REGULARITY THEORY FOR A CLASS OF 2-DIMENSIONAL ALMOST AREA MINIMIZING CURRENTS

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ABSTRACT

In this thesis we deal with interior regularity issues for area minimizing surfaces. In particular, we consider a special class of almost area minimizing, 2-dimensional integral currents, with bounded mean curvature, and we prove that their interior singular set is discrete. More specifically, we treat area minimzing currents in riemannian manifolds, semicalibrated currents and spherical cross sections of 3-dimensional area minimizing cones. In all these three situations our result is sharp. Moreover, a nice corollary of our theorem is the fact that the singular set of 3-dimensional area minimizing cones consists of at most a finite number of lines.

Our result is inspired by the approach of Almgren-Chang (cf. [9]) for area minimizing currents, which we revisit and complete, adding also some new cases. In particular we use a lot of techniques coming from De Lellis and Spadaro's new proof of Almgren's Big Regularity paper (cf. [17, 18, 19, 20, 21]). Other important known results that we manage to cover are Tian-Riviére regularity theorem for almost complex curves (cf. [53]) and Bellettini-Riviére extension to a class of semicalibrated 3-dimensional cones (cf. [7]). Our result for general semicalibrated currents and general 3-dimensional area minimizing cones is entirely new.

It is worth mentioning that, among the various steps in the proof of our main result, we give a unified and much shorter proof of already existing results concerning the uniqueness of the tangent cone to 2-dimensional area minimizing and semicalibrated currents (cf. [66, 46]), generalizing it to the larger class of almost area minimizing 2-dimensional currents. This is done relying heavily on [66]. Moreover, we also generalize a Lipschitz approximation result for area minimizing currents, proved first by Almgren (cf. [3]) and recently revisited by De Lellis and Spadaro (cf. [19]). In particular this result is independent from the dimension of the current.

The other two fundamental tools are the so called Center Manifold and the Frequency function, for which we were inspired by [3, 9, 20, 21], and which we combine in an inductive argument to conclude our main theorem.

All the results of this thesis where obtained in collaboration with Camillo De Lellis and Emanuele Spadaro, to whom I deeply grateful for guiding me step by step in the beautiful (and hard) world of geometric measure theory.

ABSTRAKT

In dieser Arbeit beschäftigen wir uns mit der inneren Regularität von Oberflächen minimalen Flächeninhalts. Insbesondere betrachten wir eine spezielle Klasse von beinahe-flächenminimierenden, 2-dimensionalen Integral-Strömen mit beschränkter mittlerer Krümmung und wir beweisen, dass die Menge ihrer inneren Singularitäten diskret ist. Etwas genauer gesagt, behandeln wir Ströme, welche den Flächeninhalt in Riemannschen Mannigfaltigkeit minimieren, semikalibrierte Ströme sowie sphärische Querschnitte von 3-dimensionalen flächenminimierenden Kegeln. In jedem dieser drei Situationen ist unser Resultat optimal. Ausserdem ergibt sich als schönes Korollar, dass die Menge der Singularitäten von 3-dimensionalen, flächenminimierenden Kegeln höchstens aus einer endlichen Menge von Geraden besteht.

Unser Resultat wurde durch den Ansatz von Almgren-Chen (cf. [9]) für flächenminimierende Ströme inspiriert. Wir greifen diesen Ansatz wieder auf, vervollständigen ihn und fügen ausserdem einige neue Fälle hinzu. Insbesondere benutzen wir viele Techniken von De Lellis and Spadaro's neuem Beweis von Almgren's Big Regularity paper (cf. [17, 18, 19, 20, 21]). Weitere wichtige bekannte Resultate welche wir mit dieser Arbeit abdecken sind Tian-Riviére's Regularität's-Thoerem für beinahe-komplexe Kurven (cf. [53]) und Bellettini-Riviére's Erweiterung auf eine Klasse von semi-kalibrierten 3-dimensionalen Kegeln (cf. [7]). Unser Resultat für semi-kalibrierte Ströme und allgemeine 3-dminesionale flächenminimierenden Kegeln ist völlig neu.

Es lohnt sich zu erwähnen, dass neben den diversen Schritten im Beweis unseres Hauptsatzes ein vereinheitlichender und sehr viel kürzerer Beweis von bereits bekannten Resultaten betreffend der Eindeutigkeit von Tangentialkegeln an 2-dimensionale flächenminimierenden and semi-kalibrierten Strömen (cf. [66, 46])) gegeben wird. Hierbei wird dieses Resultat zugleich auf die grössere Klasse von beinahe-flächenminimierenden 2-dimensionalen Strömen gegeben. In diesem Abschnitt stützen wir uns stark auf [66]. Ausserdem verallgemeinern wir das Lipschitz-Approximations-Resultat für flächenminimierende Ströme, welches erstmals von Almgren (cf. [3]) bewiesen wurde und kürzlich von De Lellis and Spadaro (cf. [19]) erneut aufgegriffen wurde. Dabei ist dieses Resultat unabhängig von der Dimension des Stromes.

Die anderen beiden fundamentalen Werkzeuge sind die so genannte Center Manifold und die Frequency function. Hierbei wurden wir von [3, 9, 20, 21] inspiriert. Wir kombinierten diese beiden Hilfsmittel in einem induktiven Argument um unseren Hauptsatz daraus folgern zu können.

Alle Resultate dieser Arbeit wurden in Zusammenarbeit mit Camillo De Lellis und Emanuele Spadaro erzielt, welchen ich zu tiefstem Dank verpflichtet bin, dafür, dass sie mich Schritt für Schritt durch die wunderbare (und beschwerliche) Welt der geometrischen Masstheorie geführt haben.

I don't know half of you half as well as I should like; and I like less than half of you half as well as you deserve

— J.R.R. Tolkien, The Fellowship of the Ring

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CONTENTS

1 INTRODUCTION

The main focus of this thesis is the study of the interior regularity properties of the following types of surfaces.

Definition 1.1. Let $\Sigma \subset \mathbb{R}^{m+n}$ be a C^2 submanifold and $U \subset \mathbb{R}^{m+n}$ an open set.

(a) An m-dimensional integral current T with finite mass and $\mathrm{spt}(T) \subset \Sigma \cap U$ is area-minimizing in $\Sigma \cap U$ if $\mathbf{M}(T + \partial S) \geqslant \mathbf{M}(T)$ for any $(m+1)$-dimensional integral current S with $\mathrm{spt}(S) \subset\subset \Sigma \cap U$.

(b) A semicalibration (in Σ) is a C^1 m-form ω on Σ such that $\|\omega_x\|_c \leqslant 1$ at every $x \in \Sigma$, where $\|\cdot\|_c$ denotes the comass norm on $\Lambda^m T_x\Sigma$. An m-dimensional integral current T with $\mathrm{spt}(T) \subset \Sigma$ is *semicalibrated* by ω if $\omega_x(\vec{T}) = 1$ for $\|T\|$-a.e. x.

(c) An m-dimensional integral current T supported in $\partial \mathbf{B}_R(x) \subset \mathbb{R}^{m+n}$ is a *spherical cross-section of an area-minimizing cone* if $x \times\!\!\!\!\times\, T$ is area-minimizing.

Given an integer rectifiable current T, we denote by $\mathrm{Reg}(T)$ the subset of $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ consisting of those points x for which there is a neighborhood U such that $T \llcorner U$ is a (costant multiple of) a regular submanifold. Correspondingly, $\mathrm{Sing}(T)$ is the set $\mathrm{spt}(T) \setminus (\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T))$. Observe that $\mathrm{Reg}(T)$ is relatively open in $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ and thus $\mathrm{Sing}(T)$ is relatively closed. The main achievement of this thesis is then the following regularity Theorem.

Theorem 1.2. *Let $m = 2$ and T be as in (a), (b) or (c) of Definition 1.1. Assume in addition that Σ is of class C^{3,ε_0} (in case (a) and (b)) and ω of class C^{2,ε_0} (in case (b)) for some positive ε_0. Then the set of points $\mathrm{Sing}(T)$ is discrete.*

1.1 MOTIVATIONS AND COMMENTS

The currents described in (a) and (c) of Definition 1.1 are particular solutions of the so called Plateau problem. Introduced first by the French mathematician Lagrange in 1762 (cf. [43]), and named after the belgian physicist Plateau, who studied it in connection with the shape of soap bubbles, the Plateau problem can be phrased as follows:

(PB) given an $(m-1)$-dimensional boundary in $\mathbb{R}^{m+n}$ (that is an object without boundary itself), find an m-dimensional surface with least area among all the surfaces spanning the given boundary.

There are several possible ways to state this problem rigorously in a mathematical sense.

- The parametric formulation: the competitor surfaces are images of map, the volume is computed with the area formula and the boundary is the trace of the chosen map. This theory was satisfactorily developed in dimension 2 by Douglas and Rado in the thirties (cf. [47, 29] and [26] for a modern introduction).

- The set-theoretical formulation: the competitor surfaces are closed sets, the volume is simply the Hausdorff measure and several notion of spanning the boundary are possible. This theory was introduced by Reifenberg and further developed by Harrison, David and others (cf. [48, 40, 36]).
- The functional-analytic formulations: the surfaces are given as action on a linear space of smooth test functions, mainly integration. The two most famous formulations of this kind are De Giorgi's theory of sets of finite perimeter (cf. [10, 11, 14]) and Federer and Fleming's theory of integral currents (cf. [34]).

All these formulations give a positive answer to (PB), thanks to powerful compactness theorems combined with the lower semicontinuity of the proper notion of volume. However, since these are very general classes of surfaces, it is natural to ask about the regularity of the solutions. In the rest of the introduction, and indeed of the thesis, surface will mean integral current and (PB) will be formulated as in case (a) of Definition 1.1, that is

(PB′) Let $\Sigma \subset R^{m+n}$ be a $(m+\bar{n})$-dimensional C^2 submanifold and $U \subset \mathbb{R}^{m+n}$ an open set. An m-dimensional integral current T with finite mass and $\mathrm{spt}(T) \subset \Sigma \cap U$ is area-minimizing in $\Sigma \cap U$ if $\mathbf{M}(T + \partial S) \geqslant \mathbf{M}(T)$ for any $(m+1)$-dimensional integral current S with $\mathrm{spt}(S) \subset\subset \Sigma \cap U$.

For an extensive treatment abount currents see [32]. In this framework we can distinguish two cases.

The codimension one case, that is $\bar{n} = 1$, is quiet well understood. Indeed we have the following result.

Theorem 1.3 (Regularity in codimension $\bar{n} = 1$)**.** *Assume U, Σ and T are as in (PB′) with $n = 1$. Then*

(i) for $m \leqslant 6$ $\mathrm{Sing}(T) \cap U$ is empty (Fleming and De Giorgi ($m = 2$), Almgren ($m = 3$), Simons ($4 \leqslant m \leqslant 6$), see [13, 35, 12, 2, 57, 49]);

(ii) for $m = 7$ $\mathrm{Sing}(T) \cap U$ consists of isolated points (Federer, see [33]);

(iii) for $m \geqslant 8$ $\mathrm{Sing}(T) \cap U$ has Hausdorff dimension at most $m - 7$ (Federer, [33]) and is countably $m - 7$ rectifiable (Simon, [56]);

(iv) the above results are optimal, indeed for every $m \geqslant 7$ there are area minimizing integral currents T in $\mathbb{R}^{m+1}$ for which $\mathrm{Sing}(T)$ has positive $\mathcal{H}^{m-7}$ measure (Bombieri-De Giorgi-Giusti, [8]).

In general codimension the situation is much more complicate, mainly because of multiplicity issues. In particular, it is possible for the limit of singular surfaces to be regular (cf. [16] for a reader-friendly introduction and [15] for a more technical treatment). The best regularity theorem available in this case is the following result.

Theorem 1.4 (Regularity in codimension $\bar{n} \geqslant 2$)**.** *Assume U, Σ and T are as in (PB′) with $\bar{n} \geqslant 2$. Then*

(i) for $m = 1$ $\mathrm{Sing}(T) \cap U$ is empty;

(ii) *for* $m = 2$ $\mathrm{Sing}(T) \cap U$ *consists of isolated points (Chang, [9]);*

(iii) *for* $m \geqslant 2$ $\mathrm{Sing}(T) \cap U$ *has Hausdorff dimension at most* $m-2$ *(Almgren, [3]);*

(iv) *the above results are optimal, indeed for every* $m \geqslant 2$ *there are area minimizing integral currents* T *in* $\mathbb{R}^{m+2}$ *for which* $\mathrm{Sing}(T)$ *has positive* $\mathcal{H}^{m-2}$ *measure (Federer, [31]).*

Some comments are now in order. Case (a) of Theorem 1.2 is exactly the same as (ii) of Theorem 1.4. The original argument of Chang is however not entirely complete since a key starting point of his analysis, the existence of the so-called "branched center manifold", is only sketched in the appendix of [9] and requires the understanding (and a suitable modification) of the most involved portion of the monograph [3]. Meanwhile Camillo De Lellis and Emanuele Spadaro revisited Almgren's theory giving a much shorter version of his program for proving point (iii) of the Theorem, cf. [17, 18, 19, 20, 21]. It seemed therefore worthy to complete and revisit Chang's result in light of this new theory.

Case (c) of Theorem 1.2 is instead entirely new and a simple consequence of it is the fact that the singular set of a 3-dimensional area minimizing cone consists of at most a finite number of lines. Notice also that this could be seen as a first step in the study of conical solutions to the Plateau problem when $m \geqslant 3$ and $\bar{n} \geqslant 2$ (cf. [57] for $\bar{n} = 1$ and [35] $m = 2, \bar{n} \geqslant 2$).

For what concerns case (b), our motivation came from a paper by Rivière and Tian ([53]), where they prove that 2-dimensional almost complex cycles in an almost complex, locally symplectic manifold (M^{2p}, J, ω), are J-holomorphic curves with multiplicity. This result is not new, indeed the locally symplectic assumption makes the almost complex cycles locally area minimizing for the metric $\omega(\cdot, J\cdot)$, and so their regularity is a consequence of Chang's theorem. However their proof is independent from [9] and is the first step in a program to generalize the statement to any almost complex manifold. Since almost complex curves are locally semicalibrated, case (b) of Theorem 1.2 completes this program. We should also remark that in dimension 2 all semicalibration admits locally an almost complex structure, cf. [5]. For further motivation about the importance of almost complex structure in geometry see [28, 44, 51, 64, 65], while for known regularity results we refer the reader to [52, 53, 63].

Later on, the approach of Rivière and Tian has been generalized by Bellettini and Rivière in [7] to handle the new case of special Legendrian cycles in $\mathbb{S}^5$. These are spherical cross sections of a class of 3-dimensional calibrated cones in $\mathbb{R}^6$, and so a subclass of both (b) and (c). However this result was not covered by Chang's result.

Finally it is worth to spend a couple of words on the notion of calibration, since it is the link between cases (a) and (b). A calibration ω is a semicalibration which is closed. Notice that if T is semicalibrated in $\mathbb{R}^{m+n}$ and S is an $(m+1)$-dimensional current, then

$$\mathbf{M}(T) = T(\omega) = T(\omega) + \partial S(\omega) - S(d\omega) \leqslant \mathbf{M}(T + \partial S) + \|d\omega\|_0\, \mathbf{M}(S)\,. \tag{1.1}$$

In particular calibrated currents are solution of the Plateau problem (PB′), although the viceversa is not true in general. An extremely important example of calibration is given by the Kähler form

$$\omega := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, with the usual identification $z_i = x_i + iy_i$. Wirtinger's inequality ([67, 32]) states that $\omega^k = \frac{1}{k!}\omega \wedge \cdots \wedge \omega$ is a calibration and complex planes are calibrated by it. It follows that complex varieties are area minimizing and, in particular, a simple generalization of the argument above allows one to prove that the following complex curves are area minimizing.

Example 1.1. *Consider the holomorphic curve*

$$\Gamma := \{(z,w) \in \mathbb{C}^2 \,:\, z = 0\} \cup \{(z,w) \in \mathbb{C}^2 \,:\, w = 0\}.$$

Then Γ is area minimizing and the origin belongs to $\mathrm{Sing}(\Gamma)$. *Moreover Γ cannot be represented as the graph of a single valued function in any neighborhood of the origin. In particular the cartesian product $\Gamma \times \mathbb{R} \subset \mathbb{R}^5$ is a 3-dimensional area minimizing cone with a line singularity.*

Example 1.2. *Consider the holomorphic curve*

$$\Theta = \{(z,w) \in \mathbb{C}^2 \,:\, (z^2 - w^3 - w^4)^2 = w^7 + w^8\}.$$

Then the same conclusions of the previous example hold for Θ and furthermore notice that Θ is a very small perturbartion of the complex curve $\{z^2 = w^3\}$ counted with multiplicity 2.

These examples prove that Theorem 1.2 is optimal, indeed notice that calibrated currents are in particular semicalibrated. For an extensive treatment of calibrated geometries we refer the reader to [41].

1.2 CONTENT OF THE THESIS

We start by explaining why it is possible to treat cases (a), (b) and (c) of Definition 1.1 together. The key properties shared by our objects are the almost minimality and the boundedness of the generalized mean curvature. In particular (1.1) holds also for (c), when we replace $\|d\omega\|$ with $(m+1)\,R^{-1}$. It doesn't hold in this form for case (a): competitors need to be supported in the manifold Σ. As a consequence of the isoperimetric inequality, a weaker form of (1.1) is true in all three cases: for any $(m+1)$-dimensional current S in $\mathbf{B}_r \subset \mathbb{R}^{n+m}$ we have

$$\mathbf{M}(T) \leqslant \mathbf{M}(T + \partial S) + C r^{m+1}\,. \tag{1.2}$$

Moreover, if δT denotes the first variation of the current T, with T as in (a), (b) or (c), then for every compactly supported vector field X,

$$|\delta T(X)| \leqslant C \int |X|\, d\|T\| < \infty\,. \tag{1.3}$$

Next we wish to discuss the strategy of the proof of Theorem 1.2. In this we follow mainly the Almgren-Chang's program, which consists of the following steps.

(i) Construct a Q-valued Lipschitz function that under suitable conditions approximates our current in a very sharp way.

(ii) Prove that the tangent cone to the current is unique at every point and consists of a union of planes with multiplicity whose support can cross only at the origin.

(iii) Construct a surface $\mathcal{M}$, which we call Center Manifold, and is basically the average of the sheets of the current. From this surface approximate very carefully the current with a map $\mathcal{N}$.

(iv) Use this new approximation to define a quantity, called frequency function $\mathbf{I}$, which enjoys some good monotonicity property. Study the asymptotic of this quantity to prove that either T coincides with $\mathcal{M}$ or there exists a rescaling of the approximation $\mathcal{N}$ which is nontrivial in the limit.

In the last part of this introduction we explain better each one of these steps, specifing in which part of the thesis they are treated and how the final result can be derived from them.

1.2.1 *Part II: Approximation of currents with* Q*-valued functions*

The first typical step of the regularity theory for objects linked to area minimization problems is an approximation result with Lipschitz function. This is due to De Giorgi's remark that the first order term in the Taylor expansion of the area of a graph of a Lipschitz function is the Dirichlet energy of the function itself, that is if $f \in \mathrm{Lip}(B_r)$ then

$$\mathrm{vol}(\mathrm{graph}(f)) = \int_{B_r} \sqrt{1+|Df|^2} \leqslant |B_r| + \frac{1}{2}\int_{B_r} |Df|^2 + C\int_{B_r} |Df|^4\,, \tag{1.4}$$

and therefore area minimizing graphs are very close to being harmonic. While in codimension 1 we can alway approximate minimal currents with vector valued functions, for higher codimensions there exist area minimizing surfaces which are not the graph of any such function in any neighborhood of a fixed point (cf. Examples 1.1 and 1.2).

For all these reasons it is important to develop a theory of Lipschitz and Sobolev multiple valued functions, that is functions taking values in the space of unordered Q-tuples of points of $\mathbb{R}^n$. This is done in Chapter 2, where, after introducing the theory of multiple valued Lipschitz functions, we use the Almgren-White's embedding of the space of Q-points in $\mathbb{R}^{N(Q)}$ to introduce Sobolev multiple valued functions, define the Dirichlet problem and study the properties of its solutions. Furthermore we explain how to associate an integral current to the image of a Q-valued map and prove that De Giorgi's remark still holds, that is the energy of a Q-valued graph is the first order term in the Taylor expansion of its mass. This chapter is mainly taken from [17] and [18]. We made the effort of proving any result that is not taken from one of these two papers. It should be observed that in [17], the authors develop the theory of multiple valued functions independently from Almgren's embedding, but in a purely intrinsic, metric way. For other interesting properties of multiple valued functions see [22] and [59].

The second chapter is devoted to the two main analytic estimates of the whole thesis. Fix a plane π and consider the cylinder $\mathbf{C}_r(x,\pi) = B_r(x,\pi)\times\pi^\perp$, where $B_r(x,\pi)$ is the ball of radius r centered in x and contained in π. The cylindrical excess $E := \mathbf{E}(T,\mathbf{C}_r(x,\pi))$ of a current with respect to π is a measure of how much the tangent space to the current in the cylinder is tilting, more precisely

$$\mathbf{E}(T,\mathbf{C}_r(x,\pi)) := (2\omega_m r^m)^{-1}\int_{\mathbf{C}_r(x,\pi)} |\vec{T}-\vec{\pi}|^2\, d\|T\|\,.$$

Assuming that E is small enough in $C_{4r}(x, \pi)$, we can prove that

Proposition 4.2: in the ball $B_r(x) \subset \pi$, there exists a Lipschitz multiple valued function $f\colon B_r(x) \subset \pi \to \pi^\perp$, whose graph and energy differ from the support and the mass of the current by $E^{1+\beta_0}$, for some $\beta_0 > 0$;

Theorem 4.8: there exists a Dir-minimizing multiple valued map $u\colon B_r(x) \subset \pi \to \pi^\perp$ whose $W^{1,2}$ norm differs from that of f by $o(E)$.

We notice that case (a) is already covered by [19], and, indeed, our original contribution is to prove the results for the cases (b) and (c). In fact, the only property that we use in this chapter is (1.1). Furthermore, the results of this part of the thesis hold for any dimension m.

For a detailed explanation of how this approximation result is proved we refer the reader to the introduction of [19, 15, 16], the only diffrence being that, whenever a comparison argument is needed, we use (1.1) instead of the minimality property, and so the choice of the filling surface S must be carefully done. This is the content of the Homotopy Lemma 4.6.

1.2.2 *Part III: Uniqueness of tangent cones*

Given a current T as in Definition 1.1 and a point $x \in \operatorname{spt}(T)$, we want to study the infinitesimal behaviour of T in x. To do this we consider the current $T_{x,r} := (\iota_{x,r})_\sharp T$, where the map $\iota_{x,r}$ is given by $\mathbb{R}^{m+n} \ni y \mapsto \frac{y-x}{r} \in \mathbb{R}^{m+n}$. Recall that an area minimizing cone S is an integral area minimizing current such that $(\iota_{0,r})_\sharp S = S$ for every $r > 0$ (cf. [54, Theorem 19.3]). Then, combining the almost monotonicity of the quantity $\mathbf{M}(T_{x,r})$ with the compactness Theorem for integral currents, one can prove that, up to subsequences, $T_{x,r} \to S$, where S is an integral cone. The difficult question is wether or not S is unique. In the 2 dimensional case we answer affirmatively to this question for all the surfaces satisfying only (1.2).

The uniqueness of tangent cones for 2-dimensional area minimizing currents has been proved first in the euclidean case by White ([66]) and then generalized to the Riemannian setting (case (a)) by Chang in [9]. The same statement for semicalibrated integral 2-dimensional cycles (case (b)) has been shown more recently by Pumberger and Rivière in [46]. As far as we know the result for spherical cross sections of 3-dimensional area minimizing cones is instead new. In codimension 1 the uniqueness of tangent cones is known at isolated singularities thanks to the pioneering work of Simon, cf. [55]. The uniqueness of tangent cones is widely open in dimension higher than 2 and general codimension. Some interesting higher dimensional cases have been recently covered by Bellettini in [5, 6].

Our approach follows very closely that of White ([66]). The key ingredient is an Epiperimetric inequality for the mass. First introduced by Reifenberg ([50]), this inequality improves the usual monotonicity inequality, and indeed the key idea is that, if in a cylinder we extend the boundary cycle of the current as an harmonic graph, then its mass is strictly less than the mass of the cone with the same boundary. This intuitevely follows again by De Giorgi's remark, that is minimizer of the Dirichlet energy in the graphical case are very close to minimizers of the area. In turn, the Epiperimetric inequality implies an exponential rate of decay of the excess, which, combined with a monotonicity identity, proves the uniqueness of the tangent cone. For a nice explanation of the proof of this inequality we refer the reader to White's paper. What is new here is a simplification in a step of its proof (although the overall procedure is the same) and its application to a larger class of surfaces, namely to all two dimensional objects satisfying only (1.2).

1.2.3 *Part I: Proof of the main result*

In order to conclude Theorem 1.2, we would like to prove that, for $x \in \mathrm{Sing}(T)$ and $r > 0$ sufficiently small, the current $T_{x,r}$ is a perturbation of a surface of the same type as the one in Example 1.2. We use therefore this example to illustrate the procedure for a general current as in Definition 1.1.

Step 1. If we rescale geometrically Θ, that is we consider the current $\Theta_{0,r}$, then in the limit for $r \to 0$ we get the complex plane $\pi := \{z = 0\}$, which is regular. By the uniqueness of the tangent cone of Part iii, combined with the structure of 2-dimensional tangent cones (they are planes whose supports intesect only at the origin, cf. Example 1.1), we can assume that this holds also for $T_{0,r}$. We call the plane π the center manifold $\mathcal{M}_0$.

Step 2. From the plane $\mathcal{M}_0$ we approximate the surface with a multiple valued function $\mathscr{N}_0$ using the Lipschitz approximation result of Part ii. We prove that, either T coincides with (a constant multiple of) $\mathcal{M}_0$, or a suitable rescaling of $\mathscr{N}_0$ converges to a unique profile g_0, which is strictly multiple valued and non-trivial, $C^{1,\alpha}$ regular in a neighborhood of the origin and $C^{3,\alpha}$ outside the origin. Moreover a horned neighborhood of this profile captures the current. In our example, the graph of the function g_0 is $\{z^2 = w^3\}$ and is obtained by rescaling Θ inhomogeneously by $z' := r^{\frac{3}{2}} z$ and $w' := r\, w$.

Step 3. We build a new center manifold surface $\mathcal{M}_1$, which is, roughly speaking, the average of the sheets of the current T restricted to the horned neighborhood of g_0. This surface enjoys the same regularity and multiplicity of the graph of g_0 and takes care of small smooth perturbations of it. In our example $\mathcal{M}_1 = \{z^2 = w^3 + w^4\}$.

Step 4. We approximate the current with a map $\mathscr{N}_1$, which is a graph on $\mathcal{M}_1$. We perform the same analisys as in Step 2, and so either $T = Q\,[\![\mathcal{M}_1]\!]$, or we find a new profile g_1 which allows to repeat Step 3. In our example we have that the graph of g_1 is $\{z^4 = w^7\}$ and is obtained by rescaling z with $r^{\frac{7}{4}}$. When we glue g_1 back on top of $\mathcal{M}_1$.

Step 5. Finally we repeat inductively Steps 2-3. Since the density of T in 0 is bounded from above, and since at each step the multiplicity of $\mathcal{M}_i$ is increasing (because each g_i is strictly multiple valued), this procedure must stop after a finite number of steps. By the regularity of each $\mathcal{M}_i$, this concludes the proof. In our examples we construct $\mathcal{M}_2 = \Theta$ and we conclude.

1.2.4 *Part IV: Center Manifold and Normal Approximation*

Here we explain how to construct $\mathcal{M}_i$, $\mathscr{N}_i$, given g_{i-1}. Assume that the graph of g_{i-1} has multiplicity $\bar{Q}$ in the singular point (for instance g_1 of the example has multiplicity 2 in 0), and that T has multiplicity $Q \cdot \bar{Q}$ ($4 = 2 \cdot 2$ in the example). We wish to construct a $\bar{Q}$-sheeted cover of the plane π which is globally the graph of a $C^{1,\alpha}$ function, $C^{3,\alpha}$ away from the singularity, and which is at every scale a good approximation of the average of the Q sheets of T captured in the horned neighborhood of g_{i-1}. First we introduce the notions of excess and height of a current. Given an m-dimensional current T in $\mathbb{R}^{m+n}$ with finite mass, its *excess* in the ball $\mathbf{B}_r(x)$ with respect to the m-plane π is

$$\mathbf{E}(T, \mathbf{B}_r(p), \pi) := (2\omega_m r^m)^{-1} \int_{\mathbf{B}_r(p)} |\vec{T} - \vec{\pi}|^2 \, d\|T\| \,. \tag{1.5}$$

In order to define the spherical excess we consider T as in Assumption 1 and we say that π *optimizes the excess* of T in a ball $\mathbf{B}_r(x)$ if

- In case (b)

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi); \tag{1.6}$$

- In case (a) and (c) $\pi \subset T_x\Sigma$ and

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau \subset T_x \Sigma} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi) \,. \tag{1.7}$$

The height of a current T in a set E with respect to a plane π is given by

$$\mathbf{h}(S, E, \pi) := \sup\{|\mathbf{p}_\pi^\perp(p - q)| : p, q \in \mathrm{spt}(S) \cap E\} \,. \tag{1.8}$$

If $E = \mathbf{C}_r(p, \pi)$ we will then set $\mathbf{h}(S, \mathbf{C}_r(p, \pi)) := \mathbf{h}(S, \mathbf{C}_r(p, \pi), \pi)$. If $E = \mathbf{B}_r(p)$, T is as in Assumption 1 and $p \in \Sigma$ (in the cases (a) and (c) of Definition 1.1), then $\mathbf{h}(T, \mathbf{B}_r(p)) := \mathbf{h}(T, \mathbf{B}_r(p), \pi)$ where π gives the minimal height among all π for which $\mathbf{E}(T, \mathbf{B}_r(p), \pi) = \mathbf{E}(T, \mathbf{B}_r(p))$ (and such that $\pi \subset T_p\Sigma$ in case (a) and (c) of Definition 1.1).

The procedure for the construction of $\mathcal{M}_i$ is then the following.

Step 1. We first make a Whitney decomposition of (a model space for) g_{i-1}. Let $L \subset \mathrm{graph}(g_{i-1})$ be a cube and let T_L be the part of the current T captured by the horned neighborhood of g_{i-1} around L. Moreover let $\ell(L)$ be the sidelength of L and $d(L)$ its distance from the singularity. Then, we ask that the refinement procedure stops if either

$$E_L := \mathbf{E}(T_L, \mathbf{B}_L) > C_e \mathbf{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2 - 2\delta_1}; \tag{1.9}$$

or

$$\mathbf{h}_L := \mathbf{h}(T_L, \mathbf{B}_L) > C_h \mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1 + \beta_2}; \tag{1.10}$$

where $\gamma_0, \beta_2, \delta_1, C_h, C_e$ are some parameters and $\mathbf{m}_0$ is a geometrical quantity as small as we want (we will also need another technical compatibility condition). One could in fact conjecture that the condition on the height is not really needed.

Step 2. By Step 1, in every final cube L the excess is very small, and so we can apply the Lipschitz approximation Theorem of Chapter 4 to get a Q-valued Lipschitz map f_L, which is a good approximation of the Q sheets of T_L.

Step 3. We then consider the average of f_L, denoted by $\boldsymbol{\eta} \circ f_L$, which is a single valued function and satisfies, in a ball around L, the inequality $|\mathcal{L}(\boldsymbol{\eta} \circ f_L)| \leqslant C\, E_L^{1+\eta_0}$, where $\mathcal{L}$ is a linear elliptic operator.

Step 4. We consider the precise solution h_L to the elliptic system $\mathcal{L}$ with boundary datum $\boldsymbol{\eta} \circ f_L$ (which is a perturbation of the Laplace equation, and so admits a solution).

Step 5. We prove quantitative regularity estimates for h_L, using the Lipschitz regularity of f_L, the decay of its energy (since E_L is decaying by Step 1), the fact that $\mathcal{L}$ is a perturbation of the laplacian and $\mathcal{L}(h_L - \boldsymbol{\eta} \circ f_L) \leqslant C\, E_L^{1+\eta_0}$.

Step 6. We patch all the h_L together and prove the regularity of the resulting function, whose image is $\mathcal{M}_i$.

For what concerns the construction of $\mathscr{N}_i$, the basic idea is to use on each cube the portion of f_L that coincides with T_L, reparametrize it on $\mathcal{M}_i$ and then extend to the whole domain. We can do this because the C^1 norm of $\mathcal{M}_i$ is small and the Lipschitz constant of f_L is also small. With this procedure, we manage to bound the height of $\mathscr{N}_i$, its average $\boldsymbol{\eta} \circ \mathscr{N}_i$ and its difference from T, in every cube L, with suitable powers of E_L and $\mathbf{h}_L$. However, we want this estimates in terms of the $W^{1,2}$ norm of $\mathscr{N}_i$, and, since by Step 1, E_L is bounded by a power of $\ell(L)$, we need to control the energy and the height of $\mathscr{N}_i$ from below with a power of $\ell(L)$. This is achieved in the sections called vertical separation and splitting before tilting.

In conclusion, if we denote with $\mathbf{D}(r) := \int_{B_r} |D\mathscr{N}_i|^2$, $\mathbf{H}(r) := \int_{\partial B_r} |\mathscr{N}_i|^2$ and we define $\mathscr{F}_i(x) := x + \mathscr{N}_i(x)$ for every $x \in \mathcal{M}_i$, we achieve, roughly speaking, the following estimates

$$\begin{aligned}
&\mathrm{Lip}(\mathscr{N}_i) \leqslant \mathbf{D}(r)^{\eta_0} \\
&\int_{B_r} \frac{|\boldsymbol{\eta} \circ \mathscr{N}_i|(z)}{|z|^{1-\alpha_0}} \leqslant \mathbf{D}(r)^{\eta_0}(\mathbf{D}(r) + \mathbf{H}(r)) \\
&\|T - \mathbf{T}_{\mathscr{F}_i}\|(\mathbf{C}_r) \leqslant \mathbf{D}(r)^{\eta_0}(\mathbf{D}(r) + \mathbf{H}(r))\,.
\end{aligned} \tag{1.11}$$

Finally we wish to point out the main differences with the construction in [20].

- The uniqueness of the tangent cone implies the decay of the excess, and this allows us to construct a single center manifold that works at every scale (this is not known in dimension higher than 2). A key consequence of this is that the sidelength of each cube is less than its distance from the singularity.

- Our surface $\mathcal{M}_i$ enjoys less regularity than the one there, indeed it is branched in the singularity to resemble the current itself.

- The elliptic PDE satisfied by $\boldsymbol{\eta} \circ f_L$ in case (b) is more complicate then the one satisfied in the minimizing case (a), and indeed the building blocks in the construction of $\mathcal{M}_i$ are defined differently than in [20].

- We need to make sure that $\mathcal{M}_i$ is captured in the horned neighborhood of g_{i-1}, so that an horned neighborhood of $\mathcal{M}_i$ contains the current T.

- At the end of the construction, we will reparametrize $\mathcal{M}_i$ in a conformal way, to make the asymptotic analysis of Part v easier.

1.2.5 *Part V: Blow up Analysis*

Finally, we explain how to construct g_i from $\mathcal{M}_i$ and $\mathscr{N}_i$. As already remarked, the idea is to rescale T inhomogeneously, that is to rescale $\mathscr{N}_i$ by a suitable power of r and consider the limit for $r \to 0$. To guess the right power, the fundamental tool introduced by Almgren is the so called Frequency Function $\mathbf{I}(r) := \frac{r\mathbf{D}(r)}{\mathbf{H}(r)}$. This quantity would be non decreasing, if $\mathscr{N}_i$ was a Dir-minimizing function. Even though this is not true, a modification of (1.4) for $\mathscr{F}_i$, leads to

$$\|\mathbf{T}_{\mathscr{F}_i}\|(\mathbf{C}_r) \leqslant Q\,|B_r| + \int_{B_r} \frac{|\boldsymbol{\eta} \circ \mathscr{N}_i|(z)}{|z|^{1-\alpha_0}} + \frac{1}{2}\mathbf{D}(r) + \mathbf{D}(r)^{1+\eta_0}\,,$$

so that, by (1.11),

$$\|T\|(\mathbf{C}_r) \leqslant Q\,|B_r| + \frac{1}{2}\mathbf{D}(r) + \mathbf{D}(r)^{1+\eta_0}\,.$$

Using this together with the almost minimality of T, one can prove an almost minimality property in terms of the energy for $\mathscr{N}_i$, that is

$$\mathbf{D}(r) \leqslant \int_{B_r} |D\mathscr{L}|^2 + C\,\mathbf{D}(r)^{1+\eta_0} \quad \text{for every Lipschitz competitor } \mathscr{L}. \tag{1.12}$$

The almost Dir-minimality condition is then used, together with a competitor argument similar to the one of Part 3, to prove a Poincaré inequality $\mathbf{H}(r) \leqslant C\,\mathbf{D}(r)$ and a sort of Epiperimetric inequality for the energy of $\mathscr{N}_i$. The Poincaré inequality implies that $\mathbf{I}(r)$ is bounded from below and also that all the errors of the form $\mathbf{H}(r)$ can be translated in terms of $\mathbf{D}(r)$, so that the estimates (1.11) become

$$\mathrm{Lip}(\mathscr{N}_i) \leqslant \mathbf{D}(r)^{\eta_0}, \quad \int_{B_r} \frac{|\boldsymbol{\eta} \circ \mathscr{N}_i|(z)}{|z|^{1-\alpha_0}} \leqslant \mathbf{D}(r)^{1+\eta_0} \quad \text{and} \quad \|T - \mathbf{T}_{\mathscr{F}_i}\|(\mathbf{C}_r) \leqslant \mathbf{D}(r)^{1+\eta_0}\,. \tag{1.13}$$

Next we observe that, thanks to 1.3, the interior and exterior variations of $\mathscr{N}_i$ are once again perturbation of the Dir-minimizing ones, with errors of type $\mathbf{D}(r)^{1+\eta_0}$, and this allows us to prove the almost monotonicity of $\mathbf{I}(r)$. Using this and the estimates (1.13), we prove the dichotomy: either T coincides with $Q\,[\![\mathcal{M}_i]\!]$, or $I_0 := \lim_{r\to 0} \mathbf{I}(r) < \infty$. On the other hand, combining this with the Epiperimetric inequality allows us to prove that, if we set $\mathscr{N}_r(z) := \frac{\mathscr{N}_i(rz)}{r^{I_0}}$, than there exists a unique limit g_i as $r \to 0$.

Finally, by (1.11), we see immediately that g_i is nontrivial and $\boldsymbol{\eta} \circ g_i = 0$, so that it must be strictly multiple valued, and, by the almost minimality of $\mathscr{N}_i$ (cf. (1.12)), g_i is a Dir-minimizer. The last part of the argument involves a careful use of the decay property of $\mathbf{D}(r), \mathbf{H}(r)$ and of the Lipschitz regularity of $\mathscr{N}_i$, to prove first that $\mathscr{N}_r$ converges to g_i uniformly with a rate depending on r, and then, using again the estimates (1.13) and the monotonicity formula for T, that T is captured in an horned neighborhood of g_i.

Finally we wish to point out that, although the structure of this part is analogous to the one in [9], there are two main differences.

- Since the PDE associated to case (b) is more complex than the one in case (a), in order to prove the dichotomy, we need to modify $\mathbf{I}(r)$ following the ideas of Garofalo-Lin in [38] and [37].
- The almost Dir-minimality of $\mathscr{N}_i$ is much more difficult to prove in cases (b) and especially (c), than in case (a).

Part I

MAIN STEPS

2 PROOF OF THE MAIN RESULT

In this chapter, after setting some basic notations, we prove Theorem 1.2 making use of the tools that will be proved in the subsequent chapters.

2.1 PRELIMINARIES

2.1.1 *Basic notations*

We use the notation $\langle , \rangle$ for: the euclidean scalar product, the naturally induced inner products on p-vectors and p-covectors and the duality pairing of p-vectors and p-covectors; we instead restrict the use of the symbol $\cdot$ to matrix products. Given a C^1 m-dimensional submanifold $\Sigma \subset \mathbb{R}^{m+n}$, a function $f : \Sigma \to \mathbb{R}^k$ and a vector field X tangent to Σ, we denote by $D_X f$ the derivative of f along X, that is $D_X f(p) = (f \circ \gamma)'(0)$ whenever γ is a smooth curve on Σ with $\gamma(0) = p$ and $\gamma'(0) = X(p)$. When $k = 1$, we denote by ∇f the vector field tangent to Σ such that $\langle \nabla f, X \rangle = D_X f$ for every tangent vector field X. For general k, $Df|_x : T_x\Sigma \to \mathbb{R}^k$ will be the linear operator such that $Df|_x \cdot X(x) = D_X f(x)$ for any tangent vector field X. We write Df for the map $x \mapsto Df|_x$ and sometimes we will also use the notation $Df(x)$ in place of $Df|_x$. Having fixed an orthonormal base $e_1, \dots e_m$ on $T_x\Sigma$ and letting $(f_1, \dots, f_k)$ be the components of f, we can write $\nabla f_i = \sum_{j=1}^m a_{ij} e_j$ and $|Df|$ for the usual Hilbert-Schmidt norm:

$$|Df|^2 = \sum_{j=1}^{m} |D_{e_j} f|^2 = \sum_{i=i}^{k} |\nabla f_i|^2 = \sum_{i,j} a_{ij}^2 \, .$$

All the notation above is extended to the differential of Lipschitz multiple valued functions at points where they are differentiable in the sense of Definition 3.5: although the definition in there is for euclidean domains, its extension to C^1 submanifolds $\Sigma \subset \mathbb{R}^{m+n}$ is done, as usual, using coordinate charts.

We will keep the same notation also when $f = Y$ is a vector field, i.e. takes values in $\mathbb{R}^{m+n}$, the same Euclidean space where Σ is embedded. In that case we define additionally $\mathrm{div}_\Sigma Y := \sum_i \langle D_{e_i} Y, e_i \rangle$. Moreover, when Y is tangent to Σ, we introduce the covariant derivative $D_\Sigma Y|_x$, i.e. a linear map from $T_x\Sigma$ into itself which gives the tangential component of $D_X Y$. Thus, if we denote by $\mathbf{p}_x : \mathbb{R}^N \to T_x\Sigma$ the orthogonal projection onto $T_x\Sigma$, we have $D_\Sigma Y|_x = \mathbf{p}_x \cdot DY(x)$. It follows that $D_\Sigma Y \cdot X = \nabla_X Y$, where we use ∇ for the connection (or covariant differentiation) on Σ compatible with its structure as Riemannian submanifold of $\mathbb{R}^{m+n}$. Such covariant differentiation is then extended in the usual way to general tensors on Σ.

When dealing with C^2 submanifolds Σ of $\mathbb{R}^{m+n}$ we will denote by A_Σ the following tensor: $A_\Sigma|_x$ as a bilinear map on $T_x\Sigma \times T_x\Sigma$ taking values on $T_x\Sigma^\perp$ (the orthogonal complement of $T_x\Sigma$) and if X and Y are vector fields tangent to Σ, then $A_\Sigma(X, Y)$ is the normal component of

$D_X Y$, which we will denote by $D_X^\perp Y$. A_Σ is called second fundamental form by some authors (cf. [54, Section 7], where the tensor is denoted by B) and we will use the same terminology, although in differential geometry it is more customary to call A_Σ "shape operator" and to use "second fundamental form" for scalar products $\langle A_\Sigma(X,Y),\eta\rangle$ with a fixed normal vector field (cf. [27, Chapter 6, Section 2] and [60, Vol. 3, Chapter 1]). In addition, H_Σ will denote the trace of A_Σ (i.e. $H_\Sigma = \sum_i A_\Sigma(e_i,e_i)$ where $e_1,\dots,e_m$ is an orthonormal frame tangent to Σ) and will be called *mean curvature*. Moreover $\mathbf{A}_\Sigma$ and $\mathbf{H}_\Sigma$ will denote respectively the L^∞ norm of A_Σ and H_Σ.

With $\mathbf{B}_r(p)$ and $B_r(x)$ we denote, respectively, the open ball with radius r and center p in $\mathbb{R}^{m+n}$ and the open ball with radius r and center x in $\mathbb{R}^m$. $\mathbf{C}_r(p)$ and $\mathbf{C}_r(x)$ will always denote the cylinder $B_r(x)\times\mathbb{R}^n$, where $p=(x,y)\in\mathbb{R}^m\times\mathbb{R}^n$. We will often need to consider cylinders whose bases are parallel to other m-dimensional planes, as well as balls in m-dimensional affine planes. We then introduce the notation $B_r(p,\pi)$ for $\mathbf{B}_r(p)\cap(p+\pi)$ and $\mathbf{C}_r(p,\pi)$ for $B_r(p,\pi)+\pi^\perp$. e_i will denote the unit vectors in the standard basis, π_0 the (oriented) plane $\mathbb{R}^m\times\{0\}$ and $\vec{\pi}_0$ the m-vector $e_1\wedge\cdots\wedge e_m$ orienting it. Given a m-dimensional plane π, we denote by $\mathbf{p}_\pi$ and $\mathbf{p}_\pi^\perp$ the orthogonal projections onto, respectively, π and its orthogonal complement $\pi^\perp$. For what concerns integral currents we use the definitions and the notation of [54]. Since π is used recurrently for m-dimensional planes, the m-dimensional area of the unit circle in $\mathbb{R}^m$ will be denoted by ω_m.

2.1.2 *First assumptions*

By the following Lemma, in case (b) of Definition 1.1, we can assume, without loss of generality, that the ambient manifold Σ coincides with the euclidean space $\mathbb{R}^{2+n}$.

Lemma 2.1. *Let $k\in\mathbb{N}\setminus\{0\}$, $\varepsilon_0\in[0,1]$, $\Sigma\subset\mathbb{R}^{m+n}$ be a C^{k+1,ε_0} $m+\bar{n}$-dimensional submanifold, $V\subset\mathbb{R}^{m+n}$ an open subset and ω a C^{k,ε_0} m-form on $V\cap\Sigma$. If T is a cycle in $V\cap\Sigma$ semicalibrated by ω, then T is semicalibrated in V by a C^{k,ε_0} form $\tilde\omega$.*

Proof. The argument is straightforward: we just need to extend ω to a form $\tilde\omega$ on the open set V in such a way that $\|\tilde\omega_x\|_c\leqslant 1$ for every x and the regularity of ω is preserved. Without loss of generality it suffices to do this on a tubular neighborhood U of $\Sigma\cap V$ on which there is a C^{k,ε_0} orthogonal projection $\mathbf{p}:U\to\Sigma\cap U$ (we then multiply this extension by a function $\varphi\in C^\infty_c(U)$ which is identically 1 on Σ and satisfies $0\leqslant\varphi\leqslant 1$; the resulting form can then be extended to V by setting it equal to 0 where it is not yet defined). For $x\in U$ we set $y:=\mathbf{p}(x)\in\Sigma$ and let $\mathbf{p}_y:\mathbb{R}^{m+n}\to T_y\Sigma$ be the orthogonal projection. We then set $\tilde\omega_x(v_1,\dots,v_m)=\omega_y(\mathbf{p}_y(v_1),\dots,\mathbf{p}_y(v_m))$. Observe that $\tilde\omega$ *is not* $\mathbf{p}^\sharp\omega$ (in general the latter would not satisfy $\|\tilde\omega_x\|_c\leqslant 1$). □

In particular, for the rest of the work we will make the following assumptions.

Assumptions 1. T *is an integral current of dimension 2 with bounded support and it satisfies one of the three conditions (a), (b) or (c) in Definition 1.1. Moreover*

- *In case (a), $\Sigma\subset\mathbb{R}^{2+n}$ is a C^{3,ε_0} submanifold of dimension $2+\bar{n}=2+n-l$, which is the graph of an entire function $\Psi:\mathbb{R}^{2+\bar{n}}\to\mathbb{R}^l$ and satisfies the bounds*

$$\|D\Psi\|_0\leqslant c_0 \quad \textit{and} \quad \mathbf{A}:=\|A_\Sigma\|_0\leqslant c_0, \tag{2.1}$$

where c_0 is a positive (small) dimensional constant and $\varepsilon_0 \in]0,1[$.

- *In case (b) we assume that $\Sigma = \mathbb{R}^{2+n}$ and that the semicalibrating form ω is a C^{2,ε_0} m-form.*
- *In case (c) we assume that T is supported in $\Sigma = \partial \mathbf{B}_R(p_0)$ for some p_0 with $|p_0| = R$, so that $0 \in \partial \mathbf{B}_R(p_0)$. We assume also that $T_0 \partial \mathbf{B}_R(p_0)$ is $\mathbb{R}^{2+n-1}$ (namely $p_0 = (0, \ldots, 0, \pm|p_0|)$) and we let $\Psi : \mathbb{R}^{2+n-1} \to \mathbb{R}$ be a smooth extension to the whole space of the function which describes Σ in $\mathbf{B}_2(0)$. We assume then that (2.1) holds, which is equivalent to the requirement that R^{-1} be sufficiently small.*

2.1.3 *Properties of (b) & (c)*

In some cases it will be convenient to regard cases (b) and (c) of Definition 1.1 as a particular type of almost area minimizing currents with bounded mean curvature.

Proposition 2.2. *Let T be as in Definition 1.1 (b) (in which case we assume $\Sigma = \mathbb{R}^{m+n}$) or (c). Then there is a constant $\mathbf{\Omega}$ such that*

$$\mathbf{M}(T) \leqslant \mathbf{M}(T + \partial S) + \mathbf{\Omega}\, \mathbf{M}(S) \qquad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}) \quad \textit{with compact support.} \tag{2.2}$$

In particular, $\mathbf{\Omega} \leqslant \|d\omega\|_0$ in case (b) and $\mathbf{\Omega} \leqslant (m+1)R^{-1}$ in case (c).

Moreover, if $\chi \in C^\infty_c(\mathbb{R}^{m+n} \setminus \mathrm{spt}(\partial T), \mathbb{R}^{m+n})$, we have

$$\delta T(\chi) = T(d\omega \,\lrcorner\, \chi) \textit{in case (b),} \tag{2.3}$$

$$\delta T(\chi) = \int mR^{-1}\, x \cdot \chi(x)\, d\|T\|(x) \textit{in case (c),} \tag{2.4}$$

where $\delta T(\chi)$ denotes the first variation of T along the vector field χ (cf. Section 3.3.2)

Proof. We first prove (2.2). Assume we are in case (c). Without loss of generality we can assume $x = 0$ and $R = 1$. Therefore fix S compactly supported and consider $W = T + \partial S$. Next, let $p : \mathbb{R}^{m+n} \to \overline{\mathbf{B}}_1(0)$ be the orthogonal projection and set $S' = p_\sharp S$ and $W' := p_\sharp W = T + \partial p_\sharp S$ (where the latter identity holds because $\mathrm{spt}(T) \subset \partial \mathbf{B}_1(0)$). The current $Z := 0 \times\!\!\!\!\times W' - S'$ is then a competitor for the minimality of $0 \times\!\!\!\!\times T$ and observe, moreover, that since $\mathrm{spt}(W') \subset \overline{B}_1(0)$, we have $\mathbf{M}(Z) \leqslant (m+1)^{-1}\mathbf{M}(W')$. Then we have

$$\begin{aligned} 0 \leqslant& (m+1)(\mathbf{M}(Z) - \mathbf{M}(0 \times\!\!\!\!\times T)) \leqslant \mathbf{M}(W') - \mathbf{M}(T) + (m+1)\mathbf{M}(S') \\ \leqslant& \mathbf{M}(W) - \mathbf{M}(T) + (m+1)\mathbf{M}(S)\,. \end{aligned}$$

In case (b), if ω is the semicalibrating form, we can then estimate

$$\mathbf{M}(T) = T(\omega) = W(\omega) - \partial S(\omega) \leqslant \mathbf{M}(W) - S(d\omega) \leqslant \mathbf{M}(W) + \|d\omega\|_0 \mathbf{M}(S)\,.$$

Next, (2.4) is simply the stationarity of T in $\partial \mathbf{B}_1(0)$. As for (2.3), the formula seems new in the literature and we provide here a simple proof. Fix χ and consider the maps $\Phi_t(x) := x + t\chi(x)$ and $\Lambda(t, x) = \Phi_t(x)$. We then denote by $[\![0, \varepsilon]\!]$ the current in $\mathbf{I}_1(\mathbb{R})$ induced by the oriented segment $\{t : 0 \leqslant t \leqslant \varepsilon\}$. We define $T_\varepsilon := (\Phi_\varepsilon)_\sharp T$ and $S_\varepsilon := \Lambda_\sharp([\![0, \varepsilon]\!] \times T)$. We then have $\partial S_\varepsilon = T_\varepsilon - T$ and hence

$$\mathbf{M}(T_\varepsilon) - \mathbf{M}(T) \geqslant T_\varepsilon(\omega) - T(\omega) = S_\varepsilon(d\omega) = [\![0, \varepsilon]\!] \times T(\Lambda^\sharp d\omega) =: h(\varepsilon)\,. \tag{2.5}$$

Since h is C^1 and $h(0)=0$, by a Taylor expansion we conclude $\varepsilon\delta T(\chi)\geqslant \varepsilon h'(0)+o(\varepsilon)$. On the other hand, since the latter inequality is valid for both positive and negative ε, we infer $\delta T(\chi)=h'(0)$. We thus only need to show the identity $h'(0)=T(d\omega\lrcorner\chi)$. Consider the set of ordered multiindices $I=\{1\leqslant i_1<i_2<\ldots<i_{m+1}\}$ and let $d\omega=\sum f_I dx^I$, where $dx^I=dx^{i_1}\wedge\ldots\wedge dx^{i_{m+1}}$. We then have

$$(\Lambda^\sharp d\omega)_{(x,t)}=\sum f_I(\Phi_t(x))d\Phi_t^{i_1}\wedge\ldots\wedge d\Phi_t^{i_{m+1}}\,.$$

Next, we will denote by $o(1)$ any continuous function of x and t which vanish at $t=0$ and we let $\pi:\mathbb{R}\times\mathbb{R}^{m+n}\to\mathbb{R}^{m+n}$ be the projection $\pi(t,x)=x$. Since $\Phi(0,x)=x$ and f_I is continuous we conclude

$$\begin{aligned}
&(\Lambda^\sharp d\omega)_{(x,t)}=\sum f_I(x)d\Phi_t^{i_1}\wedge\ldots\wedge d\Phi_t^{i_{m+1}}+o(1)=\\
&\sum_I f_I(x)\Big(dx^I+\sum_{1\leqslant j\leqslant m+1} f_I(x)\chi^{i_j}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_{j-1}}\wedge dt\wedge dx^{i_{j+1}}\wedge\ldots\wedge dx^{m+1}\Big)+o(1)\\
&=\pi^\sharp d\omega+dt\wedge\sum_I f_I(x)\sum_j(-1)^j\chi^{i_j}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_{j-1}}\wedge dx^{i_{j+1}}\wedge\ldots\wedge dx^{m+1}+o(1)\,.
\end{aligned}$$

Thus,

$$(\Lambda^\sharp d\omega)_{(x,t)}=\pi^\sharp d\omega+dt\wedge\pi^\sharp(d\omega\lrcorner\chi)+o(1)\,.$$

In particular, since $d\omega$ is orthogonal to dt, we have $[\![0,\varepsilon]\!]\times T(\pi^\sharp d\omega)=0$. Thus we can write

$$h(\varepsilon)=[\![0,\varepsilon]\!]\times T(dt\wedge\pi^\sharp(d\omega\lrcorner\chi))+o(1)\varepsilon\mathbf{M}(T)=\varepsilon T(d\omega\lrcorner\chi)+o(\varepsilon)\,,$$

from which we finally conclude $h'(0)=T(d\omega\lrcorner\chi)$. □

As an easy consequence of this proposition and the regularity of Σ we can prove that all the objects of definition 1.1 are almost minimizers in a classical sense.

Proposition 2.3. *Under the assumptions of Definition 1.1, any m-dimensional current T as in (a), (b) or (c) is almost minimizing in the sense that for every $x\notin \operatorname{spt}(\partial T)$ there are constants $C_0, r_0, \alpha_0>0$ such that*

$$\|T\|(\mathbf{B}_r(x))\leqslant\|T+\partial S\|(\mathbf{B}_r(x))+C_0\,r^{m+\alpha_0}\tag{2.6}$$

for all $0<r<r_0$ and for all integral $(m+1)$-dimensional currents S supported in $\mathbf{B}_r(x)$.

Proof. **Case (a).** Consider $x\in\Sigma$ and a ball $\mathbf{B}_r(x)\subset\mathbb{R}^{m+n}$. If $\bar r$ is sufficiently small there is a well-defined C^1 orthogonal projection $\mathbf{p}:\mathbf{B}_{\bar r}(x)\to\Sigma$ with the property that $\operatorname{Lip}(\mathbf{p})\leqslant 1+C\mathbf{A}r$, where C is a geometric constant and $\mathbf{A}$ denotes the L^∞ norm of the second fundamental form of Σ. Consider T area-minimizing in Σ and assume $\bar r<\operatorname{dist}(x,\operatorname{spt}(\partial T))$. Let $r\leqslant\bar r$ and $S\in\mathbf{I}_{m+1}(\mathbb{R}^{m+n})$ be such that $\operatorname{spt}(S)\subset\mathbf{B}_r(x)$. We set $W:=T+\partial S$. If $\|W\|(\mathbf{B}_r(x))\geqslant\|T\|(\mathbf{B}_r(x))$ there is nothing to prove, otherwise by the standard monotonicity formula we have $\|W\|(\mathbf{B}_r(x))\leqslant\|T\|(\mathbf{B}_r(x))\leqslant Cr^m$. Then $W':=\mathbf{p}_\sharp W$ is an admissible competitor for the minimality property of T and we have

$$\|T\|(\mathbf{B}_r(x))\leqslant\|W'\|(\mathbf{B}_r(x))\leqslant(\operatorname{Lip}(\mathbf{p}))^m\|W\|(\mathbf{B}_r(x))\leqslant\|W\|(\mathbf{B}_r(x))+Cr^{m+1}\,.$$

Case (b)&(c). First observe that, by Lemma 2.1, in case (b) we can assume, w.l.o.g., that $\Sigma = \mathbb{R}^{m+n}$. Fix $r < \mathrm{dist}(x, \mathrm{spt}(\partial T))$ and let $S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n})$ be such that $\mathrm{spt}(S) \subset \mathbf{B}_r(x)$. As above, either $\|W\|(\mathbf{B}_r(x)) \geqslant \|T\|(\mathbf{B}_r(x))$, in which case there is nothing to prove, otherwise by the standard monotonicity formula we have $\|W\|(\mathbf{B}_r(x)) \leqslant \|T\|(\mathbf{B}_r(x)) \leqslant Cr^m$ (observe that, by (2.3) and (2.4), T induces a varifold with bounded mean curvature, which in turn implies Allard's monotonicity formula, cf. [54, Section 17]). In the latter case, by the isoperimetric inequality there exists $S' \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n})$ such that

$$\partial S' = \partial S \quad \text{and} \quad \mathbf{M}(S') \leqslant Cr^{m+1}\,.$$

Applying now (4.1) to this current S' we get the desired conclusion, with $C_1 = C\mathbf{\Omega}$. □

Remark 2.4. Observe that we have achieved (2.6) with any fixed $r_0 < \frac{1}{2}\mathrm{dist}(x, \mathrm{spt}(\partial T))$, $\alpha_0 = 1$ and $C_0 = C\mathbf{A}$, in case (a), $C_0 = C\mathbf{\Omega}$, in the cases (b) and (c), where the constant C depends only upon $\|T\|(B_{2r_0})(x)$.

Finally they preserve their property under opportune decompositions.

Proposition 2.5. *Let* T *be as in Definition 1.1*($\diamondsuit$), *with* $\diamondsuit = a, b$ *or* c, *and suppose that there are* $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$, $\bar r > 0$ *and* J *currents* $T^1, \dots, T^J$ *such that*

$$T \llcorner \mathbf{B}_{\bar r}(x) = \sum_{j=1}^{J} T^j, \quad \partial T^j \llcorner \mathbf{B}_{\bar r}(x) = 0 \quad \textit{and} \quad \|T\|(\mathbf{B}_{\bar r}(x)) = \sum_{j=1}^{J} \|T^j\|(\mathbf{B}_{\bar r}(x))\,.$$

Then each T^j *satisfies* ($\diamondsuit$) *in Definition 1.1.*

Proof. We divide the proof in the three cases of Definition 1.1.

(a) Suppose by contradiction that there exist $j \in \{1, \dots, J\}$ and $S \in \mathbf{I}_{m+1}(\Sigma)$ with $\mathrm{spt}(T) \subset \mathbf{B}_{\bar r}(x)$ such that $\mathbf{M}(T^j \llcorner \mathbf{B}_{\bar r}(x)) > \mathbf{M}(T^j \llcorner \mathbf{B}_{\bar r}(x) + \partial S)$. Then it is straightforward to check that $\mathbf{M}(T \llcorner \mathbf{B}_{\bar r}(x) + \partial S) < \mathbf{M}(T \llcorner \mathbf{B}_{\bar r}(x))$, which contradicts the minimality of T.

(b) By contradiction, suppose there exists $j \in \{1, \dots, J\}$ such that T^j is not semicalibrated by ω. Assume $j = 1$. Then since $\|\omega\|_c \leqslant 1$, we have $T^1(\omega) < \|T^1\|(B_{\bar r}(x))$ and $T^j(\omega) \leqslant \|T^j\|(B_{\bar r}(x))$, for every $j \in \{2, \dots, J\}$. It follows that

$$\|T\|(\mathbf{B}_{\bar r}(x)) = T(\omega) = \sum_{j=1}^{J} T^j(\omega) < \sum_{j=1}^{J} \|T^j\|(\mathbf{B}_{\bar r}(x)) = \|T\|(\mathbf{B}_{\bar r}(x)))$$

which gives a contradiction and concludes the proof.

(c) Without loss of generality we can assume $x = 0$ and $R = 1$. Again by contradiction assume there exist $j \in \{1, \dots, J\}$ and $S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n})$ such that $\partial(S \llcorner C) = \partial(0 \mathbin{⨳} T^j \llcorner C)$ and $\mathbf{M}(S \llcorner C) < \mathbf{M}(0 \mathbin{⨳} T^j \llcorner C)$, where

$$C := \left\{\lambda z \,:\, z \in B_{\bar r}(x) \cap \partial \mathbf{B}_1(0),\ \lambda \in]0, 1[\right\}.$$

We can assume $j = 1$. Notice also that

$$\mathbf{M}((0 \mathbin{⨳} T) \llcorner C) = \frac{1}{m}\|T\|(\mathbf{B}_{\bar r}(x)) = \frac{1}{m}\sum_{j=1}^{J} \|T^j\|(\mathbf{B}_{\bar r}(x)) = \sum_{j=1}^{J} \mathbf{M}((0 \mathbin{⨳} T^j) \llcorner C). \tag{2.7}$$

Then we have

$$\begin{aligned}\mathbf{M}((0 \times\!\!\!\!\times T)\llcorner C) &\leqslant \mathbf{M}\Big(\Big(S + \sum_{j=2}^{J} 0 \times\!\!\!\!\times T^j\Big)\llcorner C\Big) \leqslant \mathbf{M}(S\llcorner C) + \mathbf{M}\Big(\sum_{j=2}^{J}(0 \times\!\!\!\!\times T^j)\llcorner C\Big)\\ &< \mathbf{M}((0 \times\!\!\!\!\times T^1)\llcorner C) + \mathbf{M}\Big(\sum_{j=2}^{J}(0 \times\!\!\!\!\times T^j)\llcorner C\Big) \overset{(2.7)}{=} \mathbf{M}((0 \times\!\!\!\!\times T)\llcorner C).\end{aligned}$$

The latter is a contradiction and thus completes the proof. □

2.2 ALMGREN'S LIPSCHITZ APPROXIMATION

Just for this section we will assume that for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leqslant 1$) and some positive integer Q,

$$\mathbf{p}_\sharp T = Q\,[\![B_{4r}(x)]\!] \quad \text{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0\,. \tag{2.8}$$

Definition 2.6 (Excess measure)**.** For a current T as in Assumption 1, which additionally satisfies (2.8), we define the *cylindrical excess* $\mathbf{E}(T, \mathbf{C}_r(x))$, the *excess measure* $\mathbf{e}_T$ and its *density* $\mathbf{d}_T$:

$$\begin{aligned}&\mathbf{E}(T, \mathbf{C}_r(x)) := \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m r^m} - Q,\\ &\mathbf{e}_T(A) := \|T\|(A \times \mathbb{R}^n) - Q\,|A| \qquad \text{for every Borel } A \subset B_r(x),\\ &\mathbf{d}_T(y) := \limsup_{s\to 0} \frac{\mathbf{e}_T(B_s(y))}{\omega_m\, s^m} = \limsup_{s \to 0} \mathbf{E}(T, \mathbf{C}_s(y)),\end{aligned}$$

where ω_m is the measure of the m-dimensional unit ball (the subscripts $_T$ will be omitted if clear from the context).

Remark 2.7. Later on we will give a different definition of cylindrical excess $\mathbf{E}$ (cf. Definition 2.13). However, if (2.8) holds, then the two notions coincide.

Although its role will not be apparent in this first chapter, a fundamental tool for the proof of Theorem 1.2 is the following strong Lipschitz approximation result. Notice that, since here the dimension 2 doesn't play any role, we state the Theorem for any dimension m.

Theorem 2.8. *There exist costants* $M, C_{21}, \beta_0, \varepsilon_{11} > 0$ *(depending on* $m, n, \bar{n}, Q$*) with the following property. Assume that* T *satisfies Assumption 1 and* (2.8) *in the cylinder* $\mathbf{C}_{4r}(x)$ *and* $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{11}$*. Then, there exist a map* $f\colon B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$*, with* $\mathrm{spt}(f(x)) \subset \Sigma$ *for every* x*, and a closed set* $K \subset B_r(x)$ *such that*

$$\mathrm{Lip}(f) \leqslant C_{21} E^{\beta_0} + C_{21}\,\mathbf{\Omega}\, r \quad \textit{in case (a) and (c)}\,, \tag{2.9}$$

$$\mathrm{Lip}(f) \leqslant C_{21} E^{\beta_0} \quad \textit{in case (b)}\,, \tag{2.10}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \textit{and} \quad |B_r(x) \setminus K| \leqslant C_{21} E^{\beta_0}\big(E + r^2 \mathbf{\Omega}^2\big) r^m\,, \tag{2.11}$$

$$\Big|\|T\|\,(\mathbf{C}_r(x)) - Q\omega_m r^m - \frac{1}{2}\int_{B_r(x)} |Df|^2\Big| \leqslant C_{21} E^{\beta_0}\big(E + r^2\mathbf{\Omega}^2\big) r^m\,, \tag{2.12}$$

where $\mathbf{\Omega} = \mathbf{A}$ *in case (a). If in addition* $\mathbf{h}(T, \mathbf{C}_{4r}(x)) := \sup\{|\mathbf{p}^\perp(x) - \mathbf{p}^\perp(y)| : x, y \in \mathrm{spt}(T) \cap \mathbf{C}_{4r}(x)\} \leqslant r$, *then*

$$\mathrm{osc}(f) \leqslant C_{21}\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21}(E^{1/2} + r\mathbf{\Omega})r \quad \textit{in case (a) and (c)}, \tag{2.13}$$

$$\mathrm{osc}(f) \leqslant C_{21}\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21}rE^{1/2} \quad \textit{in case (b).} \tag{2.14}$$

Notice that the case of area minimizing current in a Riemannian manifold (case (a) of Definition 1.1) is already covered by [19, Theorem 1.4], and indeed in Chapter 4 we will only prove it for the cases (b) and (c).

2.3 UNIQUENESS OF TANGENT CONE AND SIMPLIFICATION OF THE PROBLEM

The following Theorem is the starting point of our analysis and it concerns the uniqueness of the tangent cones and the subsequent splitting of the current. To state it we introduce the current $(\iota_{x,r})_\sharp T$, where the map $\iota_{x,r}$ is given by $\mathbb{R}^{m+n} \ni y \mapsto \frac{y-x}{r} \in \mathbb{R}^{m+n}$. Recall that an area minimizing cone S is an integral area minimizing current such that $(\iota_{0,r})_\sharp S = S$ for every $r > 0$ (cf. [54, Theorem 19.3]). Furthermore, for any given $R \in \mathbf{I}_m(\mathbb{R}^{m+n})$ we define $\mathcal{F}(R) := \inf\{\mathbf{M}(Z) + \mathbf{M}(W) : Z \in \mathbf{I}_m, W \in \mathbf{I}_{m+1}, Z + \partial W = R\}$.

Theorem 2.9 (Uniqueness of tangent cones for almost minimizers). *Let* T *be as in Definition 1.1*($\diamondsuit$), *with* $\diamondsuit = a, b$ *or* c, *and* $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. *Then there is a* $\gamma_0 > 0$, J *2-dim. distinct planes* π_i, *each pair of which intersect only at* 0, *and* J *integers* n_i *such that, if we set* $S := \sum_i n_i [\![\pi_i]\!]$, *then*

$$\mathcal{F}\big((T_{x,r} - S) \llcorner \mathbf{B}_1\big) \leqslant C_{11}\, r^{\gamma_0}, \tag{2.15}$$

$$\mathrm{dist}\big(\mathrm{spt}(T \llcorner \mathbf{B}_r(x)), \mathrm{spt}(S)\big) \leqslant C_{11}\, r^{1+\gamma_0}. \tag{2.16}$$

Moreover, there are $\bar{r} > 0$ *and* $J \geqslant 1$ *currents* $T^j \in \mathbf{I}_2(\mathbf{B}_{\bar{r}}(x))$ *such that*

(i) $\partial T^j \llcorner \mathbf{B}_{\bar{r}}(x) = 0$ *and each* T^j *satisfies Definition 1.1*($\diamondsuit$);

(ii) $T \llcorner \mathbf{B}_{\bar{r}}(x) = \sum_j T^j$ *and* $\mathrm{spt}(T_j) \cap \mathrm{spt}(T_i) = \{x\}$ *for every* $i \neq j$;

(iii) $n_j [\![\pi_j]\!]$ *is the unique tangent cone to each* T^j *at* x.

As an immediate consequence of this Theorem we can make the following ulterior assumptions.

Assumptions 2. *In addition to Assumption 1 we assume the following:*

(i) $\partial T \llcorner \mathbf{C}_2(0, \pi_0) = 0$;

(ii) $0 \in \mathrm{spt}(T)$ *and the tangent cone at* 0 *is given by* $\Theta(T, 0)\, [\![\pi_0]\!]$ *where* $\Theta(T, 0) \in \mathbb{N} \setminus \{0\}$;

(iii) T *is irreducible in any neighborhood* U *of* 0 *in the following sense: it is not possible to find* S, Z *non-zero integer rectifiable currents in* U *with* $\partial S = \partial Z = 0$ *(in* U*),* $T = S + Z$ *and* $\mathrm{spt}(S) \cap \mathrm{spt}(Z) = \{0\}$.

In order to justify point (iii), observe that if in a certain neighborhood U there is a decomposition $T = S + Z$ as above, it follows from Proposition 2.5 that both S and Z fall in one of the classes of Definition 1.1. In turn this implies that $\Theta(S, 0), \Theta(Z, 0) \in \mathbb{N} \setminus \{0\}$ and thus $\Theta(S, 0) < \Theta(T, 0)$. We can then replace T with either S or Z. Assume without loss of generality that $T_1 = S$: if it is not irreducibile we can argue as above and find a T_2 which satisfies all the requirements and has $0 < \Theta(T_2, 0) < \Theta(T_1, 0)$. This process must stop after at most $Q = \Theta(T, 0)$ steps: the final current is then necessarily irreducible.

2.4 THE MAIN INDUCTION STATEMENT AND THE PROOF OF THE MAIN THEOREM

2.4.1 *Branching model*

We next introduce an object which will play a key role in the rest of our work, because it is the basic local model of the singular behavior of a 2-dimensional area-minimizing current: for each positive natural number Q we will denote by $\mathfrak{B}_{Q,\rho}$ the flat Riemann surface which is a disk with a conical singularity, in the origin, of angle $2\pi Q$ and radius $\rho > 0$. More precisely we have

Definition 2.10. $\mathfrak{B}_{Q,\rho}$ is topologically an open 2-dimensional disk, which we identify with the topological space $\{(z, w) \in \mathbb{C}^2 : w^Q = z, |z| < \rho\}$. For each $(z_0, w_0) \neq 0$ in $\mathfrak{B}_{Q,\rho}$ we consider the connected component $\mathfrak{D}(z_0, w_0)$ of $\mathfrak{B}_{Q,\rho} \cap \{(z, w) : |z - z_0| < |z_0|/2\}$ which contains (z_0, w_0). We then consider the smooth manifold given by the atlas

$$\{(\mathfrak{D}(z, w)), (x_1, x_2)) : (z, w) \in \mathfrak{B}_{Q,\rho} \setminus \{0\}\},$$

where (x_1, x_2) is the function which gives the real and imaginary part of the first complex coordinate of a generic point of $\mathfrak{B}_{Q,\rho}$. On such smooth manifold we consider the following flat Riemannian metric: on each $\mathfrak{D}(z, w)$ with the chart (x_1, x_2) the metric tensor is the usual euclidean one $dx_1^2 + dx_2^2$. Such metric will be called the *canonical flat metric*. The coordinates $(x_1, x_2) = z$ will be called *standard flat coordinates*.

When $Q = 1$ we can extend smoothly the metric tensor to the origin and we obtain the usual euclidean 2-dimensional disk. For $Q > 1$ the metric tensor does not extend smoothly to 0, but we can nonetheless complete the induced geodesic distance on $\mathfrak{B}_{Q,\rho}$ in a neighborhood of 0: for $(z, w) \neq 0$ the distance to the origin will then correspond to $|z|$. The resulting metric space is a well-known object in the literature, namely a flat Riemann surface with an isolated conical singularity at the origin (see for instance [68]). Note that for each z_0 and $0 < r \leqslant \min\{\rho/2, \rho - |z_0|\}$ the set $\mathfrak{B}_{Q,\rho} \cap \{|z - z_0| < r\}$ consists then of Q nonintersecting 2-dimensional disks, each of which is a geodesic ball of $\mathfrak{B}_{Q,\rho}$ with radius r and center (z_0, w_i) for some $w_i \in \mathbb{C}$ with $w_i^Q = z_0$. We then denote each of them by $B_r(z_0, w_i)$ and treat it as a standard disk in the euclidean 2-dimensional plane (which is correct from the metric point of view). We use however the same notation for the distance disk $B_r(0)$, namely for the set $\{(z, w) : |z| < 0\}$, although the latter is *not isometric* to the standard euclidean disk.

When Q (and/or ρ) are clear from the context, (one of or both) the subscripts will be omitted. We will consider repeatedly functions u defined on $\mathfrak{B}$. We will always treat each point of $\mathfrak{B}$ as an element of $\mathbb{C}^2$, mostly using z and w for the horizontal and vertical complex coordinates. Often $\mathbb{C}$ will be identified with $\mathbb{R}^2$ and thus the coordinate z will be treated as a two-dimensional real vector, avoiding the more cumbersome notation (x_1, x_2).

Definition 2.11 (Q-branchings)**.** Let $\alpha \in]0,1[$, $b > 1$, $Q \in \mathbb{N}\setminus\{0\}$ and $n \in \mathbb{N}\setminus\{0\}$. An admissible α-smooth and b-separated Q-branching in $\mathbb{R}^{2+n}$ (shortly a Q-branching) is the graph

$$\mathrm{Gr}(u) := \{(z, u(z,w)) : (z,w) \in \mathfrak{B}_{Q,2\rho}\} \subset \mathbb{R}^{2+n} \tag{2.17}$$

of a map $u : \mathfrak{B}_{Q,2\rho} \to \mathbb{R}^n$ satisfying the following assumptions. For some constants $C_i > 0$ we have

- u is continuous, $u \in C^{3,\alpha}$ on $\mathfrak{B}_{Q,\rho}\setminus\{0\}$ and $u(0) = 0$;
- $|D^j u(z,w)| \leqslant C_i |z|^{1-j+\alpha}$ $\forall (z,w) \neq 0$ and $j \in \{0,1,2,3\}$;
- $[D^3 u]_{\alpha, B_r(z,w)} \leqslant C_i |z|^{-2}$ for every $(z,w) \neq 0$ with $|z| = 2r$;
- If $Q > 1$, then there is a positive constant $c_s \in]0,1[$ such that

$$\min\{|u(z,w) - u(z,w')| : w \neq w'\} \geqslant 4c_s|z|^b \qquad \text{for all } (z,w) \neq 0. \tag{2.18}$$

The map $\mathbf{\Phi}(z,w) := (z, u(z,w))$ will be called the *graphical parametrization* of the Q-branching.

Any Q-branching as in the Definition above is an immersed disk in $\mathbb{R}^{2+n}$ and can be given a natural structure as integer rectifiable current, which will be denoted by $\mathbf{G}_u$. For $Q = 1$ a map u as in Definition 2.11 is a (single valued) $C^{1,\alpha}$ map $u : B_2(0) \to \mathbb{R}^n$. Although the term branching is not appropriate in this case, the advantage of our setup is that $Q = 1$ will not be a special case in the induction statement of Theorem 2.14 below. Observe that for $Q > 1$ the map u can be thought as a Q-valued map $u : B_\rho(0) \to \mathcal{A}_Q(\mathbb{R}^n)$, setting $u(z) = \sum_{(z,w_i)\in\mathfrak{B}} [\![u(z,w_i)]\!]$ for $z \neq 0$ and $u(0) = Q\,[\![0]\!]$. The notation $\mathrm{Gr}(u)$ and $\mathbf{G}_u$ is then coherent with the corresponding objects defined in Section 3.2 for general Q-valued maps.

2.4.2 *Inductive step*

Before coming to the key inductive statement, we need to introduce some more terminology.

Definition 2.12 (Horned Neighborhood)**.** Let $\mathrm{Gr}(u)$ be a b-separated Q-branching. For every $a > b$ we define the *horned neighborhood* $\mathbf{V}_{u,a}$ of $\mathrm{Gr}(u)$ to be

$$\mathbf{V}_{u,a} := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^n : \exists (x,w) \in \mathfrak{B}_{Q,2\rho} \text{ with } |y - u(x,w)| < c_s|x|^a\}, \tag{2.19}$$

where c_s is the constant in (2.18).

Definition 2.13 (Excess)**.** Given an m-dimensional current T in $\mathbb{R}^{m+n}$ with finite mass, its *excess* in the ball $\mathbf{B}_r(x)$ and in the cylinder $\mathbf{C}_r(p,\pi')$ with respect to the m-plane π are

$$\mathbf{E}(T, \mathbf{B}_r(p), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{B}_r(p)} |\vec{T} - \vec{\pi}|^2 \, d\|T\| \tag{2.20}$$

$$\mathbf{E}(T, \mathbf{C}_r(p,\pi'), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{C}_r(p,\pi')} |\vec{T} - \vec{\pi}|^2 \, d\|T\|\,. \tag{2.21}$$

For cylinders we omit the third entry when $\pi = \pi'$, i.e. $\mathbf{E}(T, \mathbf{C}_r(p,\pi)) := \mathbf{E}(T, \mathbf{C}_r(p,\pi),\pi)$. In order to define the spherical excess we consider T as in Assumption 1 and we say that π *optimizes the excess* of T in a ball $\mathbf{B}_r(x)$ if

- In case (b)

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi); \tag{2.22}$$

- In case (a) and (c) $\pi \subset T_x\Sigma$ and

$$\mathbf{E}(T, \mathbf{B}_r(x)) := \min_{\tau \subset T_x\Sigma} \mathbf{E}(T, \mathbf{B}_r(x), \tau) = \mathbf{E}(T, \mathbf{B}_r(x), \pi)\,. \tag{2.23}$$

Note in particular that, in case (a) and (c), $\mathbf{E}(T, \mathbf{B}_r(x))$ differs from the quantity defined in [21, Definition 1.1], where, although Σ does not coincide with the ambient euclidean space, τ is allowed to vary among *all* planes, as in case (b). Thus a notation more consistent with that of [21] would be, in case (a) and (c), $\mathbf{E}^{\Sigma}(T, \mathbf{B}_r(x))$. However, the difference is a minor one and we prefer to keep our notation simpler.

Our main induction assumption is then the following

Assumptions 3 (Inductive Assumption). *T is as in Assumption 1 and 2. For some constants $\bar{Q} \in \mathbb{N} \setminus \{0\}$ and $0 < \bar{\alpha} < \frac{1}{2\bar{Q}}$ there is an $\bar{\alpha}$-admissible $\bar{Q}$-branching $\mathrm{Gr}(u)$ with $u : \mathfrak{B}_{\bar{Q},2} \to \mathbb{R}^n$ such that*

(Sep) *If $\bar{Q} > 1$, u is b-separated for some $b > 1$; a choice of some $b > 1$ is fixed also in the case $\bar{Q} = 1$, although in this case the separation condition is empty.*

(Hor) $\mathrm{spt}(T) \subset \mathbf{V}_{u,a} \cup \{0\}$ *for some $a > b$;*

(Dec) *There exist $\gamma > 0$ and a $C_i > 0$ with the following property. Let $p = (x_0, y_0) \in \mathrm{spt}(T) \cap \mathbf{C}_{\sqrt{2}}(0)$ and $4d := |x_0| > 0$, let V be the connected component of $\mathbf{V}_{u,a} \cap \{(x,y) : |x - x_0| < d\}$ containing p and let $\pi(p)$ be the plane tangent to $\mathrm{Gr}(u)$ at the only point of the form $(x_0, u(x_0, w_i))$ which is contained in V. Then*

$$\mathbf{E}(T \llcorner V, \mathbf{B}_\sigma(p), \pi(p)) \leqslant C_i^2 d^{2\gamma - 2} \sigma^2 \qquad \forall \sigma \in \left[\tfrac{1}{2} d^{(b+1)/2}, d\right]\,. \tag{2.24}$$

The main inductive step is then the following theorem, where we denote by $T_{p,r}$ the rescaled current $(\iota_{p,r})_\sharp T$, through the map $\iota_{p,r}(q) := (q-p)/r$.

Theorem 2.14 (Inductive statement). *Let T be as in Assumption 3 for some $\bar{Q} = Q_0$. Then,*

(a) *either T is, in a neighborhood of 0, a Q multiple of a $\bar{Q}$-branching $\mathrm{Gr}(v)$;*

(b) *or there are $r > 0$ and $Q_1 > Q_0$ such that $T_{0,r}$ satisfies Assumption 3 with $\bar{Q} = Q_1$.*

Theorem 1.2 follows then easily combining Theorem 2.9 and Theorem 2.14.

2.4.3 *Proof of Theorem 1.2*

As already mentioned, without loss of generality we can assume that Assumption 1 holds (the bounds on $\mathbf{A}$ and Ψ can be achieved by a simple scaling argument). Fix now a point p in $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. Our aim is to show that T is regular in a punctured neighborhood of p. Without loss of generality we can assume that p is the origin. By Theorem 2.9 , we can assume that Assumption 2 is satisfied, that is T is irreducible in some neighborhood of 0 and, upon suitably rescaling and rotating T, π_0 is the unique tangent cone to T at 0. In fact, T satisfies Assumption 3 with $\bar{Q} = 1$: it suffices to choose $u \equiv 0$ as admissible smooth branching. If T were not regular in any punctured neighborhood of 0, we could then apply Theorem 2.14 inductively to find a sequence of rescalings T_{0,ρ_j} with $\rho_j \downarrow 0$ which satisfy Assumption 3 with $\bar{Q} = Q_j$ for some strictly increasing sequence of integers. It is however elementary that the density $\Theta(0, T)$ bounds Q_j from above, which is a contradiction.

2.5 THE TWO FUNDAMENTAL TOOLS: THE BRANCHED CENTER MANIFOLD AND THE BLOW-UP THEOREM

From now on we fix T satisfying Assumption 3. Observe that, without loss of generality, we are always free to rescale homothetically our current T with a factor larger than 1 and ignore whatever portion falls outside $\mathbf{C}_2(0)$. We will do this several times, with factors which will be assumed to be sufficiently large. Hence, if we can prove that something holds in a sufficiently small neighborhood of 0, then we can assume, withouth loss of generality, that it holds on $\mathbf{C}_2$. For this reason we can assume that the constants C_i in Definition 2.11 and Assumption 3 are as small as we want. In turns this implies that there is a well-defined orthogonal projection $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_1 \to \mathrm{Gr}(u) \cap \mathbf{C}_2$, which is a $C^{2,\alpha}$ map.

By the constancy theorem, $(\mathbf{P}_\sharp(T \llcorner \mathbf{C}_1)) \llcorner \mathbf{C}_{1/2}$ coincides with the current $Q\mathbf{G}_u \llcorner \mathbf{C}_{1/2}$ (again, we are assuming C_i in Definition 2.11 sufficiently small), where $Q \in \mathbb{Z}$. If Q were 0, condition (Dec) in Assumption 3 and a simple covering argument would imply that $\|T\|(\mathbf{C}_{1/2}(0)) \leqslant C_0 C_i^2$, where C_0 is a geometric constant. In particular this would violate, by the monotonicity formula, the assumption $0 \in \mathrm{spt}(T)$. Thus $Q \neq 0$. On the other hand condition (Dec) in Assumption 3 implies also that Q must be positive (again, provided C_i is smaller than a geometric constant).

Now, recall that from Theorem 2.9 the density $\Theta(p, T)$ is a positive integer at any $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. Moreover, the rescaled currents $T_{0,r}$ converge to $\Theta(0, T) [\![\pi_0]\!]$. It is easy to see that the rescaled currents $(\mathbf{G}_u)_{0,r}$ converge to $\bar{Q} [\![\pi_0]\!]$ and that $(\mathbf{P}_\sharp T)_{0,r}$ converges to $\Theta(0, T) [\![\pi_0]\!]$. We then conclude that $\Theta(0, T) = \bar{Q}Q$.

We summarize these conclusions in the following lemma, where we also claim an additional important bound on the density of T outside 0, which will be proved in the appendix to this chapter.

