the ball of radius $r_{||}$ in $\mathbb{R}$ and we keep the normalizations $\|\overline{\phi_{r_\perp}}\|_{L^2}^2 = 4\pi^2$ and $\|\overline{\psi_{r_{||}}}\|_{L^2}^2 = 2\pi$.

We then periodize the previous functions

$$\phi_{r_\perp}(x_1+2\pi n, x_2+2\pi m) = \overline{\phi_{r_\perp}}(x_1,x_2),$$
$$\Phi_{r_\perp}(x_1+2\pi n, x_2+2\pi m) = \overline{\Phi_{r_\perp}}(x_1,x_2),$$
$$\psi_{r_{||}}(x_3+2\pi n) = \overline{\psi_{r_{||}}}(x_3).$$

For every $\xi \in \Lambda$ (recalling the notations in Lemma 8.6), we introduce the functions defined on $\mathbb{T}^3\times\mathbb{R}$

$$\psi_\xi(x,t) := \psi_{r_{||}}(n_* r_\perp \lambda_{q+1}(x\cdot\xi + \mu t)), \tag{8.18a}$$

$$\Phi_\xi(x) := \Phi_{r_\perp}(n_* r_\perp \lambda_{q+1}(x-\alpha_\xi)\cdot A_\xi, n_* r_\perp \lambda_{q+1}(x-\alpha_\xi)\cdot(\xi\times A_\xi)), \tag{8.18b}$$

$$\phi_\xi(x) := \phi_{r_\perp}(n_* r_\perp \lambda_{q+1}(x-\alpha_\xi)\cdot A_\xi, n_* r_\perp \lambda_{q+1}(x-\alpha_\xi)\cdot(\xi\times A_\xi)), \tag{8.18c}$$

where α_ξ are shifts which ensure that the functions $\{\Phi_\xi\}$ have mutually disjoint support.

In order for such shifts α_ξ to exist, it is sufficient to assume that $r_\perp$ is smaller than a universal constant, which depends only on the geometry of the finite set Λ.

It is important to note that the function ψ_ξ oscillates at frequency proportional to $r_\perp r_{||}^{-1}\lambda_{q+1}$, whereas ϕ_ξ and Φ_ξ oscillate at frequency proportional to λ_{q+1}.

Definition 8.7. *The intermittent jets are vector fields $W_\xi : \mathbb{T}^3\times\mathbb{R}\to\mathbb{R}^3$ defined as*

$$W_\xi(x,t) := \xi\,\psi_\xi(x,t)\phi_\xi(x).$$

If $\sigma := r_\perp n_* \in \mathbb{N}$, thanks to the choice of n_* in Lemma 8.6 we have that W_ξ has zero average in $\mathbb{T}^3$ and is $\left(\frac{\mathbb{T}}{\sigma}\right)^3$ periodic. Moreover, by our choice of α_ξ, we have that

$$W_\xi \otimes W_{\xi'} \equiv 0,$$

whenever $\xi \neq \xi' \in \Lambda$, i.e. $\{W_\xi\}_{\xi\in\Lambda}$ have mutually disjoint support. The essential identities obeyed by the intermittent jets are

$$\begin{aligned} \|W_\xi\|^p_{L^p(\mathbb{T}^3)} &= \frac{1}{8\pi^3}\|\psi_\xi\|^p_{L^p(\mathbb{T}^3)}\|\phi_\xi\|^p_{L^p(\mathbb{T}^3)} \\ \operatorname{div}(W_\xi\otimes W_\xi) = 2(W_\xi\cdot\nabla\psi_\xi)\phi_\xi\xi &= \frac{1}{\mu}\partial_t(\phi_\xi^2\psi_\xi^2\xi) \\ \fint_{\mathbb{T}^3} W_\xi\otimes W_\xi &= \xi\otimes\xi, \end{aligned} \tag{8.19}$$

where the last identity will be useful to apply Lemma 8.6.

We denote by $\mathbb{P}_{\neq 0}$ the operator which projects a function onto its non-zero frequencies $\mathbb{P}_{\neq 0} f = f - \fint_{\mathbb{T}^3} f$, and by $\mathbb{P}_H$ we will denote the usual Helmholtz projector onto divergence-free vector fields, $\mathbb{P}_H f = f - \nabla(\Delta^{-1}\operatorname{div} f)$. Motivated by (8.19), we define

$$W_\xi^{(t)}(x,t) := -\frac{1}{\mu}\mathbb{P}_H\mathbb{P}_{\neq 0}\phi_\xi^2(x)\psi_\xi^2(x,t)\xi. \tag{8.20}$$

Lastly, we note that the intermittent jets W_ξ are not divergence free, then we introduce the following two functions $W_\xi^{(c)}, V_\xi : \mathbb{T}^3\times\mathbb{R}\to\mathbb{R}^3$

$$\begin{aligned} V_\xi(x,t) &:= \frac{1}{n_*\lambda_{q+1}^2}\xi\,\psi_\xi(x,t)\Phi_\xi(x), \\ W_\xi^{(c)}(x,t) &:= \frac{1}{n_*\lambda_{q+1}^2}\nabla\psi_\xi(x,t)\times(\nabla\times\Phi_\xi(x)\xi). \end{aligned}$$

Using $\Delta\Phi_\xi = -\lambda_{q+1}^2 n_*^2\phi_\xi$ we compute the intermittent jets in terms of V_ξ

$$\begin{aligned} \lambda_{q+1}^2 n_*^2 W_\xi &= \lambda_{q+1}^2 n_*^2\xi\phi_\xi\psi_\xi = -\Delta\Phi_\xi\psi_\xi\xi \\ &= \nabla\times(\psi_\xi\nabla\times(\Phi_\xi\xi)) - \nabla\psi_\xi\times(\nabla\times\Phi_\xi\xi) \\ &= \nabla\times\nabla\times(\psi_\xi\Phi_\xi\xi) - \nabla\times(\nabla\psi_\xi\times\Phi_\xi\xi) - \nabla\psi_\xi\times(\nabla\times\Phi_\xi\xi) \\ &= \nabla\times\nabla\times(\psi_\xi\Phi_\xi\xi) - \nabla\psi_\xi\times(\nabla\times\Phi_\xi\xi) \\ &= \lambda_{q+1}^2 n_*^2\left(\nabla\times\nabla\times V_\xi - W_\xi^{(c)}\right), \end{aligned} \tag{8.22}$$

from which we deduce

$$\operatorname{div}(W_\xi + W_\xi^{(c)}) \equiv 0.$$

Moreover, since $r_\perp \ll r_{||}$, the correction W_ξ^c is comparatively small in L^2 with respect to W_ξ, more precisely we state the following lemma (see [10, Section 7.4]).

Lemma 8.8. *For any $N,M \geq 0$ and $p \in [1,\infty]$ the following inequalities hold*

$$\|\nabla^N \partial_t^M \psi_\xi\|_{L^p} \lesssim r_{||}^{1/p-1/2} \left(\frac{r_\perp \lambda_{q+1}}{r_{||}}\right)^N \left(\frac{r_\perp \lambda_{q+1} \mu}{r_{||}}\right)^M \tag{8.23a}$$

$$\|\nabla^N \phi_\xi\|_{L^p} + \|\nabla^N \Phi_\xi\|_{L^p} \lesssim r_\perp^{2/p-1} \lambda_{q+1}^N \tag{8.23b}$$

$$\|\nabla^N \partial_t^M W_\xi\|_{L^p} \lesssim r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^N \left(\frac{r_\perp \lambda_{q+1} \mu}{r_{||}}\right)^M \tag{8.23c}$$

$$\frac{r_{||}}{r_\perp} \|\nabla^N \partial_t^M W_\xi^{(c)}\|_{L^p} \lesssim r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^N \left(\frac{r_\perp \lambda_{q+1} \mu}{r_{||}}\right)^M \tag{8.23d}$$

$$\lambda_{q+1}^2 \|\nabla^N \partial_t^M V_\xi\|_{L^p} \lesssim r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^N \left(\frac{r_\perp \lambda_{q+1} \mu}{r_{||}}\right)^M . \tag{8.23e}$$

The implicit constants are independent of $\lambda_{q+1}, r_\perp, r_{||}, \mu$.

8.4 Proof of the iterative proposition

Given $(v_q, p_q, \mathring{R}_q)$ a triple solving the Navier-Stokes-Reynolds system (8.5) in $\mathbb{T}^3 \times B_{2s}(t_0)$ satisfying the inductive estimates (8.10) and (8.11) at step q, we have to construct $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ which still solves the Navier-Stokes-Reynolds system (8.5) in $\mathbb{T}^3 \times B_{2s}(t_0)$ and satisfies the estimates (8.10) and (8.11) at step $q+1$ and the estimate (8.12) holds.

Mollification

In order to avoid a loss of derivatives in the iterative scheme, we replace v_q by a mollified velocity field $\tilde{v}_\ell$. For this purpose we choose a small parameter $\ell \in (0,1)$ which lies between λ_q^{-1} and λ_{q+1}^{-1} and that satisfies

$$\begin{aligned} \ell \lambda_q^4 &\leq \lambda_{q+1}^{-\alpha} \\ \ell^{-1} &\leq \lambda_{q+1}^{2\alpha}, \end{aligned}$$

where $0 < \alpha \ll 1$. This can be done since $\alpha b > 4$.

For instance, we may define ℓ as the geometric mean of the two bounds imposed before, namely

$$\ell = \lambda_{q+1}^{-3\alpha/2} \lambda_q^{-2}.$$

With this choice we also have that $\ell \ll s_{q+1}$. Let $\{\theta_\ell\}_{\ell>0}$ and $\{\varphi_\ell\}_{\ell>0}$ be two standard families of Friedrichs mollifiers on $\mathbb{R}^3$ (space) and $\mathbb{R}$ (time) respectively. We define the mollification of v_q and $\mathring{R}_q$ in space and time, at length scale ℓ by

$$\overline{v}_\ell := (v_q *_x \theta_\ell) *_t \varphi_\ell,$$
$$\mathring{\overline{R}}_\ell := (\mathring{R}_q *_x \theta_\ell) *_t \varphi_\ell,$$

where we possibly extend to 0 the definition of v_q outside $B_{2s}(t_0)$. We have that $\overline{v}_\ell$ solves

$$\begin{cases} \partial_t \overline{v}_\ell + \operatorname{div}(\overline{v}_\ell \otimes \overline{v}_\ell) + \nabla p_\ell - \mu \Delta \overline{v}_\ell = \operatorname{div}(\mathring{\overline{R}}_\ell + \mathring{\overline{R}}_{com}) \\ \operatorname{div} \overline{v}_\ell = 0, \end{cases} \tag{8.25}$$

where $\mathring{\overline{R}}_{com}$ is defined by

$$\mathring{\overline{R}}_{com} = (\overline{v}_\ell \mathring{\otimes} \overline{v}_\ell) - ((v_q \mathring{\otimes} v_q) *_x \theta_\ell) *_t \varphi_\ell.$$

We introduce the following notations $y + I_q := (t_0 - S_q - y, t_0 + S_q + y)$ and $\tilde{I}_q := \frac{s_{q+1}}{2} + I_q$. Let $\eta \in C_c^\infty(\tilde{I}_q; \mathbb{R}^+)$ such that

$$\eta(t) \equiv 1 \text{ for all } t \in I_q,$$
$$\|\eta\|_{C^N} \leq C \left(\frac{2}{s}\right)^{Nq},$$

Moreover, we define

$$\tilde{v}_\ell = \eta \overline{v}_\ell + (1-\eta) v_q.$$

Note that $\tilde{v}_\ell$ satisfies

$$\operatorname{Supp}_T(\tilde{v}_\ell - v_q) \subset \tilde{I}_q \subset I_{q+1},$$

that will be crucial in order to guarantee (8.11c) at step $q+1$.

Moreover, using (8.25) and that $(v_q, p_q, \mathring{R}_q)$ is a Navier–Stokes–Reynolds solution, we have that $\tilde{v}_\ell$ satisfies

$$\begin{aligned} \partial_t \tilde{v}_\ell + \operatorname{div}(\tilde{v}_\ell \otimes \tilde{v}_\ell) - \Delta \tilde{v}_\ell = {} & (\overline{v}_\ell - v_q)\partial_t \eta + \eta(1-\eta) \operatorname{div}(\overline{v}_\ell \mathring{\otimes} (v_q - \overline{v}_\ell)) \\ & + \eta(1-\eta) \operatorname{div}(v_q \mathring{\otimes} (\overline{v}_\ell - v_q)) \\ & + \eta \operatorname{div}(\overline{R}_\ell + \overline{R}_{com}) + (1-\eta) \operatorname{div}(\mathring{R}_q) - \nabla \pi_\ell, \end{aligned}$$

for some pressure π_ℓ.