Lemma 2.15. *Let T and u be as in Assumption 3 for some $\bar{Q}$. Then the nearest point projection $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_1 \to \mathrm{Gr}(u)$ is a well-defined $C^{0,\alpha}$ map, $C^{2,\alpha}$ outside the origin. In addition there is $Q \in \mathbb{N} \setminus \{0\}$ such that $\Theta(0, T) = Q\bar{Q}$ and the unique tangent cone to T at 0 is $Q\bar{Q} [\![\pi_0]\!]$. Finally, after possibly rescaling T, $\Theta(p, T) \leqslant Q + \frac{1}{2}$ for every $p \in \mathbf{C}_2 \setminus \{0\}$ and, for every $x \in B_2(0)$, each connected component of $(x \times \mathbb{R}^n) \cap \mathbf{V}_{u,a}$ contains at least one point of $\mathrm{spt}(T)$.*

Since we will assume during the rest of the paper that the above discussion applies, we summarize the relevant conclusions in the following

Assumptions 4. T *satisfies Assumption 3 for some* $\bar{Q}$ *and with* C_i *sufficiently small.* $Q \geqslant 1$ *is an integer,* $\Theta(0, T) = Q\bar{Q}$ *and* $\Theta(p, T) \leqslant Q$ *for all* $p \in \mathbf{C}_2 \setminus \{0\}$*.*

The overall plan to prove Theorem 2.14 is then the following:

(CM) We construct first a branched center manifold, i.e. a second admissible smooth branching $\boldsymbol{\varphi}$ on $\mathfrak{B}_{\bar{Q}}$, and a corresponding Q-valued map N defined on the normal bundle of $\mathrm{Gr}(\boldsymbol{\varphi})$, which approximates T with a very high degree of accuracy (in particular more accurately than u) and whose average $\boldsymbol{\eta} \circ N$ is very small;

(BU) Assuming that alternative (a) in Theorem 2.14 does not hold, we study the asymptotic behavior of N around 0 and use it to build a new admissible smooth branching ν on some $\mathfrak{B}_{k\bar{Q}}$ where $k \geqslant 2$ is a factor of Q: this map will then be the one sought in alternative (b) of Theorem 2.14 and a suitable rescaling of T will lie in a horned neighborhood of its graph.

The first part of the program is the one achieved in Part iv, whereas the second part is completed in Part v : after stating both of them we will finish this section with the proof of Theorem 2.14. Note that, when $Q = 1$, from (BU) we will conclude that alternative (a) necessarily holds: this will be a simple corollary of the general case, but we observe that it could also be proved resorting to the classical Allard's regularity theorem.

2.5.1 *Smallness condition*

In several occasions we will need that the ambient manifold Σ is suitably flat and that the excess of the current T is suitably small. This can, however, be easily achieved after scaling.

Lemma 2.16. *Let* T *be as in the Assumptions 3 and 4. After possibly rescaling, rotating and modifying* Σ *outside* $\mathbf{C}_2(0)$ *we can assume that, in case (a) and (c) of Definition 1.1,*

(i) Σ *is a complete submanifold of* $\mathbb{R}^{2+n}$*;*

(ii) $T_0\Sigma = \mathbb{R}^{2+\bar{n}} \times \{0\}$ *and,* $\forall p \in \Sigma$*,* Σ *is the graph of a* C^{3,ε_0} *map* $\Psi_p : T_p\Sigma \to (T_p\Sigma)^\perp$*.*

Under these assumptions, we denote by $\mathbf{c}$ *and* $\mathbf{m}_0$ *the following quantities*

$$\mathbf{c} := \sup\{\|D\Psi_p\|_{C^{2,\varepsilon_0}} : p \in \Sigma\} \qquad \text{in the cases (a) and (c) of Definition 1.1} \tag{2.25}$$

$$\mathbf{c} := \|d\omega\|_{C^{1,\varepsilon_0}} \qquad \text{in case (b) of Definition 1.1} \tag{2.26}$$

$$\mathbf{m}_0 := \max\left\{\mathbf{c}^2, \mathbf{E}(T, \mathbf{C}_2, \pi_0), C_i^2, c_s^2\right\}\,, \tag{2.27}$$

where C_i *and* c_s *are the constants appearing in Definition 2.11 and Assumption 3. Then, for any* $\varepsilon_{41} > 0$*, after possibly rescaling the current by a large factor, we can assume*

$$\mathbf{m}_0 \leqslant \varepsilon_{41}\,. \tag{2.28}$$

In order to carry on the plan outlined in the previous subsection, it is convenient to use a different parametrization of Q-branchings.

If we remove the origin, any admissible Q-branching is a Riemannian submanifold of $\mathbb{R}^{2+n}$: this gives a Riemannian tensor $g := \mathbf{\Phi}^\sharp e$ (where e denotes the euclidean metric on $\mathbb{R}^{2+n}$) on the punctured disk $\mathfrak{B}_{Q,2\rho} \setminus \{0\}$. Note that in (z,w), the difference between the metric tensor g and the canonical flat metric is estimated by (a constant times) $|z|^{2\alpha}$: thus, as it happens for the flat metric, when $Q > 1$ it is not possible to extend the metric g to the origin. However, using well-known arguments in differential geometry, we can find a conformal map from $\mathfrak{B}_{Q,r}$ onto a neighborhood of 0 which maps the conical singularity of $\mathfrak{B}_{Q,r}$ in the conical singularity of the Q-branching. In fact, we need the following accurate estimates for such a map, whose proof will be given in the appendix to the chapter.

Proposition 2.17 (Conformal parametrization)**.** *Given an admissible* b*-separated* α*-smooth* Q*-branching* $\mathrm{Gr}(u)$ *with* $\alpha < 1/(2Q)$ *there exist a constant* $C_0(Q,\alpha) > 0$*, a radius* $r > 0$ *and functions* $\mathbf{\Psi}\colon \mathfrak{B}_{Q,r} \to \mathrm{Gr}(u)$ *and* $\lambda\colon \mathfrak{B}_{Q,r} \to \mathbb{R}_+$ *such that*

(i) $\mathbf{\Psi}$ *is a homeomorphism of* $\mathfrak{B}_{Q,r}$ *with a neighborhood of* 0 *in* $\mathrm{Gr}(u)$*;*

(ii) $\mathbf{\Psi} \in C^{3,\alpha}(\mathfrak{B}_{Q,r} \setminus \{0\})$*, with the estimates*

$$|\mathbf{\Psi}(z,w) - (z,0)| \leqslant C_0 C_i |z|^{1+\alpha}\,, \tag{2.29}$$

$$|D^l\big(\mathbf{\Psi}(z,w) - (z,0)\big)| \leqslant C_0 C_i |z|^{\alpha - l} \qquad \text{for } l = 1,\dots,3,\ z \neq 0\,, \tag{2.30}$$

$$[D^3\mathbf{\Psi}]_{\alpha, B_r(z,w)} \leqslant C_0 C_i |z|^{-2} \qquad \text{for } z \neq 0 \text{ and } r = |z|/2\,; \tag{2.31}$$

(iii) Ψ *is a conformal map with conformal factor* λ*, namely, if we denote by* e_{2+n} *the ambient euclidean metric in* $\mathbb{R}^{2+n}$ *and by* e_Q *the canonical euclidean metric of* $\mathfrak{B}_{Q,r}$*,*

$$g := \mathbf{\Psi}^\sharp e_{2+n} = \lambda\, e_Q \qquad \text{on } \mathfrak{B}_{Q,r} \setminus \{0\}. \tag{2.32}$$

(iv) The conformal factor λ *satisfies*

$$|D^l(\lambda - 1)(z,w)| \leqslant C_0 C_i |z|^{2\alpha - l} \qquad \text{for } l = 0,1,\dots,2 \tag{2.33}$$

$$[D^2\lambda]_{\alpha, B_r(z,w)} \leqslant C_0 C_i |z|^{\alpha-2} \qquad \text{for } z \neq 0 \text{ and } r = |z|/2\,. \tag{2.34}$$

2.5.2 *The center manifold and the approximation*

We are now ready to state the two "halves" of Theorem 2.14. The first one is the construction of a surface which at every inductive step will play the role of a wedge between the sheets of the current, together with a very careful approximation map on top of it.

Theorem 2.18 (Center Manifold Approximation)**.** *Let* T *be as in Assumptions 3 and 4. Then there exist* $\eta_0, \gamma_0, r_0, C > 0$*,* $b > 1$*, an admissible* b*-separated* γ_0*-smooth* $\bar{Q}$*-branching* $\mathcal{M}$*, a corresponding conformal parametrization* $\mathbf{\Psi} : \mathfrak{B}_{\bar{Q},2} \to \mathcal{M}$ *and a* Q*-valued map* $\mathscr{N} : \mathfrak{B}_{\bar{Q},2} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *with the following properties:*

(i) $\bar{Q}Q = \Theta(T, 0)$ *and*

$$|D(\Psi(z,w) - (z,0))| \leqslant C\mathbf{m}_0^{1/2}|z|^{\gamma_0} \tag{2.35}$$

$$|D^2\Psi(z,w)| + |z|^{-1}|D^3\Psi(z,w)| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|z|^{\gamma_0 - 1}\,; \tag{2.36}$$

in particular, if we denote by $A_{\mathcal{M}}$ *the second fundamental form of* $\mathcal{M}\setminus\{0\}$*,*

$$|A_{\mathcal{M}}(\Psi(z,w))| + |z|^{-1}|D_{\mathcal{M}}A_{\mathcal{M}}(\Psi(z,w))| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|z|^{\gamma_0-1}\,.$$

(ii) $\mathscr{N}_i(z,w)$ *is orthogonal to the tangent plane, at* $\Psi(z,w)$*, to* $\mathcal{M}$*.*

(iii) If we define $S := T_{0,r_0}$*, then* $\mathrm{spt}(S)\cap \mathbf{C}_1\setminus\{0\}$ *is contained in the closure of a suitable horned neighborhood of the* $\bar{Q}$*-branching, where the orthogonal projection* $\mathbf{P}$ *onto it is well-defined. Moreover, for every* $r\in]0,1[$ *we have*

$$\|\mathscr{N}|_{B_r}\|_0 + \sup_{p\in \mathrm{spt}(S)\cap \mathbf{P}^{-1}(\Psi(B_r))} |p - \mathbf{P}(p)| \leqslant C\mathbf{m}_0^{\frac{1}{4}} r^{1+\frac{\gamma_0}{2}}\,. \tag{2.37}$$

(iv) If we define

$$\mathbf{D}(r) := \int_{B_r} |D\mathscr{N}|^2 \quad \textit{and} \quad \mathbf{H}(r) := \int_{\partial B_r} |\mathscr{N}|^2\,,$$

$$\mathbf{F}(r) := \int_0^r \frac{\mathbf{H}(t)}{t^{2-\gamma_0}}\,dt \quad \textit{and} \quad \mathbf{\Lambda}(r) := \mathbf{D}(r) + \mathbf{F}(r)\,,$$

then the following estimates hold for every $r\in]0,1[$*:*

$$\mathrm{Lip}(\mathscr{N}|_{B_r}) \leqslant C\min\{\mathbf{\Lambda}^{\eta_0}(r), \mathbf{m}_0^{\eta_0} r^{\eta_0}\} \tag{2.38}$$

$$\mathbf{m}_0^{\eta_0}\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\mathscr{N}(z,w)| \leqslant C\,\mathbf{\Lambda}^{\eta_0}(r)\,\mathbf{D}(r) + C\,\mathbf{F}(r)\,. \tag{2.39}$$

(v) Finally, if we set

$$\mathscr{F}(z,w) := \sum_i [\![\Psi(z,w) + \mathscr{N}_i(z,w)]\!]\,,$$

then

$$\|S - \mathbf{T}_{\mathscr{F}}\|\big(\mathbf{P}^{-1}(\Psi(B_r))\big) \leqslant C\,\mathbf{\Lambda}^{\eta_0}(r)\,\mathbf{D}(r) + C\,\mathbf{F}(r)\,. \tag{2.40}$$

2.5.3 *The asymptotic analysis*

The second main step is the analysis of the asymptotic behaviour of $\mathscr{N}$ around the origin, in particular the mode of convergence of a suitable rescaling of it to its unique limit and the properties of this limit.

Remark 2.19. In order to state it, we agree to define $W^{1,2}$ functions on $\mathfrak{B}$ in the following fashion: removing the origin 0 from $\mathfrak{B}$ we have a C^3_{loc} (flat) Riemannian manifold embedded in $\mathbb{R}^4$ and we can define $W^{1,2}$ maps on it following Definition 3.9. Alternatively we can use the conformal parametrization $\mathbf{W} : \mathbb{R}^2 = \mathbb{C} \to \mathfrak{B}_{\bar{Q}}$ given by $\mathbf{W}(z) = (z^{\bar{Q}}, z)$ and agree that $u \in W^{1,2}(\mathfrak{B}_{\bar{Q}})$ if $u \circ \mathbf{W}$ is in $W^{1,2}(\mathbb{R}^2)$. Since discrete sets have zero 2-capacity, it is immediate to verify that these two definitions are equivalent.

In a similar fashion, we will ignore the origin when integrating by parts Lipschitz vector fields, treating $\mathfrak{B}_{\bar{Q}}$ as a C^1 Riemannian manifold. It is straightforward to show that our assumption is correct, for instance removing a disk of radius ε centered at the origin, integrating by parts and then letting $\varepsilon \downarrow 0$.

Theorem 2.20 (Blowup Analysis)**.** *Under the assumptions of Theorem 2.18, the following dichotomy holds:*

(i) either there exists $s > 0$ such that $\mathscr{N}|_{B_s} \equiv Q\,[\![0]\!]$,

(ii) or there exist constants $I_0 > 1$, $a_0, \bar{r}, C > 0$ and an I_0-homogeneous nontrivial Dir-minimizing function $g : \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ such that $\boldsymbol{\eta} \circ g \equiv 0$, $\mathrm{spt}(g(z,w)) \subset \{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}$, for every $(z,w) \in \mathfrak{B}_{\bar{Q}}$, and

$$\mathcal{G}\big(\mathscr{N}(z,w), g(z,w)\big) \leqslant C|z|^{I_0+a_0} \quad \forall\, (z,w) \in \mathfrak{B}_Q,\ |z| < \bar{r}, \tag{2.41}$$

and moreover the following estimates hold

$$\int_{B_{r+2\rho}\setminus B_{r-2\rho}} |D\mathscr{N}|^2 \leqslant C\, r^{2I_0+a_0} + C\, r^{2I_0-1}\,\rho \quad \forall\, 4\,\rho \leqslant r < 1, \tag{2.42}$$

$$\mathbf{H}(r) \leqslant C\, r\, \mathbf{D}(r) \quad \forall\, r < 1. \tag{2.43}$$

Remark 2.21. Note that, when $\bar{Q} = \Theta(T, 0)$, we necessarily have $Q = 1$ and the second alternative is excluded. In particular we conclude that T coincides with $[\![\mathcal{M}]\!]$ in a neighborhood of 0 and thus that it is a regular submanifold in a *punctured neighborhood* of 0.

Remark 2.22. By a simple dyadic argument it follows from (2.42) and (2.43) that

$$\int_{B_r} |D\mathscr{N}|^2 \leqslant C\, r^{2I_0} \quad \text{and} \quad \mathbf{F}(r) \leqslant C\, r^{2I_0+\gamma_0} \quad \forall\, r < 1. \tag{2.44}$$

so that, in particular

$$\boldsymbol{\Lambda}(r) \leqslant C\, r^{2I_0} \quad \text{and} \quad \boldsymbol{\Lambda}^{\eta_0}(r) \leqslant C\, r^{2I_0\,\eta_0}\,.$$

2.6 PROOF OF THE INDUCTIVE STEP

We start observing that if case (a) of Theorem 2.14 does not hold, then we are necessarily in case (ii) of Theorem 2.20. Therefore we only need to prove that Theorem 2.20(ii) implies Theorem 2.14(b).

We divide the proof in different steps.

Step 1. For a reason which will become clear later, it is convenient to slightly modify the map g to a multivalued map $n(z,w) = \sum_i [\![n_i(z,w)]\!]$ in such a way that $n_i(z,w)$ is orthogonal to $\mathcal{M}$ at $\boldsymbol{\Psi}(z,w)$. To achieve this it suffices to project $g_i(z,w) = (0, \bar{g}_i(z,w), 0)$ on the normal bundle. Observe that, by the estimates on $|A_{\mathcal{M}}|$ and $\boldsymbol{\Psi}$, we easily have (cf. the proof of Lemma 10.14)

$$|g_i(z,w) - n_i(z,w)| \leqslant C C_i |z|^{\gamma_0} |g_i(z,w)|\,, \tag{2.45}$$

$$|Dn|(z,w) \leqslant |Dg|(z,w) + C C_i |z|^{\gamma_0 - 1} |g|(z,w)\,. \tag{2.46}$$

We introduce the function $H : \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ given by

$$H(z,w) = \sum_{i=1}^{Q} [\![H_i(z,w)]\!] := \sum_{i=1}^{Q} [\![\boldsymbol{\Psi}(z,w) + n_i(z,w)]\!]\ .$$

Note that, since g is I_0-homogeneous, by (2.45) there exists a constant $C > 0$ such that

$$|H_i(z,w) - H_j(z,w)| \geqslant C\,|z|^{I_0} \quad \text{whenever } H_i(z,w) \neq H_j(z,w). \tag{2.47}$$

Let $0 < \bar{a} < a_0$ be a constant to be fixed momentarily and $\zeta := I_0 + \frac{\bar{a}}{2} > 1$. Set

$$\mathbf{V}_{H,\zeta} := \left\{ H_i(z,w) + p \in \mathbb{R}^{2+n} : |p| < |z|^{\zeta},\ i = 1, \ldots, Q \right\}.$$

We claim that there exists $s > 0$ such that $\mathrm{spt}(T) \cap \mathbf{B}_s \subset \mathbf{V}_{H,\zeta}$.

In order to prove this claim, we distinguish two cases. First we consider any point $p \in \mathrm{spt}(T) \cap \mathrm{spt}(\mathbf{T}_F)$. In this case $p = \boldsymbol{\Psi}(z,w) + \mathscr{N}_i(z,w)$ for some $(z,w) \in \mathfrak{B}_Q$ and for some $i = 1, \ldots, Q$. Without loss of generality, by (2.41) we can assume $|\mathscr{N}_i(z,w) - g_i(z,w)| \leqslant C|z|^{I_0 + \bar{a}}$, i.e.

$$\begin{aligned} |p - H_i(z,w)| = |\mathscr{N}_i(z,w) - n_i(z,w)| &\leqslant |\mathscr{N}_i(z,w) - g_i(z,w)| + |g_i(z,w) - n_i(z,w)| \\ &\leqslant C|z|^{I_0 + \bar{a}} + C\,|z|^{I_0 + \gamma_0}, \end{aligned} \tag{2.48}$$

which in particular implies $\mathrm{spt}(T) \cap \mathrm{spt}(\mathbf{T}_{\mathscr{F}}) \cap \mathbf{B}_s \subset \mathbf{V}_{H,\zeta}$ if s is sufficiently small and we impose $\frac{\bar{a}}{2} < \gamma_0$.

For the second case we consider a point $p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\mathbf{T}_{\mathscr{F}})$ and assume by contradiction that $p \notin \mathbf{V}_{H,\zeta}$. In particular, in view of (2.48) we have that

$$B := \mathbf{B}_{\frac{|z|^{\zeta}}{2}}(p) \cap \mathrm{spt}(\mathbf{T}_{\mathscr{F}}) = \emptyset$$

if $|z|$ is sufficiently small. By the monotonicity formula we know that $\|T\|(B) \geqslant C\,|z|^{2\zeta}$; nevertheless since $B \subset \mathbf{P}^{-1}(B_{2|z|} \setminus B_{\frac{|z|}{2}})$, we deduce from (2.40) and (2.44) that $\|T\|(B) \leqslant C\,|z|^{2I_0 + 2\kappa}$ with $\kappa = \min\{2\eta_0\, I_0, \gamma_0\}$, which gives a contradiction if $\bar{a} < 2\kappa$.

Step 2. From the previous step we can infer that g is a constant multiple of an irreducible function, namely there exists $Q' > 0$ such that $\mathrm{card}(g(z,w)) = Q'$ for every $(z,w) \neq (0,0)$ and there exists a continuous map $h : \mathfrak{B}_{\bar{Q}Q'} \to \mathbb{R}^{2+n}$ such that

$$g(z,w) = \frac{Q}{Q'} \sum_{\tilde{z}=z,\ \tilde{w}^{Q'}=w} [\![h(\tilde{z},\tilde{w})]\!]\,. \tag{2.49}$$

If this is not the case, by a straightforward generalization of [17, Proposition 5.1] we can decompose g in the superposition of irreducible functions, i.e. there exists a unique decomposition $g = \sum_{j=1}^{J} k_j g_j$ where $g_j : \mathfrak{B}_Q \to \mathcal{A}_{q_j}(\mathbb{R}^n)$ are Dir-minimizing I_0-homogeneous functions, for some choice of positive integers J, k_j, q_j such that $\sum_{j=1}^{J} k_j q_j = Q$.

Denoting by H^j the corresponding maps (recall that n is the projection of g on the normal bundle to $\mathcal{M}$)

$$H^j(z,w) := \sum_{l=1}^{q_j} \llbracket \mathbf{\Psi}(z,w) + (n^j)_l(z,w) \rrbracket$$

and by $\mathbf{V}_{H^j,\zeta}$ the corresponding horned neighborhoods

$$\mathbf{V}_{H^j,\zeta} := \left\{ (H^j)_l(z,w) + p \in \mathbb{R}^{2+n} : |p| \leqslant |z|^{\zeta},\ l = 1,\dots,q_j \right\},$$

it follows from (2.47) that $V_{\zeta,H_i} \cap V_{\zeta,H_j} = \{0\}$. Setting $T_i := T \llcorner V_{\zeta,H_i}$, we infer that $T = \sum_i T_i$ with $\mathrm{spt}(T_i) \cap \mathrm{spt}(T_j) = \{0\}$, against the irreducibility of T. Note that, since $\boldsymbol{\eta} \circ g = 0$ it also follows that $Q' > 1$.

Having established (2.49), let us define $\mathbf{\Theta} : \mathfrak{B}_{\bar{Q}Q'} \to \mathbb{R}^n$ as

$$\mathbf{\Theta}(\tilde{z},\tilde{w}) := \mathbf{\Psi}(\tilde{z},\tilde{w}^{Q'}) + h^n(\tilde{z},\tilde{w}) \quad \forall\, (\tilde{z},\tilde{w}) \in \mathfrak{B}_{\bar{Q}Q'},$$

where $h^n(\tilde{z},\tilde{w})$ is the projection of $h(\tilde{z},\tilde{w})$ on the space normal to $\mathcal{M}$ at the point $\mathbf{\Psi}(\tilde{z},\tilde{w}^{Q'})$. It follows that $\mathrm{Im}(H) = \mathrm{Im}(\mathbf{\Theta})$ is an admissible $\bar{Q}Q'$-branching (the Hölder regularity for the graphical parametrization follow from the fact that $I_0 > 1$). Moreover, from the homogeneity of g we easily infer that $\mathrm{Im}(\mathbf{\Theta})$ is I_0-separated (for a suitable constant c_s). Note that for $\zeta' := I_0 + \frac{a}{4}$ and s sufficiently small $\mathbf{V}_{H,\zeta} \cap \mathbf{B}_s \subset \mathbf{V}_{\mathbf{\Theta},\zeta'} \cap \mathbf{B}_s$.

Step 3. Finally we prove the condition (Dec) of Assumption 3. Let $(z,w) \in \mathfrak{B}_{\bar{Q}}$ with $0 < |z| < \sqrt{2}$, let V be the connected component of $\mathbf{V}_{\mathbf{\Theta},\zeta'} \cap \{(x,y) : |x-z| < d\}$ with $d := \frac{|z|}{4}$ containing $\mathbf{\Theta}(z,w)$, and $p \in \mathrm{spt}(T) \cap V$ with co-ordinates $p = (z,y)$. Denote by π the oriented two-vector for $\mathrm{Im}(\mathbf{\Theta})$ at $\mathbf{\Theta}(z,w)$, and consider $\rho \in [\frac{1}{2} d^{\frac{(I_0+1)}{2}}, d]$.

Since $\mathbf{B}_\rho(p) \subset \mathbf{P}^{-1}(\mathbf{\Psi}(B_{|z|+2\rho} \setminus B_{|z|-2\rho}))$, we start estimating as follows

$$\begin{aligned}\int_{\mathbf{B}_\rho(p)} |\vec{T} - \vec{\pi}|^2\, d\|T\| &\leqslant \int_{\mathbf{B}_\rho(p)} |\vec{\mathbf{T}}_{\mathscr{F}} - \vec{\pi}|^2\, d\|\mathbf{T}_{\mathscr{F}}\| + \|T - \mathbf{T}_{\mathscr{F}}\|(\mathbf{p}^{-1}(B_{|x_0|+2\rho})) \\ &\overset{(2.40)}{\leqslant} \int_{\mathbf{B}_\rho(p)} |\vec{\mathbf{T}}_{\mathscr{F}} - \vec{\pi}|^2\, d\|\mathbf{T}_{\mathscr{F}}\| + C\,|z|^{2I_0+2\kappa}. \end{aligned} \tag{2.50}$$

Next, note that for $|z|$ small enough $\mathbf{P}(\mathbf{B}_\rho(p) \cap \mathbf{V}_{\mathbf{\Theta},\zeta'}) \subset \mathbf{\Psi}(B_{2\rho}(z,w))$.

We can consider the set of indices $A \subset \{1,\dots,Q\}$ such that $\mathscr{F}_i(z,w) \in V$ for $i \in A$ and estimate as follows

$$\begin{aligned}\int_{\mathbf{B}_\rho(p)} |\vec{\mathbf{T}}_{\mathscr{F}} - \vec{\pi}|^2\, d\|\mathbf{T}_{\mathscr{F}}\| &\leqslant C \sum_{i\in A} \int_{B_{2\rho}(z,w)} |\vec{\mathbf{T}}_{\mathscr{F}_i} - \vec{\mathbf{T}}_{\mathbf{\Theta}}|^2 \ + C\,\rho^2\, \mathrm{Lip}(D\mathbf{\Theta}|_{B_{2\rho}(z,w)})^2 \\ &\leqslant C \sum_{i\in A} \int_{B_{2\rho}(z,w)} |\vec{\mathbf{T}}_{\mathscr{F}_i} - \vec{\mathbf{T}}_{\mathbf{\Psi}}|^2 \\ &\quad + C \int_{B_{2\rho}(z,w)} |\vec{\mathbf{T}}_{\mathbf{\Psi}} - \vec{\mathbf{T}}_{\mathbf{\Theta}}|^2 + C\,\rho^4\,|z|^{2\theta-2}, \end{aligned} \tag{2.51}$$

where $\theta := \min\{\gamma_0, I_0 - 1\}$ and we used the fact that $|D^2\boldsymbol{\Theta}|(z,w) \leqslant C\,|z|^{\theta-1}$.

We can finally use the computation of the excess in curvilinear coordinates in Proposition 3.50 to get

$$\sum_i \int_{B_{2\rho}(z,w)} |\vec{\mathbf{T}}_{\mathscr{F}_i} - \vec{\mathbf{T}}_{\boldsymbol{\Psi}}|^2 \leqslant C \int_{B_{2\rho}(z,w)} \left(|D\mathscr{N}|^2 + |z|^{2\gamma_0-2}|\mathscr{N}|^2\right)$$

$$\overset{(2.44)}{\leqslant} C \int_{B_{|z|+2\rho}\setminus B_{|z|-2\rho}} |D\mathscr{N}|^2 + C\,|z|^{2I_0+2\gamma_0} \tag{2.52}$$

$$\overset{(2.42)}{\leqslant} C\,|z|^{2I_0+a_0} + C\,|z|^{2I_0-1}\,\rho\,, \tag{2.53}$$

and similarly

$$\int_{B_{2\rho}(z,w)} |\vec{\mathbf{T}}_{\boldsymbol{\Theta}} - \vec{\mathbf{T}}_{\boldsymbol{\Psi}}|^2 \leqslant C \int_{B_{2\rho}(z,w)} \left(|Dn|^2 + |z|^{2\gamma_0-2}|n|^2\right)$$

$$\leqslant C \int_{B_{2\rho}(z,w)} \left(|Dg|^2 + |z|^{2\gamma_0-2}|g|^2\right)$$

$$\leqslant C\,|z|^{2I_0-2}\rho^2 + C\,|z|^{2I_0+2\gamma_0} \tag{2.54}$$

(observe that, in order to apply Proposition Proposition 3.50 we need that n takes value into the normal bundle).

Collecting all the estimates together, we have that there exists a suitable constant ϖ such that

$$\int_{\mathbf{B}_\rho(p)} |\vec{T} - \vec{\pi}|^2 \, d\|T\| \leqslant C\,|z|^{2I_0+2\varpi} + C\,\rho\,|z|^{2I_0-1} + C\,\rho^4\,|z|^{2\varpi-2} \leqslant |z|^{\gamma-2}\rho^4, \tag{2.55}$$

where the last inequality is verified for a suitable $\gamma > 0$, and for every $\rho \in \left[\frac{1}{2}\left(\frac{|z|}{4}\right)^{\frac{(I_0+1)}{2}}, \frac{|z|}{4}\right]$ and $|z|$ small enough.

2.7 APPENDIX A: PROOF OF THE TECHNICAL LEMMAS

In this section we prove the two technical Lemmas 2.15 and 2.16.

Proof of Lemma 2.15. Consider $x_0 \in \pi_0$ with $2\rho = |x_0|$, a smooth C^2 function $\phi : B_\rho(x_0) \to \mathbb{R}^n$ and the open set $\mathbf{V}_\rho := \{(x,y) : x \in B_{\rho/2}(x_0), |y - \phi(x)| \leqslant \rho\}$. Recall that there is a geometric constant C such that, if $\rho \leqslant C/\|D^2\phi\|_{B_\rho(x_0)}$, then for each $p \in \mathbf{V}_\rho$ there is a unique nearest point $\mathbf{P}(p) \in \mathrm{Gr}(\phi)$ (which defines a C^1 map $\mathbf{P} : \mathbf{V}_\rho \to \mathrm{Gr}(\phi)$). In particular, if $\|D^2\phi\|_{B_\rho(x_0)} \leqslant C\rho^{\alpha-1}$, the existence of such point is guaranteed under the assumption that $\rho \leqslant c\rho^{1-\alpha}$ (where c is a, possibly small but positive, constant). Consider now an admissible smooth branching $u : \mathfrak{B}_{\bar{Q}} \to \mathbb{R}^n$. If $\bar{Q} = 1$, the above discussion shows easily the existence of a well defined C^1 map $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_{2r} \to \mathrm{Gr}(u)$, provided r is sufficiently small. If $\bar{Q} >!1$, the same conclusion holds under the assumption that u is b-separated and $a > b > 1$. Indeed consider $p = (z,y) \in \mathbf{V}_{u,a}$ and $(z, w_i) \in \mathfrak{B}_Q$ such that $|y - u(z,w_i)| \leqslant c_s|z|^a$. The assumptions of being well-separated implies easily that $|p - u(\zeta, \omega)| \geqslant c_s |z|^b$ whenever $z \notin B_{|z|/2}(z, w_i)$ and thus we can argue locally on the sheet $\mathrm{Gr}(u|_{B_{|z|/2}(z,w_i)})$.

Next, up to rescaling we can assume that $\mathbf{P}$ is well-defined on $\mathbf{V}_{u,a} \cap \mathbf{C}_2$. The discussion before Lemma 2.15 applies now verbatim and we conclude the first sentence of the Lemma.

To reach the other two conclusions of the Lemma we argue by contradiction: if they were wrong, then we would find a sequence of points $\{x_k\} \subset B_2(0)$ converging to 0 for which one of the following two conditions hold:

- either $\{x_k\} \times \mathbb{R}^n$ contains a point $p_k \in \mathrm{spt}(T)$ with $\Theta(p_k, T) \geqslant Q + \frac{1}{2}$;
- or one connected component Ω of $(\{x_k\} \times \mathbb{R}^n) \cap \mathbf{V}_{u,a}$ does not intersect spt(T).

Set $2r_k := |x_k|$ and consider the connected component $\mathbf{V}_k$ of $\mathbf{V}_{u,a} \cap \mathbf{C}_{r_k}(x_k)$ which contains p_k (in the first case) or Ω_k (in the second). Let $S_k := T_k \llcorner \mathbf{V}_k$ and let $q_k = (x_k, u(x_k, w_k))$ be such that $q_k \in \mathbf{V}_k$. Finally set $Z_k := (S_k)_{q_k, r_k}$. Observe that $\mathrm{spt}(Z_k)$ is contained in a neighborhood of height Cr_k^{a-1} of π_0 and we therefore conclude that Z_k converges to a current Z which is an integer multiple of $[\![B_1(0)]\!]$. On the other hand, since $\mathbf{P}_\sharp(S_k) \llcorner \mathbf{C}_{r_k/2}(x_k) = Q\mathbf{G}_u \llcorner \mathbf{C}_{r_k/2}(x_k)$ for k large enough, we conclude that $Z = Q\,[\![B_1(0)]\!]$. Now, either $\mathrm{spt}(Z_k) \cap (\{0\} \times \mathbb{R}^n)$ contains a point $\bar{q}_k$ of multiplicity $Q + \frac{1}{2}$ or it is empty. Since however $(\mathbf{p}_{\pi_0})_\sharp Z_k = Q_k\,[\![B_1(0)]\!] \to (\mathbf{p}_{\pi_0})_\sharp Z$ (by the constancy theorem), for k large enough we would have $(\mathbf{p}_{\pi_0})_\sharp Z_k = Q\,[\![B_1(0)]\!]$, contradic! ting the emptyness of $\mathrm{spt}(Z_k) \cap (\{0\} \times \mathbb{R}^n) = \emptyset$ because $Q \geqslant 1$. As for the other alternative, we must have, by the almost minimality of Z_k (see Proposition 5.8)

$$\limsup_{k \to \infty} \|Z_k\|(\mathbf{B}_{1/2-|\bar{q}_k|}(\bar{q}_k)) \leqslant \lim_{k \to \infty} \|Z_k\|(\mathbf{B}_{1/2}(0)) = \frac{Q}{4}\omega_2\,.$$

Since $\bar{q}_k \to 0$, the almost monotonicity formula (see Proposition 5.8) would imply $\Theta(\bar{q}_k, Z_k) \leqslant Q + o(1)$. $\square$

Proof of Lemma 2.16. Since $Q\bar{Q}\,[\![\pi_0]\!]$ is tangent to T at 0, we obviously must have $T_0\Sigma \supset \pi_0$ and thus $T_0\Sigma = \mathbb{R}^{2+\bar{n}} \times \{0\}$ can be achieved suitably rotating the coordinates. To achieve the other two conclusions we scale Σ and intersect it with $\mathbf{C}_4(0, T_0\Sigma)$ to reach that $\Sigma \cap \mathbf{C}_4(0, T_0\Sigma)$ is the graph of some Ψ with very small C^{3,ε_0} norm. We can then extend Ψ outside $B_4(0, T_0\Sigma)$ without increasing the C^{3,ε_0} norm by more than a factor: this gives (i) and (ii) and also shows that $\mathbf{c}$ can be assumed smaller than ε_{41} in case (a) and (c) of Definition 1.1. For the details we refer the reader to the proof of [20, Lemma 1.5]. The rest of the Lemma is a simple scaling argument. $\square$

2.8 APPENDIX B: CONFORMAL COORDINATES FOR BRANCHED SURFACES

In order to prove the Proposition we recall the following classical fact about the existence of conformal coordinates. As in the rest of the paper, e denotes the standard euclidean metric.

Lemma 2.23. *For every $k \in \mathbb{N}$ and $\alpha, \beta \in]0,1[$ there are positive constants C_0 and c_0 with the following properties. Let g be a $C^{k,\beta}$ Riemannian metric on the unit disk $B_2 \subset \mathbb{R}^2$ with $\|g - e\|_{C^{0,\alpha}} \leqslant c_0$. Then there exists an orientation preserving diffeomorphism $\Lambda : \Omega \to B_2$ and a positive function $\lambda : \Omega \to \mathbb{R}$ such that* *(i)* $\Lambda^\sharp g = \lambda e$;

(ii) $\|\Lambda - \mathrm{Id}\|_{C^{1,\alpha}} + \|\lambda - 1\|_{C^{0,\alpha}} \leqslant C_0 \|g - e\|_{C^{0,\alpha}}$;

(ii) $\|\Lambda - \mathrm{Id}\|_{C^{k+1,\beta}} + \|\lambda - 1\|_{C^{k,\beta}} \leqslant C_0 \|g - e\|_{C^{k,\beta}}$.

Although the statement above is a well-known fact (and it follows, for instance, from the treatment of the problem given in [61, Addendum 1 to Chapter 9]), we have not been able to find a classical reference for it. However a complete proof can be found in the Appendix of [23].

Proof of Proposition 2.17. After rescaling we can assume that $\rho \geqslant 2^Q$. We fix Q and drop subscripts in $\mathfrak{B}_{Q,2}$. Observe also that, if we rescale by a large factor R, the constants C_i in Definition 2.11 can then replaced by the constants $C_i R^{-\alpha}$. Hence, without loss of generality we can assume that C_i is sufficiently small.

Let $\mathbf{\Phi} : \mathfrak{B} \to \mathbb{R}^{n+2}$ be the graphical parametrization of the branching and recall that $g = \mathbf{\Phi}^\sharp e$. Fix a point $(z_0, w_0) \in \mathfrak{B} \setminus \{0\}$, let $r := |z_0|/2$ and observe that on $B_r(z_0, w_0)$ we can use z as a chart and compute the metric tensor explicitely as

$$g_{ij}(z,w) = \delta_{ij} + \partial_i u(z,w) \partial_j u(z,w) =: \delta_{ij} + \sigma_{ij} .$$

It then follows easily that

$$|D^j \sigma(z)| \leqslant C_0 C_i^2 |z|^{2\alpha - j} \qquad \text{for } j \in \{0,1,2\} \tag{2.56}$$

$$[D^2 \sigma]_{\alpha, B_r(z_0,w_0)} \leqslant C_0 C_i^2 r^{\alpha-2} . \tag{2.57}$$

Step 1. Next consider the map $\boldsymbol{W} \colon \mathbb{R}^2 \supset B_2 \to \mathfrak{B}$ defined by $\boldsymbol{W}(z) := (z^Q, z)$. We set

$$\bar g = \boldsymbol{W}^\sharp g = (\mathbf{\Phi} \circ \boldsymbol{W})^\sharp e .$$

We then infer that (following Einstein's convention on repeated indices)

$$\bar g_{ij}(z) = Q^2 |z|^{2Q-2} \delta_{ij} + \sigma_{kl}(z^Q) \partial_i \boldsymbol{W}_l \partial_j \boldsymbol{W}_k ,$$

and we set

$$\tau(z) := (Q^2 |z|^{2Q-2})^{-1} \, \bar g(z) .$$

We then easily see that

$$|\tau(z) - e| \leqslant C_0 |z|^{-(2Q-2)} |D\boldsymbol{W}(z)|^2 |\sigma(z^Q)| \leqslant C_0 C_i^2 |z|^{2Q\alpha} .$$

Differentiating the identity which defines τ we also get

$$\begin{aligned} |D\tau(z)| \leqslant & C_0 |z|^{-(2Q-1)} |D\boldsymbol{W}(z)|^2 |\sigma(z^Q)| + C_0 |z|^{-(2Q-2)} |D^2\boldsymbol{W}(z)| |D\boldsymbol{W}(z)| |\sigma(z^Q)| \\ & + C_0 |z|^{-(2Q-2)} |D\boldsymbol{W}(z)|^2 |D\sigma(z^Q)| |z|^{Q-1} \\ \leqslant & C_0 C_i^2 |z|^{2Q\alpha - 1} . \end{aligned}$$

Analogous computations lead then to the estimates

$$|D^j(\tau - e)|(z) \leqslant C_0 C_i^2 |z|^{2Q\alpha - j} \qquad \text{for } j \in \{0,1,2\} \tag{2.58}$$
$$[D^2\tau]_{\alpha, B_s(z)} \leqslant C_0 C_i^2 |z|^{2Q\alpha - 2 - \alpha} \qquad \text{for } s = |z|/2. \tag{2.59}$$

Interpolating between the C^1 and the C^0 bound, we easily conclude that

$$[\tau]_{2Q\alpha, B_{2r}\setminus B_r} \leqslant C_0 C_i^2 .$$

Note in particular that τ (unlike g) can be extended to a nondegenerate $C^{0,Q\alpha}$ metric to the origin.

Since C_i can be assumed sufficiently small, we can apply Lemma 2.23 to find an orientation preserving diffeomorphism $\Lambda\colon \Omega \to B_2$ and a function $\lambda : \Omega \to \mathbb{R}^+$ such that

$$\Lambda^\sharp \tau = \bar\lambda e \tag{2.60}$$
$$\|\Lambda - \mathrm{Id}\|_{C^{1,2Q\alpha}} + \|\bar\lambda - 1\|_{C^{0,2Q\alpha}} \leqslant C_0 C_i . \tag{2.61}$$

Observe that, without loss of generality, we can assume that $0 \in \Omega$ and $\Lambda(0) = 0$. In particular (2.61) implies that, for C_i suitably small, $B_1 \subset \Omega$ and hence we will regard Λ and λ as defined on B_1. Next divide Λ by $\bar\lambda(0)^{\frac{1}{2}}$ and keep, by abuse of notation, the same symbols for the resulting map and the resulting conformal factor in (2.60). After this normalization we achieve that $\bar\lambda(0) = 1$ and that the estimates (2.61) still hold with a larger C_0. Moreover, $\bar\lambda(0) = 1$ implies that $D\Lambda(0) \in SO(2)$: composing Λ with an appropriate rotation we can then assume that $D\Lambda(0)$ is the identity. This implies that

$$|\bar\lambda(z) - 1| \leqslant C_0 C_i |z|^{Q\alpha} \tag{2.62}$$
$$|D^j(\Lambda(z) - z)| \leqslant C_0 C_i |z|^{1+Q\alpha - j} \qquad \text{for } j \in \{0,1\}. \tag{2.63}$$

Step 2. We next wish to estimates the higher derivatives of both Λ and λ. We adopt the following procedure. We fix a point $p \neq 0$ and let $r := |p|/2$. We then apply a simple scaling argument to rescale $B_r(p)$ to a ball of radius 2 so that we can apply Lemma 2.23. If we rescale back to $B_r(p)$ it is then easy to see that we find maps $\Lambda_p : \Omega_p \to B_r(p)$, $\lambda_p : \Omega \to \mathbb{R}^+$ with the properties properties:

$$\Lambda_p^\sharp \tau = \lambda_p g \tag{2.64}$$
$$\|\Lambda_p - \mathrm{Id}\|_{C^{1,2Q\alpha}} + \|\lambda_p - 1\|_{C^{0,2Q\alpha}} \leqslant C_0 C_i \tag{2.65}$$
$$[\Lambda_p - \mathrm{Id}]_{3,\alpha} + [\lambda_p - 1]_{2,\alpha} \leqslant C_0 C_i r^{2Q\alpha - 2 - \alpha} . \tag{2.66}$$

Note that $\Xi := \Lambda \circ \Lambda_p^{-1}$ Moreover, its domain is a disk of radius r. Since

$$\sup_z |\partial_z(\Xi(z) - z)| \leqslant C_0 r^{2Q\alpha} ,$$

we easily conclude the higher derivative estimates

$$\|\partial_k^z(\Xi - z)\| \leqslant C_0 C_i r^{2Q\alpha - k} \qquad \text{for } k \in \{1,2,3,4\},$$

which, by holomorphicity, are actually estimates on the full derivatives. Since $\Lambda = \Xi \circ \Lambda_p$ we then easily conclude that

$$|D^{j+1}\Lambda(z)| + |D^j(\bar\lambda(z) - 1)| \leqslant C_0 C_i |z|^{2Q\alpha - j} \qquad \text{for } j \in \{0,1,2\} \tag{2.67}$$

$$[D^3\Lambda]_{\alpha, B_r(z)} + [D^2\bar\lambda]_{\alpha, B_r(z)} \leqslant C_0 C_i r^{2Q\alpha - 2 - \alpha} \qquad \text{for } r = |z|/2 > 0\,. \tag{2.68}$$

Finally notice that

$$\left(\Lambda^\sharp \bar g\right)(z) = Q^2 |\Lambda(z)|^{2Q-2} \bar\lambda(z) e\,. \tag{2.69}$$

Step 3. We are finally ready to define $\boldsymbol\Psi := \boldsymbol\Phi \circ \boldsymbol W \circ \Lambda \circ \boldsymbol W^{-1}$. First of all observe that

$$(\boldsymbol\Psi^\sharp e)(z,w) = ((\boldsymbol W^{-1})^\sharp \Lambda^\sharp \bar g)(z,w) = \frac{|\Lambda(\boldsymbol W^{-1}(z,w))|^{2Q-2}}{|z|^{2-2/Q}} \bar\lambda(\boldsymbol W^{-1}(z,w)) e =: \lambda(z,w) e\,.$$

Since $|\boldsymbol W^{-1}(z,w)| = |z|^{1/Q}$, we can also estimate

$$\begin{aligned}
|\lambda(z,w) - 1| \leqslant & \frac{|\Lambda(\boldsymbol W^{-1}(z,w))|^{Q-2}}{|z|^{2-2/Q}} |\bar\lambda(\boldsymbol W^{-1}(z,w)) - 1| + C \frac{|\Lambda(\boldsymbol W^{-1}(z,w))|^{Q-2} - |z|^{2-2/Q}}{|z|^{2-2/Q}} \\
\leqslant & C_0 C_i^2 |\boldsymbol W^{-1}(z,w)|^{2Q\alpha} + C_0 |z|^{-1/Q} \left(|\Lambda(\boldsymbol W^{-1}(z,w))| - |\boldsymbol W^{-1}(z,w)|\right) \\
\leqslant & C_0 C_i^2 |z|^{2\alpha} + C_0 C_i^2 |z|^{-1/Q} |\boldsymbol W^{-1}(z,w)|^{1+2Q\alpha} \leqslant C_0 C_i^2 |z|^{2\alpha}\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
|D\lambda(z,w)| \leqslant & C_0 |D\bar\lambda(\boldsymbol W^{-1}(z,w))| |z|^{-1} + C_0 \left| D \frac{|\Lambda(\boldsymbol W^{-1}(z,w))|^{2Q-2}}{|\boldsymbol W^{-1}(z,w)|^{2Q-2}} \right| \\
\leqslant & C_0 C_i^2 |z|^{2\alpha - 1} + C_0 \left| D \frac{|\Lambda(\boldsymbol W^{-1}(z,w)|}{|\boldsymbol W^{-1}(z,w)|} \right|
\end{aligned}$$

and observe that

$$\begin{aligned}
\left| D \frac{|\Lambda(\boldsymbol W^{-1})|}{|\boldsymbol W^{-1}|} \right| = & \left| \left(\frac{D\Lambda(\boldsymbol W^{-1})}{|\Lambda(\boldsymbol W^{-1})||\boldsymbol W^{-1}|} - \frac{|\Lambda(\boldsymbol W^{-1})|}{|\boldsymbol W^{-1}|^3} \mathrm{Id} \right) D\boldsymbol W^{-1} \boldsymbol W^{-1} \right| \\
\leqslant & C_0 |D\boldsymbol W^{-1}| |\boldsymbol W^{-1}|^{-1} \left(|D\Lambda(\boldsymbol W^{-1}) - \mathrm{Id}| + |\boldsymbol W^{-1}| \left(|\Lambda(\boldsymbol W^{-1}) - (\boldsymbol W^{-1})| \right) \right) \\
\leqslant & C_0 C_i^2 |D\boldsymbol W^{-1}| |\boldsymbol W^{-1}|^{2Q\alpha - 2}\,.
\end{aligned}$$

Recalling that $|D\boldsymbol W^{-1}(z,w)| \leqslant |z|^{1/Q - 1}$, $|\boldsymbol W^{-1}(z,w)| = |z|^{1/Q}$, we conclude

$$|D\lambda(z,w)| \leqslant C_0 C_i^2 |z|^{2\alpha - 1}\,.$$

The estimates on the second derivative and its Hölder norm follow from similar computations.

We now come to the estimates on $\boldsymbol\Psi$. Let $\bar\Lambda := \boldsymbol W \circ \Lambda \circ \boldsymbol W^{-1}$. Fix $(z_0, w_0) \neq 0$, let $r := |z_0|/2$ and use z as a local chart. It will then suffice to show that

$$|D^j(\bar\Lambda(z) - z)| \leqslant C_0 C_i |z|^{1+\alpha - l} \qquad \text{for } j \in \{0,1,2,3\} \tag{2.70}$$

$$[D^3 \bar\Lambda]_{\alpha, B_r(z_0, w_0)} \leqslant C_0 C_i |z|^{-2}\,. \tag{2.71}$$

On the other hand since $\bar{\Lambda}(0,0) = (0,0)$, it actually suffces to show the first estimate for $j = 1$ to obtain it in the case $j = 0$.

We start computing the first derivatives:

$$D\bar{\Lambda} = D\boldsymbol{W}(\Lambda \circ \boldsymbol{W}^{-1}) D\Lambda(\boldsymbol{W}^{-1}) D\boldsymbol{W}^{-1} .$$

Recalling that $D\boldsymbol{W}(\boldsymbol{W}^{-1}) D\boldsymbol{W}^{-1} = \mathrm{Id}$, we estimate

$$\begin{aligned}
|D\bar{\Lambda}(z) - \mathrm{Id}| \leqslant & |D\boldsymbol{W}(\Lambda(\boldsymbol{W}^{-1}(z))) - D\boldsymbol{W}(\boldsymbol{W}^{-1}(z))||D\Lambda(\boldsymbol{W}^{-1}(z))||D\boldsymbol{W}^{-1}(z)| \\
& + |D\boldsymbol{W}(\boldsymbol{W}^{-1}(z))||D\Lambda(\boldsymbol{W}^{-1}(z)) - \mathrm{Id}||D\boldsymbol{W}^{-1}(z)| \\
\leqslant & C_0 |\boldsymbol{W}^{-1}(z)|^{Q-1} |\Lambda(\boldsymbol{W}^{-1}(z)) - \boldsymbol{W}^{-1}(z)||z|^{1/Q-1} \\
\leqslant & + C_0 C_i^2 |\boldsymbol{W}^{-1}(z)|^{Q-1} ||\boldsymbol{W}^{-1}(z)|^{2Q\alpha} |z|^{1/Q-1} \\
\leqslant & C_0 C_i^2 |\boldsymbol{W}^{-1}(z)|^{Q+2Q\alpha} |z|^{1/Q-1} + C_0 C_i^2 |z|^{2\alpha} \leqslant C_0 C_i^2 |z|^{2\alpha} .
\end{aligned}$$

Similar computations give the estimates on the higher derivatives. □

Part II

STEP 1: APPROXIMATION OF CURRENTS WITH Q-VALUED FUNCTIONS

3 MULTIPLE VALUED FUNCTIONS AND INTEGRAL CURRENTS

The content of this Chapter is taken mainly from the works of De Lellis and Spadaro in their proof of Almgren's regularity result for Area Minimizing currents. In particular the main references are [17], [18], and [19]. The chapter is organized in three sections each addressing useful tools from the theory of multiple valued maps and their link with integral currents. The first section deals with the theory of multiple valued functions. In particular, after giving the basic definitions, we address the questions of existence and regularity of energy minimizing maps, together with some useful properties such as higher integrability of their gradient and unique continuation. Moreover we give a reparametrization criterium and a very general construction of competitors for the energy.

The second section deals with the identification of the image of a Q-valued function with an integral current of multiplicity Q, the good behaviour of the usual boundary operation and an explicit formula to compute the mass.

In the third and final section we recall the Taylor expansion for the mass of the image of a multiple valued function in terms of the energy, in the graphical case and in a slightly more general situation. As a consequence we derive the corresponding expansions for the excess and the first variations.

3.1 TUTORIAL ON MULTIPLE VALUED FUNCTIONS AND DIR-MINIMIZERS

In this section we recall some basic results from the theory of multiple valued maps developed in [17] and the main properties of Dir-minimizing functions that will be needed in the sequel.

Definition 3.1. We denote by $[\![P]\!]$ the Dirac mass centered in $P \in \mathbb{R}^n$ and we define the space of Q-points as

$$\mathcal{A}_Q(\mathbb{R}^n) :=\left\{\sum_{i=1}^{Q} [\![P_i]\!] \,:\, P_i \in \mathbb{R}^n \text{ for every } i = 1, \dots, Q\right\}.$$

Moreover for every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we define

$$\mathcal{G}(T_1, T_2) := \min_{\sigma\in\mathscr{P}_Q} \sqrt{\sum_i |P_i - S_{\sigma(i)}|^2},$$

where $\mathscr{P}_Q$ denotes the group of permutations of $\{1, \dots, Q\}$. We adopt the convention that $|T| = \mathcal{G}(T, Q\,[\![0]\!])$.

If $T = \sum_{i=1}^{Q} [\![P_i]\!] \in \mathcal{A}_Q$ we define the *diameter* and the *separation* of T by

$$d(T) := \max_{i,j} |P_i - P_j| \quad \text{and} \quad s(T) := \min\{|P_i - P_j| \,:\, P_i \neq P_j\}$$

with the convention that $s(T) = \infty$ if $T = Q\,[\![P]\!]$.
Finally we define the map $\boldsymbol{\eta}\colon \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ which takes each measure $T = \sum_{i=1}^{Q} [\![P_i]\!]$ to its center of mass $\boldsymbol{\eta}(T) := \frac{\sum_i P_i}{Q}$.

The couple $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ is a metric space so the usual functional spaces (Continuous, Lipschitz, Hölder, Measurable, L^p) are well defined, in particular $L^p(\Omega, \mathcal{A}_Q)$ consits of those map $u\colon \Omega \to \mathcal{A}_Q$ such that $\|\mathcal{G}(u, Q\,[\![0]\!])\|_{L^p}$ is finite. Furthermore we have the following easy decomposition result.

Lemma 3.2 (Measurable selection [17, Proposition 0.4]). *Let $B \subset \mathbb{R}^m$ be a measurable set and let $f\colon B \to \mathbb{R}^n$ be a measurable function. Then, there exist $f_1, \ldots, f_Q$ measurable $\mathbb{R}^n$-valued functions such that*

$$f(x) = \sum_{i=1}^{Q} [\![f_i(x)]\!] \quad \textit{for a.e. } x \in B\,.$$

3.1.1 *Lipschitz Multiple valued maps*

Multiple valued Lipschitz maps enjoy similar properties to their vector valued counterparts. This is a consequence of the following decomposition Lemma, which allows us to perform inductive reasoning on the multiplicity Q.

Lemma 3.3 (Lipschitz decomposition [17, Proposition 1.2]). *Let $f\colon B \subset \mathbb{R}^m \to \mathcal{A}_Q$ be a Lipschitz function, $f = \sum_{i=1}^{Q} [\![f_i]\!]$. Suppose that there exist $x_0 \in B$ and $i, j \in \{1, \ldots, Q\}$ such that*

$$|f_i(x_0) - f_j(x_0)| > 3(Q-1)\,\mathrm{Lip}(f)\,\mathrm{diam}(B)\,.$$

Then, there is a decomposition of f into two simpler Lipschitz functions f_K and f_L with $\mathrm{Lip}(f_K), \mathrm{Lip}(f_L) \leqslant \mathrm{Lip}(f)$ and $\mathrm{spt}(f_K(x)) \cap \mathrm{spt}(f_L(x)) = \emptyset$ for every x.

Using this result one can prove the following extension result.

Proposition 3.4 (Lipschitz extension [17, Theorem 1.7]). *Let $f\colon B \subset \mathbb{R}^m \to \mathcal{A}_Q$ be Lipschitz. Then, there exists an extension $\bar{f}\colon \mathbb{R}^m \to \mathcal{A}_Q$ of f, with $\mathrm{Lip}(\bar{f}) \leqslant C(m, Q)\,\mathrm{Lip}(f)$. Moreover, if f is bounded, then*

$$\sup_{x \in \mathbb{R}^m} |\bar{f}(x)| \leqslant C(m, Q) \sup_{x \in B} |f(x)|\,.$$

Next we study the differentiability properties of Lipschitz maps.

Definition 3.5. Let $f\colon B \subset \mathbb{R}^m \to \mathcal{A}_Q$ and $x_0 \in B$. We say that f is differentiable at x_0 if there exist Q matrices L_i satisfying:

(i) $\mathcal{G}(f(x), T_{x_0} f) = o(|x - x_0|)$, where

$$T_{x_0} f(x) := \sum_i [\![L_i \cdot (x - x_0) + f_i(x_0)]\!]\ ;$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The point $\sum_i [\![L_i]\!] \in \mathcal{A}_Q(\mathbb{R}^{n\times m})$ will be called the differential of f at x_0 and denoted by $Df(x_0)$. Moreover we define the directional derivative in direction ν by $\partial_\nu f(x) := \sum_i [\![Df_i(x)\cdot\nu]\!]$.

Differentiable functions enjoy a chain rule formula.

Proposition 3.6 (Chain rules [17, Proposition 1.12]). *Let $f\colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at x_0.*

(i) Consider $\mathbf{\Phi}\colon \tilde{\Omega} \to \Omega$ such that $\mathbf{\Phi}(y_0) = x_0$ and assume that $\mathbf{\Phi}$ is differentiable at y_0. Then, $f\circ\mathbf{\Phi}$ is differentiable at y_0 and

$$D(f\circ\mathbf{\Phi})(y_0) = \sum_i [\![Df_i(x_0)\cdot D\mathbf{\Phi}(y_0)]\!] \ .$$

(ii) Consider $\mathbf{\Psi}\colon \Omega_x \times \mathbb{R}^n_u \to \mathbb{R}^k$ such that $\mathbf{\Psi}$ is differentiable at $(x_0, f_i(x_0))$ for every i. Then, $\mathbf{\Psi}(x, f(x))$ fulfills (i) of Definition 3.5 and, if (ii) holds, then

$$D\mathbf{\Psi}(x,f)(x_0) = \sum_i [\![D_u\mathbf{\Psi}(x_0, f_i(x_0))\cdot Df_i(x_0) + D_x\mathbf{\Psi}(x_0, f_i(x_0))]\!] \ .$$

Moreover the analogous of Rademacher Theorem holds.

Proposition 3.7 (Rademacher [17, Theorem 1.13]). *Let $f\colon \Omega \to \mathcal{A}_Q$ be a Lipschitz function. Then, f is differentiable almost everywhere in Ω.*

3.1.2 *Sobolev Multiple Valued Maps*

In order to define Sobolev spaces we are going to use Almgren's extrinsic theory and immerse $\mathcal{A}_Q$ in a big $\mathbb{R}^N$ using a bilipschitz homeomorphism. It should be noted that it was an original contribution of De Lellis and Spadaro to carry out a theory of Sobolev multiple valued functions completely independent from this immersion and which relays on modern techniques for general metric spaces. Since we will need the immersion later on however, we prefer to adopt here Almgren's point of view.

Lemma 3.8 (Bilipschitz embedding [17, Theorem 2.1 & Corollary 2.2]). *There exists $N = N(Q,n)$ and an injective map $\boldsymbol{\xi}\colon \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ such that:*

(i) $\mathrm{Lip}(\boldsymbol{\xi}) \leqslant 1$;

(ii) if $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$, then $\mathrm{Lip}(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leqslant C(n,Q)$;

(iii) for every $T \in \mathcal{A}_Q(\mathbb{R}^n)$ there exists $\delta > 0$ such that

$$|\boldsymbol{\xi}(T) - \boldsymbol{\xi}(S)| = \mathcal{G}(S,T) \quad \forall\, S \in B_\delta(T) \subset \mathcal{A}_Q(\mathbb{R}^n)\,.$$

Moreover there exists a Lipschitz map $\boldsymbol{\rho}\colon \mathbb{R}^N \to \mathcal{Q}$ which is the identity on $\mathcal{Q}$.

Using this embedding we can give meaning to the notion of Sobolev spaces and trace operator.

Definition 3.9. Let ξ be the map of Lemma 3.8. Then a Q-valued function f belongs to the Sobolev space $W^{1,p}(\Omega, \mathcal{A}_Q)$ if $\xi \circ f$ belongs to $W^{1,p}(\Omega, \mathbb{R}^N)$. Furthermore for every $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, we define

$$\int_\Omega |Df|^p : = \int_\Omega |D(\xi \circ f)|^p \,.$$

Definition 3.10. Let $f \in W^{1,p}(\Omega, \mathcal{A}_Q)$. The trace of f is the unique function $g \in L^p(\partial\Omega, \mathcal{A}_Q)$ such that $\xi \circ f|_{\partial\Omega} = \xi \circ g$. Moreover the space

$$W^{1,p}_g(\Omega, \mathcal{A}_Q) := \{f \in W^{1,p}(\Omega, \mathcal{A}_Q) \,:\, f|_{\partial\Omega} = g\}$$

is sequentially weakly closed in $W^{1,p}$.

As in the classical theory, we can approximate Sobolev functions with Lipschitz functions.

Lemma 3.11 (Lipschitz approximation [19, Lemma 3.5])**.** *Let $f \in W^{1,p}(B, \mathcal{A}_Q)$. Then, for every $\varepsilon > 0$, there exists $f_\varepsilon \in \mathrm{Lip}(B, \mathcal{A}_Q)$ such that*

$$\int_B \mathcal{G}(f, f_\varepsilon)^p + \int_B (|Df| - |Df_\varepsilon|)^p + \int_B (|D(\boldsymbol{\eta} \circ f)| - |D(\boldsymbol{\eta} \circ f_\varepsilon)|)^p \leqslant \varepsilon \,. \tag{3.1}$$

If $f|_{\partial B} \in W^{1,2}(\partial B, \mathcal{A}_Q)$, then f_ε can be chosen to satisfy also

$$\int_{\partial B} \mathcal{G}(f, f_\varepsilon)^p + \int_{\partial B} (|Df| - |Df_\varepsilon|)^p \leqslant \varepsilon \,.$$

Remark 3.12. As a consequence of this, Sobolev functions are approximately differentiables and the chain rule of Proposition 3.6 holds at a.e. point. In particular it is possible to prove that

$$\int_\Omega |Df|^2 = \sum_{i,j} \int_\Omega |\partial_j f_i(x)|^2 \,dx \,, \tag{3.2}$$

where $\partial_j f_i$ are the approximate partial derivatives of f.

Another simple consequence of Lemma 3.8 is the validity of the usual Sobolev immersions for multiple valued functions and a sort of Poincaré inequality.

Proposition 3.13 (Sobolev Embeddings [17, Proposition 2.11])**.** *For $p < m$ set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{m}$. Then, the following embeddings hold:*

(i) if $p < m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^Q(\Omega, \mathcal{A}_Q)$ for every $q \in [1, p^*]$, and the inclusion is compact when $q < p^*$;

(ii) if $p = m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^Q(\Omega, \mathcal{A}_Q)$ for every $q \in [1, \infty)$, with compact inclusion;

(iii) if $p > m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$, for $\alpha = 1 - \frac{m}{p}$, with compact inclusion if $\alpha < 1 - \frac{m}{p}$.

Proposition 3.14 (Poincaré inequality [17, Proposition 2.12]). *Let M be a connected bounded Lipschitz open set of an m-dimensional Riemannian manifold and let $p < m$. There exists a constant $C = C(p, m, n, Q, M)$ with the following property: for every $f \in W^{1,p}(M, \mathcal{A}_Q)$, there exists a point $\bar{f} \in \mathcal{A}_Q$ such that*

$$\left(\int_M \mathcal{G}(f, \bar{f})^{p^*}\right)^{\frac{1}{p^*}} \leqslant C \left(\int_M |Df|^p\right)^{\frac{1}{p}},$$

where p^* is the Sobolev exponent of p, that is $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{m}$.

Finally we state two very useful technical Lemmas about $W^{1,2}$ multiple valued functions.

Lemma 3.15 (Interpolation lemma [19, Lemma 3.6]). *There exists a constant $C = C(m, n, Q) > 0$ with the following property. Assume $r \in]1, 3[$, $f \in W^{1,2}(B_r, \mathcal{A}_Q)$ and $g \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ are given maps such that $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$. Then, for every $\varepsilon \in]0, r[$ there exists a function $h \in W^{1,2}(B_r, \mathcal{A}_Q)$ such that $h|_{\partial B_r} = g$ and*

$$\int_{B_r} |Dh|^2 \leqslant \int_{B_r} |Df|^2 + \varepsilon \int_{\partial B_r} \left(|D_\tau f|^2 + |D_\tau g|^2\right) + \frac{C_0}{\varepsilon} \int_{\partial B_r} \mathcal{G}(f, g)^2, \tag{3.3}$$

$$\mathrm{Lip}(h) \leqslant C_0 \left\{\mathrm{Lip}(f) + \mathrm{Lip}(g) + \varepsilon^{-1} \sup_{\partial B_r} \mathcal{G}(f, g)\right\}, \tag{3.4}$$

$$\int_{B_r} |\boldsymbol{\eta} \circ h| \leqslant C_0 \int_{\partial B_r} |\boldsymbol{\eta} \circ g| + C_0 \int_{B_r} |\boldsymbol{\eta} \circ f|, \tag{3.5}$$

(here D_τ denotes the tangential derivative).

Lemma 3.16 (Irreducible selection [17, Proposition 1.2]). *$f \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$ is called* irreducible *if there is no decomposition of f into two simpler $W^{1,2}$ functions. For every Q-function $g \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$, there exists a decomposition $g = \sum_{j=1}^{J} [\![g_j]\!]$, where each g_j is an irreducible $W^{1,2}$ map. Moreover g is irreducible if and only if*

(i) $\mathrm{card}(\mathrm{spt}(g(z))) = Q$ *for every $z \in \mathbb{S}^1$;*

(ii) there exists a $W^{1,2}$ map $h \colon \mathbb{S}^1 \to \mathbb{R}^n$ with the property that $f(z) = \sum_{\zeta^Q = z} [\![h(\zeta)]\!]$.

3.1.3 *Reparametrization Lemmas*

The following two results will allow us to reparametrize Lipschitz functions both in the classical and the Q-valued cases on different domains whose tangent planes are sufficiently close.

Lemma 3.17 (Change of coordinates for classical functions [20, Lemma B.1]). *For any $m, n \in \mathbb{N} \setminus \{0\}$ and radii $0 < s < \rho$, there are constants $c_0, C_0 > 0$ depending on the ratio $\frac{\rho}{s}$ with the following properties. Assume that*

(i) $\varkappa, \varkappa_0 \subset \mathbb{R}^{m+n}$ are m-dim. planes with $|\varkappa - \varkappa_0| \leqslant c_0$;

(ii) $p = (q, u) \in \varkappa \times \varkappa^\perp$ *and* $f, g : B^m_\rho(q, \varkappa) \to \varkappa^\perp$ *are Lipschitz functions such that*

$$\mathrm{Lip}(f), \mathrm{Lip}(g) \leqslant c_0 \quad \textit{and} \quad |f(q) - u| + |g(q) - u| \leqslant c_0\, \rho.$$

Then there are two maps $f', g' : B_s(p, \varkappa_0) \to \varkappa_0^\perp$ *such that*

(a) $\mathbf{G}_{f'} = \mathbf{G}_f \llcorner \mathbf{C}_s(p, \varkappa_0)$ *and* $\mathbf{G}_{g'} = \mathbf{G}_g \llcorner \mathbf{C}_s(p, \varkappa_0)$;

(b) $\|f' - g'\|_{L^1(B_s(p,\varkappa_0))} \leqslant C_0\, \|f - g\|_{L^1(B_\rho(p,\varkappa))}$;

(c) if $f \in C^{3,\kappa}(B_\rho(p, \varkappa))$ *then* $f' \in C^{3,\kappa}(B_s(p, \varkappa_0))$ *with the estimates*

$$\|f' - u'\|_{C^0} \leqslant C\|f - u\|_{C^0} + C|\varkappa - \varkappa_0|r \tag{3.6}$$

$$\|Df'\|_{C^0} \leqslant C\|Df\|_{C^0} + C|\varkappa - \varkappa_0| \tag{3.7}$$

$$\|D^2 f'\|_{C^{1,\kappa}} \leqslant \Phi(|\varkappa - \varkappa_0|, \|D^2 f\|_{C^{1,\kappa}}) \tag{3.8}$$

where $(q', u') \in \varkappa_0 \times \varkappa_0^\perp$ *coincides with the point* $(q, u) \in \varkappa \times \varkappa^\perp$ *and* Φ *is a smooth functions with* $\Phi(\cdot, 0) \equiv 0$;

(d) $\|f' - g'\|_{W^{1,2}(B_s(p,\varkappa_0))} \leqslant C_0(1 + \|D^2 f\|_{C^0})\|f - g\|_{W^{1,2}(B_\rho(p,\varkappa))}$.

We should remark that the proof of the next Theorem exploits the interpretation of the graph of a Q-valued map as an integral current. This notion will be made clear in the next section.

Theorem 3.18 (Q-valued parametrizations [18, Theorem 5.1]). *Let* $Q, m, n \in \mathbb{N}$ *and* $s < r < 1$. *Then, there are constants* $c_0, C > 0$ *(depending on* Q, m, n *and* $\frac{r}{s}$*) with the following property. Let* $\boldsymbol{\varphi}$, $\mathcal{M}$ *and* $\mathbf{U}$ *be such that*

(M) $\mathcal{M} \subset \mathbb{R}^{m+n}$ *is an open submanifold of dimension* m *with* $\mathcal{H}^m(\mathcal{M}) < \infty$*, which is the graph of a function* $\boldsymbol{\varphi} : \mathbb{R}^m \supset B_s \to \mathbb{R}^n$ *with* $\|\boldsymbol{\varphi}\|_{C^3} \leqslant \bar{c}$;

(U) $\mathbf{U}$ *is a regular tubular neighborhood of* $\mathcal{M}$*, i.e. the set of points* $\{x + y : x \in \mathcal{M}, y \perp T_x\mathcal{M}, |y| < c_0\}$*, where the thickness* c_0 *is sufficiently small so that the nearest point projection* $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ *is well defined and* C^2*; the thickness is supposed to be larger than a fixed geometric constant (which depends on* $\bar{c}$*).*

Let $f : B_r \to \mathcal{A}_Q(\mathbb{R}^n)$ *be such that*

$$\|\boldsymbol{\varphi}\|_{C^2} + \mathrm{Lip}(f) \leqslant c_0 \qquad \textit{and} \qquad \|\boldsymbol{\varphi}\|_{C^0} + \|f\|_{C^0} \leqslant c_0\, r. \tag{3.9}$$

Set $\mathbf{\Phi}(x) := (x, \boldsymbol{\varphi}(x))$*. Then, there is a map* $F : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ *of the form*

$$\sum_{i=1}^{Q} [\![F_i(x)]\!] = \sum_{i=1}^{Q} [\![x + N_i(x)]\!]\,,$$

where $N : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ *satisfies* $x + N_i(x) \in \mathbf{U}$, $N_i(x) \perp T_x\mathcal{M}$ *for every* x *and* $\mathrm{Lip}(N) \leqslant \bar{c}$*, such that* $\mathbf{T}_F = \mathbf{G}_f \llcorner \mathbf{U}$ *and*

$$\mathrm{Lip}(N) \leqslant C\big(\|D^2\boldsymbol{\varphi}\|_{C^0}\|N\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0} + \mathrm{Lip}(f)\big)\,, \tag{3.10}$$

$$\frac{1}{2\sqrt{Q}}|N(\mathbf{\Phi}(p))| \leqslant \mathcal{G}(f(p), Q\,[\![\boldsymbol{\varphi}(p)]\!]) \leqslant 2\sqrt{Q}\,|N(\mathbf{\Phi}(p))| \qquad \forall p\, \in B_s\,, \tag{3.11}$$

$$|\boldsymbol{\eta} \circ N(\mathbf{\Phi}(p))| \leqslant C|\boldsymbol{\eta} \circ f(p) - \boldsymbol{\varphi}(p)| + C\mathrm{Lip}(f)|D\boldsymbol{\varphi}(p)||N(\mathbf{\Phi}(p))| \qquad \forall p\, \in B_s. \tag{3.12}$$

Finally, assume $p \in B_s$ *and* $(p, \boldsymbol{\eta} \circ f(p)) = \xi + q$ *for some* $\xi \in \mathcal{M}$ *and* $q \perp T_\xi \mathcal{M}$. *Then,*

$$\mathcal{G}(N(\xi), Q[\![q]\!]) \leqslant 2\sqrt{Q}\,\mathcal{G}(f(p), Q[\![\boldsymbol{\eta} \circ f(p)]\!]) . \tag{3.13}$$

For further reference, we state the following immediate corollary of Theorem 3.18, corresponding to the case of a linear $\boldsymbol{\varphi}$.

Proposition 3.19 (Q-valued graphical reparametrization [18, Proposition 5.2]). *Let* $Q, m, n \in \mathbb{N}$ *and* $s < r < 1$. *There exist positive constants* c, C *(depending only on* Q, m, n *and* $\frac{r}{s}$*) with the following property. Let* π_0 *and* π *be* m*-planes with* $|\pi - \pi_0| \leqslant c$ *and* $f : B_r(\pi_0) \to \mathcal{A}_Q(\pi_0^\perp)$ *with* $\mathrm{Lip}(f) \leqslant c$ *and* $|f| \leqslant cr$. *Then, there is a Lipschitz map* $g : B_s(\pi) \to \mathcal{A}_Q(\pi^\perp)$ *with* $\mathbf{G}_g = \mathbf{G}_f \llcorner \mathbf{C}_s(\pi)$ *and such that the following estimates hold on* $B_s(\pi)$*:*

$$\|g\|_{C^0} \leqslant Cr|\pi - \pi_0| + C\|f\|_{C^0}, \tag{3.14}$$

$$\mathrm{Lip}(g) \leqslant C|\pi - \pi_0| + C\mathrm{Lip}(f) . \tag{3.15}$$

3.1.4 *Main Regularity results about Dir-minimizing Maps*

We list in this subsection the main results about existence and regularity of Dir-minimizing function. We will not really need these results, but they are the analogous of our result on 2-dimensional currents for multiple valued maps.

Definition 3.20 (Dir-minimizing map). $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is said to be *Dir-minimizing* if

$$\int_\Omega |Df|^2 \leqslant \int_\Omega |Dh|^2 \quad \text{for all } h \in W^{1,2}(\Omega, \mathcal{A}_Q) \text{ with } f|_{\partial\Omega} = h|_{\partial\Omega} .$$

Theorem 3.21 (Existence for the Dirichlet Problem [17, Theorem 0.8]). *Let* $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$. *Then, there exists a Dir-minimizing function* $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *such that* $f|_{\partial\Omega} = g|_{\partial\Omega}$.

Proposition 3.22 (Harmonicity and compactness [17, Lemma 3.23 & Proposition 3.20]). *The following properties hold.*

(i) if $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ *is Dir-minimizing, then* $\boldsymbol{\eta} \circ f \in W^{1,2}(\Omega, \mathbb{R}^n)$ *is harmonic.*

(ii) Let $f_k \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *be Dir-minimizing Q-functions weakly converging to* f. *Then, for every open* $\Omega' \subset\subset \Omega$, $f|_{\Omega'}$ *is Dir-minimizing and it holds*

$$\lim_{k\to\infty} \int_{\Omega'} |Df_k|^2 = \int_{\Omega'} |Df|^2 .$$

Theorem 3.23 (Hölder regularity [17, Theorem 0.9]). *There exists a positive constant* $\alpha = \alpha(m, Q) > 0$ *with the following property. If* $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ *is Dir-minimizing, then* $f \in C^{0,\alpha}(\Omega')$ *for every* $\Omega' \subset\subset \Omega \subset \mathbb{R}^n$. *For two dimensional domains, we have the explicit constant* $\alpha(2, Q) = 1/Q$.

For the second regularity theorem we need the definition of singular set of f.

Definition 3.24 (Regular and Singular points)**.** A Q-valued function f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i \colon B \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \quad \text{for almost every } y \in B$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The singular set Σ_f of f is the complement of the set of regular points.

Theorem 3.25 (Estimate of the singular set [17, Theorem 0.11])**.** *Let f be a Dir-minimizing function. Then, the singular set Σ_f of f is relatively closed in Ω. Moreover, if $m = 2$, then Σ_f is at most countable, and if $m \geqslant 3$, then the Hausdorff dimension of Σ_f is at most $m - 2$.*

The next result is the analogous of Theorem 1.2 in the case of multiple valued maps.

Theorem 3.26 (Improved estimate of the singular set [17, Theorem 0.12])**.** *Let f be Dir-minimizing and $m = 2$. Then, the singular set Σ_f of f consists of isolated points.*

3.1.5 *Competitor construction*

In this section we show a concentration compactness principle for Q-valued functions, and give an algorithm to construct suitable competitors for the Dirichlet energy. All the results of this section come from [19].

Definition 3.27 (Translating sheets)**.** Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. A sequence of maps $\{h_k\}_{i\in\mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is called a sequence of *translating sheets* if there are:

(a) integers $J \geqslant 1$ and $Q_1, \dots, Q_J \geqslant 1$ satisfying $\sum_{j=1}^J Q_j = Q$,

(b) vectors $y_k^j \in \mathbb{R}^n$ (for $j \in \{1, \dots, J\}$ and $k \in \mathbb{N}$) with

$$\lim_k |y_k^j - y_k^i| = +\infty \qquad \forall i \neq j, \tag{3.16}$$

(c) and maps $\zeta^j \in W^{1,2}(\Omega, \mathcal{A}_{Q_j})$ for $j \in \{1, \dots, J\}$,

such that $h_k = \sum_{j=1}^J [\![\tau_{y_k^j} \circ \zeta^j]\!]$, where for any generic $y \in \mathbb{R}^n$ we denote by $\tau_y : \mathcal{A}_Q(\mathbb{R}^n) \to \mathcal{A}_Q(\mathbb{R}^n)$ the translation map (cp. [17, Section 3.3.3])

$$\mathcal{A}_Q(\mathbb{R}^n) \ni T = \sum_i [\![P_i]\!] \mapsto \tau_y(T) := \sum_i [\![P_i - y]\!] \in \mathcal{A}_Q(\mathbb{R}^n).$$

Remark 3.28. Assume that h_k, Q_j, y_k^j and ζ^k satisfy all the requirements of Definition 3.27 except for (3.16). Up to subsequences and relabellings, assume that $y_k^1 - y_k^2$ converges to a vector $2\bar{y}$. We can replace

- the integers Q_1 and Q_2 with $Q' = Q_1 + Q_2$;
- the vectors y_k^1 and y_2^k with $y_k' = (y_k^1 + y_k^2)/2$;

- the maps ζ^1 and ζ^2 with $\zeta' := [\![\tau_{\bar y}\circ\zeta^1]\!] + [\![\tau_{-\bar y}\circ\zeta^2]\!]$.

The new collections $Q', Q_3, \ldots, Q_J$, $y_k', y_k^3, \ldots, y_k^J$ and $\zeta', \zeta^3, \ldots, \zeta^J$, and the function $h_k' := [\![\zeta']\!] + \sum_{j=3}^J [\![\zeta^j]\!]$, satisfy again all the requirements of Definition 3.27 except, possibly, for (3.16). Moreover, $\|\mathcal{G}(h_k', h_k)\|_{L^2} \to 0$ and $|Dh_k'| = |Dh_k|$. Obviously, we can iterate this procedure only a finite number of times, obtaining a subsequence of translating sheets $\hat h_k$ asymptotic to h_k in the L^2 distance with $|D\hat h_k| = |Dh_k|$.

Concentration compactness

Translating sheets give a useful device to recover a suitable "compactness statement" for sequences of maps with equi-bounded energy.

Proposition 3.29 (Concentration compactness [19, Proposition 3.3]). *Let $\Omega \subset \mathbb{R}^m$ be a Lipschitz bounded open set and $(g_k)_{k\in\mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q)$ a sequence of functions with $\sup_k \int_\Omega |Dg_k|^2 < \infty$. Then, there exist a subsequence (not relabeled) and a sequence of translating sheets h_k such that $\|\mathcal{G}(g_k, h_k)\|_{L^2} \to 0$ and the following inequalities hold for every open $\Omega' \subset \Omega$ and any sequence of measurable sets J_k with $|J_k| \to 0$:*

$$\liminf_{k\to+\infty}\left(\int_{\Omega'\setminus J_k}|Dg_k|^2 - \int_{\Omega'}|Dh_k|^2\right) \geqslant 0 \tag{3.17}$$

$$\limsup_{k\to+\infty}\int_\Omega \left(|Dg_k| - |Dh_k|\right)^2 \leqslant \limsup_k \int_\Omega \left(|Dg_k|^2 - |Dh_k|^2\right)\,. \tag{3.18}$$

Proof. We start proving, by induction on Q, the existence of translating sheets $\{h_k\}$ (and a subsequence) with $\|\mathcal{G}(h_k, g_k)\|_{L^2} \to 0$ and satisfying the following additional property. If J, Q_j, y_k^j and ζ^j are as in Definition 3.27, then there are Q_j valued functions w_k^j such that, after setting $f_k = \sum_j [\![w_k^j]\!]$, we have

$$\|\mathcal{G}(f_k, g_k)\|_{L^2} + |\{g_k \neq f_k\}| \to 0, \quad \|\mathcal{G}(\tau_{-y_k^j}\circ w_k^j, \zeta^j)\|_{L^2} \to 0 \quad \text{and} \quad |Df_k| \leqslant |Dg_k|\,. \tag{3.19}$$

If $Q = 1$ the claim with $f_k = g_k$ is an easy corollary of the Poincaré inequality and the compact embedding $W^{1,2} \hookrightarrow L^2$. Assuming that the claim holds for any $Q^* < Q$, we prove it for Q. By the generalized Poincaré inequality Proposition 3.14:, there exist points $\bar g_k \in \mathcal{A}_Q(\mathbb{R}^n)$ and a real number M such that

$$\int_\Omega \mathcal{G}(g_k, \bar g_k)^2 \leqslant C\int_\Omega |Dg_k|^2 \leqslant M < \infty \qquad \forall\, k \in \mathbb{N}\,.$$

Recall the separation $s(T)$ and the diameter $d(T)$ of a point $T = \sum_i [\![P_i]\!]$ introduced in Definition 3.1: $s(T) := \min\left\{|P_i - P_j| : P_i \neq P_j\right\}$ and $d(T) := \max\{|P_i - P_j|\}$. We distinguish between to cases.

Case 1: $\liminf_k d(\bar g_k) < \infty$. After passing to a subsequence, we find $y_k \in \mathbb{R}^n$ such that the functions $\tau_{y_k}\circ g_k$ are equi-bounded in the $W^{1,2}$-metric. By the Sobolev embedding, Proposition 3.13, there exists a Q-valued map $\zeta \in W^{1,2}$ such that $\tau_{y_k}\circ g_k \to \zeta$ in $L^2(\Omega)$.

Case 2: $\lim_k d(\bar g_k) = +\infty$. By [17, Lemma 3.8] there are points $S_k \in \mathcal{A}_Q$ such that

$$\beta\, d(\bar g_k) \leqslant s(S_k) < +\infty \quad \text{and} \quad \mathcal{G}(S_k, \bar g_k) \leqslant s(S_k)/32,$$

where β is a dimensional constant. Write $S_k = \sum_{i=1}^{J} \kappa_i [\![P_k^i]\!]$, with $P_k^i \neq P_k^j$ for $i \neq j$. Both J and κ_i may depend on k but they have a finite range: therefore, after extracting a subsequence, we can assume that they do not depend on k. Set next $r_k = \frac{s(S_k)}{16}$ and let ϑ_k be the retraction of $\mathcal{A}_Q(\mathbb{R}^n)$ into $\overline{B_{r_k}(S_k)}$ provided by [17, Lemma 3.7]. Clearly, the functions $\hat{f}_k = \vartheta_k \circ g_k$ satisfy $|D\hat{f}_k| \leqslant |Dg_k|$ and there are κ_i-valued functions z_k^i such that

$$\hat{f}_k = \sum_{i=1}^{J} [\![z_k^i]\!], \quad \text{with} \quad \|\mathcal{G}(z_k^i, \kappa_i [\![P_k^i]\!])\|_\infty \leqslant r_k.$$

Since $\kappa_i < Q$, we apply the inductive hypothesis to each sequence $(z_k^i)_k$ and, using Remark 3.28 reach a subsequence (not relabeled) of $\hat{f}_k$, a sequence of translating sheets h_k and corresponding functions f_k which satisfy (3.19) with $\hat{f}_k$ replacing g_k.

We next claim that (3.19) holds even for g_k, i.e. that $\lim_k \left(\|\mathcal{G}(f_k, g_k)\|_{L^2} + |\{f_k \neq g_k\}|\right) = 0$. To this aim, recall first that

$$\left\{g_k \neq \hat{f}_k\right\} = \{\mathcal{G}\left(g_k, S_k\right) > r_k\} \subseteq \{\mathcal{G}\left(g_k, \bar{g}_k\right) > r_k/2\}.$$

Thus,

$$\left|\left\{g_k \neq \hat{f}_k\right\}\right| \leqslant |\{\mathcal{G}\left(g_k, \bar{g}_k\right) > r_k/2\}| \leqslant \frac{C}{r_k^2} \int_{\{\mathcal{G}(g_k, \bar{g}_k) > \frac{r_k}{2}\}} \mathcal{G}\left(g_k, \bar{g}_k\right)^2 \leqslant \frac{CM}{(d(\bar{g}_k))^2}. \tag{3.20}$$

Since $d(\bar{g}_k) \to +\infty$ and (3.19) holds with $\hat{f}_k$ replacing g_k, we conclude $|\{f_k \neq g_k\}| \to 0$. Next, since $\vartheta_k(\bar{g}_k) = \bar{g}_k$ and $\mathrm{Lip}(\vartheta_k) = 1$, we have $\mathcal{G}(\hat{f}_k, \bar{g}_k) \leqslant \mathcal{G}(g_k, \bar{g}_k)$. Therefore, by the Sobolev embedding and the Poincaré inequality, for any $p \in]2, 2^*[$, we infer

$$\begin{aligned}\int_\Omega \mathcal{G}(\hat{f}_k, g_k)^2 &= \int_{\{g_k \neq \hat{f}_k\}} \mathcal{G}(\hat{f}_k, g_k)^2 \leqslant 2 \int_{\{\hat{f}_k \neq g_k\}} \mathcal{G}(\hat{f}_k, \bar{g}_k)^2 + 2 \int_{\{\hat{f}_k \neq g_k\}} \mathcal{G}(\bar{g}_k, g_k)^2 \\ &\leqslant 4 \int_{\{\hat{f}_k \neq g_k\}} \mathcal{G}(\bar{g}_k, g_k)^2 \leqslant C \left\|\mathcal{G}\left(g_k, \bar{g}_k\right)\right\|_{L^p}^2 \left|\left\{\hat{f}_k \neq g_k\right\}\right|^{1-\frac{2}{p}} \stackrel{(3.20)}{\leqslant} \frac{CM^{1-\frac{2}{p}}}{d(\bar{g}_k)^{2-\frac{4}{p}}} \int_\Omega |Dg_k|^2.\end{aligned}$$

Since $d(\bar{g}_k)$ diverges, this shows $\|\mathcal{G}(\hat{f}_k, g_k)\|_{L^2} \to 0$ and by inductive hypothesis that $\|\mathcal{G}(f_k, g_k)\|_{L^2} \to 0$.