Using (8.11b) and that $\eta(t) \equiv 1$ on I_q, we have

$$(1-\eta)\operatorname{div}(\mathring{R}_q) \equiv 0.$$

Thus $\tilde{v}_\ell$ solves

$$\partial_t \tilde{v}_\ell + \operatorname{div}(\tilde{v}_\ell \otimes \tilde{v}_\ell) - \Delta \tilde{v}_\ell + \nabla \pi_\ell = \operatorname{div}(R_\ell + R_{com} + R_{loc}),$$

where $R_\ell = \eta \mathring{\overline{R}}_\ell$, $R_{com} = \eta \mathring{\overline{R}}_{com}$ and

$$R_{loc} := \eta(1-\eta)\overline{v}_\ell \mathring{\otimes}(v_q - \overline{v}_\ell) + \eta(1-\eta) v_q \mathring{\otimes}(\overline{v}_\ell - v_q) + \mathscr{R}\left((\overline{v}_\ell - v_q)\partial_t \eta\right).$$

A simple bound on $\overline{v}_\ell - v_q$ on $L^\infty_t L^2$ is given by

$$\|\overline{v}_\ell - v_q\|_{L^2} \lesssim \ell \|v_q\|_{C^1} \le \ell \lambda_q^4 \ll \frac{1}{10}\lambda_{q+1}^{-4\zeta}\delta_{q+2},$$

where the last holds if $4\zeta + 2\beta b < \alpha$. Then using the previous bound, (8.10a) and that $\|\mathscr{R}\|_{L^2 \to L^2} \lesssim 1$ by Proposition D.1, we have

$$\|R_{com}\|_{L^1} + \|R_{loc}\|_{L^1} \ll \frac{1}{3}\lambda_{q+1}^{-3\zeta}\delta_{q+2},$$

where we used that $\lambda_{q+1}^{\zeta} \gg C\left(\frac{2}{s}\right)^q$, unless to possibly enlarge $a_0(s,\zeta)$. Note that we also have the property on the compact supports of the errors

$$\operatorname{Supp}(R_\ell) \cup \operatorname{Supp}(R_{com}) \cup \operatorname{Supp}(R_{loc}) \subset \tilde{I}_q \subset I_{q+1}.$$

The mollified functions satisfy

$$\|\tilde{v}_\ell\|_{C^N_{x,t}(\mathbb{T}^3 \times B_{2s}(t_0))} \lesssim \lambda_q^4 \ell^{-N+1} \lesssim \lambda_q^{-\alpha}\ell^{-N},\ N \ge 1, \tag{8.26a}$$

$$\|\tilde{v}_\ell\|_{L^2} \le \|v_q\|_{L^2} + \|v_q - \overline{v}_\ell\|_{L^2} \le 2\|v_0\|_{L^2} - \delta_q^{1/2} + \lambda_q^{-\alpha}, \tag{8.26b}$$

$$\|\tilde{v}_\ell - v_q\|_{L^2} \lesssim \ell \lambda_q^4 \le \lambda_{q+1}^{-\alpha}, \tag{8.26c}$$

$$\|R_\ell\|_{L^1} \le \lambda_q^{-3\zeta}\delta_{q+1}, \tag{8.26d}$$

$$\|R_\ell\|_{C^N_{x,t}} \lesssim \lambda_q^{-3\zeta}\delta_{q+1}\ell^{-4-N},\ N \ge 0. \tag{8.26e}$$

We are now ready to go to the perturbation step, in which we will add a small perturbation to $\tilde{v}_\ell$ in order to cancel the bigger error R_ℓ proving (8.10b), (8.11b) and satisfying all the other estimates (8.10), (8.11) and (8.12).

Amplitudes

Here we define the amplitudes of the perturbation, namely the functions needed to apply Lemma 8.6 and cancel the Reynolds error R_ℓ. We define $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, a smooth function such that

$$\chi(z) := \begin{cases} 1 & if \quad 0 \le z \le 1 \\ z & if \quad z \ge 2 \end{cases}$$

and $z \le 2\chi(z) \le 4z$ for $z \in (1,2)$ and $\chi(z) \ge 1$ for all $z \in [0,\infty)$. We define for all $t \in I_0 = [t_0 - \frac{s}{2}, t_0 + \frac{s}{2}]$

$$\overline{\rho}(t) := \frac{1}{3\int_{\mathbb{T}^3} \chi\left(\frac{|R_\ell(x,t)| 4\lambda_q^\zeta \delta_1}{\delta_{q+1}}\right) dx} \left(e(t) - \int_{\mathbb{T}^3} |\tilde{v}_\ell(x,t)|^2 dx - \frac{\delta_{q+2}}{2}\right) \tag{8.27}$$

and with a little abuse of notation we define

$$\overline{\rho}(t) := \overline{\rho}\left(t_0 + \frac{s}{2}\right) \text{ for all } t > t_0 + \frac{s}{2},$$
$$\overline{\rho}(t) := \overline{\rho}\left(t_0 - \frac{s}{2}\right) \text{ for all } t < t_0 - \frac{s}{2}.$$

Now, we consider another local cut-off in time $\tilde{\eta} \in C_c^\infty(I_{q+1}; \mathbb{R}^+)$ such that

$$\tilde{\eta}(t) \equiv 1 \text{ for all } t \in \tilde{I}_q,$$
$$\|\tilde{\eta}\|_{C^N} \le C\left(\frac{2}{s}\right)^{Nq},$$

and we define

$$\rho(x,t) := \tilde{\eta}^2(t)\overline{\rho}(t)\chi\left(\frac{|R_\ell(x,t)| 4\lambda_q^\zeta \delta_1}{\delta_{q+1}}\right). \tag{8.28}$$

Lemma 8.9. *The following estimates hold*

$$\frac{\delta_{q+1}}{\delta_1 \lambda_q^\zeta} \le \overline{\rho}(t) \le \frac{\varepsilon_1 \delta_{q+1}}{\delta_1}, \tag{8.29}$$

$$\left|\frac{R_\ell(x,t)}{\rho(x,t)}\right| \le \frac{1}{2}, \tag{8.30}$$

$$\|\rho\|_{L^1} \le 16\pi^3 \varepsilon_1 \frac{\delta_{q+1}}{\delta_1}. \tag{8.31}$$

Proof. Note that

$$\left|\|v_q\|_{L^2}^2 - \|\tilde{v}_\ell\|_{L^2}^2\right| \le \|v_q - \tilde{v}_\ell\|_{L^2}\|v_q + \tilde{v}_\ell\|_{L^2} \lesssim \ell\|v_q\|_{C^1}\|v_q\|_{L^2} \lesssim \ell\lambda_q^4 \le \lambda_q^{-\zeta}\delta_{q+1}, \tag{8.32}$$

where in the last inequality we used that $2\beta + \frac{\zeta}{b} < \alpha$. Moreover, thanks to the construction of χ and (8.10b) we have

$$(2\pi)^3 \le \int_{\mathbb{T}^3} \chi\left(\frac{|R_\ell(x,t)|4\lambda_q^\zeta\delta_1}{\delta_{q+1}}\right)dx \le 2(2\pi)^3. \tag{8.33}$$

Thus, thanks to (8.11a), (8.32) and (8.33) we get

$$\begin{aligned}\overline{\rho}(t) &\le \frac{1}{3\cdot(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3}|v_q(x,t)|^2dx\right) + \frac{1}{3\cdot(2\pi)^3}\left(\int_{\mathbb{T}^3}|v_q(x,t)|^2dx - \int_{\mathbb{T}^3}|\tilde{v}_\ell(x,t)|^2dx - \frac{\delta_{q+2}}{2}\right)\\ &\le \frac{1}{3\cdot(2\pi)^3}\left(2\frac{\delta_{q+1}\varepsilon_1}{\delta_1}\right) \le \frac{\varepsilon_1\delta_{q+1}}{\delta_1}.\end{aligned}$$

and similarly

$$\begin{aligned}\overline{\rho}(t) &\ge \frac{1}{6\cdot(2\pi)^3}\left(e(t) - \int_{\mathbb{T}^3}|v_q(x,t)|^2dx\right) + \frac{1}{6\cdot(2\pi)^3}\left(\int_{\mathbb{T}^3}|v_q(x,t)|^2dx - \int_{\mathbb{T}^3}|\tilde{v}_\ell(x,t)|^2dx - \frac{\delta_{q+2}}{2}\right)\\ &\ge \frac{1}{6\cdot(2\pi)^3}\left(\frac{\delta_{q+1}}{\delta_1\lambda_q^{\zeta/2}} - \frac{\delta_{q+1}}{\lambda_q^\zeta} - \frac{\delta_{q+2}}{2}\right) \ge \frac{\delta_{q+1}}{\delta_1\lambda_q^\zeta},\end{aligned}$$

where the last holds if we choose $a_0(\zeta)$ sufficiently large. Thus (8.29) holds.

The proof of (8.30) follows from the following computation, observing that $\mathrm{Supp}_T(R_\ell) \subset \tilde{I}_q$, $\tilde{\eta}(t) \equiv 1$ for all $t \in \tilde{I}_q$ and that $\chi(z) \ge z/2$ for all $z \ge 0$

$$\left|\frac{R_\ell(x,t)}{\rho(x,t)}\right| \le \frac{|R_\ell(x,t)|}{\overline{\rho}(t)\frac{|R_\ell(x,t)|}{2\delta_{q+1}}4\lambda_q^\zeta\delta_1} = \frac{\delta_{q+1}}{2\overline{\rho}(t)\lambda_q^\zeta\delta_1} \le 1/2.$$

We conclude the proof by estimating

$$\begin{aligned}\int_{\mathbb{T}^3}|\rho(x,t)|dx &\le \int_{\frac{|R_\ell(x,t)|4\lambda_q^\zeta\delta_1}{\delta_{q+1}}<1}|\rho(x,t)|dx + \int_{\frac{|R_\ell(x,t)|4\lambda_q^\zeta\delta_1}{\delta_{q+1}}\ge 1}|\rho(x,t)|dx\\ &\le 8\pi^3\left(\delta_{q+1}\frac{\varepsilon_1}{\delta_1}\right) + \int_{\mathbb{T}^3}|8\lambda_q^\zeta\varepsilon_1R_\ell|dx\\ &\le 8\pi^3\left(\delta_{q+1}\frac{\varepsilon_1}{\delta_1}\right) + 8\varepsilon_1\lambda_q^{2\zeta}\|R_\ell\|_{L^1}\\ &\le 8\pi^3\varepsilon_1\left(\frac{1}{\delta_1} + \lambda_q^{-\zeta}\right)\delta_{q+1} \le 16\pi^3\varepsilon_1\frac{\delta_{q+1}}{\delta_1}.\end{aligned}$$

□

We can now define the amplitudes functions $a_\xi : \mathbb{T}^3 \times (0,T) \to \mathbb{R}$ as

$$a_\xi(x,t) := a_{\xi,q+1}(x,t) := \rho^{1/2}(x,t)\gamma_\xi\left(Id - \frac{R_\ell(x,t)}{\rho(x,t)}\right), \tag{8.34}$$

where γ_ξ are defined in Lemma 8.6, hence we also get the identity

$$\rho(x,t)Id - R_\ell(x,t) = \sum_{\xi\in\Lambda} a_\xi^2(x,t)\xi\otimes\xi. \tag{8.35}$$

Lemma 8.10. *The following estimates hold*

$$\|a_\xi\|_{L^2} \le \frac{\delta_{q+1}^{1/2}}{2C_0|\Lambda|}\frac{\varepsilon}{4\pi\delta_1^{1/2}}, \tag{8.36}$$

$$\|a_\xi\|_{C^N_{x,t}} \lesssim \ell^{-8-5N}, \tag{8.37}$$

where C_0 is the universal constant for which Lemma 8.2 holds.