We now show that (3.17) and (3.18) are consequences of (3.19). For each j we consider the corresponding embedding $\boldsymbol{\xi}_j : \mathcal{A}_{Q_j}(\mathbb{R}^n) \to \mathbb{R}^{N(Q_j,n)}$ and, by a slight abuse of notation, we drop the j subscript. Then, we conclude that $\boldsymbol{\xi} \circ \tau_{-y_k^j} \circ w_k^j \to \boldsymbol{\xi} \circ \zeta^j$ in L^2 and $\|D(\boldsymbol{\xi} \circ \tau_{-y_k^j} \circ w_k^j)\|_{L^2}$ is a bounded sequence, from which

$$D(\boldsymbol{\xi} \circ \tau_{-y_k^j} \circ w_k^j) \rightharpoonup D(\boldsymbol{\xi} \circ \zeta^j) \quad \text{in } L^2(\Omega). \tag{3.21}$$

If J_k is a sequence of measurable sets with $|J_k| \downarrow 0$, then $\mathbf{1}_{\Omega' \setminus J_k} \to \mathbf{1}_{\Omega'}$ in $L^2(\Omega)$ and it follows from (3.21) that

$$D(\boldsymbol{\xi} \circ \tau_{-y_k^j} \circ w_k^j)\mathbf{1}_{\Omega' \setminus J_k} \rightharpoonup D(\boldsymbol{\xi} \circ \zeta^j)\mathbf{1}_{\Omega'} \quad \text{in } L^2(\Omega),$$

and, hence,

$$\mathrm{Dir}(\zeta^j,\Omega') = \int_{\Omega'} |D(\xi\circ\zeta^j)|^2 \leqslant \liminf_k \int_{\Omega'\setminus J_k} |D(\xi\circ\tau_{-y_k^j}\circ w_k^j)|^2 = \liminf_k \int_{\Omega'\setminus J_k} |Dw_k^j|^2.$$

Summing over j, we obtain (3.17). As for (3.18), set $J_k := \{g_k \neq f_k\}$. Thus,

$$\begin{aligned}
&\int_{\Omega\setminus J_k} (|Dg_k| - |Dh_k|)^2 \leqslant \sum_j \int_{\Omega\setminus J_k} (|Dw_k^j| - |D\zeta^j|)^2 \\
&= \sum_j \int_{\Omega\setminus J_k} \left(|D(\xi\circ\tau_{-y_k^j}\circ w_k^j)| - |D(\xi\circ\zeta^j)|\right)^2 \leqslant \sum_j \int_{\Omega\setminus J_k} |D(\xi\circ\tau_{-y_k^j}\circ w_k^j) - D(\xi\circ\zeta^j)|^2 \\
&= \sum_j \int_{\Omega\setminus J_k} \left(|D(\xi\circ\tau_{-y_k^j}\circ w_k^j)|^2 + |D(\xi\circ\zeta^j)|^2 - 2\,D(\xi\circ\tau_{-y_k^j}\circ w_k^j)\cdot D(\xi\circ\zeta^j)\right).
\end{aligned} \tag{3.22}$$

Therefore, by (3.21) (and taking into account that $|J_k| \to 0$) one gets

$$\begin{aligned}
&\limsup_{k\to+\infty} \int_{\Omega\setminus J_k} (|Dg_k| - |Dh_k|)^2 \\
&\leqslant \lim_{k\to+\infty} \sum_j \int_{\Omega\setminus J_k} \left(|D(\xi\circ\tau_{-y_k^j}\circ w_k^j)|^2 + |D(\xi\circ\zeta^j)|^2 - 2\,D(\xi\circ\tau_{-y_k^j}\circ w_k^j)\cdot D(\xi\circ\zeta^j)\right) \\
&= \limsup_{k\to+\infty} \int_{\Omega\setminus J_k} \sum_j |D(\xi\circ\tau_{-y_k^j}\circ w_k^j)|^2 - \int_\Omega \sum_j |D(\xi\circ\zeta^j)|^2 \\
&= \limsup_{k\to+\infty} \int_{\Omega\setminus J_k} |Dg_k|^2 - \int_\Omega |Dh_k|^2.
\end{aligned} \tag{3.23}$$

On the other hand, since $|J_k| \to 0$ we conclude

$$\limsup_{k\to\infty} \int_{J_k} (|Dg_k| - |Dh_k|)^2 = \limsup_{k\to\infty} \int_{J_k} |Dg_k|^2 .$$

Observe that, after passing to a subsequence, we can actually assume that all limsups are in fact limits. Summing (3.23) and the last equation we then conclude (3.18). □

Dirichlet competitors

We consider next a standard procedure to construct competitors for the Dirichlet energy of a sequence of functions with equi-bounded energy. A similar procedure will be repeated at the end of the last chapter to prove that a certain map is a minimizer of the energy.

Proposition 3.30 (Construction of a competitor [19, Proposition 3.4]). *Consider two radii $1 \leqslant r_0 < r_1 < 4$ and maps $g_k, h_k \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ such that $\{h_k\}_k$ is a sequence of translating sheets,*

$$\sup_k \mathrm{Dir}(g_k, B_{r_1}) < +\infty \quad \textit{and} \quad \|\mathcal{G}(g_k, h_k)\|_{L^2(B_{r_1}\setminus B_{r_0})} \to 0.$$

For every $\eta > 0$*, there exist* $r \in]r_0, r_1[$*, a subsequence of* $\{g_k\}_k$ *(not relabeled) and functions* $H_k \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ *such that* $H_k|_{B_{r_1}\setminus B_r} = g_k|_{B_{r_1}\setminus B_r}$ *and* $\mathrm{Dir}(H_k, B_{r_1}) \leqslant \mathrm{Dir}(h_k, B_{r_1}) + \eta$*. In addition, there is a dimensional constant* C *and a constant* C^* *(depending on* η *and the two sequences,* but not on k*) such that*

$$\mathrm{Lip}(H_k) \leqslant C^* \left(\mathrm{Lip}(g_k) + 1\right), \tag{3.24}$$

$$\|\mathcal{G}(H_k, h_k)\|_{L^2(B_r)} \leqslant C\mathrm{Dir}(g_k, B_r) + C\mathrm{Dir}(H_k, B_r)\,, \tag{3.25}$$

$$\|\boldsymbol{\eta} \circ H_k\|_{L^1(B_{r_1})} \leqslant C^* \|\boldsymbol{\eta} \circ g_k\|_{L^1(B_{r_1})} + C\|\boldsymbol{\eta} \circ h_k\|_{L^1(B_{r_1})}\,. \tag{3.26}$$

Proof. Set for simplicity $A_k := \|\mathcal{G}(g_k, h_k)\|_{L^2(B_{r_1}\setminus B_{r_0})}$ and $B_k := \|\boldsymbol{\eta} \circ g_k\|_{L^1(B_{r_1})}$. If $A_k \equiv 0$, then there is nothing to prove and so we can assume that, for a subsequence, not relabeled, $A_k > 0$. Assuming that for yet another subsequence (not relabeled) $B_k > 0$, we consider the function

$$\psi_k(r) := \int_{\partial B_r} \left(|Dg_k|^2 + |Dh_k|^2\right) + A_k^{-2} \int_{\partial B_r} \mathcal{G}(g_k, h_k)^2 + B_k^{-1} \int_{\partial B_r} |\boldsymbol{\eta} \circ g_k|. \tag{3.27}$$

By assumption $\liminf_k \int_{r_0}^{r_1} \psi_k(r)\, dr < \infty$. So, by Fatou's Lemma, there is $r \in]r_0, r_1[$ and a subsequence, not relabeled, such that $\lim_k \psi_k(r) < \infty$. Thus, for some $M > 0$ we have

$$\int_{\partial B_r} \mathcal{G}(g_k, h_k)^2 \to 0, \tag{3.28}$$

$$\mathrm{Dir}(h_k, \partial B_r) + \mathrm{Dir}(g_k, \partial B_r) \leqslant M, \tag{3.29}$$

$$\int_{\partial B_r} |\boldsymbol{\eta} \circ g_k| \leqslant M \,\|\boldsymbol{\eta} \circ g_k\|_{L^1(B_{r_1})}. \tag{3.30}$$

In case $B_k = 0$ for all k large enough, we define ψ_k dropping the last summand in (3.27) and reach the same conclusion.

Let ζ^j be the blocks of the translating sheets h_k as in Definition 3.27. We apply Lemma 3.11 to each ζ^j and find Lipschitz functions ζ^j_η satisfying the conclusion of the lemma with $\bar\varepsilon_1 = \bar\varepsilon_1(\eta, M) > 0$ (which will be chosen later). We also choose a standard radial convolution kernel φ in $\mathbb{R}^m$ and a small parameter $\bar\rho$ (also to be chosen later). Then, set

$$h_{k,\eta} := \sum_{j=1}^{J} [\![\tau_{y^j_k} \circ \zeta^j_\eta]\!] \quad \text{and} \quad \bar h_{k,\eta} := \sum_{i=1}^{Q} [\![(h_{k,\eta})_i - \boldsymbol{\eta} \circ h_{k,\eta} + (\boldsymbol{\eta} \circ h_k) * \varphi_{\bar\rho}]\!],$$

and choose $\bar\rho$ so small that

$$Q^2 \|\boldsymbol{\eta} \circ h_k - (\boldsymbol{\eta} \circ h_k) * \varphi_{\bar\rho}\|^2_{L^2} \leqslant \bar\varepsilon_1, \tag{3.31}$$

$$\int_{B_r} \left(|D(\boldsymbol{\eta} \circ h_k)|^2 - |D(\boldsymbol{\eta} \circ h_k * \varphi_{\bar\rho})|^2\right) \leqslant \bar\varepsilon_1. \tag{3.32}$$

Note that this is possible because, from the fact that h_k is a sequence of translating sheets, it follows that $\eta \circ h_k(x) = F(x) + p_k$ for some $F \in W^{1,2}$ and a sequence of vectors $p_k \in \mathbb{R}^n$. Therefore $(\eta \circ h_k) * \varphi_{\bar\rho} = F * \varphi_{\bar\rho} + p_k$ and $D(\eta \circ h_k) * \varphi_{\bar\rho} = DF * \varphi_{\bar\rho}$, and (3.31) and (3.32) follows if $\bar\rho$ is sufficiently small by the usual convolution estimates. In particular by very rough estimates,

$$\|\mathcal{G}(g_k, \bar h_{k,\eta})\|_{L^2} \overset{(3.31)}{\leqslant} \|\mathcal{G}(g_k, h_k)\|_{L^2} + 2\|\mathcal{G}(h_k, h_{k,\eta})\|_{L^2} + \bar\varepsilon_1 \leqslant o(1) + 3\,\bar\varepsilon_1, \tag{3.33}$$

$$\mathrm{Dir}(\bar h_{k,\eta}, \partial B_r) \leqslant 2\,M + 2\,\bar\varepsilon_1 \tag{3.34}$$

and

$$\begin{aligned}
\mathrm{Dir}(\bar h_{k,\eta}, B_r) &= \sum_i \int_{B_r} \left|D(h_{k,\eta})_i - D(\boldsymbol{\eta}\circ h_{k,\eta}) + D(\boldsymbol{\eta}\circ h_k * \varphi_{\bar\rho})\right|^2 \\
&= \int_{B_r} \left(|Dh_{k,\eta}|^2 - Q|D(\boldsymbol{\eta}\circ h_{k,\eta})|^2 + Q|D(\boldsymbol{\eta}\circ h_k * \varphi_{\bar\rho})|^2\right) \\
&= \mathrm{Dir}(h_{k,\eta}, B_r) + Q\int_{B_r} \left(|D(\boldsymbol{\eta}\circ h_k)|^2 - |D(\boldsymbol{\eta}\circ h_{k,\eta})|^2\right) \\
&\quad + Q\int_{B_r} \left(|D(\boldsymbol{\eta}\circ h_k * \varphi_{\bar\rho})|^2 - |D(\boldsymbol{\eta}\circ h_k)|^2\right) \\
&\overset{(3.1),(3.32)}{\leqslant} \mathrm{Dir}(h_{k,\eta}, B_r) + 2\,Q\,\bar\varepsilon_1 .
\end{aligned} \tag{3.35}$$

We can then apply Lemma 3.15 to $\bar h_{k,\eta}$ and g_k with $\bar\varepsilon_2 = \bar\varepsilon_2(\eta, M) > 0$, and get (up to subsequences) maps H_k satisfying $H_k|_{\partial B_r} = g_k|_{\partial B_r}$ and

$$\begin{aligned}
\mathrm{Dir}\,(H_k, B_r) &\leqslant \mathrm{Dir}\left(\bar h_{k,\eta}, B_r\right) + \bar\varepsilon_2\,\mathrm{Dir}\left(\bar h_{k,\eta}, \partial B_r\right) + \bar\varepsilon_2\,\mathrm{Dir}(g_k, \partial B_r) + \frac{C_0}{\bar\varepsilon_2}\int_{\partial B_r} \mathcal{G}\left(\bar h_{k,\eta}, g_k\right)^2 \\
&\leqslant \mathrm{Dir}(h_k, B_r) + Q\bar\varepsilon_1 + 3\,\bar\varepsilon_2\,(M + \bar\varepsilon_1) + 3\,C_0\,\bar\varepsilon_2^{-1}\,\bar\varepsilon_1
\end{aligned}$$

where in the last line we have used (3.28), (3.29) and (3.33) - (3.35). An appropriate choice of the parameters ε_1 and ε_2 gives the desired bound $\mathrm{Dir}\,(H_k, B_r) \leqslant \mathrm{Dir}(h_k, B_r) + \eta$.

Observe next that, by construction, $\limsup_k \mathrm{Lip}(\bar h_{k,\eta}) \leqslant C^*$, for some constant which depends on η and the two sequences, but not on k. Moreover,

$$\|\mathcal{G}(\bar h_{k,\eta}, g_k)\|_{L^\infty(\partial B_r)} \leqslant \|\mathcal{G}(\bar h_{k,\eta}, g_k)\|_{L^2(\partial B_r)} + C\mathrm{Lip}(g_k) + C\mathrm{Lip}(\bar h_{k,\eta})\,.$$

Thus (3.24) follows from (3.4).

Finally, (3.25) follows from the Poincaré inequality applied to $\mathcal{G}(H_k, g_k)$ (which vanishes identically on ∂B_r), and (3.26) follows from (3.5), because of (3.30) and $\|\boldsymbol{\eta}\circ \bar h_{k,\eta}\|_{L^1(B_r)} = \|(\boldsymbol{\eta}\circ h_k) * \varphi_{\bar\rho}\|_{L^1(B_r)} \leqslant \|\boldsymbol{\eta}\circ h_k\|_{L^1(B_{r_1})}$ if $\bar\rho$ is also chosen small enough such that $r + \bar\rho < r_1$. □

3.1.6 *Higher Integrability of the Gradient of* Dir*-minimizers*

Most of the energy of a Dir-minimizer lies where the gradient is relatively small. We prove indeed the following a priori estimate (cf. [58] for a different proof and some improvements).

Theorem 3.31 (Higher integrability of Dir-minimizers [19, Theorem 5.1])**.** *There exists* $p_{10} > 2$ *such that, for every* $\Omega' \subset\subset \Omega \subset \mathbb{R}^m$ *open domains, there is a constant* $C > 0$ *such that*

$$\|Du\|_{L^{p_{10}}(\Omega')} \leqslant C\,\|Du\|_{L^2(\Omega)} \quad \textit{for every } \mathrm{Dir}\textit{-minimizing } u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)). \tag{3.36}$$

Proof. The statement is a corollary of Proposition 3.32 below and a Gehring type lemma, cf. [39, Proposition 5.1]. □

Proposition 3.32 ([19, Propostion 5.2]). *Let* $\frac{2(m-1)}{m} < p_{11} < 2$. *Then, there exists* $C = C(m, n, Q, p_{11})$ *such that, for every* $u : \Omega \to \mathcal{A}_Q$ Dir*-minimizing, the following holds*

$$\left(\fint_{B_r(x)} |Du|^2\right)^{\frac{1}{2}} \leqslant C \left(\fint_{B_{2r}(x)} |Du|^{p_{11}}\right)^{\frac{1}{p_{11}}} \quad \forall\, x \in \Omega,\ \forall\, r < \min\left\{1, \operatorname{dist}(x, \partial\Omega)/2\right\}.$$

Proof. Since the estimate is invariant under translations and rescalings, it is enough to prove it for $x = 0$ and $r = 1$. We assume, therefore $\Omega = B_2$. Let $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing and let $F = \boldsymbol{\xi} \circ u : \Omega \to \mathcal{Q} \subset \mathbb{R}^N$. Denote by $\bar{F} \in \mathbb{R}^N$ the average of F on B_2. By Fubini's theorem and the Poincaré inequality, there exists $s \in [1, 2]$ such that

$$\int_{\partial B_s} \left(|F - \bar{F}|^{p_{11}} + |DF|^{p_{11}}\right) \leqslant C \int_{B_2} \left(|F - \bar{F}|^{p_{11}} + |DF|^{p_{11}}\right) \leqslant C \|DF\|^{p_{11}}_{L^{p_{11}}(B_2)}.$$

Consider $F|_{\partial B_s}$. Since $\frac{1}{2} > \frac{1}{p_{11}} - \frac{1}{2(m-1)}$, we can use the embedding $W^{1,p_{11}}(\partial B_s) \hookrightarrow H^{1/2}(\partial B_s)$ (see, for example, [1]). Hence, we infer that

$$\left\|F - \bar{F}\right\|_{H^{1/2}(\partial B_s)} \leqslant C\, \|DF\|_{L^{p_{11}}(B_2)}. \tag{3.37}$$

Let $\hat{F}$ be the harmonic extension of $F|_{\partial B_s}$ in B_s. It is well known (one could, for example, use the result in [1] on the half-space together with a partition of unity) that

$$\|D\hat{F}\|_{L^2(B_s)} \leqslant C(m) \min_{p \in \mathbb{R}^N} \|\hat{F} - p\|_{H^{1/2}(\partial B_s)} \overset{(3.37)}{\leqslant} C\, \|DF\|_{L^{p_{11}}(B_2)}\,. \tag{3.38}$$

Consider the map $\boldsymbol{\rho}$ of Lemma 3.8. Since $\boldsymbol{\rho} \circ \hat{F}|_{\partial B_s} = u|_{\partial B_s}$ and $\boldsymbol{\rho} \circ \hat{F}$ takes values in $\mathcal{Q}$, by the minimizing property of u and the Lipschitz continuity of $\boldsymbol{\xi}$, $\boldsymbol{\xi}^{-1}$ and $\boldsymbol{\rho}$, we conclude:

$$\left(\int_{B_1} |Du|^2\right)^{\frac{1}{2}} \leqslant C \left(\int_{B_s} |D\hat{F}|^2\right)^{\frac{1}{2}} \leqslant C \left(\int_{B_2} |DF|^{p_{11}}\right)^{\frac{1}{p_{11}}} = C \left(\int_{B_2} |Du|^{p_{11}}\right)^{\frac{1}{p_{11}}}. \qquad \square$$

3.1.7 *Unique continuation for Dir-minimizers*

We want to prove a De Giorgi-type decay estimate for Dir-minimizing Q-valued maps which are close to a classical harmonic function with multiplicity Q. The argument involves a unique continuation-type result for Dir-minimizers.

Lemma 3.33 (Unique continuation for Dir-minimizers [20, Lemma 7.1]). *For every* $\eta \in (0, 1)$ *and* $c > 0$, *there exists* $\gamma > 0$ *with the following property. If* $w : \mathbb{R}^m \supset B_{2r} \to \mathcal{A}_Q(\mathbb{R}^n)$ *is Dir-minimizing,* $\operatorname{Dir}(w, B_r) \geqslant c$ *and* $\operatorname{Dir}(w, B_{2r}) = 1$, *then*

$$\operatorname{Dir}(w, B_s(q)) \geqslant \gamma \quad \textit{for every } B_s(q) \subset B_{2r} \textit{ with } s \geqslant \eta\, r.$$

Proof. We start showing the following claim:

(UC) if Ω is a connected open set and $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dir-minimizing in any open $\Omega' \subset\subset \Omega$, then either w is constant or $\int_J |Dw|^2 > 0$ on any open $J \subset \Omega$.

We prove (UC) by induction on Q. If $Q = 1$, this is the classical unique continuation for harmonic functions. Assume now it holds for all $Q^* < Q$ and we prove it for Q-valued maps. Assume $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $J \subset \Omega$ is an open set on which $|Dw| \equiv 0$. Without loss of generality, we can assume J connected and $w|_J \equiv T$ for some $T \in \mathcal{A}_Q$. Let J' be the interior of $\{w = T\}$ and $K := \overline{J'} \cap \Omega$. We prove now that K is open, which in turn by connectedness of Ω concludes (UC). We distinguish two cases.

Case (a): the diameter of T is positive. Since w is continuous, for every $x \in K$ there is $B_\rho(x)$ where w separates into $[\![w_1]\!] + [\![w_2]\!]$ and each w_i is a Q_i-valued Dir-minimizer. Since $J' \cap B_\rho(x) \neq \emptyset$, each w_i is constant in a (nontrivial) open subset of $B_\rho(x)$. By inductive hypothesis each w_i is constant in $B_\rho(x)$ and therefore $w = T$ in $B_\rho(x)$, that is $B_\rho(x) \subset J' \subset K$.

Case (b): $T = Q[\![p]\!]$ **for some** p. In this case let J'' be the interior of $\{w = Q[\![\boldsymbol{\eta} \circ w]\!]\}$. By Definition 3.24, $\partial J'' \cap \Omega$ is contained in the singular set of w. By Theorem 3.25, $\mathcal{H}^{m-2+\varepsilon}(\Omega \cap \partial J'') = 0$ for every $\varepsilon > 0$. Consider now a point $p \in \partial J'' \cap \Omega$ and a small ball $B_\rho(x) \subset \Omega$. Since $\mathcal{H}^{m-1}(\partial J'' \cap B_\rho(x)) = 0$, by the isoperimetric inequality, either $|B_\rho(x) \setminus J''| = 0$ or $|J''| = 0$. The latter alternative is impossible because J'' is open and has nonempty intersection with $B_\rho(x)$. It then turns out that $|B_\rho(x) \setminus J''| = 0$ and thus the closure of J'' contains $B_\rho(x)$. But then $w = Q[\![\boldsymbol{\eta} \circ w]\!]$ on $B_\rho(x)$ and thus x cannot belong to $\partial J''$. So $\partial J'' \cap \Omega$ is empty and thus $w = Q[\![\boldsymbol{\eta} \circ w]\!]$ on Ω. On the other hand $\boldsymbol{\eta} \circ w$ is an harmonic function (cf. Proposition 3.22). Being $\boldsymbol{\eta} \circ w|_{J'} \equiv p$, by the classical unique continuation $\boldsymbol{\eta} \circ w \equiv p$ on Ω.

We now come to the proof of the lemma. Without loss of generality, we can assume $r = 1$. Arguing by contradiction, there exists sequences $\{w_k\}_{k\in\mathbb{N}} \subset W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ and $\{B_{s_k}(q_k)\}_{k\in\mathbb{N}}$ with $s_k \geqslant \eta$ and such that $\mathrm{Dir}(w_k, B_{s_k}(q_k)) \leqslant \frac{1}{k}$. By the compactness of Dir-minimizers (cp. Proposition 3.22), a subsequence (not relabeled) converges to $w \in W^{1,2}(B_2, \mathcal{A}_Q(\mathbb{R}^n))$ Dir-minimizing in every open $\Omega' \subset\subset B_2$. Up to subsequences, we can also assume that $q_k \to q$ and $s_k \to s \geqslant \eta > 0$. Thus, $B_s(q) \subset B_2$ and $\mathrm{Dir}(w, B_s(q)) = 0$. By (UC) this implies that w is constant. On the other hand, by 3.22, $\mathrm{Dir}(w, B_1) = \lim_k \mathrm{Dir}(w_k, B_1) \geqslant c > 0$ gives the desired contradiction. □

As a consequence of the Unique Continuation, we show that if the energy of a Dir-minimizer w does not decay appropriately, then w must split. In order to simplify the exposition, in the sequel we fix $\lambda > 0$ such that

$$(1+\lambda)^{(m+2)} < 2^{\delta_2}\,. \tag{3.39}$$

Proposition 3.34 (Decay estimate for Dir-minimizers [20, Proposition 7.2])**.** *For every* $\eta > 0$, *there is* $\gamma > 0$ *with the following property. Let* $w : \mathbb{R}^m \supset B_{2r} \to \mathcal{A}_Q(\mathbb{R}^n)$ *be Dir-minimizing in every* $\Omega' \subset\subset B_{2r}$ *such that*

$$\int_{B_{(1+\lambda)r}} \mathcal{G}\big(Dw, Q[\![D(\boldsymbol{\eta} \circ w)(0)]\!]\big)^2 \geqslant 2^{\delta_2 - m - 2}\mathrm{Dir}(w, B_{2r})\,. \tag{3.40}$$

Then, if we set $\bar{w} = \sum_i [\![w_i - \boldsymbol{\eta} \circ w]\!]$, *the following holds:*

$$\gamma\,\mathrm{Dir}(w, B_{(1+\lambda)r}) \leqslant \mathrm{Dir}(\bar{w}, B_{(1+\lambda)r}) \leqslant \frac{1}{\gamma r^2}\int_{B_s(q)} |\bar{w}|^2 \quad \forall\, B_s(q) \subset B_{2\,r} \text{ with } s \geqslant \eta\, r\,. \tag{3.41}$$

Before coming to the proof of the Proposition we point out an elementary fact which will be used repeatedly in this section. Since its proof is completely straightforward, it is left to the reader.

Lemma 3.35. *Let Ω be a bounded open set, $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, $\bar{w} = \sum_i [\![w_i - \boldsymbol{\eta} \circ w]\!]$ and $A = \fint_\Omega D(\boldsymbol{\eta} \circ w)$. We then have*

$$\begin{aligned}\int_\Omega |Dw|^2 &= \int_\Omega (|D\bar{w}|^2 + Q|D(\boldsymbol{\eta} \circ w)|^2) = \int_\Omega (|D\bar{w}|^2 + Q|D(\boldsymbol{\eta} \circ w) - A|^2) + Q|A|^2|\Omega| \\ &= \int_\Omega \mathcal{G}(Dw, Q[\![A]\!])^2 + Q|A|^2|\Omega|\,. \end{aligned} \tag{3.42}$$

Proof of Proposition 3.34. By a simple scaling argument we can assume $r = 1$ and we argue by contradiction. Let w_k be a sequence of local Dir-minimizers which satisfy (3.40), $\mathrm{Dir}(w_k, B_2) = 1$ and

(a) either $\int_{B_{s_k}(q_k)} |\bar{w}_k|^2 \to 0$ for some sequence of balls $B_{s_k}(q_k) \subset B_{2r}$ with $s_k \geqslant \eta$;

(b) or $\mathrm{Dir}(\bar{w}_k, B_{1+\lambda}) \to 0$.

Up to subsequences, w_k converges locally in $W^{1,2}$ to w locally Dir-minimizing. If (a) holds, we can appeal to Lemma 3.33 and conclude that $\bar{w} = \sum_i [\![w_i - \boldsymbol{\eta} \circ w]\!]$ vanishes identically on B_2. This means in particular that $\mathrm{Dir}(\bar{w}_k, B_{1+\lambda}) \to \mathrm{Dir}(\bar{w}, B_{1+\lambda}) = 0$, i.e. (b) holds.

Therefore, we can assume to be always in case (b). Let next $u_k := \boldsymbol{\eta} \circ w_k$. Since u_k is harmonic, $\fint_{B_{1+\lambda}} Du_k = Du_k(0)$. Thus from (3.40) and Lemma 3.35 we get

$$\begin{aligned}\int_{B_{1+\lambda}} Q|Du_k - Du_k(0)|^2 &= \int_{B_{1+\lambda}} \left(\mathcal{G}(Dw_k, Q[\![Du_k(0)]\!])^2 - |D\bar{w}_k|^2\right) \\ &\geqslant 2^{\delta_2 - m - 2} \int_{B_2} |Dw_k|^2 - \int_{B_{1+\lambda}} |D\bar{w}_k|^2\,. \end{aligned} \tag{3.43}$$

As $k \uparrow \infty$, by (b) and $\mathrm{Dir}(w_k, B_2) = 1$, we then conclude

$$\int_{B_{1+\lambda}} |Du - Du(0)|^2 \geqslant 2^{\delta_2 - m - 2} \geqslant 2^{\delta_2 - m - 2} \int_{B_2} |Du|^2\,. \tag{3.44}$$

Since $(1 + \lambda)^{m+2} < 2^{\delta_2}$, (3.44) violates the decay estimate for classical harmonic functions:

$$\int_{B_{1+\lambda}} |Du - Du(0)|^2 \leqslant 2^{-m-2}(1 + \lambda)^{m+2} \int_{B_2} |Du|^2\,, \tag{3.45}$$

thus concluding the proof. In order to show (3.45) it suffices to decompose Du in series of homogeneous harmonic polynomials $Du(x) = \sum_{i=0}^\infty P_i(x)$, where i is the degree. In particular the restriction of this decomposition on any sphere $S := \partial B_\rho$ gives the decomposition of $Du|_S$ in spherical harmonics, see [62, Chapter 5, Section 2]. It turns out, therefore, that the P_i are $L^2(B_\rho)$ orthogonal. Since the constant polynomial P_0 is $Du(0)$ and $\int_{B_{1+\lambda}} |P_i|^2 \leqslant 2^{-m-2i} \int_{B_2} |P_i|^2$, (3.45) follows at once. □

3.2 PUSH-FORWARD THROUGH MULTIPLE VALUED FUNCTIONS OF C^1 SUBMANIFOLDS

In what follows we consider an m-dimensional C^1 submanifold Σ of $\mathbb{R}^N$ and use the word *measurable* for those subsets of M which are $\mathcal{H}^m$-measurable. Any time we write an integral over (a measurable subset of) Σ we understand that this integral is taken with respect to the $\mathcal{H}^m$ measure. We start with a refinement of Lemma 3.3.

Lemma 3.36 (Decomposition [18, lemma 1.1])**.** *Let $M \subset \Sigma$ be measurable and $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ Lipschitz. Then there are a countable partition of M in bounded measurable subsets M_i ($i \in \mathbb{N}$) and Lipschitz functions $f_i^j : M_i \to \mathbb{R}^n$ ($j \in \{1, \ldots, Q\}$) such that*

(a) $F|_{M_i} = \sum_{j=1}^Q \left[\!\left[f_i^j \right]\!\right]$ *for every* $i \in \mathbb{N}$ *and* $\mathrm{Lip}(f_i^j) \leqslant \mathrm{Lip}(F)$ $\forall i, j$*;*

(b) $\forall i \in \mathbb{N}$ *and* $j, j' \in \{1, \ldots, Q\}$*, either* $f_i^j \equiv f_i^{j'}$ *or* $f_i^j(x) \neq f_i^{j'}(x)$ $\forall x \in M_i$*;*

(c) $\forall i$ *we have* $DF(x) = \sum_{j=1}^Q \left[\!\left[Df_i^j(x) \right]\!\right]$ *for a.e.* $x \in M_i$*.*

When $F : M \subset \Sigma \to \mathbb{R}^n$ is a proper Lipschitz function and $\Sigma \subset \mathbb{R}^N$ is oriented, the current $S = F_\sharp [\![M]\!]$ in $\mathbb{R}^n$ is given by

$$S(\omega) = \int_M \langle \omega(F(x)), DF(x)_\sharp \vec{e}(x) \rangle \, d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n),$$

where $\vec{e}(x) = e_1(x) \wedge \ldots \wedge e_m(x)$ is the orienting m-vector of Σ and

$$DF(x)_\sharp \vec{e} = (DF|_x \cdot e_1) \wedge \ldots \wedge (DF|_x \cdot e_m),$$

(cf. [54, Remark 26.21(3)]; as usual $\mathcal{D}^m(\Omega)$ denotes the space of smooth m-forms compactly supported in Ω). Using the Decomposition Lemma 3.36 it is possible to extend this definition to multiple valued functions. To this purpose, we give the definition of *proper* multiple valued functions.

Definition 3.37 (Proper Q-valued maps)**.** A measurable $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ is called *proper* if there is a measurable selection $F^1, \ldots, F^Q$ as in Lemma 3.2 (i.e. $F = \sum_i [\![F^i]\!]$) such that $\bigcup_i \overline{(F^i)^{-1}(K)}$ is compact for every compact $K \subset \mathbb{R}^n$. It is then obvious that if there exists such a selection, then *every* measurable selection shares the same property.

We warn the reader that the terminology might be slightly misleading, as the condition above is effectively *stronger* than the usual properness of maps taking values in the metric space $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$, even when F is continuous: the standard notion of *properness* would not ensure the well-definition of the multiple-valued push-forward.

Definition 3.38 (Q-valued push-forward)**.** Let $\Sigma \subset \mathbb{R}^N$ be a C^1 oriented manifold, $M \subset \Sigma$ a measurable subset and $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ a proper Lipschitz map. Then, we define the push-forward of M through F as the current $\mathbf{T}_F = \sum_{i,j} (f_i^j)_\sharp [\![M_i]\!]$, where M_i and f_i^j are as in Lemma 3.36: that is,

$$\mathbf{T}_F(\omega) := \sum_{i \in \mathbb{N}} \sum_{j=1}^{Q} \underbrace{\int_{M_i} \langle \omega(f_i^j(x)), Df_i^j(x)_\sharp \vec{e}(x) \rangle \, d\mathcal{H}^m(x)}_{T_{ij}(\omega)} \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{3.46}$$

We first want to show that T is well-defined. Since F is proper, we easily deduce that

$$|T_{ij}(\omega)| \leqslant \mathrm{Lip}(F)\|\omega\|_\infty \mathcal{H}^m((f_i^j)^{-1})(\mathrm{spt}(\omega)) < \infty.$$

On the other hand, upon setting $F^j(x) := f_i^j(x)$ for $x \in M_i$, we have $\cup_i (f_i^j)^{-1}(\mathrm{spt}(\omega)) = (F^j)^{-1}(\mathrm{spt}(\omega))$ and $(f_i^j)^{-1}(\mathrm{spt}(\omega)) \cap (f_{i'}^j)^{-1}(\mathrm{spt}(\omega)) = \emptyset$ for $i \neq i'$, thus leading to

$$\sum_{i,j} |T_{ij}(\omega)| \leqslant \mathrm{Lip}(F)\, \|\omega\|_\infty \sum_{j=1}^{Q} \mathcal{H}^m((F^j)^{-1}(\mathrm{spt}(\omega))) < +\infty.$$

Therefore, we can pass the sum inside the integral in (3.46) and, by Lemma 3.36, get

$$\mathbf{T}_F(\omega) = \int_M \sum_{l=1}^{Q} \langle \omega(F^l(x)), DF^l(x)_\sharp \vec{e}(x) \rangle \, d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{3.47}$$

In particular, recalling the standard theory of rectifiable currents (cf. [54, Section 27]) and the area formula (cf. [54, Section 8]), we have achieved the following proposition.

Proposition 3.39 (Representation of the push-forward [18, Proposition 1.4]). *The definition of the action of $\mathbf{T}_F$ in (3.46) does not depend on the chosen partition M_i nor on the chosen decomposition $\{f_i^j\}$, (3.47) holds and, hence, $\mathbf{T}_F$ is a (well-defined) integer rectifiable current given by $\mathbf{T}_F = (\mathrm{Im}(F), \Theta, \vec{\tau})$ where:*

(R1) $\mathrm{Im}(F) = \bigcup_{x\in M} \mathrm{spt}(F(x)) = \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{Q} f_i^j(M_i)$ *is an m-dimensional rectifiable set;*

(R2) *$\vec{\tau}$ is a Borel unitary m-vector orienting $\mathrm{Im}(F)$; moreover, for $\mathcal{H}^m$-a.e. $p \in \mathrm{Im}(F)$, we have $Df_i^j(x)_\sharp \vec{e}(x) \neq 0$ for every i, j, x with $f_i^j(x) = p$ and*

$$\vec{\tau}(p) = \pm \frac{Df_i^j(x)_\sharp \vec{e}(x)}{|Df_i^j(x)_\sharp \vec{e}(x)|}; \tag{3.48}$$

(R3) *for $\mathcal{H}^m$-a.e. $p \in \mathrm{Im}(F)$, the (Borel) multiplicity function Θ equals*

$$\Theta(p) := \sum_{i,j,\{x: f_i^j(x) = p\}} \left\langle \vec{\tau}, \frac{Df_i^j(x)_\sharp \vec{e}(x)}{|Df_i^j(x)_\sharp \vec{e}(x)|} \right\rangle.$$

3.2.1 *Push-forward of Lipschitz submanifolds*

As for the classical push-forward, Definition 3.38 can be extended to domains Σ which are Lipschitz submanifolds using the fact that such Σ can be "chopped" into C^1 pieces. Recall indeed the following fact.

Theorem 3.40 ([54, Theorem 5.3]). *If Σ is a Lipschitz m-dimensional oriented submanifold, then there are countably many C^1 m-dimensional oriented submanifolds Σ_i which cover $\mathcal{H}^m$-a.s. Σ and such that the orientations of Σ and Σ_i coincide on their intersection.*

Definition 3.41 (Q-valued push-forward of Lipschitz submanifolds)**.** Let $\Sigma \subset \mathbb{R}^N$ be a Lipschitz oriented submanifold, $M \subset \Sigma$ a measurable subset and $F : M \to \mathcal{A}_Q(\mathbb{R}^n)$ a proper Lipschitz map. Consider the $\{\Sigma_i\}$ of Theorem 3.40 and set $F_i := F|_{M\cap\Sigma_i}$. Then, we define the push-forward of M through F as the integer rectifiable current $\mathbf{T}_F := \sum_i \mathbf{T}_{F_i}$.

The following conclusion is a simple consequence of Theorem 3.40 and classical arguments in geometric measure theory (cf. [54, Section 27]).

Lemma 3.42 ([18, Lemma 1.7])**.** *Let M, Σ and F be as in Definition 3.41 and consider a Borel unitary m-vector $\vec{e}$ orienting Σ. Then $\mathbf{T}_F$ is a well-defined integer rectifiable current for which all the conclusions of Proposition 3.39 hold.*

As for the classical push-forward, $\mathbf{T}_F$ is invariant under bilipschitz change of variables.

Lemma 3.43 (Bilipschitz invariance [18, Lemma 1.8])**.** *Let $F : \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a Lipschitz and proper map, $\Phi : \Sigma' \to \Sigma$ a bilipschitz homeomorphism and $G := F \circ \Phi$. Then, $\mathbf{T}_F = \mathbf{T}_G$.*

We will next use the area formula to compute explicitely the mass of $\mathbf{T}_F$. Following standard notation, we will denote by $\mathbf{J}F^j(x)$ the Jacobian determinant of DF^j, i.e. the number

$$\left|DF^j(x)_\sharp\vec{e}\right| = \sqrt{\det((DF^j(x))^T \cdot DF^j(x))}$$

Lemma 3.44 (Q-valued area formula [18, Lemma 1.9])**.** *Let Σ, M and $F = \sum_j [\![F^j]\!]$ be as in Definition 3.41. Then, for any bounded Borel function $h : \mathbb{R}^n \to [0, \infty[$, we have*

$$\int h(p)\, d\|\mathbf{T}_F\|(p) \leqslant \int_M \sum_j h(F^j(x))\, \mathbf{J}F^j(x)\, d\mathcal{H}^m(x)\,. \tag{3.49}$$

Equality holds in (3.49) *if there is a set $M' \subset M$ of full measure for which*

$$\langle DF^j(x)_\sharp\vec{e}(x), DF^i(y)_\sharp\vec{e}(y)\rangle \geqslant 0 \qquad \forall x, y \in M' \text{ and } i, j \text{ with } F^i(x) = F^j(y)\,. \tag{3.50}$$

If (3.50) *holds the formula is valid also for bounded* real*-valued Borel h with compact support.*

A particular class of push-forwards are given by graphs.

Definition 3.45 (Q-graphs)**.** Let Σ, M and $f = \sum_i [\![f_i]\!]$ be as in Definition 3.41. Define the map $F : M \to \mathcal{A}_Q(\mathbb{R}^{N+n})$ as $F(x) := \sum_{i=1}^Q [\![(x, f_i(x))]\!]$. $\mathbf{T}_F$ is the *current associated to the graph* $\mathrm{Gr}(f)$ and will be denoted by $\mathbf{G}_f$.

Observe that, if Σ, f and F are as in Definition 3.45, then the condition (3.50) is always trivially satisfied. Moreover, when $\Sigma = \mathbb{R}^m$ the well-known Cauchy-Binet formula gives

$$(\mathbf{J}F^j)^2 = 1 + \sum_{k=1}^m \sum_{A\in M^k(DF^j)} (\det A)^2\,,$$

where $M^k(B)$ denotes the set of all $k \times k$ minors of the matrix B. Lemma 3.44 gives then the following corollary in the case of Q-graphs

Corollary 3.46 (Area formula for Q-graphs [18, Corollary 1.11]). *Let $\Sigma = \mathbb{R}^m$, $M \subset \mathbb{R}^m$ and f be as in Definition 3.45. Then, for any bounded compactly supported Borel $h : \mathbb{R}^{m+n} \to \mathbb{R}$, we have*

$$\int h(p)\, d\|\mathbf{G}_f\|(p) = \int_M \sum_i h(x, f_i(x)) \Big(1 + \sum_{k=1}^{m} \sum_{A \in M^k(Df^j)} (\det A)^2 \Big)^{\frac{1}{2}} dx. \tag{3.51}$$

In the classical theory of currents, when Σ is a Lipschitz manifold with Lipschitz boundary and $F : \Sigma \to \mathbb{R}^N$ is Lipschitz and proper, then $\partial(F_\sharp [\![\Sigma]\!]) = F_\sharp [\![\partial\Sigma]\!]$ (see [32, 4.1.14]). This result can be extended to multiple-valued functions.

Theorem 3.47 (Boundary of the push-forward [18, Theorem 2.1]). *Let Σ be a Lipschitz submanifold of $\mathbb{R}^N$ with Lipschitz boundary, $F : \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ a proper Lipschitz function and $f = F|_{\partial\Sigma}$. Then, $\partial \mathbf{T}_F = \mathbf{T}_f$.*

3.3 AREA FORMULA AND TAYLOR EXPANSIONS OF THE RELEVANT QUANTITIES

In this section we compute the Taylor expansion of the area functional in several forms. To this aim, we fix the following notation and hypotheses.

Assumptions 5. *We consider:*

- *(M) an open submanifold $\mathcal{M} \subset \mathbb{R}^{m+n}$ of dimension m with $\mathcal{H}^m(\mathcal{M}) < \infty$, which is the graph of a function $\boldsymbol{\varphi} : \mathbb{R}^m \supset \Omega \to \mathbb{R}^n$ with $\|\boldsymbol{\varphi}\|_{C^3} \leqslant \bar{c}$; A and H will denote, respectively, the second fundamental form and the mean curvature of $\mathcal{M}$;*
- *(U) a regular tubular neighborhood $\mathbf{U}$ of $\mathcal{M}$, i.e. the set of points $\{x + y : x \in \mathcal{M}, y \perp T_x\mathcal{M}, |y| < c_0\}$, where the thickness c_0 is sufficiently small so that the nearest point projection $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ is well defined and C^2; the thickness is supposed to be larger than a fixed geometric constant (which depends on $\bar{c}$);*
- *(N) a Q-valued map $F : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ of the form*

$$\sum_{i=1}^{Q} [\![F_i(x)]\!] = \sum_{i=1}^{Q} [\![x + N_i(x)]\!],$$

where $N : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ satisfies $x + N_i(x) \in \mathbf{U}$, $N_i(x) \perp T_x\mathcal{M}$ for every x and $\mathrm{Lip}(N) \leqslant \bar{c}$.

We recall the notation $\boldsymbol{\eta} \circ F := \frac{1}{Q} \sum_i F_i$, for every multiple valued function $F = \sum_i [\![F_i]\!]$.

Theorem 3.48 (Expansion of $\mathbf{M}(\mathbf{T}_F)$ [18, Theorem 3.2]). *If $\mathcal{M}$, F and N are as in Assumption 5 and $\bar{c}$ is smaller than a geometric constant, then*

$$\begin{aligned} \mathbf{M}(\mathbf{T}_F) = Q\,\mathcal{H}^m(\mathcal{M}) - Q \int_{\mathcal{M}} \langle H, \boldsymbol{\eta} \circ N \rangle + \frac{1}{2} \int_{\mathcal{M}} |DN|^2 \\ + \int_{\mathcal{M}} \sum_i \Big(P_2(x, N_i) + P_3(x, N_i, DN_i) + R_4(x, DN_i) \Big), \end{aligned} \tag{3.52}$$

where P_2, P_3 and R_4 are C^1 functions with the following properties:

(i) $v \mapsto P_2(x, v)$ *is a quadratic form on the normal bundle of* $\mathcal{M}$ *satisfying*

$$|P_2(x,v)| \leqslant C|A(x)|^2|v|^2 \qquad \forall\, x \in \mathcal{M},\ \forall\, v \perp T_x\mathcal{M}; \tag{3.53}$$

(ii) $P_3(x, v, D) = \sum_i L_i(x, v)Q_i(x, D)$, *where* $v \mapsto L_i(x, v)$ *are linear forms on the normal bundle of* $\mathcal{M}$ *and* $D \mapsto Q_i(x, D)$ *are quadratic forms on the space of* $(m+n) \times (m+n)$*-matrices, satisfying*

$$\begin{aligned} &|L_i(x,v)| \leqslant C|A(x)||v| && \forall x \in \mathcal{M}, \forall v \perp T_x\mathcal{M},\\ &|Q_i(x,D)| \leqslant C|D|^2 && \forall x \in \mathcal{M}, \forall D \in \mathbb{R}^{(m+n)\times(m+n)}; \end{aligned}$$

(iii) $|R_4(x, D)| = |D|^3L(x, D)$, *for some function* L *with* $\mathrm{Lip}(L) \leqslant C$, *which satisfies* $L(x, 0) = 0$ *for every* $x \in \mathcal{M}$ *and is independent of* x *when* $A \equiv 0$.

Moreover, for any Borel function $h : \mathbb{R}^{m+n} \to \mathbb{R}$,

$$\left|\int h\, d\|\mathbf{T}_F\| - \int_{\mathcal{M}} \sum_i h \circ F_i\right| \leqslant C \int_{\mathcal{M}} \Big(\sum_i |A||h \circ F_i||N_i| + \|h\|_\infty(|DN|^2 + |A|^2|N|^2)\Big), \tag{3.54}$$

and, if $h(p) = g(\mathbf{p}(p))$ *for some* g, *we have*

$$\left|\int h\, d\|\mathbf{T}_F\| - \int_{\mathcal{M}} (Q - Q\langle H, \boldsymbol{\eta} \circ N\rangle + \tfrac{1}{2}|DN|^2)\, g\right| \leqslant C \int_{\mathcal{M}} \big(|A|^2|N|^2 + |DN|^4\big)|g|. \tag{3.55}$$

In particular, as a simple corollary of the theorem above, we have the following fact.

Corollary 3.49 (Expansion of $\mathbf{M}(\mathbf{G}_f)$ [18, Corollary 3.3]). *Assume* $\Omega \subset \mathbb{R}^m$ *is an open set with bounded measure and* $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ *a Lipschitz map with* $\mathrm{Lip}(f) \leqslant \bar{c}$. *Then,*

$$\mathbf{M}(\mathbf{G}_f) = Q|\Omega| + \frac{1}{2}\int_\Omega |Df|^2 + \int_\Omega \sum_i \bar{R}_4(Df_i)\,, \tag{3.56}$$

where $\bar{R}_4 \in C^1$ *satisfies* $|\bar{R}_4(D)| = |D|^3\bar{L}(D)$ *for* $\bar{L}$ *with* $\mathrm{Lip}(\bar{L}) \leqslant C$ *and* $\bar{L}(0) = 0$.

Proof. The corollary is reduced to Theorem 3.48 by simply setting $\mathcal{M} = \Omega \times \{0\}$,

$$N = \sum_i [\![N_i(x)]\!] := \sum_i [\![(0, f_i(x))]\!] \quad \text{and} \quad F(x) = \sum_i [\![F_i(x)]\!] = \sum_i [\![(x, f_i(x))]\!]\,.$$

Since in this case A vanishes, (3.52) gives precisely (3.56). □

3.3.1 *Taylor expansion for the excess in a cylinder*

The last results of this section concern estimates of the excess in different systems of coordinates, in particular with respect to tilted planes and curvilinear coordinates.

Proposition 3.50 (Expansion of a curvilinear excess [18, Proposition 3.4]). *There exists a dimensional constant $C > 0$ such that, if $\mathcal{M}$, F and N are as in Assumption 5 with $\bar{c}$ small enough, then*

$$\left| \int |\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2 \, d\|\mathbf{T}_F\|(x) - \int_{\mathcal{M}} |DN|^2 \right| \leqslant C \int_{\mathcal{M}} (|A|^2|N|^2 + |DN|^4) \,, \tag{3.57}$$

where $\vec{\mathbf{T}}_F$ and $\vec{\mathcal{M}}$ are the unit m-vectors orienting $\mathbf{T}_F$ and $T\mathcal{M}$, respectively.

Next we compute the excess of a Lipschitz graph with respect to a tilted plane. We use here the notation $\mathbf{C}_s$ for the open set $B_s(0) \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$.

Theorem 3.51 (Expansion of a cylindrical excess [18, Theorem 3.5]). *There exist dimensional constants $C, c > 0$ with the following property. Let $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ be a Lipschitz map with $\mathrm{Lip}\,(f) \leqslant c$. For any $0 < s$, set $L := \fint_{B_s} D(\boldsymbol{\eta} \circ f)$ and denote by $\vec{\tau}$ the unitary m-dimensional simple vector orienting the graph of the linear map $y \mapsto A \cdot y$. Then, we have*

$$\left| \int_{\mathbf{C}_s} \left| \vec{\mathbf{G}}_f - \vec{\tau} \right|^2 d\|\mathbf{G}_f\| - \int_{B_s} \mathcal{G}(Df, Q\,[\![L]\!])^2 \right| \leqslant C \int_{B_s} |Df|^4 \,. \tag{3.58}$$

3.3.2 *First variations*

In this section we compute the first variations of the currents induced by multiple valued maps. These formulae are ultimately the link between the stationarity of area minimizing currents and the partial differential equations satisfied by suitable approximations. We use here the following standard notation: given a current T in $\mathbb{R}^N$ and a vector field $X \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, we denote the first variation of T along X by $\delta T(X) := \frac{d}{dt}\big|_{t=0} \mathbf{M}(\Phi_{t\sharp} T)$, where $\Phi :]-\eta, \eta[\times U \to \mathbb{R}^N$ is any C^1 isotopy of a neighborhood U of $\mathrm{spt}(T)$ with $\Phi(0, x) = x$ for any $x \in U$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Phi_\varepsilon = X$ (in what follows we will often use Φ_ε for the map $x \mapsto \Phi(\varepsilon, x)$). It would be more appropriate to use the notation $\delta T(\Phi)$ (see, for instance, [32, Section 5.1.7]), but since the currents considered in this paper are rectifiable, it is well known that the first variation depends only on X and is given by the formula

$$\delta T(X) = \int \mathrm{div}_{\vec{T}} X \, d\|T\|, \tag{3.59}$$

where $\mathrm{div}_{\vec{T}} X = \sum_i \langle D_{e_i} X, e_i \rangle$ for any orthonormal frame $e_1, \ldots, e_m$ with $e_1 \wedge \ldots \wedge e_m = \vec{T}$ (see [32, 5.1.8] and cf. [54, Section 2.9]). We begin with the expansion for the first variation of graphs.

Theorem 3.52 (Expansion of $\delta\mathbf{G}_f(X)$ [18, Theorem 4.1]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ a map with $\mathrm{Lip}(f) \leqslant \bar{c}$. Consider a function $\zeta \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and the corresponding vector field $\chi \in C^1(\Omega \times \mathbb{R}^n, \mathbb{R}^{m+n})$ given by $\chi(x, y) = (0, \zeta(x, y))$. Then,*

$$\left| \delta\mathbf{G}_f(\chi) - \int_\Omega \sum_i \big(D_x\zeta(x, f_i) + D_y\zeta(x, f_i) \cdot Df_i \big) : Df_i \right| \leqslant C \int_\Omega |D\zeta||Df|^3 \,. \tag{3.60}$$

The next two theorems deal with general $\mathbf{T}_F$ as in Assumption 5. However we restrict our attention to "outer and inner variations", where we borrow our terminology from the elasticity theory and the literature on harmonic maps. Outer variations result from deformations of the normal bundle of $\mathcal{M}$ which are the identity on $\mathcal{M}$ and map each fiber into itself, whereas inner variations result from composing the map F with isotopies of $\mathcal{M}$.

Theorem 3.53 (Expansion of outer variations [18, Theorem 4.2]). *Let $\mathcal{M}$, $\mathbf{U}$, $\mathbf{p}$ and F be as in Assumption 5 with $\bar{c}$ sufficiently small. If $\varphi \in C^1_c(\mathcal{M})$ and $X(p) := \varphi(\mathbf{p}(p))(p - \mathbf{p}(p))$, then*

$$\delta \mathbf{T}_F(X) = \int_{\mathcal{M}} \Big(\varphi\, |DN|^2 + \sum_i (N_i \otimes D\varphi) : DN_i\Big) - \underbrace{Q \int_{\mathcal{M}} \varphi \langle H, \boldsymbol{\eta} \circ N \rangle}_{\mathrm{Err}_1} + \sum_{i=2}^{3} \mathrm{Err}_i \tag{3.61}$$

where

$$|\mathrm{Err}_2| \leqslant C \int_{\mathcal{M}} |\varphi| |A|^2 |N|^2 \tag{3.62}$$

$$|\mathrm{Err}_3| \leqslant C \int_{\mathcal{M}} \Big(|\varphi| \big(|DN|^2 |N| |A| + |DN|^4\big) + |D\varphi| \big(|DN|^3 |N| + |DN| |N|^2 |A|\big)\Big) . \tag{3.63}$$

Let Y be a C^1 vector field on $T\mathcal{M}$ with compact support and define X on $\mathbf{U}$ setting $X(p) = Y(\mathbf{p}(p))$. Let $\{\Psi_\varepsilon\}_{\varepsilon \in]-\eta, \eta[}$ be any isotopy with $\Psi_0 = \mathrm{id}$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Psi_\varepsilon = Y$ and define the following isotopy of $\mathbf{U}$: $\Phi_\varepsilon(p) = \Psi_\varepsilon(\mathbf{p}(p)) + (p - \mathbf{p}(p))$. Clearly $X = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Phi_\varepsilon$.

Theorem 3.54 (Expansion of inner variations [18, Theorem 4.3]). *Let $\mathcal{M}$, $\mathbf{U}$ and F be as in Assumption 5 with $\bar{c}$ sufficiently small. If X is as above, then*

$$\delta \mathbf{T}_F(X) = \int_{\mathcal{M}} \Big(\frac{|DN|^2}{2} \mathrm{div}_{\mathcal{M}}\, Y - \sum_i DN_i : (DN_i \cdot D_{\mathcal{M}} Y)\Big) + \sum_{i=1}^{3} \mathrm{Err}_i, \tag{3.64}$$

where

$$\mathrm{Err}_1 = -Q \int_{\mathcal{M}} \big(\langle H, \boldsymbol{\eta} \circ N \rangle\, \mathrm{div}_{\mathcal{M}} Y + \langle D_Y H, \boldsymbol{\eta} \circ N \rangle\big) , \tag{3.65}$$

$$|\mathrm{Err}_2| \leqslant C \int_{\mathcal{M}} |A|^2 \big(|DY| |N|^2 + |Y| |N|\, |DN|\big) , \tag{3.66}$$

$$|\mathrm{Err}_3| \leqslant C \int_{\mathcal{M}} \Big(|Y| |A| |DN|^2 \big(|N| + |DN|\big) + |DY| \big(|A|\, |N|^2 |DN| + |DN|^4\big)\Big) . \tag{3.67}$$

4 STRONG LIPSCHITZ APPROXIMATION FOR ALMOST MINIMIZING CURRENTS

The aim of this chapter is to prove a Lipschitz approximation result for a wide class of almost area minimizing currents, which we will call $\mathbf{\Omega}$-minima.

Definition 4.1 ($\mathbf{\Omega}$-minimality)**.** A current $T \in \mathbf{I}_m(\mathbb{R}^{n+m})$ is called $\mathbf{\Omega}$-minimum if there exists a constant $\mathbf{\Omega} > 0$ such that

$$\mathbf{M}(T) \leqslant \mathbf{M}(T + \partial S) + \mathbf{\Omega}\,\mathbf{M}(S) \qquad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}) \quad \text{with compact support.} \tag{4.1}$$

The main result is the following Lipschitz-type approximation result for $\mathbf{\Omega}$-minimal currents.

Proposition 4.2. *Assume that $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is $\mathbf{\Omega}$-minimal and for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leqslant 1$) and some positive integer Q,*

$$\mathbf{p}_\sharp T = Q\,[\![B_{4r}(x)]\!] \quad \textit{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0\,. \tag{4.2}$$

There exist constants $M, C_{21}, \beta_0, \varepsilon_{21} > 0$ (depending on m, n, Q), such that if $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{21}$ then the following holds. There exist a map $f \colon B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$ and a closed set $K \subset B_r(x)$ such that

$$\mathrm{Lip}(f) \leqslant C_{21} E^{\beta_0} \tag{4.3}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \textit{and} \quad |B_r(x) \setminus K| \leqslant C_{21} E^{\beta_0} \big(E + r^2 \mathbf{\Omega}^2\big) r^m \tag{4.4}$$

$$\left| \|T\| \left(\mathbf{C}_r(x)\right) - Q\omega_m r^m - \frac{1}{2}\int_{B_r(x)} |Df|^2 \right| \leqslant C_{21} E^{\beta_0} \big(E + r^2 \mathbf{\Omega}^2\big) r^m\,. \tag{4.5}$$

If in addition $\mathbf{h}(T, \mathbf{C}_{4r}(x)) := \sup\{|\mathbf{p}^\perp(x) - \mathbf{p}^\perp(y)| \,:\, x, y \in \mathrm{spt}(T) \cap \mathbf{C}_{4r}(x)\} \leqslant r$, then

$$\mathrm{osc}(f) \leqslant C_{21}\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21} r E^{1/2}\,. \tag{4.6}$$

Proof of Theorem 2.8. As already pointed out in Chapter 2, Theorem 2.8 case (a) follows from [19, Theorem 1.4]. Note also that case (b) follows directly from Proposition 4.2. It remains to handle case (c), because the graph of the map f given by Proposition 4.2 is not necessarily contained in Σ. We show here how to modify it in such a way to fulfill the requirements of Theorem 2.8.

We assume that Ψ is a function whose graph coincides with Σ (the connected component of $\partial\mathbf{B}_R(p) \cap \mathbf{C}_{4r}(x)$ containing $\mathrm{spt}(T)$) and arguing as in [19, Remark 1.5] we can assume that $\|\Psi_0\| \leqslant CE^{\frac{1}{2}} r + C\mathbf{\Omega} r^2$, $\|D\Psi\|_0 \leqslant CE^{\frac{1}{2}} + C\mathbf{\Omega} r$ and $\|D^2\Psi\|_0 \leqslant C\mathbf{\Omega}$. The domain of Ψ is a subset of $B_{4r}(x) \times \mathbb{R}^{n-1}$. Let now $f = \sum_i [\![f_i]\!]$ be the function given by Proposition 4.2 and let $\bar{f} = \sum_i [\![\bar{f}_i]\!]$, where $\bar{f}_i(y)$ gives the first $n-1$ coordinates of $f_i(y)$. Observe that on the set K we necessarily have

$$f(y) = \sum_i [\![(\bar{f}_i(y), \Psi(y, \bar{f}_i(y)))]\!]\ .$$

We then can extend $\bar f$ to $B_r(x)\setminus K$ with $\mathrm{Lip}(\bar f) \leqslant C\mathrm{Lip}(f)$ and $\mathrm{osc}\,(\bar f) \leqslant C\mathrm{osc}\,(f)$ and hence define $\hat f(y) = \sum_i [\![(\bar f_i(y), \Psi(y, \bar f_i(y))]\!]$ for *every* $y \in B_r(x)$ (it must be shown that $(y, \bar f_i(y))$ belongs to the domain of definition of Ψ, but this follows easily from the smallness of $\mathrm{osc}\,(\bar f)$). Obviously $f = \hat f$ on K. On the other hand it is straightforward to check that

$$\mathrm{Lip}(\hat f) \leqslant C\,\mathrm{Lip}(\bar f) + C(\mathrm{Lip}(\bar f)+1)\|D\Psi_0\| \leqslant CE_0^{\beta} + C\mathbf{\Omega} r \tag{4.7}$$

$$\mathrm{osc}\,(\hat f) \leqslant C\,\mathrm{osc}\,(f) + \|\Psi\|_0 \leqslant C\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C(E^{\frac{1}{2}} + \mathbf{\Omega} r)r\,. \tag{4.8}$$

In addition we conclude

$$\left|\int_{B_r(x)} |Df|^2 - \int_{B_r(x)} |D\hat f|^2\right| \leqslant (\mathrm{Lip}(f)^2 + \mathrm{Lip}(\hat f)^2)|B_r(x)\setminus K| \leqslant C|K|\,.$$

Thus the estimates in Proposition 4.2 complete the proof. □

The rest of the chapter is devoted to the proof of Proposition 4.2. This will be achieved in four sections. In the first we recall the standard Lipschitz approximation result for integral currents satisfying (4.2), which can be applied in our case without any modification (cp. [19]). In the second we improve upon the almost minimality condition under the assumption that the cylindrical excess is small: this section contains, indeed, the most significant new ideas compared to [19]. Finally, in the last two sections we modify accordingly the computations of [19] to prove Proposition 4.2.

4.1 LIPSCHITZ APPROXIMATION

We start with the following definition. Recall that the notion of excess we are using here is the one of Definition 2.6.

Definition 4.3 (Excess measure)**.** For a current T as in Proposition 4.2 we define the *excess measure* $\mathbf{e}_T$ and its *density* $\mathbf{d}_T$:

$$\mathbf{e}_T(A) := \|T\|(A\times\mathbb{R}^n) - Q\,|A| \qquad \text{for every Borel } A \subset B_r(x),$$
$$\mathbf{d}_T(y) := \limsup_{s\to 0} \frac{\mathbf{e}_T(B_s(y))}{\omega_m\, s^m} = \limsup_{s\to 0} \mathbf{E}(T, \mathbf{C}_s(y)),$$

where ω_m is the measure of the m-dimensional unit ball (the subscripts $_T$ will be omitted if clear from the context). Moreover we introduce the "non-centered" maximal function of $\mathbf{e}_T$:

$$\mathbf{me}_T(y) := \sup_{y\in B_{\frac{s}{2}}(w)\subset B_{4r}(x)} \frac{\mathbf{e}_T(B_{\frac{s}{2}}(w))}{\omega_m\, s^m} = \sup_{y\in B_{\frac{s}{2}}(w)\subset B_{4r}(x)} \mathbf{E}(T, \mathbf{C}_{\frac{s}{2}}(w)).$$

Notice that we take the supremum over balls of radius $\frac{s}{2}$ instead of s: this is to achieve the following result in a ball of radius bigger than $3r$.

Proposition 4.4 (Lipschitz approximation; cf. [19, Proposition 2.2])**.** *There exists a constant* $C_{22}(m, n, Q) > 0$ *with the following property. Let* T *be as in Proposition 4.2 in the cylinder* $\mathbf{C}_{4s}(x)$*. Set* $E = \mathbf{E}(T, \mathbf{C}_{4r}(x))$*, let* $0 < \delta < 1$ *be such that*

$$r_0 := 16\,\sqrt[m]{\frac{E}{\delta}} < 1,$$

and define $K := \{\mathbf{m}\boldsymbol{e}_T < \delta\} \cap B_{\frac{7r}{2}}(x)$. *Then, there is* $u \in \mathrm{Lip}(B_{\frac{7r}{2}}(x), \mathcal{A}_Q(\mathbb{R}^n))$ *such that*

$$\mathrm{Lip}(u) \leqslant C_{22}\,\delta^{\frac{1}{2}},$$

$$\mathbf{G}_u \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n),$$

$$|B_s(x) \setminus K| \leqslant \frac{10^m}{\delta}\,\boldsymbol{e}_T\Big(\{\mathbf{m}\boldsymbol{e}_T > 2^{-m}\delta\} \cap B_{s+r_0 r}(x)\Big) \quad \forall\, r \leqslant \frac{7r}{2}. \tag{4.9}$$

When $\delta = E^{2\beta}$, we will call the map u given by the proposition E^β*-Lipschitz approximation of* T *in* $\mathbf{C}_{\frac{7r}{2}}(x)$.

For the sake of completeness we give here the same proof as in [19, Proposition 2.2]. The proof of the proposition is based on a BV estimate which differs from the ones of [4, 42]. Note that we do not assume that T is area minimizing.

The modified Jerrard–Soner estimate

Recall that each element $S \in \mathbf{I}_0(\mathbb{R}^{m+n})$ is simply a finite sum of Dirac delta, $S = \sum_{i=1}^h w_i\,\delta_{z_i}$, where $h \in \mathbb{N}$, $w_i \in \{-1, 1\}$ and the z_i's are (not necessarily distinct) points in $\mathbb{R}^{m+n}$. Let T be a current as in Assumption 1 in the cylinder $\mathbf{C}_4$. The slicing map $x \mapsto \langle T, \mathbf{p}, x\rangle$ takes values in $\mathbf{I}_0(\mathbb{R}^{m+n})$ and is characterized by (cf. [54, Section 28]):

$$\int_{B_4} \langle T, \mathbf{p}, x\rangle(\varphi)\,dx = T(\varphi\,dx) \quad \text{for every } \varphi \in C_c^\infty(\mathbf{C}_4). \tag{4.10}$$

Moreover $\mathrm{spt}(\langle T, \mathbf{p}, x\rangle) \subseteq \mathbf{p}^{-1}(\{x\})$ and therefore $\langle T, \mathbf{p}, x\rangle = \sum_i w_i\,\delta_{(x,y_i)}$. The assumption (4.2) guarantees that $\sum_i w_i = Q$ for almost every x. In order to state our BV estimate, we consider the push-forwards of $\langle T, \mathbf{p}, x\rangle$ into the vertical directions:

$$T_x := \mathbf{p}^\perp_\sharp\big(\langle T, \mathbf{p}, x\rangle\big) \in \mathbf{I}_0(\mathbb{R}^n)\,. \tag{4.11}$$

It follows from (4.10) that the currents T_x are characterized through the identity:

$$\int_{B_4} T_x(\psi)\varphi(x)\,dx = T(\varphi\,\psi\,dx) \quad \text{for every } \varphi \in C_c^\infty(B_4),\ \psi \in C_c^\infty(\mathbb{R}^n). \tag{4.12}$$

Proposition 4.5 (BV estimate)**.** *Assume* T *satisfies Assumption 1 in* $\mathbf{C}_4$. *For every* $\psi \in C_c^\infty(\mathbb{R}^n)$, *set* $\Phi_\psi(x) := T_x(\psi)$. *If* $\|D\psi\|_\infty \leqslant 1$, *then* $\Phi_\psi \in BV(B_4)$ *and satisfies*

$$\big(|D\Phi_\psi|(A)\big)^2 \leqslant 2m^2\,\boldsymbol{e}_T(A)\,\|T\|(A \times \mathbb{R}^n) \quad \textit{for every Borel set } A \subseteq B_4. \tag{4.13}$$

Note that in the usual Jerrard-Soner estimate the RHS of (4.13) would be $(\|T\|(A \times \mathbb{R}^n))^2$.

Proof. It is enough to prove (4.13) for every open set $A \subseteq B_4$. To this aim, recall that:

$$|D\Phi_\psi|(A) = \sup\left\{\int_A \Phi_\psi(x)\,\mathrm{div}\,\varphi(x)\,dx\ :\ \varphi \in C_c^\infty(A, \mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1\right\}. \tag{4.14}$$

For any smooth vector field φ, it holds that $(\mathrm{div}\,\varphi(x))\,dx = d\Xi$, where

$$\Xi = \sum_j \varphi_j\, d\hat{x}^j \quad \text{and} \quad d\hat{x}^j = (-1)^{j-1} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^m.$$

From (4.12) and the assumption $\partial T \llcorner \mathbf{C}_4 = 0$ in (4.2), we conclude that

$$\begin{aligned} \int_A \Phi_\psi(x)\,\mathrm{div}\,\varphi(x)\,dx &= \int_{B_4} T_x(\psi)\mathrm{div}\,\varphi(x)\,dx = T(\psi\,\mathrm{div}\,\varphi\,dx) \\ &= T(\psi\,d\Xi) = T(d(\psi\,\Xi)) - T(d\psi \wedge \Xi) = -T(d\psi \wedge \Xi)\,. \end{aligned} \tag{4.15}$$

Observe that the m-form $d\psi \wedge \Xi$ has no dx component, since

$$d\psi \wedge \Xi = \sum_{j=1}^m \sum_{i=1}^n (-1)^{j-1} \frac{\partial \psi}{dy^i}(y)\,\varphi_j(x)\,dy^i \wedge d\hat{x}^j. \tag{4.16}$$

Write $\vec{T} = \langle \vec{T}, \vec{\pi}_0\rangle\,\vec{\pi}_0 + \vec{S}$. Then,

$$(T(d\psi \wedge \Xi))^2 = \left(\int \langle \vec{S}, d\psi \wedge \Xi\rangle\, d\|T\|\right)^2 \leqslant \||d\psi \wedge \Xi|\|_\infty^2 \|T\|(A \times \mathbb{R}^n) \int_{A\times\mathbb{R}^n} |\vec{S}|^2\, d\,\|T\|\,,$$

($|\cdot|$ denotes the norms on Λ_m and Λ^m induced by the natural inner products $\langle,\rangle$). Since $|\vec{S}|^2 = 1 - \langle \vec{T}, \vec{\pi}_0\rangle^2 \leqslant 2 - 2\langle \vec{T}, \vec{\pi}_0\rangle$, we have

$$\int_{A\times\mathbb{R}^n} |\vec{S}|^2\, d\,\|T\| \leqslant 2\int_{A\times\mathbb{R}^n} \left(1 - \langle \vec{T}, \vec{\pi}_0\rangle\right)\, d\,\|T\| = 2\,\boldsymbol{e}_T(A).$$

Moreover, by (4.16), $\||d\psi \wedge \Xi|\|_\infty \leqslant m\,\|D\psi\|_\infty\ \|\varphi\|_\infty \leqslant m$. Summarizing, we get

$$\int_A \Phi_\psi(x)\,\mathrm{div}\,\varphi(x)\,dx \leqslant \left(2m^2\,\boldsymbol{e}_T(A)\,\|T\|(A\times\mathbb{R}^n)\right)^{\frac{1}{2}}\,. \tag{4.17}$$

Taking the supremum in (4.17) over φ's with $\|\varphi\|_\infty \leqslant 1$, we conclude (4.13) through (4.14). □

Proof of Proposition 4.4

Since the statement is invariant under translations and dilations, without loss of generality we assume $x = 0$ and $s = 1$. Consider the slices $T_x := \mathbf{p}^\perp_\sharp\langle T, \mathbf{p}, x\rangle \in \mathbf{I}_0(\mathbb{R}^n)$ and recall that $\|T\|(A\times\mathbb{R}^n) \geqslant \int_A \mathbf{M}(T_x)\,dx$ for every open set A (cf. [54, Lemma 28.5]). Therefore,

$$\mathbf{M}(T_x) \leqslant \lim_{r\to 0} \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m\, r^m} \leqslant \boldsymbol{m}\boldsymbol{e}_T(x) + Q \quad \text{for almost every } x.$$

Since $\delta_{11} < 1$, we infer $\mathbf{M}(T_x) < Q+1$ for a.e. $x \in K$. There are, then, Q functions $g_i : K \to \mathbb{R}^n$ such that $T_x = \sum_{i=1}^Q \delta_{g_i(x)}$ for a.e. $x \in K$. Define $g : K \mapsto \mathcal{A}_Q(\mathbb{R}^n)$ as $g := \sum_i [\![g_i]\!]$ and fix $\psi \in C^\infty_c(\mathbb{R}^n)$. Proposition 4.5 gives

$$\left(|D\Phi_\psi|(B_r(y))\right)^2 \leqslant 2m\left(\boldsymbol{e}_T(B_r(y))\right)\|T\|(\mathbf{C}_r(y)) = 2m\left(\boldsymbol{e}_T(B_r(y))\right)(Q|B_r(y)| + \boldsymbol{e}_T(B_r(y)))\,.$$

Hence, if we define the maximal function

$$\mathbf{m}|D\Phi_\psi|(x) := \sup_{x\in B_{\frac{r}{2}}(y)\subset B_{4r}} \frac{|D\Phi_\psi|(B_{\frac{r}{2}}(y))}{|B_{\frac{r}{2}}(y)|},$$

we conclude that

$$(\mathbf{m}|D\Phi_\psi|(x))^2 \leqslant 2m\,\mathbf{m}\boldsymbol{e}_T(x)^2 + 2m\,Q\,\mathbf{m}\boldsymbol{e}_T(x) \leqslant C\delta_{11} \quad \text{for every } x\in K.$$

Therefore, the theory of BV functions gives a dimensional constant C such that

$$|\Phi_\psi(x)-\Phi_\psi(y)| \leqslant C\,\delta_{11}^{\frac{1}{2}}\,|x-y| \qquad \forall x,y\in K \text{ Lebesgue points of } \Phi_\psi, \tag{4.18}$$

(see for instance [30, Section 6.6.2]: although in that reference the authors use the *centered* maximal function, the proof works obviously also in our context). Consider next the Wasserstein distance of exponent 1:

$$\mathcal{W}_1(S_1,S_2) := \sup\left\{\langle S_1 - S_2,\psi\rangle \,:\, \psi\in C^1(\mathbb{R}^n),\ \|D\psi\|_\infty \leqslant 1\right\}. \tag{4.19}$$

Obviously, when $S_1 = \sum_i [\![S_{1i}]\!], S_2 = \sum_i [\![S_{2i}]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$, the supremum in (4.19) can be taken over a suitable countable subset of $\psi\in C^\infty_c(\mathbb{R}^n)$, chosen independently of the S_i's. Moreover, it follows easily from the definition in (4.19) that

$$\mathcal{W}_1(S_1,S_2) = \inf_{\sigma\in\mathscr{P}_Q}\sum_i |S_{1i}-S_{2\sigma(i)}| \geqslant \inf_{\sigma\in\mathscr{P}_Q}\Big(\sum_i |S_{1i}-S_{2\sigma(i)}|^2\Big)^{\frac{1}{2}} = \mathcal{G}(S_1,S_2).$$

So $\mathcal{G}(g(x),g(y)) \leqslant C\,\delta_{11}^{\frac{1}{2}}\,|x-y|$ for a.e. $x,y\in K$.

Next, write $g(x) = \sum_i [\![(h_i(x),\Psi(x,h_i(x)))]\!]$. Obviously $x\mapsto h(x) := \sum_i [\![h_i(x)]\!] \in \mathcal{A}_Q(\mathbb{R}^{\bar n})$ is a Lipschitz map. Recalling Proposition 3.4, we can extend it to a map $\bar u\in \mathrm{Lip}(B_{\frac{7r}{2}},\mathcal{A}_Q(\mathbb{R}^{\bar n}))$ satisfying $\mathrm{Lip}(\bar u)\leqslant C\,\delta_{11}^{\frac{1}{2}}$ and $\mathrm{osc}\,(\bar u)\leqslant C\mathrm{osc}\,(h)$. Set finally $u(x) = \sum_i [\![(\bar u_i(x),\Psi(x,\bar u_i(x)))]\!]$. The estimates claimed on u follow easily.

The identity $\mathbf{G}_u\llcorner(K\times\mathbb{R}^n) = T\llcorner(K\times\mathbb{R}^n)$ is a consequence of $u(x) = T_x$ for a.e. $x\in K$. Indeed, recall that both T and $\mathbf{G}_u$ are rectifiable and observe that $\langle\vec T,\vec\pi_0\rangle\neq 0$ $\|T\|$-a.e. on $K\times\mathbb{R}^n$, because $\mathbf{m}\boldsymbol{e}_T<\infty$ on K. Similarly, $\langle\vec{\mathbf{G}}_u,\vec\pi_0\rangle\neq 0$ $\|\mathbf{G}_u\|$-a.e. on $K\times\mathbb{R}^n$, by Proposition 3.39. Thus, $(\mathbf{G}_u - T)\llcorner K\times\mathbb{R}^n = 0$ if and only if $(\mathbf{G}_u-T)\llcorner dx\mathbf{1}_{K\times\mathbb{R}^n} = 0$. The latter identity follows from the slicing formula and the property $\langle T,\mathbf{p},x\rangle = \langle\mathbf{G}_u,\mathbf{p},x\rangle = \sum_i \delta_{(x,u_i(x))}$, valid for a.e. $x\in K$.

Finally, for each $x\in B_r\setminus K$ choose a ball $x\in B^x = B_{r(x)}(y(x))\subset B_4$ such that $\boldsymbol{e}_T(B^x)\geqslant 2^{-m}\delta_{11}\omega_m r(x)^m$. By the 5r-Covering theorem, we choose balls $\hat B^i = B_{5r(x_i)}(y(x_i))$ which cover $B_r\setminus K$ and such that the balls B^{x_i} are pairwise disjoint. We then conclude

$$|B_r\setminus K|\leqslant 10^m\delta_{11}^{-1}\boldsymbol{e}_T\left(\bigcup_i B^{x_i}\right). \tag{4.20}$$

Fix $y\in B^{x_i}$. Since $B^{x_i}\subset B_4$, we have $2^{-m}\delta_{11}\omega_m r(x_i)^m \leqslant \boldsymbol{e}_T(B^{x_i})\leqslant \boldsymbol{e}_T(B_4) = 4^m\omega_m E$, which implies $2r(x_i)\leqslant r_0 < 1$. Thus, $y\in B_{r+r_0}\subset B_4$. By definition of $\mathbf{m}\boldsymbol{e}_T$ we obviously have $\mathbf{m}\boldsymbol{e}_T(y)\geqslant 2^{-m}\delta_{11}$. So $\cup_i B^{x_i}\subset B_{r+r_0}\cap\{\mathbf{m}\boldsymbol{e}_T > 2^{-m}\delta_{11}\}$ and (4.20) implies (4.9).

4.2 HOMOTOPY LEMMA

Before proving the main Lipchitz approximation theorem we need a lemma which estimates carefully the difference of mass between an $\mathbf{\Omega}$-almost minimizer and a competitor in terms of a power of the excess and the costant $\mathbf{\Omega}$. The key idea is to choose the surface S in (4.1) to be an homotopy between the E^β approximation of T and that of S.

Lemma 4.6 (Homotopy Lemma). *Let T be an $\mathbf{\Omega}$-almost minimizer which satisfies Assumption 1 in $\mathbf{C}_{4r}(x)$. There are positive dimensional constants ε_{22} and C_{25} such that, if $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) \leqslant \varepsilon_{22}$, then the following holds. For every $R \in \mathbf{I}_m(\mathbf{C}_{3r}(x))$ such that $\partial R = \partial(T \llcorner \mathbf{C}_{3r}(x))$, we have*

$$\|T\|(\mathbf{C}_{3r}(x)) \leqslant \mathbf{M}(R) + C_{25} r^{m+1} \mathbf{\Omega} E^{\frac{1}{2}}\,. \tag{4.21}$$

Moreover, let $\beta \leqslant \frac{1}{2m}$, $s \in]r, 2r[$, $R = \mathbf{G}_g \llcorner \mathbf{C}_s(x)$ for some Lipschitz map $g\colon B_s \to \mathcal{A}_Q(\mathbb{R}^n)$ with $\mathrm{Lip}(g) \leqslant 1$ and f be the E^β-approximation of T in $\mathbf{C}_{3r}$. If $f = g$ on ∂B_s and $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is such that $\partial P = \partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s)$, then

$$\|T\|(\mathbf{C}_s(x)) \leqslant \mathbf{M}(\mathbf{G}_g) + \mathbf{M}(P) + C_{25}\mathbf{\Omega}\Big(E^{\frac{3}{4}} r^{m+1} + (\mathbf{M}(P))^{1+\frac{1}{m}} + \int_{B_s(x)} \mathcal{G}(f,g)\Big)\,. \tag{4.22}$$

Proof. We will show (4.21): the reader will notice that (4.22) follows easily from a portion of the argument.

Without loss of generality we assume $x = 0$. If $\|T\|(\mathbf{C}_{3r}) \leqslant \mathbf{M}(R)$ then there is nothing to prove. Hence we can suppose

$$\mathbf{M}(R) \leqslant \|T\|(\mathbf{C}_{3r}). \tag{4.23}$$

Define the current $R' \in \mathbf{I}_m(\mathbf{C}_{4r})$ by $R' := R + T \llcorner (\mathbf{C}_{4r} \setminus \mathbf{C}_{3r})$. Observe that $\partial(T - R') = 0$. So $\partial(\mathbf{p}_\sharp(T - R')) = 0$. On the other hand $\mathbf{p}_\sharp(T - R') = k\,[\![B_{4r}]\!]$ for some constant k and thus we conclude $\mathbf{p}_\sharp(T - R') = 0$. Therefore R' satisfies (4.2). Moreover we notice that, thanks to (4.23), the cylindrical excess of R' enjoys the following bound:

$$\mathbf{E}(R', C_{4r}) = \frac{\mathbf{M}(R')}{\omega_m r^m} - Q \overset{(4.23)}{\leqslant} \frac{\mathbf{M}(T)}{\omega_m r^m} - Q = \mathbf{E}(T, C_{4r}) =: E.$$

Let $f, h\colon B_{\frac{7r}{2}} \to \mathcal{A}_Q(\mathbb{R}^n)$ be the E^β-Lipschitz approximations of T and R' respectively, in the cylinders $\mathbf{C}_{\frac{7r}{2}}$ (where the choice of the exponent β will be specified later). Then there exist sets $K_T, K_{R'} \subset B_{\frac{7r}{2}}(x)$ such that

$$\mathbf{M}((T - \mathbf{G}_f) \llcorner C_{\frac{7r}{2}}) \leqslant C_{21} r^m E^{1-2\beta} \quad \text{and} \quad \mathbf{M}((R' - \mathbf{G}_h) \llcorner C_{\frac{7r}{2}}) \leqslant C_{21} r^m E^{1-2\beta}, \tag{4.24}$$

$$|B_{\frac{7r}{2}} \setminus K_T| \leqslant C_{21} r^m E^{1-2\beta} \quad \text{and} \quad |B_{\frac{7r}{2}} \setminus K_{R'}| \leqslant C_{21} r^m E^{1-2\beta}, \tag{4.25}$$

$$\mathrm{Lip}(f) \leqslant C_{21} E^\beta \quad \text{and} \quad \mathrm{Lip}(h) \leqslant C E^\beta. \tag{4.26}$$

Next we set $K := K_T \cap K_{R'}$ and we notice that by (4.25)

$$|B_{\frac{7r}{2}} \setminus K| \leqslant C_{21} r^m E^{1-2\beta}. \tag{4.27}$$

Let $|\cdot|$ be the cylindrical euclidean norm, that is $|(x,y)| := |x|$ for every $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$. By slicing theory, (4.24), (4.27) and Fubini's Theorem there exist $I_1, I_2 \subset (3r, \frac{7r}{2})$ such that $|(3r, \frac{7r}{2}) \setminus I_j| \leqslant r/8$ and

$$\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s\rangle) \leqslant C_{21} r^{m-1} E^{1-2\beta} \quad \text{and} \quad \mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s) \leqslant C_{21} r^{m-1} E^{1-2\beta}\,.$$

and

$$|\partial B_s \setminus K| \leqslant C_{21} r^{m-1} E^{1-2\beta}\,,$$

for every $s \in I_j$, $j = 1, 2$. Therefore there exists $s \in (3r, 7/2r)$ such that

$$\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s\rangle) \leqslant C_{21} r^{m-1} E^{1-2\beta} \quad \text{and} \quad \mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s\rangle) \leqslant C_{21} r^{m-1} E^{1-2\beta} \tag{4.28}$$

and

$$|\partial B_s \setminus K| \leqslant C_{21} r^{m-1} E^{1-2\beta}\,. \tag{4.29}$$

By the Isoperimetric Inequality, there exists $P_T, P_R \in \mathbf{I}_m(\mathbb{R}^{m+n})$ such that

$$\partial P_T = \langle T - \mathbf{G}_f, |\cdot|, s\rangle \qquad \partial P_R = \langle R' - \mathbf{G}_h, |\cdot|, s\rangle$$

and

$$\begin{aligned}\mathbf{M}(P_T) + \mathbf{M}(P_R) &\leqslant C\big(\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s\rangle)\big)^{\frac{m}{(m-1)}} + C\big(\mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s\rangle)\big)^{\frac{m}{(m-1)}} \\ &\leqslant C r^m E^{m(1-2\beta)/(m-1)}.\end{aligned}$$

Choosing $\beta = \frac{1}{2m}$, we can conclude that

$$\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s) = \partial P_T \qquad \partial((R' - \mathbf{G}_h) \llcorner \mathbf{C}_s) = \partial P_R \tag{4.30}$$

with

$$\mathbf{M}(P_T) + \mathbf{M}(P_R) \leqslant C r^m E\,. \tag{4.31}$$

Next consider the functions

$$f' := \boldsymbol{\xi} \circ f \colon B_{\frac{7r}{2}} \to \mathcal{Q} \subset \mathbb{R}^{N(Q,n)} \quad \text{and} \quad h' := \boldsymbol{\xi} \circ h \colon B_{\frac{7r}{2}} \to \mathcal{Q} \subset \mathbb{R}^{N(Q,n)}\,.$$

and the homotopy between them, defined by

$$\tilde{H}(x,t) \colon [0,1] \times B_{\frac{7r}{2}}(x) \ni (t,x) \to (x, tf'(x) + (1-t)h'(x)) \in \mathbb{R}^m \times \mathbb{R}^N\,.$$

Consider the Lipschitz map

$$\phi \colon \mathbb{R}^m \times \mathbb{R}^N \ni (x,y) \to (x, \boldsymbol{\xi}^{-1}(\boldsymbol{\rho}(y))) \in \mathbb{R}^m \times \mathcal{A}_Q(\mathbb{R}^n)$$

and define $H := \phi \circ \tilde{H}$. H can be seen as a Q-valued map $H : B_{2r} \times [0,1] \to \mathcal{A}_Q(\mathbb{R}^{2+m})$. Without changing notation for H we restrict it to $[0,1] \times B_s$ and following the notation of Definition 3.38 we define $S := \mathbf{T}_H$. If we set $G := H|_{[0,1]\times\partial B_s}$ we can use Theorem 3.47 to conclude that

$$\partial S = (\mathbf{G}_f - \mathbf{G}_h)\llcorner \mathbf{C}_s + \mathbf{T}_G = (\mathbf{G}_f - \mathbf{G}_h)\llcorner \mathbf{C}_s + P\,, \tag{4.32}$$

where $P := \mathbf{T}_G$. We now want to estimate $\mathbf{M}(S)$ and $\mathbf{M}(P)$ and we will do it using the Q-valued area formula in Lemma 3.44. We start with $\mathbf{M}(S)$. We fix a point of differentiability p where $DH = \sum [\![DH_i]\!]$. On $[0,1] \times B_s$ we use the coordinates (t,x) and on the target space $\mathbb{R}^{m+n}$ the coordinates (x,y). Let $p = (t_0, x_0)$. It is then obvious that the matrix DH_i can be decomposed as

$$DH_i(p) = \begin{pmatrix} I_{m\times m} & 0_{m\times 1} \\ A_{n\times m} & v_{n\times 1}\,. \end{pmatrix}$$

where the matrices A and v can be bound using the following observation. If we consider the map $t \mapsto \Phi(t) := H(x_0, t)$ and $x \mapsto \Lambda(x) := H(t_0, x)$, we then have $|v| \leqslant C\mathrm{Lip}(\Phi)$ and $|A| \leqslant C\mathrm{Lip}(\Lambda)$, where the constant C depends only on n and Q. On the other hand, it is easy to see that $\mathrm{Lip}(\Phi) \leqslant C\mathcal{G}(f(x_0), h(x_0))$ and $\mathrm{Lip}(\Lambda) \leqslant C(\mathrm{Lip}(h) + \mathrm{Lip}(f)) \leqslant E^{\beta} = E^{\frac{1}{2m}}$. Thus we can estimate

$$JH_i := \sqrt{\det(DH_i^* \cdot DH_i)} \leqslant C\mathcal{G}(f(x_0), g(x_0))\,.$$

Using Lemma 3.44 we then conclude

$$\mathbf{M}(S) \leqslant C\int_{B_s} \mathcal{G}(f,h)$$

and, arguing in a similar fashion,

$$\mathbf{M}(P) \leqslant C\int_{\partial B_s} \mathcal{G}(f,h)\,.$$

Observe that f and h coincides, respectively, with the slices of the currents T and R' on any $x_0 \in K$. On the other hand, $s > 3r$ and $T\llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r} = R'\llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r}$. We thus conclude that $h = f$ on $K \cap \partial B_s$. Let $x \in \partial B_s \setminus K$. By (4.29), there exists $x_0 \in K \cap \partial B_s$ such that $|x - x_0| \leqslant CrE^{(1-2\beta)/(m-1)} = CrE^{2\beta}$ (recall that $\beta = \frac{1}{2m}$). Thus

$$\mathcal{G}(f(x), h(x)) \leqslant (\mathrm{Lip}(f) + \mathrm{Lip}(h))\,|x - x_0| \leqslant CrE^{3\beta}\,,$$

and so we conclude

$$\mathbf{M}(P) \leqslant C\int_{\partial B_s} \mathcal{G}(f,h) \leqslant CrE^{3\beta}|\partial B_s \setminus K| \leqslant Cr^m E^{1+\beta} \leqslant Cr^m E\,. \tag{4.33}$$

On the other hand, we recall that, by a standard variant of the Poincaré inequality,

$$\int_{B_s} \mathcal{G}(f,h) \leqslant Cr\|\mathcal{G}(f,h)\|_{L^1(\partial B_s)} + Cr\|D(\mathcal{G}(f,h))\|_{L^1(B_s)}$$

$$\overset{(4.33)}{\leqslant} Cr^{m+1}E + Cr^{1+\frac{m}{2}}\left(\int (|Df|^2 + |Dh|^2)\right)^{\frac{1}{2}} \leqslant Cr^{m+1}E^{\frac{1}{2}}\,. \tag{4.34}$$

Thus,

$$\mathbf{G}_f - \mathbf{G}_h = \partial S + P \tag{4.35}$$

with

$$\mathbf{M}(P) \leqslant C r^m E \quad \text{and} \quad \mathbf{M}(S) \leqslant C r^{m+1} E^{\frac{1}{2}}. \tag{4.36}$$

Now observe that

$$0 = \partial(T - R') = \partial((\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s) + \partial(P_T - P_R) = \partial\partial S + \partial P + \partial(P_T - P_R)\,.$$

Hence, by the isoperimetric inequality, there is an S' with $\mathbf{M}(S') \leqslant C r^{m+1} E^{1+\frac{1}{m}}$ and $\partial S' = \partial(P + P_T - P_R)$. Additionally, again using the isoperimetric inequality, there are currents S_T and S_R such that

$$\begin{aligned}
\partial S_T &= (T - \mathbf{G}_f) \llcorner \mathbf{C}_s - P_T \\
\partial S_R &= (R' - \mathbf{G}_h) \llcorner \mathbf{C}_s - P_R
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{M}(S_T) &\leqslant C\left(\|T - \mathbf{G}_f\|(\mathbf{C}_s) + \mathbf{M}(P_T)\right)^{\frac{(m+1)}{m}} \leqslant C E^{\frac{3}{4}} r^{m+1} \\
\mathbf{M}(S_R) &\leqslant C\left(\|T - \mathbf{G}_h\|(\mathbf{C}_s) + \mathbf{M}(P_R)\right)^{\frac{(m+1)}{m}} \leqslant C E^{\frac{3}{4}} r^{m+1}\,.
\end{aligned}$$

In the latter inequalities we have used $\|T - \mathbf{G}_h\|(\mathbf{C}_s) + \|T - \mathbf{G}_f\|(\mathbf{C}_s) \leqslant C E^{1-2\beta} r^m = C E^{(m-1)/m} r^m$: in particular $(1-2\beta)(m+1)/m = 1 - 1/m^2 \geqslant 3/4$; observe that this estimate is valid even if $\beta < 1/(2m)$ and explains the exponent of E in the third summand of the right hand side of (4.22).

Thus, setting $S'' = S + S_T - S_R - S'$ we finally achieve $(T - R) \llcorner \mathbf{C}_{3r} = \partial S''$ and $\mathbf{M}(S'') \leqslant C r^{m+1} E^{\frac{1}{2}}$. Applying now the $\mathbf{\Omega}$-minimality of T we conclude

$$\|T\|(\mathbf{C}_{3r}) \leqslant \mathbf{M}(R) + C_{25} r^{m+1} \mathbf{\Omega} E^{\frac{1}{2}}\,.$$

For the proof of (4.22) we conclude with the same computations, except that this time $f = g$ on ∂B_s and the current R is already given by $\mathbf{G}_g \llcorner \mathbf{C}$. The modifications to the argument are then straightforward, given the remark of the previous paragraph. □

4.3 HARMONIC APPROXIMATION AND GRADIENT L^p ESTIMATES

In this and in the next section we follow largely [19] with minor modifications: on the one hand we have the additional $\mathbf{\Omega}$-error terms, but on the other hand the ambient Riemannian manifold is the euclidean space. Thus the arguments are somewhat less technical.

4.3.1 *Harmonic Approximation*

In this subsection we prove that if T is an almost minimizer then its E^β-Lipschitz approximation is close to a Dir-minimizing function w. This comes with an $o(E)$-improvement of the estimates in Proposition 4.4.

Remark 4.7. There exists a dimensional constant $c > 0$ such that, if $E \leqslant c$, then the E^β-Lipschitz approximation satisfies the following estimates:

$$\mathrm{Lip}(f) \leqslant C\, E^\beta, \tag{4.37}$$

$$\int_{B_{3s}(x)} |Df|^2 \leqslant C\, E\, s^m. \tag{4.38}$$

Indeed (4.37) follows from Proposition 4.4, while (4.38) follows from the Taylor expansion of the mass of $\mathbf{G}_u$:

$$\mathbf{M}(\mathbf{G}_u) = Q\,|V| + \int_V \frac{|Du|^2}{2} + \int_V \sum_i R(Du_i),$$

where $R : \mathbb{R}^{n\times m} \to R$ is a C^1 function satisfying $|R(D)| = |D|^3\, L(D)$ for some positive function L such that $L(0) = 0$ and $\mathrm{Lip}(L) \leqslant C$ (cp. Corollary 3.49). Indeed, for E sufficiently small we have

$$\int_{B_{3s}(x)} \sum_i R(Df_i) \leqslant C\, E^{2\beta} \int_{B_{3s}(x)} |Df|^2 < \frac{1}{4} \int_{B_{3s}(x)} |Df|^2,$$

and therefore, since $T\llcorner(K\times\mathbb{R}^n) = \mathbf{G}_f\llcorner(K\times\mathbb{R}^n)$,

$$\begin{aligned}
\int_{B_{3s}(x)} |Df|^2 &\leqslant C\,(\mathbf{M}(\mathbf{G}_f\llcorner\mathbf{C}_{3s}(x)) - Q\,\omega_m\,(3\,s)^m)\\
&\leqslant C\,(\mathbf{M}(T\llcorner(K\times\mathbb{R}^n)) - Q\,\omega_m\,(3\,s)^m) + C\,\mathbf{M}(\mathbf{G}_f\llcorner(B_{3s}(x)\setminus K)\times\mathbb{R}^n)\\
&\leqslant C\,(\mathbf{M}(T\llcorner\mathbf{C}_{3s}(x)) - Q\,\omega_m\,(3\,s)^m) + C\,E^{2\beta}\,|B_{3s}(x)\setminus K| \leqslant C\,E\,s^m.
\end{aligned}$$

Theorem 4.8 (First harmonic approximation)**.** *For every $\eta_1, \delta > 0$ and every $\beta \in (0, \frac{1}{2m})$, there exists a constant $\varepsilon_{23} > 0$ with the following property. Let T be an $\boldsymbol{\Omega}$-almost minimizer which satisfies Assumption 1 in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leqslant \varepsilon_{23}$ and $s\boldsymbol{\Omega} \leqslant \varepsilon_{23}E^{\frac{1}{2}}$, then the E^β-Lipschitz approximation f in $\mathbf{C}_{3s}(x)$ satisfies*

$$\int_{B_{2s}(x)\setminus K} |Df|^2 \leqslant \eta_1\, E\,\omega_m\,(4\,s)^m = \eta_1\,\boldsymbol{e}_T(B_{4s}(x)). \tag{4.39}$$

Moreover, there exists a Dir*-minimizing function w such that*

$$s^{-2}\int_{B_{2s}(x)} \mathcal{G}(f,w)^2 + \int_{B_{2s}(x)} \big(|Df| - |Dw|\big)^2 \leqslant \eta_1\, E\,\omega_m\,(4\,s)^m = \eta_1\,\boldsymbol{e}_T(B_{4s}(x))\,, \tag{4.40}$$

$$\int_{B_{2s}(x)} |D(\boldsymbol{\eta}\circ f) - D(\boldsymbol{\eta}\circ w)|^2 \leqslant \eta_1\, E\,\omega_m\,(4\,s)^m = \eta_1\,\boldsymbol{e}_T(B_{4s}(x))\,. \tag{4.41}$$

Proof of Theorem 4.8. By rescaling and translating, it is not restrictive to assume that $x = 0$ and $s = 1$. We proceed by contradiction. Assume there exist a constant $c_1 > 0$, a sequence of positive real numbers $(\varepsilon_l)_l$, a sequence of currents $\boldsymbol{\Omega}_l$-minimal currents $(T_l)_{l\in\mathbb{N}}$ and corresponding E_l^β-Lipschitz approximations $(f_l)_{l\in\mathbb{N}}$ such that

$$E_l := \mathbf{E}(T_l, \mathbf{C}_4) \leqslant \varepsilon_l \to 0, \quad \boldsymbol{\Omega}_l \leqslant \varepsilon_l E^{\frac{1}{2}} \quad \text{and} \quad \int_{B_2\setminus K_l} |Df_l|^2 \geqslant c_1\, E_l, \tag{4.42}$$

where $K_l := \{x \in B_3 : \boldsymbol{m}\boldsymbol{e}_{T_l}(x) < E_l^{2\beta}\}$. Set $\Gamma_l := \{x \in B_4 : \boldsymbol{m}\boldsymbol{e}_{T_l}(x) \leqslant 2^{-m} E_l^{2\beta}\}$ and observe that $\Gamma_l \cap B_3 \subset K_l$. From Proposition 4.4, it follows that

$$\mathrm{Lip}(f_l) \leqslant C_{22} E_l^{\beta}, \tag{4.43}$$

$$|B_r \setminus K_l| \leqslant C_{22} E_l^{-2\beta} \boldsymbol{e}_T\big(B_{r+r_0(l)} \setminus \Gamma_l\big) \quad \text{for every } r \leqslant 3\,, \tag{4.44}$$

where $r_0(l) = 16\, E_l^{(1-2\beta)/m} < \frac{1}{2}$. Then, (4.42), (4.43) and (4.44) give

$$c_1\, E_l \leqslant \int_{B_2 \setminus K_l} |Df_l|^2 \leqslant C_{22}\, \boldsymbol{e}_{T_l}(B_s \setminus \Gamma_l) \quad \forall\, s \in \big[\tfrac{5}{2}, 3\big]\,.$$

Setting $c_2 := c_1/(2C_{22})$, we have $2c_2 E_l \leqslant \boldsymbol{e}_{T_l}(B_s \setminus \Gamma_l) = \boldsymbol{e}_{T_l}(B_s) - \boldsymbol{e}_{T_l}(B_s \cap \Gamma_l)$, thus leading to

$$\boldsymbol{e}_{T_l}(\Gamma_l \cap B_s) \leqslant \boldsymbol{e}_{T_l}(B_s) - 2\, c_2\, E_l\,, \tag{4.45}$$

for l large enough Next observe that $\omega_m 4^m E_l = \boldsymbol{e}_{T_l}(B_4) \geqslant \boldsymbol{e}_{T_l}(B_s)$. Therefore, by the Taylor expansion in Corollary 3.49, (4.45) and $E_l \downarrow 0$, it follows that, for every $s \in [5/2, 3]$,

$$\begin{aligned} \int_{\Gamma_l \cap B_s} \frac{|Df_l|^2}{2} &\leqslant (1 + C\, E_l^{2\beta})\, \boldsymbol{e}_{T_l}(\Gamma_l \cap B_s) \\ &\leqslant (1 + C\, E_l^{2\beta}) \Big(\boldsymbol{e}_{T_l}(B_s) - 2\, c_2\, E_l\Big) \leqslant \boldsymbol{e}_{T_l}(B_s) - c_2\, E_l. \end{aligned} \tag{4.46}$$

Our aim is to show that (4.46) contradicts the $\boldsymbol{\Omega}$-almost minimizing property (4.1) of T_l. To construct a competitor consider $g_l := E_l^{-\frac{1}{2}} f_l$. Observe that from the estimates of Remark 4.7, we easily infer $\mathrm{Dir}(f_l, B_3) \leqslant C E_l$. Hence, $\sup_l \mathrm{Dir}(g_l, B_3) < \infty$. Since $|B_3 \setminus \Gamma_l| \to 0$, by Proposition 3.29 we can find a subsequence (not relabelled) of translating sheets h_l satisfying (3.17) - (3.18) and $\|\mathcal{G}(g_l, h_l)\|_{L^2(B_3)} \to 0$. In particular, we are in the position to apply Proposition 3.30 to g_l and h_l, with $r_0 = \frac{5}{2}$, $r_1 = 3$ and $\eta = \frac{c_2}{2}$, and find $r \in \big(\frac{5}{2}, 3\big)$ and competitor functions H_l satisfying $H_l|_{B_3 \setminus B_r} = g_l|_{B_3 \setminus B_r}$,

$$\mathrm{Dir}(H_l, B_r) \leqslant \mathrm{Dir}(g_l, B_r \cap \Gamma_l) + \frac{c_2}{2}, \tag{4.47}$$

$$\mathrm{Lip}(H_l) \leqslant C^*\, E_l^{\beta - \frac{1}{2}} \tag{4.48}$$

$$\|\mathcal{G}(H_l, g_l)\|_{L^2(B_r)} \leqslant C_{23}\, \mathrm{Dir}(g_l, B_r) + C_{23}\, \mathrm{Dir}(H_l, B_r) \leqslant M < \infty. \tag{4.49}$$

Note that (4.48) follows from (3.24) observing that $E_l^{\beta - \frac{1}{2}} \uparrow \infty$: thus C^* depends on c_2 and the two chosen sequences, but not on l. From now on, although this and similar constants are not dimensional, we will keep denoting them by C, with the understanding that they do not depend on l. Note that, from (4.43) and (4.44), one gets

$$\begin{aligned} \|T_l - \mathbf{G}_{f_l}\|(\mathbf{C}_3) &= \|T_l\|((B_3 \setminus K_l) \times \mathbb{R}^n) + \|\mathbf{G}_{f_l}\|((B_3 \setminus K_l) \times \mathbb{R}^n) \\ &\leqslant Q\, |B_3 \setminus K_l| + E_l + Q\, |B_3 \setminus K_l| + C\, |B_3 \setminus K_l|\, \mathrm{Lip}(f_l) \\ &\leqslant E_l + C\, E_l^{1-2\beta} \leqslant C\, E_l^{1-2\beta}. \end{aligned} \tag{4.50}$$

Consider the function $\varphi(z, y) = |z|$ and the slice $\langle T_l - \mathbf{G}_{f_l}, \varphi, r\rangle$. For every l, there exists $r_l \in (r, 3)$ such that $\mathbf{M}\big(\langle T_l - \mathbf{G}_{f_l}, \varphi, r_l\rangle\big) \leqslant C\, E_l^{1-2\beta}$.

Let now $u_l := E_l^{\frac{1}{2}} H_l|_{B_{r_l}}$, and consider the current $Z_l := \mathbf{G}_{u_l} \llcorner \mathbf{C}_{r_l}$. Since $u_l|_{\partial B_{r_l}} = f_l|_{\partial B_{r_l}}$, one gets $\partial Z_l = \langle \mathbf{G}_{f_l}, \varphi, r_l\rangle$ and, hence, $\mathbf{M}(\partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l)) \leqslant C E_l^{1-2\beta}$. By the Isoperimetric Inequality there is an integral current R_l such that

$$\partial R_l = \partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l) \quad \text{and} \quad \mathbf{M}(R_l) \leqslant C E_l^{m(1-2\beta)/(m-1)}.$$

Set $S_l = T_l \llcorner (\mathbf{C}_4 \setminus \mathbf{C}_{r_l}) + Z_l + R_l$. Notice that $\partial S_l = \partial T_l$. We assume from now on $\beta < \frac{1}{2m}$ and we set $1 + \gamma = m(1 - 2\beta)/(m - 1) > 1$. We want to compare the mass of S_l with that of T_l to achieve a contradiction in the limit for $l \to \infty$.

$$\int_{B_{r_l}} |Du_l|^2 - \int_{B_r \cap \Gamma_l} |Df_l|^2 = \mathrm{Dir}(B_{r_l}, u_l) - \mathrm{Dir}(B_{r_l} \cap \Gamma_l, f_l) \overset{(4.47)}{\leqslant} \frac{c_2}{2} E_l$$

where the factor E_l in the last inequality comes from the renormalizations $u_l = E_l^{\frac{1}{2}} H_l$ and $f_l = E_l^{\frac{1}{2}} g_l$. By possibly changing γ so that $2\beta \geqslant \gamma$, we can then write

$$\begin{aligned}
\mathbf{M}(S_l) - \mathbf{M}(T_l) &\leqslant \mathbf{M}(Z_l) + C\,\mathbf{M}(R_l) - \mathbf{M}(T_l \llcorner \mathbf{C}_r) \\
&\leqslant Q\,|B_r| + \int_{B_r} \frac{|Du_l|^2}{2} + C\, E_l^{1+\gamma} - Q|B_r| - \boldsymbol{e}_{T_l}(B_r) \\
&\leqslant \int_{B_r \cap \Gamma_l} \frac{|Df_l|^2}{2} + \frac{c_2}{2} E_l + C\, E_l^{1+\gamma} - \boldsymbol{e}_{T_l}(B_r) \\
&\overset{(4.46)}{\leqslant} -\frac{c_2\, E_l}{4} + C\, E_l^{1+\beta} + C\, E_l^{1+\gamma}.
\end{aligned} \tag{4.51}$$

Hence,

$$\mathbf{M}(S_l) < \mathbf{M}(T_l) \qquad \text{for } l \text{ large enough.} \tag{4.52}$$

This would be already a contradiction if T were area-minimizing. In our case, by (4.21) of Lemma 4.6 we have the upper bound

$$\mathbf{M}(S_l) - \mathbf{M}(T_l) \geqslant -C_{25}\boldsymbol{\Omega}_l E_l^{\frac{1}{2}} \geqslant -C_{25}\varepsilon_l E_l.$$

Combining this inequality with (4.51) we obtain

$$\frac{c_2 E_l}{4} \leqslant C E_l^{1+\gamma} + C\varepsilon_l E_l$$

which for E_l, ε_l sufficiently small (and hence for l large enough) provides the desired contradiction.

For what concerns (4.40), we argue similarly. Let $(T_l)_l$ be a sequence with vanishing $E_l := \mathbf{E}(T_l, \mathbf{C}_4)$, contradicting the second part of the statement and perform the same analysis as before. Up to subsequences, one of the following statement must be false:

(i) $\lim_l \int_{B_2} |Dg_l|^2 = \int_{B_2} |Dh_{l_0}|^2$, for any l_0 (recall that $\int_{B_2} |Dh_l|^2$ is constant);

(ii) h_l is Dir-minimizing in B_2.

If (i) is false, then there is a positive constant c_2 such that, for every $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|Dh_l|^2}{2} \leqslant \int_{B_r} \frac{|Dg_l|^2}{2} - c_2 \leqslant \frac{\mathbf{e}_{T_l}(B_r)}{E_l} - \frac{c_2}{2},$$

for l large enough (where the last inequality is again an effect of the Taylor expansion of Remark 4.7. Therefore we can argue exactly as in the proof of (4.39) (using h_l instead of H_l to construct the competitors) and reach a contradiction. If (ii) is false, then h_l is not Dir-minimizing in $B_{5/2}$. This implies that one of the ζ^j in the translating sheets h_l is not Dir-minimizing in B_2. Indeed, in the opposite case, by Theorem 3.23, $\|\mathcal{G}(\zeta^j, Q[\![0]\!])\|_{C^0(B_2)} < \infty$ and, since $h_l = \sum_i [\![\tau_{y_l^i} \circ \zeta^i]\!]$ and $|y_l^i - y_l^j| \to \infty$ for $i \neq j$, by the maximum principle of [17, Proposition 3.5], h_l would be Dir-minimizing. Thus, we can find a competitor $\hat{\zeta}^j$ for some ζ^j with less energy in the ball B_2. So the functions $F_l = \sum_j [\![\tau_{y_l^j} \circ \hat{\zeta}^j]\!]$ satisfy, for any $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|DF_l|^2}{2} \leqslant \int_{B_r} \frac{|Dh_l|^2}{2} - c_2 \leqslant \lim_l \int_{B_r} \frac{|Dg_l|^2}{2} - 2\,c_2 \leqslant \frac{\mathbf{e}_T(B_r)}{E_l} - \frac{c_2}{2},$$

provided l is large enough (where $c_2 > 0$ is a constant indepedent of r and l). On the other hand $F_l = h_l$ on $B_3 \setminus B_{5/2}$ and therefore $\|\mathcal{G}(F_l, g_l)\|_{L^2(B_3 \setminus B_{5/2})} \to 0$. We then argue as above with F_l in place of H_l and reach a contradiction in this case as well. $\square$

4.3.2 *Improved excess estimate.*

The higher integrability of the Dir-minimizing functions and the harmonic approximation lead to the following estimate, which we call "weak" since we will improve it in the next section with Theorem 4.11.

Proposition 4.9 (Weak excess estimate). *For every $\eta_2 > 0$, there exist $\varepsilon_{24}, C_{26} > 0$ with the following property. Let T be an $\mathbf{\Omega}$-almost minimizer and assume it satisfies Assumption 1 in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leqslant \varepsilon_{24}$, then*

$$\mathbf{e}_T(A) \leqslant \eta_2\, \mathbf{e}_T(B_{4s}(x)) + C_{26}\, \mathbf{\Omega}^2\, s^{m+2}, \tag{4.53}$$

for every $A \subset B_s(x)$ Borel with $|A| \leqslant \varepsilon_{24}|B_s(x)|$ (observe that the constant C_{26} depends on η_2).

Proof. Without loss of generality, we can assume $s = 1$ and $x = 0$. We distinguish the two regimes: $\hat{\varepsilon}^2 E \leqslant \mathbf{\Omega}^2$ and $\mathbf{\Omega}^2 \leqslant \hat{\varepsilon}^2 E$, where $\hat{\varepsilon} \leqslant \varepsilon_{24}$ is a parameter whose choice will be specified later. In the former, clearly $\mathbf{e}_T(A) \leqslant C\,E \leqslant C\,\mathbf{\Omega}^2$. In the latter, we let f be the $E^{\frac{1}{4m}}$-Lipschitz approximation of T in $\mathbf{C}_3$. By a Fubini-type argument as the ones already used in the previous secions, we find a radius $r \in (1, 2)$ and a current P with $\mathbf{M}(P) \leqslant CE^{1+\gamma}$ and $\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_r) = \partial P$ for some $\gamma(m) > 0$. We can thus apply Lemma 4.6 to $R = \mathbf{G}_f \llcorner \mathbf{C}_r + P + T \llcorner (\mathbf{C}_3 \setminus \mathbf{C}_r)$. Recalling the Taylor expansion in Corollary 3.49, we have

$$\begin{aligned}\|T\|(\mathbf{C}_r) &\leqslant \mathbf{M}(R \llcorner \mathbf{C}_r) + C\mathbf{\Omega} E^{\frac{1}{2}} \leqslant \|\mathbf{G}_f\|(\mathbf{C}_r) + C\hat{\varepsilon}E + CE^{1+\gamma} \\ &\leqslant Q\,|B_r| + \int_{B_r} \frac{|Df|^2}{2} + C\hat{\varepsilon}E + C\,E^{1+\gamma},\end{aligned} \tag{4.54}$$

for some positive γ (possibly smaller than the previous one). On the other hand, using again the Taylor expansion for the part of the current which coincides with the graph of f, we deduce as well that

$$\begin{aligned}\|T\|(\mathbf{C}_r) &= \|T\|((B_r\setminus K)\times\mathbb{R}^n) + \|T\|((B_r\cap K)\times\mathbb{R}^n)\\ &\geqslant \|T\|((B_r\setminus K)\times\mathbb{R}^n) + Q\,|B_r\cap K| + \int_{B_r\cap K}\frac{|Df|^2}{2} - C\,E^{1+\gamma}.\end{aligned} \tag{4.55}$$

Subtracting (4.55) from (4.54), we deduce

$$\mathbf{e}_T(B_r\setminus K) \leqslant \int_{B_r\setminus K}\frac{|Df|^2}{2} + C\hat\varepsilon E + CE^{1+\gamma}. \tag{4.56}$$

If ε_{24} is chosen small enough, we infer from (4.56) and (4.39) in Theorem 4.8 that

$$\mathbf{e}_T(B_r\setminus K) \leqslant \eta\,\mathbf{e}_T(B_4) + CE^{1+\gamma}, \tag{4.57}$$

for a suitable $\eta = \hat\varepsilon/2C$ to be specified later. Let now $A\subset B_1$ be such that $|A|\leqslant \varepsilon_{24}\,\omega_m$. Combining (4.57) with the Taylor expansion, we have

$$\mathbf{e}_T(A) \leqslant \mathbf{e}_T(A\setminus K) + \int_A\frac{|Df|^2}{2} + C\,E^{1+\gamma} \leqslant \int_A\frac{|Df|^2}{2} + \eta\,\mathbf{e}_T(B_4) + CE^{1+\gamma}. \tag{4.58}$$

If ε_{24} is small enough, we can again use Theorem 4.8 and Theorem 3.31 in (4.58) to get, for a Dir-minimizing w,

$$\mathbf{e}_T(A) \overset{(4.40)}{\leqslant} \int_A\frac{|Dw|^2}{2} + 2\eta\,\mathbf{e}_T(B_4) + CE^{1+\gamma} \leqslant \left(C_{24}|A|^{1-\frac{2}{p_1}} + 2\eta\right)\,\mathbf{e}_T(B_4) + CE^{1+\gamma}. \tag{4.59}$$

Hence, if ε_{24} and η are suitably chosen, (4.53) follows from (4.59). □

4.3.3 *Gradient* L^p *estimate.*

The density $\mathbf{d}$ of the excess measure is naturally an L^1 function. We prove here that for $\boldsymbol{\Omega}$-almost minimizer this function is in fact L^p, for some $p>1$.

Theorem 4.10 (Gradient L^p estimate)**.** *There exist constants $p_2>1$ and $C,\varepsilon_{25}>0$ (depending on n,Q) with the following property. Let T be as in Assumption 1 in the cylinder $\mathbf{C}_4$. If T is an $\boldsymbol{\Omega}$-almost minimizer and $E=\mathbf{E}(T,\mathbf{C}_4)<\varepsilon_{25}$, then*

$$\int_{\{\mathbf{d}\leqslant 1\}\cap B_2}\mathbf{d}^{p_2} \leqslant C\,E^{p_2-1}\left(E+\boldsymbol{\Omega}^2\right). \tag{4.60}$$

Proof. We assume without loss of generality that $E>0$ and divide the proof into two steps.

Step 1. There exist constants $\gamma\geqslant 2^m$ and $\rho>0$ such that, for every $c\in[1,(\gamma\,E)^{-1}]$ and $s\in[2,4]$ with $\bar s = s+2\,c^{-\frac{1}{m}}\leqslant 4$, we have

$$\int_{\{\gamma\,c\,E\leqslant d\leqslant 1\}\cap B_s} d \leqslant \gamma^{-\rho}\int_{\left\{\frac{c\,E}{\gamma}\leqslant d\leqslant 1\right\}\cap B_{\bar s}} d + C\,c^{-\frac{2}{m}}\,\boldsymbol{\Omega}^2. \tag{4.61}$$

In order to prove it, let N_B be the constant in Besicovich's covering theorem [30, Section 1.5.2] and choose $N \in \mathbb{N}$ so large that $N_B < 2^{N-1}$. Let ε_{24} be as in Proposition 4.9 when we choose $\eta_2 = 2^{-2m-N}$, and set

$$\gamma = \max\{2^m, \varepsilon_{24}^{-1}\} \quad \text{and} \quad \rho = \min\left\{-\log_\gamma(N_B/2^{N-1}), \frac{1}{4}\right\}.$$

Let c and s be any real numbers as above. For almost every $x \in \{\gamma\, c\, E \leqslant d \leqslant 1\} \cap B_s$, there exists r_x such that

$$\mathbf{E}(T, \mathbf{C}_{4r_x}(x)) \leqslant c\, E \qquad \text{and} \qquad \mathbf{E}(T, \mathbf{C}_t(x)) \geqslant c\, E \quad \forall t \in]0, 4\, r_x[. \tag{4.62}$$

Indeed, since $d(x) = \lim_{r\to 0} \mathbf{E}(T, \mathbf{C}_r(x)) \geqslant \gamma\, c\, E \geqslant 2^2 c\, E$ and

$$\mathbf{E}(T, \mathbf{C}_t(x)) = \frac{\boldsymbol{e}_T(B_t(x))}{\omega_m\, t^m} \leqslant \frac{4^m\, E}{t^m} \leqslant c\, E \quad \text{for} \quad t \geqslant \frac{4}{\sqrt[m]{c}},$$

we just choose $4r_x = \min\{t \leqslant 4/\sqrt[m]{c} : \mathbf{E}(T, \mathbf{C}_t(x)) \leqslant cE\}$. Note also that $r_x \leqslant 1/\sqrt[m]{c}$. Consider the current T in $\mathbf{C}_{4r_x}(x)$. Setting $A = \{\gamma\, c\, E \leqslant d\} \cap B_{4r_x}(x)$, we have that

$$\mathbf{E}(T, \mathbf{C}_{4r_x}(x)) \leqslant c\, E \leqslant \frac{E}{\gamma\, E} \leqslant \varepsilon_{24} \quad \text{and} \quad |A| \leqslant \frac{c\, E\, |B_{4r_x}(x)|}{\gamma\, c\, E} \leqslant \varepsilon_{24} |B_{4r_x}(x)|.$$

Hence, we can apply Proposition 4.9 to $T \llcorner \mathbf{C}_{4r_x}(x)$ to get

$$\begin{aligned} \int_{B_{r_x}(x)\cap\{\gamma\, c\, E \leqslant d \leqslant 1\}} d &\leqslant \int_A d \leqslant \boldsymbol{e}_T(A) \leqslant 2^{-2m-N}\, \boldsymbol{e}_T(B_{4r_x}(x)) + C\, r_x^{m+2} \boldsymbol{\Omega}^2 \\ &\leqslant 2^{-2m-N}\, (4\, r_x)^m\, \omega_m\, \mathbf{E}(T, \mathbf{C}_{4r_x}(x)) + C\, r_x^{m+2} \boldsymbol{\Omega}^2 \overset{(4.62)}{\leqslant} 2^{-N}\, \boldsymbol{e}_T(B_{r_x}(x)) + C\, r_x^{m+2} \boldsymbol{\Omega}^2. \end{aligned} \tag{4.63}$$

Thus,

$$\begin{aligned} \boldsymbol{e}_T(B_{r_x}(x)) &= \int_{B_{r_x}(x)\cap\{d>1\}} d + \int_{B_{r_x}(x)\cap\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}} d + \int_{B_{r_x}(x)\cap\{d < \frac{c\,E}{\gamma}\}} d \\ &\leqslant \int_A d + \int_{B_{r_x}(x)\cap\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}} d + \frac{c\, E}{\gamma}\, \omega_m\, r_x^m \\ &\overset{(4.62),(4.63)}{\leqslant} \left(2^{-N} + \gamma^{-1}\right)\, \boldsymbol{e}_T(B_{r_x}(x)) + C\, r_x^{m+2} \boldsymbol{\Omega}^2 + \int_{B_{r_x}(x)\cap\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}} d. \end{aligned} \tag{4.64}$$

Therefore, recalling that $\gamma \geqslant 2^m \geqslant 4$, from (4.63) and (4.64) we infer:

$$\begin{aligned} \int_{B_{r_x}(x)\cap\{\gamma\, c\, E \leqslant d \leqslant 1\}} d &\leqslant \frac{2^{-N}}{1 - 2^{-N} - \gamma^{-1}} \int_{B_{r_x}(x)\cap\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}} d + C\, r_x^{m+2} \boldsymbol{\Omega}^2 \\ &\leqslant 2^{-N+1} \int_{B_{r_x}(x)\cap\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}} d + C\, r_x^{m+2} \boldsymbol{\Omega}^2. \end{aligned}$$

By Besicovich's covering theorem, we choose N_B families of disjoint balls $\overline{B}_{r_x}(x)$ whose union covers $\{\gamma\, c\, E \leqslant d \leqslant 1\} \cap B_s$ and, since as already noticed $r_x \leqslant 1/\sqrt[m]{c}$ for every x, we conclude:

$$\int_{\{\gamma\, c\, E \leqslant d \leqslant 1\}\cap B_s} d \leqslant N_B\, 2^{-N+1} \int_{\{\frac{c\,E}{\gamma} \leqslant d \leqslant 1\}\cap B_{s+2/\sqrt[m]{c}}} d + C\, c^{-\frac{2}{m}}\, \boldsymbol{\Omega}^2,$$

which, for the above defined ρ, implies (4.61).

Step 2. We iterate (4.61) in order to conclude (4.60). Denote by L the largest integer smaller than $2^{-1}\log_\gamma\left(E^{-1}-1\right)$, $s_L = 2$ and recursively $s_k = s_{k+1} + 2\gamma^{-\frac{2k}{m}}$ for $k \in \{L, L-1, \dots, 1\}$. Notice that, since $\gamma \geqslant 2^m$, $s_k < 4$ for every k. Thus, we can apply (4.61) with $c = \gamma^{2k}$, $s = s_k$ and $\bar{s} = s_{k-1}$ to conclude

$$\int_{\{\gamma^{2k+1}E \leqslant d \leqslant 1\}\cap B_{s_k}} d \leqslant \gamma^{-\rho} \int_{\{\gamma^{2k-1}E \leqslant d \leqslant 1\}\cap B_{s_{k-1}}} d + C\gamma^{-\frac{4k}{m}}\mathbf{\Omega}^2 \quad \forall k \in \{1, \dots, L\}.$$

In particular, iterating this estimate we get

$$\int_{\{\gamma^{2k+1}E \leqslant d \leqslant 1\}\cap B_2} d \leqslant \gamma^{-k\rho} \int_{\{\gamma E \leqslant d \leqslant 1\}\cap B_{s_0}} d + C\,\mathbf{\Omega}^2 \sum_{l=0}^{k-1} \gamma^{-\left(\frac{4(k-l)}{m}+l\rho\right)}. \tag{4.65}$$

Set $A_0 = \{d < \gamma E\}$, $A_k = \{\gamma^{2k-1}E \leqslant d < \gamma^{2k+1}E\}$ for $k = 1, \dots, L$, and $A_{L+1} = \{\gamma^{2L+1}E \leqslant d \leqslant 1\}$. Since $\cup A_k = \{d \leqslant 1\}$, for $p_2 < 1 + \frac{\rho}{2} \leqslant 1 + \frac{1}{2}$, we conclude:

$$\begin{aligned}
\int_{B_2\cap\{d\leqslant 1\}} d^{p_2} &= \sum_{k=0}^{L+1}\int_{A_k\cap B_2} d^{p_2} \leqslant \sum_k \gamma^{(2k+1)(p_2-1)}E^{p_2-1}\int_{A_k\cap B_2} d \\
&\overset{(4.65)}{\leqslant} C\sum_k \gamma^{k(2(p_2-1)-\rho)}E^{p_2} + C\sum_k\sum_{l=0}^{k-1}\gamma^{k(2(p_2-1)-\frac{4}{m})+l(\frac{4}{m}-\rho)}E^{p_2-1}\,\mathbf{\Omega}^2 \\
&\leqslant CE^{p_2} + C\sum_k \gamma^{k(2(p_2-1)-\rho)}\,\mathbf{\Omega}^2.
\end{aligned}$$
□

4.4 STRONG EXCESS ESTIMATE AND CONCLUSION OF THE PROOF

4.4.1 *Almgrem's strong excess estimate.*

Thanks to the higher integrability of Theorem 4.10, we can control the excess where $\mathbf{d} \leqslant 1$. To control it outside this region, we will need the following estimate.

Theorem 4.11 (Almgren's strong excess estimate). *There are constants $\varepsilon_{21}, \gamma_2, C_{27} > 0$ (depending on n, Q) with the following property. Assume T satisfies Assumption 1 in $\mathbf{C}_4$ and is $\mathbf{\Omega}$ almost minimizing. If $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{21}$, then*

$$\mathbf{e}_T(A) \leqslant C_{27}\left(E^{\gamma_2} + |A|^{\gamma_3}\right)\left(E + \mathbf{\Omega}^2\right) \quad \textit{for every Borel } A \subset B_1. \tag{4.66}$$

Proof. Since the proof of this result is rather involved we split it into two parts.

Regularization by convolution

In this first part we construct a competitor via convolution. To do that we will need the following Proposition, whose highly nontrivial proof can be found in [19].

Proposition 4.12 (Cf. [19, Proposition 6.2]). *For every $n, Q \in \mathbb{N} \setminus \{0\}$ there are geometric constants $\delta_0, C_{24} > 0$ with the following property. For every $\delta \in]0, \delta_0[$ there is $\boldsymbol{\rho}^\star_\delta : \mathbb{R}^{N(Q,n)} \to \mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ such that $|\boldsymbol{\rho}^\star_\delta(P) - P| \leqslant C_{24}\, \delta^{8^{-nQ}}$ for all $P \in \mathcal{Q}$ and, for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, it holds*

$$\int |D(\boldsymbol{\rho}^\star_\delta \circ u)|^2 \leqslant \left(1 + C_{24}\, \delta^{8^{-nQ-1}}\right) \int_{\{\mathrm{dist}(u,\mathcal{Q}) \leqslant \delta^{nQ+1}\}} |Du|^2 + C_{24} \int_{\{\mathrm{dist}(u,\mathcal{Q}) > \delta^{nQ+1}\}} |Du|^2 \,. \tag{4.67}$$

The precise claim about the smoothed competitor is contained in the following proposition.

Proposition 4.13. *Let $\beta_1 \in \left(0, \frac{1}{2m}\right)$ and T be an $\boldsymbol{\Omega}$-almost minimizing current satisfying Assumption 1 in $\mathbf{C}_4$. Let f be its E^{β_1}-Lipschitz approximation. Then, there exist constants $\gamma_3, C_{28} > 0$ and a subset of radii $B \subset [1,2]$ with $|B| > 1/2$ with the following properties. For every $\sigma \in B$, there exists a Q-valued function $g \in \mathrm{Lip}(B_\sigma, \mathcal{A}_Q)$ such that*

$$g|_{\partial B_\sigma} = f|_{\partial B_\sigma}, \quad \mathrm{Lip}(g) \leqslant C_{28}\, E^{\beta_1}$$

and

$$\int_{B_\sigma} |Dg|^2 \leqslant \int_{B_\sigma \cap K} |Df|^2 + C_{28}\, E^{1+\gamma_3}. \tag{4.68}$$

Proof. Since $|Df|^2 \leqslant C\, d_T \leqslant C E^{2\,\beta_1} \leqslant 1$ on K, by Theorem 4.10 there exists $q_2 = 2\,p_2 > 2$ such that

$$\||Df|\|^2_{L^{q_2}(K \cap B_2)} \leqslant C\, E^{1 - \frac{1}{p_1}} (E + \boldsymbol{\Omega}^2)^{\frac{1}{p_1}} \leqslant C(E + \boldsymbol{\Omega}^2)\,. \tag{4.69}$$

Given two (vector-valued) functions h_1 and h_2 and two radii $0 < s < r$, we denote by $\mathrm{lin}(h_1, h_2)$ the linear interpolation in $B_r \setminus \bar{B}_s$ between $h_1|_{\partial B_r}$ and $h_2|_{\partial B_s}$. More precisely, if $(\theta, t) \in \mathbb{S}^{m-1} \times [0, \infty)$ are spherical coordinates, then

$$\mathrm{lin}(h_1, h_2)(\theta, t) = \frac{r-t}{r-s}\, h_2(\theta, s) + \frac{t-s}{r-s}\, h_1(\theta, r)\,.$$

Next, let $\delta > 0$ and $\varepsilon > 0$ be two parameters and let $1 < r_1 < r_2 < r_3 < 2$ be three radii, all to be chosen later. To keep the notation simple, we will write $\boldsymbol{\rho}^\star$ in place of $\boldsymbol{\rho}^\star_\delta$. Let $\varphi \in C^\infty_c(B_1)$ be a standard (nonnegative!) mollifier. We set $f' := \boldsymbol{\xi} \circ f$. Recall the map $\boldsymbol{\rho}$ of Lemma 3.8 and define:

$$g' := \begin{cases} \sqrt{E}\, \boldsymbol{\rho} \circ \mathrm{lin}\left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right)\right) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \sqrt{E}\, \boldsymbol{\rho} \circ \mathrm{lin}\left(\boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right), \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right)\right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \sqrt{E}\, \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\right) & \text{in } B_{r_1}. \end{cases} \tag{4.70}$$

Finally set $g := \boldsymbol{\xi}^{-1} \circ g'$. We claim that, for $\sigma := r_3$ in a suitable set $B \subset [1,2]$ with $|B| > 1/2$, we can choose $r_2 = r_3 - s$ and $r_1 = r_2 - s$ so that g satisfies the conclusion of the proposition.

Some computations will be simplified taking into account that our choice of the parameter will imply the following inequalities:

$$\delta^{2\cdot 8^{-nQ}} \leqslant s\,, \quad \varepsilon \leqslant s \quad \text{and} \quad E^{1-2\beta_1} \leqslant \varepsilon^2\,. \tag{4.71}$$

We start noticing that clearly $g|_{\partial B_{r_3}} = f|_{\partial B_{r_3}}$. Moreover we have $\mathrm{Lip}(g) \leqslant CE^{\beta_1}$, indeed

$$\begin{cases} \mathrm{Lip}(g) \leqslant C\,\mathrm{Lip}(f' * \varphi_\varepsilon) \leqslant C\,\mathrm{Lip}(f) \leqslant C\,E^{\beta_1} & \text{in } B_{r_1}, \\ \mathrm{Lip}(g) \leqslant C\,\mathrm{Lip}(f') + C\,\frac{\|f' - f' * \varphi_\varepsilon\|_{L^\infty}}{s} \leqslant C(1 + \frac{\varepsilon}{s})\,\mathrm{Lip}(f') \leqslant C\,E^{\beta_1} & \text{in } B_{r_2} \setminus B_{r_1}, \\ \mathrm{Lip}(g) \leqslant C\,\mathrm{Lip}(f') + C\,E^{1/2}\,\frac{\delta^{8^{-nQ}}}{s} \leqslant C\,E^{\beta_1} + C\,E^{1/2} \leqslant C\,E^{\beta_1} & \text{in } B_{r_3} \setminus B_{r_2}\,. \end{cases}$$

In the second inequality of the last line we have used that, since $\mathcal{Q}$ is a cone, $E^{-\frac{1}{2}} f'(x) \in \mathcal{Q}$ for every x: therefore $|\boldsymbol{\rho}^\star(f'/E^{\frac{1}{2}}) - f'/E^{\frac{1}{2}}| \leqslant C\delta^{8^{-\bar{n}Q}}$. We pass now to estimate the Dirichlet energy of g.

Step 1. Energy in $B_{r_3} \setminus B_{r_2}$. By Proposition 4.12, $|\boldsymbol{\rho}^\star(P) - P| \leqslant C_{24}\,\delta^{8^{-\bar{n}Q}}$ for all $P \in \mathcal{Q}$. Thus, elementary estimates on the linear interpolation give

$$\begin{aligned} \int_{B_{r_3}\setminus B_{r_2}} |Dg|^2 &\leqslant \frac{C\,E}{(r_3 - r_2)^2} \int_{B_{r_3}\setminus B_{r_2}} \left|\frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^\star\left(\frac{f'}{\sqrt{E}}\right)\right|^2 + C \int_{B_{r_3}\setminus B_{r_2}} |Df'|^2 \\ &\quad + C \int_{B_{r_3}\setminus B_{r_2}} |D(\boldsymbol{\rho}^\star \circ f')|^2 \leqslant C \int_{B_{r_3}\setminus B_{r_2}} |Df|^2 + C\,E\,s^{-1}\,\delta^{2\cdot 8^{-\bar{n}Q}}\,. \end{aligned} \tag{4.72}$$

Step 2. Energy in $B_{r_2} \setminus B_{r_1}$. Here, using the same interpolation inequality and a standard estimate on convolutions of $W^{1,2}$ functions, we get

$$\begin{aligned} \int_{B_{r_2}\setminus B_{r_1}} |Dg|^2 &\leqslant C \int_{B_{r_2}\setminus B_{r_1}} |Df|^2 + \frac{C}{(r_2 - r_1)^2} \int_{B_{r_2}\setminus B_{r_1}} |f' - \varphi_\varepsilon * f'|^2 \\ \leqslant& C \int_{B_{r_2}\setminus B_{r_1}} |Df|^2 + C\,\varepsilon^2 s^{-2} \int_{B_3} |Df'|^2 = C \int_{B_{r_2}\setminus B_{r_1}} |Df|^2 + C\,\varepsilon^2\,E\,s^{-2}\,. \end{aligned} \tag{4.73}$$

Step 3. Energy in B_{r_1}. Define $Z := \left\{\mathrm{dist}\left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon, \mathcal{Q}\right) > \delta^{nQ+1}\right\}$ and use (4.67) to get

$$\int_{B_{r_1}} |Dg|^2 \leqslant \left(1 + C\,\delta^{8^{-\bar{n}Q-1}}\right) \int_{B_{r_1}\setminus Z} \left|D\left(f' * \varphi_\varepsilon\right)\right|^2 + C \int_Z \left|D\left(f' * \varphi_\varepsilon\right)\right|^2 =: I_1 + I_2. \tag{4.74}$$

We consider I_1 and I_2 separately. For I_1 we first observe the elementary inequality

$$\begin{aligned} \|D(f' * \varphi_\varepsilon)\|_{L^2}^2 \leqslant& \||Df'| * \varphi_\varepsilon\|_{L^2}^2 \leqslant \|(|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\|_{L^2}^2 + \|(|Df'|\,\mathbf{1}_{K^c}) * \varphi_\varepsilon\|_{L^2}^2 \\ &+ 2\|(|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\|_{L^2} \|(|Df'|\,\mathbf{1}_{K^c}) * \varphi_\varepsilon\|_{L^2}\,, \end{aligned} \tag{4.75}$$

where K^c is the complement of K in B_3. Recalling $r_1 + \varepsilon \leqslant r_1 + s \leqslant r_2$ we estimate the first summand in (4.75) as follows:

$$\|(|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\|_{L^2(B_{r_1})}^2 \leqslant \int_{B_{r_1+\varepsilon}} \left(|Df'|\,\mathbf{1}_K\right)^2 \leqslant \int_{B_{r_3}\cap K} |Df|^2\,. \tag{4.76}$$

To treat the other terms recall that $\mathrm{Lip}(f) \leqslant C\, E^{\beta_1}$ and $|K^c| \leqslant C\, E^{1-2\beta_1}$:

$$\|(|Df'|\mathbf{1}_{K^c}) * \varphi_\varepsilon\|^2_{L^2(B_{r_1})} \leqslant CE^{2\beta_1}\|\mathbf{1}_{K^c} * \varphi_\varepsilon\|^2_{L^2} \leqslant CE^{2\beta_1}\,\|\mathbf{1}_{K^c}\|^2_{L^1}\,\|\varphi_\varepsilon\|^2_{L^2} \leqslant \frac{CE^{2-2\beta_1}}{\varepsilon^2}. \tag{4.77}$$

Putting (4.76) and (4.77) in (4.75) and recalling $E^{1-2\beta_1} \geqslant \varepsilon^m$ and $\int |Df'|^2 \leqslant CE$, we get

$$I_1 \leqslant \int_{B_{r_2}\cap K} |Df|^2 + C\,\delta^{8^{-nQ-1}}\, E + C\,\varepsilon^s\, E^{s\frac{3}{2}-\beta_1}\,. \tag{4.78}$$

For what concerns I_2, first we argue as for I_1, splitting in K and K^c, to deduce that

$$I_2 \leqslant C\int_Z \left((|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\right)^2 + C\,\varepsilon^{-\frac{m}{2}}E^{\frac{3}{2}-\beta_1}. \tag{4.79}$$

Then, regarding the first summand in (4.79), we note that

$$|Z|\,\delta^{2nQ+2} \leqslant \int_{B_{r_1}} \left|\tfrac{f'}{\sqrt{E}} * \varphi_\varepsilon - \tfrac{f'}{\sqrt{E}}\right|^2 \leqslant C\,\varepsilon^2. \tag{4.80}$$

Recalling that $q_2 = 2p_2 > 2$, we use (4.69) to obtain

$$\begin{aligned}\int_Z \left((|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\right)^2 &\leqslant |Z|^{\frac{p_1-1}{p_1}}\,\|(|Df'|\,\mathbf{1}_K) * \varphi_\varepsilon\|^2_{L^{q_2}} \leqslant C\,\left(\frac{\varepsilon}{\delta^{nQ+1}}\right)^{\frac{2(p_1-1)}{p_1}}\,\||Df'|\|^2_{L^{q_2}(K)}\\ &\leqslant C\,\left(\frac{\varepsilon}{\delta^{nQ+1}}\right)^{\frac{2(p_1-1)}{p_1}}\,(E+\mathbf{\Omega}^2)\,.\end{aligned} \tag{4.81}$$

Gathering all the estimates together, (4.74), (4.78), (4.79) and (4.81) give

$$\int_{B_{r_1}} |Dg|^2 \leqslant \int_{B_{r_2}\cap K} |Df|^2 + C\Big(E\delta^{8^{-nQ-1}} + \frac{E^{\frac{3}{2}-\beta_1}}{\varepsilon} + (E+\mathbf{\Omega}^2)\left(\frac{\varepsilon}{\delta^{nQ+1}}\right)^{\frac{2(p_1-1)}{p_1}}\Big). \tag{4.82}$$

Final estimate. Summing (4.72), (4.73) and (4.82) (and recalling $\varepsilon < s$), we conclude

$$\begin{aligned}\int_{B_{r_3}} |Dg|^2 \leqslant \int_{B_{r_1}\cap K} |Df|^2 &+ C\int_{B_{r_1+3s}\setminus B_{r_1}} |Df|^2\\ &+ C\,E\left(\frac{\varepsilon^2}{s^2} + \frac{\delta^{2\cdot 8^{-Q}}}{s} + \frac{E^{\frac{1}{2}-\beta_1}}{\varepsilon} + \left(1+\mathbf{\Omega}^2\,E^{-1}\right)\left(\frac{\varepsilon}{\delta^{nQ+1}}\right)^{\frac{2(p_1-2)}{p_1}}\right).\end{aligned}$$

We set $\varepsilon = E^a$, $\delta = E^b$ and $s = E^c$, where

$$a = \frac{1-2\,\beta_1}{4}, \quad b = \frac{1-2\,\beta_1}{8\,(n\,Q+1)} \quad \text{and} \quad c = \frac{1-2\,\beta_1}{8^{nQ}\,8\,(n\,Q+1)}.$$

This choice respects (4.71). Assume E is small enough so that $s \leqslant \frac{1}{8}$. Now, if $C > 0$ is a sufficiently large constant, there is a set $B' \subset [1, \frac{7}{8}]$ with $|B'| > 1/2$ such that,

$$\int_{B_{r_1+3s}\setminus B_{r_1}} |Df|^2 \leqslant C\,s\int_{B_2} |Df|^2 \leqslant C\,E^{1+c} \quad \text{for every } r_1 \in B'.$$

For $\sigma = r_3 \in B = s + B'$ we then conclude, for some $\gamma(\beta_1, n, N, Q) > 0$,

$$\int_{B_\sigma} |Dg|^2 \leqslant \int_{B_\sigma\cap K} |Df|^2 + CE^{1+\gamma}. \qquad \square$$

Proof of (4.66)

Using the isoperimetric inequality and a slicing argument, we find a radius $\sigma \in (1,2)$ for which Proposition 4.13 applies and such that there is $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ with $\partial P = \partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s)$ and $\mathbf{M}(P) \leqslant CE^{1+\gamma}$. We can therefore apply both Lemma 4.6 to conclude that

$$\|T\|(\mathbf{C}_\sigma) \leqslant \|\mathbf{G}_g\|(\mathbf{C}_\sigma) + C\boldsymbol{\Omega} \int_{B_\sigma} \mathbf{G}(g,f) + CE^{1+\gamma}\,. \tag{4.83}$$

In order to estimate $\int_{B_\sigma} \mathbf{G}(g,f)$, we recall how g is constructed, and in particular, using the notation of the previous section

$$\begin{aligned}\int_{B_\sigma} \mathcal{G}(f,g) \leqslant{}& \underbrace{C\int_{B_\sigma \setminus B_{\sigma-s}} \Big| f' - \sqrt{E}\boldsymbol{\rho} \circ \operatorname{lin}\Big(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big)\Big)\Big|}_{I_1} + \\ &+\underbrace{C\int_{B_{\sigma-s} \setminus B_{\sigma-2s}} \Big| f' - \sqrt{E}\boldsymbol{\rho} \circ \operatorname{lin}\Big(\boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big), \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\big)\Big)\Big|}_{I_2} + \\ &+\underbrace{C\int_{B_{s-2\eta}} \Big| f' - \sqrt{E}\boldsymbol{\rho}^\star\Big(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\Big)\Big|}_{I_3}\,.\end{aligned}$$

We will estimate I_1, I_2, I_3 separately. Recall that $\boldsymbol{\rho} \circ f' = f'$, $\boldsymbol{\rho}$ is Lipschitz and moreover $\lambda\boldsymbol{\rho}(P) = \boldsymbol{\rho}(\lambda P)$, for every $\lambda > 0, P \in \mathcal{Q}$, since $\mathcal{Q}$ is a cone.

$$\begin{aligned} I_1 &\leqslant C\int_{\sigma-s}^{\sigma}\int_{\partial B_t} \sqrt{E}\Big|\frac{f'}{\sqrt{E}} - \frac{t+s-\sigma}{s}\frac{f'}{\sqrt{E}} - \frac{\sigma-t}{s}\boldsymbol{\rho}^\star\Big(\frac{f'}{\sqrt{E}}\Big)\Big|\,dt \\ &= C\sqrt{E}\int_{\sigma-s}^{\sigma}\frac{\sigma-t}{s}\int_{\partial B_t}\Big|\frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big)\Big|\,dt \leqslant C\sqrt{E}\delta^{8-nQ}\,|B_\sigma \setminus B_{\sigma-s}| \leqslant CE^{\frac{1}{2}+c}\end{aligned}$$

where we used $|B_\sigma \setminus B_{\sigma-s}| \leqslant Cs \leqslant CE^c$. We next bound I_2.

$$\begin{aligned} I_2 &\leqslant C\sqrt{E}\int_{\sigma-2s}^{\sigma-s}\int_{\partial B_t}\Big|\frac{f'}{\sqrt{E}} - \frac{t+2s-\sigma}{s}\boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big) - \frac{\sigma-s-t}{s}\boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\big)\Big| \\ &\leqslant C\sqrt{E}\int_{\sigma-2s}^{\sigma-s}\int_{\partial B_t}\Big(\Big|\frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big)\Big| + \frac{\sigma-s-t}{s}\Big|\boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big) - \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\big)\Big|\Big)\,dt \\ &\leqslant CE^{\frac{1}{2}+c} + C\int_{B_{\sigma-s}\setminus B_{\sigma-2s}}\big|f' - f' * \varphi_\varepsilon\big|\end{aligned}$$

where we have used the fact that $\boldsymbol{\rho}^\star$ is Lipschitz. The estimate for I_3 is then

$$\begin{aligned} I_3 &\leqslant C\sqrt{E}\int_{B_{\sigma-2s}}\Big(\Big|\frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big)\Big| + \Big|\boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}}\big) - \boldsymbol{\rho}^\star\big(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon\big)\Big|\Big) \\ &\leqslant CE^{\frac{1}{2}+c} + C\int_{B_{\sigma-2s}} |f' - f' * \varphi_\varepsilon|\,.\end{aligned}$$

We therefore achieve the estimate

$$I_2 + I_3 \leqslant CE^{\frac{1}{2}+c} + \int_{B_{\sigma-s}} |f' - f' * \phi_\varepsilon|$$

and to conclude, we compute

$$\begin{aligned}
&\int_{B_{\sigma-s}} \left|f' - f' * \varphi_\varepsilon\right| \leqslant \int_{B_{\sigma-s}} \int_{B_\varepsilon} \varphi_\varepsilon(x) |f'(y-x) - f'(y)|\, dy\, dx \\
&\leqslant \int_{B_{\sigma-s}} \int_{B_\varepsilon} \int_0^1 \varphi_\varepsilon(x) |Df'(y-tx)\cdot x|\, dt\, dy\, dx \\
&\leqslant \int_0^1 \int_{B_\varepsilon} \varphi_\varepsilon(x) \varepsilon \int_{B_{\sigma-s}} |Df(y-tx)|\, dy\, dx\, dt \leqslant \varepsilon \, \|Df\|_{L^1(B_\sigma)} \leqslant CE^{\frac{1}{2}+a},
\end{aligned}$$

(where we have used the fact that $\varepsilon \leqslant s$). Putting everything together we conlude that

$$\mathbf{M}(S) \leqslant CE^{\frac{1}{2}+\gamma}$$

for a suitable $\gamma > 0$. Then, from (4.83), the Taylor expansion for $\mathbf{M}(\mathbf{G}_g)$ and Proposition 4.13 we achieve

$$\|T\|(\mathbf{C}_\sigma) \;\leqslant\; Q\,|B_\sigma| + \int_{B_\sigma \cap K} \frac{|Df|^2}{2} + CE^\gamma (E + \boldsymbol{\Omega}^2)\,. \tag{4.84}$$

On the other hand, by the Taylor's expansion in Corollary 3.49,

$$\begin{aligned}
\|T\|(\mathbf{C}_s) &= \|T\|((B_s \setminus K) \times \mathbb{R}^n) + \|\mathbf{G}_f\|((B_s \cap K) \times \mathbb{R}^n) \\
&\geqslant \|T\|((B_s \setminus K) \times \mathbb{R}^n) + Q\,|K \cap B_s| + \int_{K \cap B_s} \frac{|Df|^2}{2} - C\,E^{1+\gamma}.
\end{aligned} \tag{4.85}$$

Hence, from (4.84) and (4.85), we get $\boldsymbol{e}_T(B_s \setminus K) \leqslant C\,E^\gamma\,(E + \boldsymbol{\Omega}^2)$.

This is enough to conclude the proof. Indeed, let $A \subset B_1$ be a Borel set. Using the higher integrability of $|Df|$ in K (and therefore possibly selecting a smaller $\gamma > 0$) we get

$$\begin{aligned}
\boldsymbol{e}_T(A) &\leqslant \boldsymbol{e}_T(A \cap K) + \boldsymbol{e}_T(A \setminus K) \leqslant \int_{A \cap K} \frac{|Df|^2}{2} + C\,E^{1+\gamma} + C\,E^\gamma\,(E + \boldsymbol{\Omega}^2) \\
&\leqslant C\,|A \cap K|^{\frac{p_1 - 1}{p_1}} \left(\int_{A \cap K} |Df|^{q_2}\right)^{\frac{2}{q_2}} + C\,E^{1+\gamma} + C\,E^\gamma\,(E + \boldsymbol{\Omega}^2) \\
&\leqslant C\,|A|^{\frac{p_1 - 1}{p_1}} \left(E + \boldsymbol{\Omega}^2\right) + C\,E^\gamma\,(E + \boldsymbol{\Omega}^2) + C\,E^{1+\gamma}.
\end{aligned}$$

□

4.4.2 *Proof of Proposition 4.2*

As usual we assume, w.l.o.g., $r = 1$ and $x = 0$. Choose $\beta_2 < \min\{\frac{1}{2m}, \frac{\gamma_3}{2(1+\gamma_3)}\}$, where γ_3 is the constant in Theorem 4.11. Let f be the E^{β_2}-Lipschitz approximation of T. Clearly (4.3) follows directly from Proposition 4.4 if $\gamma_1 < \beta_2$. Set next $A := \left\{\mathbf{m}\boldsymbol{e}_T > 2^{-m}E^{2\,\beta_2}\right\} \cap B_{\frac{9}{8}}$. By Proposition 4.4, $|A| \leqslant C E^{1-2\beta_2}$. Apply estimate (4.66) to A to conclude:

$$|B_1 \setminus K| \leqslant C\, E^{-2\,\beta_2}\, \boldsymbol{e}_T\,(A) \leqslant C\, E^{\gamma_3 - 2\beta_2(1+\gamma_3)}(E + \boldsymbol{\Omega}^2).$$

By our choice of γ_3 and β_2, this gives (4.4) for some positive β_0. Finally, set $S = \mathbf{G}_f$. Recalling the strong Almgren's estimate (4.66) and the Taylor expansion in Corollary 3.49, we conclude:

$$\begin{aligned}&\left|\|T\|(\mathbf{C}_1) - Q\,\omega_m - \int_{B_1} \frac{|Df|^2}{2}\right| \leqslant \boldsymbol{e}_T(B_1 \setminus K) + \boldsymbol{e}_S(B_1 \setminus K) + \left|\boldsymbol{e}_S(B_1) - \int_{B_1} \frac{|Df|^2}{2}\right| \\ &\leqslant C\, E^{\gamma_3}(E + \boldsymbol{\Omega}^2) + C\,|B_1 \setminus K| + C\,\mathrm{Lip}(f)^2 \int_{B_1} |Df|^2 \leqslant C\, E^{\gamma_1}(E + \boldsymbol{\Omega}^2).\end{aligned}$$

The L^∞ bound follows from Proposition 4.4.

Part III

STEP 2: TANGENT CONES

5 UNIQUENESS OF TANGENT CONES FOR 2-DIMENSIONAL ALMOST MINIMIZING CURRENTS

In this chapter we consider 2-dimensional integer rectifiable currents T in the euclidean space $\mathbb{R}^{n+2}$ which are almost (area) minimizing, in the following sense.

Definition 5.1. An m-dimensional integer rectifiable current T in $\mathbb{R}^{m+n}$ is *almost (area) minimizing* if for every $x \notin \mathrm{spt}(\partial T)$ there are constants $C_0, r_0, \alpha_0 > 0$ such that

$$\|T\|(\mathbf{B}_r(x)) \leqslant \|T + \partial S\|(\mathbf{B}_r(x)) + C_0\, r^{m+\alpha_0} \tag{5.1}$$

for all $0 < r < r_0$ and for all integral $(m+1)$-dimensional currents S supported in $\mathbf{B}_r(x)$.

Our aim is to extend Brian White's classical result (cf. [66]) on the uniqueness of tangent cones for area minimizing 2-dimensional currents to almost minimizers,.

To state the main theorem we recall the definition of the current $T_{x,r} := (\iota_{x,r})_\sharp T$, where the map $\iota_{x,r}$ is given by $\mathbb{R}^{m+n} \ni y \mapsto \frac{y-x}{r} \in \mathbb{R}^{m+n}$. Recall that an area minimizing cone S is an integral area minimizing current such that $(\iota_{0,r})_\sharp S = S$ for every $r > 0$ (cf. [54, Theorem 19.3]). Furthermore, for any given $R \in \mathbf{I}_m(\mathbb{R}^{m+n})$ we define $\mathcal{F}(R) := \inf\{\mathbf{M}(Z) + \mathbf{M}(W) : Z \in \mathbf{I}_m, W \in \mathbf{I}_{m+1}, Z + \partial W = R\}$.

Theorem 5.2 (Uniqueness of tangent cones for almost minimizers). *Let $T \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an almost minimizer. Then there is a $\gamma_0 > 0$, J 2-dim. distinct planes π_i, each pair of which intersect only at 0, and J integers n_i such that, if we set $S := \sum_i n_i \llbracket \pi_i \rrbracket$, then*

$$\mathcal{F}\big((T_{x,r} - S)\llcorner \mathbf{B}_1\big) \leqslant C_{11}\, r^{\gamma_0}, \tag{5.2}$$

$$\mathrm{dist}\big(\mathrm{spt}(T\llcorner \mathbf{B}_r(x)), \mathrm{spt}(S)\big) \leqslant C_{11}\, r^{1+\gamma_0}. \tag{5.3}$$

Moreover, there are $\bar r > 0$ and $J \geqslant 1$ currents $T^j \in \mathbf{I}_2(\mathbf{B}_{\bar r}(x))$ such that

(i) $\partial T^j \llcorner \mathbf{B}_{\bar r}(x) = 0$ *and each* T^j *is an almost minimizer;*

(ii) $T\llcorner \mathbf{B}_{\bar r}(x) = \sum_j T^j$ *and* $\mathrm{spt}(T_j) \cap \mathrm{spt}(T_i) = \{x\}$ *for every* $i \neq j$*;*

(iii) $n_j \llbracket \pi_j \rrbracket$ *is the unique tangent cone to each* T^j *at* x.

From the latter theorem, Proposition 2.3 and Proposition 2.5 we easily deduce Theorem 2.9

The rest of the chapter is dedicated to the proof of Theorem 5.2. This will be achieved in three sections organized as follows. In the first section we recall an important property of 2-dimensional area minimizing cones due to White and give a simplified proof of it. The second section contains a generalization of White's epiperimetric inequality to the case of almost minimizers and an almost monotonicity formula for almost minimizers. Finally, in the third section we give the proof of Theorem 5.2.

5.1 WHITE'S EPIPERIMETRIC INEQUALITY (WEI)

As already mentioned, the key ingredient in the proof of Theorem 5.2 is a suitable generalization of White's epiperimetric inequality [66]. We record the main ingredient of White's argument in the following lemma. Since however the paper [66] does not state this lemma explicitely, we provide a brief argument, referring to propositions and lemmas which are instead explicitely stated in [66] (the only difference is in a technical point, namely the estimate (5.4), for which we point out a shorter argument).

Lemma 5.3. *Let $S \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an area minimizing cone. There exists a constant $\varepsilon_{31} > 0$ with the following property. If $R := \partial(S \llcorner \mathbf{B}_1)$ and $Z \in \mathbf{I}_1(\partial \mathbf{B}_1)$ is a cycle with*

(i) $\mathcal{F}(Z - R) < \varepsilon_{31}$,

(ii) $\mathbf{M}(Z) - \mathbf{M}(R) < \varepsilon_{31}$,

(iii) $\operatorname{dist}\big(\operatorname{spt}(Z), \operatorname{spt}(R)\big) < \varepsilon_{31}$,

then there exists $H \in \mathbf{I}_2(\mathbf{B}_1)$ such that $\partial H = Z$ and

$$\|H\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) \leqslant (1 - \varepsilon_{31})\big[\|0 \times\!\!\!\!\times Z\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\big].$$

Proof. We start by recalling a well known result about the decomposition of 1-dimensional integral cycles.

Lemma 5.4 ([35, Lemma 2.1])**.** *Let R be an integral 1-cycle. Then, there is a decomposition into simple closed curves R_i such that $\mathbf{M}(R) + \sum_i \mathbf{M}(R_i)$, $\operatorname{spt}(R_i) \subset \operatorname{spt}(R)$. The sum is either finite or convergent.*

A consequence of this lemma, the fact that an area minimazing two dimensional cone which has a simple closed curve as a boundary must be a disk, and easy comparison arguments, we conclude

Lemma 5.5 (Characterization of 2-dimensional area minimizng cones (cf. [35]))**.** *Any 2-dimensional area-minimizing cone S is the sum of (integer multiples of) finitely many oriented planes, each pair of which intersects only at the origin.*

Therefore the support of the cycle $R := \partial(S \llcorner \mathbf{B}_1)$ of the statement of Lemma 5.3 consists of a finite number (say N) of disjoint equatorial circles of $\partial \mathbf{B}_1$. By condition (iii), we can thus assume that Z splits into N cycles, each close (in the sense of (i), (ii) and (iii)) to an integer multiple of an equatorial circle of ∂B_1. Thus, without loss of generality, from now on we assume that S is given by $Q\,[\![\pi_0]\!]$, where π_0 is the (oriented) plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^{n+2}$ and Q is a positive integer. Correspondingly, we can assume that

(iv) $R = Q\,[\![\gamma_0]\!]$ for some integer $Q > 0$, where γ_0 is the oriented equatorial circle $\pi_0 \cap \partial \mathbf{B}_1$.

STEP 1. REDUCTION TO A LIPSCHITZ WINDING CURVE. We next recall the notation $B_r(x, \pi)$ for the 2-dimensional disk $x + \mathbf{B}_r(0) \cap \pi$ and $\mathbf{C}_r(x, \pi)$ for the cylinder $B_r(x, \pi) + \pi^\perp$, omitting x when it is the origin and π when it is the plane π_0. Given any 1-dimensional cycle W we consider the infinite 2-dimensional cone T with vertex 0 and spherical cross section

W, namely $\lim_{R\to\infty}(\iota_{0,R})_\sharp(0 ⨳ W)$ and denote it by $(0 ⨳ W)_\infty$. The *cylindrical excess* of any infinite 2-dimensional cone T in $\mathbf{C}_1(\tau)$ is then given by

$$\mathbf{E}(T,\tau) := \frac{1}{2}\int_{\mathbf{C}_1(\tau)} |\vec{T}(x) - \tau|^2 \, d\|T\|(x)$$

whereas the cylindrical excess of Z is denoted by

$$\mathbf{E}(Z) := \min_\tau \mathbf{E}((0 ⨳ Z)_\infty, \tau)\,.$$

It is simple to see that under the assumptions (i), (ii) and (iii), any minimum plane τ for $(0 ⨳ Z)_\infty$ in the expression above must be close to π_0.

Let now $\mathbf{P}$ be the orthogonal projection onto $\partial\mathbf{B}_1$ (which obviously is defined in $\mathbb{R}^{n+2}\setminus\{0\}$). For each π, such projection is invertible when we restrict its domain of definition to $\partial\mathbf{C}_1(\pi)$ and its target to $\partial\mathbf{B}_1\setminus\pi^\perp$. We then let $\mathbf{P}_\pi^{-1}$ be its inverse. Note also that, under the assumptions (i), (ii) and (iii), when τ is close enough to π_0, $\mathrm{spt}(Z)\subset\mathbf{B}_1\setminus\tau^\perp$. Therefore, for any such τ we have

$$(0 ⨳ Z)_\infty \llcorner \mathbf{C}_1(\tau) = 0 ⨳ (\mathbf{P}_\tau^{-1})_\sharp Z\,.$$

In particular such identity is valid for the π which minimizes $\mathbf{E}((0 ⨳ Z)_\infty,\tau)$.

If Z is as in the statement of the lemma, by Lemma 5.4, Z can be written as the sum of (at most countably many) 1-dimensional cycles Z_i, where each Z_i is a simple closed Lipschitz curve and $\sum \mathbf{M}(Z_i) = \mathbf{M}(Z)$. Observe also that, if ε_{31} is sufficiently small, then $(\mathbf{p}_{\pi_0})_\sharp(\mathbf{P}_{\pi_0}^{-1})_\sharp Z_i$ (where $\mathbf{p}_{\pi_0}$ is the orthogonal projection onto π_0) equals $k_i\,[\![\gamma_0]\!]$ for some nonnegative integer k_i. We thus have $\sum k_i = Q$ and it follows by standard arguments that each Z_i fulfills the assumptions (i), (ii) and (iii) of the Lemma with k_i in place of Q and with $\varepsilon_{32} > 0$ in place of ε_{31}, where the constant $\varepsilon_{32}\downarrow 0$ as $\varepsilon_{31}\downarrow 0$. Thus, it suffices to prove the main estimate for each Z_i and sum it over i. Observe next that assumption (ii) in the Lemma excludes the possibility that $k_i < 0$ for some i. Moreover, the case $k_i = 0$ corresponds to the trivial situation in which the minimizing cone S is 0. In this case $\mathbf{M}(Z_i) < \varepsilon_{31}$ and we can use the the isoperimetric inequality to find an H such that $\partial H = Z_i$ and

$$\|H\|(\mathbf{B}_1) \leqslant C(\mathbf{M}(Z))^2 \leqslant C\varepsilon_{31}\mathbf{M}(Z) \leqslant C\varepsilon_{31}\tfrac{1}{2}\|0 ⨳ Z\|(\mathbf{B}_1)\,.$$

It suffices therefore to consider the case $k_i > 0$.

Summarizing, in addition to (i), (ii), (iii) and (iv) we can also assume, w.l.o.g., the following:

(v) $Z = \eta_\sharp\,[\![[0,\mathbf{M}(Z)]]\!]$, where $\eta : [0,\mathbf{M}(Z)] \to \partial\mathbf{B}_1$ is Lipschitz and $\eta(0) = \eta(\mathbf{M}(Z))$;

(vi) If $\mathbf{E}((0 ⨳ Z)_\infty,\tau) = \mathbf{E}(Z)$, then $\mathbf{E}((0 ⨳ Z)_\infty,\tau) < \varepsilon_{33}$ and $(\mathbf{p}_\tau)_\sharp(\mathbf{P}_\tau^{-1})_\sharp Z = Q\,[\![\gamma_0]\!]$ (where $\varepsilon_{33}(\varepsilon, Q)\downarrow 0$ as $\varepsilon_{31}\downarrow 0$).

For any fixed $\delta > 0$, we can find a second curve $\zeta' : [0, 2Q\omega_2] \to \partial\mathbf{C}_1(\tau)$ with the following properties (recall that $2\omega_2$ is the length of the unit circle in $\mathbb{R}^2$):

(a1) $\zeta'(\vartheta) = (\cos\vartheta, \sin\vartheta, f'(\vartheta)) \in \tau\times\tau^\perp$ for some Lipschitz function $f' : [0, 2Q\omega_2] \to \tau^\perp$ with $f'(0) = f'(2Q\omega_2)$ and $\|f'\|_\infty + \mathrm{Lip}(f') \leqslant \delta$;

(a2) If we set $Z' = \zeta'_\sharp [\![[0, 2Q\omega_2]]\!]$, then $\mathbf{M}((\mathbf{P}_\tau)_\sharp Z - Z') \leqslant \mathbf{E}(Z)/C(\delta)$;

(a3) $\mathbf{E}((0 \times\!\!\!\!\times Z')_\infty, \tau) \leqslant \mathbf{E}((0 \times\!\!\!\!\times Z)_\infty, \tau) = \mathbf{E}(Z)$.

δ will be chosen (sufficiently small) later. Indeed assume this is not true, than we can find a sequence of Lipschitz curves $Z_i = \eta_\sharp [\![[0, \mathbf{M}(Z_i)]]\!]$ such that $(\mathbf{p}_\tau)_\sharp(\mathbf{P}_\tau^{-1})_\sharp Z = Q [\![\gamma_0]\!]$ and $\mathbf{E}(Z_i) \to 0$. Therefore we are in position to apply the following Proposition and get a contradiction for ε_{31} sufficiently small.

Proposition 5.6 ([66, Proposition 2.8]). *Given a sequence of Lipschitz curves $(Z_i)_i$ as above, there exist Lipschitz functions $f_i\colon [0, 2\omega_2 Q] \to \mathbb{R}^n$ such that $\|f_i\|_\infty + \mathrm{Lip}(f_i) \to 0$, and, if we set $Z'_i := (\cos, \sin, f_i)_\sharp [\![[0, 2Q\omega_2]]\!]$, then, $\mathbf{E}(Z'_i) \leqslant \mathbf{E}(Z_i)$ and $\mathbf{M}((\mathbf{P}_\tau)_\sharp Z_i - Z'_i) \leqslant \mathbf{E}(Z_i)/C(\delta)$.*

Since from (a2) we conclude easily

$$\mathbf{M}(((0 \times\!\!\!\!\times Z)_\infty - (0 \times\!\!\!\!\times Z')_\infty) \llcorner \mathbf{C}_2(\tau')) \leqslant \mathbf{E}(Z)/C(\delta)\,,$$

we also infer

$$\mathbf{M}(\partial((0 \times\!\!\!\!\times Z - 0 \times\!\!\!\!\times Z') \llcorner \mathbf{C}_{1/2}(\tau'))) \leqslant \mathbf{E}(Z)/C(\delta)\,.$$

After applying a rotation we can assume that $\tau' = \pi_0$. We thus achieve, in addition to (i)-(vi), the condition

(vii) $\mathbf{E}((0 \times\!\!\!\!\times Z')_\infty, \pi_0) = \mathbf{E}(Z')$ and $\mathbf{M}(\partial((0 \times\!\!\!\!\times Z - 0 \times\!\!\!\!\times Z') \llcorner \mathbf{C}_{1/2})) \leqslant \mathbf{E}(Z)/C(\delta)$.

Next, observe that if τ' minimizes $\mathbf{E}((0 \times\!\!\!\!\times Z')_\infty, \tau')$, then

$$|\tau' - \tau| \leqslant 2\mathbf{E}((0 \times\!\!\!\!\times Z')_\infty, \tau) \leqslant 2\mathbf{E}(Z) \leqslant 2\varepsilon_{31}$$

so that we can apply the reparametrization Lemma 3.17 and deduce easily that

(viii) the cycle $Z'' := \partial((0 \times\!\!\!\!\times Z') \llcorner \mathbf{C}_{1/2})$ is of the form $\zeta_\sharp [\![[0, 2Q\omega_2]]\!]$ for some $\zeta(\vartheta) = \frac{1}{2}(\cos\vartheta, \sin\vartheta, f(\vartheta))$, where $|f| + \mathrm{Lip}(f) \leqslant C\delta$ (C being a geometric constant);

(ix) $\mathbf{E}((0 \times\!\!\!\!\times Z'')_\infty, \pi_0) = \mathbf{E}(Z'') \leqslant \mathbf{E}(Z) < \bar{\varepsilon}_{31}$.

Step 2. Cylindrical epiperimetric inequality and conclusion. Consider the Fourier expansion of f as

$$f(\vartheta) = \alpha_0 + \sum_{k=0}^{\infty} \left(\alpha_k \cos\left(\tfrac{k}{Q}\vartheta\right) + \beta_k \sin\left(\tfrac{k}{Q}\vartheta\right)\right)$$

and let

$$P(f) := \alpha_Q \cos + \beta_Q \sin\,.$$

We first claim the existence of a constant K (depending only upon Q) such that, provided δ is smaller than some geometric constant, then

$$K\,\|(f - P(f))\|_{W^{1,2}} \geqslant \|f\|_{W^{1,2}}\,. \tag{5.4}$$

Indeed consider the 2-dimensional plane τ which contains the image of the map $\vartheta \mapsto (\cos\vartheta, \sin\vartheta, P(f)(\vartheta))$. It is then straightforward to check that

- $\mathbf{C}_1(\tau) \cap \operatorname{spt}((0 \times\!\!\!\!\times Z'')_\infty) \subset \mathbf{C}_2$
- If $x = r\zeta(\vartheta) \in \operatorname{spt}(Z'')$ and $r > 0$, then

$$|\vec{T}(x) - \pi_0| \geqslant \frac{1}{C}\left(|Df(\vartheta)| + |f(\vartheta)|\right) \tag{5.5}$$
$$|\vec{T}(x) - \tau| \leqslant C\left(|D(f - P(f))(\vartheta)| + |(f - P(f))(\vartheta)|\right), \tag{5.6}$$

where C is just a geometric constant.

Using that $\operatorname{Lip}(f) \leqslant \delta$, by the area formula we easily conclude that

$$\mathbf{E}((0 \times\!\!\!\!\times Z'')_\infty, \pi_0) \geqslant \frac{1}{C}\|f\|^2_{W^{1,2}} \tag{5.7}$$
$$\mathbf{E}((0 \times\!\!\!\!\times Z'')_\infty, \tau) \leqslant C\|f - P(f)\|^2_{W^{1,2}}. \tag{5.8}$$

Since C is a fixed geometric constant, (5.4) follows easily from

$$\mathbf{E}((0 \times\!\!\!\!\times Z''), \pi_0) = \mathbf{E}(Z'') \leqslant \mathbf{E}((0 \times\!\!\!\!\times Z'')_\infty, \tau).$$

Next, following [66, Proposition 2.4] we consider the map $g :]0, \frac{1}{2}] \times [0, 2Q\omega_2] \to \mathbb{R}^n$ given by

$$g(r, \vartheta) = \alpha_0 + \sum_{k=1}^{\infty} r^{\frac{k}{Q}}\left(\alpha_k \cos\left(\tfrac{k}{Q}\vartheta\right) + \beta_k \sin\left(\tfrac{k}{Q}\vartheta\right)\right)$$

and let $H' = g_\sharp\, [\![ ]0, \frac{1}{2}] \times [0, 2Q\omega_2]]\!]$. It is clear that

$$\partial H' = Z''.$$

Let us compute $\mathbf{M}(H')$. By the area formula we have

$$\begin{aligned}
\mathbf{M}(H') &= \frac{1}{4}\int_0^1 \int_0^{2\omega_2 Q} \sqrt{1 + |g_r|^2 + |r^{-1}g_\theta|^2 + |g_r \wedge r^{-1}g_\theta|^2}\, d\theta\, r\, dr \\
&\leqslant \frac{1}{4}\int_0^1 \int_0^{2\omega_2 Q} \sqrt{1 + |g_r|^2}\,\sqrt{1 + |r^{-1}g_\theta|^2}\, d\theta\, r\, dr \\
&\leqslant \frac{1}{4}\sup\left\{\int_0^1 \int_0^{2\omega_2 Q} (1 + |g_r|^2)\, d\theta\, r\, dr, \int_0^1 \int_0^{2\omega_2 Q} (1 + |r^{-1}g_\theta|^2)\, d\theta\, r\, dr\right\} \\
&= \frac{Q}{4}\omega_2 + \frac{Q}{8}\omega_2 \sum_{k=1}^{\infty} \frac{k}{Q}(|\alpha_k|^2 + |\beta_k|^2),
\end{aligned}$$

where in the last equality we have used

$$\|g_r\|^2_{L^2(0,2\omega_2 Q)} = \|r^{-1}g_\theta\|_{L^2(0,2\omega_2 Q)} = \omega_2\, Q \sum_{k=1}^{\infty}\left(\frac{k}{Q}\right)^2 r^{\frac{2k}{Q}-2}(|\alpha_k|^2 + |\beta_k|^2).$$

In particular, we have proved that

$$\mathbf{M}(H') - \frac{Q}{4}\omega_2 \leqslant \frac{Q}{8}\omega_2 \sum_{k=1}^{\infty} \frac{k}{Q}(|\alpha_k|^2 + |\beta_k|^2) \tag{5.9}$$

Next notice that, if $k \neq Q$, then there exists $\varepsilon_Q = \varepsilon(Q) > 0$ such that $\frac{k}{Q} \leqslant (1-\varepsilon_Q)\left(1+\left(\frac{k}{Q}\right)^2\right)$. In particular, combining this with (5.9), we deduce

$$\mathbf{M}(H') - \frac{Q}{4}\omega_2 \leqslant \frac{Q}{8}\omega_2 \left(\|P(f)\|^2_{W^{1,2}} + (1-\varepsilon_Q)\|f-P(f)\|^2_{W_{1,2}}\right) . \tag{5.10}$$

On the other hand, again by the area formula and the fact that $\|f\|_\infty + \mathrm{Lip}(f) \leqslant C\,\delta$, we have, choosing $\varepsilon_{31} = C\,\delta^2$,

$$\begin{aligned}
\mathbf{M}(0 \mathbin{⨳} Z'') &= \frac{1}{8}\int_0^{2\omega_2 Q} \sqrt{1+|f|^2+|f'|^2+|f\wedge f'|^2}\, d\theta \geqslant \frac{1}{8}\int_0^{2\omega_2 Q} \sqrt{1+|f|^2+|f'|^2}\, d\theta \\
&\geqslant \frac{1}{8}\int_0^{2\omega_2 Q} \left(1+\frac{1}{2}(|f|^2+|f'|^2) - \frac{1}{8}(|f|^4+|f'|^4) - \frac{1}{4}(|f|^2\,|f'|^2)\right) d\theta \\
&\geqslant \frac{1}{8}\int_0^{2\omega_2 Q} \left(1+\frac{1}{2}|f|^2(1-C\delta^2) + \frac{1}{2}|f'|^2(1-C\delta^2)\right) d\theta \\
&\geqslant \frac{Q}{4}\omega_2 + \frac{1}{8}(1-\varepsilon_{31})Q\,\omega_2\,\|f\|^2_{W^{1,2}}\,,
\end{aligned}$$

where from the first to the second line we have used the inequality $\sqrt{1+x} \geqslant 1+\frac{x}{2}-\frac{x^2}{8}$. In particular, we get

$$\mathbf{M}(0 \mathbin{⨳} \partial H') - \frac{Q}{4}\omega_2 \geqslant \frac{Q}{8}\omega_2\,(1-\varepsilon_{31})\|f\|^2_{W^{1,2}}\ . \tag{5.11}$$

Finally, combining (5.10) and (5.11), we achieve

$$\begin{aligned}
&(1-\varepsilon_{31})\left[\mathbf{M}(0 \mathbin{⨳} \partial H') - \frac{Q}{4}\omega_2\right] - \mathbf{M}(H') - \frac{Q}{4}\omega_2 \\
&\qquad \geqslant \frac{Q}{8}\omega_2\left\{(1-\varepsilon_{31})^2\|f\|^2_{W^{1,2}} - (1-\varepsilon_Q)]\,\|f-P(f)\|^2_{W^{1,2}} - \|P(f)\|^2_{W^{1,2}}\right\} \\
&\qquad \geqslant \frac{Q}{8}\omega_2\left\{(1-\varepsilon_{31})^2\|f\|^2_{W^{1,2}} - \|f-P(f)\|^2_{W^{1,2}} - \|P(f)\|^2_{W^{1,2}} + \varepsilon_Q\|f-P(f)\|^2_{W^{1,2}}\right\} \\
&\qquad \overset{(5.4)}{\geqslant} \frac{Q}{8}\omega_2\left\{(1-\varepsilon_{31})^2 - 1 + \frac{\varepsilon_Q}{K}\right\}\,\|f\|^2_{W^{1,2}} > 0
\end{aligned}$$

for $\varepsilon_{31} > 0$ sufficiently small. Therefore we can conclude that

$$\mathbf{M}(H') - \frac{Q}{4}\omega_2 \leqslant \frac{1}{4}(1-8\varepsilon_{31})\mathbf{E}(Z'') \leqslant \frac{1}{4}(1-8\varepsilon_{31})\mathbf{E}(Z)\,,$$

for some $\varepsilon_{31}(Q,K) > 0$.