Proof. We define

$$\begin{aligned}
\rho_1(x,t) &:= \overline{\rho}(t)\chi\left(\frac{|R_\ell(x,t)|4\lambda_q^\zeta\delta_1}{\delta_{q+1}}\right),\\
\overline{a}_\xi(x,t) &:= \rho_1^{1/2}(x,t)\gamma_\xi\left(Id - \frac{R_\ell(x,t)}{\rho(x,t)}\right),\\
a_\xi(x,t) &= \tilde{\eta}(t)\overline{a}_\xi(x,t).
\end{aligned}$$

The first estimate follows from (8.31) and the definition of ε_1

$$\|a_\xi\|_{L^2} \le \|\rho\|_{L^1}^{1/2}\|\gamma_\xi\|_{C^0}\|\tilde{\eta}\|_{C^0} \le \left(16\pi^3\delta_{q+1}\frac{\varepsilon_1}{\delta_1}\right)^{1/2}\|\gamma_\xi\|_{C^0} \le \frac{\delta_{q+1}^{1/2}}{2C_0|\Lambda|}\frac{\varepsilon}{4\pi\delta_1^{1/2}}.$$

We prove the second estimate. We introduce the notation $\tilde{\gamma}_\xi(x,t) := \gamma_\xi\left(\mathrm{Id} - \frac{R_\ell(x,t)}{\rho(x,t)}\right)$ and thanks to (A.1) we have

$$\|\overline{a}_\xi\|_{C^N_{x,t}} \lesssim \|\rho_1^{1/2}\|_{C^N}\|\tilde{\gamma}\|_{C^0} + \|\rho_1^{1/2}\|_{C^0}\|\tilde{\gamma}\|_{C^N}.$$

We now estimate every piece. Using Proposition A.2 and (8.11a)

$$\|\overline{\rho}\|_{C^N_t} \lesssim \ell^{-5N}.$$

Thanks to the previous inequality, Proposition A.2 and (A.1) we get

$$\|\rho_1\|_{C^N_{x,t}} \lesssim \ell^{-4-5N}. \tag{8.38}$$

Using Proposition A.2, estimate (8.26e), the previous estimate and that ρ is bounded from below by $\frac{\delta_{q+1}}{\delta_1 \lambda_q^{\zeta}}$, we have

$$\|\tilde{\gamma}\|_{C^N} \lesssim \left\| \frac{R_\ell}{\rho} \right\|_{C^N} \lesssim \ell^{-8-5N}$$

and using also that $\frac{\delta_{q+1}}{\delta_1 \lambda_q^{\zeta}} \geq \ell$ (choosing $\zeta = \zeta(\alpha)$ sufficiently small), we have

$$\|\rho_1^{1/2}\|_{C^N_{x,t}} \lesssim \ell^{-5-5N}.$$

Hence

$$\|\overline{a}_\xi\|_{C^N_{x,t}} \lesssim \ell^{-8-5N}.$$

Moreover, by applying again (A.1) we get

$$\|a_\xi\|_{C^N_{x,t}} \lesssim \|\overline{a}_\xi\|_{C^N_{x,t}} \|\tilde{\eta}\|_{C^0} + \|\tilde{\eta}\|_{C^N} \|\overline{a}_\xi\|_{C^0_{x,t}} \lesssim \|\overline{a}_\xi\|_{C^N_{x,t}},$$

since $s_{q+1}^{-1} \ll \lambda_q \ll \ell^{-1}$, up to enlarge $a_0(s,\alpha)$. □

8.4.1 Perturbation

The principal part of w_{q+1} is defined as

$$w^{(p)}_{q+1} := \sum_{\xi \in \Lambda} a_\xi W_\xi. \tag{8.39}$$

The incompressibility corrector $w^{(c)}_{q+1}$, that we define in order to have the incompressibility of w_{q+1}, is defined as

$$w^{(c)}_{q+1} := \sum_{\xi \in \Lambda} \operatorname{curl}(\nabla a_\xi \times V_\xi) + \nabla a_\xi \times \operatorname{curl} V_\xi + a_\xi W^{(c)}_\xi.$$

Note that

$$w^{(p)}_{q+1} + w^{(c)}_{q+1} = \sum_{\xi \in \Lambda} \nabla \times \nabla \times (a_\xi V_\xi),$$
$$\operatorname{div}(w^{(p)}_{q+1} + w^{(c)}_{q+1}) = 0,$$

where the first equation follows from a direct computation similar to (8.22) with amplitudes functions

$$\begin{aligned}
a_\xi W_\xi &= a_\xi \nabla\times\nabla\times V_\xi - a_\xi W_\xi^{(c)} \\
&= \nabla\times(a_\xi\nabla\times V_\xi) - \nabla a_\xi \times (\nabla\times V_\xi) - a_\xi W_\xi^{(c)} \\
&= \nabla\times\nabla\times(a_\xi V_\xi) - \nabla\times(\nabla a_\xi\times V_\xi) - \nabla a_\xi\times(\nabla\times V_\xi) - a_\xi W_\xi^{(c)}.
\end{aligned}$$

Moreover, we introduce a temporal corrector similar to (8.20) with amplitude functions

$$w_{q+1}^{(t)} := -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_H\mathbb{P}_{\neq 0}\left(a_\xi^2\phi_\xi^2\psi_\xi^2\xi\right). \tag{8.40}$$

Note that $w_{q+1}^{(t)}$ satisfies

$$\begin{aligned}
&\partial_t w_{q+1}^{(t)} + \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(a_\xi^2 \operatorname{div}(W_\xi\otimes W_\xi)\right) \\
&\quad= -\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_H\mathbb{P}_{\neq 0}\partial_t\left(a_\xi^2\phi_\xi^2\psi_\xi^2\xi\right) + \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(a_\xi^2\partial_t\left(\phi_\xi^2\psi_\xi^2\xi\right)\right) \\
&\quad= \underbrace{(\mathrm{Id}-\mathbb{P}_H)\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\partial_t\left(a_\xi^2\phi_\xi^2\psi_\xi^2\xi\right)}_{=:\nabla P_{q+1}} - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t a_\xi^2\left(\phi_\xi^2\psi_\xi^2\xi\right)\right).
\end{aligned}$$

From this computation and the identity (8.35), it follows that

$$\begin{aligned}
\operatorname{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} + R_\ell) + \partial_t w_{q+1}^{(t)} &= \sum_{\xi\in\Lambda}\operatorname{div}\left(a_\xi^2\mathbb{P}_{\neq 0}\left(W_\xi\otimes W_\xi\right)\right) + \nabla\rho + \partial_t w_{q+1}^{(t)} \\
= \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\nabla a_\xi^2\mathbb{P}_{\neq 0}\left(W_\xi\otimes W_\xi\right)\right) &+ \nabla\rho + \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(a_\xi^2\operatorname{div}\left(W_\xi\otimes W_\xi\right)\right) + \partial_t w_{q+1}^{(t)} \\
= \sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\nabla a_\xi^2\mathbb{P}_{\neq 0}\left(W_\xi\otimes W_\xi\right)\right) &+ \nabla\rho + \nabla P_{q+1} - \frac{1}{\mu}\sum_{\xi\in\Lambda}\mathbb{P}_{\neq 0}\left(\partial_t a_\xi^2\left(\phi_\xi^2\psi_\xi^2\xi\right)\right).
\end{aligned} \tag{8.41}$$

We now define the total increment

$$w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \tag{8.42}$$

and the new vector field is then given by

$$v_{q+1} := \tilde{v}_\ell + w_{q+1}. \tag{8.43}$$

In this section we verify that the inductive estimates (8.10) hold with q replaced by $q+1$, and that (8.12) is satisfied.

Proof of (8.12)

We want to apply Lemma 8.2 in L^2 with $f = a_\xi$ and $g_\sigma = W_\xi$, which is by construction $\left(\frac{\mathbb{T}}{\sigma}\right)^3$ −periodic with $\sigma \sim \lambda_{q+1} r_\perp$, where $\sim$ means up to a constant depending only on n_* and $\xi \in \Lambda$. For this purpose, note that by (8.10) we get

$$\|D^j a_\xi\|_{L^2} \leq \frac{\delta_{q+1}^{1/2}}{2C_0|\Lambda|} \frac{\varepsilon}{4\pi\delta_1^{1/2}} \ell^{-13j},$$

and thus we can take $C_f = \frac{\delta_{q+1}^{1/2}}{2C_0|\Lambda|} \frac{\varepsilon}{4\pi\delta_1^{1/2}}$. By conditions on ℓ we have $\ell^{-13} \leq \lambda_{q+1}^{26\alpha}$, whereas by (8.17) we have that $\lambda_{q+1} r_\perp = \left(\frac{\lambda_{q+1}}{2\pi}\right)^{1/7}$. Thus, since $\alpha < \frac{1}{7\cdot 70}$ and a is huge, Lemma 8.2 is applicable. Combining the resulting estimate with the normalization $\|W_\xi\|_{L^2} = 1$ we obtain

$$\|w_{q+1}^{(p)}\|_{L^2} \leq \sum_{\xi \in \Lambda} \frac{C_0 \delta_{q+1}^{1/2}}{2C_0|\Lambda|} \frac{\varepsilon}{4\pi\delta_1^{1/2}} \|W_\xi\|_{L^2} \leq \frac{\varepsilon}{4\pi\delta_1^{1/2}} \frac{1}{2} \delta_{q+1}^{1/2}. \tag{8.44}$$

For the correctors $w_{q+1}^{(c)}$ and $w_{q+1}^{(t)}$ we can use rougher estimates since they are considerably smaller than $w_{q+1}^{(p)}$. The following estimates are consequence of Proposition D.1, estimates (8.17), (8.23) and Lemma 8.10

$$\|w_{q+1}^{(p)}\|_{L^p} \lesssim \sum_{\xi \in \Lambda} \|a_\xi\|_{C^0} \|W_\xi\|_{L^p} \lesssim \ell^{-8} r_\perp^{2/p-1} r_{||}^{1/p-1/2} \tag{8.45a}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{L^p} &\lesssim \sum_{\xi \in \Lambda} \|a_\xi\|_{C^2} \|V_\xi\|_{W^{1,p}} + \|a_\xi\|_{C^0} \|W_\xi^{(c)}\|_{L^p} \\ &\lesssim \ell^{-18} r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^{-1} + \ell^{-8} r_\perp^{2/p-1} r_{||}^{1/p-1/2} \frac{r_\perp}{r_{||}} \\ &\lesssim \ell^{-18} r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^{-2/7}\end{aligned} \tag{8.45b}$$

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{L^p} &\lesssim \mu^{-1} \sum_{\xi \in \Lambda} \|a_\xi\|_{C^0}^2 \|\phi_\xi\|_{L^{2p}}^2 \|\psi_\xi\|_{L^{2p}}^2 \lesssim \ell^{-16} r_\perp^{2/p-2} r_{||}^{1/p-1} \mu^{-1} \\ &\lesssim \ell^{-16} r_\perp^{2/p-1} r_{||}^{1/p-1/2} \lambda_{q+1}^{-1/7},\end{aligned} \tag{8.45c}$$

where in the last inequality we used also the continuity of $\mathbb{P}_H$ in L^p (for any $1 < p < \infty$) and the fact that $\|\phi_\xi^2 \psi_\xi^2\|_{L^p} = \|\phi_\xi^2\|_{L^p} \|\psi_\xi^2\|_{L^p}$, thanks to Fubini. Combining (8.44), with the last two estimates of

(8.45) for $p = 2$, and using (8.17), we obtain for a constant $C > 0$ (which is independent of q) that[3]

$$\|w_{q+1}\|_{L^2} \leq \left(\frac{\varepsilon}{4\pi\delta_1^{1/2}} \frac{1}{2}\delta_{q+1}^{1/2} + C\ell^{-18}\frac{r_\perp}{r_{||}} + C\ell^{-16}\lambda_{q+1}^{-1/7} \right)$$
$$\leq \frac{\varepsilon}{4\pi\delta_1^{1/2}} \left(\frac{\delta_{q+1}^{1/2}}{2} + C\lambda_{q+1}^{36\alpha-2/7} + C\lambda_{q+1}^{32\alpha-1/7} \right) \leq \frac{3}{4}\frac{\varepsilon}{4\pi\delta_1^{1/2}}\delta_{q+1}^{1/2}.$$

Moreover from (8.26), by choosing a_0 sufficiently large we get

$$\|v_{q+1} - v_q\|_{L^2} \leq \|w_{q+1}\|_{L^2} + \|\tilde{v}_\ell - v_q\|_{L^2} \leq \frac{\varepsilon}{4\pi\delta_1^{1/2}}\delta_{q+1}^{1/2},$$

thus (8.12) is satisfied.