Step 3. Conclusion. Using the isoperimetric inequality we find a 2-dimensional current K such that $\partial K = \partial((0 \mathbin{⨳} Z)\llcorner \mathbf{C}_{1/2}) - Z'' = \partial((0 \mathbin{⨳} Z - 0 \mathbin{⨳} Z')\llcorner \mathbf{C}_{1/2})$ and

$$\mathbf{M}(K) \leqslant C(\mathbf{M}(\partial((0 \mathbin{⨳} Z)\llcorner \mathbf{C}_{1/2}) - Z''))^2 \overset{(vii)}{\leqslant} C(\delta)\mathbf{E}(Z)^2\,.$$

Thus, if we set $H := H' + K + 0 \mathbin{⨳} Z\llcorner \mathbf{B}_1 \setminus \mathbf{C}_{1/2}$, we have $\partial H = Z$ and

$$\mathbf{M}(H) \leqslant \frac{Q}{4}\omega_2 + \frac{1}{4}(1-8\varepsilon_{31})\mathbf{E}(Z) + C(\delta)\mathbf{E}(Z)^2 + \mathbf{M}((0 \mathbin{⨳} Z)\llcorner \mathbf{B}_1 \setminus \mathbf{C}_{1/2})\,.$$

Since $\mathbf{E}((0 \times\!\!\!\!\times Z)_\infty < \bar{\varepsilon}$, it suffices to choose ε sufficiently small to achieve

$$\mathbf{M}(H) \leqslant \frac{Q}{4}\omega_2 + \frac{1}{4}(1-4\varepsilon_{31})\mathbf{E}(Z) + \mathbf{M}((0 \times\!\!\!\!\times Z) \llcorner \mathbf{B}_1 \setminus \mathbf{C}_{1/2}) .$$

Next recall that

$$\begin{aligned}\frac{1}{4}\mathbf{E}(Z) \leqslant & \frac{1}{4}(\mathbf{E}((0 \times\!\!\!\!\times Z)_\infty, \pi_0) = \frac{1}{8}\int_{\mathbf{C}_1} |\vec{T} - \pi_0|^2 d\|0 \times\!\!\!\!\times Z\| \\ = & \frac{1}{4}(\mathbf{M}((0 \times\!\!\!\!\times Z) \llcorner \mathbf{C}_1) - Q\omega_2) = \mathbf{M}((0 \times\!\!\!\!\times Z) \llcorner \mathbf{C}_{1/2}) - \frac{Q\omega_2}{4} ,\end{aligned}$$

where the first equality in the last line is due to $\mathbf{p}_{\pi_0 \sharp}(0 \times\!\!\!\!\times Z) = Q [\![B_1(0,\pi_0)]\!]$. We therefore infer

$$\begin{aligned}\mathbf{M}(H) - Q\omega_2 &\leqslant \mathbf{M}(0 \times\!\!\!\!\times Z) + \varepsilon_{31} Q\omega_2 - 4\varepsilon_{31}\mathbf{M}((0 \times\!\!\!\!\times Z) \llcorner \mathbf{C}_{1/2}) - Q\omega_2 \\ &\leqslant \mathbf{M}(0 \times\!\!\!\!\times Z) + \varepsilon_{31} Q\omega_2 - 4\varepsilon_{31}\mathbf{M}((0 \times\!\!\!\!\times Z) \llcorner \mathbf{B}_{1/2}) - Q\omega_2 \\ &= \mathbf{M}(0 \times\!\!\!\!\times Z) + \varepsilon_{31} Q\omega_2 - \varepsilon_{31}\mathbf{M}(0 \times\!\!\!\!\times Z) - Q\omega_2 \\ &= (1-\varepsilon_{31})(\mathbf{M}(0 \times\!\!\!\!\times Z) - Q\omega_2) .\end{aligned}$$

□

5.2 (wei) and almost monotonicity for almost minimizers

As already mentioned, the key ingredient in the proof of Theorem 5.2 is a suitable generalization of White's epiperimetric inequality [66]. This inequality is a simple consequence of Lemma 5.3 and a compactness argument.

Proposition 5.7. *Let $S \in \mathbf{I}_2(\mathbb{R}^{n+2})$ be an area minimizing cone. For every $C_{12} > 0$ there exists a constant $\varepsilon_{34} > 0$, depending only on the constants C_{01} and α_0 of Definition 5.1 and upon S, with the following property. Assume that $T \in \mathbf{I}_2(\mathbb{R}^{n+2})$ is an almost minimizer with $0 \in \mathrm{spt}(T)$ and set $T_\rho := (\iota_{0,\rho})_\sharp T$. If r is a positive number with*

- $0 < 2r < \min\{2^{-1}\mathrm{dist}(0, \mathrm{spt}(\partial T)), 2\varepsilon_{34}\}$,
- $\mathcal{F}\big((T_{2r} - S) \llcorner \mathbf{B}_1\big) < 2\,\varepsilon_{34}$, $\|T\|(\mathbf{B}_{2r}) \leqslant C_{12} r^2$
- *and* $\partial(T \llcorner \mathbf{B}_r) \in \mathbf{I}_1(\mathbb{R}^{n+2})$,

then

$$\|T_r\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) \leqslant (1-\varepsilon_{35})\Big(\|0 \times\!\!\!\!\times \partial(T_r \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\Big) + \bar{c}\, r^{\alpha_0}. \tag{5.12}$$

$\bar{c}$ depends only on C_{01}, α_0 and $\Theta(0,S)$ and $\varepsilon_{35} > 0$ is any number smaller than some $\bar{\varepsilon} > 0$, which also depends on C_0, α_0 and $\Theta(0,S)$. Moreover $\bar{c}$ depends linearly on C_{01}. In particular, if T is as in Definition 1.1, then $\alpha_0 = 1$ and: $\bar{c}$ depends linearly on $\mathbf{A} := \|A_\Sigma\|_\infty$ in case (a), it depends linearly on $\mathbf{\Omega} := \|d\omega\|_\infty$ in case (b) and it quals $C_0 R^{-1}$ for some geometric constant C_0 in case (c) (in the sense of Remark 2.4).

Proof. We argue by contradiction and assume there exist sequences of almost minimizers $(T^k)_{k\in\mathbb{N}} \subset \mathbf{I}_2(\mathbb{R}^{2+n})$ and radii $r_k \downarrow 0$ with $0 < 2r_k < \operatorname{dist}(0, \operatorname{spt}(\partial T^k))$ such that $R^k := (T^k)_{r_k}$ satisfies $\mathcal{F}((R^k - S)\llcorner \mathbf{B}_2) < \frac{1}{k}$ and

$$\|R^k\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) > \left(1 - \frac{1}{k}\right)\Big(\|0 \mathbin{⨳} \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\Big) + k\, r_k^{\alpha_0}. \tag{5.13}$$

It is important to notice that, in contradicting the statement of Proposition 5.7, the currents T^k satisfy (5.1) for some constants C_0 and α_0 which are fixed, i.e. *independent of* k. First of all, without loss of generality we can assume

$$\|0 \mathbin{⨳} \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) \geqslant 0\,; \tag{5.14}$$

indeed if $\|0 \mathbin{⨳} \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) < 0$ we could use the almost minimality and the appropriate rescaling to conclude

$$\begin{aligned}\|R^k\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) &\leqslant \|0 \mathbin{⨳} \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) + C_1 r_k^{\alpha_0}\\ &\leqslant \left(1 - \frac{1}{k}\right)\Big(\|0 \mathbin{⨳} \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\Big) + C_1\, r_k^{\alpha_0}\,,\end{aligned}$$

contradicting (5.13) for k large enough.

Observe that we have a uniform bound for $\|R^k\|(\mathbf{B}_2)$. Thus, by the usual slicing theorem, passing to a subsequence there is a radius $\rho \in]\frac{3}{2}, 2[$ such $\mathbf{M}(\partial((R^k - S)\llcorner \mathbf{B}_\rho))$ is uniformly bounded. On the other hand $R^k - S$ is converging to 0 in the sense of currents and hence, by [54, Theorem 31.2], $\mathcal{F}((R^k - S)\llcorner \mathbf{B}_\rho) \to 0$. This means that there are integral currents H^k, G^k with $\mathbf{M}(H^k) + \mathbf{M}(G^k) \to 0$ such that

$$(R^k - S)\llcorner \mathbf{B}_\rho = \partial H^k + G^k\,.$$

Taking the boundary of the latter identity we conclude that $\partial G^k = \partial((R^k - S)\llcorner \mathbf{B}_\rho)$. Now, rescaling the almost minimality property of T^k, we conclude that

$$\|R^k\|(\mathbf{B}_\rho) \leqslant \|S\|(\mathbf{B}_\rho) + \mathbf{M}(G_k) + C_1 r_k^{\alpha_0}\,.$$

On the other hand, since $(\mathbf{M}(G^k) + r_k) \downarrow 0$, we infer

$$\limsup_{k\to\infty} \|R^k\|(\mathbf{B}_\rho) \leqslant \|S\|(\mathbf{B}_\rho)\,.$$

Since however $R^k \to S$ in $\mathbf{B}_2$, we also have

$$\|S\|(\mathbf{B}_\rho) \leqslant \liminf_{k\to\infty} \|R^k\|(\mathbf{B}_\rho)\,.$$

We thus conclude that $\|R^k\| \overset{*}{\rightharpoonup} \|S\|$ on $\mathbf{B}_\rho$ in the sense of measures and, since $\|S\|(\partial\mathbf{B}_1) = 0$ by the conical property of S, we infer that $\|R^k\|(\mathbf{B}_1) \to \|S\|(\mathbf{B}_1)$. Thus (5.13) and (5.14) imply

$$\lim_{k\to\infty} \mathbf{M}(\partial(R^k \llcorner \mathbf{B}_1)) = \mathbf{M}(\partial(S \llcorner \mathbf{B}_1))\,. \tag{5.15}$$

The almost monotonicity formula for T^k (in the rescaled version for R^k) implies through standard arguments that $\mathrm{spt}(R^k)$ converges to $\mathrm{spt}(S)$ in the Hausdorff sense: one can follow, for instance, the proof of [54, Lemma 17.11]. Finally, again by [54, Theorem 31.2], we conclude that $\mathcal{F}((R^k - S) \llcorner \mathbf{B}_1) \to 0$ and hence, arguing as above, we infer the existence of integer rectifiable currents G^k such that $\partial G^k = \partial((R^k - S) \llcorner \mathbf{B}_1)$ and $\mathbf{M}(G^k) \to 0$. In turn this implies $\mathcal{F}(\partial(R^k \llcorner \mathbf{B}_1) - \partial(S \llcorner \mathbf{B}_1)) \to 0$. So all the assumptions of Lemma 5.3 are satisfied, and there exist integral currents H^k such that $\partial H^k = \partial(R^k \llcorner \mathbf{B}_1)$ and

$$\|H^k\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) \leqslant (1 - \varepsilon_{31})\big(\|0 \times\!\!\!\!\times\, \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\big). \tag{5.16}$$

By the almost minimality of T^k and the usual rescaling, we conclude

$$\|R^k\|(\mathbf{B}_1) \leqslant \|H^k\|(\mathbf{B}_1) + C_0 r_k^{\alpha_0}.$$

Thus,

$$\begin{aligned}\|R^k\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) &\leqslant \|H^k\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1) + C_0 r_k^{\alpha_0} \\ &\overset{(5.16)}{\leqslant} (1 - \varepsilon_{31})\big(\|0 \times\!\!\!\!\times\, \partial(R^k \llcorner \mathbf{B}_1)\|(\mathbf{B}_1) - \|S\|(\mathbf{B}_1)\big) + C_0 r_k^{\alpha_0}.\end{aligned}$$

However, when k is so large that $\frac{1}{k} < \varepsilon_{31}$ and $k > C_0$, the latter inequality contradicts (5.13) (recall (5.14)). □

It is known that the almost minimizing condition of Definition 5.1 is alone sufficient to derive a monotonicity formula. However, we have been unable to find a reference and we therefore provide the proof below. Note also that in the geometric cases (a), (b) and (c), a more precise form of the monotonicity formula could be derived directly appealing to the fact that the corresponding induced varifolds have bounded mean curvature.

Proposition 5.8 (Almost Monotonicity). *Let* $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ *be an almost minimizer and* $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. *There are constants* $C_{02}, \bar{r}, \alpha_0 > 0$ *such that*

$$\int_{\mathbf{B}_r(x)\setminus \mathbf{B}_s(x)} \frac{|(z-x)^\perp|^2}{|z-x|^{m+2}} \, d\|T\|(z) \leqslant C_{02}\Big(\frac{\|T\|(\mathbf{B}_r(x))}{\omega_k\, r^m} - \frac{\|T\|(\mathbf{B}_s(x))}{\omega_m\, s^m} + r^{\alpha_0}\Big) \tag{5.17}$$

for all $0 < s < r < \bar{r}$ *(in (5.17)* $(z-x)^\perp$ *denotes the projection of the vector* $z - x$ *on the orthogonal complement of the approximate tangent to* T *at* x*). In particular the function* $r \to \dfrac{\|T\|(\mathbf{B}_r(x))}{\omega_k\, r^m} + C\, r^\alpha$ *is nondecreasing.*

Proof. Assume without loss of generality $x = 0$. For a.e. r the current $\partial(T \llcorner \mathbf{B}_r)$ is integral (cf. [54, Section 28]) and we have, by (5.1) with $W = 0 \times\!\!\!\!\times\, \partial(T \llcorner \mathbf{B}_r)$,

$$\|T\|(\mathbf{B}_r) \leqslant \|W\|(\mathbf{B}_r) + C_0 r^{m+\alpha_0} = \frac{r}{m}\mathbf{M}(\partial(T \llcorner \mathbf{B}_r)) + C_0 r^{m+\alpha_0}. \tag{5.18}$$

Set $f(r) := \|T\|(\mathbf{B}_r)$ and observe that f is an nondecreasing function and so a function of bounded variation. As such it has left and right limits at each point and in fact $f(r) = f(r^-)$. In particular we can decompose its distributional derivative Df, which is a nonnegative measure, as $Df = f'\mathscr{L} + \mu_s$, where $\mathscr{L}$ denotes the Lebesgue one-dimensional measure and μ_s is the singular part of Df. We multiply (5.18) by mr^{-m-1} and add $\frac{f'(r)}{r^m} + \frac{\mu_s}{r^m}$:

$$\frac{\mu_s}{r^m} + \frac{1}{r^m} f'(r) - \frac{1}{r^m}\mathbf{M}(\partial(T \llcorner \mathbf{B}_r)) \leqslant \frac{Df}{r^m} - \frac{mf(r)}{r^{m+1}} + C_{01} r^{\alpha_0 - 1}.$$

Integrating on the interval $[s, r[$ (where $r_0 > r > s$) we reach

$$\underbrace{\int_{[s,r[} \frac{1}{\rho^m} d\mu_s(\rho)}_{I^s} + \underbrace{\int_s^r \frac{1}{\rho^m}(f'(\rho) - \mathbf{M}(\partial(T \llcorner \mathbf{B}_\rho)))\, d\rho}_{I^a} \leqslant \frac{f(r)}{r^m} - \frac{f(s)}{s^m} + C_0 r^{\alpha_0}.$$

To conclude we only need to prove that $I := I^s + I^a$ bounds a multiple of the left hand side of (5.17). Denote by $x^{\|}$ the projection of x on the approximate tangent space to T at x. Recall first (cf. [54, eq. (28.6)]) that

$$T_\rho := \langle T, |\cdot|, \rho \rangle = \partial(T \llcorner \mathbf{B}_\rho) - (\partial T) \llcorner \mathbf{B}_\rho = \partial(T \llcorner \mathbf{B}_\rho) \text{ for a.e. } \rho.$$

Next introduce the Borel set $E := \{|x^{\|}| > 0\}$ and its complementary E^c and recall that, by the coarea formula (cf. [54, Lemma 28.1 & Lemma 28.5]), for any Borel map g we have

$$\int_{\mathbf{B}_r \setminus \mathbf{B}_s} g(y) \frac{|y^{\|}|}{|y|} d\|T\|(y) = \int_s^r \int g(x) d\|T_\rho\|(x)\, d\rho\,. \tag{5.19}$$

Let R be the countable rectifiable set such that $\|T\| = \Theta(T, \cdot)\mathcal{H}^m \llcorner R$. It then follows from the slicing theory that $\|T_\rho\| = \Theta(T, \cdot)\mathcal{H}^{m-1} \llcorner (R \cap \partial \mathbf{B}_\rho)$ for a.e. ρ and thus inserting $g = \mathbf{1}_{E^c}$ in (5.19) above we derive

$$\mathcal{H}^{m-1}(E^c \cap \partial \mathbf{B}_\rho) \leqslant \|T_\rho\|(E^c) = 0 \qquad \text{for a.e. } \rho. \tag{5.20}$$

Thus, since $|x^{\|}| > 0$ for every $x \in (\mathbf{B}_r \setminus \mathbf{B}_s) \cap E$, we conclude

$$\begin{aligned} I^a &= \int_s^r \frac{1}{\rho^m} \int_E \frac{|x| - |x^{\|}|}{|x^{\|}|} d\|T_\rho\|(x)\, d\rho = \int_s^r \frac{1}{\rho^m} \int_E \frac{|x|^2 - |x^{\|}|^2}{|x^{\|}|(|x| + |x^{\|}|)} d\|T_\rho\|(x)\, d\rho \\ &\geqslant \int_s^r \frac{1}{2\rho^{m+2}} \int_E \frac{|x^\perp|^2 |x|}{|x^{\|}|} d\|T_\rho\|(x)\, d\rho = \int_{(\mathbf{B}_r \setminus \mathbf{B}_s) \cap E} \frac{|x^\perp|^2}{2|x|^{m+2}} d\|T\|(x)\,. \end{aligned} \tag{5.21}$$

Now observe that on E^c we have $|x^\perp| = |x|$ and thus

$$\int_{(\mathbf{B}_r \setminus \mathbf{B}_s) \cap E^c} \frac{|x^\perp|^2}{2|x|^{m+2}} d\|T\|(x) = \int_{(\mathbf{B}_r \setminus \mathbf{B}_s) \cap E^c} \frac{1}{2|x|^m} d\|T\|(x)\,. \tag{5.22}$$

Next, denote by S the set of radii r such that $\mathcal{H}^{m-1}(E^c \cap \partial \mathbf{B}_r) > 0$. We then must have

$$\|T\|(E^c \cap (\mathbf{B}_\rho \setminus \mathbf{B}_\tau)) \leqslant \|T\| \left(\cup_{s \in S \cap [\tau, \rho[} \partial \mathbf{B}_s \right) \leqslant Df(S \cap [\tau, \rho[) \overset{(5.20)}{\leqslant} \mu_s([\tau, \rho[)$$

for every $0 < \tau < \rho$ (in fact the inequalities above are all identities, but this is not really needed). Thus for every $N \in \mathbb{N} \setminus 0$ we can estimate

$$\int_{(\mathbf{B}_r \setminus \mathbf{B}_s) \cap E^c} \frac{1}{2|x|^m} d\|T\|(x) \leqslant \sum_{i=1}^N \frac{1}{2 s_{i-1}^m} \|T\|(E^c \cap (\mathbf{B}_{s_i} \setminus \mathbf{B}_{s_{i-1}})) \leqslant \sum_{i=1}^N \frac{1}{2 s_{i-1}^m} \int_{[s_{i-1}, s_i[} d\mu_s$$

where $s_i := s + \frac{i}{N}(r-s)$. In particular letting $N \uparrow \infty$ we conclude

$$\int_{(\mathbf{B}_r\setminus\mathbf{B}_s)\cap E^c} \frac{1}{2|x|^m}\, d\|T\|(x) \leqslant \int_{[s,r[} \frac{1}{2\rho^m}\, d\mu_s(\rho) = I^s\,. \tag{5.23}$$

From (5.21), (5.22) and (5.23) we conclude that $I^a + I^s$ bounds the right hand side of (5.17). □

To conclude this section we prove a simple consequence of the Area Formula: that is how to compute the mass of the pushforward through the radial map of the portion of the current in a shell.

Lemma 5.9. *Let* $R \in \mathbf{I}_m(\mathbb{R}^n)$ *and let* $x \in \mathrm{spt}(T)\setminus \mathrm{spt}(\partial T)$. *Moreover let* $F(z) := \frac{z}{|z|}$ *for every* $z \in \mathbb{R}^n$ *and* $0 < r < s$. *Then*

$$\mathbf{M}(F_\sharp(T\llcorner(\mathbf{B}_t\setminus\mathbf{B}_s))) \leqslant \int_{\mathbf{B}_t\setminus\mathbf{B}_s} \frac{|x^\perp|}{|x|^{m+1}}\, d\|T\|\,.$$

Proof. Let $\vec{T}(x) := T_1(x)\wedge\cdots\wedge T_m(x)$ and notice that by the Area Formula for a push-forward (cf. [54, Remark 27.2]), we have

$$\begin{aligned}\mathbf{M}(F_\sharp(T\llcorner(\mathbf{B}_t\setminus\mathbf{B}_s))) &= \int_{B_t\setminus B_s} |DF(x)\,T_1(x)\wedge\cdots\wedge DF(x)\,T_m(x)|\, d\|T\|(x)\\ &= \int_{B_t\setminus B_s} \sqrt{\det((DF(x)\,T_j(x))\cdot(DF(x)\,T_i(x)))}\, d\|T\|(x)\,,\end{aligned}\tag{5.24}$$

that is we need to compute $\det((DF(x)\,T_j(x))\cdot(DF(x)\,T_i(x)))$. To this aim set $F_i(x) := \frac{x_i}{|x|}$ and notice that, if $x \neq 0$, we have

$$\frac{\partial F_i}{\partial x_j}(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i\,x_j}{|x|^3}\,,$$

so that

$$DF(x)\,T_j(x) = \frac{1}{|x|}\left(T_j(x) - (T_j\cdot x)\,\frac{x}{|x|^2}\right)\,.$$

It follows from this that

$$\begin{aligned}(DF(x)\,T_j(x))\cdot(DF(x)\,T_i(x)) &= \frac{1}{|x|^2}\left(T_j(x)\cdot T_i(x) - \frac{1}{|x|^2}(T_j\cdot x)\,(T_i\cdot x)\right)\\ &= \frac{1}{|x|^2}\left(1 - \frac{1}{|x|^2}(T_j\cdot x)\,(T_i\cdot x)\right)\,,\end{aligned}$$

where in the last line we used the orthonormality of $(T_j)_j$. Set $t := \frac{1}{|x|^2}$ and $A_{ij} := (T_j\cdot x)\,(T_i\cdot x)$, then, by the usual formula for $\det(I + tA)$ (cf. [32, 1.4.5]), we deduce

$$\begin{aligned}\det(1+tA) &= \sum_{k=0}^{m}\left(-\frac{1}{|x|^2}\right)^k \sum_{1\leqslant\lambda_1<\cdots<\lambda_k\leqslant m)} \det((x\cdot T_{\lambda_i}(x))\,(x\cdot T_{\lambda_j}(x)))\\ &= 1 - \frac{1}{|x|^2}\sum_{l=1}^{m}(x\cdot T_l(x))^2 + \sum_{k=2}^{m}\left(-\frac{1}{|x|^2}\right)^m \sum_{1\leqslant\lambda_1<\cdots<\lambda_k\leqslant m} \det((x\cdot T_{\lambda_i}(x))\,(x\cdot T_{\lambda_j}(x)))\,.\end{aligned}$$

Since the column of the matrix with entries $(x \cdot T_{\lambda_i}(x))\,(x \cdot T_{\lambda_j}(x))$ are linearly dependent, we conclude that

$$\det((DF(x)\,T_j(x)) \cdot (DF(x)\,T_i(x))) = \frac{1}{|x|^{2m}}\left(1 - \frac{1}{|x|^2}\sum_{l=1}^{m+1}(x \cdot T_l(x))^2\right) = \frac{|x^\perp|^2}{|x|^{2(m+1)}}\,. \quad (5.25)$$

Combining (5.24) and (5.25) we reach the conclusion. □

5.3 CONCLUSION OF THE PROOF

Without loss of generality from now on we assume that $x = 0$ and that $\mathrm{dist}(0, \mathrm{spt}(\partial T)) \geqslant 2$. Moreover we set $T_r := (\iota_{0,r})_\sharp T$.

STEP 1. BLOW-UP. By the almost monotonicity, the family $\{T_r\}_{0<r\leqslant 1} \subset \mathbf{I}_2(\mathbb{R}^{n+2})$ enjoys a uniform bound for $\|T_r\|(K)$ whenever $K \subset \mathbb{R}^{n+2}$ is a compact set. Moreover, for any $U \subset\subset \mathbb{R}^{n+2}$ open, $\partial T_r \llcorner U = 0$, provided r is large enough. It follows that we can apply the compactness theorem of integral currents, and for every sequence $r_k \downarrow 0$ we can extract a subsequence T_{ρ_k} converging to an integral current S with $\partial S = 0$. Observe also that we can argue as in the proof of Proposition 5.7 to conclude that for every $N_0 \in \mathbb{N}$ there is a subsequence, not relabeled, and a $\bar r \in]N_0, N_0 + 1[$ with the following properties

- $\|T_{\rho_k}\|(\mathbf{B}_{\bar r}) \to \|S\|(\mathbf{B}_{\bar r})$;
- There are currents $H_k \in \mathbf{I}_2(\mathbb{R}^{n+2})$ with $\mathbf{M}(H^k) \downarrow 0$ and $\partial H^k = \partial((T_{\rho_k} - S)\llcorner \mathbf{B}_{\bar r})$.

We then easily conclude that S is area minimizing in $\mathbf{B}_{\bar r}$ and that $\|T_{\rho_k}\|(V) \to \|S\|(V)$ for any open set $V \subset\subset \mathbf{B}_{\bar r}$ with $\|S\|(\partial V) = 0$. A standard argument shows that these properties remain then true for every ball and for the *entire sequence* $\{T_{\rho_k}\}$. As a consequence of the fact that $\Theta(0, T)$ exists, we then conclude that

$$\|S\|(\mathbf{B}_r(0)) = \Theta(T, 0)r^2 := Q\omega_2 r^2$$

for all radii but an (at most) countable family (recall that ω_2 denotes the area of the unit disk in $\mathbb{R}^2$). It is then a standard fact, using the monotonicity formula for area-minimizing currents, that S is a cone (see for instance [54]). Finally, it is well known that 2-dimensional area minimizing cones are all sum of planes intersecting only at the origin (see for instance [35]). So we conclude from the standard theory of currents (see for instance the proof of Proposition 5.7) that $\mathcal{F}((T_{\rho_k} - S)\llcorner \mathbf{B}_r) \to 0$ for every $r > 0$.

Let ε_{34} be the constant of Proposition 5.7. We then conclude the existence of a radius $r_0 > 0$ such that, for every $r < r_0$ there is an an area minimizing cone S such that $\mathcal{F}((T_{2r} - S)\llcorner \mathbf{B}_1) \leqslant 2\,\varepsilon_{34}$. We can then apply (5.12) for every $0 < r < r_0$ such that $\partial(T\llcorner \mathbf{B}_r) \in \mathbf{I}_1(\partial \mathbf{B}_r)$ (which holds for a.e. r). After scaling back and multiplying by r^2, we get

$$\mathbf{M}(T\llcorner \mathbf{B}_r) - Q\,\omega_2\,r^2 \leqslant (1 - \varepsilon_{35})\left(\mathbf{M}(0 \mathbin{⨳} \partial(T\llcorner \mathbf{B}_r)) - Q\,\omega_2\,r^2\right) + \bar c\, r^{2+\alpha_0} \quad \text{for a.e. } r < r_0\,. \quad (5.26)$$

Set $f(r) := \mathbf{M}(T\llcorner \mathbf{B}_r) - Q\,\omega_2\,r^2$. Since $r \mapsto \mathbf{M}(T\llcorner \mathbf{B}_r)$ is monotone, the function f is differentiable a.e. and its distributional derivative is a measure. Its absolutely continuous part coincides a.e. with the classical differential and its singular part is nonnegative. Note also that we can assume $2+\alpha_0 > \varepsilon + \frac{2}{1-\varepsilon} =: \varepsilon + a$ for some $\varepsilon > 0$.

Therefore, by the well-known expansion for the mass of a cone, (5.26) reads

$$-a\,\bar{c}\,r^{\varepsilon-1} \leqslant \frac{d}{dr}\big(r^{-a}f(r)\big), \tag{5.27}$$

Integrating (5.27) we get $-\frac{a}{\varepsilon}\,\bar{c}\,\left(r^{\varepsilon}-s^{\varepsilon}\right) \leqslant r^{-a}f(r) - s^{-a}f(s)$ for all $0 < s < r < r_0$. Setting $e(r) := \frac{f(r)}{\omega_2 r^2}$ this implies

$$e(s) \leqslant \left(\frac{s}{r}\right)^a e(r) + C\,r^{\varepsilon} \qquad \forall\, 0 < s < r < r_0. \tag{5.28}$$

STEP 2. Consider now the map $F(x) := \frac{x}{|x|}$ and radii $0 < \frac{t}{2} \leqslant s \leqslant t < r_0$. By Lemma 5.9,

$$\begin{aligned}\mathbf{M}(F_\sharp(T\llcorner(\mathbf{B}_t\setminus\mathbf{B}_s))) &= \int_{\mathbf{B}_t\setminus\mathbf{B}_s} \frac{|x^\perp|}{|x|^3}\, d\|T\| \\ &\leqslant \underbrace{\left(\int_{\mathbf{B}_t\setminus\mathbf{B}_s} \frac{|x^\perp|^2}{|x|^4}\, d\|T\|\right)^{\frac{1}{2}}}_{:=I_1} \cdot \underbrace{\left(\int_{\mathbf{B}_t\setminus\mathbf{B}_s} \frac{1}{|x|^2}\, d\|T\|\right)^{\frac{1}{2}}}_{I_2}.\end{aligned}$$

I_1 and I_2 can be easily estimated using the almost monotonicity formula

$$I_1^2 \overset{(5.17)}{\leqslant} e(t) - e(s) + C_1\,t^{\alpha_0} \leqslant e(t) + 2\,C_1\,t^{\alpha_0} \overset{(5.28)}{\leqslant} C\,t^{\frac{\varepsilon}{2}}, \tag{5.29}$$

$$I_2^2 \leqslant \frac{\|T\|(\mathbf{B}_t)}{s^2} \overset{(5.17)}{\leqslant} \left(\frac{t}{s}\right)^2 \left[\frac{\|T\|(\mathbf{B}_{r_0})}{r_0^2} + C_1\,r_0^{\alpha_0}\right] \leqslant C, \tag{5.30}$$

where we took into account that, by (5.17), $e(s) > -C_1 s^{\alpha}$ for every $s > 0$ and that $C > 0$ is a constant depending on r_0. In particular we conclude that

$$\mathbf{M}(F_\sharp(T\llcorner(\mathbf{B}_t\setminus\mathbf{B}_s))) \leqslant C\,t^{\frac{\varepsilon}{2}} \qquad \forall\, 0 < \frac{t}{2} \leqslant s \leqslant t < r_0,$$

and, by iteration on diadic intervals,

$$\mathbf{M}(F_\sharp(T\llcorner(\mathbf{B}_r\setminus\mathbf{B}_s))) \leqslant C\,r^{\frac{\varepsilon}{2}} \qquad \forall\, 0 < s < r < r_0. \tag{5.31}$$

Since $\partial F_\sharp(T\llcorner(\mathbf{B}_r\setminus\mathbf{B}_s)) = \partial(T_r\llcorner\mathbf{B}_1) - \partial(T_s\llcorner\mathbf{B}_1)$ for a.e. $0 < s < r$, from the definition of $\mathcal{F}$ we get:

$$\mathcal{F}\big(\partial(T_r\llcorner\mathbf{B}_1) - \partial(T_s\llcorner\mathbf{B}_1)\big) \overset{(5.31)}{\leqslant} C\,r^{\frac{\varepsilon}{2}}. \tag{5.32}$$

This implies that the currents $\partial(T_r\llcorner\mathbf{B}_1)$ converges to a unique current Z. On the other hand, by the almost monotonicity formula it follows easily that $T_r\llcorner\mathbf{B}_1$ converge to the cone $0 \mathbin{⨳} Z$. Since we already know that an appropriate sequence converges to $S = \sum_i n_i[\![\pi_i]\!]$, we conclude that T_r converges to S.

STEP 3. PROOF OF (5.2) AND (5.3). In order to prove (5.2), it is enough to find integral currents V and W such that $T_r - T_s = \partial H + W$ and $\mathbf{M}(H) + \mathbf{M}(W) \leqslant Cr^{\frac{\varepsilon}{2}}$. To this aim, fix a small parameter $a > 0$. Let $[\![p, q]\!]$ denote the current in $\mathbf{I}_1(\mathbb{R})$ induced by the oriented segment $\{t : p \leqslant t \leqslant q\}$. Similarly $[\![p]\!] \in \mathbf{I}_0(\mathbb{R})$ is the Dirac mass at the point p. Consider the currents $V_a \in \mathbf{I}_3(\mathbb{R} \times \mathbb{R}^{n+2})$ defined by

$$V_a := \Big([\![0, 1]\!] \times T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_a)\Big) \llcorner \Big\{(t, x) \in \mathbb{R} \times \mathbb{R}^{n+2} \,:\, r^{-1}|x| \leqslant t \leqslant s^{-1}|x|\Big\}.$$

Next, we consider the map $h \colon \mathbb{R} \times (\mathbb{R}^{n+2} \setminus \{0\}) \ni (t, x) \to \frac{tx}{|x|} \in \mathbb{R}^{n+2}$ and the currents $H_a := h_\sharp V_a$. If $d_1, d_2 : \mathbb{R} \times \mathbb{R}^{n+2} \to \mathbb{R}$ denote the functions $d_1(t, x) := t - s^{-1}|x|$ and $d_2(t, x) := t - r^{-1}|x|$, then for a.e. $a > 0$ we have

$$\begin{aligned}\partial V_a = &\; [\![1]\!] \times T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_s) - [\![\tfrac{a}{r}, \tfrac{a}{s}]\!] \times \partial(T \llcorner \mathbf{B}_a) \\ &+ \langle [\![0, 1]\!] \times T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_a), d_1, 0\rangle - \langle [\![0, 1]\!] \times T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_a), d_2, 0\rangle\,.\end{aligned}$$

Since ∂ commutes with the push-forward, we also get

$$\partial H_a = F_\sharp(T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_s) - \underbrace{h_\sharp \left([\![\tfrac{a}{r}, \tfrac{a}{s}]\!] \times \partial(T \llcorner \mathbf{B}_a)\right)}_{Z_a} - T_r \llcorner (\mathbf{B}_1 \setminus \mathbf{B}_{\frac{a}{r}}) + T_s \llcorner (\mathbf{B}_1 \setminus \mathbf{B}_{\frac{a}{s}}), \quad (5.33)$$

where we have used the fact that $h(t, x) \equiv s^{-1}x$ and $h(t, x) \equiv r^{-1}x$ respectively in the sets $\{(t, x) \in \mathbb{R} \times (\mathbb{R}^{n+2} \setminus \{0\}) \,:\, t = s^{-1}|x|\}$ and $\{(t, x) \in \mathbb{R} \times (\mathbb{R}^{n+2} \setminus \{0\}) \,:\, t = r^{-1}|x|\}$. It is simple to see that there exists H such that $H_a \to -H$ as $a \downarrow 0$. Thus (5.33) gives

$$-\partial H = F_\sharp(T \llcorner (\mathbf{B}_r \setminus \mathbf{B}_s)) - T_r \llcorner \mathbf{B}_1 + T_s \llcorner \mathbf{B}_1,$$

because $\mathbf{M}(Z_a) \leqslant a\,|s^{-1} - r^{-1}|\mathbf{M}(\partial(T_a \llcorner \mathbf{B}_1)) \leqslant C\,a\,|s^{-1} - r^{-1}|\mathbf{M}(\partial(T_0 \llcorner \mathbf{B}_1)) \to 0$. To conclude (5.2) we only need to estimate the mass of H. To this extent, note that $h_\sharp(\frac{\partial}{\partial t} \wedge \vec{T}) = dh(\frac{\partial}{\partial t}) \wedge h_\sharp(\vec{T})$ and, since $dh\left(\frac{\partial}{\partial t}\right) = \frac{x}{|x|}$,

$$\begin{aligned} H(\omega) &= \int_0^1 \int_{\mathbf{B}_{rt} \setminus \mathbf{B}_{st}} \langle h_\sharp \left(\tfrac{\partial}{\partial t} \wedge \vec{T}\right), \omega_{h(x)}\rangle d\|R\|(x)\, dt \\ &= \int_0^1 \int_{\mathbf{B}_{rt} \setminus \mathbf{B}_{st}} \langle \tfrac{tx}{|x|} \wedge (F_\sharp \vec{T}), \omega_{tx/|x|}\rangle\, d\|T\|(x)\, dt \\ &= \int_0^1 \int_{\mathbf{B}_{rt} \setminus \mathbf{B}_{st}} \langle (tF)_\sharp \vec{T}, \omega_{tx/|x|} \lrcorner \tfrac{x}{|x|}\rangle d\|T\|(x)\, dt \\ &= \int_0^1 (tF)_\sharp\, (T \llcorner (\mathbf{B}_{rt} \setminus \mathbf{B}_{st}))\, (\omega \lrcorner \tfrac{x}{|x|})\, dt \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{M}(H) &\leqslant \int_0^1 \mathbf{M}((tF)_\sharp\, (T \llcorner (\mathbf{B}_{rt} \setminus \mathbf{B}_{st}))\, dt = \int_0^1 t^2 \mathbf{M}(F_\sharp(T \llcorner (\mathbf{B}_{rt} \setminus \mathbf{B}_{st})))\, dt \\ &\overset{(5.31)}{\leqslant} C \int_0^1 r^{\varepsilon/2} t^{2+\varepsilon/2}\, dt \leqslant C r^{\varepsilon/2}\,. \end{aligned}$$

(5.3) follows then from the lower bound on the density of T which is a consequence of the almost monotonicity formula, see for instance [54, Lemma 17.11].

STEP 4. DECOMPOSITION. We first introduce the following notation: we call T irreducible in $\mathbf{B}_r(x)$ if it is not possible to find two (integral) currents with $T \llcorner \mathbf{B}_r(x) = T^1 + T^2$ and $\mathrm{spt}(T^1) \cap \mathrm{spt}(T^2) = \{0\}$ (cf. to the notion of *indecomposabality* as in [32, 4.2.25]: T is indecomposable if it is impossible to write it as $T^1 + T^2$ with $\partial T_1 \llcorner \mathbf{B}_r(x) = \partial T_2 \llcorner \mathbf{B}_r(x) = 0$ and $\mathbf{M}(T_1) + \mathbf{M}(T_2) = \|T\|(\mathbf{B}_r(x))$). If T is reducible, then clearly $\Theta(\|T\|, x) = \Theta(\|T^1\|, x) + \Theta(\|T^2\|, x)$. Since each T^i would be almost minimizing, $\Theta(\|T^i\|, x) \in \mathbb{N} \setminus \{0\}$ and we can only decompose T finitely many times. Next suppose by contradiction that T is irreducible in x but its tangent cone $T_{x,0}$ is not a plane. Then, since $T_{x,0}$ is area minimizing, by [35], there exists $J \geqslant 2$ such that $T_{x,0} = \sum_{i=1}^{J} Q_i [\![V_i]\!]$, where $V_i \subset \mathbb{R}^{n+2}$ are 2-dimensional linear subspaces such that $V_i \cap V_j = \{0\}$ for every $i \neq j$ and $Q_i \in \mathbb{N}$ satisfy $\sum_{i=1}^{J} Q_i = Q$. Then consider the currents

$$T^i := T \llcorner \{y \in \mathbb{R}^{m+n} : \mathrm{dist}(y - x, V_i) \leqslant C r^{1+\gamma}\} \quad \text{for } i = 1, 2, \ldots, J\,.$$

By (5.3) this is a decomposition of T in two non-zero currents whose supports intersect each other only in $\{0\}$, which is a contradiction.

Part IV

STEP 3: CENTER MANIFOLD AND NORMAL APPROXIMATION

6

CENTER MANIFOLD

In this chapter we construct the center manifold.

6.1 THE CONSTRUCTION ALGORITHM

6.1.1 *Choice of some parameters and smallness of some other constants*

As in [20] the construction of the center manifold involves several parameters. We start by choosing three of them which will appear as exponents of (two) lenghtscales in several estimates.

Assumptions 6. *Let* T *be as in Assumptions 3 and 4 and in particular recall the exponents* $\bar{\alpha}, b, a$ *and* γ *defined therein. We choose the positive exponents* γ_0, β_2 *and* δ_1 *(in the given order) so that*

$$\gamma_0 < \min\{\gamma, \bar{\alpha}, a-b, b-\tfrac{b+1}{2}, \log_2 \tfrac{6}{5}\} \tag{6.1}$$

$$\beta_2 < \min\{\varepsilon_0, \tfrac{\gamma_0}{4}, \tfrac{a}{b}-1, \tfrac{\bar{\alpha}}{2}, \tfrac{\beta_0}{2}, \beta_0\gamma_0\} \qquad b > \tfrac{1+b}{2}(1+\beta_2) \tag{6.2}$$

$$\beta_2 - 2\delta_1 \geqslant \tfrac{\beta_2}{3} \qquad \beta_0(2-2\delta_1) - 2\delta_1 \geqslant 2\beta_2 \tag{6.3}$$

(where β_0 *is the constant of Theorem 2.8 and* ε_0 *the exponent in the regularity of* Σ*)*

Having fixed γ_0, β_2 and δ_1 we introduce five further parameters: M_0, N_0, C_e, C_h and ε_{41}. We will impose several inequalities upon them, but following a very precise hierarchy, which ensures that all the conditions required in the remaining statements can be met. We will use the term "geometric" when such conditions depend only upon $\bar{n}, n, Q, \bar{Q}, \gamma_0, \beta_2$ and δ_1, whereas we keep track of their dependence on M_0, N_0, C_e and C_h using the notation $C = C(M_0), C(M_0, N_0)$ and so on. ε_{41} is always the last parameter to be chosen: it will be small depending upon all the other constants, but constants will never depend upon it.

Assumptions 7 (Hierarchy of the parameters)**.** *In all the statements of the paper*

- $M_0 \geqslant 4$ *is larger than a geometric constant and* N_0 *is a natural number larger than* $C(M_0)$*; one such condition is recurrent and we state it here:*

$$\sqrt{2}M_0 2^{10-N_0} \leqslant 1\,; \tag{6.4}$$

- C_e *is larger than* $C(M_0, N_0)$*;*
- C_h *is larger than* $C(M_0, N_0, C_e)$*;*
- $\varepsilon_{41} > 0$ *is smaller than* $c(M_0, N_0, C_e, C_h) > 0$*.*

6.1.2 *Whitney decomposition of* $\mathfrak{B}_{\bar{Q},2}$

From now on we will use $\mathfrak{B}$ for $\mathfrak{B}_{\bar{Q},2}$, since the positive natural number $\bar{Q}$ is fixed for the rest of the paper. In this section we decompose $\mathfrak{B}\setminus\{0\}$ in a suitable way. More precisely, a closed subset L of $\mathfrak{B}$ will be called a dyadic square if it is a connected component of $\mathfrak{B}\cap(H\times\mathbb{C})$ for some euclidean dyadic square $H=[a_1,a_1+2\ell]\times[a_2,a_2+2\ell]\subset\mathbb{R}^2=\mathbb{C}$ with

- $\ell=2^{-j}$, $j\in\mathbb{N}, j\geqslant 2$, and $a\in 2^{1-j}\mathbb{Z}^2$;
- $H\subset[-1,1]^2$ and $0\notin H$.

Observe that L is truly a square, both from the topological and the metric point of view. 2ℓ is the sidelength of both H and L. Note that $\mathfrak{B}\cap(H\times\mathbb{C})$ consists then of Q distinct squares $L_1,\dots,L_{\bar{Q}}$. $z_H:=a+(\ell,\ell)$ is the center of the square H. Each L lying over H will then contain a point (z_H,w_L), which is the center of L. Depending upon the context we will then use z_L rather than z_H.

The family of all dyadic squares of $\mathfrak{B}$ defined above will be denoted by $\mathscr{C}$. We next consider, for $j\in\mathbb{N}$, the dyadic closed annuli

$$\mathcal{A}_j:=\mathfrak{B}\cap\left(([-2^{-j},2^{-j}]^2\setminus]-2^{-j-1},2^{-j-1}[^2)\times\mathbb{C}\right).$$

Each dyadic square L of $\mathfrak{B}$ is then contained in exactly one annulus $\mathcal{A}_j$ and we define $d(L):=2^{-j-1}$. Moreover $\ell(L)=2^{-j-k}$ for some $k\geqslant 2$. We then denote by $\mathscr{C}^{k,j}$ the family of those dyadic squares L such that $L\subset\mathcal{A}_j$ and $\ell(L)=2^{-j-k}$. Observe that, for each $j\geqslant 1, k\geqslant 2$, $\mathscr{C}^{k,j}$ is a covering of $\mathcal{A}_j$ and that two elements of $\mathscr{C}^{k,j}$ can only intersect at their boundaries. Moreover, any element of $\mathscr{C}^{k,j}$ can intersect at most 8 other elements of $\mathscr{C}^{k,j}$. Finally, we set $\mathscr{C}^k:=\bigcup_{j\geqslant 2}\mathscr{C}^{k,j}$. Observe now that $\mathscr{C}$ covers a punctured neighborhood of 0 and that if $L\in\mathscr{C}^k$, then

- L intesects at most 9 other elements $J\in\mathscr{C}^k$;
- If $L\cap J\neq\emptyset$, then $\ell(J)/2\leqslant\ell(L)\leqslant 2\ell(L)$ and $L\cap J$ is either a vertex or a side of the smallest among the two.

More in general if the intersection of two distinct elements L and J in $\mathscr{C}$ has nonempty interior, then one is contained in the other: if $L\subset J$ we then say that L is a descendant of J and J an ancestor of L. If in addition $\ell(L)=\ell(J)/2$, then we say that L is a son of J and J is the father of L. When L and J intersect only at their boundaries, we then say that L and J are adjacent.

Next, for each dyadic square L we set $r_L:=\sqrt{2}M_0\ell(L)$. Note that, by our choice of N_0, we have that:

$$\text{if } L\in\mathscr{C}^{k,j} \text{ and } k\geqslant N_0, \text{ then } \mathbf{C}_{64r_L}(z_L)\subset\mathbf{C}_{2^{1-j}}\setminus\mathbf{C}_{2^{-2-j}}. \tag{6.5}$$

In particular $\mathbf{V}_{u,a}\cap\mathbf{C}_{64r_L}(z_L)$ consists of Q connected components and we can select the one containing $(z_L,u(z_L,w_L))$, which we will denote by $\mathbf{V}_L$. We will then denote by T_L the current $T\llcorner\mathbf{V}_L$. According to Lemma 2.15, $\mathbf{V}_L\cap\{z_L\}\times\mathbb{R}^n$ contains at least one point of spt(T): we select any such point and denote it by $p_L=(z_L,y_L)$. Correspondingly we will denote by $\mathbf{B}_L$ the ball $\mathbf{B}_{64r_L}(p_L)$.

Definition 6.1. The height of a current S in a set E with respect to a plane π is given by

$$\mathbf{h}(S,E,\pi) := \sup\{|\mathbf{p}_\pi^\perp(p-q)| : p,q \in \mathrm{spt}(S)\cap E\}. \tag{6.6}$$

If $E = \mathbf{C}_r(p,\pi)$ we will then set $\mathbf{h}(S,\mathbf{C}_r(p,\pi)) := \mathbf{h}(S,\mathbf{C}_r(p,\pi),\pi)$. If $E = \mathbf{B}_r(p)$, T is as in Assumption 1 and $p \in \Sigma$ (in the cases (a) and (c) of Definition 1.1), then $\mathbf{h}(T,\mathbf{B}_r(p)) := \mathbf{h}(T,\mathbf{B}_r(p),\pi)$ where π gives the minimal height among all π for which $\mathbf{E}(T,\mathbf{B}_r(p),\pi) = \mathbf{E}(T,\mathbf{B}_r(p))$ (and such that $\pi \subset T_p\Sigma$ in case (a) and (c) of Definition 1.1). Moreover, for such π we say that it optimizes the excess and the height in $\mathbf{B}_r(p)$.

We are now ready to define the dyadic decomposition of $\mathfrak{B}\setminus 0$.

Definition 6.2 (Refining procedure)**.** We build inductively the families of squares $\mathscr{S}, \mathscr{W} = \mathscr{W}_e \cup \mathscr{W}_h \cup \mathscr{W}_n$ and their subfamilies $\mathscr{S}^k = \mathscr{S}\cap\mathscr{C}^k$, $\mathscr{S}^{k,j} = \mathscr{S}\cap\mathscr{C}^{k,j}$ and so on. First of all, we set $\mathscr{S}^k = \mathscr{W}^k = \emptyset$ if $k < N_0$. For $k \geqslant N_0$ we use a double induction. Having defined $\mathscr{S}^{k'}, \mathscr{W}^{k'}$ for all $k' < k$ and $\mathscr{S}^{k,j'}, \mathscr{W}^{k,j'}$ for all $j' < j$, we pick all squares L of $\mathscr{C}^{k,j}$ which do not have any ancestor in $\mathscr{W}$ and we proceed as follows.

(EX) We assign L to $\mathscr{W}_e^{k,j}$ if

$$\mathbf{E}(T_L,\mathbf{B}_L) > C_e\mathbf{m}_0 d(L)^{2\gamma_0-2+2\delta_1}\ell(L)^{2-2\delta_1}; \tag{6.7}$$

(HT) We assign L to $\mathscr{W}_h^{k,j}$ if we have not assigned it to $\mathscr{W}_e$ and

$$\mathbf{h}(T_L,\mathbf{B}_L) > C_h\mathbf{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}; \tag{6.8}$$

(NN) We assign L to $\mathscr{W}_n^{k,j}$ if we have not assigned it to $\mathscr{W}_e\cup\mathscr{W}_h$ and it intersects a square J already assigned to $\mathscr{W}$ with $\ell(J) = 2\ell(L)$.

(S) We assign L to $\mathscr{S}^{k,j}$ if none of the above occurs.

We finally set

$$\Gamma := ([-1,1]^2\times\mathbb{R}^2)\cap\mathfrak{B}\setminus\bigcup_{L\in\mathscr{W}} L = \{0\}\cup\bigcap_{k\geqslant N_0}\bigcup_{L\in\mathscr{S}^k} L. \tag{6.9}$$

Proposition 6.3 (Whitney decomposition)**.** *Let T, γ_0, β_2 and δ_1 be as in the Assumptions 3, 4 and 6. If $M_0 \geqslant C$, $N_0 \geqslant C(M_0)$, $C_e, C_h \geqslant C(M_0,N_0)$ (for suitably large constants) and ε_{41} is sufficiently small then:*

(i) $\ell(L) \leqslant 2^{-N_0+1}|z_L|$ $\forall L \in \mathscr{S}\cup\mathscr{W}$;

(ii) $\mathscr{W}^k = \emptyset$ *for all* $k \leqslant N_0+6$;

(iii) Γ *is a closed set and* $\mathrm{sep}(\Gamma,L) := \inf\{|x-x'| : x\in\Gamma, x'\in L\} \geqslant 2\ell(L)$ $\forall L\in\mathscr{W}$.

Moreover, the following estimates hold with $C = C(M_0,N_0,C_e,C_h)$:

$$\mathbf{E}(T_J,\mathbf{B}_J) \leqslant C_e\mathbf{m}_0 d(J)^{2\gamma_0-2+2\delta_1}\ell(J)^{2-2\delta_1} \qquad \forall J\in\mathscr{S}, \tag{6.10}$$

$$\mathbf{h}(T_J,\mathbf{B}_J) \leqslant C_h\mathbf{m}_0^{\frac14} d(J)^{\frac{\gamma_0}{2}-\beta_2}\ell(J)^{1+\beta_2} \qquad \forall J\in\mathscr{S}, \tag{6.11}$$

$$\mathbf{E}(T_H,\mathbf{B}_H) \leqslant C\,\mathbf{m}_0 d(H)^{2\gamma_0-2+2\delta_1}\ell(H)^{2-2\delta_1} \qquad \forall H\in\mathscr{W}, \tag{6.12}$$

$$\mathbf{h}(T_H,\mathbf{B}_H) \leqslant C\,\mathbf{m}_0^{\frac14} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2} \qquad \forall H\in\mathscr{W}. \tag{6.13}$$

6.1.3 Approximating functions and construction algorithm

We will see below that in (a suitable portion of) each $\mathbf{B}_L$ the current T_L can be approximated efficiently with a graph of a Lipschitz multiple-valued map. The average of the sheets of this approximating map will then be used as a local model for the center manifold.

Definition 6.4 (π-approximations)**.** Let $L \in \mathscr{S} \cup \mathscr{W}$ and π be a 2-dimensional plane. If $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi)$ fulfills the assumptions of Theorem 2.8 in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi)$, then the resulting map $f : B_{8r_L}(p_L, \pi) \to \mathcal{A}_Q(\pi^\perp)$ given by Theorem 2.8 is called a *π-approximation of* T_L *in* $\mathbf{C}_{8r_L}(p_L, \pi)$.

As in [20], we wish to find a suitable smoothing of the average of the π-approximation $\boldsymbol{\eta} \circ f$. However the smoothing procedure is more complicated in the case (b) of Definition 1.1: rather than smoothing by convolution, we need to solve a suitable elliptic system of partial differential equations. This approach can in fact be used in cases (a) and (c) as well. In several instances regarding case (a) and (c) we will have to manipulate maps defined on some affine space $q + \pi$ and taking value on $\pi^\perp$, where $q \in \Sigma$ and $\pi \subset T_q\Sigma$. In such cases it is convenient to introduce the following conventions: the maps will be regarded as maps defined on π (requiring a simple translation by q), the space $\pi^\perp$ will be decomposed into $\varkappa := \pi^\perp \cap T_q\Sigma$ and its orthogonal complement $T_q\Sigma^\perp$ and we will regard Ψ_q as a map defined on $\pi \times \varkappa$ and taking values in $T_q\Sigma^\perp$. Similarly, elements of $\pi^\perp$ will be decomposed as $(\xi, \eta) \in \varkappa \times T_q\Sigma^\perp$.

Lemma 6.5. *Let the assumptions of Proposition 6.3 hold and assume* $C_e \geqslant C^\star$ *and* $C_h \geqslant C^\star C_e$ *for a suitably large* $C^\star(M_0, N_0)$*. For each* $L \in \mathscr{W} \cup \mathscr{S}$ *we choose a plane* π_L *which optimizes the excess and the height in* $\mathbf{B}_L$*. For any choice of the other parameters, if* ε_{41} *is sufficiently small, then* $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_L)$ *satisfies the assumptions of Theorem 2.8 for any* $L \in \mathscr{W} \cup \mathscr{S}$*.*

Definition 6.6 (Smoothing)**.** Let L and π_L be as in Lemma 6.5 and denote by f_L the corresponding π_L-approximation. In case of Definition 1.1 (a)&(c) we let $\bar f(x) := \sum_i \left[\!\left[\mathbf{p}_{T_{p_L}\Sigma}(f_i) \right]\!\right]$ be the projection of f_L on the tangent $T_{p_L}\Sigma$, whereas in the other case (Definition 1.1(b)) we set $\bar f = f$. We let $\bar h_L$ be a solution (provided it exists) of

$$\begin{cases} \mathscr{L}_L \bar h_L = \mathscr{F}_L \\ \\ \bar h_L|_{\partial B_{8r_L}(p_L, \pi_L)} = \boldsymbol{\eta} \circ \bar f_L\,, \end{cases} \tag{6.14}$$

where $\mathscr{L}_L$ is a suitable second order linear elliptic operator with constant coefficients and $\mathscr{F}_L$ a suitable affine map: the precise expressions for $\mathscr{L}_L$ and $\mathscr{F}_L$ depend on a careful Taylor expansion of the first variations formulae and are given in Proposition 6.16. We then set $h_L(x) := (\bar h_L(x), \Psi_{p_L}(x, \bar h_L(x)))$ in case (a) and (c) and $h_L(x) = \bar h_L(x)$ in case (b). The map h_L is the *tilted interpolating function* relative to L.

In what follows we will deal with graphs of multivalued functions f in several system of coordinates. These objects can be naturally seen as currents $\mathbf{G}_f$ (see Section 3.2 of Part ii) and in this respect we will use extensively the notation and results of Section 3.2 (therefore $\mathrm{Gr}(f)$ will denote the "set-theoretic" graph).

Lemma 6.7. *Let the assumptions of Proposition 6.3 hold and assume $C_e \geqslant C^\star$ and $C_h \geqslant C^\star C_e$ (where $C^\star$ is the constant of Lemma 6.5). For any choice of the other parameters, if ε_{41} is sufficiently small the following holds. For any $L \in \mathscr{W} \cup \mathscr{S}$, there is a unique solution $\bar{h}_L$ of (6.14) and there is a smooth $g_L : B_{4r_L}(z_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$, where h_L is the tilted interpolating function of Definition 6.6. Using the charts introduced in Definition 2.10, the map g_L will be considered as defined on the ball $B_{4r_L}(z_L, w_L) \subset \mathfrak{B}$.*

The center manifold is defined by gluing together the maps g_L.

Definition 6.8 (Interpolating functions)**.** The map g_L in Lemma 6.5 will be called the *L-interpolating function*. Fix next a $\vartheta \in C_c^\infty\big([-\frac{17}{16}, \frac{17}{16}]^m, [0,1]\big)$ which is nonnegative and is identically 1 on $[-1,1]^m$. For each j let $\mathscr{P}^j := \mathscr{S}^j \cup \bigcup_{i=N_0}^{j} \mathscr{W}^i$ and for $L \in \mathscr{P}^j$ define $\vartheta_L((z,w)) := \vartheta(\frac{z-z_L}{\ell(L)})$. Set

$$\hat{\varphi}_j := \frac{\sum_{L\in\mathscr{P}^j} \vartheta_L g_L}{\sum_{L\in\mathscr{P}^j} \vartheta_L} \qquad \text{on } \{(z,w) \in \mathfrak{B} : z \in [-1,1]^2 \setminus \{0\}\} \tag{6.15}$$

and extend the map to 0 defining $\hat{\varphi}_j(0) = 0$. In case (b) of Definition 1.1 we set $\varphi_j := \hat{\varphi}_j$. In cases (a) and (c) we let $\bar{\varphi}_j(z,w)$ be the first $\bar{n}$ components of $\hat{\varphi}_j(z,w)$ and define $\varphi_j(z,w) = \big(\bar{\varphi}_j(z,w), \Psi(z, \bar{\varphi}_j(z,w))\big)$. φ_j will be called the *glued interpolation* at step j.

We now come to the first main theorem, which yields the surface which we call "branched center manifold" (again notice that for $\bar{Q} = 1$ there is certainly no branching, since the surface is a classical $C^{1,\alpha}$ graph). In the statement we will need to "enlarge" slightly dyadic squares: given $L \in \mathscr{C}$ let H be dyadic square of $\mathbb{R}^2$ so that L is a connected component of $\mathfrak{B} \cap (H \times \mathbb{C})$. Given $\ell < |z_L| = |z_H|$, we let H' be the closed euclidean square of $\mathbb{R}^2$ which has the same center as H and sides of length $2\ell(L)$, parallel to the coordinate axes. The square L' concentric to L and with sidelength $2\ell(L) = 2\ell$ is that connected component of $\mathfrak{B} \cap (H' \times \mathbb{C})$ which contains L.

Theorem 6.9. *Under the same assumptions of Lemma 6.5, the following holds provided ε_{41} is sufficiently small.*

(i) For $\kappa := \beta_2/4$ and $C = C(M_0, N_0, C_e, C_h)$ we have (for all j)

$$|\varphi_j(z,w)| \leqslant C\mathbf{m}_0^{\frac{1}{4}} |z|^{1+\frac{\gamma_0}{2}} \qquad \textit{for all } (z,w) \tag{6.16}$$

$$|D^l \varphi_j(z,w)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} |z|^{1+\gamma_0 - l} \qquad \textit{for } l = 1, \ldots, 3 \textit{ and } (z,w) \neq 0 \tag{6.17}$$

$$[D^3 \varphi_j]_{\mathcal{A}_j, \kappa} \leqslant C\mathbf{m}_0^{\frac{1}{2}} 2^{2j}\,. \tag{6.18}$$

(ii) The sequence φ_j stabilizes on every square $L \in \mathscr{W}$: more precisely, if $L \in \mathscr{W}^i$ and H is the square concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_k = \varphi_j$ on H for every $j, k \geqslant i+2$. Moreover there is an admissible smooth branching $\boldsymbol{\varphi} : \mathfrak{B} \cap ([-1,1]^2 \times \mathbb{C}) \to \mathbb{R}^n$ such that $\varphi_j \to \boldsymbol{\varphi}$ uniformly on $\mathfrak{B} \cap ([-1,1]^2 \times \mathbb{C})$ and in $C^3(\mathcal{A}_j)$ for every $j \geqslant 0$.

(iii) For some constant $C = C(M_0, N_0, C_e, C_h)$ and for $a' := b + \gamma_0 > b$ we have

$$|u(z,w) - \boldsymbol{\varphi}(z,w)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} |z|^{a'}\,. \tag{6.19}$$

Definition 6.10 (Center manifold, Whitney regions)**.** The manifold $\mathcal{M} := \mathrm{Gr}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi}$ is as in Theorem 6.9, is called *a branched center manifold for* T *relative to* $\mathbf{G}_u$. It is convenient to introduce the map $\mathbf{\Phi} : \mathfrak{B} \cap ([-1,1]^2 \times \mathbb{C}) \to \mathbb{R}^{2+n}$ given by $\mathbf{\Phi}(z,w) = (z, \boldsymbol{\varphi}(z,w))$. If we neglect the origin, $\mathbf{\Phi}$ is then a classical (C^3) parametrization of $\mathcal{M}$. $\mathbf{\Phi}(\Gamma)$ will be called the contact set. Moreover, to each $L \in \mathscr{W}$ we associate a *Whitney region* $\mathcal{L}$ on $\mathcal{M}$ as follows:

(WR) $\mathcal{L} := \mathbf{\Phi}(H \cap ([-1,1]^2 \times \mathbb{C}))$, where H is the square concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$.

6.2 TECHNICAL PRELIMINARIES

In this section we prove the two technical Lemmas 2.15 and 2.16.

Proof of Lemma 2.15. Consider $x_0 \in \pi_0$ with $2\rho = |x_0|$, a smooth C^2 function $\phi : B_\rho(x_0) \to \mathbb{R}^n$ and the open set $\mathbf{V}_\rho := \{(x,y) : x \in B_{\rho/2}(x_0), |y - \phi(x)| \leqslant \rho\}$. Recall that there is a geometric constant C such that, if $\rho \leqslant C/\|D^2\phi\|_{B_\rho(x_0)}$, then for each $p \in \mathbf{V}_\rho$ there is a unique nearest point $\mathbf{P}(p) \in \mathrm{Gr}(\phi)$ (which defines a C^1 map $\mathbf{P} : \mathbf{V}_\rho \to \mathrm{Gr}(\phi)$). In particular, if $\|D^2\phi\|_{B_\rho(x_0)} \leqslant C\rho^{\alpha-1}$, the existence of such point is guaranteed under the assumption that $\rho \leqslant c\rho^{1-\alpha}$ (where c is a, possibly small but positive, constant). Consider now an admissible smooth branching $u : \mathfrak{B}_{\bar{Q}} \to \mathbb{R}^n$. If $\bar{Q} = 1$, the above discussion shows easily the existence of a well defined C^1 map $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_{2r} \to \mathrm{Gr}(u)$, provided r is sufficiently small. If $\bar{Q} > 1$, the same conclusion holds under the assumption that u is b-separated and $a > b > 1$. Indeed consider $p = (z,y) \in \mathbf{V}_{u,a}$ and $(z,w_i) \in \mathfrak{B}_Q$ such that $|y - u(z,w_i)| \leqslant c_s|z|^a$. The assumptions of being well-separated implies easily that $|p - u(\zeta,\omega)| \geqslant c_s|z|^b$ whenever $z \notin B_{|z|/2}(z,w_i)$ and thus we can argue locally on the sheet $\mathrm{Gr}(u|_{B_{|z|/2}(z,w_i)})$.

Next, up to rescaling we can assume that $\mathbf{P}$ is well-defined on $\mathbf{V}_{u,a} \cap \mathbf{C}_2$. The discussion before Lemma 2.15 applies now verbatim and we conclude the first sentence of the Lemma.

To reach the other two conclusions of the Lemma we argue by contradiction: if they were wrong, then we would find a sequence of points $\{x_k\} \subset B_2(0)$ converging to 0 for which one of the following two conditions hold:

- either $\{x_k\} \times \mathbb{R}^n$ contains a point $p_k \in \mathrm{spt}(T)$ with $\Theta(p_k, T) \geqslant Q + \frac{1}{2}$;
- or one connected component Ω of $(\{x_k\} \times \mathbb{R}^n) \cap \mathbf{V}_{u,a}$ does not intersect $\mathrm{spt}(T)$.

Set $2r_k := |x_k|$ and consider the connected component $\mathbf{V}_k$ of $\mathbf{V}_{u,a} \cap \mathbf{C}_{r_k}(x_k)$ which contains p_k (in the first case) or Ω_k (in the second). Let $S_k := T_k \llcorner \mathbf{V}_k$ and let $q_k = (x_k, u(x_k, w_k))$ be such that $q_k \in \mathbf{V}_k$. Finally set $Z_k := (S_k)_{q_k,r_k}$. Observe that $\mathrm{spt}(Z_k)$ is contained in a neighborhood of height Cr_k^{a-1} of π_0 and we therefore conclude that Z_k converges to a current Z which is an integer multiple of $[\![B_1(0)]\!]$. On the other hand, since $\mathbf{P}_\sharp(S_k) \llcorner \mathbf{C}_{r_k/2}(x_k) = Q\mathbf{G}_u \llcorner \mathbf{C}_{r_k/2}(x_k)$ for k large enough, we conclude that $Z = Q\,[\![B_1(0)]\!]$. Now, either $\mathrm{spt}(Z_k) \cap (\{0\} \times \mathbb{R}^n)$ contains a point $\bar{q}_k$ of multiplicity $Q + \frac{1}{2}$ or it is empty. Since however $(\mathbf{p}_{\pi_0})_\sharp Z_k = Q_k\,[\![B_1(0)]\!] \to (\mathbf{p}_{\pi_0})_\sharp Z$ (by the constancy theorem), for k large enough we would have $(\mathbf{p}_{\pi_0})_\sharp Z_k = Q\,[\![B_1(0)]\!]$, contradic ting the emptyness of $\mathrm{spt}(Z_k) \cap (\{0\} \times \mathbb{R}^n) = \emptyset$ because $Q \geqslant 1$. As for the other alternative, we must have, by the almost minimality of Z_k (see Proposition 5.8)

$$\limsup_{k\to\infty} \|Z_k\|(\mathbf{B}_{1/2-|\bar{q}_k|}(\bar{q}_k)) \leqslant \lim_{k\to\infty} \|Z_k\|(\mathbf{B}_{1/2}(0)) = \tfrac{Q}{4}\omega_2\,.$$

Since $\bar{q}_k \to 0$, the almost monotonicity formula (see Proposition 5.8) would imply $\Theta(\bar{q}_k, Z_k) \leqslant Q + o(1)$. □

Proof of Lemma 2.16. Since $Q\bar{Q}[\![\pi_0]\!]$ is tangent to T at 0, we obviously must have $T_0\Sigma \supset \pi_0$ and thus $T_0\Sigma = \mathbb{R}^{2+\bar{n}} \times \{0\}$ can be achieved suitably rotating the coordinates. To achieve the other two conclusions we scale Σ and intersect it with $\mathbf{C}_4(0, T_0\Sigma)$ to reach that $\Sigma \cap \mathbf{C}_4(0, T_0\Sigma)$ is the graph of some Ψ with very small C^{3,ε_0} norm. We can then extend Ψ outside $B_4(0, T_0\Sigma)$ without increasing the C^{3,ε_0} norm by more than a factor: this gives (i) and (ii) and also shows that $\mathbf{c}$ can be assumed smaller than ε_{41} in case (a) and (c) of Definition 1.1. For the details we refer the reader to the proof of [20, Lemma 1.5]. The rest of the Lemma is a simple scaling argument. □

6.2.1 *Proof of Proposition 6.3*

In this section we prove several estimates on the excess, height and tilting of planes π_L in the cubes $L \in \mathscr{W} \cup \mathscr{S}$. Proposition 6.3 will then be a simple corollary of these more general statements.

Proposition 6.11 (Tilting of optimal planes). *Let T be as in Assumptions 3 and 4 and assume the various parameters satisfy Assumption 6. If $C_e, C_h \geqslant C(M_0, N_0)$ and ε_{41} is sufficiently small then:*

(i) The conclusions (i), (ii) and (iii) of Proposition 6.3 hold.

(ii) $\mathbf{B}_H \subset \mathbf{B}_L \subset \mathbf{B}_{d(L)/10}(p_L)$ *and* $T_H = T_L \llcorner \mathbf{V}_H$ *for all* $H, L \in \mathscr{W} \cup \mathscr{S}$ *with* $H \subset L$;

Moreover, if $H, L \in \mathscr{W} \cup \mathscr{S}$ *and either* $H \subset L$ *or* $H \cap L \neq \emptyset$ *and* $\frac{\ell(L)}{2} \leqslant \ell(H) \leqslant \ell(L)$, *then the following holds, for* $\bar{C} = \bar{C}(M_0, N_0, C_e)$ *and* $C = C(M_0, N_0, C_e, C_h)$:

(iii) $d(L)/2 \leqslant d(H) \leqslant 2d(L)$ *(and* $d(L) = d(H)$ *when* $H \subset L$*);*

(iv) $|\pi_H - \pi_L| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(L)^{\gamma_0 - 1 + \delta_1} \ell(L)^{1-\delta_1}$;

(v) $|\pi_H - \pi_0| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0}$;

(vi) $\mathbf{h}(T_H, \mathbf{C}_{36 r_H}(p_H, \pi_0)) \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}} \ell(H)$ *and* $\mathrm{spt}(T_H) \cap \mathbf{C}_{36 r_H}(p_H, \pi_0) \subset \mathbf{B}_H$;

(vii) $\mathbf{h}(T_L, \mathbf{C}_{36 r_L}(p_L, \pi_H)) \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}$ *and* $\mathrm{spt}(T_L) \cap \mathbf{C}_{36 r_L}(p_L, \pi_H) \subset \mathbf{B}_L$.

In particular, the estimates (6.12) *and* (6.13) *hold.*

The proof of the proposition will use repeatedly a few elementary observations concerning the excess and the height, which we collect in the following lemma.

Lemma 6.12. *If T is as in Proposition 6.11 there is a geometric constant C_0 with the following properties. Assume the points p, q belong to* $\mathrm{spt}(T) \cap \mathbf{C}_{\sqrt{2}}$, $\mathbf{B}_r(p) \subset \mathbf{B}_\rho(q) \subset \mathbf{C}_2$ *and* $r \geqslant \rho/4$. *Then, if* $\varepsilon_{41} \leqslant C_0^{-1}$

(i) $\mathbf{E}(T, \mathbf{B}_\rho(q)) \leqslant C_0 \min_\tau \mathbf{E}(T, \mathbf{B}_\rho(q), \tau) + C_0 \mathbf{m}_0 \rho^2$;

(ii) $\mathbf{E}(T, \mathbf{B}_r(p)) \leqslant C_0 \mathbf{E}(T, \mathbf{B}_\rho(q)) + C_0 \mathbf{m}_0 r^2$;

(iii) $|\pi - \tau|^2 \leqslant C_0[\mathbf{E}(T, \mathbf{B}_r(p), \pi) + \mathbf{E}(T, \mathbf{B}_\rho(q), \tau)]$;

(iv) $\mathbf{h}(T, F, \pi) \leqslant \mathbf{h}(T, F, \tau) + C_0|\pi - \tau| \mathrm{diam}(\mathrm{spt}(T) \cap F)$ *for any set* F;

(v) $\mathbf{h}(T, \mathbf{C}_r(0, \pi)) \leqslant C_0 \mathbf{m}_0^{\frac{1}{2}} r^{1+\gamma_0} + C_0|\pi - \pi_0| r$ *whenever* $|\pi - \pi_0| \leqslant C_0^{-1}$ *and* $r < 7/4$.

Proof. Recall that, by Lemma 6.21 and Allard's monotonicity formula (which can be applied by Proposition 2.2), we have

$$\frac{3\omega_2}{4} \rho^2 \leqslant \|T\|(\mathbf{B}_\rho(p)) \leqslant C_0 \rho^2 . \tag{6.20}$$

(i) is trivial in (b) of Definition 1.1, since $\mathbf{E}(T, \mathbf{B}_\rho(q)) = \min_\tau \mathbf{E}(T, \mathbf{B}_\rho(q), \tau)$. In the cases (a) and (c) recall that

$$\mathbf{E}(T, \mathbf{B}_\rho(q)) = \min_{\tau \subset T_q \Sigma} \mathbf{E}(T, \mathbf{B}_\rho(q), \tau) .$$

Let now π be such that $\mathbf{E}(T, \mathbf{B}_\rho(q), \pi) = \min_\tau \mathbf{E}(T, \mathbf{B}_\rho(q), \tau) =: E$. Then, by the Chebyshev inequality there is a point $q' \in \mathbf{B}_\rho(q) \cap \mathrm{spt}(T)$ such that

$$|\vec{T}(q') - \vec{\pi}|^2 \leqslant \frac{\omega_2 \rho^2}{\|T\|(\mathbf{B}_\rho(q))} E \leqslant C_0 E .$$

Observe that $\vec{T}(q')$ is the orienting 2-vector of some space $\xi \subset T_{q'}\Sigma$ and that

$$|T_{q'}\Sigma - T_q\Sigma|^2 \leqslant C_0 \|A_\Sigma\|^2_{C^0} \rho^2 \leqslant C_0 \mathbf{m}_0 \rho^2 .$$

Thus there is a 2-plane $\tau \subset T_q\Sigma$ such that $|\tau - \pi|^2 \leqslant CE + C_0 \mathbf{m}_0 \rho^2$. Hence

$$\mathbf{E}(T, \mathbf{B}_\rho(p)) \leqslant \mathbf{E}(T, \mathbf{B}_\rho(q), \tau) \leqslant C(E + C_0 \mathbf{m}_0 \rho^2)\|T\|(\mathbf{B}_\rho(q))/(\omega_2 \rho^2) \leqslant C_0 E + C_0 \mathbf{m}_0 \rho^2 .$$

Keeping the notation of the argument above, in the case (b) of Definition 1.1 statement (ii) follows from the simple observation

$$\mathbf{E}(T, \mathbf{B}_r(p)) \leqslant \mathbf{E}(T, \mathbf{B}_r(p), \pi) \leqslant 4^2 \mathbf{E}(T, \mathbf{B}_\rho(q), \pi) = 16 \mathbf{E}(T, \mathbf{B}_\rho(q)) .$$

In the cases (a) and (c) of Definition 1.1 we combine the same idea with (i).

(iii) is a simple consequence of

$$\begin{aligned} |\pi - \tau|^2 &\leqslant \frac{2}{\|T\|(\mathbf{B}_\rho(q))} \int_{\mathbf{B}_\rho(q)} (|\vec{T} - \vec{\pi}|^2 + |\vec{\tau} - \vec{T}|^2) d\|T\| \\ &\overset{(6.20)}{\leqslant} C_0(\mathbf{E}(T, \mathbf{B}_\rho(q), \pi) + \mathbf{E}(T, \mathbf{B}_\rho(q), \tau) , \end{aligned} \tag{6.21}$$

and $\mathbf{E}(T, \mathbf{B}_\rho(q), \pi) \leqslant 16 \mathbf{E}(T, \mathbf{B}_r(p), \pi)$. Next, for $p, q \in \mathrm{spt}(T) \cap F$ we compute

$$|\mathbf{p}_\pi^\perp(p - q)| \leqslant |\mathbf{p}_\tau^\perp(p - q)| + |(\mathbf{p}_\tau^\perp - \mathbf{p}_\pi^\perp)(p - q)| \leqslant \mathbf{h}(T, F, \tau) + C|\pi - \tau||p - q| .$$

Taking the supremum over $p, q \in F \cap \mathrm{spt}(T)$ we reach (iv).

We finally argue for (v). Fix $r < 7/8$, π with $|\pi - \pi_0| \leqslant C_0^{-1}$ and the cylinder $\mathbf{C} := \mathbf{C}_r(0, \pi)$. Observe that, by Assumption 3, for every $p = (x, y) \in \operatorname{spt}(T) \cap (\mathbb{R}^2 \times \mathbb{R}^n)$ we have $|y| \leqslant \varepsilon_{41}^{\frac{1}{2}} |x|^{1+\alpha} \leqslant \varepsilon_{41}^{\frac{1}{2}} |x|^{1+\gamma_0}$. It follows easily that, for a sufficiently small ε_{41} and a sufficiently large C_0, this implies that $\operatorname{spt}(T) \cap \mathbf{C} \subset \mathbf{C}_{8r/7}(0, \pi_0)$. Hence, $\mathbf{h}(T, \mathbf{C}, \pi_0) \leqslant \mathbf{h}(T, \mathbf{C}_{8r/7}(0, \pi_0)) \leqslant C_0 \mathbf{m}_0^{\frac{1}{2}} r^{1+\gamma_0}$. As a consequence $\operatorname{diam}(T \cap \mathbf{C}) \leqslant C_0 r$ and (v) follows from (iv). □

Proof of Proposition 6.11. In this proof we will use the following convention: geometric constants will be denoted by C_0 or c_0, constants depending upon M_0, N_0, C_e will be denoted by $\bar{C}$ or $\bar{c}$ and constants depending upon M_0, N_0, C_e and C_h will be denoted by C or c. Next observe that the second inclusion in (ii) is in fact correct for any cube $L \in \mathscr{C}^j$ with $j \geqslant N_0$, provided N_0 is chosen sufficiently large compared to M_0. Similarly (iii) holds for N_0 larger than a geometric constant.

Proof of (i), (ii) and (iii) in Proposition 6.3. The conclusion (i) is obvious since indeed it also holds for every $L \in \mathscr{C}^{N_0}$. (iii) is a simple consequence of the fact that, because of (NN) in the refining procedure, given any pair $H, L \in \mathscr{W}$ with nonempty intersection, $\frac{1}{2}\ell(H) \leqslant \ell(L) \leqslant 2\ell(H)$. Consider now any $L \in \mathscr{C}^j$ with $N_0 \leqslant j \leqslant N_0 + 6$. Observe first that $C(N_0)^{-1} d_L \leqslant \ell(L) \leqslant d_L$. We thus can use (2.24) to estimate

$$\mathbf{E}(T_L, \mathbf{B}_L, \pi(p)) \leqslant C(M_0, N_0) \mathbf{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2 - 2\delta_1} .$$

By Lemma 6.12(i) we conclude

$$\mathbf{E}(T_L, \mathbf{B}_L) \leqslant C(M_0, N_0) \mathbf{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2 - 2\delta_1} + C(M_0) \mathbf{m}_0 \ell(L)^2 .$$

Hence, for C_e sufficiently large, condition (EX) of Definition 6.2 cannot be a reason to stop the refinining procedure of any cube $L \in \mathscr{C}^j$ when $N_0 \leqslant j \leqslant n_0 + 6$.

Recall next the chosen plane π_L such that $\mathbf{E}(T_L, \mathbf{B}_L, \pi_L) = \mathbf{E}(T_L, \mathbf{B}_L)$ and $\mathbf{h}(T_L, \mathbf{B}_L) = \mathbf{h}(T_L, \mathbf{B}_L, \pi_L)$. By Lemma 6.12(iii) we easily conclude that

$$|\pi_L - \pi(p)| \leqslant C(M_0, N_0) C_e \mathbf{m}_0^{\frac{1}{2}} d(L)^{\gamma_0} .$$

On the other hand $|\pi(p) - \pi_0| \leqslant C_0 [Du]_{0,\alpha, B_{C_0 d(L)}} d(L)^\alpha \leqslant C_0 \mathbf{m}_0^{\frac{1}{2}} d(L)^{\gamma_0}$ and thus

$$|\pi_L - \pi_0| \leqslant C(M_0, N_0)\, C_e \mathbf{m}_0^{\frac{1}{2}} d(L)^{\gamma_0} \qquad \forall L \in \mathscr{C}^{N_0} . \tag{6.22}$$

Since

$$\mathbf{B}_L \subset \mathbf{C}_{d(L)/10}(p_L, \pi_0) \subset \mathbf{C}_{(2\sqrt{2} + \frac{1}{10}) d(L)}(0, \pi_0)$$

and $(2\sqrt{2} + \frac{1}{10}) d(L) \leqslant (2\sqrt{2} + \frac{1}{10}) \frac{1}{2} \leqslant \frac{3}{2}$, we infer from Lemma 6.12(v):

$$\mathbf{h}(T_L, \mathbf{B}_L) \leqslant \bar{C} \mathbf{m}_0^{\frac{1}{2}} d(L)^{1+\gamma_0} \leqslant \bar{C} \mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2} .$$

Thus, choosing C_h large depending upon M_0, N_0 and C_e, we conclude that condition (HT) in Definition 6.2 cannot be a reason to stop the refining procedure of a cube $L \in \mathscr{C}^j$ when $N_0 \leqslant j \leqslant N_0 + 6$.

This means that: for $k = N_0$ and $j = 0$ all cubes of $\mathscr{C}^{N_0,0}$ are refined (the condition (NN) is empty here). But then the same happens for $k = N_0$ and $j = 1$, since $\mathscr{W}^{N_0,0}$ is empty. Proceeding inductively we conclude this for every j and thus obtain that $\mathscr{W}^{N_0}$ is empty. We now repeat the argument with $\mathscr{W}^{N_0+1,j}$ to conclude that $\mathscr{W}^{N_0+1}$ is also empty. Proceeding for other 5 steps we conclude then that (ii) holds.

Proof of (ii)-(iv)-(v)-(vi)-(vii) when $H \subset L$. The proof is by induction over i, where $H \in \mathscr{C}^i$. We thus prove first the claims when $i = N_0$. Under this assumption $H = L$ and hence (iv) is trivial. The second inclusion in (ii) has already been proved above and the remaining assertions of (ii) are obvious because $H = L$. (v) has been shown above, cf. (6.22). The first conclusion in (vi) follows easily, since $\mathbf{h}(T_H, \mathbf{C}_{36r_H}(p_H, \pi_0)) \leqslant C_0 \mathbf{m}_0^{\frac{1}{2}} d(H)^{1+\gamma_0}$ by Lemma 6.12(v) and $\ell(H) \geqslant d(H)/C(N_0)$. The inclusion in (vi) follows then trivially from this bound when $\mathbf{m}_0 \leqslant \varepsilon_{41}$ is small enough, because $p_H \in \mathrm{spt}(T_H)$. As for (vii), recall that $L = H$ in our case. First observe that $|\pi_H - \pi_0| \leqslant C_0 C_e d(L)^{\gamma_0}$, simply by (6.22) (assuming $C_e \geqslant C(M_0, N_0)$). Thus we can apply Lemma 6.12(v): since $d(L)$ and $\ell(L)$ are comparable up to a constant $C(N_0)$, we conclude that $\mathbf{h}(T_L, \mathbf{C}_{36r_L}(p_L, \pi_H)) \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2} \ell(L)^{1+\beta_2}$. As we already argued for (vi), the inclusion is a consequence of the bound.

We now pass to the inductive step. Thus fix some $H_{i+1} \in \mathscr{S}^{i+1} \cup \mathscr{W}^{i+1}$ and consider a chain $H_{i+1} \subset H_i \subset \ldots \subset H_{N_0}$ with $H_l \in \mathscr{S}^l$ for $l \leqslant i$. We wish to prove all the conclusions (ii)-(iv)-(v)-(vi)-(vii) when $H = H_{i+1}$ and $L = H_j$ for some $j \leqslant i + 1$, recalling that, by inductive assumption, all the statements hold when $H = H_k$ and $L = H_l$ for $l \leqslant k \leqslant i$. Note also that $d(H_k) = d(H_{i+1})$ for all k.

With regard to (ii), it is enough to prove that $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$ and $\mathbf{V}_{H_{i+1}} \subset \mathbf{V}_{H_i}$. Note that $|z_{H_i} - z_{H_{i+1}}| \leqslant 2\sqrt{2}\,\ell(H_i)$ (recall the notation $p_H = (z_H, y_H)$). In particular notice that $\mathbf{C}_{r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{C}_{r_{H_i}}(p_{H_i}, \pi_0)$. Recall the open sets $\mathbf{V}_{H_i}$ and $\mathbf{V}_{H_{i+1}}$ defined in Section 6.1.2. Since H_i and H_{i+1} are nearby cubes in $\mathfrak{B}$, it is clear that $p_{H_{i+1}} = (z_{H_{i+1}}, u(z_{H_{i+1}}, w_{H_{i+1}}))$ and $p_{H_i} = (z_{H_i}, u(z_{H_i}, w_{H_i}))$ must be in the same connected component of $\mathbf{V}_{u,a} \cap \mathbf{C}_{r_{H_i}}(p_{H_i}, \pi_0)$. It then follows that $\mathbf{V}_{H_{i+1}} \subset \mathbf{V}_{H_i}$. In particular $p_{H_{i+1}} \in \mathrm{spt}(T_{H_i})$ and (vi) applied to $H = H_i$ implies then that $|p_{H_{i+1}} - p_{H_i}| \leqslant 2(\sqrt{2} + C\mathbf{m}_0^{\frac{1}{4}})\ell(H_{i+1})$. In particular, assuming that $\varepsilon_{41} \leqslant c$ for some positive constant $c = c(M_0, N_0, C_e, C_h)$, we conclude $|p_{H_{i+1}} - p_{H_i}| \leqslant 3\sqrt{2}\ell(H_i)$ and $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$ follows from the fact that M_0 is assumed larger than a suitable geometric constant.

We now come to (iv). Notice next that H_{i+1} is a son of H_i and thus H_i cannot belong to $\mathscr{W}$: it must therefore belong to $\mathscr{S}$. Hence, from the inclusion $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$, from the identity $T_{H_{i+1}} = T_{H_i} \llcorner \mathbf{B}_{H_{i+1}}$ and from Lemma 6.12(ii) we easily infer that

$$\mathbf{E}(T_{H_{i+1}}, \mathbf{B}_{H_{i+1}}) \leqslant C_0 \mathbf{E}(T_{H_i}, \mathbf{B}_{H_i}) + C_0 \mathbf{m}_0 \ell(H_{i+1})^2 \leqslant \bar{C}\mathbf{m}_0 d(H_{i+1})^{2\gamma_0-2+2\delta_1} \ell(H_{i+1})^{2-2\delta_1} .$$

We thus have, from Lemma 6.12(iii),

$$|\pi_{H_i} - \pi_{H_{i+1}}| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(H_{i+1})^{\gamma_0-1+\delta_1} \ell(H_{i+1})^{1-\delta_1} .$$

On the other hand, since $d(H_l) = d(H_j)$ for every $l \geqslant j$, by the same argument with l in place of i we also get

$$|\pi_{H_l} - \pi_{H_{l+1}}| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(H_{i+1})^{\gamma_0-1+\delta_1} \ell(H_{l+1})^{1-\delta_1} .$$

Summing the latter estimates for l between i and j, we easily reach (iv) for $H = H_{i+1}$ and $L = H_j$.

As for (v), note that it holds for H_{N_0} and moreover we just proved (iv) for $H = H_{i+1}$ and $L = H_{N_0}$, and thus, by triangular inequality, we get (v) (with a constant independent of the index i!).

As for (vi), note first that $\mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{C}_{36r_{H_i}}(p_{H_i}, \pi_0) \subset \mathbf{B}_{H_i}$ (the latter because (vi) holds for $H = H_i$ by inductive hypothesis). Thus we can apply Lemma 6.12(iv) to conclude

$$\mathbf{h}(T_{H_{i+1}}, \mathbf{C}_{36r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)) \leqslant C\mathbf{h}(T_{H_i}, \mathbf{B}_{H_i}) + C_0|\pi_{H_i} - \pi_0|\,\mathrm{diam}(\mathrm{spt}(T_{H_i}) \cap \mathbf{B}_{H_i})\,.$$

On the other hand we already noticed that $H_i \in \mathscr{S}$. Taking into account (v) we then conclude the inequality of (vi) for $H = H_{i+1}$ and, as already noticed in other cases, the inclusion follows from the estimate and $p_{H_{i+1}} \in \mathbf{B}_{H_{i+1}} \cap \mathrm{spt}(T_{H_{i+1}})$.

We finally come to (vii). Fix $H = H_{i+1}$. First we prove it for $L = H_{N_0}$. Observe that by the bound on $|\pi_H - \pi_0|$, we can bound $\mathbf{h}(T_{H_{N_0}}, \mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_H))$ with the same argument used for $\mathbf{h}(T_{H_{N_0}}, \mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}))$. As already argued several times, we then conclude the inclusion $\mathbf{C}_{36r_{H_{N_0}}}(p_{H_{N_0}}, \pi_H) \subset \mathbf{B}_{H_{N_0}}$. We now argue inductively on j: assuming that we know (vii) for H and $L = H_j$, we now wish to conclude it for $L = H_{j+1}$. Notice that $\mathbf{C}_{36r_{H_{j+1}}}(p_{H_{j+1}}, \pi_H) \subset \mathbf{C}_{36r_{H_j}}(p_{H_j}, \pi_H)$. Then the inductive assumption gives $\mathbf{C}_{36r_{H_{j+1}}}(p_{H_{j+1}}, \pi_H) \subset \mathbf{B}_{H_j}$ and recalling that $T_{H_{j+1}} = T_{H_j} \llcorner \mathbf{B}_{H_{j+1}}$ and that $H_j \in \mathscr{S}$, we can use Lemma 6.12(iv) to bound

$$\mathbf{h}(T_{H_{j+1}}, \mathbf{C}_{36r_{H_{j+1}}}(p_{H_{j+1}}, \pi_H)) \leqslant \mathbf{h}(T_{H_j}, \mathbf{B}_{H_j}) + C_0|\pi_H - \pi_{H_j}|\mathrm{diam}(\mathrm{spt}(T_{H_j}) \cap \mathbf{B}_{H_j})\,.$$

However, having already shown (iv), this easily shows the bound in (vii). The inclusion then follows with the usual argument used above.

Proof of (6.12) **and** (6.13)**.** Fix $H \in \mathscr{W}$ and let L be its father. Having shown (ii), we know that $\mathbf{B}_H \subset \mathbf{B}_L$. We then use $d(L) = d(H)$, $\ell(L) \leqslant 2\ell(H)$ the estimate

$$\mathbf{E}(T_L, \mathbf{B}_L) \leqslant C_e \boldsymbol{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2-2\delta_1}$$

and Lemma 6.12(i) to conclude (6.12) as follows

$$\mathbf{E}(T_H, \mathbf{B}_H) \leqslant \mathbf{E}(T_H, \mathbf{B}_H, \pi_L) + C\boldsymbol{m}_0 r_L^2 \leqslant \mathbf{E}(T_L, \mathbf{B}_L) + C\boldsymbol{m}_0 \ell(L)^2 \leqslant C\boldsymbol{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2-2\delta_1}$$

Next, we use Lemma 6.12, (iii), (iv) and

$$\mathbf{h}(T_L, \mathbf{B}_L) \leqslant C_h \boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}$$

to conclude (6.13).

Proof of (iv) and (vii) when H **and** L **are neighbors.** Without loss of generality assume $\ell(L) \geqslant \ell(H)$. If $L \notin \mathscr{C}^{N_0}$, then let J be the father of L. Observe that $|z_H - z_J|, |z_L - z_J| \leqslant 2\sqrt{2}\ell(J)$. On the other hand, observe that p_H, p_L are both elements of $\mathbf{C}_{36r_J}(p_J, \pi_0)$ (provided M_0 is larger than a geometric constant). Thus, by (vi) (applied to J), for ε_{41} sufficiently small we easily conclude $|p_H - p_J|, |p_L - p_J| \leqslant 3\sqrt{2}\ell(J)$. Since $\ell(L), \ell(H) \leqslant \ell(J)/2$, again assuming that

M_0 is larger than a geometric constant we have the inclusion $\mathbf{B}_H \cup \mathbf{B}_L \subset \mathbf{B}_J$. It is also easy to see that $\mathbf{V}_H \cup \mathbf{V}_L \subset \mathbf{V}_J$. Now, we can use (6.10), (6.12), (iii) and Lemma 6.12(ii) to achieve

$$|\pi_H - \pi_J|, |\pi_L - \pi_J| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(J)^{\gamma_0 - 1 + \delta_1} \ell(J)^{1-\delta_1}\,.$$

Next we use again (iii), the triangle inequality and $\ell(H) \leqslant \ell(L) \leqslant \ell(J) \leqslant 4\ell(H)$ to show (iv). The case $L \in \mathscr{C}^{N_0}$ can be handled similarly, just using a ball concentric to $\mathbf{B}_L$ and slightly larger so to include $\mathbf{B}_H$: the excess and the height in this ball is then estimated with the same argument used for estimating them in $\mathbf{B}_L$.

As for (vii) we fix a chain of ancestors $L = L_j, L_{j-1}, \ldots, L_i, \ldots\ldots, L_{N_0}$ and, as in the proof of (vii) for the case $H \subset L$, we argue inductively over i. The argument is precisely the same and can be applied because, using (iv) for H and L and for L_i and L_{i+1}, we can sum the corresponding estimate to show that

$$|\pi_H - \pi_{L_i}| \leqslant \bar{C}\mathbf{m}_0^{\frac{1}{2}} d(L_i)^{\gamma_0 - 1 + \delta_1} \ell(L_i)^{1-\delta_1}\,. \qquad \square$$

6.3 π-APPROXIMATIONS AND ELLIPTIC REGULARIZATIONS

In this section we introduce the π-approximations and define the corresponding elliptic regularizations of their averages, which in turn will be the building blocks of the center manifold. We begin with the following:

Proposition 6.13. *Assume the hypotheses and the conclusions of Proposition 6.11 apply and let ε_{41} be sufficiently small. If $H, L \in \mathscr{W} \cup \mathscr{S}$ and either $H \subset L$ or $H \cap L \neq \emptyset$ and $\frac{\ell(L)}{2} \leqslant \ell(H) \leqslant \ell(L)$, then*

$$(\mathbf{p}_{\pi_H})_\sharp(T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)) = Q\,[\![B_{32r_L}(\mathbf{p}_{\pi_H}(p_L), \pi_H)]\!]\,, \tag{6.23}$$
$$\partial T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H) = 0\,. \tag{6.24}$$

Moreover Theorem 2.8 applies to the current $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)$ in $\mathbf{C}_{32r_L}(p_L, \pi_H)$.

Proof. (6.24) is rather straightforward: by the height estimate in Proposition 6.11 we conclude easily $\mathrm{spt}(T_L) \cap \mathbf{C}_{32r_L}(p_L, \pi_H) \subset \mathbf{C}_{36r_L}(p_L, \pi_0)$. On the other hand by definition of $T_L = T \llcorner \mathbf{V}_L$ and by Assumption 3, we have $\mathrm{spt}(\partial T_L) \subset \partial \mathbf{C}_{64r_L}(p_L, \pi_0)$, implying $\mathrm{spt}(\partial T_L) \cap \mathbf{C}_{36r_L}(p_L, \pi_0) = \emptyset$ and thus also $\mathrm{spt}(\partial T_L) \cap \mathbf{C}_{32r_L}(p_L, \pi_H) = \emptyset$.

In order to prove (6.23) we argue as follows. First consider the chain of ancestors of $L := L = L_j \subset L_{j-1} \subset \ldots \subset L_{N_0} =: J$, where $J \in \mathscr{S}^{N_0}$. We first show that $(\mathbf{p}_{\pi_0})_\sharp(T_J \llcorner \mathbf{C}_{36r_J}(p_J, \pi_0)) = Q\,[\![B_{36r_J}(z_J, \pi_0)]\!]$. This is done in the following way: consider that $\mathrm{Gr}(u) \cap \mathbf{C}_{64r_J}(p_J, \pi_0)$ is the graph of a $C^{1,\alpha}$ function v with $\|v\|_{C^{1,\alpha}} \leqslant C_0 \mathbf{m}_0^{\frac{1}{2}}$. Define the function $v_t(x) := tv(x)$ and let $\mathbf{p}_t$ be the orthogonal projection onto $\mathrm{Gr}(v_t)$, which is well-defined on $\mathbf{V}_J$ provided $\mathbf{m}_0$ is sufficiently small (the smallness being independent of J). The currents $S_t := (\mathbf{p}_t)_\sharp(T_J \llcorner \mathbf{C}_{64r_J}(p_J, \pi_0))$ are easily seen to coincide with $Q_t \mathbf{G}_v \llcorner \mathbf{C}_{36r_J}(z_J, \pi_0)$ in the cylinder $\mathbf{C}_{36r_J}(p_J, \pi_0)$ by the constancy theorem. On the other hand such currents vary continuously and thus the integer Q_t must be constant. This implies that $Q_0 = Q_1 = Q$. On the other hand $\mathbf{p}_0 = \mathbf{p}_{\pi_0}$ and we have thus proved our claim.

Observe that $(\mathbf{p}_{\pi_0})_\sharp(T_L \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0)) = Q\,[\![B_{36r_L}(z_L, \pi_0)]\!]$ because $T_L \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0) = T_J \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0)$. Choose next a continuous path of planes π_t which connects π_0 and π_H

and satisfies the bound $|\pi_t - \pi_0| \leqslant C_0|\pi_H - \pi_0|$ for some geometric constant C_0. We then look at $Z_t = (\mathbf{p}_{\pi_t})_\sharp(T_L \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0))$ and conclude, similarly to the previous paragraph, that $((\mathbf{p}_{\pi_H})_\sharp(T_L \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0))) \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H) = [\![B_{32r_L}(\mathbf{p}_{\pi_H}(p_L), \pi_H)]\!]$. On the other hand since $(T_L \llcorner \mathbf{C}_{36r_L}(p_L, \pi_0))) \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H) = T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)$, this concludes the proof of (6.24).

Now, by the estimates of Proposition 6.11 in order to apply Theorem 2.8 we just need to choose ε_{41} sufficiently small. □

We next generalize slightly the terminology of Section 6.1.2.

Definition 6.14. Let H and L be as in Proposition 6.13. After applying Theorem 2.8 to $T_L \llcorner \mathbf{C}_{32r_L}(p_L, \pi_H)$ in the cylinder $\mathbf{C}_{32r_L}(p_L, \pi_H)$ we denote by f_{HL} the corresponding π_H-approximation. However, rather then defining f_{HL} on the disk $B_{8r_L}(p_L, \pi_H)$, by applying a translation we assume that the domain of f_{HL} is the disk $B_{8r_L}(p_{HL}, \pi_H)$ where $p_{HL} = p_H + \mathbf{p}_{\pi_H}(p_L - p_H)$. Note in particular that $\mathbf{C}_r(p_{HL}, \pi_H)$ equals $\mathbf{C}_r(p_L, \pi_H)$, whereas $B_{8r_L}(p_{HL}, \pi_H) \subset p_H + \pi_H$ and $p_h \in \mathbf{B}_{8r_L}(p_{HL}, \pi_H)$.

Observe that $f_{LL} = f_L$.

6.3.1 *First variations*

The next proposition is the core in the construction of the center manifold and it is the main reason behind the $C^{3,\alpha}$ estimate for the glued interpolation. It is also the place where our proof differs most from that of [20].

Definition 6.15. Let H and L be as in Proposition 6.13. In the cases (a) and (c) of Definition 1.1 we denote by $\varkappa_H$ the orthogonal complement in $T_{p_H}\Sigma$ of π_H and we denote by $\bar{f}_{HL}$ the map $\mathbf{p}_{\varkappa_H} \circ f_{HL}$.

In what follows we will consider elliptic systems of the following form. Given a vector valued map $v : p_H + \pi_H \supset \Omega \to \varkappa_H$ and after introducing an orthonormal system of coordinates x^1, x^2 on π_H and $y^1, \ldots, y^{\bar{n}}$ on $\varkappa_H$, the system is given by the $\bar{n}$ equations

$$\Delta v^k + \underbrace{(\mathbf{L}_1)^k_{ij}\partial_j v^i + (\mathbf{L}_2)^k_i v^i}_{=:\mathscr{E}^k(v)} = \underbrace{(\mathbf{L}_3)^k_i (x - x_H)^i + (\mathbf{L}_4)^k}_{=:\mathscr{F}^k}, \tag{6.25}$$

where we follow Einstein's summation convention and the tensors $\mathbf{L}_i$ have constant coefficients. After introducing the operator $\mathscr{L}(v) = \Delta v + \mathscr{E}(v)$ we summarize the corresponding elliptic system (6.25) as

$$\mathscr{L}(v) = \mathscr{F}\,. \tag{6.26}$$

We then have a corresponding weak formulation for $W^{1,2}$ solutions of (6.26), namely v is a weak solution in a domain D if the integral

$$\mathscr{I}(v, \zeta) := \int (Dv : D\zeta + (\mathscr{F}(v) - \mathscr{E}(v)) \cdot \zeta) \tag{6.27}$$

vanishes for smooth test functions ζ with compact support in D.

Proposition 6.16. *Let* H *and* L *be as in Proposition 6.13 (including the possibility that* $H = L$*) and let* f_{HL}, $\bar f_{HL}$ *and* $\varkappa_H$ *be as in Definition 6.14 and Definition 6.15. Then, there exist tensors with constant coefficients* $\mathbf{L}_1, \ldots, \mathbf{L}_4$ *and a constant* $C = C(M_0, N_0, C_e, C_h)$*, with the following properties:*

(i) The tensors depend upon H *and* Σ *(in the cases (a) and (c) of Definition 1.1) or* ω *(in case (b) of Definition 1.1) and* $|\mathbf{L}_1| + |\mathbf{L}_2| + |\mathbf{L}_3| + |\mathbf{L}_4| \leqslant C\mathbf{m}_0^{\frac12}$.

(ii) If $\mathscr{I}_H$, $\mathscr{L}_H$ *and* $\mathscr{F}_H$ *are defined through* (6.25), (6.26) *and* (6.27)*, then*

$$\mathscr{I}_H(\boldsymbol{\eta} \circ \bar f_{HL}, \zeta) \leqslant C\mathbf{m}_0\, d(H)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{4+\beta_2}\, \|D\zeta\|_0 \tag{6.28}$$

for all $\zeta \in C_c^\infty(B_{8r_L}(p_{HL}, \pi_H), \varkappa_H)$.

Proof. Set for simplicity $\pi = \pi_H$, $\varkappa := \varkappa_H$ $r = r_L$, $p = p_{HL}$, $f = f_{HL}$, $B = B_{8r}(p, \pi)$ and $T = T_L$.

Cases (a) and (b) of Definition 1.1. The proof is very similar to the one of [20, Proposition 5.2]. Nevertheless, for the sake of completeness, we give here all the details. We fix a system of coordinates $(x, y, w) \in \pi \times \varkappa \times (T_{p_H}\Sigma)^\perp$ so that $p_H = (0,0,0)$. We drop the subscript p_H for the map Ψ_{p_H}. Recall that

$$\Psi(0,0) = 0, \quad D\Psi(0,0) = 0 \quad \text{and} \quad \|D\Psi\|_{C^{2,\varepsilon_0}} \leqslant C\mathbf{m}_0^{\frac12}.$$

Let $\zeta \in C_c(B_{8r}(p, \pi), \varkappa)$ be a test function. We consider the vector field $\chi : \Sigma \to \mathbb{R}^{2+n}$ given by $\chi(q) = (0, \zeta(x), D_y\Psi(x,y)\cdot\zeta(x))$ for every $q = (x, y, \Psi(x,y)) \in \Sigma$. Note that χ is tangent to Σ. Therefore we infer that $\delta T(\chi) = 0$ and

$$|\delta \mathbf{G}_f(\chi)| \leqslant |\delta \mathbf{G}_f(\chi) - \delta T(\chi)| \leqslant C \int_{\mathbf{C}_{8r}(p,\pi)} |D\chi|\, d\|\mathbf{G}_f - T\|\,. \tag{6.29}$$

Observe also that $|\chi| \leqslant C|\zeta|$ and $|D\chi| \leqslant C|\zeta| + C|D\zeta| \leqslant C|D\zeta|$. Set $E := \mathbf{E}\big(T, \mathbf{C}_{32r}(p,\pi)\big)$. By Proposition 6.11, $\mathbf{C}_{32r}(p,\pi) \subset \mathbf{B}_L$. Thus, by Proposition 6.3 and Proposition 6.11(iv) we have

$$E \leqslant C\mathbf{m}_0 d(H)^{2\gamma_0 - 2 + 2\delta_1} \ell(L)^{2-2\delta_1}\,. \tag{6.30}$$

Similarly

$$\mathbf{h}(T, \mathbf{C}_{32r}(p,\pi)) \leqslant C\mathbf{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}\,. \tag{6.31}$$

Recall that, by Theorem 2.8 we have

$$|Df| \leqslant CE^{\beta_0} + C\mathbf{m}_0 r \leqslant C\mathbf{m}_0^{\beta_0}\, d(L)^{(2\gamma_0 - 2 + 2\delta_1)\beta_0} r^{\beta_0(2-2\delta_1)} \tag{6.32}$$

$$|f| \leqslant C\mathbf{h}(T, \mathbf{C}_{32r}(p,\pi)) + (E^{\frac12} + r\,\mathbf{m}_0^{\frac12})r \leqslant C\mathbf{m}_0^{\frac14}\, d(L)^{\frac{\gamma_0}{2} - \beta_2} r^{1+\beta_2}, \tag{6.33}$$

$$\int_B |Df|^2 \leqslant C\, r^2 E \leqslant C\mathbf{m}_0\, d(L)^{2\gamma_0 - 2 + 2\delta_1}\, r^{4 - 2\delta_1}, \tag{6.34}$$

and

$$|B \setminus K| \leqslant C\mathbf{m}_0^{1+\beta_0}\, d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} r^{2+(1+\beta_0)(2-2\delta_1)}\,, \tag{6.35}$$

$$\left| \|T\|(\mathbf{C}_{8r}(p_L,\pi)) - |B| - \frac{1}{2}\int_B |Df|^2 \right| \leqslant C\mathbf{m}_0^{1+\beta_0}\, d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} r^{2+(1+\beta_0)(2-2\delta_1)}\,, \tag{6.36}$$

where $K \subset B$ is the set

$$B \setminus K = \mathbf{p}_\pi\left((\mathrm{spt}(T)\Delta \mathrm{spt}(\mathbf{G}_f)) \cap \mathbf{C}_{8r_L}(p_L,\pi)\right)\,. \tag{6.37}$$

Writing $f = \sum_i [\![f_i]\!]$ and $\bar f = \sum_i [\![\bar f_i]\!]$, since $\mathrm{Gr}(f) \subset \Sigma$, we have $f = \sum_i [\![(\bar f_i, \Psi(x,\bar f_i))]\!]$. From Theorem 3.52 we can infer that

$$\begin{aligned}\delta\mathbf{G}_f(\chi) = \int_B \sum_i \Big(&\underbrace{D_{xy}\Psi(x,\bar f_i)\cdot\zeta}_{(A)} + \underbrace{(D_{yy}\Psi(x,\bar f_i)\cdot D\bar f_i)\cdot\zeta}_{(B)} + \underbrace{D_y\Psi(x,\bar f_i)\cdot D_x\zeta}_{(C)}\Big)\\ &: \Big(\underbrace{D_x\Psi(x,\bar f_i)}_{(D)} + \underbrace{D_y\Psi(x,\bar f_i)\cdot D\bar f_i}_{(E)}\Big) + \int_B \sum_i D\zeta : D\bar f_i + \mathrm{Err}\,,\end{aligned} \tag{6.38}$$

where, the error term Err in (6.38) satisfies the inequality

$$\begin{aligned}|\mathrm{Err}| &\leqslant C\int |D\chi||Df|^3 \leqslant \|D\zeta\|_{L^\infty}\int |Df|^3\\ &\leqslant C\|D\zeta\|_0 \mathbf{m}_0^{1+\beta_0}\, d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} r^{4-2\delta_1+\beta_0(2-2\delta_1)}\,.\end{aligned} \tag{6.39}$$

The second integral in (6.38) is $Q\int_B D\zeta : D(\boldsymbol{\eta}\circ\bar f)$. We therefore expand the product in the first integral and estimate all terms separately, using the Taylor expansion

$$D\Psi(x,y) = D_x D\Psi(0,0)\cdot x + D_y D\Psi(0,0)\cdot y + O\big(\mathbf{m}_0^{\frac{1}{2}}(|x|^2+|y|^2)\big)$$

so that

$$\begin{aligned}&|D\Psi(x,\bar f_i)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} r\\ &D\Psi(x,\bar f_i) = D_x D\Psi(0,0)\cdot x + O\big(\mathbf{m}_0^{\frac{1}{2}+\frac{1}{4}} d(L,0)^{\frac{\gamma_0}{2}-\beta_2} r^{1+\beta_2}\big),\\ &|D^2\Psi(x,\bar f_i)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} \quad \text{and} \quad D^2\Psi(x,\bar f_i) = D^2\Psi(0,0) + O\big(\mathbf{m}_0^{\frac{1}{2}} r\big)\,.\end{aligned}$$

We compute as follows:

$$\begin{aligned}\int\sum_i (A):(D) &= \int\sum_i (D_{xy}\Psi(0,0)\cdot\zeta) : D_x\Psi(x,\bar f_i) + O\Big(\mathbf{m}_0\, r^2\int|\zeta|\Big)\\ &= \int Q(D_{xy}\Psi(0,0)\cdot\zeta) : (D_{xx}\Psi(0,0)\cdot x)\\ &\quad + O\Big(\mathbf{m}_0\, d(L,0)^{\frac{\gamma_0}{2}-\beta_2}\, r^{1+\beta_2}\int|\zeta|\Big).\end{aligned} \tag{6.40}$$

The integral in (6.40) has the form $\int \mathbf{L}_{AD}\, x \cdot \zeta$. Next, we estimate

$$\int \sum_i \big((A):(E)+(B):(D)+(B):(E)\big)$$
$$= O\Big(\mathbf{m}_0^{1+\beta_0} d(L)^{\beta_0(2\gamma_0-2+2\delta_1)} r^{1+\beta_0(2-2\delta_1)} \int |\zeta|\Big) \tag{6.41}$$

and

$$\int \sum_i (C):(E) = O\Big(\mathbf{m}_0^{1+\beta_0} d(L)^{\beta_0(2\gamma_0-2+2\delta_1)} r^{2+\beta_0(2-2\delta_1)} \int |D\zeta|\Big). \tag{6.42}$$

Finally we compute

$$\begin{aligned}\int \sum_i (C):(D) &= \int \sum_i ((D_{xy}\Psi(0,0)\cdot x)\cdot D_x\zeta) : D_x\Psi(x,\bar{f}_i)\\
&\quad + O\Big(\mathbf{m}_0\, d(L)^{\frac{\gamma_0}{2}-\beta_2} r^{2+\beta_2} \int |D\zeta|\Big)\\
&= Q\int (D_{xy}\Psi(0,0)\cdot x)\cdot D_x\zeta) : (D_{xx}\Psi(0,0)\cdot x)\\
&\quad + O\Big(\mathbf{m}_0\, d(L)^{\frac{\gamma_0}{2}-\beta_2} r^{2+\beta_2} \int |D\zeta|\Big).\end{aligned}$$

Integrating by parts in the last integral we reach

$$\int \sum_i (C):(D) = \int \mathbf{L}_{CD}\, x\cdot\zeta + O\Big(\mathbf{m}_0\, d(L,0)^{\frac{\gamma_0}{2}-\beta_2} r^{2+\beta_2} \int |D\zeta|\Big). \tag{6.43}$$

Set next $\mathbf{L}_3 := \mathbf{L}_{AD} + \mathbf{L}_{CD}$. Clearly $\mathbf{L}_3$ is a quadratic function of $D^2\Psi(0,0)$, i.e. a quadratic function of the tensor A_Σ at the point p_H. From (6.29), (6.39), (6.40) – (6.43), we infer (6.28) and (i). Indeed we have to compare the following three types of errors

$$\mathcal{E}_1 := \mathbf{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} r^{4-2\delta_1+\beta_0(2-2\delta_1)} \tag{6.44}$$
$$\mathcal{E}_2 := \mathbf{m}_0^{1+\beta_0} d(L)^{\beta_0(2\gamma_0-2+2\delta_1)} r^{4+\beta_0(2-2\delta_1)} \tag{6.45}$$
$$\mathcal{E}_3 := \mathbf{m}_0 d(L)^{\frac{\gamma_0}{2}-\beta_2} r^{4+\beta_2}. \tag{6.46}$$

It is easy to see that if

$$-2\delta_1 + \beta_0(2-2\delta_1) - \beta_2 > 0 \tag{6.47}$$

then

$$\begin{aligned}\mathcal{E}_2 \leqslant \mathcal{E}_1 &\leqslant \mathbf{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)-2\delta_1+\beta_0(2-2\delta_1)-\beta_2} r^{4+\beta_2}\\
&\leqslant \mathbf{m}_0^{1+\beta_0} d(L)^{2(1+\beta_0)\gamma_0-2-\beta_2} r^{4+\beta_2}\end{aligned} \tag{6.48}$$

Therefore

$$\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \leqslant \mathbf{m}_0 d(L)^{2(1+\beta_0)\gamma_0-2-\beta_2} r^{4+\beta_2}. \tag{6.49}$$

To conclude the proof we observe that, by the bound on E,

$$\int_{\mathbf{C}_{8r}(p,\pi)} |D\chi|\, d\|\mathbf{G}_f - T\| \leqslant C\|D\zeta\|_0 \mathbf{M}(T\llcorner \mathbf{C} - \mathbf{G}_f) \leqslant C_0\|D\zeta\|_0 r^2 E^{\beta_0}(E + \mathbf{m}_0 r^2) \leqslant C\mathcal{E}_2\,.$$

Case (c) of Definition 1.1. Fix coordinates $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^n$ such that $p_H = (0, 0)$. Consider the vector field $\chi(x, y) := (0, \zeta(x))$ for some ζ as in the statement. Recalling Proposition 2.2 we infer

$$\delta\mathbf{G}_f(\chi) = \delta T(\chi) + \mathrm{Err}_0 = T(d\omega \lrcorner \chi) + \mathrm{Err}_0 = \mathbf{G}_f(d\omega \lrcorner \chi) + \mathrm{Err}_0 + \mathrm{Err}_1$$

with

$$\begin{aligned}
|\mathrm{Err}_0 + \mathrm{Err}_1| &= |\delta T(\chi) - \delta\mathbf{G}_f(\chi)| + \left|T(d\omega \lrcorner \chi) - \mathbf{G}_f(d\omega \lrcorner \chi)\right| \\
&\leqslant C\left(\|D\zeta\|_0 + \|d\omega \lrcorner \chi\|_0\right)\|T - \mathbf{G}_f\|(\mathbf{C}_{8r}(p, \pi)) \\
&\leqslant C\left(\|D\zeta\|_0 + \|\zeta\|_0\right) E^{\beta_0}\,(E + r^2\,\mathbf{m}_0)\, r^2 \\
&\leqslant C\,\|D\zeta\|_0\, \mathbf{m}_0^{1+\beta_0}\, d(H)^{(2\gamma_0-2+2\delta_1)(1+\beta_0)}\, r^{2+(2-2\delta_1)(1+\beta_0)}.
\end{aligned} \tag{6.50}$$

From Theorem 3.52

$$\delta\mathbf{G}_f(\chi) = Q\int D(\boldsymbol{\eta}\circ f) : D\zeta + \mathrm{Err}_2$$

with

$$\begin{aligned}
|\mathrm{Err}_2| &\leqslant C\int |D\zeta|\,|Df|^3 \leqslant C\,\|D\zeta\|_0 E^{1+\beta_0}\, r^2 \\
&\leqslant C\,\|D\zeta\|_0\, \mathbf{m}_0^{1+\beta_0}\, d(H)^{(2\gamma_0-2+2\delta_1)(1+\beta_0)}\, r^{2+(2-2\delta_1)(1+\beta_0)}.
\end{aligned}$$

Next we proceed to expand $\mathbf{G}_f(dw \lrcorner \chi)$. To this aim we write

$$\begin{aligned}
d\omega(x, y) = &\sum_{l=1}^{n} a_l(x, y)\, dy^l \wedge dx^1 \wedge dx^2 + \sum_{j=1,2}\sum_{l<k} b_{lk,j}(x, y)\, dy^l \wedge dy^k \wedge dx^j \\
&+ \sum_{l<k<j} c_{lkj}(x, y)\, dy^l \wedge dy^k \wedge dy^j
\end{aligned} \tag{6.51}$$

and get

$$d\omega \lrcorner \chi = \underbrace{\sum_{l=1}^{n} a_l\, \zeta^l\, dx^1 \wedge dx^2}_{\omega^{(1)}} + \underbrace{\sum_{j=1,2}\sum_{l<k} b_{lk,j}\, \zeta^l dy^k \wedge dx^j}_{\omega^{(2)}} + \underbrace{\sum_{l<k<j} c_{lkj}\, \zeta^l\, dy^k \wedge dy^j}_{\omega^{(3)}}\,. \tag{6.52}$$

We consider separately $\mathbf{G}_f(\omega^{(1)}), \mathbf{G}_f(\omega^{(2)}), \mathbf{G}_f(\omega^{(3)})$. We start with the latter

$$\mathbf{G}_f(\omega^{(3)}) \leqslant C\,\|d\omega\|_0\,\|\zeta\|_0 \int_B |Df|^2 \leqslant C\,\mathbf{m}_0^2\, d(H)^{2\gamma_0-2+2\delta_1}\, r^{5-2\delta_1}\|D\zeta\|_0. \tag{6.53}$$

Next

$$\begin{aligned}
\mathbf{G}_f(\omega^{(2)}) &= \sum_{l<k}\sum_{i=1}^{Q_2}\int \zeta^l(x)\,\left(b_{lk,2}(x,f_i(x))\,\frac{\partial f_i^k}{\partial x^1} - b_{lk,1}(x,f_i(x))\,\frac{\partial f_i^k}{\partial x^2}\right)dx\\
&= Q_2\sum_{l<k}\int \zeta^l(x)\,\left(b_{lk,2}(0,0)\,\frac{\partial(\boldsymbol{\eta}\circ f)^k}{\partial x^1} - b_{lk,1}(0,0)\,\frac{\partial(\boldsymbol{\eta}\circ f)^k}{\partial x^2}\right)dx + \mathrm{Err}_3,\\
&= \int \mathbf{L}_1\, D(\boldsymbol{\eta}\circ f)\cdot\zeta + \mathrm{Err}_3 \qquad (6.54)
\end{aligned}$$

with

$$\begin{aligned}
|\mathrm{Err}_3| &\leqslant C\|\zeta\|_0\,\|D(d\omega)\|_0\int_B \left(r\,|Df| + |f|\,|Df|\right)\,dx\\
&\leqslant C\|D\zeta\|_0\,\mathbf{m}_0\left(r + \mathrm{osc}(f) + \mathbf{h}(T,\mathbf{C}_{8r}(0,\pi))\right) r^3\,E^{\beta_0}\\
&\leqslant C\|D\zeta\|_0\,\mathbf{m}_0^{1+\beta_0}\,r^{4+(2-2\delta_1)\beta_0}\,d(H)^{(2\gamma_0-2+2\delta_1)\beta_0} \qquad (6.55)
\end{aligned}$$

and $\mathbf{L}_1 : \mathbb{R}^{n\times 2}\to\mathbb{R}^n$ given by

$$\mathbf{L}_1 A\cdot e_l := Q\sum_{k=1}^{n}\left(b_{lk,2}(0,0)\,A_{k1} - b_{lk,1}(0,0)\,A_{k2}\right)\quad \forall\, A = (A_{kj})_{k=1,\dots,n}^{j=1,2}\in\mathbb{R}^{n\times 2}.$$

Finally

$$\begin{aligned}
\mathbf{G}_f(\omega^{(1)}) &= \sum_{l}\sum_{i=1}^{Q_2}\int \zeta^l(x)\,a_l(x,f_i(x))\,dx\\
&= Q\sum_{l}\int \zeta^l(x)\,\left(a_l(0,0) + D_x a_l(0,0)\cdot x + D_y a_l(0,0)\cdot(\boldsymbol{\eta}\circ f)\right)dx + \mathrm{Err}_4,\\
&= \int \left(\mathbf{L}_2\,(\boldsymbol{\eta}\circ f) + \mathbf{L}_3\,x + \mathbf{L}_4\right)\cdot\zeta + \mathrm{Err}_4 \qquad (6.56)
\end{aligned}$$

where $\mathbf{L}_2 : \mathbb{R}^n\to\mathbb{R}^n$, $\mathbf{L}_3 : \mathbb{R}^2\to\mathbb{R}^n$ $\mathbf{L}_4\in\mathbb{R}^n$ are given by

$$\mathbf{L}_2\,v\cdot e_l := \sum_{k=1}^{n}\frac{\partial a_l}{\partial y^k}(0,0)\,v^k\quad \forall\, v\in\mathbb{R}^n,\ \forall\, l = 1,\dots,n \qquad (6.57)$$

$$\mathbf{L}_3\,w\cdot e_l := \sum_{j=1}^{2}\frac{\partial a_l}{\partial x^j}(0,0)\,w^j\quad \forall\, w\in\mathbb{R}^n,\ \forall\, l = 1,\dots,n \qquad (6.58)$$

$$\mathbf{L}_4\cdot e_l := a_l(0,0)\quad \forall\, l = 1,\dots,n \qquad (6.59)$$

and arguing as above

$$|\mathrm{Err}_4| \leqslant C\|\zeta\|_0\,[D(d\omega)]_{\varepsilon_0}\int_B \left(r^{1+\varepsilon_0} + |f|^{1+\varepsilon_0}\right)\,dx \leqslant C\|D\zeta\|_0\,\mathbf{m}_0\,r^{4+\varepsilon_0}. \qquad (6.60)$$

In order to deduce (6.28) we need to compare

$$|\mathrm{Err}_0 + \mathrm{Err}_1 + \mathrm{Err}_2| \leqslant \|D\zeta\|_0 \mathcal{E}_1 \leqslant \|D\zeta\|_0 \boldsymbol{m}_0 d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2} r^{4+\beta_2}$$
$$\mathcal{E}_2 = C\, \boldsymbol{m}_0^{1+\beta_0}\, d(H)^{(2\gamma_0 - 2 + 2\delta_1)\beta_0}\, r^{4+(2-2\delta_1)\beta_0}$$
$$\mathcal{E}_{2.5} = C\, \boldsymbol{m}_0^2\, d(H)^{2\gamma_0 - 2 + 2\delta_1}\, r^{5-2\delta_1}$$
$$\mathcal{E}_4 = C\, \boldsymbol{m}_0\, r^{4+\varepsilon_0}.$$

As before, if (6.47) holds, then $\mathcal{E}_2 \leqslant \mathcal{E}_1$. Moreover, since $\mathcal{E}_4 \leqslant r^{4+\beta_2}$, to conclude (6.28) it is enough to observe that if

$$1 \geqslant \beta_0(2 - 2\delta_1) \tag{6.61}$$

then $0 > 2\gamma_0 - 2 + 2\delta_1 > (2\gamma_0 - 2 + 2\delta_1)(1+\beta_0)$ and $5 - 2\delta_1 > 2 + (2 - 2\delta_1)(1+\beta_0)$, so that

$$d(H)^{2\gamma_0 - 2 + 2\delta_1}\, r^{5-2\delta_1} \leqslant d(H)^{(2\gamma_0 - 2 + 2\delta_1)(1+\beta_0)}\, r^{2+(2-2\delta_1)(1+\beta_0)},$$

that is $\mathcal{E}_{2.5\&4} \leqslant \mathcal{E}_1$. □

6.3.2 *Tilted interpolating functions,* L^1 *and* L^∞ *estimates*

In this subsection we generalize the definition of the tilted interpolating functions h_L. More precisely we consider

Definition 6.17. Let H and L be as in Proposition 6.13, assume that the conclusions of Proposition 6.16 applies and let $\mathscr{L}_H$ and $\mathscr{F}_H$ be the corresponding operator and map as given by Proposition 6.16 in combination with (6.25), (6.26) and (6.27). Let f_{HL} be as in Definition 6.14, $\varkappa_H$ and $\bar{f}_{HL}$ be as in Definition 6.15 and fix coordinates $(x, y, z) \in \pi_H \times \varkappa_H \times T_{p_H}\Sigma^\perp$ as in the proof of Proposition 6.16. We then let $\bar{h}_{HL}$ be the solution of

$$\begin{cases} \mathscr{L}_H \bar{h}_{HL} = \mathscr{F}_H \\ \\ \bar{h}_{HL}\big|_{\partial B_{8r_L}(p_{HL}, \pi_H)} = \boldsymbol{\eta} \circ \bar{f}_{HL}\,. \end{cases} \tag{6.62}$$

In case (b) of Definition 1.1 we then define $h_{HL} = \bar{h}_{HL}$, whereas in the other cases we define $h_{HL}(x) = (\bar{h}_{HL}(x), \Psi_{p_H}(x, \bar{h}_{HL}(x)))$.

In order to show that the maps $\bar{h}_{HL}$ are well defined, we need to show that there is a solution of the system (6.62).

Lemma 6.18. *Under the assumptions of Definition 6.17, if ε_{41} is sufficiently small, then the elliptic system*

$$\begin{cases} \mathscr{L}_H v = F \\ \\ v|_{\partial B_{8r_L}(p_{HL}, \pi_H)} = g\,. \end{cases} \tag{6.63}$$

has a unique solution for every $F \in W^{-1,2}$ and every $g \in W^{1,2}(B_{8r_L}(p_{HL}, \pi_H))$. Observe moreover that we have the estimate $\|Dv\|_{L^2} \leqslant C_0 r_L (\|F\|_{L^2} + \boldsymbol{m}_0^{\frac{1}{2}} \|g\|_{L^2}) + C_0 \|Dg\|_{L^2}$.

Proof. As for the first assertion, it suffices to show the Lemma for $g = 0$, since we can define $w = v - g$ and solve $\mathscr{L}_H(w) = F + \mathscr{L}_H(g)$. Setting $B = B_{8r_L}(p_{HL}, \pi_H)$, the existence and uniqueness for the latter case reduces, by Lax-Milgram, to the coercivity of the suitable quadatic form $\mathscr{Q}(v,v)$ on $W^{1,2}_0(B)$. The latter follows easily from

$$\begin{aligned}\mathscr{Q}(w,w) &:= \int (|Dw|^2 - \mathbf{L}_1\, Dw \cdot w - \mathbf{L}_2\, w\cdot w)\\ &\geqslant \|Dw\|^2_{L^2(B)} - \frac{|\mathbf{L}_1|}{2}\|Dw\|^2_{L^2(B)} - \left(\frac{|\mathbf{L}_1|}{2} + |\mathbf{L}_2|\right)\|w\|^2_{L^2(B)}\,.\end{aligned}$$

Since $r_L \leqslant 1$, by the Poincaré inequality $\|w\|^2_{L^2} \leqslant C_0\|Dw\|^2_{L^2}$ for every $w \in W^{1,2}_0(B)$. The coercivity follows then from $|\mathbf{L}_1| + |\mathbf{L}_2| \leqslant C\boldsymbol{m}_0^{\frac12} \leqslant C\varepsilon_{41}$, where the constant C depends only upon M_0, N_0, C_e and C_h. In particular we can assume the coercivity factor to be $\frac12$.

On the other hand, multiplying the equation by w and integrating by parts we easily see (using the coercivity) that

$$\begin{aligned}\frac12\int |Dw|^2 &\leqslant \int (|Dw||Dg| + |F||w|) + C\boldsymbol{m}_0^{\frac12}\int (|g||w| + |w||Dg|)\\ &\leqslant \frac14\int |Dw|^2 + \frac{r_L^2}{\gamma}\int |F|^2 + \frac{2\gamma}{r_L^2}\int |w|^2 + C\int \left(|Dg|^2 + \tfrac{\boldsymbol{m}_0}{\gamma} r_L^2|g|^2\right),\end{aligned}$$

where γ is any fixed positive number and C does not depend upon it.

We choose γ smaller than a geometric constant, so that we can use the Poincaré inequality to absorb the terms $\int |w|^2$ on the right hand side. We then conclude the desired estimate $\|Dw\|_{L^2} \leqslant C(\|Dg\|_{L^2} + \boldsymbol{m}_0^{\frac12} r_L\|g\|_{L^2} + r_L\|F_L\|_{L^2})$. Since $v = w + g$, we then conclude $\|Dv\|_{L^2} \leqslant C(\|Dg\|_{L^2} + \boldsymbol{m}_0^{\frac12} r_L\|g\|_{L^2} + Cr_L\|F_L\|_{L^2})$. □

Observe that $h_{HH} = h_H$. We next record three fundamental estimates, which regard, respectively, the L^∞ norms of derivatives of solutions of $\mathscr{L}_H(v) = F$, the L^∞ norm of $\bar h_{HL} - \boldsymbol{\eta}\circ \bar f_{HL}$ and the L^1 norm of $\bar h_{HL} - \boldsymbol{\eta}\circ\bar f_{HL}$.

Proposition 6.19. *Let* H *and* L *be as in Proposition 6.16 and assume the conclusions in there apply. Then the following estimates hold for a constant* $C = C(m_0, N_0, C_e, C_h)$ *for* $\hat B := B_{8r_L}(p_{HL}, \pi_H)$ *and* $\tilde B := B_{6r_L}(p_{HL}, \pi_H)$*:*

$$\|\bar h_{HL} - \boldsymbol{\eta}\circ\bar f_{HL}\|_{L^1(\hat B)} \leqslant C\boldsymbol{m}_0 d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\ell(L)^{5+\beta_2} \tag{6.64}$$

$$\|\bar h_{HL} - \boldsymbol{\eta}\circ\bar f_{HL}\|_{L^\infty(\tilde B)} \leqslant C\boldsymbol{m}_0 d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\ell(L)^{3+\beta_2} + C\boldsymbol{m}_0^{\frac12}\ell(L)^2\,. \tag{6.65}$$

Moreover, if $\mathscr{L}_H$ *is the operator of Proposition 6.16,* r *a positive number no larger than* 1 *and* v *a solution of* $\mathscr{L}_H(v) = F$ *in* $B_{8r}(q, \pi_H)$*, then*

$$\|v\|_{L^\infty(B_{6r}(q,\pi_H))} \leqslant \frac{C_0}{r^2}\|v\|_{L^1(B_{8r}(q,\pi_H))} + Cr^2\|F\|_{L^\infty(B_{8r}(q,\pi_H))} \tag{6.66}$$

and, for $l \in \mathbb{N}$

$$\|D^l v\|_{L^\infty(B_{6r}(q,\pi_H))} \leqslant \frac{C_0}{r^{2+l}}\|v\|_{L^1(B_{8r}(q,\pi_H))} + Cr^2\sum_{j=0}^{l} r^{j-l}\|D^j F\|_{L^\infty(B_{8r}(q,\pi_H))}, \tag{6.67}$$

where the latter constants depend also upon l*.*

Proof. **Proof of** (6.66). The estimate will be proved for a linear constant coefficient operator of the form $\mathscr{L} = \Delta + \mathbf{L}_1 \cdot D + \mathbf{L}_2$ when $\mathbf{L}_1$ and $\mathbf{L}_2$ are sufficiently small. We can then assume $\pi_H = \mathbb{R}^2$ and $q = 0$. Besides, if we define $u(x) := v(rx)$ we seee that u just satisfies $\Delta u + r\mathbf{L}_1 \cdot Du + r^2\mathbf{L}_2 \cdot u = 0$ and thus, without loss of generality, we can assume $r = 1$. We thus set $B = B_8(0) \subset \mathbb{R}^2$.

We recall the following interpolation estimate on the ball of radius 1, see [45, Theorem 1]. For $0 \leqslant j \leqslant m$ and $\frac{j}{m} \leqslant a \leqslant 1$ we have, for a constant $C_0 = C_0(m, j, q, r)$,

$$\|D^j u\|_{L^p(B_1)} \leqslant C\|D^m u\|^a_{L^s(B_1)} \|u\|^{1-a}_{L^q(B_1)} + C\,\|u\|_{L^q(B_1)}\,, \tag{6.68}$$

where

$$\tfrac{1}{p} = \tfrac{j}{2} + a\left(\tfrac{1}{s} - \tfrac{m}{2}\right) + (1-a)\tfrac{1}{q}\,.$$

We apply the estimate (6.68) for $j = 1$, $m = 2$, $q = 1$ and $p = s = 2$, $a = \frac{2}{3}$ and use Young's inequality and a simple scaling argument to achieve the inequality

$$\|Du\|_{L^2(B_\rho(x))} \leqslant C_0\rho\|D^2u\|_{L^2(B_\rho(x))} + C_0\rho^{-2}\|u\|_{L^1(B_\rho(x))}\,. \tag{6.69}$$

Moreover, by Sobolev embedding:

$$\|u\|_{L^2(B_\rho(x))} \leqslant C_0\rho\|Du\|_{L^2(B_\rho(x))} + C_0\rho^{-1}\|u\|_{L^1(B_\rho(x))}\,. \tag{6.70}$$

Next, recall the standard L^2 esimates for second order derivatives of solutions of the Laplace equations: if $B_{2\rho}(x) \subset B$, then

$$\|D^2u\|_{L^2(B_\rho(x))} \leqslant C_0\|\Delta u\|_{L^2(B_{2\rho}(x))} + C_0\rho^{-3}\|u\|_{L^1(B_{2\rho}(x))}\,. \tag{6.71}$$

Now, recall that $\Delta u = -\mathbf{L}_1 \cdot Du - \mathbf{L}_2 \cdot u + F$. Using the fact that $|\mathbf{L}_1| + |\mathbf{L}_2| \leqslant C_0\mathbf{m}_0^{\frac{1}{2}}$, we can combine all the inequalities above to conclude

$$\rho^6\|D^2u\|^2_{L^2(B_\rho(x))} \leqslant C_0\rho^6\mathbf{m}_0^{\frac{1}{2}}\|D^2u\|^2_{L^2(B_{2\rho}(x))} + C_0\|u\|^2_{L^1(B_8)} + C_0\|F\|^2_{L^\infty}\,. \tag{6.72}$$

Define next

$$S := \sup\{\rho^3\|D^2u\|_{L^2(B_\rho(x))} : B_{2\rho(x)} \subset B_8\} \tag{6.73}$$

and let ρ and ξ be such that $B_{2\rho}(\xi) \subset B_8$ and

$$\rho^3\|D^2u\|_{L^2(B_\rho(x))} \geqslant \frac{S}{2}\,. \tag{6.74}$$

We can cover $B_\rho(\xi)$ with N_0 balls $B_{\rho/2}(x_i)$ with $x_i \in B_\rho(\xi)$, where N_0 is only a geometric constant. We then can apply (6.72) to conclude that

$$\frac{S}{2} \leqslant C_0N_0\mathbf{m}_0^{\frac{1}{2}}S + C_0N_0\|u\|_{L^1(B_8)} + C_0N_0\|F\|_{L^\infty(B_8)}\,.$$

Therefore, when $\mathbf{m}_0^{\frac{1}{2}}$ is smaller than a geometric constant we conclude $S \leqslant C_0\|u\|_{L^1(B_8)} + C_0\|F\|_{L^\infty(B_8)}$. By definition of S, we have reached the estimate

$$\rho^3\|D^2u\|_{L^2(B_\rho(x))} \leqslant C_0\|u\|_{L^1(B_8)} + C_0\|F\|_{L^\infty(B_8)} \qquad \text{whenever } B_{2\rho}(x) \subset B_8.$$

Of course, with a simple covering argument, this implies

$$\|D^2u\|_{L^2(B_6)} \leqslant C_0\|u\|_{L^1(B_8)} + C_0\|F\|_{L^\infty(B_8)}\,. \tag{6.75}$$

Next, again using the interpolation inequality (6.69) we get

$$\|Du\|_{L^2(B_6)} \leqslant C_0\|u\|_{L^1(B_8)} + C_0\|F\|_{L^\infty(B_8)}\,.$$

So, by Sobolev embedding

$$\|Du\|_{L^4(B_6(0))} \leqslant C_0\|Du\|_{W^{1,2}(B_6)} \leqslant C_0\|u\|_{L^1(B_8(0))} + C_0\|F\|_{L^\infty(B_8(0))}\,.$$

Again using interpolation and Sobolev we finally achieve

$$\|u\|_{L^\infty(B_6)} \leqslant C_0\|u\|_{W^{1,4}(B_6)} \leqslant C_0\|u\|_{L^1(B_8(0))} + C_0\|F\|_{L^\infty(B_8(0))}\,.$$

Proof of (6.67)**.** As in the previous step, we can, without loss of generality, assume $r = 1$. Note that a byproduct of the argument given above is also the estimate

$$\|Du\|_{L^1(B_6)} \leqslant C_0\|u\|_{L^1(B_8)} + C_0\|F\|_{L^\infty(B_8)}\,.$$

In fact, by a simple covering and scaling argument one can easily see that

$$\|Du\|_{L^1(B_\tau)} \leqslant C_0(\tau)\|u\|_{L^1(B_8)} + C_0(\tau)\|F\|_{L^\infty(B_8)} \qquad \text{for every } \tau < 8.$$

We can then differentiate the equation and use the proof of the previous paragraph to show

$$\|Du\|_{L^\infty(B_\sigma)} \leqslant C_0(\sigma,\tau)\|Du\|_{L^1(B_\tau)} + C_0(\sigma,\tau)\|DF\|_{L^\infty(B_\tau)}\,.$$

Again, arguing as above, a byproduct of the proof is also the estimate

$$\|D^2u\|_{L^1(B_\sigma)} \leqslant C_0(\sigma,\tau)\|Du\|_{L^1(B_\tau)} + C_0(\sigma,\tau)\|DF\|_{L^\infty(B_\tau)}\,.$$

This can be applied inductively to get estimates for all higher derivatives.

Proof of (6.64)**.** Let $B := B_{8r_L}(p_{HL}, \pi_H)$. We use the coordinates introduced in the proof of Proposition 6.16. We set $w := \bar{h}_{HL} - \boldsymbol{\eta} \circ \bar{f}_{HL}$ and observe that

$$\begin{cases} \mathscr{L} w = \mathscr{F}_H - \mathscr{L}_H(\boldsymbol{\eta} \circ \bar{f}_{HL}) \\ \\ w|_{\partial B} = 0 \end{cases}$$

Next, for $1 < p < \infty$, we define the continuous (by Calderon-Zygmund theory) linear operator $T : L^p(B) \to W^{1,p}_0(B) \cap W^{2,p}$ by $T(g) = \psi$ if

$$\begin{cases} -\Delta\psi = g & \text{in } B \\ \\ \psi = 0 & \text{on } B. \end{cases}$$

Applying the Sobolev embedding $W_0^{1,3}(B) \hookrightarrow C^0(B)$ to the derivative of $\zeta \in W^{2,3} \cap W_0^{1,3}$ and using (6.28) we get

$$\int_B (Dw : D\zeta - \mathbf{L}_1\, Dw \cdot \zeta - \mathbf{L}_2\, w \cdot \zeta) \leqslant C\mathbf{m}_0\, d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{4+\beta_2}\, r_L^{1-\frac{2}{3}}\, \|D^2\zeta\|_{L^3}.$$

Then, we can estimate the $L^{\frac{3}{2}}$-norm of w as follows:

$$\begin{aligned}
\|w\|_{L^{\frac{3}{2}}(B)} &= \sup_{\|h\|_{L^3(B)}=1} \int_B w\, h = - \sup_{\|h\|_{L^3(B)}=1} \int_B w\, \Delta T(h) \\
&\leqslant \sup_{\|h\|_{L^3(B)}=1} \int_B D\, w \cdot DT(h) \\
&\leqslant C\mathbf{m}_0\, d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{5+\beta_2-\frac{2}{3}} \sup_{\|h\|_{L^3(B)}=1} \|D^2 T(h)\|_{L^3} \\
&\quad + \sup_{\|h\|_{L^3(B)}=1} \int_B (-\mathbf{L}_1\, Dw \cdot T(h) - \mathbf{L}_2\, w \cdot T(h))\,.
\end{aligned}$$

Recalling the Calderon-Zygmund estimates we have

$$\begin{aligned}
&\|D^2 T(h)\|_{L^3} \leqslant C_0 \|h\|_{L^3} \\
&\|DT(h)\|_{L^3} \leqslant C_0 r_L \|h\|_{L^3} \\
&\|T(h)\|_{L^3} \leqslant C_0 r_L^2 \|h\|_{L^3}\,.
\end{aligned}$$

Integrating by parts we then achieve

$$\begin{aligned}
\|w\|_{L^{\frac{3}{2}}(B)} &\leqslant C\mathbf{m}_0\, d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{5+\beta_2-\frac{2}{3}} + \sup_{\|h\|_{L^3(B)}=1} \int_B w \cdot (\mathbf{L}_1 DT(h) - \mathbf{L}_2 T(h)) \\
&\leqslant C\mathbf{m}_0\, d(L)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{5+\beta_2-\frac{2}{3}} + C\mathbf{m}_0^{\frac{1}{2}} \|w\|_{L^{3/2}(B)}\,.
\end{aligned}$$

Therefore, if $\mathbf{m}_0^{\frac{1}{2}}$ is sufficiently small, that is ε is sufficiently small, we deduce that

$$\|w\|_{L^1} \leqslant C\, r_L^{\frac{2}{3}} \|w\|_{L^{\frac{3}{2}}(B)} \leqslant C\, \mathbf{m}_0\, \mathrm{dist}(H)^{2(1+\beta_0)\gamma_0 - 2 - \beta_2}\, r_L^{5+\beta_2}.$$

Proof of (6.65)**.** The estimate follows easily from (6.64) and (6.66), recalling that $\|\mathscr{F}_H\|_0 \leqslant C\mathbf{m}_0^{\frac{1}{2}}$. □

6.4 MAIN ESTIMATES ON THE INTERPOLATING FUNCTIONS

In this section we adopt the terminology of the previous subsection and we show that

Proposition 6.20. *Assume the conclusions of Proposition 6.13 applies, let $\kappa := \frac{\beta_2}{4}$ and assume ε_{41} is sufficiently small, depending upon the other parameters. Then there exists a constant $C = C(M_0, N_0, C_e, C_h)$ such that for any cube $H \in \mathscr{W} \cup \mathscr{S}$, the following conclusions hold.*

(i) *Lemma 6.7 applies and thus* g_H *is well-defined.*

(ii) *The following estimates hold:*

$$\|h_H - \mathbf{p}^{\perp}_{\pi_H}(p_H)\|_{C^0(B_{6r_H}(p_H,\pi_H))} \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2} \tag{6.76}$$

$$\|g_H\|_{C^0} \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{1+\frac{\gamma_0}{2}} \tag{6.77}$$

$$\|Dg_H\|_{C^0} + d(H)\|D^2 g_H\|_{C^0} + d(H)^2\|D^3 g_H\|_{C^\kappa} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0} \tag{6.78}$$

$$\|g_H - u(z_H, w_H)\|_{C^0} \leqslant C\,\mathbf{m}_0^{\frac{1}{4}}\, d(H)^{\frac{\gamma_0}{2}}\,\ell(H) + c_s d(H)^a \tag{6.79}$$

$$|\pi_H - T_{(x,g_H(x))}\mathbf{G}_{g_H}| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\ell(H)^{1-\delta_1} \qquad \forall x \in B_{4r_H}(z_H, w_H)\,. \tag{6.80}$$

(iii) *If* $L \in \mathscr{W} \cup \mathscr{S}$, $L \cap H \neq \emptyset$ *and* $\ell(H) \leqslant \ell(L) \leqslant 2\ell(H)$, *then, for every* $l = 0, \ldots, 3$,

$$\|D^l g_L - D^l g_H\|_{C^0(B_{r_L}(z_L, w_L))} \leqslant C\,\mathbf{m}_0^{\frac{1}{2}}\, d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2}\,\ell(H)^{3+\kappa-l}\,. \tag{6.81}$$

(iv) *If* $L \in \mathscr{W} \cup \mathscr{S}$ *and* $d(H) \leqslant d(L) \leqslant 2d(H)$, *then*

$$|D^3 g_H(z_H, w_H) - D^3 g_L(z_L, w_L)| \leqslant C\,\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2}\,|z_H - z_L|^\kappa\,, \tag{6.82}$$

where $d(\cdot,\cdot)$ *denotes the distance in* $\mathfrak{B}$.

6.4.1 *Proof of (i) and (ii) in Proposition 6.20*

We start by fixing H, L, J so that $H \in \mathscr{S} \cup \mathscr{W}$, L is an ancestor of H (possibly H itself) and J is the father of L. We denote by B' the ball $B_{8r_J}(p_{HJ}, \pi_H)$, by B the ball $B_{8r_L}(p_{HL}, \pi_H)$, by $\mathbf{C}'$ the cylinder $\mathbf{C}_{8r_J}(p_J, \pi_H)$ and by $\mathbf{C}$ the cylinder $\mathbf{C}_{8r_L}(p_L, \pi_H)$. Observe that $B \subset B'$ (this just requires M_0 sufficiently large, given the estimate $|p_J - p_L| \leqslant 2\sqrt{2}\ell(J)$) and thus $\mathbf{C} \subset \mathbf{C}'$. We let $A \subset B$ be the projection onto π_H of $\mathrm{spt}(T_J) \cap \mathrm{Gr}(f_{HL}) \cap \mathrm{Gr}(f_{HJ})$. Next, set $E := \mathbf{E}(T_L, \mathbf{C}_{32r_L}(p_L, \pi_H))$ and $E' := \mathbf{E}(T_J, \mathbf{C}_{32r_J}(p_J, \pi_H))$ and recalling the argument in the proof of Proposition 6.16, we get

$$E \leqslant C\mathbf{m}_0 d(L)^{2\gamma_0-2+2\delta_1}\ell(L)^{2-2\delta_1} \leqslant C\mathbf{m}_0 d(H)^{2\gamma_0-2+2\delta_1}\ell(J)^{2-2\delta_1} \tag{6.83}$$

$$E' \leqslant C\mathbf{m}_0 d(J)^{2\gamma_0-2+2\delta_1}\ell(J)^{2-2\delta_1} \leqslant C\mathbf{m}_0 d(H)^{2\gamma_0-2+2\delta_1}\ell(J)^{2-2\delta_1} \tag{6.84}$$

$$\mathbf{h}(T, \mathbf{C}) \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2} \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(J)^{1+\beta_2} \tag{6.85}$$

$$\mathbf{h}(T, \mathbf{C}') \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(J)^{\frac{\gamma_0}{2}-\beta_2}\ell(J)^{1+\beta_2} \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(J)^{1+\beta_2}\,. \tag{6.86}$$

Next let $\bar{K}$ be the projection of $\mathrm{Gr}(f_{HL}) \cap \mathrm{Gr}(f_{HJ})$ onto $p_H + \pi_H$ and, recalling the estimates of Theorem 2.8 we achieve

$$|B \setminus \bar{K}| \leqslant C_0 r_J^2 (E^{\beta_0}(E + C_0\mathbf{m}_0 r_J^2) + E'^{\beta_0}(E' + C_0\mathbf{m}_0 r_J^2)) \leqslant C\mathbf{m}_0^{1+\beta_0} d(H)^{2(1+\beta_0)\gamma_0-2}\ell(J)^4\,.$$

In particular K is certainly nonempty, provided ε_{41} is small enough, and thus we can use the estimates of Theorem 2.8 on the oscillation of f_{HL} and f_{HJ} to conclude that

$$\|\boldsymbol{\eta} \circ f_{HL} - \boldsymbol{\eta} \circ f_{HJ}\|_{L^\infty(B)} \leqslant C\mathbf{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(J)^{1+\beta_2}\,.$$

Set therefore $\zeta := \boldsymbol{\eta}\circ\bar f_{HL} - \boldsymbol{\eta}\circ\bar f_{HJ}$ and conclude that

$$\|\zeta\|_{L^1(B)} \leqslant \|\boldsymbol{\eta}\circ f_{HL} - \boldsymbol{\eta}\circ f_{HJ}\|_{L^\infty(B)}\,|B\setminus\bar K| \leqslant C\boldsymbol{m}_0^{1+\beta_0+\frac14} d(H)^{\frac{\gamma_0}{2}-\beta_2+2(1+\beta_0)\gamma_0-2}\ell(J)^{5+\beta_2}\,.$$

If we define $\xi := \bar h_{HL} - \bar h_{HJ}$ we can use (6.64) of Proposition 6.19 and the triangular inequality to infer

$$\|\xi\|_{L^1(B)} \leqslant C\boldsymbol{m}_0 d(H)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(J)^{5+\beta_2}\,.$$

In turn, again by Proposition 6.19, this time using the fact that $\mathscr{L}_H\xi = 0$ and (6.67), we infer

$$\begin{aligned}\|D^l(\bar h_{HL} - \bar h_{HJ})\|_{C^0(\hat B)} &\leqslant C d(H)^{2(1+\beta_0)-2-\beta_2}\ell(J)^{3+\beta_2-l}\\ &\leqslant C d(H)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(J)^{3+2\kappa-l} \qquad \text{for } l = 0,1,2,3,4,\end{aligned} \tag{6.87}$$

where $\hat B = B_{6r}(p_{HL},\pi_H)$. Interpolating we get easily also

$$[D^3(\bar h_{HL} - \bar h_{HJ})]_{0,\kappa,\hat B} \leqslant C d(H)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(J)^{3+\kappa}\,. \tag{6.88}$$

In case (b) of Definition 1.1 we have $h_{HL} = \bar h_{HL}$ and $h_{HJ} = \bar h_{HJ}$. In case (a) and (c), using the system of coordinates introduced in the proof of Proposition 6.16 we have

$$\begin{aligned} h_{HL}(x) &= (\bar h_{HL}(x), \Psi_{p_H}(x, \bar h_{HL}(x)))\\ h_{HJ}(x) &= (\bar h_{HJ}(x), \Psi_{p_H}(x, \bar h_{HJ}(x)))\end{aligned}$$

and we use the chain rule and the regularity of Ψ_{p_H} to achieve the corresponding estimates

$$\|D^l(h_{HL} - h_{HJ})\|_{C^0(\hat B)} \leqslant C\, d(H)^{2(1+\beta_0)\gamma_0-2}\ell(J)^{3-l} \qquad \text{for } l = 0,1,2,3. \tag{6.89}$$

$$[D^3(h_{HL} - h_{HJ})]_{0,\kappa,\hat B} \leqslant C\, d(H)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(J)^{3+\kappa}\,. \tag{6.90}$$

Fix now a chain of cubes $H = H_j \subset H_{j-1} \subset \ldots \subset H_{N_0} =: L$, where each H_{j+1} is the father of H_j. Summing the estimates above and using the fact that $\ell(H_j) = 2^{-j}$ and $\ell(H) \leqslant d(H) = d(H_{N_0})$, we infer

$$\|D^l(h_{HL} - h_H)\|_{C^0(\tilde B)} \leqslant C\, d(H)^{2(1+\beta_0)\gamma_0+1-l} \qquad \text{for } l = 0,1,2,3 \tag{6.91}$$

$$[D^3(h_{HL} - h_H)]_{0,\kappa,\tilde B} \leqslant C\, d(H)^{2(1+\beta_0)\gamma_0-\beta_2+\kappa-2}\,, \tag{6.92}$$

where $\tilde B = B_{6r_H}(p_H,\pi_H)$. Observe that, assuming that we have fixed coordinates so that $p_H = (0,0,0)$ we also know, arguing as in the proof of Proposition 6.16, that, if we set $\bar B := B_{8r_L}(p_{HL},\pi_H)$, then

$$\|\boldsymbol{\eta}\circ\bar f\|_{L^\infty(\bar B)} \leqslant C\boldsymbol{m}_0^{\frac14} d(H)^{1+\frac{\gamma_0}{2}}\,.$$

In particular, applying (6.66) of Proposition 6.19, we conclude

$$\|\bar h_{HL}\|_{C^0(\hat B)} \leqslant C\boldsymbol{m}_0^{\frac14} d(H)^{1+\frac{\gamma_0}{2}}\,.$$

Since the graph of f_H and the support of T coincide on $K \times \pi^\perp$ for a set $K \subset B_{8r_H}(p_H, \pi_H)$ whose complement has very small measure, on such set we have

$$|\boldsymbol{\eta} \circ f_H| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}$$

(recall that $p_H = 0 \in \mathrm{spt}(T)$). On the other hand, given the bound on $|K|$ and the oscillation of f_H, we conclude that

$$\|\boldsymbol{\eta} \circ f_H\|_{L^1(B_{8r_H}(p_H,\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{3+\beta_2}$$

Using (6.64) we conclude

$$\|\bar{h}_H\|_{L^1(B_{8r_H}(p_H,\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{3+\beta_2}\,.$$

Using next (6.66) we achieve

$$\|\bar{h}_H\|_{L^\infty(B_{6r_H}(p_H,\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}\,. \tag{6.93}$$

Using the estimates upon Ψ_{p_H} and the fact that $\Psi_{p_H}(0) = 0$, $D\Psi_{p_H}(0) = 0$ we easily conclude

$$\|h_H\|_{L^\infty(B_{6r_H}(p_H,\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}\,. \tag{6.94}$$

Next, let $p_H = (\xi, \eta) \in \pi_H \times \pi_H^\perp$. Observe that $p_H \in \mathrm{spt}(T)$ and thus, for every $q \in \mathrm{spt}(T) \cap \mathbf{C}_{8r_H}(p_H, \pi_H)$, we must have $|\mathbf{p}_{\pi_H}(q) - \eta| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}$. Since the graph of f_H and the support of T coincide on $K \times \pi^\perp$ for a set $K \subset B_{8r_H}(p_H, \pi_H)$ whose complement has very small measure, on such set we have

$$|\boldsymbol{\eta} \circ f_H - \eta| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}\,.$$

Given the Lipschitz bound on $\boldsymbol{\eta} \circ f_H$, actually this bound is true over all $B_{8r_H}(p_H, \pi_H)$. Next, by the smallness of $\|h_H - \boldsymbol{\eta} \circ f_H\|_{L^1}$ there is at least one point $x \in B_{8r_H}(p_H, \pi_H)$ such that $|h_H(x) - \eta| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}$ and we can the extend the same estimate to all points in $B_{7r_H}(p_H, \pi_H)$ using the C^0 bound on h_H. We namely achieve

$$\|h_H - \eta\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}, \tag{6.95}$$

that is (6.76)

We next estimate the derivatives of h_{HL}. Let $E := \mathbf{E}(T_L, \mathbf{C}_{r_L}(p_L, \pi_H))$ and recall the discussion above and the estimates of Theorem 2.8 to conclude that

$$\int_{\bar{B}} |Df_{HL}|^2 \leqslant C_0 r_L^2 E \leqslant C\boldsymbol{m}_0 d(H)^{2\gamma_0-2+2\delta_1}\ell(L)^{4-2\delta_1}\,. \tag{6.96}$$

We thus conclude that $\|D\boldsymbol{\eta} \circ \bar{f}_{HL}\|_{L^2(\bar{B})} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\,\ell(H)^{2-\delta_1}$. We can now use the Lemma 6.18 to estimate $\|D\bar{h}_{HL}\|_{L^2} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\,\ell(H)^{2-\delta_1}$ and thus $\|D\bar{h}_{HL}\|_{L^1} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\,\ell(H)^{3-\delta_1}$. If we differentiate the equation defining $\bar{h}_{HL}$ we then find

$$\mathscr{L}_H \partial_j \bar{h}^i_{HL} = (\mathbf{L}_2)_{ij}$$

and we can thus apply (6.66) of Proposition 6.19, with $v = D\bar{h}_{HL}$, to conclude that

$$\|D^l \bar{h}_{HL}\|_{L^\infty(B_{6r_L})} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0 - 1 + \delta_1} \ell(L)^{2-\delta_1 - l} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0 + 1 - l} \qquad \text{for } l = 1, 2, 3, 4, \tag{6.97}$$

where we used the fact, for the starting cubes $L = H_{N_0}$, $d(H) = d(L) \leqslant C\ell(L)$.

Arguing as above we achieve a similar estimate for h_{HL}. We observe however that the condition $D\Psi_{p_H}(0, 0) = 0$ plays an important role (assuming to have moved the origin so that it coincides with p_H). For instance we have

$$Dh_{HL} = (D\bar{h}_{HL}, D_x\Psi_{p_H}(x, \bar{h}_{HL}(x)) + D_y\Psi_{p_H}(x, \bar{h}_{HL}(x))D\bar{h}_{HL}(x)) .$$

Thus we can easily estimate

$$|Dh_{HL}(x)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0} + |D\Psi_{p_H}(x, \bar{h}_{HL}(x))| . \tag{6.98}$$

Now, the second summand in (6.98) is estimated with $\|D^2\Psi_{p_H}\|\ell(H) \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)$, precisely because $D\Psi_{p_H}(0, 0) = 0$.

It follows by (6.89), (6.90), (6.97) and the triangular inequality that we have the uniform estimates

$$\|Dh_H\|_{C^0(B)} + d(H)\|D^2 h_H\|_{C^0(B)} + d(H)^2\|D^3 h_H\|_{C^\kappa(B)} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0} \tag{6.99}$$

Recall now that, by Proposition 6.11 we have $|\pi_H - \pi_0| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0}$. We can therefore apply Lemma 3.17 to the rescaling $k_H(x) := d(H)^{-1} h_H(d(H)x)$ and conclude the existence of the interpolating functions g_H and that the estimates (6.78) hold.

Using now Lemma 3.17, together with (6.76), we finally get

$$\|g_H - \mathbf{p}_{\pi_0}^\perp(p_H)\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2}} \ell(H) . \tag{6.100}$$

On the other hand $\mathbf{p}_{\pi_0}(p_H) = z_H$ and since $p_H \in \mathrm{spt}(T_H) \cup \mathbf{V}_{u,a}$, we conclude immediately $|\mathbf{p}_{\pi_0}^\perp(p_H) - u(z_H, w_H)| \leqslant c_s d(H)^a$. Combining this last estimate with (6.100) we conclude (6.79).

Finally, recall that, if $E := \mathbf{E}(T, \mathbf{C}_{32r_H}(p_H, \pi_H))$, then

$$\int_{B_{8r_H}(p_H, \pi_H)} |Df_H|^2 \leqslant C\boldsymbol{m}_0 d(H)^{2\gamma_0 - 2 + 2\delta_2} \ell(H)^{4-2\delta_1} ,$$

from which clearly we get

$$\int_{B_{8r_H}(p_H, \pi_H)} |D\boldsymbol{\eta} \circ f_H|^2 \leqslant C\boldsymbol{m}_0 d(H)^{2\gamma_0 - 2 + 2\delta_1} \ell(H)^{4-2\delta_1} .$$

By the estimate in Lemma 6.18, we deduce

$$\int_{B_{8r_H}(p_H, \pi_H)} |D\bar{h}_H|^2 \leqslant C\boldsymbol{m}_0 d(H)^{2\gamma_0 - 2 + 2\delta_1} \ell(H)^{4-2\delta_1} .$$

Thus we conclude the existence of a point p such that

$$|D\bar{h}_H(p)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0 - 1 + \delta_1} \ell(H)^{1-\delta_1}\,. \tag{6.101}$$

Assume now to be in the case (a) or (c) of Definition 1.1 and shift the origin so that it coincides with p_H. Given the bound on $D^2\bar{h}_H$ we then conclude

$$|D\bar{h}_H(0)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0 - 1 + \delta_1} \ell(H)^{1-\delta_1}$$

and, since $D\Psi_{p_H}(0) = 0$, we also have $|Dh_H(0)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\ell(H)^{1-\delta_1}$. Hence using the bound on $\|D^2 h_H\|_0$, we finally conclude $|D\bar{h}_H(q)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\ell(H)^{1-\delta_1}$ for all q's in the domain of $\bar{h}_H$. This implies the estimate

$$|T_q \mathbf{G}_{h_H} - \pi_H| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-1+\delta_1}\ell(H)^{1-\delta_1} \qquad \forall p \in \mathrm{Gr}(h_H) \cap \mathbf{C}_{6r_H}(p_H, \pi_H)\,.$$

Since however $\mathrm{Gr}(g_H) \subset \mathrm{Gr}(h_H) \cap \mathbf{C}_{6r_H}(p_H, \pi_H)$, we then conclude (6.80). The same conclusion for case (b) in Definition 1.1 follows directly from (6.101).

6.4.2 *Proof of (iii) and (iv)*

We observe first that (iv) is a rather simple consequence of (iii). Indeed fix H and L as in the statements and consider $H = H_i \subset H_{i-1} \subset \ldots \subset H_{N_0}$ and $L = L_j \subset L_{j-1} \subset \ldots \subset L_{N_0}$ so that H_l is the father of H_{l+1} and L_l is the father of L_{l+1}. We distinguish two cases:

(A) If $H_{N_0} \cap L_{N_0} \neq \emptyset$, we let i_0 and j_0 be the smallest indices so that $H_{i_0} \cap L_{j_0} \neq \emptyset$;

(B) $H_{N_0} \cap L_{N_0} = \emptyset$.

In case (A) observe that $\max\{\ell(H_{i_0}), \ell(L_{j_0})\} \leqslant d((z_H, w_H), (z_L, w_L)) := d$. On the other hand, recalling that $d(H_l) = d(H)$, $d(L_l) = d(L)$ and $d(L) \leqslant 2d(H)$, by (iii) with $l = 3$ we have

$$|D^3 g_H(z_H, w_H) - D^3 g_{H_{i_0}}(z_{H_{i_0}}, w_{H_{i_0}})| \leqslant \sum_{l=i_0}^{i-1} |D^3 g_{H_l}(z_{H_l}, w_{H_l}) - D^3 g_{H_{l+1}}(z_{H_{l+1}}, w_{H_{l+1}})|$$

$$\leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2} \ell(H_{i_0})^{\kappa} \sum_{l=i_0}^{i-1} 2^{(i_0 - l)\kappa} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^{\kappa}$$

$$|D^3 g_L(z_L, w_L) - D^3 g_{L_{j_0}}(z_{L_{j_0}}, w_{L_{j_0}})| \leqslant \sum_{l=j_0}^{j-1} |D^3 g_{L_l}(z_{L_l}, w_{L_l}) - D^3 g_{L_{l+1}}(z_{L_{l+1}}, w_{L_{l+1}})|$$

$$\leqslant C\mathbf{m}_0^{\frac{1}{2}} d(L)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2} \ell(L_{j_0})^{\kappa} \sum_{l=j_0}^{j-1} 2^{(j_0 - l)\kappa} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^{\kappa}$$

$$|D^3 g_{L_{j_0}}(z_{L_{j_0}}, w_{L_{j_0}}) - D^3 g_{H_{i_0}}(z_{H_{i_0}}, w_{H_{i_0}})| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H_{i_0})^{2(1+\beta_0)\gamma_0-\beta_2-2} \ell(H_{i_0})^{\kappa}$$

$$\leqslant \mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^{\kappa}\,.$$

The triangle inequality implies then the desired estimate.

In case (B) we first notice that by the very same argument we have the estimates

$$|D^3 g_H(z_H, w_H) - D^3 g_{H_{N_0}}(z_{H_{N_0}}, w_{H_{N_0}})| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^\kappa$$
$$|D^3 g_L(z_L, w_L) - D^3 g_{L_{N_0}}(z_{L_{N_0}}, w_{L_{N_0}})| \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^\kappa .$$

Next we find a chain of cubes $H_{N_0} = J_0, J_1, \ldots, J_N = L_{N_0}$, all distinct and belonging to $\mathscr{S}^{N_0}$, so that

- $d(H) \leqslant d(J_l) \leqslant d(L) \leqslant 2d(H)$;
- $J_l \cap J_{l+1} \neq \emptyset$ and thus $\ell(H_{N_0}) \leqslant \ell(J_l) \leqslant \ell(L_{N_0})$;
- N is smaller than a constant $C(N_0, \bar{Q})$.

Using again (iii) and arguing as above we conclude

$$|D^3 g_{H_{N_0}}(z_{H_{N_0}}, w_{H_{N_0}}) - D^3 g_{L_{N_0}}(z_{L_{N_0}}, w_{L_{N_0}})|$$
$$\leqslant \sum_{l=1}^{N} |D^3 g_{J_l}(z_{J_l}, w_{J_l}) - D^3 g_{J_{l-1}}(z_{J_{l-1}}, w_{J_{l-1}})| \leqslant CN\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} d^\kappa .$$

Again, using the triangular inequality we conclude (iv).

We now come to (iii). Fix therefore two cubes H and L as in the statement and set $r := r_H$. Observe that, by (i) and Lemma 6.24, it suffices to show that $\|g_H - g_L\|_{L^1(B)} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{\frac{\gamma_0}{2}-2} \ell(H)^{5+\kappa}$. where $B = B_r(z_L, \pi_0)$. Consider now the two corresponding tilted interpolating functions, namely h_L and h_H. Given the estimate upon h_L proved in the previous paragraph, we can find a function $\hat{h}_L : B_{7r}(p_L, \pi_H) \to \pi_H^\perp$ such that $\mathbf{G}_{\hat{h}_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{6r}(p_L, \pi_H)$ (in this paragraph $\hat{\cdot}$ will always denote the riparametrization on π_L). Obviously $\mathbf{G}_{\hat{h}_L} \llcorner \mathbf{C}_r(z_L, \pi_0) = \mathbf{G}_{g_L}$. We can therefore apply Lemma 3.17 to conclude that

$$\|g_H - g_L\|_{L^1(B)} \leqslant C\|h_H - \hat{h}_L\|_{L^1(B_{5r}(p_L, \pi_H))} .$$

Consider next the tilted interpolating function h_{HL} and observe that, by (6.87) and the usual estimates on Ψ, we know

$$\|h_H - h_{HL}\|_{L^1(B_{5r}(p_H, \pi_H))} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} \ell(H)^{5+\beta_2} .$$

Hence, since $\beta_2 \geqslant \kappa$, we are reduced to show

$$\|h_{HL} - \hat{h}_L\|_{L^1(B_{5r}(p_H, \pi_H))} \leqslant C\mathbf{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2} \ell(H)^{5+\kappa} . \tag{6.102}$$

In turn, consider the π_H-approximating function f_{HL} and the π_L-approximating function $f_{LL} = f_L$. In the $\pi_H \times \varkappa_H \times T_{p_H}\Sigma^\perp$ coordinates we set

$$\mathbf{f}_{HL}(x) = (\mathbf{p}_{\varkappa_H}(\boldsymbol{\eta} \circ f_{HL}(x)), \Psi_{p_H}(x, \mathbf{p}_{\varkappa_H}(\boldsymbol{\eta} \circ f_{HL}(x))))$$

and recall that, by Proposition 6.19, we have

$$\|h_{HL} - \mathbf{f}_{HL}\|_{L^1(B_{8r_L}(p_{HL},\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(L)^{5+\beta_2}\,. \tag{6.103}$$

Similarly, in the $\pi_L \times \varkappa_L \times T_{p_L}\Sigma^\perp$ coordinates we set

$$\mathbf{f}_L(x) = (\mathbf{p}_{\varkappa_L}(\boldsymbol{\eta}\circ f_L(x)), \Psi_{p_L}(x, \mathbf{p}_{\varkappa_L}(\boldsymbol{\eta}\circ f_L(x))))$$

and get

$$\|h_L - \mathbf{f}_L\|_{L^1(B_{8r_L}(p_L,\pi_L))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(L)^{5+\beta_2}\,.$$

Next we denote by $\hat{\mathbf{f}}_L$ the map $\hat{\mathbf{f}}_L : B_{6r_L}(p_{HL}, \pi_H) \to \pi_H^\perp$ such that $\mathbf{G}_{\hat{\mathbf{f}}_L} = \mathbf{G}_{\mathbf{f}_L} \llcorner \mathbf{C}_{6r_L}(p_L, \pi_H)$ and we use again Lemma 3.17 to infer

$$\|\hat{h}_L - \hat{\mathbf{f}}_L\|_{L^1(B_{6r_L}(p_{HL},\pi_H))} \leqslant C\|h_L - \mathbf{f}_L\|_{L^1(B_{8r_L}(p_L,\pi_L)} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(L)^{5+\beta_2}\,. \tag{6.104}$$

In view of (6.103) and (6.104), (6.102) is then reduced to

$$\|\mathbf{f}_{HL} - \hat{\mathbf{f}}_L\|_{L^1(B_{5r_L}(p_{HL},\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(H)^{5+\kappa}\,. \tag{6.105}$$

Consider now the map $\hat{f}_L : \mathbf{B}_{6r_L}(p_{HL}, \pi_H) \to \mathcal{A}_Q(\pi_H^\perp)$ such that $\mathbf{G}_{\hat{f}_L} = \mathbf{G}_{f_L} \llcorner \mathbf{C}_{6r_L}(p_L, \pi_H)$. Let A and $\hat{A}$ be the projections on $p_H + \pi_H$ of the Borel sets $\mathrm{Gr}(f_{HL})) \setminus \mathrm{spt}(T)$ and $\mathrm{Gr}(\hat{f}_L) \setminus \mathrm{spt}(T) \subset \mathrm{Gr}(f_L) \setminus \mathrm{spt}(T)$. We know that

$$\begin{aligned}
|A \cup A'| \leqslant& \|\mathbf{G}_{f_{HL}} - T\|(\mathbf{C}_{8r_L}(p_L, \pi_H)) + \|\mathbf{G}_{f_L} - T\|(\mathbf{C}_{8r_L}(p_L, \pi_L))\\
\leqslant& C\boldsymbol{m}_0^{1+\beta_0} d(H)^{2(1+\beta_0)\gamma_0 - 2}\ell(H)^4\,.
\end{aligned}$$

On the other hand, it is not difficult to see, thanks to the height bound, that $\|\boldsymbol{\eta}\circ f_{HL} - \boldsymbol{\eta}\circ \hat{f}_L\|_\infty \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2} - \beta_2}\ell(H)^{1+\beta_2}$. We thus conclude that

$$\|\boldsymbol{\eta}\circ f_{HL} - \boldsymbol{\eta}\circ \hat{f}_L\|_{L^1(B_{6r_L}(p_{HL},\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(H)^{5+\beta_2}\,.$$

Define in the $\pi_H \times \varkappa_H \times T_{p_H}\Sigma^\perp$ co-ordinates the function

$$\mathbf{g}(x) := (\mathbf{p}_{\varkappa_H}(\boldsymbol{\eta}\circ \hat{f}_L(x)), \Psi_{p_H}(x, \mathbf{p}_{\varkappa_H}(\boldsymbol{\eta}\circ \hat{f}_L(x))))\,.$$

We can thus conclude that

$$\|\mathbf{f}_{HL} - \mathbf{g}\|_{L^1(B_{6r_L}(p_{HL},\pi_L))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(L)^{5+\beta_2}\,. \tag{6.106}$$

Thus, (6.105) is now reduced to

$$\|\mathbf{g} - \hat{\mathbf{f}}_L\|_{L^1(B_{5r_L}(p_{HL},\pi_H))} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0 - \beta_2 - 2}\ell(H)^{5+\kappa}\,. \tag{6.107}$$

Denoting by An the distance $|\pi_H - \pi_L|$, by $\hat{B}$ the ball $B_{6r_L}(p_{HL}, \pi_H)$ and by $\tilde{B}$ the ball $B_{8r_L}(p_L, \pi_L)$, we then have, by Lemma 6.23

$$\|g - \hat{f}_L\|_{L^1(\hat{B})} \leqslant C_0(\operatorname{osc}(f_L) + r_L \mathrm{An}) \left(\int |Df_L|^2 + r_L^2(\|D\Psi_{p_L}\|^2_{C^0(\tilde{B})} + \mathrm{An}^2)\right).$$

Recall that $D\Psi_{p_L}(p_L) = 0$ and thus $\|D\Psi_{p_L}\|^2_{C^0(\tilde{B})} \leqslant C_0 \boldsymbol{m}_0 r_L^2$. Recalling the estimate on $|\pi_H - \pi_L|$ and upon the Dirichlet energy of f_L, we then conclude

$$\int |Df_L|^2 + r_L^2(\|D\Psi_{p_L}\|^2_{C^0(\tilde{B})} + \mathrm{An}^2) \leqslant C\boldsymbol{m}_0 d(L)^{2\gamma_0 - 2 + 2\delta_1} \ell(H)^{4 - 2\delta_1}.$$

On the other hand

$$\operatorname{osc}(f_L) + r_L \mathrm{An} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\frac{\gamma_0}{2} - \beta_2} \ell(H)^{1+\beta_2}.$$

Thus (6.107) follows by our choice of the various parameters, in particular $\beta_2 - 2\delta_1 \geqslant \frac{\beta_2}{4} = \kappa$.