Proof of (8.10a)

The bound (8.10a) follows easily from and the previous estimates (if $q \neq 0$)

$$\|v_{q+1}\|_{L^2} = \|v_{q+1} - v_q + v_q\|_{L^2} \leq \|v_q\|_{L^2} + \|v_{q+1} - v_q\|_{L^2}$$
$$\leq 2\|v_0\|_{L^2} - \frac{\varepsilon}{4\pi}\delta_q^{1/2} + \frac{\varepsilon}{\delta_1^{1/2}4\pi}\delta_{q+1}^{1/2} \leq 2\|v_0\|_{L^2} - \frac{\varepsilon}{4\pi}\delta_{q+1}^{1/2},$$

where in the last inequality we have used that a is taken sufficiently large and $b \gg 1$. If $q = 0$, then (8.10a) is trivial.

Proof of (8.11c)

The property (8.11c) is verified since

$$v_{q+1} - v_q = \tilde{v}_\ell - v_q + w_{q+1}$$

and $\mathrm{Supp}_T(\tilde{v}_\ell - v_q) \subset \mathrm{Supp}_T \eta \subset I_{q+1}$, $\mathrm{Supp}_T w_{q+1} \subset \mathrm{Supp}_T a_\xi \subset \mathrm{Supp}_T \tilde{\eta} \subset I_{q+1}$.

[3] In the last inequality, we have implicitly used that $\alpha < 1/(7 \cdot 74)$ and a_0 be sufficiently large.

Proof of (8.10c)

Taking either a spatial or a temporal derivative, using Lemma 8.8, Lemma 8.10, (8.17) and (8.24), we have

$$\begin{aligned}
\|w^{(p)}_{q+1}\|_{C^1_{x,t}} &\lesssim \|a_\xi\|_{C^1_{x,t}}\|W_\xi\|_{C^0_{x,t}} + \|a_\xi\|_{C^0_{x,t}}\|W_\xi\|_{C^1_{x,t}} \\
&\lesssim \ell^{-13} r_\perp^{-1} r_{||}^{-1/2} + \ell^{-8} r_\perp^{-1} r_{||}^{-1/2}\lambda_{q+1}^2 \lesssim \lambda_{q+1}^{2+8/7+26\alpha},
\end{aligned}$$

$$\begin{aligned}
\|w^{(p)}_{q+1}\|_{C^1_{x,t}} &\lesssim \|a_\xi\|_{C^2_{x,t}}\|V_\xi\|_{C^1_{x,t}} + \|a_\xi\|_{C^1_{x,t}}\|W^{(c)}_\xi\|_{C^1_{x,t}}, \\
&\lesssim \ell^{18} r_\perp^{-1} r_{||}^{-1/2}\lambda_{q+1}^{-2}\lambda_{q+1}^2 + \ell^{-13}\frac{r_\perp}{r_{||}} r_\perp^{-1} r_{||}^{-1/2}\lambda_{q+1}^2 \lesssim \lambda_{q+1}^{2+6/7+36\alpha},
\end{aligned}$$

$$\begin{aligned}
\|w^{(t)}_{q+1}\|_{C^1_{x,t}} &\lesssim \|w^{(t)}_{q+1}\|_{C^{1,\alpha}_{x,t}} \lesssim \frac{1}{\mu}\|a_\xi^2\phi_\xi^2\psi_\xi^2\|_{C^{1,\alpha}_{x,t}} \\
&\lesssim \frac{1}{\mu}\|a_\xi^2\|_{C^0_{x,t}}\|\phi_\xi^2\|_{C^0_{x,t}}\|\psi_\xi^2\|_{C^{1,\alpha}_t} \lesssim \frac{1}{\mu}\ell^{-16} r_\perp^{-2} r_{||}^{-1/2}\lambda_{q+1}^2\lambda_{q+1}^{2\alpha} \lesssim \lambda_{q+1}^{3-2/7+34\alpha}.
\end{aligned}$$

In the latter inequality we have used that $\mathbb{P}_H$ is continuous on Hölder spaces. Therefore, using that $\alpha < 1/40$, that a_0 is sufficiently large and thanks to estimate (8.26a), we have

$$\|v_{q+1}\|_{C^1_{x,t}(B_{2s}(t_0)\times\mathbb{T}^3)} \le \|\tilde{v}_\ell\|_{C^1_{x,t}(B_{2s}(t_0)\times\mathbb{T}^3)} + \|w_{q+1}\|_{C^1_{x,t}} \le \lambda_{q+1}^4.$$

8.4.2 The new Reynolds stress

Here we will define the new Reynolds stress $\mathring{R}_{q+1}$. By definitions, $\tilde{v}_{q+1}$ solves

$$\begin{aligned}
&\operatorname{div}\mathring{R}_{q+1} - \nabla p_{q+1} \\
&\quad = \partial_t(\tilde{v}_\ell + w_{q+1}) + \operatorname{div}((\tilde{v}_\ell + w_{q+1})\otimes(\tilde{v}_\ell + w_{q+1})) - \Delta(\tilde{v}_\ell + w_{q+1}) \\
&\quad = \underbrace{-\Delta w_{q+1} + \partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1}) + \operatorname{div}(\tilde{v}_\ell\otimes w_{q+1} + w_{q+1}\otimes\tilde{v}_\ell)}_{\operatorname{div}(R_{lin})+\nabla p_{lin}} \\
&\quad + \underbrace{\operatorname{div}\Big((w^{(c)}_{q+1} + w^{(t)}_{q+1})\otimes w_{q+1} + w^{(p)}_{q+1}\otimes(w^{(c)}_{q+1} + w^{(t)}_{q+1})\Big)}_{\operatorname{div}(R_{cor})+\nabla p_{cor}} \\
&\quad + \underbrace{\operatorname{div}(w^{(p)}_{q+1}\otimes w^{(p)}_{q+1} + R_\ell) + \partial_t w^{(t)}_{q+1}}_{\operatorname{div}(R_{osc})+\nabla p_{osc}} + \operatorname{div}(R_{com}) - \nabla p_\ell.
\end{aligned}$$

More precisely

$$
\begin{aligned}
R_{lin} &:= -\mathscr{R}\Delta w_{q+1} + \mathscr{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \tilde{v}_\ell \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} \tilde{v}_\ell, \\
R_{cor} &:= \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} \left(w_{q+1}^{(c)} + w_{q+1}^{(t)}\right), \\
R_{osc} &:= \sum_{\xi\in\Lambda} \mathscr{R}\left(\nabla a_\xi^2 \mathbb{P}_{\neq 0}(W_\xi \otimes W_\xi)\right) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathscr{R}\left(\partial_t a_\xi^2 (\phi_\xi^2 \psi_\xi^2 \xi)\right), \\
p_{lin} &:= 2\tilde{v}_\ell \cdot w_{q+1}, \\
p_{cor} &:= |w_{q+1}|^2 - |w_{q+1}^{(p)}|^2, \\
p_{osc} &:= \rho + P_{q+1},
\end{aligned}
$$

where the definitions of p_{osc} and R_{osc} are justified by the previous computation (8.41). Hence we define

$$p_{q+1} := p_\ell - p_{cor} - p_{lin} - p_{osc}$$

and

$$\mathring{R}_{q+1} := R_{lin} + R_{cor} + R_{osc} + R_{com} + R_{loc},$$

where the last two were defined during the mollification step. We observe that the new Reynolds-stress $\mathring{R}_{q+1}$ is traceless, this property will be crucial in the energy estimates.

We need to estimate the new stress $\mathring{R}_{q+1}$ in L^1. However, since the Calderón-Zygmund operator $\nabla\mathscr{R}$ fails to be bounded on L^1, we introduce an integrability parameter,

$$p \in (1,2] \text{ such that } p-1 \ll 1.$$

Recalling the parameters choice (8.17), we fix p to obey

$$r_\perp^{2/p-2} r_{||}^{1/p-1} \leq (2\pi)^{1/7} \lambda_{q+1}^{16(p-1)/(7p)} \leq \lambda_{q+1}^{\alpha}, \tag{8.46}$$

where we recall that $0 < \alpha < \frac{1}{7\cdot 74}$. For instance, we take $p = \frac{32}{32-7\alpha}$.

Linear error Reynolds stress

By using Proposition D.1 we get that

$$
\begin{aligned}
\|R_{lin}\|_{L^p} &\lesssim \|\mathscr{R}\Delta w_{q+1}\|_{L^p} + \|\tilde{v}_\ell \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} \tilde{v}_\ell\|_{L^p} + \|\mathscr{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{L^p} \\
&\lesssim \|\nabla w_{q+1}\|_{L^p} + \|\tilde{v}_\ell\|_{L^\infty}\|w_{q+1}\|_{L^p} + \sum_{\xi\in\Lambda} \|\partial_t \operatorname{curl}(a_\xi V_\xi)\|_{L^p} \\
&\lesssim \sum_{\xi\in\Lambda} \|a_\xi\|_{C^1}\|W_\xi\|_{W^{1,p}} + \|\tilde{v}_\ell\|_{C^1} \sum_{\xi\in\Lambda} \|a_\xi\|_{C^1}\|W_\xi\|_{W^{1,p}} \\
&\quad + \sum_{\xi\in\Lambda} (\|a_\xi\|_{C^1}\|\partial_t V_\xi\|_{W^{1,p}} + \|\partial_t a_\xi\|_{C^1}\|V_\xi\|_{W^{1,p}}).
\end{aligned}
$$

Thus, by appealing to Lemma 8.8, Lemma 8.10, estimates (8.26) and to the choice of $p = \frac{32}{32-7\alpha}$, we conclude

$$\begin{aligned}\|R_{lin}\|_{L^p} &\lesssim \ell^{-13} r_\perp^{\frac{2}{p}-1} r_{||}^{\frac{1}{p}-1/2}\lambda_{q+1} + \ell^{-18} r_\perp^{\frac{2}{p}-1} r_{||}^{\frac{1}{p}-\frac{1}{2}}\lambda_{q+1} + \ell^{-18}\lambda_{q+1}^{-1} r_\perp^{\frac{2}{p}-1} r_{||}^{\frac{1}{p}-\frac{1}{2}} \\ &\lesssim \ell^{-18}\lambda_{q+1}^{\alpha}\lambda_{q+1} r_\perp r_{||}^{1/2} \lesssim \lambda_{q+1}^{37\alpha-\frac{1}{7}} \ll \frac{1}{6}\lambda_{q+1}^{-3\zeta}\delta_{q+2},\end{aligned}$$

where for the last inequality we used that $\alpha < \frac{1}{7\cdot 74}$ and $2\beta b + 3\zeta < \frac{1}{14}$.

Corrector error

The estimate on the corrector error is a consequence of (8.45) and our choice of p

$$\begin{aligned}\|R_{cor}\|_{L^p} &\leq \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^{2p}}\|w_{q+1}\|_{L^{2p}} + \|w_{q+1}^{(p)}\|_{L^{2p}}\|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^{2p}} \\ &\leq 2\|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^{2p}}\|w_{q+1}\|_{L^{2p}} \\ &\lesssim \ell^{-18} r_\perp^{1/p-1} r_{||}^{\frac{1}{2p}-\frac{1}{2}}\lambda_{q+1}^{-1/7} \lesssim \lambda_{q+1}^{36\alpha+\frac{\alpha}{2}-1/7} \ll \frac{1}{6}\lambda_{q+1}^{-3\zeta}\delta_{q+2},\end{aligned}$$

where the last inequality is justified as before.

Oscillation error

By using the boundedness on L^p of the Reynolds operator $\mathscr{R}$, Lemma 8.8, Lemma 8.10, (8.17), Fubini (to separate ϕ_ξ and ψ_ξ) and the choice of p we can estimate the second summand in the definition of R_{osc} as

$$\left\|\frac{1}{\mu}\sum_{\xi\in\Lambda}\mathscr{R}\left(\partial_t a_\xi^2(\phi_\xi^2\psi_\xi^2\xi)\right)\right\|_{L^p} \leq \mu^{-1}\sum_{\xi\in\Lambda}\|a_\xi\|_{C^1}^2\|\phi_\xi\|_{L^2p}^2\|\psi_\xi\|_{L^2p}^2 \lesssim \mu^{-1}\ell^{-26}\lambda_{q+1}^{\alpha} \ll \frac{1}{6}\lambda_{q+1}^{-3\zeta}\delta_{q+2}.$$

To estimate the remaining summand we will use Lemma 8.3. We apply it with $a = \nabla a_\xi^2$, $\kappa = \sigma = \lambda_{q+1} r_\perp$ and $\mathbb{P}_{\geq\sigma}(f) = \mathbb{P}_{\neq 0}(W_\xi \otimes W_\xi)$, that is a $\frac{\mathbb{T}^3}{\sigma}$−periodic function. Then we have

$$\begin{aligned}\left\|\sum_{\xi\in\Lambda}\mathscr{R}\left(\nabla a_\xi^2\mathbb{P}_{\neq 0}(W_\xi\otimes W_\xi)\right)\right\|_{L^p} &\lesssim (\lambda_{q+1}r_\perp)^{-1}\|\mathbb{P}_{\neq 0}(W_\xi\otimes W_\xi)\|_{L^p}\|\nabla a_\xi^2\|_{C^1} \\ &\lesssim \ell^{-21}\lambda_{q+1}^{-1/7}\|W_\xi\|_{L^{2p}}^2 \lesssim \ell^{-21}\lambda_{q+1}^{-1/7} r_\perp^{\frac{2}{p}-1} r_{||}^{\frac{1}{p}-\frac{1}{2}} \\ &\lesssim \lambda_{q+1}^{42\alpha+\alpha-1/7} \ll \frac{1}{6}\lambda_{q+1}^{-3\zeta}\delta_{q+2}.\end{aligned}$$

Then (8.10b) at step $q+1$ follows easily using also the previous estimates for R_{com} and R_{loc}

$$\begin{aligned}\|\mathring{R}_{q+1}\|_{L^1} &\leq \|R_{lin}\|_{L^1} + \|R_{cor}\|_{L^1} + \|R_{osc}\|_{L^1} + \|R_{com}\|_{L^1} + \|R_{loc}\|_{L^1} \\ &\leq \frac{2}{3}\lambda_{q+1}^{-3\zeta}\delta_{q+2} + \frac{1}{3}\lambda_{q+1}^{-3\zeta}\delta_{q+2} \leq \lambda_{q+1}^{-3\zeta}\delta_{q+2},\end{aligned}$$

where in the last inequality we have used that $2\beta b + 3\zeta < \alpha$. Finally, since $\mathrm{Supp}_T w_{q+1} \subset I_{q+1}$, then also (8.11b) holds at step $q+1$.

8.4.3 Energy iteration

In order to complete the proof of Proposition 8.5 we only need to prove the energy estimate (8.11a) at step $q+1$.

Lemma 8.11. *The following estimate holds for all $t \in I_0$*

$$\frac{\delta_{q+2}}{\lambda_{q+1}^{\zeta/2}} \leq e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x,t)|^2 dx \leq \frac{\delta_{q+2}\varepsilon_1}{\delta_1}. \tag{8.47}$$

Proof. Recalling (8.39) and the mutually disjoint supports of $\{W_\xi\}_{\xi\in\Lambda}$ we notice that

$$\begin{aligned}|w_{q+1}^{(p)}|^2 &= \left|\sum_{\xi\in\Lambda} a_\xi W_\xi\right|^2 = \sum_{\xi\in\Lambda} \mathrm{Tr}(a_\xi W_\xi \otimes a_\xi W_\xi) \\ &= \sum_{\xi\in\Lambda} a_\xi^2 \mathrm{Tr}\left(\fint_{\mathbb{T}^3} W_\xi\otimes W_\xi\right) + \sum_{\xi\in\Lambda} a_\xi^2 \mathrm{Tr}\left(W_\xi\otimes W_\xi - \fint_{\mathbb{T}^3} W_\xi\otimes W_\xi\right) \\ &= 3\rho + \sum_{\xi\in\Lambda} a_\xi^2 \mathrm{Tr}\left(W_\xi\otimes W_\xi - \fint_{\mathbb{T}^3} W_\xi\otimes W_\xi\right),\end{aligned} \tag{8.48}$$

where in the last equation we used the traceless property of R_ℓ and (8.35).

Applying Lemma 8.4 with f replaced by a_ξ^2 (which oscillates at frequency $\sim \ell^{-5}$), the constant $C_f \sim \ell^{-16}$ (thanks to the estimate of Lemma 8.10) and g_σ replaced with $W_\xi \otimes W_\xi - \fint_{\mathbb{T}^3} W_\xi\otimes W_\xi$ (where $\sigma = \lambda_{q+1} r_\perp$), we get

$$\left|\int_{\mathbb{T}^3} \sum_{\xi\in\Lambda} a_\xi^2 \mathrm{Tr}\left(W_\xi\otimes W_\xi - \fint_{\mathbb{T}^3} W_\xi\otimes W_\xi\right)\right| \lesssim \ell^{-21}\frac{1}{\lambda_{q+1} r_\perp} \ll \frac{\delta_{q+2}}{6}, \tag{8.49}$$

where in the last inequality we used that $\alpha < \frac{1}{7\cdot 74}$ and $2\beta b < \frac{1}{14}$. We write the identity

$$\begin{aligned}e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 &= e(t) - \left(\int_{\mathbb{T}^3} |\tilde{v}_\ell|^2 + \int_{\mathbb{T}^3} |w_{q+1}^{(p)}|^2\right) - \left(\int_{\mathbb{T}^3} |w_{q+1}^{(c)} + w_{q+1}^{(t)}|^2 + 2\int_{\mathbb{T}^3} \tilde{v}_\ell \cdot w_{q+1}\right) \\ &\quad - \left(2\int_{\mathbb{T}^3} w_{q+1}^{(p)} \cdot (w_{q+1}^{(c)} + w_{q+1}^{(t)})\right)\end{aligned} \tag{8.50}$$

and thanks to (8.48), (8.49) and to the definition of ρ (8.28), using also that $\tilde{\eta} \equiv 1$ in I_0, we have

$$\frac{\delta_{q+2}}{\lambda_{q+1}^{\zeta/4}} \leq e(t) - \left(\int_{\mathbb{T}^3} |\tilde{v}_\ell|^2 + \int_{\mathbb{T}^3} |w_{q+1}^{(p)}|^2 \right) \leq \frac{2\delta_{q+2}}{3}, \text{ for all } t \in I_0,$$

up to possibly enlarge $a_0(\zeta)$. Moreover, by using (8.26) and (8.45) we can estimate

$$\left| \int_{\mathbb{T}^3} |w_{q+1}^{(c)} + w_{q+1}^{(t)}|^2 + 2 \int_{\mathbb{T}^3} \tilde{v}_\ell \cdot w_{q+1} \right| \leq \frac{\delta_{q+2}}{\lambda_{q+1}^{\zeta/3}},$$

$$\left| 2 \int_{\mathbb{T}^3} w_{q+1}^{(p)} \cdot (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \right| \leq \frac{\delta_{q+2}}{\lambda_{q+1}^{\zeta/3}},$$

from which (8.47) follows. □

Appendices

Appendix A

Main functional spaces and norms

Here we define the main functional spaces used in this work, namely Hölder, Besov and Sobolev spaces. We define them for a general $d-$dimensional Lipschitz domain $\Omega \subset \mathbb{R}^d$. In what follows $\theta \in (0,\infty)$, $n \in \mathbb{N}$, $r,s \in [1,\infty]$ and β is a multi-index, f is a (scalar or vector valued) function defined on Ω. Moreover, for any $\theta \in (0,\infty)$, let θ^- to be the biggest integer which is strictly less than θ.

Hölder spaces

For any $n \geq 0$ we define the usual $C^n(\Omega)$ norms

$$\|f\|_{C^0(\Omega)} = \sup_{x\in\Omega} |f(x)|,$$

$$[f]_{C^n(\Omega)} = \sup_{|\beta|=n} \|D^\beta f\|_{C^0(\Omega)},$$

$$\|f\|_{C^n(\Omega)} = \|f\|_{C^0(\Omega)} + \sum_{j=1}^{n} [f]_{C^j(\Omega)}.$$

Moreover, for any $\theta \in (0,1]$ we define the Hölder norms as

$$[f]_{C^\theta(\Omega)} = \sup_{x\neq y, x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\theta},$$

$$\|f\|_{C^{n,\theta}(\Omega)} = \|f\|_{C^n(\Omega)} + \sup_{|\beta|=n} [D^\beta f]_{C^\theta(\Omega)}.$$

To lighten the notation we will often denote the Hölder norms just introduced as $\|f\|_\theta$. Moreover, for time dependent functions $f = f(x,t)$ we will write $\|f(t)\|_\theta$ when the Hölder norm is computed for a fixed time slice t. Finally, when the time t is not explicit in the norm of a time dependent function we mean that the supremum is taken. More precisely

$$\|f\|_\theta = \sup_t \|f(t)\|_\theta.$$

We also recall the following elementary inequalities

Proposition A.1. *Let f,g be two smooth functions. For any $r \geq s \geq 0$ we have*

$$[fg]_{C^r(\Omega)} \leq C\big([f]_{C^r(\Omega)}\|g\|_{C^0(\Omega)} + \|f\|_{C^0(\Omega)}[g]_{C^r(\Omega)}\big) \tag{A.1}$$

$$[f]_{C^s(\Omega)} \leq C\|f\|^{1-s/r}_{C^0(\Omega)}[f]^{s/r}_{C^r(\Omega)}. \tag{A.2}$$

Proposition A.2. *Let $\Psi : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^n \to \Omega$ be two smooth functions, with $\Omega \subset \mathbb{R}^N$. Then, for every $m \in \mathbb{N}^+$, there exists a constant $C > 0$ (depending only on m, N, n) such that*

$$[\Psi \circ u]_{C^m(\mathbb{R}^n)} \leq C\left([\Psi]_{C^1(\Omega)}[u]_{C^m(\mathbb{R}^n)} + \|D\Psi\|_{C^{m-1}(\Omega)}\|u\|^{m-1}_{C^0(\mathbb{R}^n)}[u]_{C^m(\mathbb{R}^n)}\right),$$
$$[\Psi \circ u]_{C^m(\mathbb{R}^n)} \leq C\left([\Psi]_{C^1(\Omega)}[u]_{C^m(\mathbb{R}^n)} + \|D\Psi\|_{C^{m-1}(\Omega)}\|u\|^{m}_{C^1(\mathbb{R}^n)}\right).$$

Sobolev spaces

Denoting by $L^r(\Omega)$ the usual Lebesgue space of $r-$summable functions, we define the integer Sobolev norms as

$$[f]_{W^{n,r}(\Omega)} = \sup_{|\beta|=n} \|D^\beta f\|_{L^r(\Omega)},$$
$$\|f\|_{W^{n,r}(\Omega)} = \|f\|_{L^r(\Omega)} + \sum_{j=1}^{n}[f]_{W^{j,r}(\Omega)}.$$

Moreover, if θ is not an integer and $r < \infty$, the fractional Sobolev spaces will be characterize by the following

$$[f]_{W^{\theta,r}(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^r}{|x-y|^{d+(\theta-\theta^-)r}}\,dxdy\right)^{\frac{1}{r}},$$
$$\|f\|_{W^{\theta,r}(\Omega)} = \|f\|_{W^{\theta^-,r}(\Omega)} + [f]_{W^{\theta,r}(\Omega)}.$$

We will use the usual identifications $W^{0,r}(\Omega) = L^r(\Omega)$ and $W^{\theta,\infty}(\Omega) = C^\theta(\Omega)$.

Besov spaces

We first define the Besov spaces on the whole $\mathbb{R}^d$, then their version on general open sets Ω will be defined by extension.