6.5 CONCLUSION OF THE PROOF: EXISTENCE OF THE CENTER MANIFOLD

6.5.1 *Proof of (i)*

As in all the proofs so far, we will use C_0 for geometric constants and C for constants which depend upon M_0, N_0, C_e and C_h. Define $\chi_H := \vartheta_H / (\sum_{L \in \mathscr{P}^j} \vartheta_L)$ for each $H \in \mathscr{P}^j$(cf. Definition 6.8) and observe that

$$\sum_{H \in \mathscr{P}^j} \chi_H = 1 \text{ on } \mathcal{A}_k \ \forall k \in \mathbb{N} \qquad \text{and} \qquad \|\chi_H\|_{C^i} \leqslant C_0\, \ell(H)^{-i} \quad \forall i \in \{0, 1, 2, 3, 4\}. \tag{6.108}$$

Fix any $H \in \mathscr{P}^j$ and let k be such that $H \subset \mathcal{A}_k$. Set $\mathscr{P}^j(H) := \{L \in \mathscr{P}^j : L \cap H \neq \emptyset\} \setminus \{H\}$ for each $H \in \mathscr{P}^j$. By construction $\frac{1}{2}\ell(L) \leqslant \ell(H) \leqslant 2\,\ell(L)$ and $2^{-k-1} \leqslant d(L) \leqslant 2^{-k+1}$ for every $L \in \mathscr{P}^j(H)$. Moreover the cardinality of $\mathscr{P}^j(H)$ is at most 13. Fix a point $p = (z, w) \in H$ and observe that $C_0^{-1} 2^{-k} \leqslant |z| \leqslant C_0 2^{-k}$. From (6.77) of Proposition 6.20 we then conclude

$$|\hat{\varphi}_j(z, w)| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{1+\frac{\gamma_0}{2}} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} |z|^{1+\frac{\gamma_0}{2}}.$$

Recall now that $\Psi(0) = 0$, $D\Psi(0) = 0$ and $\|D^2\Psi\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}$. Considering that

$$\varphi_j(z, w) = (\bar{\varphi}_j(z, w), \Psi(z, \bar{\varphi}_j(z, w))) \tag{6.109}$$

(where $\bar{\varphi}_j(z, w)$ is the vector consisting of the first $\bar{n}$ components of $\hat{\varphi}_j(z, w)$), we easily conclude

$$|\hat{\varphi}_j(z, w)| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} |z|^{1+\frac{\gamma_0}{2}} + C\|D^2\Psi\|_{C^0} |z|^2 \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} |z|^{1+\kappa}.$$

This gives (6.16) and the continuity of φ_j, since by definition $\varphi_j(0, 0) = 0$.

For $(z,w) \in H$ we next write

$$\hat{\varphi}_j(z,w) = \Big(g_H\chi_H + \sum_{L\in\mathscr{P}^j(H)} g_L\chi_L\Big)(z,w) = g_H(x) + \sum_{L\in\mathscr{P}^j(H)} (g_L - g_H)\chi_L\,(z,w)\,, \tag{6.110}$$

because H does not meet the support of ϑ_L for any $L \in \mathscr{P}^j$ which does not meet H. Using the Leibniz rule, (6.108) and the estimates of Proposition 6.20, for $l \in \{1,2,3\}$ we get

$$\begin{aligned}
\|D^l\hat{\varphi}_j\|_{C^0(H)} &\leqslant \|D^l g_H\|_{C^0} + C_0 \sum_{0\leqslant i\leqslant l}\sum_{L\in\mathscr{P}^j(H)} \|g_L - g_H\|_{C^i(H)}\ell(L)^{i-l}\\
&\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0+1-l} + C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{2(1+\beta_0)\gamma_0-\beta_2-2}\sum_{0\leqslant i\leqslant l}\ell(H)^{3+\beta_2-i}\ell(H)^{i-l}\\
&\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0+1-l}\,.
\end{aligned}$$

Again using the formula (6.109) and the estimate $\|\Psi\|_{C^{3,\varepsilon_0}} \leqslant \boldsymbol{m}_0^{\frac{1}{2}}$ (together with $D\Psi(0) = 0$ and $\Psi(0) = 0$) we easily reach (6.17). With an argument entirely similar we obtain

$$[D^3\varphi_j]_{\kappa,H} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0-2}\,. \tag{6.111}$$

Thus, pick any two points $(z,w), (z'w') \in \mathcal{A}_k$. If they belong to the same cube $H \in \mathscr{P}^j$ with $H \subset \mathcal{A}_k$, then

$$\begin{aligned}
|D^3\varphi_j(z,w) - D^3\varphi_j(z',w')| &\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{-2} d((z',w'),(z,w))^\kappa\\
&\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} 2^{2k} d((z',w'),(z,w))^\kappa\,.
\end{aligned} \tag{6.112}$$

If they do not belong to the same cube, then let $H, L \in \mathscr{P}^j$ be two cubes contained in $\mathcal{A}_k$ such that $(z,w) \in H$ and $(z',w') \in L$. Next observe that, by our choice of the cut-off functions ϑ_J, $\varphi_j = g_H$ in a neighborhood of (z_H, w_H) and $\varphi_j = g_L$ in a neighborhood of (z_L, w_L). We can then estimate, using Proposition 6.20(iv) and (6.111)

$$\begin{aligned}
|D^3\varphi_j(z,w) - D^3\varphi_j(z',w')| &\leqslant |D^3\varphi_j(z,w) - D^3 g_H(z_H,w_H)|\\
&\quad + |D^3 g_H(z_H,w_H) - D^3 g_L(z_L,w_L)| + |D^3\varphi_j(z_L,w_L) - D^3\varphi_j(z',w')|\\
&\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{-2}\left(\ell(H)^\kappa + d((z_H,w_H),(z_L,w_L))^\kappa + \ell(L)^\kappa)\right)\\
&\leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{-2} d((z,w),(z',w'))^\kappa \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} 2^{2k} d((z',w'),(z,w))^\kappa\,.
\end{aligned} \tag{6.113}$$

From (6.112) and (6.113) we conclude (6.18) and thus the proof of Theorem 6.9(i).

6.5.2 *Proof of (ii)*

The first statement is an obvious consequence of the construction algorithm: indeed note that, if i, j, k, L and H are as in the statement then $\mathscr{P}^j(L) = \mathscr{P}^k(L)$ and moreover $\chi_J = 0$ on H for any $J \in \mathscr{P}^j \setminus \mathscr{P}^j(L)$ and for any $J \in \mathscr{P}^k \setminus \mathscr{P}^k(L)$. Then it turns out that $\hat{\varphi}_j = \hat{\varphi}_k$ on H, which in turn obviously implies that φ_j and φ_k coincide on H.

As for the second statement, if we can show that there is a uniform limit $\boldsymbol{\varphi}$ for φ_j, the C^3 convergence and the regularity of $\boldsymbol{\varphi}$ will follow from the estimates of point (i). Fix a point $(z, w) \neq 0$ and let $H \in \mathscr{P}^j$ which contains it. If $H \in \mathscr{W}^i$ and $i \leqslant j-2$, then $\hat{\varphi}_{j+1}$ and $\hat{\varphi}_j$ coincide on it. Otherwise we can assume that $H \in \mathscr{C}^{j-1} \cup \mathscr{C}^j$. In this case we can estimate

$$|\varphi_j(z,w) - \varphi_j(z_H, w_H)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^\kappa \ell(H) \leqslant C2^{-j}\,.$$

A similar estimate holds for φ_{j+1}: notice that we can choose $L \in \mathscr{P}^{j+1}$ such that $(z, w) \in L$ and L is either H or a son of H. Moreover, we can estimate

$$|\varphi_{j+1}(z,w) - \varphi_{j+1}(z_L, w_L)| \leqslant C2^{-j}\,.$$

Next, recall that $\varphi_j(z_H, w_H) = g_H(z_H, w_H)$ and that $\varphi_{j+1}(z_L, w_L) = g_L(z_L, w_L)$. Since moreover $L = H$ or L is a son of H, by Proposition 6.20 we achieve

$$|\varphi_{j+1}(z_L, w_L) - \varphi_j(z_H, w_H)| \leqslant C_0 \|Dg_H\|_{C^0} \ell(H) + C\|g_H - g_L\|_{C^0} \leqslant C2^{-j}\,.$$

Summarizing, we conclude that

$$\|\varphi_{j+1} - \varphi_j\|_{C^0} \leqslant C2^{-j}\,.$$

The latter estimate gives that φ_j is a Cauchy sequence in C^0 and thus that it converges uniformly to some $\boldsymbol{\varphi}$.

6.5.3 *Proof of (iii)*

Observe first that, if (z, w) does not belong to some $H \in \mathscr{W}$, then $\boldsymbol{\varphi}(z, w)$ is necessarily a point in the support of T and we can estimate

$$|\boldsymbol{\varphi}(z,w) - u(z,w)| \leqslant c_s |z|^a\,. \tag{6.114}$$

In fact, in this case for every $j \geqslant N_0$ there is $H_j \in \mathscr{S}^j$ such that $(z, w) \in H_j$. Observe that $\varphi_j(z_{H_j}, w_{H_j}) = g_{H_j}(z_{H_j}, w_{H_j})$ and that

$$\lim_{j\to\infty} \big(d((z_{H_j}, w_{H_j}), (z,w)) + |g_{H_j}(z_{H_j}, w_{H_j}) - \boldsymbol{\varphi}(z,w)|\big) = 0\,.$$

But we also have

$$\lim_{j\to\infty} |(z_{H_j}, g_{H_j}(z_{H_j}, w_{H_j})) - p_{H_j}| = 0\,.$$

On the other hand, since

$$|p_{H_j} - (z_{H_j}, u(z_{H_j}, w_{H_j}))| \leqslant c_s |z_{H_j}|^a\,,$$

we then conclude (6.114) taking the limit in $j \to \infty$.

From now on we therefore assume that $(z, w) \in H$ for some $H \in \mathscr{W}$.

Step 1. In this step we show that

$$\ell(H) \leqslant C_0 d(H)^{(b+1)/2}\,. \tag{6.115}$$

In fact we claim that this is the case for any $H \in \mathscr{W}$. First of all we observe that it suffices to show it for $H \in \mathscr{W}_e \cup \mathscr{W}_h$: given indeed any $H \in \mathscr{W}_n$ we find a chain of cubes $H = H_l, H_{l-1}, \dots, H_i$ with the properties that

- $H_k \cap H_{k+1} \neq \emptyset$;
- $\ell(H_k) = 2\ell(H_{k+1})$;
- $H_l \in \mathscr{W}_n$ for any $l \geqslant i+1$ and $H_i \in \mathscr{W}_e \cup \mathscr{W}_h$.

It is easy to see that, provided M_0 is larger than a geometric constant, $\frac{1}{2}d(H) \leqslant d(H_i) \leqslant 2d(H)$. Since $\ell(H) \leqslant \frac{1}{2}\ell(H_i)$, it suffices to show $\ell(H_i) \leqslant C_0 d(H_i)^{(b+1)/2}$.

Next, assume $H \in \mathscr{W}_e$. Then we know that

$$\mathbf{E}(T_H, \mathbf{B}_H) > C_e \boldsymbol{m}_0 d(H)^{2\gamma_0 - 2 + 2\delta_1} \ell(H)^{2-2\delta_1} \geqslant C_e \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{2\gamma_0 - 2} \ell(H)^2 . \tag{6.116}$$

Now recall that $d = |z_H| \leqslant 2\sqrt{2} d(H)$. Moreover, if r_H were larger than $\frac{1}{2} d^{(b+1)/2}$, then by (2.24) there would be a π such that (recall that $C_i^2 \leqslant \boldsymbol{m}_0$)

$$\mathbf{E}(T_H, \mathbf{B}_H, \pi) \leqslant \boldsymbol{m}_0 d(H)^{2\gamma - 2} r_H^2 .$$

By Lemma 6.12(i), we then would have

$$\mathbf{E}(T_H, \mathbf{B}_H(\pi)) \leqslant \boldsymbol{m}_0 d(H)^{2\gamma - 2} r_H^2 + C_0 \boldsymbol{m}_0 r_H^2 \leqslant C(M_0) \boldsymbol{m}_0 d(H)^{2\gamma_0 - 2} \ell(H)^2 \tag{6.117}$$

(recall that $\gamma_0 < \gamma$). Thus we conclude that (6.117) contradicts (6.116).

It remains to show (6.115) when $H \in \mathscr{W}_h$. Assume therefore that $r_H \geqslant \frac{1}{2} d^{(b+1)/2}$. As above observe that we know

$$\mathbf{E}(T_H, \mathbf{B}_H, \pi_H) = \mathbf{E}(T_H, \mathbf{B}_H) \leqslant \bar{C} \boldsymbol{m}_0 d(H)^{2\gamma - 2} \ell(H)^2 \tag{6.118}$$

whete the constant $\bar{C}$ does not depend on H. We thus conclude from Lemma 6.12 that

$$|\pi - \pi_H| \leqslant \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma - 1} \ell(H) . \tag{6.119}$$

We next wish to estimate $\mathbf{h}(T_H, \mathbf{B}_H, \pi)$. π is tangent to $\mathbf{G}_u$ at $q_H := (z_H, u(z_H, w_H))$. For simplicity shift the coordinates so that $q_H = 0$ and recall that $|p_H| = |p_H - q_H| \leqslant c_s |d|^a$. Fix a point $p \in \mathbf{B}_H \cap \mathrm{spt}(T_H)$ and recall that there is a point p' in $\mathrm{Gr}(u) \cap \mathbf{V}_H$ such that $|p - p'| \leqslant 2^a d^a$, since $|\mathbf{p}_{\pi_0}(p')| \geqslant \frac{d}{2}$. Obviously $|\mathbf{p}_\pi(p')| \leqslant 2r_H$ and since π is tangent to $\mathrm{Gr}(u)$ at 0, we have the estimate

$$|\mathbf{p}_\pi^\perp(p')| \leqslant C_0 \boldsymbol{m}_0^{\frac{1}{2}} d^{\alpha - 1} |\mathbf{p}_\pi(p')|^2 \leqslant \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\alpha - 1} \ell(H)^2 .$$

We can therefore estimate

$$|\mathbf{p}_\pi^\perp(p)| \leqslant \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\alpha - 1} \ell(H)^2 + \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^a .$$

This implies the estimate

$$\mathbf{h}(T_H, \mathbf{B}_H, \pi) \leqslant \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\alpha - 1} \ell(H)^2 + \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^a . \tag{6.120}$$

Using now Lemma 6.12 and (6.119) we then estimate

$$\mathbf{h}(T_H, \mathbf{B}_H) \leqslant \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\alpha - 1} \ell(H)^2 + \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^a + \bar{C} \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma - 1} \ell(H)^2 , \tag{6.121}$$

where $\bar{C}$ depends upon M_0, N_0 and C_e, but not upon C_h.

On the other hand, since $H \in \mathscr{W}_h$, we then have

$$\mathbf{h}(T_H, \mathbf{B}_H) > C_h \boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\gamma_0 - \beta_2} \ell(H)^{1+\beta_2}. \tag{6.122}$$

By our choice of the exponents it is obvious that the first and third summand in (6.121) are smaller than a fraction (say $\frac{1}{4}$) of $C_h \boldsymbol{m}_0^{\frac{1}{4}} d(H)^{\gamma_0 - \beta_2} \ell(H)^{1+\beta_2}$, provided that C_h is chosen large enough. Recalling that we are assuming $\ell(H) \geqslant \bar{C} d(H)^{(1+b)/2}$, to achieve the same conclusion with the second summand we need

$$\frac{1+b}{2}(1+\beta_2) - \beta_2 + \gamma_0 < a\,.$$

However, since $a > b$, the latter inequality is implied by (6.2), and we reach a contradiction.

Step 2. Recall that we have fixed $(z, w) \in H$ with $H \in \mathscr{W}$ and that our aim is to establish (6.19). From the previous step we know that $\ell(H) \leqslant C_0 |z|^{(1+b)/2}$ and that $d(H) \leqslant C_0 |z|$. Assume $H \in \mathscr{W}^j$ and pick any $k \geqslant j+2$. By (ii)Theorem 6.9, we know that $\boldsymbol{\varphi} = \varphi_k$ on H. Recalling the arguments above (in particular (6.81)), we also have

$$\|\varphi_j - g_H\|_{C^0} \leqslant \sum_{L \in \mathscr{P}^k(H)} \|g_H - g_L\|_{C^0} \leqslant C \boldsymbol{m}_0^{\frac{1}{2}} d(H)^{1\gamma_0 - 2 - \beta_2} \ell(H)^{3+\kappa} \leqslant C \boldsymbol{m}_0^{\frac{1}{2}} d^{\gamma_0 - 2 + \frac{(3+\kappa)(b+1)}{2}}$$

Since $\gamma_0 - 2 + (3+\kappa)(b+1)/2 > \gamma_0 + 3\frac{b}{2} - \frac{1}{2} > \gamma_0 + b$, it suffices then to show that

$$|u(z,w) - g_H(z,w)| \leqslant C \boldsymbol{m}_0^{\frac{1}{4}} |z|^{a'}. \tag{6.123}$$

We next consider both u and g_H as two functions defined on π_0 and having defined the ball $B := B_{r_H}(z_H, \pi_0)$, our goal is indeed to show that

$$\|u - g_H\|_{C^0(B)} \leqslant C \boldsymbol{m}_0^{\frac{1}{4}} d(H)^{a'}.$$

Recall next that the graph of g_H is indeed a subset of the graph of the tilted interpolating function h_H. If $v : B_{8r_H}(p_H, \pi_H) \to \pi_H^\perp$ is the function which gives the graph of u in the system of coordinates $\pi_H \times \pi_H^\perp$ and we set $B' := B_{6r_H}(p_H, \pi_H)$, we then claim that it suffices to show

$$\|v - h_H\|_{C^0(B')} \leqslant C \boldsymbol{m}_0^{\frac{1}{4}} d(H)^{a'}. \tag{6.124}$$

In fact let $p = (\zeta, g_H(\zeta)) \in \pi_0 \times \pi_0^\perp$ and let $\omega \in \pi_H$be such that $p = (\omega, h_H(\omega)) \in \pi_H \times \pi_H^\perp$. Consider also $q = (\zeta, u(z))$ and $q' = (\omega, v(\omega))$ and let $\zeta' \in \pi_0$ such that $q' = (\zeta', u(\zeta'))$. Let $\mathcal{T}$ be the triangle with vertices q, p and q'. The angle θ_p at p can be assumed to be small, because $|\pi_H - \pi_0| \leqslant C \boldsymbol{m}_0^{\frac{1}{2}}$. On the other hand the angle θ_q in q is close to $\frac{\pi}{2}$, since the Lipschitz constant of u is small. Thus the angle $\theta_{q'}$ is also close to $\frac{\pi}{2}$. From the sinus theorem applied to the triangle $\mathcal{T}$ we then conclude

$$|u(\zeta) - g_H(\zeta)| = |p - q| = \frac{\sin\theta_{q'}}{\sin\theta_q} |p - q'|\,. \tag{6.125}$$

By choosing ε_{41} small we then reach

$$\|u - g_H\|_{C^0(B)} \leqslant 2\|v - h_H\|_{C^0(B')}\,.$$

As usual, we assume now to have shifted the origin so that $p_H = 0$. Recall that $\Psi_{p_H}(0) = 0$ and $D\Psi_{p_H}(0) = 0$, so that we can estimate

$$\|h_H - \boldsymbol{\eta} \circ f_H\|_{C^0(B')} \leqslant C_0\|\bar{h}_H - \boldsymbol{\eta} \circ \bar{f}_H\|_{C^0} + C\boldsymbol{m}_0^{\frac{1}{2}}\ell(H)^2\,.$$

Using now Proposition 6.19 we then conclude

$$\|h_H - \boldsymbol{\eta} \circ f_H\|_{C^0(B')} \leqslant C\boldsymbol{m}_0 d(H)^{2\gamma_0 - 2}\ell(H)^3 + C\boldsymbol{m}_0^{\frac{1}{2}}\ell(H)^2\,. \tag{6.126}$$

Since $\ell(H) \leqslant d(H)^{(1+b)/2}$, we again see that (6.124) can be reduced to the estimate

$$\|\boldsymbol{\eta} \circ f_H - v\|_{C^0(B')} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(H)^{a'}\,. \tag{6.127}$$

We will in fact show such estimate in the ball $\hat{B} := B_{8r_H}(p_H, \pi_H)$. Consider a point $p \in \mathrm{spt}(T_H) \cap \mathbf{C}_{8r_H}(p_H, \pi_H)$ and let $p = (\zeta, \eta) \in \pi_0 \times \pi_0^\perp$. We also let q be the point $(\zeta, u(\zeta))$ and $q' = (\omega, v(\omega)) \in \pi_H \times \pi_H^\perp$, where $\omega = \mathbf{p}_H(p)$. The argument above can be applied literally to the triangle $\mathcal{T}$ with vertices p, q and q' to conclude that

$$|p - q'| \leqslant 2|p - q| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^a\,.$$

Recall that, except for a set of points $\omega \in A$ of measure no larger than $C\boldsymbol{m}_0 d(H)^{2\gamma_0 - 2}\ell(H)^4$, the slice $\langle T, \mathbf{p}_{\pi_H}, \omega\rangle$ coincides with the slice $\langle \mathbf{G}_{f_H}, \mathbf{p}_{\pi_H}, \omega\rangle$. Thus on the set A we obviously have

$$|\boldsymbol{\eta} \circ f_H(\omega) - v(\omega)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^a\,.$$

Now, for any point $\omega \notin A$ there is a point $\omega' \in A$ at distance at most $d(H)^{\gamma_0 - 1}\ell(H)^2$. Since both $\mathrm{Lip}(v)$ and $\mathrm{Lip}(\boldsymbol{\eta} \circ f_H)$ are controlled by $\boldsymbol{m}_0^{\frac{1}{2}}$, this gives the estimate

$$\|\boldsymbol{\eta} \circ f_H - v\|_{C^0(B')} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^a + C d(H)^{\gamma_0 - 1}\ell(H)^2\,.$$

On the other hand, since $\ell(H) \leqslant C d(H)^{(b+1)/2}$ and $a \geqslant b + 1$, we easily see that

$$\|\boldsymbol{\eta} \circ f_H - v\|_{C^0(B')} \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(H)^{\gamma_0 + b}\,.$$

This completes the proof of (6.127) and hence of (6.19)

6.6 APPENDIX A: DENSITY AND HEIGHT BOUND

In this appendix we record two estimates which are standard for area-minimizing currents and can be extended with routine arguments to the three cases of Definition 1.1. Both statements are valid for general m without additional efforts and we therefore do not restrict to $m = 2$ here. Consistently with [24, 18] we introduce the parameter $\boldsymbol{\Omega}$, which equals

- $\boldsymbol{A} = \|A_\Sigma\|_{C^0}$ in case (a) of Definition 1.1;
- $\max\{\|d\omega\|_{C^0}, \|A_\Sigma\|_{C^0}\}$ in case (b);
- $C_0 R^{-1}$ in case (c).

Lemma 6.21. *There is a positive geometric constant $c(m,n)$ with the following property. If T is a current as in Definition 1.1, where $\boldsymbol{\Omega} \leqslant c(m,n)$, then*

$$\|T\|(\mathbf{B}_\rho(x)) \geqslant \omega_m(\Theta(T,p) - \tfrac{1}{4})\rho^m \geqslant \omega_m \tfrac{3}{4}\rho^m \qquad \forall p \in \operatorname{spt}(T), \forall r \in \operatorname{dist}(p, \partial U)\,. \quad (6.128)$$

Proof. By [24, Proposition 1.2] $\|T\|$ is an integral varifold with bounded mean curvature in the sense of Allard, where $C\boldsymbol{\Omega}$ bounds the mean curvature for some geometric constant C. It follows from Allard's monotonicity formula that $e^{C\boldsymbol{\Omega} r}\|T\|(\mathbf{B}_r(x))$ is monotone non-decreasing in r, from which the first inequality in (6.128) follows. The second inequality is implied by $\Theta(T,p) \geqslant 1$ for every $p \in \operatorname{spt}(T)$: this holds because the density is an upper semicontinuous function which takes integer values $\|T\|$-almost everywhere. □

For the proof of the next statement we refer to [20, Theorem A.1]: in that Theorem T satisfies the stronger assumption of being area-minimizing (thus covering only case (a) of Definition 1.1), but a close inspection of the proof given in [20] shows that the only property of area-minimizing currents relevant to the arguments is the validity of the density lower bound (6.128).

Theorem 6.22. *Let Q, m and n be positive integers. Then there are $\varepsilon > 0, c > 0$ and C geometric constants with the following property. Assume that $\pi_0 = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$ and that:*

(h1) *T is an integer rectifiable m-dimensional current as in Definition 1.1 with $U = \mathbf{C}_r(x_0)$ and $\boldsymbol{\Omega} \leqslant c$;*

(h2) $\partial T \llcorner \mathbf{C}_r(x_0) = 0$, $(\mathbf{p}_{\pi_0})_\sharp T \llcorner \mathbf{C}_r(x_0) = Q\,[\![B_r(\mathbf{p}_{\pi_0}(x_0))]\!]$ *and* $E := \mathbf{E}(R, \mathbf{C}_r(x_0)) < \varepsilon$.

aThen there are $k \in \mathbb{N}$, points $\{y_1, \ldots, y_k\} \subset \mathbb{R}^{m+n}$ and positive integers $Q_1, \ldots, Q_k$ such that:

(i) *having set $\sigma := CE^{\frac{1}{2m}}$, the open sets $\mathbf{S}_i := \mathbb{R}^m \times (y_i +]-r\sigma, r\sigma[^n)$ are pairwise disjoint and* $\operatorname{spt}(T) \cap \mathbf{C}_{r(1-\sigma|\log E|)}(x_0) \subset \cup_i \mathbf{S}_i$;

(ii) $(\mathbf{p}_{\pi_0})_\sharp[T \llcorner (\mathbf{C}_{r(1-\sigma|\log E|)}(x_0) \cap \mathbf{S}_i)] = Q_i \left[\!\!\left[B_{r(1-\sigma|\log E|)}(\mathbf{p}_{\pi_0}(x_0), \pi_0)\right]\!\!\right]$ $\forall i \in \{1, \ldots, k\}$.

(iii) *for every $p \in \operatorname{spt}(T) \cap \mathbf{C}_{r(1-\sigma|\log E|)}(x_0)$ we have $\Theta(T,p) < \max\{Q_i\} + \frac{1}{2}$.*

6.7 APPENDIX B: TWO TECHNICAL LEMMAS

Lemma 6.23 ([20, Lemma 5.6])**.** *Fix m, n, l and Q. There are geometric constants c_0, C_0 with the following property. Consider two triples of planes $(\pi, \varkappa, \varpi)$ and $(\bar\pi, \bar\varkappa, \bar\varpi)$, where*

- *π and $\bar\pi$ are m-dimensional;*
- *$\varkappa$ and $\bar\varkappa$ are $\bar n$-dimensional and orthogonal, respectively, to π and $\bar\pi$;*

- *ϖ and $\bar\varpi$ l-dimensional and orthogonal, respectively, to $\pi \times \varkappa$ and $\bar\pi \times \bar\varkappa$.*

Assume $\mathrm{An} := |\pi - \bar\pi| + |\varkappa - \bar\varkappa| \leqslant c_0$ *and let* $\Psi : \pi \times \varkappa \to \varpi$, $\bar\Psi : \bar\pi \times \bar\varkappa \to \bar\varpi$ *be two maps whose graphs coincide and such that* $|\bar\Psi(0)| \leqslant c_0 r$ *and* $\|D\bar\Psi\|_{C^0} \leqslant c_0$. *Let* $u : B_{8r}(0, \bar\pi) \to \mathcal{A}_Q(\bar\varkappa)$ *be a map with* $\mathrm{Lip}(u) \leqslant c_0$ *and* $\|u\|_{C^0} \leqslant c_0 r$ *and set* $f(x) = \sum_i [\![(u_i(x), \bar\Psi(x, u_i(x)))]\!]$ *and* $\mathbf{f}(x) = (\boldsymbol{\eta} \circ u(x), \bar\Psi(x, \boldsymbol{\eta} \circ u(x)))$. *Then there are*

- *a map* $\hat u : B_{4r}(0,\pi) \to \mathcal{A}_Q(\varkappa)$ *such that the map* $\hat f(x) := \sum_i [\![(\hat u_i(x), \Psi(x, \hat u_i(x)))]\!]$ *satisfies* $\mathbf{G}_{\hat f} = \mathbf{G}_f \llcorner \mathbf{C}_{4r}(0,\pi)$
- *and a map* $\hat{\mathbf{f}} : B_{4r}(0,\pi) \to \varkappa \times \varpi$ *such that* $\mathbf{G}_{\hat{\mathbf{f}}} = \mathbf{G}_{\mathbf{f}} \llcorner \mathbf{C}_{4r}(0,\pi)$.

Finally, if $\mathbf{g}(x) := (\boldsymbol{\eta} \circ \hat u(x), \Psi(x, \boldsymbol{\eta} \circ \hat u(x)))$, *then*

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1} \leqslant C_0 \left(\|f\|_{C^0} + r\mathrm{An}\right) \left(\mathrm{Dir}(f) + r^m \left(\|D\bar\Psi\|^2_{C^0} + \mathrm{An}^2\right)\right). \tag{6.129}$$

The proof of this Lemma can be found in [20, Appendix D].

Lemma 6.24 ([20, Lemma C.2]). *For every* m, $r < s$ *and* κ *there is a positive constant* C *(depending on* m, κ *and* $\frac{s}{r}$*) with the following property. Let* f *be a* $C^{3,\kappa}$ *function in the ball* $B_s \subset \mathbb{R}^m$. *Then*

$$\|D^j f\|_{C^0(B_r)} \leqslant C r^{-m-j} \|f\|_{L^1(B_s)} + C r^{3+\kappa-j} [D^3 f]_{\kappa, B_s} \qquad \forall j \in \{0,1,2,3\}. \tag{6.130}$$

Proof. A simple covering argument reduces the lemma to the case $s = 2r$. Moreover, define $f_r(x) := f(rx)$ to see that we can assume $r = 1$. So our goal is to show

$$\sum_{j=0}^{3} |D^j f(y)| \leqslant C\|f - g\|_{L^1} + C[D^3 f]_\kappa \qquad \forall y \in B_1, \forall f \in C^{3,\kappa}(B_2). \tag{6.131}$$

By translating it suffices then to prove the estimate

$$\sum_{j=0}^{3} |D^j f(0)| \leqslant C\|f\|_{L^1(B_1)} + C[D^3 f]_{\kappa, B_1} \qquad \forall f \in C^{3,\kappa}(B_1). \tag{6.132}$$

Consider now the space of polynomials R in m variables of degree at most 3, which we write as $R = \sum_{j=0}^3 A_j x^j$. This is a finite dimensional vector space, on which we can define the norms $|R| := \sum_{j=0}^3 |A_j|$ and $\|R\| := \int_{B_1} |R(x)|\,dx$. These two norms must then be equivalent, so there is a constant C (depending only on m), such that $|R| \leqslant C\|R\|$ for any such polynomial. In particular, if P is the Taylor polynomial of third order for f at the point 0, we conclude

$$\begin{aligned}\sum_{j=0}^{3} |D^j f(0)| = |P| \leqslant C\|P\| = C \int_{B_1} |P(x)|\,dx &\leqslant C\|f\|_{L^1(B_1)} + C\|f - P\|_{L^1(B_1)} \\ &\leqslant C\|f\|_{L^1} + C[D^3 f]_\kappa .\end{aligned}$$

□

7

THE NORMAL APPROXIMATION

In what follows we assume that the conclusions of Theorem 6.9 apply and denote by $\mathcal{M}$ the corresponding center manifold. For any Borel set $\mathcal{V} \subset \mathcal{M}$ we will denote by $|\mathcal{V}|$ its $\mathcal{H}^2$-measure and will write $\int_{\mathcal{V}} f$ for the integral of f with respect to $\mathcal{H}^2$. $\mathcal{B}_r(q)$ denotes the geodesic balls in $\mathcal{M}$. Moreover, we refer to Chapter 3 for all the relevant notation pertaining to the differentiation of (multiple valued) maps defined on $\mathcal{M}$, induced currents, differential geometric tensors and so on.

7.1 ESTIMATES, SEPARATION AND SPLITTING

We next define the open set

(V) $\mathbf{V} := \{(x, y) : x \in [-1, 1]^2 \text{ and } |\boldsymbol{\varphi}(x, w) - y| \leqslant c_s |x|^b/2\}$.

$\mathbf{V}$ is clearly an horned neighborhood of the graph of $\boldsymbol{\varphi}$. By (2.18), Assumption 3 and Theorem 6.9 it is clear that the following corollary holds

Corollary 7.1. *Under the hypotheses of Theorem 6.9, there is $r > 0$ such that*

(i) For every $x \in \mathbb{R}^2$ with $0 < |x| = 2\rho < 2r$, the set $\mathbf{C}_\rho(x) \cap \mathbf{V}$ consists of $\bar{Q}$ distinct connected components and $\mathrm{spt}(T) \subset \mathbf{V}$.

(ii) There is a well-defined nearest point projection $\mathbf{p} : \mathbf{V} \cap \mathbf{C}_{4r} \to \mathrm{Gr}(\boldsymbol{\varphi})$, which is a $C^{2,\kappa}$ map.

(iii) For every $L \in \mathscr{W}$ with $d(L) \leqslant 2r$ and every $q \in L$ we have $\mathrm{spt}(\langle T, \mathbf{p}, \boldsymbol{\Phi}(q)\rangle) \subset \big\{y : |\boldsymbol{\Phi}(q) - y| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}\big\}$.

(iv) $\langle T, \mathbf{p}, p\rangle = Q\,[\![p]\!]$ for every $p \in \boldsymbol{\Phi}(\Gamma) \cap \mathbf{C}_{2r} \setminus \{0\}$.

The main goal of this paper is to couple the branched center manifold of Theorem 6.9 with a good map defined on $\mathcal{M}$ and taking values in its normal bundle, which approximates accurately T in a neighborhood of the origin.

Definition 7.2 ($\mathcal{M}$-normal approximation)**.** Let r be as in Corollary 7.1 and define

(U) $\mathbf{U} := \mathbf{p}^{-1}(\mathbf{C}_{2r} \cap \mathfrak{B}_Q)$.

An *$\mathcal{M}$-normal approximation* of T is given by a pair $(\mathcal{K}, F)$ such that

(A1) $F : \mathbf{C}_{2r} \cap \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbf{U})$ is Lipschitz and takes the form $F(x) = \sum_i [\![x + N_i(x)]\!]$, with $N_i(x) \perp T_x\mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x and i.

(A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\boldsymbol{\Phi}\big(\Gamma \cap \mathbf{C}_{2r}\big)$ and $\mathbf{T}_F \llcorner \mathbf{p}^{-1}(\mathcal{K}) = T \llcorner \mathbf{p}^{-1}(\mathcal{K})$.

The map $N = \sum_i [\![N_i]\!] : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ is *the normal part* of F.

In the definition above it is not required that the map F approximates efficiently the current outside the set $\mathbf{\Phi}(\Gamma)$. However, all the maps constructed here will approximate T with a high degree of accuracy in each Whitney region: such estimates are detailed in the next theorem. In order to simplify the notation, we will use $\|N|_{\mathcal{V}}\|_{C^0}$ (or $\|N|_{\mathcal{V}}\|_0$) to denote the number $\sup_{x\in\mathcal{V}} \mathcal{G}(N(x), Q[\![0]\!]) = \sup_{x\in\mathcal{V}} |N(x)|$.

Theorem 7.3 (Local estimates for the $\mathcal{M}$-normal approximation). *Let* r *be as in Corollary 7.1 and* $\mathbf{U}$ *as in Definition 7.2. Then there is an* $\mathcal{M}$*-normal approximation* $(\mathcal{K}, F)$ *such that the following estimates hold on every Whitney region* $\mathcal{L}$ *associated to* $L \in \mathscr{W}$ *with* $d(L) \leqslant r$*:*

$$\mathrm{Lip}(N|_{\mathcal{L}}) \leqslant C\boldsymbol{m}_0^{\beta_0} d(L)^{\beta_0\gamma_0}\ell(L)^{\beta_0\gamma_0} \quad \textit{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}, \tag{7.1}$$

$$|\mathcal{L}\setminus\mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L}_i)) \leqslant C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\,\ell(L)^{2+(1+\beta_0)(2-2\delta_1)}, \tag{7.2}$$

$$\int_{\mathcal{L}} |DN|^2 \leqslant C\boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\,\ell(L)^{4-2\delta_1}\,. \tag{7.3}$$

Moreover, for every Borel $\mathcal{V}\subset\mathcal{L}$*, it holds*

$$\int_{\mathcal{V}} |\boldsymbol{\eta}\circ N| \leqslant C\boldsymbol{m}_0 d(L)^{2(1+\beta_0)\gamma_0-2-\beta_2}\,\ell(L)^{5+\beta_2/4} + C\boldsymbol{m}_0^{\frac{1}{2}+\beta_0} d(L)^{2\beta_0\gamma_0+\gamma_0-1-\beta_2}\,\ell(L)^{1+\beta_2}\int_{\mathcal{V}}\mathcal{G}\big(N, Q[\![\boldsymbol{\eta}\circ N]\!]\big)\,. \tag{7.4}$$

The constant $C = C(M_0, N_0, C_e, C_h)$ *does not depend on* ε_{41}.

7.1.1 *Separation and splitting*

We conclude this section with two theorems which allow us to estimate the sidelengths of the squares of type $\mathscr{W}_h$ and $\mathscr{W}_e$. The squares in $\mathscr{W}_n$ do not enjoy similar bounds, but they can be partitioned in families, each of which consists of squares sufficiently close to an element of $\mathscr{W}_e$.

Proposition 7.4 (Separation). *There is a dimensional constant* $C^\sharp > 0$ *with the following property. Assume the hypotheses of Theorem 7.3, and in addition* $C_h^4 \geqslant C^\sharp C_e$. *If* ε_{41} *is sufficiently small, then the following conclusions hold for every* $L \in \mathscr{W}_h$ *with* $d(L) \leqslant r$ *:*

(S1) $\Theta(T_L, p) \leqslant Q - \frac{1}{2}$ *for every* $p \in \mathbf{B}_{16r_L}(p_L)$.

(S2) $L \cap H = \emptyset$ *for every* $H \in \mathscr{W}_n$ *with* $\ell(H) \leqslant \frac{1}{2}\ell(L)$.

(S3) $\mathcal{G}\big(N(x), Q[\![\boldsymbol{\eta}\circ N(x)]\!]\big) \geqslant \frac{1}{4} C_h \boldsymbol{m}_0^{\frac{1}{4}}\, d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}\ \forall x \in \mathbf{\Phi}(B_{4\ell(L)}(z_L, w_L))$.

A simple corollary of the previous proposition is the following.

Corollary 7.5 (Domains of influence). *For any* $H \in \mathscr{W}_n$ *there is a chain* $L = L_0, \ldots, L_n = H$ *such that* *(a)* $L_0 \in \mathscr{W}_e$ *and* $L_k \in \mathscr{W}_n$ *for all* $k > 0$*;*

(b) $L_k \cap L_{k-1} \neq \emptyset$ *and* $\ell(L_k) = \frac{\ell(L_{k-1})}{2}$ *for all* $k > 0$*.*

In particular $H \subset B_{3\sqrt{2}\ell(L)}(z_L, w_L)$*.*

We use this last corollary to partition $\mathscr{W}_n$.

Definition 7.6 (Domains of influence)**.** We first fix an ordering of the squares in $\mathscr{W}_e$ as $\{J_i\}_{i\in\mathbb{N}}$ so that their sidelengths do not increase. Then $H \in \mathscr{W}_n$ belongs to $\mathscr{W}_n(J_0)$ (the domain of influence of J_0) if there is a chain as in Corollary 7.5 with $L_0 = J_0$. Inductively, $\mathscr{W}_n(J_r)$ is the set of squares $H \in \mathscr{W}_n \setminus \cup_{i<r}\mathscr{W}_n(J_i)$ for which there is a chain as in Corollary 7.5 with $L_0 = J_r$.

Proposition 7.7 (Splitting)**.** *There are constants* C_1,$C_2(M_0)$, $\bar{r}(M_0, N_0, C_e)$ *such that, if* $M_0 \geqslant C_1$, $C_e \geqslant C_0(M_0)$*, if the hypotheses of Theorem 7.3 hold and* ε_{41} *is chosen sufficiently small, then the following holds. If* $L \in \mathscr{W}_e$ *with* $d(L) \leqslant \bar{r}$, $q \in \mathfrak{B}$ *with* $\operatorname{dist}(L, q) \leqslant 4\sqrt{2}\,\ell(L)$ *and* $\Omega := \mathbf{\Phi}(B_{\ell(L)/8}(q))$*, then:*

$$C_e \boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell(L)^{4-2\delta_2} \leqslant \ell(L)^2\mathbf{E}(T_L, \mathbf{B}_L) \leqslant C\int_\Omega |DN|^2\,, \tag{7.5}$$

$$\int_{\mathcal{L}} |DN|^2 \leqslant C\ell(L)^2\mathbf{E}(T, \mathbf{B}_L) \leqslant C\ell(L)^{-2}\int_\Omega |N|^2\,, \tag{7.6}$$

where $C = C(M_0, N_0, C_e, C_h)$*.*

7.2 THE CONSTRUCTION OF THE APPROXIMATING MAP n

In this section we prove Corollary 7.1 and Theorem 7.3.

7.2.1 *Proof of Corollary 7.1*

Statement (i) is an obvious consequence of (2.18) and (6.19). As for statement (ii), the argument is the same given in the proof of Lemma 2.15 for the existence of the nearest point projection $\mathbf{P} : \mathbf{V}_{u,a} \cap \mathbf{C}_1 \to \operatorname{Gr}(u)$.

For what concerns (iii), let $L \in \mathscr{W}$, denote by (z_L, w_L) its center and set $p := \mathbf{\Phi}(q)$ We start by observing that $\operatorname{spt}(\langle T, \mathbf{p}, p\rangle) \subset \operatorname{spt}(T_J)$ for some ancestor J of L, given the thickness of the horned neighborhood $\mathbf{V}$ and the estimates in Theorem 6.9. We next claim that

$$\operatorname{spt}(\langle T, \mathbf{p}, p\rangle) \subset \mathbf{B}_{r_L}(p)\,. \tag{7.7}$$

Assuming this for the moment, recall that we have already shown the estimate

$$\|\boldsymbol{\varphi} - g_L\|_{C^0(L)} \leqslant C\boldsymbol{m}_0^{\frac{\gamma_0}{2}}\ell(L)^{1+\beta_2}\,,$$

cf. the previous section. Recall also that the graph of g_L coincides with that of h_L and that we have shown

$$\|h_L - \eta\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}\,,$$

where $(\xi, \eta) \in \pi_L \times \pi_L^\perp$ are the coordinates for p_L, cf. (6.95). Since $\mathrm{spt}(T_J) \cap \mathbf{C}_{8r_L}(p_L, \pi_L) \subset \mathrm{spt}(T_L)$, we must then have $\mathrm{spt}(\langle T, \mathbf{p}, p\rangle) \subset \mathrm{spt}(\langle T, \mathbf{p}, p\rangle) \cap \mathbf{B}_{r_L}(p) \subset \mathrm{spt}(T_L) \cap \mathbf{C}_{8r_L}(p_L, \pi_L)$. Recalling that $p_L \in \mathrm{spt}(T_L)$ and that we have the bound

$$\mathbf{h}(T_L, \mathbf{C}_{8r_L}(p_L, \pi_L)) \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2},$$

we conclude that no point of $\mathrm{spt}(\langle T, \mathbf{p}, p\rangle)$ can be at distance larger than

$$\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}$$

from the graph of h_L. Putting all these estimates together, no point of $\mathrm{spt}(\langle T, \mathbf{p}, p\rangle)$ can be at a distance larger than $\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}$ from $\mathrm{Gr}(\boldsymbol{\varphi})$. Since for every $p' \in \mathrm{spt}(\langle T, \mathbf{p}, p\rangle)$ the point p is the closest in the graph of $\boldsymbol{\varphi}$, this completes the proof of (iii), provided we show (7.7).

If (7.7) is false, there is a $p' \in \mathrm{spt}(\langle T, \mathbf{p}, p)$ and an ancestor J with largest sidelength among those for which $|p' - p| \geqslant r_J$. Let π be the tangent to $\mathcal{M}$ at p and observe that we have the estimates $|\pi - \pi_J| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}$ and $|\pi - \pi_0| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}$. If J were an element of $\mathscr{S}^{N_0}$, the height bound would imply $|p' - p| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} r_J^{1+\gamma_0}$. If $J \notin \mathscr{S}^{N_0}$ and we let H be the father of J, we then conclude that $q \in \mathbf{B}_H$ and thus we have $|p' - p| \leqslant C\mathbf{h}(T, \mathbf{B}_H) \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} \ell(H)^{1+\beta_2}$. In both cases this would be incompatible with $|p' - p| \geqslant r_J = \frac{r_H}{2}$, provided $\varepsilon_{41} \leqslant c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$

We next prove (iv). Fix a point $(z, w) \in \mathfrak{B}$ which belongs to $\boldsymbol{\Gamma}$ and set $p := (z, \boldsymbol{\varphi}(z, w)) = \boldsymbol{\Phi}(z, w)$. To prove our statement we claim in fact that:

$$Q\,[\![T_p\mathcal{M}]\!] \text{ is the unique tangent cone to } T \text{ at } p \tag{7.8}$$

$$\mathrm{spt}(T) \cap \mathbf{p}^{-1}(\{p\}) = \{p\}. \tag{7.9}$$

By construction there is an infinite chain $L_{N_0} \supset L_{N_0 - 1} \supset \ldots \supset L_i \supset \ldots$ where $(z, w) \in L_i \in \mathscr{S}^i$ for every i. Set $\pi_i := \pi_{L_i}$. By our construction and the estimates of the previous sections, it is obvious that $\pi_{L_i} \to \pi = T_p\mathcal{M}$. In fact since $|\pi_{L_i} - \pi_{L_{i+1}}| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} |z|^{\gamma_0 + \delta_1 - 1} \ell(L_i)^{1+\delta_1}$, we easily infer

$$|\pi - \pi_{L_i}| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} |z|^{\gamma_0 + \delta_1 - 1} \ell(L_i)^{1+\delta_1}. \tag{7.10}$$

On the other hand by the height and excess bounds, it also obvious that $T_{p_{L_i}, r_{L_i}}$ converges, in $\mathbf{B}_1$, to $Q\,[\![\pi]\!]$. Since $r_{L_i}/r_{L_{i+1}} = 2$ and $p_{L_i} \to p$ (in fact $|\boldsymbol{\Phi}(z, w) - p_{L,i}| \leqslant C2^{-i}$), (7.8) is then obvious.

Assume now that (7.9) is false and let $p' \in \mathrm{spt}(\langle T, \mathbf{p}, p\rangle)$. Again by the height of $\mathbf{V}$ it turns out that $p' \in \mathrm{spt}(T_{L_{N_0}})$. Let j be the integer such that $2^{-j-1}|z| \leqslant |p - p'| \leqslant 2^{-j}|z|$. By the height bound in $\mathcal{C}_{2|z|}(0, \pi_0)$ it follows that, if ε_{41} is sufficiently small, then certainly $j \geqslant N_0 + 2$. This means that there is an L_i such that $p' \in \mathbf{B}_{L_i}$ and obviously $\ell(L_i) \leqslant C|z|2^{-j}$. Recall that $\mathrm{spt}(T_{L_{N_0}}) \cap \mathbf{B}_{L_i} \subset \mathrm{spt}(T_{L_i})$ On the other hand, by (7.10), we have

$$|p - p'| \leqslant (1 + C|\pi_{L_i} - \pi|)\mathbf{h}(T_{L_i}, \mathbf{B}_{L_i}) \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L_i)^{\frac{\gamma_0}{2} - \beta_2} \ell(L_i)^{1+\beta_2} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} |z|^{1+\frac{\gamma_0}{2}} 2^{-j}.$$

Since the constant C depends upon the parameters C_h, C_e, M_0 and N_0, but not upon ε_{41}, the latter bound contradicts $|p - p'| \geqslant 2^{-j-1}$ provided ε_{41} is chosen sufficiently small.

7.2.2 *Proof of Theorem 7.3: Part I*

We set $F(p) = Q\,[\![p]\!]$ for $p \in \mathbf{\Phi}(\mathbf{\Gamma})$. For every $L \in \mathscr{W}^j$ consider the π_L-approximating function $f_L : \mathbf{C}_{8r_L}(p_L, \pi_L) \to \mathcal{A}_Q(\pi_L^\perp)$ of Definition 6.4 and $K_L \subset B_{8r_L}(p_L, \pi_L)$ the projection on π_L of $\mathrm{spt}(T_L) \cap \mathrm{Gr}(f_L)$. In particular we have $\mathbf{G}_{f_L|_{K_L}} = T_L \llcorner (K_L \times \pi_L^\perp)$. We then denote by $\mathscr{D}(L)$ the portions of the supports of T_L and $\mathrm{Gr}(f_L)$ which differ:

$$\mathscr{D}(L) := (\mathrm{spt}(T_L) \cup \mathrm{Gr}(f_L)) \cap \left[(B_{8r_L}(p_L, \pi_L) \setminus K_L) \times \pi_L^\perp\right].$$

Observe that, by Theorem 2.8 and our choice of the parameters, for $E := \mathbf{E}(T_L, \mathbf{C}_{32r_L}(p_L, \pi_L))$, we have

$$\begin{aligned}\mathcal{H}^m(\mathscr{D}(L)) &\leqslant C E^{\beta_0}(E + \ell(L)^2 \boldsymbol{m}_0)\ell(L)^2 \\ &\leqslant C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0 - 2 + 2\delta_1)} \ell(L)^{2+(1+\beta_0)(2-2\delta_1)} \end{aligned} \tag{7.11}$$

Let $\mathcal{L}$ be the Whitney region in Definition 6.10 and set $\mathcal{L}' := \mathbf{\Phi}(J)$ where J is the cube concentric to L with $\ell(J) = \frac{9}{8}\ell(L)$. Observe that the graphical structure of $\mathbf{\Phi}$, our choice of the constants and condition (NN) ensure that

$$L \cap H = \emptyset \iff \mathcal{L}' \cap \mathcal{H}' = \emptyset \qquad \forall H, L \in \mathscr{W}, \tag{7.12}$$

$$\mathbf{\Phi}(\mathbf{\Gamma}) \cap \mathcal{L}' = \emptyset \qquad \forall L \in \mathscr{W}. \tag{7.13}$$

We then apply Theorem 3.18 to the map f_L, the plane π_L the center manifold $\boldsymbol{\varphi}$ as a graph over π_L to obtain maps $F_L : \mathcal{L}' \to \mathcal{A}_Q(\mathbf{U})$, $N_L : \mathcal{L}' \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ with the following poperties:

- $F_L(p) = \sum_i [\![p + (N_L)_i(p)]\!]$,
- $(N_L)_i(p) \perp T_p\mathcal{M}$ for every $p \in \mathcal{L}'$
- and $\mathbf{G}_{f_L} \llcorner (\mathbf{p}^{-1}(\mathcal{L}')) = \mathbf{T}_{F_L} \llcorner (\mathbf{p}^{-1}(\mathcal{L}'))$.

For each L consider the set $\mathscr{W}(L)$ of elements in $\mathscr{W}$ which have a nonempty intersection with L. We then define the set $\mathcal{K}$ in the following way:

$$\mathcal{K} = (\mathcal{M} \cap \mathbf{C}_{2r}) \setminus \Big(\bigcup_{L \in \mathscr{W}} \Big(\mathcal{L}' \cap \bigcup_{M \in \mathscr{W}(L)} \mathbf{p}(\mathscr{D}(M)) \Big) \Big). \tag{7.14}$$

In other words $\mathcal{K}$ is obtained from $\mathcal{M}$ by removing in each $\mathcal{L}'$ those points x for which there is a neighboring cube M such that the slice of $\mathbf{T}_{F_M}$ at x (relative to the projection $\mathbf{p}$) does not coincide with the slice of T. Observe that, by (7.13), $\mathcal{K}$ contains necessarily $\mathbf{\Gamma}$. Moreover, recall that $\mathrm{Lip}(\mathbf{p}) \leqslant C$, that the cardinality $\mathscr{W}(L)$ is bounded by a geometric constant and that each element of $\mathscr{W}(L)$ has side-length at most twice that of L. Thus (7.11) implies

$$\begin{aligned}|\mathcal{L} \setminus \mathcal{K}| \leqslant |\mathcal{L}' \setminus \mathcal{K}| &\leqslant \sum_{M \in \mathscr{W}(L)} \sum_{H \in \mathscr{W}(M)} \|T_H\|(\mathscr{D}(H)) \\ &\leqslant C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0 - 2 + 2\delta_1)} \ell(L)^{2+(1+\beta_0)(2-2\delta_1)}. \end{aligned} \tag{7.15}$$

By (7.12), if J and L are such that $\mathcal{J}' \cap \mathcal{L}' \neq \emptyset$, then $J \in \mathscr{W}(L)$ and therefore $F_L = F_J$ on $\mathcal{K} \cap (\mathcal{J}' \cap \mathcal{L}')$. We can therefore define a unique map on $\mathcal{K}$ by simply setting $F(p) = F_L(p)$ if $p \in \mathcal{K} \cap \mathcal{L}'$. Notice that $\mathbf{T}_F = T \llcorner \mathbf{p}^{-1}(\mathcal{K})$, which implies two facts. First, by Corollary 7.1(iii) we also have that $N(p) := \sum_i [\![F_i(p) - p]\!]$ enjoys the bound

$$\|N|_{\mathcal{L}\cap\mathcal{K}}\|_{C^0} \leqslant C\boldsymbol{m}_0^{\frac{1}{4}}\, d(L)^{\frac{\gamma_0}{2}-\beta_2}\, \ell(L)^{1+\beta_2}.$$

Secondly,

$$\begin{aligned}\|T\|(\mathbf{p}^{-1}(\mathcal{L}\setminus\mathcal{K})) \leqslant & \sum_{M\in\mathscr{W}(L)}\ \sum_{H\in\mathscr{W}(M)} \|T_H\|(\mathscr{D}(H)) \\ \leqslant & C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} \ell(L)^{2+(1+\beta_0)(2-2\delta_1)}\,. \end{aligned} \tag{7.16}$$

Finally, since $\mathcal{M}$ is given on π_L as the graph of h_L, the Lipschitz constant of N_L can be estimated, using Theorem 3.18 and (6.97) with $L = H$, by

$$\mathrm{Lip}(N_L) \leqslant C\left(\|D^2 h_L\|_{C^0}\|N\|_{C^0} + \|Dh_L\|_{C^0} + \mathrm{Lip}(f_L)\right) \leqslant C\left(\boldsymbol{m}_0\, d(L)^{\gamma_0}\,\ell(L)^{\gamma_0}\right)^{\beta_0}\,, \tag{7.17}$$

so that our map has the Lipschitz bound of (7.1). Hence, F and N satisfy the bounds (7.1) on $\mathcal{K}$. We next extend them to the whole center manifold and conclude (7.2) from (7.16) and (7.15). The extension is achieved in three steps:

- we first extend the map F to a map $\bar{F}$ taking values in $\mathcal{A}_Q(\mathbf{V})$;
- we then modify $\bar{F}$ to achieve the form $\hat{F}(x) = \sum_i [\![x + \hat{N}_i(x)]\!]$ with $\hat{N}_i(x) \perp T_x\mathcal{M}$ for every x;
- in the cases (a) and (c) of Definition 1.1 we finally modify $\hat{F}$ to reach the desired extension $F(x) = \sum_i [\![x + N_i(x)]\!]$, with $N_i(x) \perp T_x\mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x.

First extension. We use on $\mathcal{M}$ the coordinates induced by its graphical structure, i.e. we work with variables in flat domains. Note that the domain parameterizing the Whitney region for $L \in \mathscr{W}$ is then the cube concentric to L and with side-length $\frac{17}{16}\ell(L)$. The multivalued map N is extended to a multivalued $\bar{N}$ inductively to appropriate neighborhoods of the skeleta of the Whitney decomposition. The extension of F will obviously be $\bar{F}(x) = \sum_i [\![x + \bar{N}_i(x)]\!]$. The neighborhoods of the skeleta are defined in this way:

1. if p belongs to the 0-skeleton, we let $L \in \mathscr{W}$ be (one of) the smallest cubes containing it and define $U^p := B_{\ell(L)/16}(p)$;
2. if $\sigma = [p, q] \subset L$ is the edge of a cube and $L \in \mathscr{W}$ is (one of) the smallest cube intersecting σ, we then define U^σ to be the neighborhood of size $\frac{1}{4}\frac{\ell(L)}{16}$ of σ minus the closure of the unions of the U^r's, where r runs in the 0-skeleton.

Denote by $\bar{U}$ the closure of the union of all these neighborhoods and let $\{V_i\}$ be the connected components of the complement. For each V_i there is a $L_i \in \mathscr{W}$ such that $V_i \subset L_i$. Moreover, V_i has distance $c_0\ell(L)$ from ∂L_i, where c_0 is a geometric constant. It is also clear that if τ and σ are two distinct facets of the same cube L with the same dimension, then the distance

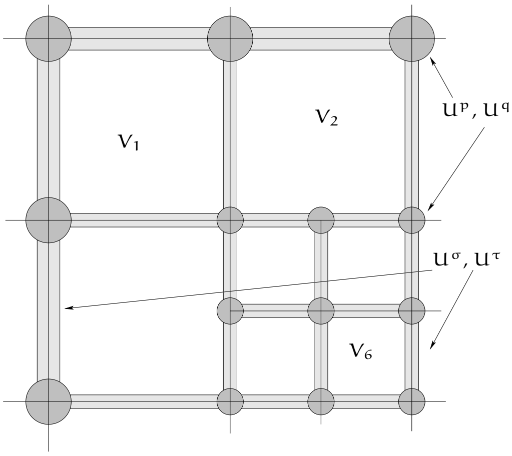

Figure 1: The sets U^p, U^σ and V_i.

between any pair of points x, y with $x \in U^\tau$ and $y \in U^\sigma$ is at least $c_0\ell(L)$. In Figure 1 the various domains are shown in a piece of a 2-dimensional decomposition.

At a first step we extend N to a new map $\bar{N}$ separately on each U^p, where p are the points in the 0-skeleton. Fix $p \in L$ and let $\mathrm{St}(p)$ be the union of all cubes which contain p. Observe that the Lipschitz constant of $N|_{\mathcal{K}\cap \mathrm{St}(p)}$ is smaller than

$$C\,(\boldsymbol{m}_0\, d(L)^{\gamma_0}\, \ell(L)^{\gamma_0})^{\beta_0}$$

and that

$$|N| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}\,.$$

We can therefore extend the map $N|_{\mathcal{K}\cap \mathrm{St}(p)}$ to $U^p \cup (\mathcal{K}\cap \mathrm{St}(p))$ at the price of slightly enlarging this Lipschitz constant and this height bound, using Proposition 3.4. Being the U^p disjoint, the resulting map, for which we use the symbol $\bar{N}$, is well-defined.

It is obvious that this map has the desired height bound in each Whitney region. We therefore want to estimate its Lipschitz constant. Consider $L \in \mathscr{W}$ and H concentric to L with side-length $\ell(H) = \frac{17}{16}\ell(L)$. Let $x, y \in H$. If $x, y \in U^p \cup (\mathcal{K}\cap \mathrm{St}(p))$ for some p, then there is nothing to check. If $x \in U^p$ and $y \in U^q$ with $p \neq q$, observe however that this would imply that p, q are both vertices of L. Given that $L \setminus \mathcal{K}$ has much smaller measure than L there is at least one point $z \in L \cap \mathcal{K}$. It is then obvious that

$$\mathcal{G}(\bar{N}(x), \bar{N}(y)) \leqslant \mathcal{G}(\bar{N}(x), \bar{N}(z)) + \mathcal{G}(\bar{N}(z), \bar{N}(y)) \leqslant C\,(\boldsymbol{m}_0 d(L)^{\gamma_0}\, \ell(L)^{\gamma_0})^{\beta_0}\, \ell(L),$$

and, since $|x-y| \geqslant c_0\ell(L)$, the desired bound readily follows. Observe moreover that, if x is in the closure of some U^q, then we can extend the map continuously to it. By the properties of the Whitney decomposition it follows that the union of the closures of the U^q and of $\mathcal{K}$ is closed and thus, w.l.o.g., we can assume that the domain of this new $\bar{N}$ is in fact closed.

We can repeat this procedure with the edges of the skeleta, that is in the argument above we simply replace points p with 1-dimensional faces σ, defining $\mathrm{St}(\sigma)$ as the union of the cubes which contain σ. In the final step we then extend over the domains V_i's: this time

$\mathrm{St}(V_i)$ will be defined as the union of the cubes which intersect the cube $L_i \supset V_i$. The correct height and Lipschitz bounds follow from the same arguments. Since the algorithm is applied 3 times, the original constants have been enlarged by a geometric factor.

Second extension. For each $x \in \mathcal{M}$ let $\mathbf{p}^\perp(x,\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the orthogonal projection on $(T_x\mathcal{M})^\perp$ and set $\hat{N}(x) = \sum_i [\![\mathbf{p}^\perp(x, \tilde{N}_i(x))]\!]$. Obviously $|\hat{N}(x)| \leqslant |\tilde{N}(x)|$, so the L^∞ bound is trivial. We now want to show the estimate on the Lipschitz constant. To this aim, fix two points p, q in the same Whitney region associated to L and parameterize the corresponding geodesic segment $\sigma \subset \mathcal{M}$ by arc-length $\gamma : [0, d(p,q)] \to \sigma$, where $d(p,q)$ denotes the geodesic distance on $\mathcal{M}$. Use Lemma 3.3 to select Q Lipschitz functions $N_i' : \sigma \to \mathbf{U}$ such that $\tilde{N}|_\gamma = \sum [\![N_i']\!]$ and $\mathrm{Lip}(N_i') \leqslant \mathrm{Lip}(\tilde{N})$. Fix a frame $\nu_1, \ldots, \nu_n$ on the normal bundle of $\mathcal{L} \subset \mathcal{M}$ with the property that $\|\nu_i\|_{C^0(\mathcal{L})} \leqslant C\|D\boldsymbol{\varphi}\|_{C^0} \leqslant C\,\mathbf{m}_0^{\frac{1}{2}}\, d(L)^{\gamma_0}$ and $\|D\nu_i\|_{C^0(\mathcal{L})} \leqslant C\|D^2\boldsymbol{\varphi}\|_{C0} \leqslant \mathbf{m}_0^{\frac{1}{2}} d(L)^{\gamma_0 - 1}$ (which is possible by [18, Appendix A], indeed we do this in $\mathcal{M}\setminus\{0\}$, where our manifold is C^{3,γ_0} and then we extend N to be 0 in the origin). We have $\hat{N}(\gamma(t)) = \sum_i [\![\hat{N}_i(t)]\!]$, where

$$\hat{N}_i(t) = \sum [\nu_j(\gamma(t)) \cdot N_i'(\gamma(t))]\, \nu_j(\gamma(t)).$$

Hence we can estimate

$$\left|\frac{d\hat{N}_i}{dt}\right| \leqslant C \sum_j [\|D\nu_j\|\|N_i'\|_{C^0} + \mathrm{Lip}(N_i')] \leqslant C\,(\mathbf{m}_0\, d(L)^{\gamma_0}\, \ell(L)^{\gamma_0})^{\beta_0}\ .$$

Integrating this inequality we find

$$\mathcal{G}(\hat{N}(p), \hat{N}(q)) \leqslant \sum_{i=1}^{Q} |\hat{N}_i(d(p,q)) - \hat{N}_i(0)| \leqslant C\,(\mathbf{m}_0\, d(L)^{\gamma_0}\, \ell(L)^{\gamma_0})^{\beta_0}\, d(p,q)\,.$$

Since $d(p,q)$ is comparable to $|p-q|$, we achieve the desired Lipschitz bound.

Third extension and conclusion. We still need to modify the map $\hat{N}$ in the cases (a) and (c) of Definition 1.1. For each $x \in \mathcal{M} \subset \Sigma$ consider the orthogonal complement $\varkappa_x$ of $T_x\mathcal{M}$ in $T_x\Sigma$. Let $\mathcal{T}$ be the fiber bundle $\bigcup_{x\in\mathcal{M}\setminus\{0\}} \varkappa_x$ and observe that, by the regularity of both $\mathcal{M}\setminus\{0\}$ and Σ, there is a C^{2,γ_0} trivialization (argue as in [18, Appendix A]). It is then obvious that there is a C^{0,γ_0} map $\Xi : \mathcal{T} \to \mathbb{R}^{m+n}$ with the following property: for each (x, v), $q := x + \Xi(x, v)$ is the only point in Σ which is orthogonal to $T_x\mathcal{M}$ and such that $\mathbf{p}_{\varkappa_x}(q - x) = v$. Let us denote by $\Omega(x,q)$ the map $\Xi(x, \mathbf{p}_{\varkappa_x}(q))$. This map extends to a C^{0,γ_0} map to the origin with the estimates

$$|D_x\Omega(x,q)| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|x|^{\gamma_0-1} \qquad \forall x \in \mathfrak{B}\setminus\{0\} \quad \forall q \text{ with } |q| \leqslant 1 \tag{7.18}$$

$$|D_x^2\Omega(x,q)| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|x|^{\gamma_0-2} \qquad \forall x \in \mathfrak{B}\setminus\{0\} \quad \forall q \text{ with } |q| \leqslant 1 \tag{7.19}$$

We then set $N(x) = \sum_i [\![\Xi(x, \mathbf{p}_{\varkappa_x}(\hat{N}_i(x)))]\!]$. Obviously, $N(x) = \hat{N}(x)$ for $x \in \mathcal{K}$, simply because in this case $x + N_i(x)$ belongs to Σ.

In order to show the Lipschitz bound, notice that, by the regularity of Σ,

$$|\Omega(x,q) - \Omega(x,p)| \leqslant C\,|q-p|\,. \tag{7.20}$$

Moreover, since $\Omega(x,0) = 0$ for every $x \in \mathcal{M} \subset \Sigma$, we have $D_x\Omega(x,0) = 0$. We therefore conclude that $|D_x\Omega(x,q)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}|x|^{\gamma_0-1}|q|$ and hence that

$$|\Omega(x,q) - \Omega(y,q)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}|x|^{\gamma_0-1}|q||y-x|\,. \tag{7.21}$$

Thus, fix two points $x, y \in \mathcal{L}_i$ and let us assume that $\mathcal{G}(\hat{N}(x),\hat{N}(y))^2 = \sum_i |\hat{N}_i(x) - \hat{N}_i(y)|^2$ (which can be achieved by a simple relabeling). We then conclude

$$\begin{aligned}
\mathcal{G}(N(x),N(y))^2 &\leqslant 2\sum_i |\Omega(x,\hat{N}_i(x)) - \Omega(x,\hat{N}_i(y))|^2 + 2\sum_i |\Omega(x,\hat{N}_i(y)) - \Omega(y,\hat{N}_i(y))|^2\\
&\leqslant C\,\boldsymbol{m}_0^{\frac{1}{2}}\,\mathcal{G}(\hat{N}(x),\hat{N}(y))^2 + C\,|x|^{2\gamma_0-2}\sum_i |\hat{N}_i(y)|^2|x-y|^2\\
&\leqslant C\,(\boldsymbol{m}_0\,d(L)^{\gamma_0}\,\ell(L)^{\gamma_0})^{2\beta_0}\,|x-y|^2 \qquad (7.22)\\
&\qquad + C\boldsymbol{m}_0 d(L)^{2\gamma_0-2+\gamma_0-2\beta_2}\ell(L)^{2+2\beta_2}|x-y|^2\\
&\leqslant C\,(\boldsymbol{m}_0\,d(L)^{\gamma_0}\,\ell(L)^{\gamma_0})^{2\beta_0}\,|x-y|^2\,, \qquad (7.23)
\end{aligned}$$

which proves the desired Lipschitz bound. Finally, using the fact that $\Omega(x,0) = 0$, we have $|\Omega(x,v)| \leqslant C|v|$ and the L^∞ bound readily follows.

7.2.3 *Proof of Theorem 7.3, Part II*

In this section we show the estimates (7.3) and (7.4). We start with the first one. Fix a Whitney region $\mathcal{L}$ and a corresponding square $L \in \mathscr{W}$. First consider the cylinder $\mathbf{C} := \mathbf{C}_{8r_L}(p_L,\pi_L)$, the tilted interpolating function g_L and the interpolating function h_L. Denote by $\vec{\mathcal{M}}$ the unit m-vector orienting $T\mathcal{M}$ and by $\vec{\tau}$ the one orienting $T\mathbf{G}_{h_L} = T\mathbf{G}_{g_L}$. Recalling that g_L and $\boldsymbol{\varphi}$ coincide in a neighborhood of (z_L, w_L) of L, by Theorem 6.9 we have

$$\sup_{p\in\mathcal{M}\cap\mathbf{C}} |\vec{\tau}(z_L, g_L(z_L,w_L)) - \vec{\mathcal{M}}(p)| \leqslant C\|D^2\boldsymbol{\varphi}_i\|_{C^0}\,\ell(L) \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}d(L)^{\gamma_0-1}\ell(L).$$

On the other hand recalling (6.80) in Proposition 6.20, we have

$$|\pi_L - \vec{\tau}(z_L, g_L(z_L,w_L))| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}d(L)^{\gamma_0-1+\delta_1}\ell(L)^{1-\delta_1}\,.$$

This in turn implies that

$$\sup_{\mathbf{C}\cap\mathcal{M}} |\vec{\mathcal{M}} - \pi_L| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}d(L)^{\gamma_0-1+\delta_1}\ell(L)^{1-\delta_1}. \tag{7.24}$$

Therefore, we can estimate

$$\begin{aligned}
&\int_{\mathbf{p}^{-1}(\mathcal{L})} |\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2\,d\|\mathbf{T}_F\|(x)\\
&\qquad\leqslant C\int_{\mathbf{p}^{-1}(\mathcal{L}_i)} |\vec{T}(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2\,d\|T\|(x) + C\boldsymbol{m}_0^{1+\beta_0}d(L)^{2(1+\beta_0)\gamma_0-2}\ell(L)^4\\
&\qquad\leqslant \int_{\mathbf{p}^{-1}(\mathcal{L}_i)} |\vec{T}(x) - \pi_L|^2\,d\|T\|(x) + C\boldsymbol{m}_0 d(L,0)^{2\gamma_0-2+2\delta_1}\ell(L)^{4-2\delta_1}\,. \qquad (7.25)
\end{aligned}$$

Since $\mathbf{p}^{-1}(\mathcal{L})\cap \mathrm{spt}(T_L)\subset \mathbf{C}$, the integral in (7.25) is bounded by $C\ell(L)^2\mathbf{E}(T_L,\mathbf{C},\pi_L)$. By Proposition 3.50 we then conclude

$$\begin{aligned}\int_{\mathcal{L}}|DN|^2 &\leqslant C\int_{\mathbf{p}^{-1}(\mathcal{L})}|\vec{\mathbf{T}}_F(x)-\vec{\mathcal{M}}(\mathbf{p}(x))|^2\,d\|\mathbf{T}_F\|(x)+C\|A_{\mathcal{M}}\|^2_{C^0(\mathbf{C}_{d(L)})\setminus\mathbf{C}_{d(L)/4})}\int_{\mathcal{L}}|N|^2\\ &\leqslant C\boldsymbol{m}_0\,d(L)^{2\gamma_0-2+2\delta_1}\,\ell(L)^{4-2\delta_1}+C\boldsymbol{m}_0\,d(L)^{2\gamma_0-2}\ell(L)^{4+2\beta_2}\,,\end{aligned}$$

where we have used $\|A_{\mathcal{M}}\|_{C^0(\mathbf{C}_{d(L)})\setminus\mathbf{C}_{d(L)/4})}\leqslant C\boldsymbol{m}_0^{\frac12}\mathrm{dist}(L,0)^{\gamma_0-1}$. This shows (7.3).

We finally come to (7.4). First observe that, by (7.1) and (7.2),

$$\begin{aligned}\int_{\mathcal{L}\setminus\mathcal{K}}|\boldsymbol{\eta}\circ N| &\leqslant C\,\boldsymbol{m}_0^{\frac14}d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}|\mathcal{L}\setminus\mathcal{K}|\\ &\leqslant C\boldsymbol{m}_0^{1+\beta_0+\frac14}d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)+(\frac{\gamma_0}{2}-\beta_2)}\ell(L)^{3+\beta_2+(1+\beta_0)(2-2\delta_1)}\,.\end{aligned}\tag{7.26}$$

Fix now $p\in\mathcal{K}$. Recalling that $F_L(p)=\sum_j [\![p+N_j(p)]\!]$ is given by Theorem 3.18 applied to the map f_L, we can conclude that

$$\begin{aligned}|\boldsymbol{\eta}\circ N_L(p)| &\leqslant C\,|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p))-\mathbf{p}^{\perp}_{\pi_L}(p)|+C\,\mathrm{Lip}(N_L|_{\mathcal{L}})\,|T_p\mathcal{M}-\pi_L|\,|N_L|(p)\\ &\overset{(7.24)}{\leqslant} C|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p))-\mathbf{p}^{\perp}_{\pi_L}(p)|\\ &\quad+C\boldsymbol{m}_0^{\frac12+\beta_0}\,d(L)^{\beta_0(2\gamma_0-2+2\delta_1)+\gamma_0-1+\delta_1}\ell(L)^{1-\delta_1+\beta_0(2-2\delta_1)}\\ &\quad(\mathcal{G}(N_L(p),Q\,[\![\boldsymbol{\eta}\circ N_L(p)]\!])+Q|\boldsymbol{\eta}\circ N_L|(p))\,.\end{aligned}\tag{7.27}$$

For ε_2 sufficiently small (depending only on $\beta_2,\gamma_2,M_0,N_0,C_e,C_h$), we then conclude that

$$\begin{aligned}|\boldsymbol{\eta}\circ N_L(p)| &\leqslant C\,|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p))-\mathbf{p}_{\pi_L^{\perp}}(p)|\\ &+C\boldsymbol{m}_0^{\frac12+\beta_0}\,d(L)^{\beta_0(2\gamma_0-2+2\delta_1)+\gamma_0-1+\delta_1}\,\ell(L)^{1-\delta_1+\beta_0(2-2\delta_1)}\mathcal{G}(N_L(p),Q\,[\![\boldsymbol{\eta}\circ N_L(p)]\!])\end{aligned}\tag{7.28}$$

Let next $\varphi':\pi_L\to\pi_L^{\perp}$ such that $\mathbf{G}_{\varphi'}=\mathcal{M}$. Applying Lemma 3.17 we conclude that

$$\int_{\mathcal{K}\cap\mathcal{V}}|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p))-\mathbf{p}_{\pi_L^{\perp}}(p))|\leqslant\int_{\mathbf{p}_{\pi_L}(\mathcal{K}\cap\mathcal{V})}|\boldsymbol{\eta}\circ f_L(x)-\varphi'(x)|\leqslant C\|g_L-\boldsymbol{\varphi}\|_{C^0(H)}\ell(L)^2\,,$$

where H is a cube concentric to L with side-length $\ell(H)=\frac98\ell(L)$. Next assume $L\in\mathscr{W}^j$ and let $k\geqslant j+2$. Consider the subset $\mathscr{P}^k(L)$ of all cubes in $\mathscr{P}^k$ which intersect L and recall that $\boldsymbol{\varphi}$ coincides with $\boldsymbol{\varphi}^k$ on H. Thus we can estimate

$$\begin{aligned}\|\boldsymbol{\varphi}_i-g_{L,i}\|_{L^1(H)} &\leqslant C\sum_{L'\in\mathscr{P}^k(L)}\|g_{L'}-g_L\|_{L^1(B_{r_L}(p_L,\pi_0))}\\ &\leqslant C\boldsymbol{m}_0\,d(L,0)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(L)^{5+\kappa}\,,\end{aligned}\tag{7.29}$$

where in the last inequality we used (6.81). We then conclude

$$\|\boldsymbol{\varphi}_i-g_{L,i}\|_{L^1(H)}\leqslant C\boldsymbol{m}_0\mathrm{dist}(L,0)^{2(1+\beta_0)\gamma_0-2-\beta_2}\ell(L)^{5+\kappa}$$

and (7.4) follows integrating (7.28) over $\mathcal{V}\cap\mathcal{K}$ and using (7.26).

7.3 separation and splitting before tilting

7.3.1 *Vertical separation*

In this section we prove Proposition 7.4 and Corollary 7.5.

Proof of Proposition 7.4. Let J be the father of L. By Lemma 6.5 and Proposition 6.3, Theorem 6.22 can be applied to the cylinder $\mathbf{C} := \mathbf{C}_{36r_J}(p_J, \pi_J)$. Moreover, $|p_J - p_L| \leqslant C\ell(J)$, where C is a geometric constant, and $r_J = 2r_L$. Thus, if M_0 is larger than a geometric constant, we have $\mathbf{B}_L \subset \mathbf{C}_{34r_J}(p_J, \pi_J)$. Denote by $\mathbf{q}_L$, $\mathbf{q}_J$ the projections $\mathbf{p}_{\hat\pi_L^\perp}$ and $\mathbf{p}_{\pi_J^\perp}$ respectively. Since $L \in \mathscr{W}_h$, there are two points $p_1, p_2 \in \mathrm{spt}(T_L) \cap \mathbf{B}_L$ such that

$$|\mathbf{q}_L(p_1 - p_2)| \geqslant C_h \boldsymbol{m}_0^{\frac14}\, d(L)^{\frac{\gamma_0}{2} - \beta_2} \ell(L)^{1+\beta_2}\,.$$

On the other hand, recalling Proposition 6.11, $|\pi_J - \hat\pi_L| \leqslant \bar C d(L)^{\gamma_0 - 1 + \delta_1} \ell(L)^{1-\delta_1}$, where $\bar C$ depends upon all the parameters except C_h and ε_{41}. Thus,

$$\begin{aligned}
|\mathbf{q}_J(p_1 - p_2)| &\geqslant |\mathbf{q}_L(p_1 - p_2)| - C_0|\pi_L - \pi_J||p_1 - p_2| \\
&\geqslant C_h \boldsymbol{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2} - \bar C \boldsymbol{m}_0^{\frac12} d(L)^{\gamma_0 - 1 + \delta_1}\, \ell(L)^{2-\delta_1} \\
&\geqslant C_h \boldsymbol{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2} - \bar C \boldsymbol{m}_0^{\frac12} d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2}\,,
\end{aligned}$$

where C_0 is a geometric constant and $\bar C$ a constant which does not depend on C_h and ε_{41}. Hence, if ε_{41} is sufficiently small, we actually conclude

$$|\mathbf{q}_J(p_1 - p_2)| \geqslant \frac{15}{16} C_h \boldsymbol{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2}\,. \tag{7.30}$$

Set $E := \mathbf{E}(T_J, \mathbf{C}_{36r_J}(p_J, \pi_J))$ and apply Theorem 6.22 to T_J and $\mathbf{C}$: the union of the corresponding "stripes" $\mathbf{S}_j$ contain the set $\mathrm{spt}(T_J) \cap \mathbf{C}_{36r_J(1 - CE^{\frac{1}{24}}|\log E|)}(p_J, \pi_J))$, where C is a geometric constant. We can therefore assume that they contain $\mathrm{spt}(T_L) \cap \mathbf{C}_{34r_J}(p_J, \pi_J)$. The width of these stripes is bounded as follows:

$$\begin{aligned}
\sup\left\{|\mathbf{q}_J(x - y)| : x, y \in \mathbf{S}_j\right\} &\leqslant C_0\, E^{\frac14} r_J \leqslant C_0\, C_e^{\frac14} \boldsymbol{m}_0^{\frac14} d(L)^{(2\gamma_0 - 2 + 2\delta_1)/4} \ell(L)^{1 + (2 - 2\delta_1)/4} \\
&\leqslant C_0 C_e^{\frac14} \boldsymbol{m}_0^{\frac14} d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2}
\end{aligned}$$

where C_0 is a geometric constant. So, if $C^\sharp$ is chosen large enough, we actually conclude that p_1 and p_2 must belong to two different stripes, say $\mathbf{S}_1$ and $\mathbf{S}_2$. By Theorem 6.22(iii) we conclude that all points in $\mathbf{C}_{34r_J}(p_J, \pi_J)$ have density Θ strictly smaller than $Q - \frac12$, thereby implying (S1). Moreover, by choosing $C^\sharp$ appropriately, we achieve that

$$|\mathbf{q}_J(x - y)| \geqslant \frac78 C_h \boldsymbol{m}_0^{\frac14}\, d(L)^{\frac{\gamma_0}{2} - \beta_2}\, \ell(L)^{1+\beta_2} \quad \forall x \in \mathbf{S}_1, y \in \mathbf{S}_2\,. \tag{7.31}$$

Assume next there is $H \in \mathscr{W}_n$ with $\ell(H) \leqslant \frac12 \ell(L)$ and $H \cap L \neq \emptyset$. From our construction it follows that $\ell(H) = \frac12 \ell(L)$, $d(H) \leqslant 2d(L)$, $\mathbf{B}_H \subset \mathbf{C}_{34r_J}(p_J, \pi_J)$ and $|\pi_H - \pi_J| \leqslant$

$\bar{C}\boldsymbol{m}_0^{\frac{1}{2}}\, d(L)^{\gamma_0-1+\delta_1}\ell(H)^{1-\delta_1}$, with $\bar{C}$ which does not depend upon C_h and ε_{41}. Hence choosing ε_{41} sufficiently small we conclude We then conclude

$$
\begin{aligned}
|\mathbf{p}_{\pi_H^\perp}(x-y)| &\geqslant \frac{3}{4}C_h\boldsymbol{m}_0^{\frac{1}{4}}\, d(L)^{\frac{\gamma_0}{2}-\beta_2}\,\ell(L)^{1+\beta_2}\\
&\geqslant \frac{3}{2}\Big(\frac{1}{2}\Big)^{\frac{\gamma_0}{2}} C_h\boldsymbol{m}_0^{\frac{1}{4}}\, d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}\\
&\geqslant \frac{5}{4}C_h\boldsymbol{m}_0^{\frac{1}{4}}\, d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2} \qquad \forall x\in \mathbf{S}_1, y\in\mathbf{S}_2\,,
\end{aligned}
\tag{7.32}
$$

where the latter inequality holds because $\gamma_0 \leqslant \log_2\frac{6}{5}$. Now, recalling Proposition 6.11, if ε_{41} is sufficiently small, $\mathbf{C}_{32r_H}(p_H,\pi_H)\cap \mathrm{spt}(T_H)\subset \mathbf{B}_H$ and $\mathrm{spt}(T_J)\cap\mathbf{B}_H\subset\mathrm{spt}(T_H)$. Moreover, by Theorem 6.22(ii) ,

$$
(\mathbf{p}_{\pi_J})_\sharp(T_J\llcorner(\mathbf{S}_j\cap\mathbf{C}_{32r_H}(p_H,\pi_J))) = Q_j\,[\![B_{32r_H}(p_H,\pi_J)]\!] \quad \text{for } j=1,2,\;\; Q_j\geqslant 1.
$$

A simple argument already used several other times allows to conclude that indeed

$$
(\mathbf{p}_{\pi_H})_\sharp(T_H\llcorner(\mathbf{S}_j\cap\mathbf{C}_{32r_H}(p_H,\pi_H))) = Q_j\,[\![B_{32r_H}(p_H,\pi_H)]\!] \quad \text{for } j=1,2,\;\; Q_j\geqslant 1.
$$

Thus, $\mathbf{B}_H$ must necessarily contain two points x, y with

$$
|\mathbf{p}_{\pi_H^\perp}(x-y)| \geqslant \frac{5}{4}C_h\boldsymbol{m}_0^{\frac{1}{4}}\, d(H)^{\frac{\gamma_0}{2}-\beta_2}\ell(H)^{1+\beta_2}.
$$

But then the refining in H should have stopped because of condition (HT) and so H cannot belong to $\mathscr{W}_n$.

Coming to (S3), set $\Omega := \boldsymbol{\Phi}(B_{2\sqrt{m}\ell(L)}((z_L,w_L))$ and observe that $\mathbf{p}_\sharp(T\llcorner(\Omega\cap\mathbf{S}_i)) = Q_i\,[\![\Omega]\!]$. Thus, for each $p\in\mathcal{K}\cap\Omega$, the support of $p+N(p)$ must contain at least one point $p+N_1(p)\in\mathbf{S}_1$ and at least one point $p+N_2(p)\in\mathbf{S}_2$. Now,

$$
|N_1(p)-N_2(p)| \geqslant \frac{7}{8}C_h\boldsymbol{m}_0^{\frac{1}{4}}d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2} - C_0\ell(L)\,|T_p\mathcal{M}-\pi_J|\,. \tag{7.33}
$$

Recalling, Proposition 6.20 and that $\mathcal{M}$ and $\mathrm{Gr}(g_J)$ coincide on a nonempty open set, we easily conclude that (see for instance the proof of (7.3)) $|T_p\mathcal{M}-\pi_J|\leqslant C\boldsymbol{m}_0^{\frac{1}{2}}d(L,0)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{-\beta_2}$ and, via (7.33),

$$
\mathcal{G}\big(N(p), Q\,[\![\boldsymbol{\eta}\circ N(p)]\!]\big) \geqslant \frac{1}{2}|N_1(p)-N_2(p)| \geqslant \frac{3}{8}C_h\boldsymbol{m}_0^{\frac{1}{4}}d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2}\,.
$$

Next observe that, by the property of the Whitney decomposition, any cube touching $B_{2\sqrt{m}\ell(L)}((z_L,w_L))$ has sidelength at most $4\ell(L)$. Thus

$$
|\Omega\setminus\mathcal{K}| \leqslant C\boldsymbol{m}_0^{1+\beta_0}d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell(L)^{2+(1+\beta_0)(2-2\delta_1)}\,.
$$

So, for every point $p \in \Omega$ there exists $q \in \mathcal{K} \cap \Omega$ which has geodesic distance to p at most $C\mathbf{m}_0^{\frac{1}{2}+\frac{\beta_0}{2}} d(L)^{(1+\beta_0)(\gamma_0-1+\delta_1)}\ell(L)^{(1+\beta_0)(1-\delta_1)}$. Given the Lipschitz bound for N and the choice $\beta_2 \leqslant \frac{1}{4}$, we then easily conclude (S3):

$$\begin{aligned}\mathcal{G}(N(q), Q[\![\eta \circ N(q)]\!]) \geqslant & \frac{3}{8} C_h \mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2} \\ & - C\mathbf{m}_0^{\frac{1}{2}+\frac{3\beta_0}{2}} d(L)^{(\frac{3\beta_0}{2}+1)\gamma_0-\beta_2}\ell(L)^{1+\beta_2} \\ \geqslant & \frac{1}{4} C_h \mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell(L)^{1+\beta_2},\end{aligned}$$

where again we need $\varepsilon_{41} < c(\beta_2, \delta_2, M_0, N_0, C_e, C_h)$ for a sufficiently small c. □

Proof of Corollary 7.5. The proof is straightforward. Consider any $H \in \mathscr{W}_n^j$. By definition it has a nonempty intersection with some cube $J \in \mathscr{W}^{j-1}$: this cube cannot belong to $\mathscr{W}_h$ by Proposition 7.4. It is then either an element of $\mathscr{W}_e$ or an element $H_{j-1} \in \mathscr{W}_n^{j-1}$. Proceeding inductively, we then find a chain $H = H_j, H_{j-1}, \dots, H_i =: L$, where $H_{\bar{l}} \cap H_{\bar{l}-1} \neq \emptyset$ for every $\bar{l}$, $H_{\bar{l}} \in \mathscr{W}_n^{\bar{l}}$ for every $\bar{l} > i$ and $L = H_i \in \mathscr{W}_e^i$. Observe also that

$$|x_H - x_L| \leqslant \sum_{\bar{l}=i}^{j-1} |x_{H_{\bar{l}}} - x_{H_{\bar{l}+1}}| \leqslant \sqrt{m}\,\ell(L) \sum_{\bar{l}=0}^{\infty} 2^{-\bar{l}} \leqslant 2\sqrt{m}\,\ell(L)\,.$$

It then follows easily that $H \subset B_{3\sqrt{m}\ell(L)}(L)$. □

7.3.2 *Splitting before tilting: Proof of Proposition 7.7*

As customary we use the convention that constants denoted by C depend upon all the parameters but ε_{41}, whereas constants denoted by C_0 depend only upon $m, n, \bar{n}$ and Q.