For any non integer $\theta \in (0,\infty)$, the Besov space $B^\theta_{r,s}(\mathbb{R}^d)$ is the space of functions $f \in W^{\theta^-,r}(\mathbb{R}^d)$ such that

$$[f]_{B^\theta_{r,s}(\mathbb{R}^d)} = \sum_{|\alpha|=\theta^-}\left(\int_{\mathbb{R}^d}\frac{1}{|h|^{d+(\theta-\theta^-)s}}\left(\int_{\mathbb{R}^d}|D^\alpha f(x+h) - D^\alpha f(x)|^r\,dx\right)^{\frac{s}{r}}dh\right)^{\frac{1}{s}} < \infty,$$

with the usual generalization when $r,s=\infty$. The full Besov norm will be then given by

$$\|f\|_{B^{\theta}_{r,s}(\mathbb{R}^d)} = \|f\|_{W^{\theta^-,r}(\mathbb{R}^d)} + [f]_{B^{\theta}_{r,s}(\mathbb{R}^d)}.$$

If instead $\theta > 0$ is an integer, the Besov space $B^{\theta}_{r,s}(\mathbb{R}^d)$ consists of all the functions $f \in W^{\theta,r}(\mathbb{R}^d)$, such that

$$[f]_{B^{\theta}_{r,s}(\mathbb{R}^d)} = \sum_{|\alpha|=\theta} \left(\int_{\mathbb{R}^d} \frac{1}{|h|^{d+s}} \left(\int_{\mathbb{R}^d} |D^{\alpha} f(x+2h) - 2D^{\alpha} f(x+h) + D^{\alpha} f(x)|^r \, dx \right)^{\frac{s}{r}} dh \right)^{\frac{1}{s}} < \infty,$$

again with the usual generalization when $r,s=\infty$. Thus the full norm will be given by

$$\|f\|_{B^{\theta}_{r,s}(\mathbb{R}^d)} = \|f\|_{W^{\theta,r}(\mathbb{R}^d)} + [f]_{B^{\theta}_{r,s}(\mathbb{R}^d)}.$$

For any open and Lipschitz set Ω we then define

$$B^{\theta}_{r,s}(\Omega) = \left\{ f : \Omega \to \mathbb{R}^d \text{ s.t. } \exists\, \tilde{f} \in B^{\theta}_{r,s}(\mathbb{R}^d),\ \tilde{f}|_{\Omega} = f \right\},$$

where the semi-norm is given by

$$[f]_{B^{\theta}_{r,s}(\Omega)} = \inf \left\{ [\tilde{f}]_{B^{\theta}_{r,s}(\mathbb{R}^d)},\ \tilde{f}|_{\Omega} = f \right\}.$$

By the definitions above we have that for any non integer $\theta \in (0,\infty)$, $B^{\theta}_{r,r}(\Omega) = W^{\theta,r}(\Omega)$ for any $r \in [1,\infty]$, which in the case $r=\infty$ gives $B^{\theta}_{\infty,\infty}(\Omega) = C^{\theta}(\Omega)$. Moreover, since the domain Ω is Lipschitz, we always have the existence of a linear extension operator to the whole space. It is well know that this operator turns out to be also continuous between every Sobolev or Besov spaces. When considering the flat d-dimensional torus $\mathbb{T}^d$, we define the Besov norm as above with $\Omega = [0,4]^d$ that is, we compute the norm in 4 copies of $\mathbb{T}^d$. This is enough to encode all the informations for a well defined periodic Besov function.

We give the following interpolation result in Besov spaces.

Proposition A.3. *Let $\Omega \subset \mathbb{R}^d$ be an open and Lipschitz set. For any $r \in [1,\infty]$, $\theta,\gamma \in (0,1)$ with $\theta \geq \gamma$, there exists a constant $C > 0$ such that*

$$[f]_{B^{\gamma}_{r,\infty}(\Omega)} \leq C \|f\|_{L^r(\Omega)}^{1-\frac{\gamma}{\theta}} \|f\|_{B^{\theta}_{r,\infty}(\Omega)}^{\frac{\gamma}{\theta}}, \tag{A.3}$$

$$[f]_{B^{\theta}_{r,\infty}(\Omega)} \leq C \|f\|_{B^{\gamma}_{r,\infty}(\Omega)}^{\frac{1-\theta}{1-\gamma}} \|f\|_{W^{1,r}(\Omega)}^{\frac{\theta-\gamma}{1-\gamma}}. \tag{A.4}$$

The constant C in the previous proposition depends only on the domain Ω, more precisely it depends on the linear operator which extends a function defined on Ω to the whole space $\mathbb{R}^d$. Note that the same inequalities hold if one replaces all the semi-norms with the full norms.

Proof. We start by proving (A.3) and (A.4) in the whole space $\mathbb{R}^d$. Note that for every $f \in B^{\theta}_{r,\infty}(\mathbb{R}^d)$ and $\theta \geq \gamma$, we have

$$[f]_{B^{\gamma}_{r,\infty}(\mathbb{R}^d)} \leq 2\left(\|f\|_{L^r(\mathbb{R}^d)} + [f]_{B^{\theta}_{r,\infty}(\mathbb{R}^d)}\right). \tag{A.5}$$

By plugging in (A.5) the rescaled function $f(\varepsilon x)$, we also get

$$\varepsilon^{\gamma}[f]_{B^{\gamma}_{r,\infty}(\mathbb{R}^d)} \leq 2\left(\|f\|_{L^r(\mathbb{R}^d)} + \varepsilon^{\theta}[f]_{B^{\theta}_{r,\infty}(\mathbb{R}^d)}\right),$$

for every $\varepsilon > 0$. Thus by choosing $\varepsilon = \|f\|^{\frac{1}{\theta}}_{L^r(\mathbb{R}^d)}[f]^{-\frac{1}{\theta}}_{B^{\theta}_{r,\infty}(\mathbb{R}^d)}$, we get (A.3) for $\Omega = \mathbb{R}^d$. Take now $\lambda \in [0,1)$ such that $(1-\lambda)\gamma + \lambda = \theta$. We estimate

$$\begin{aligned}\frac{\|f(\cdot+y)-f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|^{\theta}} &= \left(\frac{\|f(\cdot+y)-f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|^{\gamma}}\right)^{1-\lambda}\left(\frac{\|f(\cdot+y)-f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|}\right)^{\lambda}\\ &\leq [f]^{1-\lambda}_{B^{\gamma}_{r,\infty}(\mathbb{R}^d)}\|\nabla f\|^{\lambda}_{L^r(\mathbb{R}^d)},\end{aligned}$$

from which, since $\lambda = \frac{\theta-\gamma}{1-\gamma}$, we conclude (A.4) for $\Omega = \mathbb{R}^d$. If $f \in B^{\theta}_{r,\infty}(\Omega)$ for Ω as in the statement, (A.3) and (A.4) easily follow from their versions in $\mathbb{R}^d$ and the existence of a (continuous) extension operator. □

Appendix B

Mollification estimates

In this section we state some useful mollification estimates that are often used in this work. We start by recalling the definition of the standard Friedrichs' mollifiers.

Let $B_1(0) \subset \mathbb{R}^d$ be the $d-$dimensional ball of radius 1 and let $\varphi \in C_c^\infty(B_1(0))$ be a standard non negative kernel such that $\int_{B_1(0)} \varphi(x)dx = 1$. For any $\delta > 0$ we define $\varphi_\delta = \delta^{-3}\varphi(\frac{x}{\delta})$ and we denote the mollification of a function f as

$$f_\delta = f * \varphi_\delta = \int_{B_1(0)} f(x-y)\varphi_\delta(y)\,dy\,.$$

Note that a direct consequence of the definition is that the mollification preserves the average

$$\int_{\mathbb{T}^d} f(x)\,dx = \int_{\mathbb{T}^d} f_\delta(x)\,dx\,, \qquad \forall \delta > 0\,. \tag{B.1}$$

The next propositions collect some elementary estimates on these regularized functions for different spaces, in particular (B.2) is the well known Constantin-E-Titi commutator estimate from [22]. The symbol $\star$ is used to denote both the tensor and the scalar product.

Proposition B.1. *For any $f, g : \mathbb{T}^3 \to \mathbb{R}^3$, $\theta \in (0,1]$ and $N \geq 0$ we have:*

$$\|f_\delta \star g_\delta - (f \star g)_\delta\|_{C^N} \leq C_N \delta^{2\theta-N}[g]_{C^\theta}[f]_{C^\theta}\,, \tag{B.2}$$

$$\|f_\delta\|_{C^{N+\theta}} \leq C_N \delta^{-N}[f]_{C^\theta}\,, \tag{B.3}$$

$$\|f_\delta\|_{C^{N+1}} \leq C_N \delta^{\theta-N-1}[f]_{C^\theta}\,, \tag{B.4}$$

$$\|f_\delta - f\|_{C^0} \leq C\delta^{\theta}[f]_{C^\theta}\,. \tag{B.5}$$

for a constant $C_N > 0$ depending only on N.

A proof of the following elementary estimates in L^p and Sobolev spaces can be found in [28].

Proposition B.2. *For any $f:\mathbb{T}^d\to\mathbb{R}^d$, $\theta\in(0,1)$, $r\in[1,\infty]$ and any integer $n\geq 0$, we have the following*

$$\|f-f_\delta\|_{L^r(\mathbb{T}^d)}\leq C\delta^{\theta}\|f\|_{B^{\theta}_{r,\infty}(\mathbb{T}^d)}, \tag{B.6}$$

$$\|f_\delta\|_{W^{n+1,r}(\mathbb{T}^d)}\leq C\delta^{\theta-n-1}\|f\|_{B^{\theta}_{r,\infty}(\mathbb{T}^d)}, \tag{B.7}$$

$$\|f_\delta\star f_\delta-(f\star f)_\delta\|_{W^{n,r}(\mathbb{T}^d)}\leq C\delta^{2\theta-n}\|f\|^2_{B^{\theta}_{2r,\infty}(\mathbb{T}^d)}, \tag{B.8}$$

for some constant $C>0$ depending on θ,r,n but otherwise independent of δ.

For any $1\leq r<\infty$ we set

$$[f]_{W^{\theta,r}(\mathbb{T}^d\llcorner B_\delta)}=\left(\int_{\mathbb{T}^d}\int_{B_\delta(x)}\frac{|f(x)-f(y)|^r}{|x-y|^{\theta r+3}}\,dxdy\right)^{\frac{1}{r}}.$$

Proposition B.3. *There exists a constant $C>0$ such that for any $f,g:\mathbb{T}^3\to\mathbb{R}^3$ and for any $\theta,\alpha\in(0,1),r\in[1,\infty)$ and for every $1\leq p,q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\|\nabla f_\delta\|_{L^r}\leq C\delta^{\alpha-1}[f]_{W^{\alpha,r}(\mathbb{T}^3\llcorner B_\delta)}, \tag{B.9}$$

$$\|\nabla \mathrm{curl}\, f_\delta\|_{L^r}\leq C\delta^{\alpha-2}[f]_{W^{\alpha,r}(\mathbb{T}^3\llcorner B_\delta)}, \tag{B.10}$$

$$\|f_\delta\star g_\delta-(f\star g)_\delta\|_{L^r}\leq C\delta^{\theta+\alpha}[f]_{W^{\theta,rp}(\mathbb{T}^3\llcorner B_\delta)}[g]_{W^{\alpha,rq}(\mathbb{T}^3\llcorner B_\delta)}. \tag{B.11}$$

Appendix C

Fractional Laplacian

Here we recall the definition and the main properties of the fractional Laplacian, that will be denoted by $(-\Delta)^{\alpha}$, where $\alpha \in (0,1)$. We give both the definitions of this non-local operator in the whole space $\mathbb{R}^3$ and in the periodic setting $\mathbb{T}^3 = [0,1]^3$.

If $f : \mathbb{R}^3 \to \mathbb{R}$ we define, for every $x \in \mathbb{R}^3$,

$$(-\Delta)^{\alpha} f(x) = C_{\alpha} \int_{\mathbb{R}^3} \frac{f(x) - f(y)}{|x-y|^{3+2\alpha}} \, dy,$$

where the normalization constant C_{α} is given by

$$C_{\alpha} = \frac{4^{\alpha}\Gamma(3/2+\alpha)}{\pi^{3/2}|\Gamma(-s)|}.$$

For a periodic function $f : \mathbb{T}^3 \to \mathbb{R}$, for every $x \in \mathbb{T}^3$, the fractional Laplacian is defined as the symbol $|k|^{2\alpha}$ in the Fourier space. More precisely

$$(-\Delta)^{\alpha} f(x) = \sum_{k \in \mathbb{Z}^3} |k|^{2\alpha} f_k e^{2\pi i k \cdot x},$$

where $f_k = \int_{\mathbb{T}^3} f(x) e^{-2\pi i k \cdot x} \, dx$ is the k-th Fourier coefficient of f.

The two definitions above coincide. Indeed in [55, Theorem 1.5] it has been proved that, if the function $f : \mathbb{T}^3 \to \mathbb{R}$ is regular enough, then for all $x \in \mathbb{T}^3$,

$$(-\Delta)^{\alpha} f(x) = \int_{\mathbb{T}^3} (f(x) - f(y)) \, K_{\alpha}(x-y) \, dy, \tag{C.1}$$

where the kernel $K_{\alpha} : \mathbb{T}^3 \to \mathbb{R}$ is given by

$$K_{\alpha}(x) = C_{\alpha} \sum_{k \in \mathbb{Z}^3} \frac{1}{|x-k|^{3+2\alpha}}.$$

Note that since the function f is periodic, one has $f(x+k)=f(x)$, for all $x\in\mathbb{R}^3$, $k\in\mathbb{Z}^3$, from which we can rewrite (C.1) as

$$(-\Delta)^\alpha f(x)=\int_{\mathbb{T}^3}(f(x)-f(y+k))\,K_\alpha(x-y)\,dy=C_\alpha\sum_{k\in\mathbb{Z}^3}\int_{[0,1]^3}\frac{f(x)-f(y+k)}{|x-y-k|^{3+2\alpha}}\,dy$$
$$=C_\alpha\sum_{k\in\mathbb{Z}^3}\int_{[0,1]^3-k}\frac{f(x)-f(y)}{|x-y|^{3+2\alpha}}\,dy=C_\alpha\int_{\mathbb{R}^3}\frac{f(x)-f(y)}{|x-y|^{3+2\alpha}}\,dy, \tag{C.2}$$

which shows that the two definitions give the same result for every, regular enough, periodic function on $\mathbb{R}^3$. In particular, the latter integral is well defined whenever $f\in C^\theta(\mathbb{T}^3)$, with $\theta>2\alpha$.

The following theorem is taken from [55, Theorem 1.4].

Theorem C.1. *Let $\gamma,\varepsilon>0$ and $\beta\geq 0$ such that $2\gamma+\beta+\varepsilon\leq 1$, and let $f:\mathbb{T}^d\to\mathbb{R}$. If $f\in C^{0,2\gamma+\beta+\varepsilon}$, then $(-\Delta)^\gamma f\in C^\beta$, moreover there exists a constant $C=C_\varepsilon>0$ such that*

$$\|(-\Delta)^\gamma f\|_{C^\beta(\mathbb{T}^3)}\leq C_\varepsilon[f]_{C^{2\gamma+\beta+\varepsilon}(\mathbb{T}^3)}. \tag{C.3}$$

Corollary C.2. *Let $\gamma\in(0,1)$, $\varepsilon>0$ be such that $0<\gamma+\varepsilon\leq 1$, and let $f:\mathbb{T}^d\to\mathbb{R}$. There exist a constant $C=C_\varepsilon>0$ such that*

$$\int_{\mathbb{T}^3}|(-\Delta)^{\gamma/2}f|^2(x)dx\leq C_\varepsilon[f]^2_{C^{\gamma+\varepsilon}(\mathbb{T}^3)}.$$

We also have the following commutator estimate

Proposition C.3. *Let k_1, k_2, $\alpha\in\left(0,\frac{1}{2}\right)$, $\beta\in(0,1)$, $f,g:\mathbb{T}^3\to\mathbb{R}^3$ and consider the non-local operator $T^\alpha(f,g)=(-\Delta)^\alpha(f\otimes g)-(-\Delta)^\alpha f\otimes g-f\otimes(-\Delta)^\alpha g$. Assume also that $k_1+k_2=2\alpha$. We have the following*

(i) if $\max(k_1,k_2)+\frac{\beta}{2}<2\alpha$ there exists a constant $C=C_{k_1,k_2,\alpha,\beta}>0$ such that

$$\|T^\alpha(f,g)\|_{C^\beta(\mathbb{T}^3)}\leq C\|f\|_{C^{k_1+\beta/2}(\mathbb{T}^3)}\|g\|_{C^{k_2+\beta/2}(\mathbb{T}^3)};$$

(ii) if $\min(k_1,k_2)+\frac{\beta}{2}\geq 2\alpha$ and $\max(k_1,k_2)+\frac{\beta}{2}<1$ then for every small $\varepsilon>0$ there exists a constant $C=C_{k_1,k_2,\alpha,\beta,\varepsilon}>0$ such that

$$\|T^\alpha(f,g)\|_{C^{\min(k_1,k_2)+\beta/2-\varepsilon}(\mathbb{T}^3)}\leq C\|f\|_{C^{k_1+\beta/2}(\mathbb{T}^3)}\|g\|_{C^{k_2+\beta/2}(\mathbb{T}^3)}.$$

An easy consequence of the previous proposition is that, taking $f=g=u$ and $k_1=k_2=\alpha$, one gets

$$\|T^\alpha(u)\|_{C^\beta(\mathbb{T}^3)}\leq C_{\alpha,\beta}\|u\|^2_{C^{\alpha+\beta/2}(\mathbb{T}^3)}\qquad \textit{if } \frac{\beta}{2}<\alpha; \tag{C.4}$$

$$\|T^\alpha(u)\|_{C^{\alpha+\beta/2-\varepsilon}(\mathbb{T}^3)}\leq C_{\alpha,\beta,\varepsilon}\|u\|^2_{C^{\alpha+\beta/2}(\mathbb{T}^3)}\qquad \textit{if } \alpha\leq\frac{\beta}{2},\ \alpha+\frac{\beta}{2}<1; \tag{C.5}$$

where we used the notation $T^\alpha(u)=T^\alpha(u,u)$.

Proof. A direct consequence of (C.2) is the following pointwise formula

$$T^{\alpha}(f,g)(x) = C_{\alpha} \int_{\mathbb{R}^3} \frac{\big(f(x)-f(y)\big)\otimes\big(g(y)-g(x)\big)}{|x-y|^{3+2\alpha}}\, dy\,.$$

The estimate $\|T^{\alpha} f\|_{C^0} \le C\|f\|_{C^{k_1+\beta/2}}\|g\|_{C^{k_2+\beta/2}}$ is easy and is left to the reader.

We fix $x_1, x_2 \in \mathbb{R}^3$ and we define $\bar{x} = \frac{x_1+x_2}{2}$ and $\lambda = |x_1 - x_2|$. For simplicity we also define the tensor $\varphi(x,y) = \big(f(x)-f(y)\big)\otimes\big(g(y)-g(x)\big)$. We now split

$$\begin{aligned}
&\frac{T^{\alpha}(f,g)(x_1) - T^{\alpha}(f,g)(x_2)}{C_{\alpha}} = \int_{B_{\lambda}(\bar{x})} \frac{\varphi(x_1,y)}{|x_1-y|^{3+2\alpha}}\, dy + \int_{B_{\lambda}(\bar{x})} \frac{\varphi(x_2,y)}{|x_2-y|^{3+2\alpha}}\, dy \\
&+ \int_{B^c_{\lambda}(\bar{x})} \frac{\varphi(x_1,y)-\varphi(x_2,y)}{|x_1-y|^{3+2\alpha}}\, dy + \int_{B^c_{\lambda}(\bar{x})} \left(\frac{1}{|x_1-y|^{3+2\alpha}} - \frac{1}{|x_2-y|^{3+2\alpha}} \right) \varphi(x_2,y)\, dy \\
&= I + II + III + IV\,.
\end{aligned}$$

The first two integrals can be estimated as

$$\begin{aligned}
|I|, |II| &\le [f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}} \int_{B_{\lambda}(\bar{x})} \left(\frac{1}{|x_1-y|^{3-\beta}} + \frac{1}{|x_2-y|^{3-\beta}} \right) dy \\
&\le C\lambda^{\beta}[f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}}\,.
\end{aligned} \tag{C.6}$$

Now we note that the difference $\varphi(x_1,y) - \varphi(x_2,y)$ can be rewritten as

$$\varphi(x_1,y) - \varphi(x_2,y) = \big(f(x_1)-f(x_2)\big)\otimes\big(g(y)-g(x_2)\big) + \big(f(y)-f(x_1)\big)\otimes\big(g(x_1)-g(x_2)\big)\,.$$

Thus, assuming $\max(k_1,k_2) + \frac{\beta}{2} < 2\alpha$, we estimate

$$\begin{aligned}
|III| &\le C[f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}} \left(\int_{B^c_{\lambda}(\bar{x})} \frac{\lambda^{k_1+\frac{\beta}{2}}\, dy}{|\bar{x}-y|^{3+2\alpha-k_2-\frac{\beta}{2}}} + \int_{B^c_{\lambda}(\bar{x})} \frac{\lambda^{k_2+\frac{\beta}{2}}\, dy}{|\bar{x}-y|^{3+2\alpha-k_1-\frac{\beta}{2}}} \right) \\
&\le C\lambda^{\beta}[f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}}
\end{aligned} \tag{C.7}$$

while in the case $\min(k_1,k_2) + \frac{\beta}{2} \ge 2\alpha$, for every small $\varepsilon > 0$ we estimate

$$\begin{aligned}
|III| &\le C\left([f]_{C^{k_1+\beta/2}}[g]_{C^{2\alpha-\varepsilon}} \int_{B^c_{\lambda}(\bar{x})} \frac{\lambda^{k_1+\frac{\beta}{2}}\, dy}{|\bar{x}-y|^{3+\varepsilon}} + [f]_{C^{2\alpha-\varepsilon}}[g]_{C^{k_2+\frac{\beta}{2}}} \int_{B^c_{\lambda}(\bar{x})} \frac{\lambda^{k_2+\frac{\beta}{2}}\, dy}{|\bar{x}-y|^{3+\varepsilon}} \right) \\
&\le C\Big(\lambda^{k_1+\frac{\beta}{2}-\varepsilon} + \lambda^{k_2+\frac{\beta}{2}-\varepsilon}\Big)\|f\|_{C^{k_1+\beta/2}}\|g\|_{C^{k_2+\beta/2}}\,,
\end{aligned} \tag{C.8}$$

where we have also used that $|x_1-y|,|x_2-y| \gtrsim |\bar{x}-y|$ for every $y \in B^c_\lambda(\bar{x})$.
We are now left with IV. We notice that for every $y \in B^c_\lambda(\bar{x})$ we have

$$\left|\frac{1}{|x_1-y|^{3+2\alpha}} - \frac{1}{|x_2-y|^{3+2\alpha}}\right| = \left|\int_0^1 \frac{d}{dt}\frac{1}{|tx_1+(1-t)x_2-y|^{3+2\alpha}}\,dt\right| \lesssim \lambda \frac{1}{|\bar{x}-y|^{4+2\alpha}}$$

from which, in the case $\max(k_1,k_2)+\frac{\beta}{2} < 2\alpha$, we get (notice that in this case $\beta < 4\alpha - 2\max(k_1,k_2) < 2\alpha < 1$)

$$|IV| \le C\lambda [f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}} \int_{B^c_\lambda(\bar{x})} \frac{1}{|\bar{x}-y|^{4-\beta}}\,dy \le C\lambda^\beta [f]_{C^{k_1+\beta/2}}[g]_{C^{k_2+\beta/2}}\,, \tag{C.9}$$

while, if $\min(k_1,k_2)+\frac{\beta}{2} \ge 2\alpha$ and $\max(k_1,k_2)+\frac{\beta}{2} < 1$ we have

$$|IV| \le C\lambda [f]_{C^{k_1+\beta/2}}[g]_{C^{2\alpha}} \int_{B^c_\lambda(\bar{x})} \frac{1}{|\bar{x}-y|^{4-k_1-\frac{\beta}{2}}}\,dy \le C\lambda^{k_1+\frac{\beta}{2}} \|f\|_{C^{k_1+\beta/2}}\|g\|_{C^{k_2+\beta/2}}\,. \tag{C.10}$$

We conclude the proof combining (C.6)-(C.10). □

Appendix D

Inverse divergence operator and stationary phase Lemma

Here we recall the inverse divergence operator from [7] acting on zero average vector fields.

$$
\begin{aligned}
(\mathcal{R}f)^{ij} &= \mathcal{R}^{ijk} f^k \\
\mathcal{R}^{ijk} &= -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k - \frac{1}{2}\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.
\end{aligned}
\tag{D.1}
$$

By standard regularity estimates on linear elliptic equations the following holds

Proposition D.1. *For every smooth zero average vector field f, the tensor $\mathcal{R}f$ is a symmetric, trace-free matrix such that*

$$\operatorname{div} \mathcal{R} f = f.$$

Moreover we have

- *for every $\alpha \in (0,1)$, $\mathcal{R}$ and $\mathcal{R}\nabla$ are bounded linear operators from C^α to C^α;*
- *for every $p \in (1,\infty)$, $\mathcal{R}$ and $\mathcal{R}\nabla$ are bounded linear operators from L^p to L^p.*

The following is a simple consequence of classical stationary phase techniques. For a detailed proof the reader might consult [25, Lemma 2.2].

Proposition D.2. *Let $\alpha \in (0,1)$ and $N \geq 1$. Let $a \in C^\infty(\mathbb{T}^3)$, $\Phi \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ be smooth functions and assume that*

$$\hat{C}^{-1} \leq |\nabla\Phi|, \left|\nabla\Phi^{-1}\right| \leq \hat{C}$$

holds on $\mathbb{T}^3$. Then

$$\left|\int_{\mathbb{T}^3} a(x) e^{ik\cdot\Phi}\, dx\right| \lesssim \frac{\|a\|_N + \|a\|_0 \|\Phi\|_N}{|k|^N}, \tag{D.2}$$

and for the operator $\mathscr{R}$ defined in (D.1)*, we have*

$$\left\|\mathscr{R}\left(a(x)e^{ik\cdot\Phi}\right)\right\|_{\alpha} \lesssim \frac{\|a\|_0}{|k|^{1-\alpha}} + \frac{\|a\|_{N+\alpha} + \|a\|_0\|\Phi\|_{N+\alpha}}{|k|^{N-\alpha}},$$

where the implicit constant depends on $\hat{C}$, α and N, but not on k.

Appendix E

Mikado flows

Here we recall the construction of Mikado flows given in [25].

Lemma E.1. *For any compact subset $\mathcal{N} \subset\subset \mathcal{S}^{3\times 3}_{+}$ there exists a smooth vector field*

$$W : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3,$$

such that, for every $R \in \mathcal{N}$

$$\begin{cases} \operatorname{div}_\xi (W(R,\xi) \otimes W(R,\xi)) = 0 \\ \operatorname{div}_\xi W(R,\xi) = 0, \end{cases} \tag{E.1}$$

and

$$\fint_{\mathbb{T}^3} W(R,\xi)\, d\xi = 0,$$

$$\fint_{\mathbb{T}^3} W(R,\xi) \otimes W(R,\xi)\, d\xi = R. \tag{E.2}$$

Using the fact that $W(R,\xi)$ is $\mathbb{T}^3$-periodic and has zero mean in ξ, we write

$$W(R,\xi) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} a_k(R) e^{ik\cdot\xi} \tag{E.3}$$

for some smooth functions $R \to a_k(R) \in \mathbb{C}^3$, satisfying $a_k(R) \cdot k = 0$. From the smoothness of W, we further infer

$$\sup_{R\in\mathcal{N}} |D^N_R a_k(R)| \leq \frac{C(\mathcal{N}, N, m)}{|k|^m} \tag{E.4}$$

for some constant C, which depends, as highlighted in the statement, on $\mathcal{N}$, N and m. If needed, the smooth, vector valued, function $a_k(R)$ can be further decomposed as $A_k \tilde{a}_k(R)$, where A_k is a unitary vector and $\tilde{a}_k(R)$ is a scalar function.

Using the Fourier representation we see that from (E.2)

$$W(R,\xi)\otimes W(R,\xi) = R + \sum_{k\neq 0} C_k(R)e^{ik\cdot\xi} \tag{E.5}$$

where

$$C_k k = 0 \quad \text{and} \quad \sup_{R\in\mathcal{N}} |D_R^N C_k(R)| \leq \frac{C(\mathcal{N},N,m)}{|k|^m} \tag{E.6}$$

for any $m,N\in\mathbb{N}$.

It will also be useful to write the Mikado flows in terms of a potential. We note

$$\operatorname{curl}_\xi\left(\left(\frac{ik\times a_k}{|k|^2}\right)e^{ik\cdot\xi}\right) = -i\left(\frac{ik\times a_k}{|k|^2}\right)\times ke^{ik\cdot\xi} = -\frac{k\times(k\times a_k)}{|k|^2}e^{ik\cdot\xi} = a_k e^{ik\cdot\xi} \tag{E.7}$$

Appendix F

Potential Theory estimates

We recall the definition of the standard class of periodic Calderón-Zygmund operators. Let K be an $\mathbb{R}^3$ kernel which obeys the properties

- $K(z) = \Omega\left(\frac{z}{|z|}\right)|z|^{-3}$, for all $z \in \mathbb{R}^3 \setminus \{0\}$
- $\Omega \in C^\infty(\mathbb{S}^2)$
- $\int_{|\hat{z}|=1} \Omega(\hat{z})d\hat{z} = 0.$

From the $\mathbb{R}^3$ kernel K, use Poisson summation to define the periodic kernel

$$K_{\mathbb{T}^3}(z) = K(z) + \sum_{\ell \in \mathbb{Z}^3 \setminus \{0\}} \left(K(z+\ell) - K(\ell)\right).$$

Then the operator

$$T_K f(x) = p.v. \int_{\mathbb{T}^3} K_{\mathbb{T}^3}(x-y) f(y) dy$$

is a $\mathbb{T}^3$-periodic Calderón-Zygmund operator, acting on $\mathbb{T}^3$-periodic functions f with zero mean on $\mathbb{T}^3$. The following proposition, proving the boundedness of periodic Calderón-Zygmund operators on periodic Hölder spaces is classical.

Proposition F.1. *Fix $\alpha \in (0,1)$. Periodic Calderón-Zygmund operators are bounded on the space of zero mean $\mathbb{T}^3$-periodic C^α functions.*

The following proposition is taken from [7].

Proposition F.2. *Let $\alpha \in (0,1)$ and $N \geq 0$. Let T_K be a Calderón-Zygmund operator with kernel K. Let $b \in C^{N+1,\alpha}\left(\mathbb{T}^3\right)$ be a divergence free vector field. Then we have*

$$\|[T_K, b\cdot\nabla] f\|_{N+\alpha} \lesssim \|b\|_{1+\alpha}\|f\|_{N+\alpha} + \|b\|_{N+1+\alpha}\|f\|_\alpha,$$

for any $f \in C^{N+\alpha}\left(\mathbb{T}^3\right)$, where the implicit constant depends on α, N and K.

Appendix G

Some stability estimates

We recall some well known results regarding smooth solutions of the transport equation

$$\begin{cases} \partial_t f + v \cdot \nabla f = g \\ f|_{t_0} = f_0, \end{cases} \tag{G.1}$$

where $v = v(t,x)$ is a given smooth vector field. We will consider solutions on the entire space $\mathbb{R}^3$ and treat solutions on the torus simply as periodic solution in $\mathbb{R}^3$.

For a detailed proof of the next proposition we refer to [6, Appendix D].

Proposition G.1. *Assume* $|t-t_0|\|v\|_1 \leq 1$. *Any solution* f *of* (G.1) *satisfies*

$$\|f(t)\|_\alpha \lesssim \|f_0\|_\alpha + \int_{t_0}^t \|g(\tau)\|_\alpha \, d\tau\,, \tag{G.2}$$

for all $0 \leq \alpha \leq 1$, *and, more generally, for any* $N \geq 1$ *and* $0 \leq \alpha \leq 1$

$$[f(t)]_{N+\alpha} \leq [f_0]_{N+\alpha} + (t-t_0)[v]_{N+\alpha}[f_0]_1 + \int_{t_0}^t \left([g(\tau)]_N + (t-\tau)[v]_N[g(\tau)]_1\right) d\tau. \tag{G.3}$$

Define $\Phi(t,\cdot)$ *to be the inverse of the flux* X *of* v *starting at time* t_0 *as the identity (i.e.* $\frac{d}{dt}X = v(X,t)$ *and* $X(x,t_0) = x$*). Under the same assumptions as above, the following holds*

$$\|\nabla\Phi(t) - \mathrm{Id}\|_0 \lesssim |t-t_0|[v]_1\,, \tag{G.4}$$

$$[\Phi(t)]_N \lesssim |t-t_0|[v]_N \quad \forall N \geq 2. \tag{G.5}$$

Using the same technique introduced in Chapter 2 to prove the time regularity for Hölder solutions of Euler, we prove the following

Proposition G.2. *Let $u,v:\mathbb{T}^3\times[0,T]\to\mathbb{R}^3$ be two weak solutions of* (6.1) *such that $u,v\in C^0(([0,T];C^\theta(\mathbb{T}^3))$ for some $\theta\in(0,1)$. Then there exists a constant $C>0$, depending only on θ, $\|u\|_\theta$ and $\|v\|_\theta$, such that*

$$\|u-v\|_{C^\theta_{x,t}}\le C\|u-v\|_\theta.$$

Proof. We define $w=u-v$. We start by noticing that the Hölder norm, in the space-time variables, decouples as follows

$$\frac{|w(x,s)-w(y,t)|}{|(x,s)-(y,t)|^\theta}\le\frac{|w(x,s)-w(y,s)|}{|x-y|^\theta}+\frac{|w(y,s)-w(y,t)|}{|t-s|^\theta}\le\|w\|_\theta+\frac{|w(y,s)-w(y,t)|}{|t-s|^\theta}.$$

Thus it is enough to show that there exists a constant $C>0$, independent of y,t,s, such that

$$\frac{|w(y,s)-w(y,t)|}{|t-s|^\theta}\le C\|w\|_\theta. \tag{G.6}$$

If p and q are the corresponding pressures associated to u and v respectively, one has that w solves

$$\partial_t w+\operatorname{div}(w\otimes u+v\otimes w)+\nabla(p-q)=0. \tag{G.7}$$

By taking the divergence of (G.7), we get

$$-\Delta(p-q)=\operatorname{div}\operatorname{div}(w\otimes u+v\otimes w),$$

from which, by Schauder estimates, we get

$$\|p-q\|_\theta\le\|w\|_\theta(\|u\|_\theta+\|v\|_\theta)\le C\|w\|_\theta. \tag{G.8}$$

Let now $w_\delta=w*\varphi_\delta$ the space mollification of w, for some $\delta>0$ that will be fixed at the end of the proof. Since $w\in C^0([0,T];C^\theta(\mathbb{T}^3))$ we have

$$|w(y,t)-w_\delta(y,t)|\le C\|w\|_\theta\delta^\theta\qquad\forall t\in[0,T],$$

from which, by adding and subtracting $w_\delta(y,s)$ and $w_\delta(y,t)$, we can estimate

$$|w(y,s)-w(y,t)|\le C\|w\|_\theta\delta^\theta+|w_\delta(y,s)-w_\delta(y,t)|. \tag{G.9}$$

Moreover, since w solves (G.7), we get

$$|w_\delta(y,s)-w_\delta(y,t)|\le|t-s|\|\partial_t w_\delta\|_{C^0_{x,t}}\le|t-s|\big(\|(w\otimes u+v\otimes w)_\delta\|_1+\|(p-q)_\delta\|_1\big). \tag{G.10}$$

By estimate (G.8) and (B.4), we have

$$\|(p-q)_\delta\|_1\le C\|w\|_\theta\delta^{\theta-1},\quad\forall\delta>0,$$

and also

$$\|(w\otimes u+v\otimes w)_\delta\|_1 \le C\delta^{\theta-1}\|w\otimes u+v\otimes w\|_\theta \le C\|w\|_\theta\delta^{\theta-1}, \quad \forall\delta>0.$$

Thus, by plugging these two last inequalities in (G.10), we get

$$|w_\delta(y,s)-w_\delta(y,t)| \le C|t-s|\delta^{\theta-1}\|w\|_\theta,\ \forall\delta>0,$$

from which, by (G.9), we conclude

$$|w(y,s)-w(y,t)| \le C(\delta^\theta+|t-s|\delta^{\theta-1})\|w\|_\theta, \quad \forall\delta>0.$$

By choosing $\delta=|t-s|$ we finally achieve (G.6), and this concludes the proof. □

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