Given $L \in \mathscr{W}_e^j$, let us consider its ancestors $H \in \mathscr{S}^{j-1}$ and $J \in \mathscr{S}^{j-6}$, which exists thanks to Proposition 6.3. Set $\ell = \ell(L)$,$\pi = \pi_H$ and $\mathbf{C} := \mathbf{C}_{8r_J}(p_J, \pi)$, and let $f : B_{8r_J}(p_J, \pi) \to \mathcal{A}_Q(\pi^\perp)$ be the π-approximation of Definition 6.4, which is the result of Theorem 2.8applied to $\mathbf{C}_{32r_J}(p_J, \pi)$ (recall that Proposition 6.11 ensures the applicability of Theorem 2.8 in the latter cylinder).

The following are simple consequences of Proposition 6.11:

$$E := \mathbf{E}(T_J, \mathbf{C}_{32r_J}(p_J, \pi)) \leqslant C\mathbf{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\, \ell^{2-2\delta_1}\,, \tag{7.34}$$

$$\mathbf{h}(T_J, \mathbf{C}, \pi_H) \leqslant C\, \mathbf{m}_0^{\frac{1}{4}}\, d(L, 0)^{\frac{\gamma_0}{2}-\beta_2}\ell^{1+\beta_2}, \tag{7.35}$$

$$c\, C_e\, \mathbf{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_1} \leqslant E, \tag{7.36}$$

where (7.36) follows from $\mathbf{B}_L \subset \mathbf{C}$, $L \in \mathscr{W}_e$ and $\frac{r_L}{r_J} = 2^{-6}$. In particular the positive constants c and C do not depend on ε_{41}. We divide the proof of Proposition 7.7 in three steps.

Step 1: decay estimate for f**.** Let $2\rho := 64r_H - \bar{C}\mathbf{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell^{1+\beta_2}$: since $p_H \in \mathrm{spt}(T_J)$, it follows from (7.35) that, upon having chosen $\bar{C}$ appropriately, $\mathrm{spt}(T_J) \cap \mathbf{C}_{2\rho}(p_H, \pi_H) \subset \mathrm{spt}(T_H) \cap \mathbf{B}_H \subset \mathbf{C}$. Observe in particular that $\bar{C}$ does not depend on ε_{41}, although it depends upon the other parameters. In particular, setting $B = B_{2\rho}(x, \pi_H)$ with $x = \mathbf{p}_{\pi_H}(p_H)$, using the Taylor expansion in Corollary 3.49 and the estimates in Theorem 2.8, we get

$$\begin{aligned} \mathrm{Dir}(B, f) &\leqslant 2|B|\,\mathbf{E}(T_J, \mathbf{C}_{2\rho}(x_H, \pi_H)) + C\boldsymbol{m}_0^{1+\beta_0}\, d(L, 0)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell^{2+(1+\beta_0)(2-2\delta_1)} \\ &\leqslant 2\omega_2\rho^2\mathbf{E}(T_H, \mathbf{B}_H) + C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell^{2+(1+\beta_0)(2-2\delta_1)}\,. \end{aligned} \tag{7.37}$$

Consider next the cylinder $\mathbf{C}_{64r_L}(p_L, \pi_H)$, and set $x' := \mathbf{p}_{\pi_H}(p_L)$. Recall that $|x - x'| \leqslant |p_H - p_L| \leqslant C_0\ell(H)$, where C_0 is a geometric constant (cf. Proposition 6.11), and set $\sigma := 64r_L + C\ell(H) = 32r_H + C\ell(H)$. If λ is the constant in (3.39) and M_0 is chosen sufficiently large (thus fixing a lower bound for M_0 which depends only on δ_1) we reach

$$\sigma \leqslant \left(\frac{1}{2} + \frac{\lambda}{4}\right) 64\, r_H \leqslant \left(1 + \frac{\lambda}{2}\right)\rho + \bar{C}\boldsymbol{m}_0^{\frac{1}{4}}\, d(L)^{\frac{\gamma_0}{2}-\beta_2}\,\ell^{1+\beta_2}\,.$$

In particular, choosing ε_{41} sufficiently small we conclude $\sigma \leqslant (1+\lambda)\rho$ and thus also $\mathbf{B}_L \subset \mathbf{C}_{(1+\lambda)\rho}(p_L, \pi_H) =: \mathbf{C}'$. Define $B' = B_{(1+\lambda)\rho}(x, \pi_H)$. Set $A := \fint_{B'} D(\boldsymbol{\eta} \circ f)$, $\bar{A} : \pi_H \to \pi_H^\perp$ the linear map $x \mapsto A \cdot x$ and π for the plane corresponding to $\mathbf{G}_{\bar{A}}$. Using Theorem 3.51, we can estimate

$$\begin{aligned} \tfrac{1}{2}\int_{B'} \mathcal{G}(Df, Q\,[\![A]\!])^2 \geqslant& |B'|\,\mathbf{E}(T_J, \mathbf{C}', \pi) - C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell^{2+(1+\beta_0)(2-2\delta_1)} \\ \geqslant& |B'|\mathbf{E}(T_J, \mathbf{B}_L, \pi) - C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell^{2+(1+\beta_0)(2-2\delta_1)} \\ \geqslant& \omega_2((1+\lambda)\rho)^2\mathbf{E}(T_L, \mathbf{B}_L) \\ &\quad - C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\ell^{2+(1+\beta_0)(2-2\delta_1)}. \end{aligned} \tag{7.38}$$

Next, considering that $\mathbf{B}_H \supset \mathbf{B}_L$ and that, by $L \in \mathscr{W}_e^j$,

$$\mathbf{E}(T_L, \mathbf{B}_L) \geqslant C_e \boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_1},$$

we conclude from (7.37) and (7.38) that

$$\mathrm{Dir}(B, f) \leqslant 2\omega_2(2\rho)^2(1 + \boldsymbol{m}_0^{\beta_0})\mathbf{E}(T_H, \mathbf{B}_H)\,. \tag{7.39}$$

$$\int_{B'} \mathcal{G}(Df, Q\,[\![A]\!])^2 \geqslant 2\omega_2((1+\lambda)\rho)^2(1 - C\boldsymbol{m}_0^{\beta_0})\mathbf{E}(T_L, \mathbf{B}_L)\,. \tag{7.40}$$

Step 2: harmonic approximation. From now on, to simplify our notation, we use $B_s(y)$ in place of $B_s(y, \pi_H)$. Set $p := \mathbf{p}_{\pi_H}(p_J)$. Consistently with [24, 25, 19] we introduce the parameter $\boldsymbol{\Omega}$, which equals

- $\boldsymbol{A} = \|A_\Sigma\|_{C^0}$ in case (a) of Definition 1.1;
- $\max\{\|d\omega\|_{C^0}, \|A_\Sigma\|_{C^0}\}$ in case (b);
- $C_0 R^{-1}$ in case (c).

Then, from (7.36) we infer that, for any $\varepsilon_{32} > 0$, if $\bar{r}$ is chosen sufficiently small, we then have

$$8r_J\,\boldsymbol{\Omega} \leqslant C\ell(L)\boldsymbol{m}_0^{\frac{1}{2}} \leqslant \varepsilon_{32}\, C_e^{\frac{1}{2}}\boldsymbol{m}_0^{\frac{1}{2}}\, d(L)^{\gamma_0-1+\delta_1}\ell(L)^{1-\delta_1} \leqslant \varepsilon_{32}E^{\frac{1}{2}}, \tag{7.41}$$

because $\ell(L) \leqslant d(L) \leqslant \bar{r}$. Therefore, for every positive $\bar{\eta}$, we can apply [19, Theorem 1.6] (in case (a) of Definition 1.1) and [25, Theorem 4.2] (in the cases (b) and (c) of Definition 1.1) to the cylinder $\mathbf{C}$ and achieve a map $w : B_{8r_J}(p,\pi_H) \to \mathcal{A}_Q(\pi_H^\perp)$ of the form $w = (u, \Psi(y,u))$ (in fact $w = u$ in case (b) of definition 1.1) for a Dir-minimizer u and such that

$$(8\,r_J)^{-2}\int_{B_{8r_J}(p)} \mathcal{G}(f,w)^2 + \int_{B_{8r_J}(p)} (|Df| - |Dw|)^2 \leqslant \bar{\eta}\,E\,(8\,r_J)^2, \tag{7.42}$$

$$\int_{B_{8r_J}(p)} |D(\boldsymbol{\eta}\circ f) - D(\boldsymbol{\eta}\circ w)|^2 \leqslant \bar{\eta}\,E\,(8\,r_J)^2\,. \tag{7.43}$$

In the cases (a) and (c) of Definition 1.1, by the chain rule we have $D(\Psi(y,u(y))) = \sum_j [\![D_x\Psi(y,u_j(y)) + D_v\Psi(y,u_j(y))\cdot Du_j(y)]\!]$, so that

$$\int_{B_{(1+\lambda)\rho}(x)} |D(\Psi(y,u))|^2 \leqslant C_0\boldsymbol{m}_0\int_{B_{(1+\lambda)\rho}(x)} |Du|^2 + C_0\boldsymbol{m}_0\rho^4,$$

where C_0 is a geometric constant. Consider now $\tilde{\mathbf{A}} := ⨍ D(\boldsymbol{\eta}\circ w)$, and observe that, since $D\boldsymbol{\eta}\circ u = \boldsymbol{\eta}\circ Du$ is harmonic, we have $D\boldsymbol{\eta}\circ u(x) = ⨍_{B'}\boldsymbol{\eta}\circ Du$. We can use (7.42) and (7.43), together with (7.40) to infer, for ε_{41} small enough,

$$\begin{aligned}
&\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Du, Q\,[\![D(\boldsymbol{\eta}\circ u)(x)]\!]\,\big)^2\\
\geqslant&\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Dw, Q[\![\tilde{\mathbf{A}}]\!]\big)^2 - C_0\boldsymbol{m}_0\rho^4\\
\geqslant&\int_{B_{(1+\lambda)\rho}(x} \mathcal{G}\big(Df, Q[\![\mathbf{A}]\!]\big)^2 - C_0\boldsymbol{m}_0\rho^4 - C_0\bar{\eta}E\rho^2\\
\geqslant& 2\omega_2((1+\lambda)\rho)^2(1-C\boldsymbol{m}_0^{\beta_0})\mathbf{E}(T_L,\mathbf{B}_L) - C_0\boldsymbol{m}_0\rho^4 - C_0\bar{\eta}E\rho^2.
\end{aligned} \tag{7.44}$$

Analogously, using (7.42) and (7.42), we easily deduce

$$\int_{B_{2\rho}(x)} |Du|^2 \leqslant 2\omega_2(2\rho)^2(1+\boldsymbol{m}_0^{\beta_0})\mathbf{E}(T_H,\mathbf{B}_H) + C_0\boldsymbol{m}_0\rho^4 + C_0\bar{\eta}E\rho^2 \tag{7.45}$$

Now recall that, since $d(L) = d(H) = d(J)$, and $L \in \mathscr{W}_e$,

$$\mathbf{E}(T_L,\mathbf{B}_L) \geqslant C_e\boldsymbol{m}_0 d(L)^{2\gamma_0-2+2\delta_1}\ell(L)^{2-2\delta_1} \geqslant 2^{2\delta_1-2}\mathbf{E}(T_H,\mathbf{B}_H)\,,$$

and combining this with (7.45) and (7.44) we achieve

$$\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Du, Q\,[\![D(\boldsymbol{\eta}\circ u)(x)]\!]\,\big)^2 \geqslant (2^{2\delta_1-4} - C\boldsymbol{m}_0^{\beta_0})\int_{B_{2\rho}(x)} |Du|^2 - C_0\boldsymbol{m}_0\rho^4 - C_0\bar{\eta}E\rho^2\,. \tag{7.46}$$

To estimate the last two errors in terms of the energy of u we use again $L \in \mathscr{W}_e$ to conclude

$$\mathrm{E}\rho^2 \leqslant C_0\, \mathbf{E}(T_L, B_L) \overset{(7.40)}{\leqslant} C_0 \int_{B_{(1+\lambda)\rho}} |Du|^2 + C_0\, \boldsymbol{m}_0\, \rho^4 + C_0 \bar{\eta} \mathrm{E}\rho^2$$

so that, for $\bar{\eta} \leqslant \frac{1}{2C_0}$ we have

$$\mathrm{E}\rho^2 \leqslant C_0 \int_{B_{(1+\lambda)\rho}} |Du|^2 + C_0\, \boldsymbol{m}_0\, \rho^4 \,. \tag{7.47}$$

Next, using once again, $L \in \mathscr{W}_e$ and this last inequality,

$$\begin{aligned} C_0 \boldsymbol{m}_0 \rho^4 &\leqslant \frac{C_0 \rho^2}{C_e} d(L)^{2-2\gamma_0-2\delta_1} \mathbf{E}(T_L, \mathbf{B}_L) \overset{(7.40)}{\leqslant} \frac{C_0}{C_e} \int_{B_{(1+\lambda)\rho}(x)} |Df|^2 \\ &\leqslant \frac{C_0}{C_e} \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 + \frac{C_0}{C_e} \boldsymbol{m}_0 \rho^4 + \frac{C_0}{C_e} \bar{\eta} \mathrm{E}\rho^2 \leqslant \frac{C_0}{C_e} \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 + \frac{C_0}{C_e} \boldsymbol{m}_0 \rho^4 \,. \end{aligned}$$

which for C_e bigger than a geometrical constant implies

$$C_0 \boldsymbol{m}_0 \rho^4 \leqslant \frac{C_0}{C_e} \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 \,. \tag{7.48}$$

We can therefore combine (7.46) with (7.47) and (7.48) to achieve

$$\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Du, Q\, [\![D(\boldsymbol{\eta} \circ u)(x)]\!]\,\big)^2 \geqslant \Big(2^{2\delta_1 - 4} - \frac{C_0}{C_e} - C \boldsymbol{m}_0^{\beta_0} - C_0 \bar{\eta}\Big) \int_{B_{2\rho}(x)} |Du|^2 \,. \tag{7.49}$$

It is crucial that the constant C, although depending upon $\beta_2, \delta_2, M_0, N_0, C_e$ and C_h, does not depend on η and ε_{41}, whereas C_0 depends only upon $Q, m, \bar{n}$ and n. So, if C_e is chosen sufficiently large, depending only upon λ (and hence upon δ_2), we can require that $2^{2\delta_1-4} - \frac{C_0}{C_e} \geqslant 2^{3\delta_1/4-4}$. We then require $\bar{\eta}$ and ε_{41} to be sufficiently small so that $2^{3\delta_1/4-4} - C\, \boldsymbol{m}_0^{\beta_0} - C\bar{\eta} \geqslant 2^{\delta_2 - 4}$.

We can now apply Lemma 3.33 and Proposition 3.34 to u and conclude

$$\hat{C}^{-1} \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 \leqslant \int_{B_{\ell/8}(q)} \mathcal{G}(Du, Q\, [\![D(\boldsymbol{\eta} \circ u]\!])^2 \leqslant \hat{C} \ell^{-2} \int_{B_{\ell/8}(q)} \mathcal{G}(u, Q\, [\![\boldsymbol{\eta} \circ u]\!])^2 \,,$$

for any ball $B_{\ell/8}(q) = B_{\ell/8}(q, \pi) \subset B_{8r_J}(p, \pi)$, where $\hat{C}$ depends upon δ_2 and M_0. In particular, being these constants independent of ε_{41} and C_e, we can use the previous estimates and reabsorb error terms (possibly choosing ε_{41} even smaller and C_e larger) to conclude

$$\begin{aligned} \boldsymbol{m}_0\, \ell^{m+2-2\delta_2} &\leqslant \tilde{C} \ell^m\, \mathbf{E}(T, \mathbf{B}_L) \leqslant \bar{C} \int_{B_{\ell/8}(q)} \mathcal{G}(Df, Q\, [\![D(\boldsymbol{\eta} \circ f)]\!])^2 \\ &\leqslant \check{C} \ell^{-2} \int_{B_{\ell/8}(q)} \mathcal{G}(f, Q\, [\![\boldsymbol{\eta} \circ f]\!])^2, \end{aligned} \tag{7.50}$$

where $\tilde{C}$, $\bar{C}$ and $\check{C}$ are constants which depend upon δ_2, M_0 and C_e, but not on ε_{41}.

Step 3: Estimate for the $\mathcal{M}$-normal approximation. We next complete the proof showing (7.5) and (7.6). Now, consider any ball $B_{\ell/4}(q,\pi_0)$ with $\mathrm{dist}(L,q) \leqslant 4\sqrt{2}\,\ell$ and let $\Omega := \mathbf{\Phi}(B_{\ell/4}(q,\pi_0))$. Observe that $\mathbf{p}_\pi(\Omega)$ must contain a ball $B_{\ell/8}(q',\pi)$, because of the estimates on $\boldsymbol{\varphi}$ and $|\pi_0 - \pi_H|$, and in turn it must be contained in $B_{8r_J}(p,\pi)$.

Let $\boldsymbol{\varphi}' : B_{8r_J}(p,\pi) \to \pi^\perp$ be such that $\mathbf{G}_{\boldsymbol{\varphi}'} = [\![\mathcal{M}]\!]$ and $\mathbf{\Phi}'(z) = (z, \boldsymbol{\varphi}'(z))$. Since $D(\boldsymbol{\eta}\circ f)(z) = \boldsymbol{\eta}\circ Df(z)$ for a.e. z, we obviously have

$$\int_{B_{\ell/8}(q',\pi_H)} \mathcal{G}(Df, Q\,[\![D(\boldsymbol{\eta}\circ f)]\!])^2 \leqslant \int_{B_{\ell/8}(q',\pi_H)} \mathcal{G}(Df, Q\,[\![D\boldsymbol{\varphi}']\!])^2\,. \tag{7.51}$$

Let now $\vec{\mathbf{G}}_f$ be the orienting tangent m-vector to $\mathbf{G}_f$ and τ the one to $\mathcal{M}$. For a.e. z we have the inequality

$$C_0 \sum_j |\vec{\mathbf{G}}_f(f_j(z)) - \vec{\tau}(\boldsymbol{\varphi}'(z))|^2 \geqslant \mathcal{G}(Df(z), Q\,[\![D\boldsymbol{\varphi}'(z)]\!])^2\,,$$

for some geometric costant C_0, because $|\vec{\mathbf{G}}_f(f_j(z)) - \vec{\tau}(\boldsymbol{\varphi}'(z))| \leqslant \boldsymbol{m}_0^{\beta_0}$. Therefore

$$\begin{aligned}
\fint_{B_{\ell/8}(q',\pi_H)} \mathcal{G}(Df, Q_2\,[\![D\boldsymbol{\varphi}']\!])^2 &\leqslant C\fint_{\mathbf{C}_{\ell/8}(q',\pi_H)} |\vec{\mathbf{G}}_f(z) - \vec{\tau}(\boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z))|^2 d\|\mathbf{G}_f\|(z)\\
&\leqslant C\fint_{\mathbf{C}_{\ell/8}(q',\pi_H)} |\vec{T_L}(z) - \vec{\tau}(\boldsymbol{\varphi}_i'(\mathbf{p}_{\pi_H}(z))|^2 d\|T_L\|(z)\\
&\quad + C\boldsymbol{m}_0^{1+\beta_0} d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_2)}\ell^{2+(2-2\delta_2)(1+\beta_0)}\,.
\end{aligned} \tag{7.52}$$

Now, thanks to the height bound and to the fact that $|\vec{\tau} - \pi_H| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{\frac{\gamma_0}{2}-1}\ell$ in the cylinder $\hat{\mathbf{C}} = \mathbf{C}_{\ell/8}(q',\pi_H)$, we have the inequality

$$|\mathbf{p}(z) - \boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z))| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}+\frac{1}{2}} d(L)^{\gamma_0-\beta_2}\ell^{2+\beta_2} \qquad \forall z \in \mathrm{spt}(T)\cap\hat{\mathbf{C}}\,.$$

Using the estimate $|D^2\boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z)| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} d(L)^{\frac{\gamma_0}{2}-1}$ (which is valid for any $z \in \mathrm{spt}(T)\cap\hat{\mathbf{C}}$) we then easily conclude from (7.52) that

$$\begin{aligned}
&\fint_{B_{\ell/8}(p,\pi_H)} \mathcal{G}(Df, Q\,[\![D\boldsymbol{\varphi}']\!])^2\\
\leqslant& C\fint_{\hat{\mathbf{C}}} |\vec{T}_L(z) - \vec{\tau}(\mathbf{p}(z))|^2 d\|T\|(z) + C\boldsymbol{m}_0^{1+\frac{1}{2}} d(L)^{2\gamma_0-2-2\beta_2}\ell^{2+2\beta_2}\\
\leqslant& C\fint_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(z) - \tau(\mathbf{p}(z))|^2 d\|\mathbf{T}_F\|(z) + C\boldsymbol{m}_0^{1+\beta_0} d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_1},
\end{aligned}$$

where we used (7.2).

Since, on the region where we are interested, namely Ω, we have the bounds $|DN| \leqslant C\boldsymbol{m}_0^{\beta_0} d(L)^{\beta_0\gamma_0}$, $|N| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}} d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell^{1+\beta_2}$ and $\|A_{\mathcal{M}}\|^2 \leqslant C\boldsymbol{m}_0 d(L)^{\gamma_0-2}$, applying now Proposition 3.50 we conclude

$$\begin{aligned}
\fint_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(x) - \tau(\mathbf{p}(x))|^2 d\|\mathbf{T}_F\|(x) \leqslant& (1 + C\boldsymbol{m}_0^{2\beta_0} d(L)^{2\gamma_0\beta_0}) \int_\Omega |DN|^2\\
&+ C\boldsymbol{m}_0^{1+\frac{1}{2}} d(L)^{2\gamma_0-2-2\beta_2}\ell^{2+2\beta_2}\,.
\end{aligned}$$

Thus, putting all these estimates together we achieve

$$\boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_2} \leqslant C(1+C\boldsymbol{m}_0^{2\beta_0}d(L)^{2\gamma_0\beta_0})\fint_\Omega |DN|^2 + C\boldsymbol{m}_0^{1+\beta_0}d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_2}\,. \tag{7.53}$$

Since the constant C might depend on the various other parameters but not on ε_{41}, we conclude that for a sufficiently small ε_{41} we have

$$\boldsymbol{m}_0 d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_1} \leqslant C\fint_\Omega |DN|^2\,. \tag{7.54}$$

But $\mathbf{E}(T_L, \mathbf{B}_L) \leqslant C\boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_2}$ and thus (7.5) follows.

We finally show (7.6). Observe that $\mathbf{p}^{-1}(\Omega)\cap \mathrm{spt}(T) \supset \mathbf{C}_{\ell/8}(q', \pi)\cap\mathrm{spt}(T)$ and, for an appropriate geometric constant C_0, Ω cannot intersect a Whitney region $\mathcal{L}'$ corresponding to an L' with $\ell(L') \geqslant C_0\ell(L)$ or $d(L') \geqslant 2d(L)$. In particular, Theorem 7.3 implies that

$$\|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\Omega)) + \|\mathbf{T}_F - \mathbf{G}_f\|(\mathbf{p}^{-1}(\Omega)) \leqslant C\boldsymbol{m}_0^{1+\beta_0}\, d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)}\,\ell^{2+(1+\beta_0)(2-2\delta_1)}\,. \tag{7.55}$$

Let now F' be the map such that $\mathbf{T}_{F'}\llcorner(\mathbf{p}^{-1}(\Omega)) = \mathbf{G}_f\llcorner(\mathbf{p}^{-1}(\Omega))$ and let N' be the corresponding normal part, i.e. $F'(x) = \sum_i [\![x + N_i'(x)]\!]$. The region over which F and F' differ is contained in the projection onto Ω of $(\mathrm{Im}(F)\setminus\mathrm{spt}(T))\cup(\mathrm{Im}(F')\setminus\mathrm{spt}(T))$ and therefore its $\mathcal{H}^m$ measure is bounded as in (7.55). Recalling the height bound on N and f, we easily conclude $|N| + |N'| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}}d(L)^{\frac{\gamma_0}{2}-\beta_2}\ell^{1+\beta_2}$, which in turn implies

$$\int_\Omega |N|^2 \geqslant \int_\Omega |N'|^2 - C\boldsymbol{m}_0^{1+\frac{1}{4}+\beta_0}d(L)^{(1+\beta_0)(2\gamma_0-2+2\delta_1)+\gamma_0-2\beta_2}\ell^{4+2\beta_2+(2-2\delta_1)(1+\beta_0)}\,. \tag{7.56}$$

On the other hand, applying Theorem 3.18, we conclude

$$|N'(\boldsymbol{\Phi}'(z))| \geqslant \frac{1}{2\sqrt{Q}}\,\mathcal{G}(f(z), Q\,[\![\boldsymbol{\varphi}'(z)]\!]) \geqslant \frac{1}{4\sqrt{Q}}\,\mathcal{G}(f(z), Q\,[\![\boldsymbol{\eta}\circ f(z)]\!])\,,$$

which in turn implies

$$\boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\,\ell^{2-2\delta_2} \overset{(7.50)}{\leqslant} C\ell^{-2}\int_{B_{\ell/8}(q',\pi)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 \leqslant C\ell^{-2}\int_\Omega |N'|^2\,. \tag{7.57}$$

For ε_{41} sufficiently small, (7.56) and (7.57) lead to the second inequality of (7.6), while the first one comes from Theorem 7.3 and $\mathbf{E}(T, \mathbf{B}_L) \geqslant C_e\boldsymbol{m}_0\, d(L)^{2\gamma_0-2+2\delta_1}\ell^{2-2\delta_2}$.

8

PROOF OF THE CENTER MANIFOLD THEOREM

This chapter is devoted to the proof of Theorem 2.18, that is

Theorem 8.1 (Center Manifold Approximation)**.** *Let* T *be as in Assumption 3. Then there exist* $\eta_0, \gamma_0, r_0, C > 0$*, an admissible* b*-separated* γ_0*-smooth* $\bar{Q}$*-branching* $\mathcal{M}$*, a corresponding conformal parametrization* $\mathbf{\Psi} : \mathfrak{B}_{\bar{Q},2} \to \mathbb{R}^{2+n}$ *and a* Q*-valued map* $\mathscr{N} : \mathfrak{B}_{\bar{Q},2} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *with the following properties*

(i) $\bar{Q}Q = \Theta(T, 0)$ *and* $|A_{\mathcal{M}}(\mathbf{\Psi}(z,w))| + |z|^{-1}|D_{\mathcal{M}}A_{\mathcal{M}}(\mathbf{\Psi}(z,w))| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|z|^{\gamma_0 - 1}$*, where* $A_{\mathcal{M}}$ *denotes the second fundamental form of* $\mathcal{M}\setminus\{0\}$*; moreover* $|D\mathbf{\Psi}(z,w) - \mathrm{Id}| \leqslant C\mathbf{m}_0^{1/2}|z|^{\gamma_0}$ *and* $|D^2\mathbf{\Psi}(z,w)| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|z|^{\gamma_0-1}$.

(ii) $\mathscr{N}^i(z,w)$ *is orthogonal to the tangent plane, at* $\Psi(z,w)$*, to* $\mathcal{M}$*;*

(iii) Setting

$$\mathscr{F}(z,w) := \sum_i [\![\mathbf{\Psi}(z,w) + \mathscr{N}(z,w)]\!] \quad \textit{and} \quad S := T_{0,r_0},$$

then $\mathrm{spt}(S) \cap \mathbf{C}_1$ *is contained in a suitable horned neighborhood of the* $\bar{Q}$*-branching, where the orthogonal projection* $\mathbf{p}$ *onto it is well-defined. Moreover, for every* $r \in]0,1[$ *we have*

$$\|\mathscr{N}|_{B_r}\|_0 + \sup_{p\in \mathrm{spt}(S)\cap \mathbf{p}^{-1}(\mathbf{\Psi}(B_r))} |p - \mathbf{p}(p)| \leqslant C\mathbf{m}_0^{\frac{1}{4}} r^{1+\frac{\gamma_0}{2}}\,; \tag{8.1}$$

(iv) If we define

$$\mathbf{D}(r) := \int_{B_r} |D\mathscr{N}|^2 \quad \textit{and} \quad \mathbf{H}(r) := \int_{\partial B_r} |\mathscr{N}|^2\,,$$

$$\mathbf{F}(r) := \int_0^r \frac{\mathbf{H}(t)}{t^{2-\gamma_0}}\,dt \quad \textit{and} \quad \mathbf{\Lambda}(r) := \mathbf{D}(r) + \mathbf{F}(r)\,,$$

then the following estimates hold for every $r \in]0,1[$*:*

$$\mathrm{Lip}(\mathscr{N}|_{B_r}) \leqslant C\min\{\mathbf{\Lambda}^{\eta_0}(r), \mathbf{m}_0^{\eta_0} r^{\eta_0}\} \tag{8.2}$$

$$\mathbf{m}_0^{\eta_0}\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\mathscr{N}(z,w)| \leqslant C\,\mathbf{\Lambda}^{\eta_0}(r)\,\mathbf{D}(r) + C\,\mathbf{F}(r) \tag{8.3}$$

$$\|S - \mathbf{T}_{\mathscr{F}}\|\big(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r))\big) \leqslant C\,\mathbf{\Lambda}^{\eta_0}(r)\,\mathbf{D}(r) + C\,\mathbf{F}(r)\,. \tag{8.4}$$

Proof. The center manifold $\mathcal{M}$ is given by Theorem 6.9: the fact that $\mathcal{M}$ is a b-separated admissible $\bar{Q}$-branching is a simple consequence of the estimates in Theorem 6.9. We then apply Proposition 2.17 to find the map $\mathbf{\Psi}$ which is a conformal parametrization of $\mathcal{M}$ in a neighborhood of 0 and, after a suitable scaling, we assume that it is defined on $\mathfrak{B}_{\bar{Q},2}$. Secondly we consider the normal approximation N of the current T on $\mathcal{M}$ constructed in Theorem 7.3. The relation $\bar{Q}Q = \Theta(T, 0)$ is obvious from the construction. Again, after scaling, we assume that:

- The radius r of Theorem 8.1 is 4;
- $\mathbf{\Psi}(\mathfrak{B}) \subset \mathbf{C}_3(0)$;

Rather than call the rescaled current S, as it is done in the statement of Theorem 8.1, we keep denoting it by T.

The maps $\mathscr{N}$ and $\mathscr{F}$ are then defined as

$$\mathscr{N}(z,w) := N(\mathbf{\Psi}(z,w)) = \sum_i [\![N_i(\mathbf{\Psi}(z,w))]\!] \tag{8.5}$$

$$\mathscr{F}(z,w) := \sum_i [\![\mathbf{\Psi}(z,w) + \mathscr{N}_i(z,w)]\!] = \sum_i [\![\mathbf{\Psi}(z,w) + N_i(\mathbf{\Psi}(z,w))]\!] \, . \tag{8.6}$$

By the estimate (6.17) it follows immediately that

$$|A_{\mathcal{M}}(\zeta,\xi)| + |\zeta|^{-1}|D_{\mathcal{M}}A_{\mathcal{M}}(\zeta,\xi)| \leqslant C\mathbf{m}_0^{\frac{1}{2}}|\zeta|^{\gamma_0 - 1}$$

at any point $p = (\zeta,\xi) \in \mathcal{M}$ with $\zeta \in \mathbb{R}^2 \setminus 0$. On the other hand by (2.29), if we set $(\zeta,\xi) := \mathbf{\Psi}(z,w)$, then we have

$$|z| - C\mathbf{m}_0^{\frac{1}{4}}|z|^{1+\gamma_0} \leqslant |\zeta| \leqslant |z| + C\mathbf{m}_0^{\frac{1}{4}}|z|^{1+\gamma_0} \tag{8.7}$$

and thus the estimates in (i) follow. By construction $\mathscr{N}_i(z,w) = N_i(\mathbf{\Psi}(z,w))$ is orthogonal to $T_{\mathbf{\Psi}(z,w)}\mathcal{M}$, which shows (ii).

The fact that T is contained in a horned neighborhood of $\mathcal{M}$ where the prejection $\mathbf{p}$ is well defined is a consequence of Corollary 7.1. Moreover, by (8.7) we can assume $\mathbf{\Psi}(B_r(0)) \subset \mathbf{C}_{2r}$ (this is true for a sufficiently small r and hence, after scaling, we can assume it holds for any $r \leqslant 1$). On the other hand, consider a cube L of $\mathscr{W}$ which intersects $B_{3/2r}(0)$. By construction its sidelength is necessarily smaller than r. Thus (8.1) is a simple consequence of (7.1).

We are left to show the three estimates claimed in point (iv) of Theorem 8.1: the rest of the section is devoted to this task.

8.0.3 *The special covering*

. First of all consider the set $\mathbf{\Psi}(B_r(0))$ and let $\mathcal{B}_r \subset \mathfrak{B}$ be defined by

$$\mathcal{B}_r := \{(z,w) \in \mathfrak{B} : \mathbf{\Phi}(z,w) \in \mathbf{\Psi}(B_r(0))\}. \tag{8.8}$$

Observe that, by the estimates on $\mathbf{\Psi}$, the following two facts are obvious for r small:

(g1) $\mathcal{B}_r$ is star-shaped with respect to the origin, more precisely if $q = (z,w) \in \partial\mathcal{B}_r$, then the geodesic segment σ in $\mathfrak{B}$ joining $(0,0)$ and q is contained in $\mathcal{B}_r$;

(g2) If $\bar{q}$ denotes the point on σ at distance $\frac{r}{4}$, the disk $B_{r/4}(\bar{q})$ is contained in $\mathcal{B}_r$.

We next select an (at most countable) family of triples $\{(L_j, B_j, U_j)\}_{j\in\mathbb{N}}$ of subsets of $\mathfrak{B}_{\bar{Q}}$ with the following properties:

(c1) The L_j's are distinct cubes of the Whitney decomposition with $L_j \in \mathscr{W}_e \cup \mathscr{W}_h$ and $L_j \subset \bar{B}_{2r+6\ell(L_j)}$;

(c2) $B_j = \mathbf{B}_{\frac{\ell(L_j)}{4}}(z_j, w_j) \subset \mathcal{B}_r$ are disjoint balls such that $|z_{L_j} - z_j| \leqslant 7\,\ell(L_j)$;

(c3) U_j is the union of an at most countable family of cubes $\mathscr{W}(L_j) \subset \mathscr{W}$ where $H \subset \mathbf{B}_{30\ell(L_j)}(z_{L_j}, w_{L_j})$ for every $H \in \mathscr{W}(L_j)$ and $\cup_j \mathscr{W}(L_j)$ consists of all cubes in $\mathscr{W}$ which intersect $\mathcal{B}_r$; in particular

$$\mathcal{B}_r \subset \Gamma \cup \bigcup_j U_j\,. \tag{8.9}$$

To this aim we start by selecting all the cubes $L \in \mathscr{W}_e \cup \mathscr{W}_h$ such that either $L \cap \mathcal{B}_r \neq \emptyset$ or there exists $H \in \mathscr{W}_n$ in the domain of influence of L with $H \cap \mathcal{B}_r \neq \emptyset$, and we denote the collection of such cubes by $\mathscr{W}(r)$. Observe that, $\ell(L) \leqslant C_0 2^{-N_0} r$ and thus, provided N_0 is chosen sufficiently large, we can assume that the ratio $\frac{\ell(L)}{r}$ is smaller than any fixed geometric constant. Moreover, by Corollary 7.5, it is obvious that $L \subset B_{2r+6\ell(L_j)}$.

The triples above are then chosen according to the following procedure:

- We start selecting recursively $\{L_j\} \subset \mathscr{W}(r)$. L_0 is a cube with the largest sidelength in $\mathscr{W}(r)$. Having chosen $\{L_0, \ldots, L_j\}$ we select L_{j+1} as a cube with the largest sidelength among those $L \in \mathscr{W}(r)$ such that $\mathbf{B}_{15\ell(L)}(z_L, w_L) \cap \mathbf{B}_{15\ell(L_i)}(z_{L_i}, w_{L_i}) = \emptyset$ for all $i \leqslant j$.
- For every L_j we use the geometric properties (g1) and (g2) to choose a ball B_j as in (c2): for instance we consider $z_j := \frac{z_{L_j}}{|z_{L_j}|}\big(|z_{L_j}| - \frac{7\sqrt{2}}{2}\,\ell_{L_j}\big)$ and let (z_j, w_j) be the unique point of $\mathfrak{B}$ that belongs to the connected component of $\mathfrak{B} \cap (B_{L_j} \times \mathbb{C})$ that contains (z_{L_j}, w_{L_j}). The B_j's are disjoint because they are contained in $\mathbf{B}_{15\ell(L_j)}(z_{L_j}, w_{L_j})$;
- For what concerns U_j, we need to define $\mathscr{W}(L_j)$; consider then $H \in \mathscr{W}$ such that $H \cap \mathcal{B}_r \neq \emptyset$:
  - (a) If $H \in \mathscr{W}_e \cap \mathscr{W}_h$, then $H \in \mathscr{W}(r)$ and we select the L_j with largest sidelength such that $\mathbf{B}_{15\ell(L_j)}(z_{L_j}, w_{L_j}) \cap \mathbf{B}_{15\ell(H)}(z_H, w_H) \neq \emptyset$;
  - (b) If $H \in \mathscr{W}_n$, then H belongs to the domain of influence $\mathscr{W}_n(L)$ of some $L \in \mathscr{W}(r)$; we then select the L_j with largest sidelength such that $\mathbf{B}_{15\ell(L_j)}(z_{L_j}, w_{L_j}) \cap \mathbf{B}_{15\ell(L)}(z_L, w_L) \neq \emptyset$.

8.0.4 *Estimates on* $\mathcal{U}_j$ *and* $\boldsymbol{\Lambda}$

Let $\mathcal{U}_| = \mathbf{\Phi}(U_j)$ and $\mathcal{B}_j := \mathbf{\Phi}(B_j)$ and set, for notational convenience, $d_j := d(L_j)$ and $\ell_j := \ell(L_j)$. As a simple consequence of Theorem 7.3 we deduce the following estimates for every $j \in \mathbb{N}$:

$$\int_{\mathcal{U}_j} |\boldsymbol{\eta} \circ N| \leqslant C\boldsymbol{m}_0\, d_j^{2\gamma_0-2+2\beta_0\gamma_0-\beta_2}\, \ell_j^{5+\frac{\beta_2}{4}} + C\boldsymbol{m}_0^{\frac{1}{2}+\beta_0}\, d_j^{\gamma_0-1}\, \ell_j^{1+\beta_2} \int_{U_j} |N| \tag{8.10}$$

$$\int_{\mathcal{U}_j} |DN|^2 \leqslant C\boldsymbol{m}_0\, d_j^{2\gamma_0-2+2\delta_1}\, \ell(L_j)^{4-2\delta_1}, \tag{8.11}$$

$$\|N\|_{C^0(\mathcal{U}_j)} + \sup_{p\in \mathrm{spt}(T)\cap \mathbf{p}^{-1}(\mathcal{U}_j)} |p-\mathbf{p}(p)| \leqslant C\boldsymbol{m}_0^{\frac{1}{4}}\, d_j^{\frac{\gamma_0}{2}-\beta_2} \ell_j^{1+\beta_2}, \tag{8.12}$$

$$\mathrm{Lip}(N|_{\mathcal{U}_j}) \leqslant C\left(\boldsymbol{m}_0 d_j^{\gamma_0} \ell_j^{\gamma_0}\right)^{\beta_0}, \tag{8.13}$$

$$\|T-\mathbf{T}_F\|(\mathbf{p}^{-1}(\mathcal{U}_j\setminus\mathcal{K})) \leqslant C\boldsymbol{m}_0^{1+\beta_0}\, d_j^{(1+\beta_0)(2\gamma_0-2+2\delta_1)} \ell_j^{2+(1+\beta_0)(2-2\delta_1)}. \tag{8.14}$$

Indeed, observe that $d(H) \leqslant d_j \leqslant 2d(H)$ for every $H \in \mathscr{W}(L_j)$ and $\sum_{H\in\mathscr{W}(J_i)} \ell(H)^2 \leqslant C\ell_j^2$, because all $H \in \mathscr{W}(J_i)$ are disjoint and contained in a ball of radius comparable to ℓ_j. This in turn implies that $\sum_{H\in\mathscr{W}(J_j)} \ell(H)^{2+\varepsilon} \leqslant C\ell_j^{2+\varepsilon}$, because $\ell(H) \leqslant \ell_j$ for any $H \in \mathscr{W}(L)$, and (8.10) - (8.14) follows in view of (i').

Next we claim the following inequality for every $t > 0$, where $\eta(t)$ and $C(t)$ are suitable positive functions,

$$\sup_j \left(\boldsymbol{m}_0\, d_j\, \ell_j\right)^t \leqslant C(t)\, \boldsymbol{\Lambda}^{\eta(t)}(r)\,, \tag{8.15}$$

Indeed, using Propositions 7.4 and 7.7 and the disjointness of $\mathcal{B}_j$ we have

$$C_e\, \boldsymbol{m}_0\, d(L_j)^{2\gamma_0-2+2\delta_1}\, \ell(L_j)^{4-2\delta_1} \leqslant C \int_{\mathcal{B}_j} |DN|^2 \quad \text{if } L_j \in \mathscr{W}_e\,, \tag{8.16}$$

$$C_h\, \boldsymbol{m}_0^{\frac{1}{2}}\, d(L_j)^{\gamma_0-2\beta_2}\, \ell(L_j)^{4+2\beta_2} \int_{\mathcal{B}_j} |N|^2 \quad \text{if } L_j \in \mathscr{W}_h\,. \tag{8.17}$$

On the other hand

$$\sum_j \int_{\mathcal{B}_j} |DN|^2 \leqslant \int_{\mathcal{B}_r} |DN|^2 = \int_{B_r} |D\mathscr{N}|^2$$

by conformality of $\mathbf{\Psi}$ and

$$\sum_j \int_{\mathcal{B}_j} |N|^2 \leqslant \int_{\mathcal{B}_r} |N|^2 \leqslant C \int_{B_r} |\mathscr{N}|^2$$

by the Lipschitz regulari of $\mathbf{\Psi}$. Thus (8.15) follows easily by suitably choosing $C(t)$ and $\eta(t)$.

Observe therefore that (8.2) is an obvious consequence of (8.15), (8.13) and the uniform bound on $|D\mathbf{\Psi}|$.

8.0.5 *Proof of* (8.3)

First of all observe that, by the bounds on $\mathbf{\Psi}$,

$$\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ \mathcal{N}(z,w)| \leqslant C\int_{\mathcal{B}_r} |\zeta|^{\gamma_0-1}|\boldsymbol{\eta}\circ N(\zeta,\xi)|\,.$$

On the other hand, since $U_j \subset B_{30\ell_j}(z_{L_j}, w_{L_j})$, $\frac{d_j}{2} \leqslant |z| \leqslant 2d_j$ and thus

$$\int_{\mathcal{B}_r} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ N(z,w)| \leqslant C\sum_j d_j^{\gamma_0-1}\int_{\mathcal{U}_j} |\boldsymbol{\eta}\circ N(z,w)|\,.$$

Now considering that $d_j^{3\gamma_0-3+2\beta_0\gamma_0-\beta_2}\,\ell_j^{5+\frac{\beta_2}{4}} \leqslant d_j^{3\gamma_0-2}\,\ell_j^{4+2\beta_2}$, for $2\beta_2 \leqslant \beta_0\gamma_0$, we have

$$\begin{aligned}\int_{\mathcal{B}_r} &|z|^{\gamma_0-1}|\boldsymbol{\eta}\circ N(z,w)|\\ &\overset{(8.10)}{\leqslant} C\sum_{j\in\mathbb{N}}\left(\mathbf{m}_0\, d_j^{3\gamma_0-2}\,\ell_j^{4+2\beta_2} + C\underbrace{\mathbf{m}_0^{1/2+\beta_0}\, d_j^{\gamma_0-1}\,\ell_j^{1+\beta_2}\int_{\mathcal{U}_j}\frac{|N|}{|z|^{1-\gamma_0}}}_{=:A}\right).\end{aligned}$$

We treat the second term in the summand above via Young's inequality inequality;

$$\begin{aligned}A &\leqslant 2\left(\mathbf{m}_0^{1/2+\beta_0}\, d_j^{\frac{3\gamma_0}{2}-1}\,\ell_j^{2+\beta_2}\right)^2 + 2\left(\ell_j^{-1}\int_{\mathcal{U}_j}\frac{|N|}{|z|^{1-\frac{\gamma_0}{2}}}\right)^2\\ &\leqslant 2\,\mathbf{m}_0^{1+2\beta_0}\, d_j^{3\gamma_0-2}\,\ell_j^{4+2\beta_2} + C\int_{\mathcal{U}_j}\frac{|N|^2}{|z|^{2-\gamma_0}}\,.\end{aligned}$$

Moreover, observe that, if $L_j \in \mathscr{W}_h$, then by (8.17) and $\frac{d_j}{2} \leqslant |z| \leqslant 2d_j$,

$$\mathbf{m}_0^{1+\eta_0}\, d_j^{3\gamma_0-2}\,\ell_j^{4+2\beta_2} \leqslant C\,\mathbf{m}_0\, d_j^{2\beta_2}\int_{U_j}\frac{|N|^2}{|z|^{2-\gamma_0}}$$

while, if $L_j \in \mathscr{W}_e$, using (8.16) and (8.15), we deduce, for a suitable choice of η_0,

$$\mathbf{m}_0^{1+\eta_0}\, d_j^{3\gamma_0-2}\,\ell_j^{4+2\beta_2} \leqslant C\,\mathbf{m}_0^{\eta_0}\, d_j^{\gamma_0}\,\ell_j^{2\beta_2}\int_{\mathcal{U}_j}|DN|^2 \leqslant C\,\mathbf{\Lambda}(r)^{\eta_0}\int_{\mathcal{U}_j}|DN|^2\,.$$

Collecting all these estimates together and using the properties of $\mathbf{\Psi}$ we conclude

$$\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ \mathcal{N}(z,w)| \leqslant C\mathbf{\Lambda}(r)^{\eta_0}\mathbf{D}(r) + C\int_{B_r}\frac{|\mathcal{N}|^2(z,w)}{|z|^{2-\gamma_0}}\,.$$

However the later integral is precisely

$$\int_0^t \frac{\mathbf{H}(t)}{t^{2-\gamma_0}}\,.$$

This shows (8.3).

8.0.6 *Proof of* (8.4)

Observe that $\mathbf{T}_F = \mathbf{T}_{\mathscr{F}}$. Thus using (8.14) we have

$$\|T - \mathbf{T}_{\mathscr{F}}\|(\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r))) \overset{(8.14)}{\leqslant} C \sum_{j\in\mathbb{N}} \boldsymbol{m}_0^{1+\beta_0}\, d_j^{(2\gamma_0-2+2\delta_1)(1+\beta_0)}\, \ell_j^{2+(2-2\delta_1)(1+\beta_0)}.$$

Now, if $L_j \in \mathscr{W}_e$, then using (8.15) with a suitable η, we have

$$\begin{aligned}\boldsymbol{m}_0^{1+\beta_0}\, d_j^{(2\gamma_0-2+2\delta_1)(1+\beta_0)}\, \ell_j^{2+(2-2\delta_1)(1+\beta_0)} &\leqslant \left(\boldsymbol{m}_0\, d_j^{\gamma_0}\, \ell_j^{\gamma_0}\right)^{\beta_0} \left(\boldsymbol{m}_0\, d_j^{2\gamma_0-2+2\delta_1}\, \ell_j^{4-2\delta_1}\right)\\ &\leqslant C\, \boldsymbol{\Lambda}^{\eta}(r) \int_{\mathcal{U}_j} |DN|^2 .\end{aligned}$$

On the other hand, if $L_j \in \mathscr{W}_h$, then by our choice of the constants,

$$\begin{aligned}&\boldsymbol{m}_0^{1+\beta_0}\, d_j^{(2\gamma_0-2+2\delta_1)(1+\beta_0)}\, \ell_j^{2+(2-2\delta_1)(1+\beta_0)}\\ &\qquad = \boldsymbol{m}_0^{1+\beta_0}\, d_j^{(2\gamma_0-2+2\delta_1)(1+\beta_0)} \ell_j^{-2\delta_1+\beta_0(2-2\delta_1)-2\beta_2} \ell_j^{4+2\beta_2}\\ &\qquad \leqslant \boldsymbol{m}_0^{\frac{1}{2}+\beta_0}\, d_j^{2\gamma_0\beta_0}\, \boldsymbol{m}_0^{\frac{1}{2}} d_j^{\gamma_0-2\beta_2+\gamma_0-2} \ell_j^{4+2\beta_2}\\ &\qquad \leqslant \boldsymbol{m}_0^{\frac{1}{2}+\beta_0}\, d_j^{2\gamma_0\beta_0} \int_{\mathcal{U}_j} \frac{|N|^2}{|z|^{2-\gamma_0}}\end{aligned}$$

where we used that $-2\delta_1 + \beta_0(2-2\delta_1) - 2\beta_2 > 0$. Summing both contributions and arguing as in the previous paragraph we conclude the proof of (8.4).

$\square$

Part V

STEP 4: ASYMPTOTIC ANALYSIS

9 ALMOST MINIMALITY OF $\mathscr{N}$ AND THE HARMONIC COMPETITOR

The normal approximation $\mathscr{N}$ inherits from T an almost minimizing property for the Dirichlet energy, where the errors involved are in fact expressed in terms of some specific norms of $\mathscr{N}$ itself and of its competitors. Combining this almost minimality with a suitably constructed harmonic competitor we will prove two very usefull inequalities that will be fundamentals in the proof of the Poincaré and epiperimetric inequalities of the next chapter.

9.1 DIRICHLET ALMOST MINIMIZING PROPERTY

For technical reasons we introduce the map $F := \sum_{i=1}^{Q} [\![p + N_i(p)]\!]$, where $N := \mathscr{N} \circ \Psi^{-1}$. In order to state the almost minimizing property of $\mathscr{N}$ we introduce an appropriate notion of competitor.

Definition 9.1. A Lipschitz map $\mathscr{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ is called a competitor for $\mathscr{N}$ in the ball B_r if

(a) $\mathscr{L}|_{\partial B_r} = \mathscr{N}|_{\partial B_r}$;

(b) $\mathrm{spt}(\mathscr{G}(z,w)) \subset \Sigma$ for all $(z,w) \in B_r$, where $\mathscr{G}(z,w) := \sum_{j=1}^{Q} [\![\boldsymbol{\Psi}(z,w) + \mathscr{L}_j(z,w)]\!]$;

We are now ready to state the almost minimizing property for $\mathscr{N}$. We use the notation $\mathbf{p}_{T_p\Sigma}$ for the orthogonal projection on the tangent space to Σ at p. We recall that, given our choice of coordinates, $\mathbf{p}_{T_0\Sigma}$ is the projection on $\mathbb{R}^{2+\bar{n}} \times \{0\}$. Since this projection will be used several times, we will denote it by $\mathbf{p}_0$. By the C^{3,ε_0} regularity of Σ, there exists a map $\Psi_0 \in C^{3,\varepsilon_0}(\mathbb{R}^{2+\bar{n}}, \mathbb{R}^l)$ such that

$$\Psi_0(0) = 0\,,\ D\Psi_0(0) = 0 \quad \text{and} \quad (p, \Psi_0(p)) \in \Sigma \text{ for every } p \in \mathbb{R}^{2+\bar{n}}\,.$$

Next, for each function $\mathscr{L}$ satisfying condition (b) in Definition 9.1, we consider the map $\bar{\mathscr{L}} := \mathbf{p}_0 \circ \mathscr{L}$, which is a multivalued $\bar{\mathscr{L}} : \mathfrak{B} \to \mathcal{A}_Q(\mathbb{R}^{2+\bar{n}})$. We observe that it is possible to determine $\mathscr{L}$ from $\bar{\mathscr{L}}$. In particular, fix coordinates $(\xi,\eta) \in \mathbb{R}^{2+\bar{n}} \times \mathbb{R}^{n-\bar{n}}$ and let $\mathscr{L} = \sum [\![\mathscr{L}_i]\!]$, $\bar{\mathscr{L}} = \sum [\![\bar{\mathscr{L}}_i]\!]$, where $\bar{\mathscr{L}}_i = \mathbf{p}_0 \circ \mathscr{L}_i$. Then the formula relating $\mathscr{L}_i$ and $\bar{\mathscr{L}}_i$ is

$$\mathscr{L}_i(z,w) = \Big(\bar{\mathscr{L}}_i(z,w), \Psi_0\Big(\mathbf{p}_0(\boldsymbol{\Psi}(z,w)) + \bar{\mathscr{L}}_i(z,w)\Big) - \Psi_0\Big(\mathbf{p}_0(\boldsymbol{\Psi}(z,w))\Big)\Big)\,. \tag{9.1}$$

Proposition 9.2. *There exists a constant $C_{9.2} > 0$ such that the following holds. If $r \in (0,1)$ and $\mathscr{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ is a Lipschitz competitor for $\mathscr{N}$ with $\|\mathscr{L}\|_\infty \leqslant r$ and $\mathrm{Lip}(\mathscr{L}) \leqslant C_{9.2}^{-1}$, then*

$$\int_{B_r} |D\mathscr{N}|^2 \leqslant (1 + C_{9.2}\, r) \int_{B_r} |D\bar{\mathscr{L}}|^2 + C_{9.2}\,\mathrm{Err}_1(\mathscr{N}, B_r) + C_{9.2}\,\mathrm{Err}_2(\mathscr{L}, B_r) + C_{9.2}\, r^2\, \mathbf{D}'(r) \tag{9.2}$$

where $\bar{\mathscr{L}} := \mathbf{p}_0 \circ \mathscr{L}$ *and the the errors terms* $\mathrm{Err}_1(\mathscr{N}, B_r)$ $\mathrm{Err}_2(\mathscr{L}, B_r)$ *are given by the following expressions:*

$$\mathrm{Err}_1(\mathscr{N}, B_r) = \boldsymbol{\Lambda}^\eta(r)\,\mathbf{D}(r) + \mathbf{F}(r) + \mathbf{H}(r) + \mathbf{m}_0^{\frac{1}{2}}\, r^{1+\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathscr{N}| \tag{9.3}$$

and

$$\mathrm{Err}_2(\mathscr{L}, B_r) = \mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathscr{L}|\,. \tag{9.4}$$

For the proof of Proposition 9.2 we consider separately the three cases:

(a) T is mass minimizing;

(b) T is the cross-section of a mass minimizing three-dimensional cone;

(c) T is semicalibrated.

For notational convenience we set $L := \mathscr{L} \circ \boldsymbol{\Psi}^{-1}$, $G := \mathscr{G} \circ \boldsymbol{\Psi}$.

Observe also that, by Lemma 10.13 and 10.14, it is enough to prove that

$$\int_{B_r} |D\mathscr{N}|^2 \leqslant (1 + C_{9.2}\, r) \int_{B_r} |D\mathscr{L}|^2 + C\,\mathrm{Err}_1(\mathscr{N}, B_r) + C\,\mathrm{Err}_2(\mathscr{L}, B_r) + \frac{C}{r} \int_{B_r} |\mathscr{L}|^2\,. \tag{9.5}$$

Indeed Lemma 10.14 implies that

$$\begin{aligned}\int_{B_r} |D\mathscr{L}|^2 \leqslant & (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + Cr \int_{\partial B_r} |\bar{\mathscr{L}}|^2 \leqslant (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + Cr \int_{\partial B_r} |\mathscr{L}|^2 \\ = & (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + Cr \int_{\partial B_r} |\mathscr{N}|^2 \leqslant (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + C\,\mathrm{Err}_1(\mathscr{N}, B_r)\,,\end{aligned}$$

whereas Lemma 10.13 implies

$$\frac{1}{r} \int_{B_r} |\mathscr{L}|^2 \leqslant Cr \int_{B_r} |D\mathscr{L}|^2 + C \int_{\partial B_r} |\mathscr{L}|^2 \leqslant Cr \int_{B_r} |D\bar{\mathscr{L}}|^2 + C\,\mathrm{Err}_1(\mathscr{N}, B_r)\,.$$

9.1.1 *Proof of Proposition 9.2 case (a):* T *mass minimizing.*

We fix $\mathscr{L}$, $\bar{\mathscr{L}}$, L, $\bar{G}$ and G as above. Let us set

$$Z := T - \mathbf{T}_{\mathscr{F}|_{B_r}} + \mathbf{T}_{\mathscr{G}}\,. \tag{9.6}$$

Since $\mathscr{F}|_{\partial B_r} = \mathscr{G}|_{\partial B_r}$, from Theorem 3.47 it follows that $\partial(\mathbf{T}_{\mathscr{G}} - \mathbf{T}_{\mathscr{F}|_{B_r}}) = 0$. Moreover $\mathrm{spt}(Z) \subset \Sigma$ and therefore we must have $\mathbf{M}(T) \leqslant \mathbf{M}(Z)$. Taking into account (8.4), we conclude that

$$\begin{aligned}\mathbf{M}(\mathbf{T}_{\mathscr{F}|_{B_r}}) &\leqslant \mathbf{M}(T \llcorner \mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r))) + \|T - \mathbf{T}_{\mathscr{F}|_{B_r}}\|(\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r))) \\ &\leqslant \mathbf{M}(\mathbf{T}_{\mathscr{G}}) + 2\,\|T - \mathbf{T}_{\mathscr{F}|_{B_r}}\|(\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r))) \\ &\leqslant \mathbf{M}(\mathbf{T}_{\mathscr{G}}) + C\,\mathrm{Err}_1(\mathscr{N}, B_r)\,.\end{aligned} \tag{9.7}$$

Observe now that $\mathbf{T}_{\mathscr{F}|_{B_r}} = \mathbf{T}_{F|_{\mathbf{\Psi}(B_r)}}$ and we can use the Taylor expansion in Theorem 3.48 to bound the mass of $\mathbf{T}_F$ with:

$$\mathbf{M}(\mathbf{T}_{\mathscr{F}|_{B_r}}) \geqslant Q\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \frac{1}{2}\int_{\mathbf{\Psi}(B_r)} |DN|^2 - Q\int_{\mathbf{\Psi}(B_r)} \langle \boldsymbol{\eta}\circ N, H_{\mathcal{M}}\rangle$$
$$- C\int_{\mathbf{\Psi}(B_r)} \left(|A_{\mathcal{M}}|^2|N|^2 + |DN|^4\right), \tag{9.8}$$

where $H_{\mathcal{M}}$ denotes the mean curvature vector of $\mathcal{M}$. Note that in order to apply the Taylor expansion in Theorem 3.48, we need the manifold $\mathcal{M}$ to be C^2, with an apriori bound on the C^2 norm. However, if we take $\mathbf{T}_F \llcorner \mathbf{B}_r \setminus \mathbf{B}_{r/2}$ and rescale by a factor $1/r$, the corresponding rescaled current, map and manifold fall under the assumptions of the Taylor expansion in Theorem 3.48. We can then scale back to find the corresponding inequalities for $\mathbf{T}_F \llcorner \mathbf{B}_r \setminus \mathbf{B}_{r/2}$ and sum over dyadic annuli to conclude (9.8).

Using the conformality of $\mathbf{\Psi}$ we conclude

$$\int_{\mathbf{\Psi}(B_r)} |DN|^2 = \int_{B_r} |D\mathscr{N}|^2,$$

As for the other terms, we recall

$$\int_{\mathbf{\Psi}(B_r)} |\langle \boldsymbol{\eta}\circ N, H_{\mathcal{M}}\rangle| \leqslant C\boldsymbol{m}_0^{\frac{1}{2}} \int_{B_r} |\boldsymbol{\eta}\circ\mathscr{N}| \overset{(8.3)}{\leqslant} C\mathrm{Err}_1(\mathscr{N}, B_r), \tag{9.9}$$

$$\int_{\mathbf{\Psi}(B_r)} |DN|^4 \leqslant C\mathrm{Lip}(\mathscr{N}|_{B_r})^2 \int_{B_r} |D\mathscr{N}|^2 \overset{(8.2)}{\leqslant} C\mathrm{Err}_1(\mathscr{N}, B_r), \tag{9.10}$$

$$\int_{\mathbf{\Psi}(B_r)} |A_{\mathcal{M}}|^2|N|^2 \leqslant C\boldsymbol{m}_0 \int_{B_r} |z|^{2\gamma_0-2}|\mathscr{N}|^2 = C\boldsymbol{m}_0 \int_0^r \frac{\mathbf{H}(s)}{s^{2\gamma_0-2}}\,ds \leqslant C\mathrm{Err}_1(\mathscr{N}, B_r). \tag{9.11}$$

Combining the latter estimates with (9.6) and (9.7) we achieve

$$\frac{1}{2}\int_{B_r} |D\mathscr{N}|^2 \leqslant C\mathrm{Err}_1(\mathscr{N}, B_r) + \mathbf{M}(\mathbf{T}_G) - Q\mathcal{H}^2(\mathbf{\Psi}(B_r(x))). \tag{9.12}$$

Next, fix an orthonormal frame ξ_1, ξ_2 on B_r and, using the area formula from Lemma 3.44, compute

$$\begin{aligned}\mathbf{M}(\mathbf{T}_G) &= \int_{\mathbf{\Psi}(B_r)} \sum_i |(\xi_1 + DL_i\cdot\xi_1)\wedge(\xi_2 + DL_i\cdot\xi_2)|\\ &\leqslant \frac{1}{2}\int_{\mathbf{\Psi}(B_r)} \sum_i \left(|\xi_1 + DL_i\cdot\xi_1)|^2 + |\xi_2 + DL_i\cdot\xi_2|^2\right)\\ &= Q\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \frac{1}{2}\int_{\mathbf{\Psi}(B_r)} |DL|^2 + Q\int_{\mathbf{\Psi}(B_r)} \left(\langle D\boldsymbol{\eta}\circ L\cdot\xi_1, \xi_1\rangle + \langle D\boldsymbol{\eta}\circ L\cdot\xi_2, \xi_2\rangle\right).\end{aligned}$$

By conformality the second summand in the last inequality equals $\frac{1}{2}\int_{B_r} |D\mathscr{L}|^2$. We integrate by parts the third summand. Recall that on $\boldsymbol{\eta}\circ L = \boldsymbol{\eta}\circ N$ on $\mathbf{\Psi}(\partial B_r) = \partial(\mathbf{\Psi}(B_r))$: since $\boldsymbol{\eta}\circ N$ is orthogonal to ξ_i the boundary term vanishes. Moreover, since the origin is a singularity, we must in fact integrate by parts in $B_r \setminus B_\varepsilon$ and then let $\varepsilon \to 0$. A specific choice of ξ_i is $\xi_i = \lambda^{-\frac{1}{2}} D\boldsymbol{\Psi} \cdot e_i$, where e_1, e_2 is the parallel frame on $\mathfrak{B}_Q$ naturally induced by the standard flat coordinates. It then turns out that

$$|D_{\xi_1}\xi_1 + D_{\xi_2}\xi_2|(\boldsymbol{\Psi}(z,w)) \leqslant C\boldsymbol{m}_0^{\frac{1}{2}}|z|^{\gamma_0 - 1}\,.$$

In particular $|D_{\xi_1}\xi_1 + D_{\xi_2}\xi_2|$ is integrable on B_r and we can therefore conclude

$$\begin{aligned}\mathbf{M}(\mathbf{T}_G) - Q\mathcal{H}^2(\boldsymbol{\Psi}(B_r)) &\leqslant \frac{1}{2}\int_{\boldsymbol{\Psi}(B_r)} |DL|^2 + Q\int_{\boldsymbol{\Psi}(B_r)} \langle \boldsymbol{\eta}\circ L, D_{\xi_1}\xi_1 + D_{\xi_2}\xi_2\rangle \\ &\leqslant \frac{1}{2}\int_{B_r} |D\mathscr{L}|^2 + C\mathrm{Err}_2(\mathscr{L}, B_r)\,. \end{aligned} \tag{9.13}$$

Combining (9.12) and (9.13) we conclude (9.5).

9.1.2 *Proof of Proposition 9.2 case (c):* T *semicalibrated*

We proceed as in the previous step and define the current Z as in (9.6). If S is any current such that

$$\partial S = T - Z = \mathbf{T}_{\mathscr{F}|_{B_r}} - \mathbf{T}_G = \mathbf{T}_{F|_{\boldsymbol{\Psi}(B_r)}} - \mathbf{T}_G\,,$$

then the semicalibrated condition gives

$$\mathbf{M}(T) \leqslant \mathbf{M}(Z) + S(d\omega)\,,$$

where ω is the calibrating form. In particular, in order to conclude the proof it suffices to find an S such that

$$|S(d\omega)| \leqslant C\,\mathrm{Err}_1(\mathscr{N}, B_r) + C\,\mathrm{Err}_2(\mathscr{L}, B_r) + \frac{C}{r}\int_{B_r} |\mathscr{L}|^2\;: \tag{9.14}$$

combining the latter inequality with the estimates of the previous subsection we reach the desired conclusion.

To do this we first define $H_i : [0,1] \times \boldsymbol{\Psi}(B_r) \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ for $i = 1, 2$ by

$$\begin{aligned}[0,1] \times \boldsymbol{\Psi}(B_r) \ni (t,p) &\mapsto H_1(t,p) := \sum_{i=1}^{Q} [\![p + t\,N_i(p)]\!] \in \mathcal{A}_Q(\mathbb{R}^{2+n})\\ [0,1] \times \boldsymbol{\Psi}(B_r) \ni (t,p) &\mapsto H_2(t,p) := \sum_{i=1}^{Q} [\![p + (1-t)\,L_i(p)]\!] \in \mathcal{A}_Q(\mathbb{R}^{2+n})\,.\end{aligned}$$

We choose $S := S_1 + S_2$, where $S_i := \mathbf{T}_{H_i}$ for $i = 1, 2$. Thanks to Theorem 3.47, we get

$$\begin{aligned}\partial S_1 &= \mathbf{T}_{F|_{\boldsymbol{\Psi}(B_r)}} - Q\,[\![\mathcal{M}]\!] - \mathbf{T}_{H_1|_{[0,1]\times\boldsymbol{\Psi}(\partial B_r)}},\\ \partial S_2 &= Q\,[\![\mathcal{M}]\!] - \mathbf{T}_G - \mathbf{T}_{H_2|_{[0,1]\times\boldsymbol{\Psi}(\partial B_r)}}.\end{aligned}$$

On the other hand since $N = L$ on $\boldsymbol{\Psi}(\partial B_r)$, we conclude $\partial S = \partial(S_1 + S_2) = T - Z$.

We next estimate $|S_1(d\omega)|$ and $|S_2(d\omega)|$. Since the estimates are analogous, we give the details only for the first. We start from the formula

$$S_1(d\omega) = \int_{\mathbf{\Psi}(B_r)} \int_0^1 \sum_{i=1}^{Q} \left\langle \vec{\zeta}_i(t,p), d\omega((H_1)_i(t,p)) \right\rangle d\mathcal{H}^2(p)\, dt,$$

with

$$\begin{aligned}\vec{\zeta}_i(t,p) &= \big(\xi_1 + t\,\nabla_{\xi_1} N_i(p)\big) \wedge \big(\xi_2 + t\,\nabla_{\xi_2} N_i(p)\big) \wedge N_i(p) \\ &=: \xi_1 \wedge \xi_2 \wedge N_i(p) + \vec{E}_i(t,p)\,,\end{aligned}$$

and

$$|\vec{E}_i(t,p)| \leqslant C\,(|DN|(p) + |DN|^2(p))\,|N|(p). \tag{9.15}$$

Next we note that

$$d\omega((H_1)_i(t,p)) = d\omega(p) + I(t,p), \tag{9.16}$$

with $I(t,p)$ naturally estimated by

$$|I(t,p)| = |d\omega((H_1)_i(t,p)) - d\omega(p)| \leqslant C\,\|D^2\omega\|_{L^\infty}\,|N|(p). \tag{9.17}$$

Therefore, we have

$$\begin{aligned}\Big|\sum_{i=1}^{Q} \left\langle \vec{\zeta}_i(t,p), d\omega((H_1)_i(t,p)) \right\rangle\Big| &\leqslant \sum_{i=1}^{Q} \langle \xi_1 \wedge \xi_2 \wedge N_i(p), d\omega(p)\rangle + \|d\omega\|_{L^\infty} \sum_{i=1}^{Q} |\vec{E}_i(t,p)| \\ &\quad + C \sum_{i=1}^{Q} \Big((|N_i| + |\vec{E}_i|)|I|\Big)(t,p) \\ &\leqslant C\mathbf{m}_0^{\frac{1}{2}}\,|\boldsymbol{\eta} \circ N| + C|N|^2(p) + C|DN|(p)\,|N|(p) + Cr|DN|^2(p)\,,\end{aligned}$$

where we have only used the bound $|N|(p) \leqslant Cr$ on $\mathbf{\Psi}(B_r)$. Arguing similarly for S_2 (observe that we have the bound $|L|(p) \leqslant Cr$) and estimating $|N||DN| + |L||DL| \leqslant r^{-1}(|N|^2 + |L|^2) + Cr(|DN|^2 + |DL|^2)$, we conclude

$$\begin{aligned}|S_1(d\omega)| + |S_2(d\omega)| &\leqslant C\,\mathbf{m}_0^{\frac{1}{2}} \int_{\mathbf{\Psi}(B_r)} \big(|\boldsymbol{\eta} \circ N| + |\boldsymbol{\eta} \circ L|\big) + C\,r^{-1} \int_{\mathbf{\Psi}(B_r)} \big(|N|^2 + |L|^2\big) \\ &\quad + Cr \int_{\mathbf{\Psi}(B_r)} \big(|DN|^2 + |DL|^2\big),\end{aligned}$$

and by a change of variable and Theorem 6.9 the claim follows.

9.1.3 *Proof of Proposition 9.2 in case (b):* T *is the cross-section of a three dimensional area minimizing cone*

Recall that in this case $\mathrm{spt}(T) \subset \partial \mathbf{B}_R(p_0)$, where $p_0 = (0, \ldots, 0, R) = Re_{n+2}$ and $R^{-1} \leqslant \mathbf{m}_0^{\frac{1}{2}}$. For the computations of this subsection it is indeed convenient to change coordinates so that

p_0 is in fact the origin, whereas $\Psi(0,0)$ is the point $(0,\dots,0,-R)$. In these new coordinates we then have $\mathcal{M}, \mathrm{spt}(T), \mathrm{Im}(\mathscr{F}) \subset \partial \mathbf{B}_R(0)$. These coordinates will however be used only in here, whereas in the next sections we will return to the usual ones.

We introduce the following notation: $\mathcal{C}(r)$ is the cone over $\mathbf{\Psi}(B_r)$ with vertex 0, i.e.

$$\mathcal{C}(r) := \left\{\rho p \in \mathbb{R}^{n+2} \,:\, \rho \in [0,1],\, p \in \mathbf{\Psi}(B_r)\right\},$$

with the orientation compatible with that of $0 \times\!\!\!\!\times\, [\![\mathcal{M}]\!]$. We extend F to $\tilde F : \mathcal{C}(r) \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ by setting $\tilde F(\rho p) := \rho\, F(p)$ for every $p \in \mathbf{\Psi}(B_r)$.

In order to estimate the Dirichlet energy of N in terms of that of L, we construct a suitable function $K : \mathcal{C}(r) \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ (depending on L and N) such that $K|_{\partial \mathcal{C}(r)} = \tilde F|_{\partial \mathcal{C}(r)}$: we can then test the minimizing property of $0 \times\!\!\!\!\times\, T$ comparing its mass with that of the current

$$Z := 0 \times\!\!\!\!\times\, T - \mathbf{T}_{\tilde F} + \mathbf{T}_K = 0 \times\!\!\!\!\times\, (T - \mathbf{T}_{F|_{\mathbf{\Psi}(B_r)}}) + \mathbf{T}_K$$

which is easily recognized to satisfy $\partial Z = \partial(0 \times\!\!\!\!\times\, T)$. In particular, using the minimality of $0 \times\!\!\!\!\times\, T$, we conclude

$$R^{-1}\mathbf{M}(0 \times\!\!\!\!\times\, \mathbf{T}_{F|_{\mathbf{\Psi}(B_r)}}) \leqslant R^{-1}\mathbf{M}(\mathbf{T}_K) + C\,\mathrm{Err}_1(\mathscr{N}, B_r)\,. \tag{9.18}$$

We consider the space of parameters $[0,1] \times B_r$ and recall that the points in $\mathfrak{B}_Q$ are identified by four co-ordinates $(z,w) \in \mathbb{R}^2 \times \mathbb{R}^2$. For the definition of K we need to introduce the following sets

$$A_1 := \left\{(\rho,z,w) \in [0,1] \times B_r : 1 - r \leqslant \rho \leqslant 1,\ |z| \leqslant \frac{\rho + 2r - 1}{2}\right\}, \tag{9.19}$$

$$A_2 := \left\{(\rho,z,w) \in [0,1] \times B_r : 1 - 2r \leqslant \rho \leqslant 1 - r,\ |z| \leqslant \frac{1-\rho}{2}\right\}, \tag{9.20}$$

$$B := [1-2r, 1] \times B_r \setminus \big(A_1 \cup A_2\big), \tag{9.21}$$

We then define the function $\mathscr{H} : [0,1] \times B_r \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ given by

$$\mathscr{H}(\rho,z,w) := \begin{cases} \rho\, \mathscr{L}(z,w) & \text{if } \rho \leqslant 1 - 2r, \\ \rho\, l_1(\rho)\, \mathscr{N}\left(\frac{2rz}{\rho+2r-1}, \frac{(2r)^{\frac1Q}}{(\rho+2r-1)^{\frac1Q}} w\right) & \text{if } (\rho,z,w) \in A_1, \\ -\rho\, l_1(\rho)\, \mathscr{L}\left(\frac{2rz}{1-\rho}, \frac{(2r)^{\frac1Q}}{(1-\rho)^{\frac1Q}} w\right) & \text{if } (\rho,z,w) \in A_2, \\ \rho\, l_2(|z|)\, \mathscr{N}\left(\frac{rz}{|z|}, \frac{r^{\frac1Q}}{|z|^{\frac1Q}} w\right) & \text{if } (\rho,z,w) \in B, \end{cases} \tag{9.22}$$

where $l_1, l_2 : \mathbb{R} \to \mathbb{R}$ are the affine functions

$$l_1(t) := \frac{t + r - 1}{r} \quad \text{and} \quad l_2(t) := \frac{2t - r}{r}. \tag{9.23}$$

The following are simple properties of $\mathscr{H}$ which can be easily verified:

(1) $\mathscr{H}(1,z,w) = \mathscr{N}(z,w)$ for every $(z,w) \in B_r$, as $(1,z,w) \in A_1$ and $l_1(1) = 1$;

(2) $\mathscr{H}(\rho, z, w) = \rho\,\mathscr{N}(z, w)$ for every $\rho \in [0,1]$ and for every z with $|z| = r$, as $\mathscr{L}|_{\partial B_r} = \mathscr{N}|_{\partial B_r}$ and $l_2(r) = 1$;

(3) $\mathscr{H}$ is well-defined and continuous, as $\mathscr{H} \equiv 0$ in $A_1 \cap A_2$ from $l_1(1-r) = 0$,

$$\mathscr{H}(\rho, z, w) = \rho\, \tfrac{\rho+r-1}{r}\, \mathscr{N}\left(\tfrac{r\,z}{|z|}, \tfrac{r^{\frac{1}{Q}}}{|z|^{\frac{1}{Q}}} z\right) \quad \text{in } A_1 \cap \partial B,$$

and

$$\mathscr{H}(\rho, z, w) = \rho\, \tfrac{\rho+r-1}{r}\, \mathscr{N}\left(\tfrac{r\,z}{|z|}, \tfrac{r^{\frac{1}{Q}}}{|z|^{\frac{1}{Q}}} w\right) \quad \text{in } A_2 \cap \partial B.$$

The competitor map $K : \mathcal{C}(r) \to \mathcal{A}_Q(\mathbb{R}^{n+2})$ is now given by

$$K(\rho\, p) := \sum_{i=1}^{Q} [\![\rho\, p + H_i(\rho\, p)]\!] \quad \text{with } H(\rho\, p) := \mathscr{H}(\rho, \mathbf{\Psi}^{-1}(p)).$$

Note that by (1) and (2) above it follows that $K|_{\partial\mathcal{C}(r)} = \tilde{F}|_{\partial\mathcal{C}(r)}$.

We start now estimating the masses of the various currents introduced above. Since $\mathrm{spt}(\mathbf{T}_F) \subset \partial\mathbf{B}_R(0)$, it follows that $\mathbf{M}(0 \mathbin{⨳} \mathbf{T}_F) = R\mathbf{M}(\mathbf{T}_F)/3$ and, by the expansion of the mass of $\mathbf{T}_F$, we have that

$$\mathbf{M}(\mathbf{T}_{F|_{\mathbf{\Psi}(B_r)}}) \geqslant Q\,\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \frac{1}{2}\int_{B_r} |D\mathscr{N}|^2 - C\mathrm{Err}_1(\mathscr{N}, B_r)\,. \tag{9.24}$$

Combining the latter estimate with (9.18) we conclude

$$\int_{B_r} |D\mathscr{N}|^2 \leqslant 6R^{-1}\mathbf{M}(\mathbf{T}_K) - 2Q\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \mathrm{Err}_1(\mathscr{N}, B_r)\,. \tag{9.25}$$

For what concerns the mass of $\mathbf{T}_K$, recalling that $p + \mathrm{spt}(L(p)) \in \partial\mathbf{B}_R(0)$ for every $p \in \mathbf{\Psi}(B_r)$, we deduce that

$$\mathbf{M}(\mathbf{T}_K \llcorner \mathbf{B}_{R(1-2r)}) = \mathbf{M}(0 \mathbin{⨳} \mathbf{T}_G \llcorner \mathbf{B}_{R(1-2r)}) = R\frac{(1-2r)^3\mathbf{M}(\mathbf{T}_G)}{3}$$

and

$$\mathbf{M}(\mathbf{T}_G) \leqslant Q\,\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \frac{1}{2}\int_{B_r} |D\mathscr{L}|^2 + \mathrm{Err}_2(\mathscr{L}, B_r)\,.$$

In particular we conclude

$$6R^{-1}\mathbf{M}(\mathbf{T}_K \llcorner \mathbf{B}_{R(1-2r)}) \leqslant 2Q(1-2r)^3\mathcal{H}^2(\mathbf{\Psi}(B_r)) + \int_{B_r} |D\mathscr{L}|^2 + \mathrm{Err}_2(\mathscr{L}, B_r)\,. \tag{9.26}$$

Next we pass to estimating $\mathbf{M}(\mathbf{T}_K \llcorner \mathbf{B}_R \setminus \mathbf{B}_{R(1-2r)})$. In order to carry on our estimates we use the area formula for multifunctions, cf. Lemma 3.44. In particular we fix an orthonormal frame ξ_1, ξ_2 for $\mathcal{M}$ as in the proof of case (a) and we let $\xi_3 = R^{-1}\partial_t$ be normal to them in $T\mathcal{C}(r)$, i.e. pointing in the radial direction of the cone. We then have

$$\mathbf{M}(\mathbf{T}_K \llcorner (\mathbf{B}_R \setminus \mathbf{B}_{R(1-r)}) = \int_{\mathcal{C}(r)} \sum_i \underbrace{|(\xi_1 + DH_i \cdot \xi_1) \wedge (\xi_2 + DH_i \cdot \xi_2) \wedge (\xi_3 + DH_i \cdot \xi_3)|}_{(A)} .$$

Using the Taylor expansion for (A), cf. [18], we can bound (recall that $\boldsymbol{\Omega} = 3\,R^{-1} \leqslant \mathbf{m}_0^{\frac{1}{2}}$)

$$\begin{aligned} R^{-1}\mathbf{M}(\mathbf{T}_K \llcorner (\mathbf{B}_R \setminus \mathbf{B}_{R(1-2r)})) &\leqslant QR^{-1}\,\mathcal{H}^3\big(\mathcal{C}(r) \cap \mathbf{B}_1 \setminus \mathbf{B}_{1-2r}\big) \\ &+ Q\,\mathbf{m}_0^{\frac{1}{2}} \int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \frac{d}{dt}[(\boldsymbol{\eta} \circ H)(tp)]t^2 dt \\ &+ Q\,\mathbf{m}_0^{\frac{1}{2}} \int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \sum_{i=1}^2 \langle \nabla_{\xi_i}(\boldsymbol{\eta} \circ H), \xi_i \rangle\, t^2 dt + C \int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} |DH|^2\, t^2 dt\,. \end{aligned} \tag{9.27}$$

The linear terms can be integrated by parts: since $\nabla_p(\boldsymbol{\eta} \circ H)(tp) = \frac{d}{dt}(\boldsymbol{\eta} \circ H)(tp)$, we have

$$\begin{aligned} \int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \frac{d}{dt}[(\boldsymbol{\eta} \circ H)(tp)]t^2 dt &= \int_{\boldsymbol{\Psi}(B_r)} \big\langle (\boldsymbol{\eta} \circ H)(p) - (1-2r)^2 (\boldsymbol{\eta} \circ H)\big((1-2r)p\big), p \big\rangle \\ &\quad - 2 \int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \langle (\boldsymbol{\eta} \circ H)(tp), p \rangle\, t dt \end{aligned} \tag{9.28}$$

$$\int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \sum_{i=1}^2 \langle \nabla_{\xi_i}(\boldsymbol{\eta} \circ H), \xi_i \rangle\, t^2 dt = -\int_{1-2r}^1 \int_{\boldsymbol{\Psi}(B_r)} \langle (\boldsymbol{\eta} \circ H), H_{\mathcal{M}} \rangle\, t^2 dt. \tag{9.29}$$

Therefore, by a simple change of coordinates we can estimate

$$\begin{aligned} R^{-1}\mathbf{M}(\mathbf{T}_K \llcorner (\mathbf{B}_R \setminus \mathbf{B}_{R(1-r)})) &\leqslant \frac{Q\,\big(1-(1-2r)^3\big)}{3}\,\mathcal{H}^2\big(\boldsymbol{\Psi}(B_r)\big) + C\,\mathbf{m}_0^{\frac{1}{2}}\, r \int_{B_r} \big(|\boldsymbol{\eta} \circ \mathscr{N}| + |\boldsymbol{\eta} \circ \mathscr{L}|\big) \\ &+ C\,\mathbf{m}_0^{\frac{1}{2}} \int_{1-2r}^1 \int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathscr{H}|(t,z,w)\, dz\, dt + C \int_{1-2r}^1 \int_{B_r} |D\mathscr{H}|^2(t,z,w)\, dz\, dt. \end{aligned} \tag{9.30}$$

In order to bound the various integrands of (9.30), we start with the following general remark. Assume that $\chi : [1-2r, 1] \times B_r \to [0, +\infty)$ has the structure

$$\chi(\rho, x, y) = \begin{cases} \chi_1\left(\frac{2\,r\,z}{\rho+2r-1}, \frac{(2\,r)^{\frac{1}{Q}}}{(\rho+2\,r-1)^{\frac{1}{Q}}} w\right) & \text{if } (\rho, z, w) \in A_1, \\ \chi_2\left(\frac{2\,r\,z}{1-\rho}, \frac{(2\,r)^{\frac{1}{Q}}}{(1-\rho)^{\frac{1}{Q}}} w\right) & \text{if } (\rho, z, w) \in A_2, \\ \chi_3\left(\frac{r\,z}{|z|}, \frac{r^{\frac{1}{Q}}}{|z|^{\frac{1}{Q}}} w\right) & \text{if } (\rho, z, w) \in B, \end{cases} \tag{9.31}$$

for some $\chi_1, \chi_2, \chi_3 : B_r^Q \to [0, +\infty)$. Then one can compute the integral of χ in the following way:

$$\int_{1-r}^1 \int_{B_r} \chi(t,z,w)\, dz\, dt = \int_{A_1} \chi(t,z,w)\, dz\, dt + \int_{A_2} \chi(t,z,w)\, dz\, dt + \int_B \chi(t,z,w)\, dz\, dt,$$

and one can easily compute that

$$\begin{aligned}
&\int_{A_1} \chi(t,z,w)\, dz\, dt = \int_{1-r}^1 \int_{B_{\frac{t+2r-1}{2}}} \chi_1(t,z,w)\, dz\, dt \\
&= \int_{1-r}^1 \int_{B_{\frac{t+2r-1}{2}}} \chi_1 \left(\frac{2\,r\,z}{t+2r-1}, \frac{(2\,r)^{\frac{1}{Q}}}{(t+2\,r-1)^{\frac{1}{Q}}} w \right) dz\, dt \\
&= \int_{1-r}^1 \left(\frac{t+2r-1}{2r} \right)^2 \int_{B_r} \chi_1(z,w) dz\, dt \leqslant r \int_{B_r} \chi_1(z,w) dz\, dt\,.
\end{aligned} \tag{9.32}$$

Similarly

$$\int_{A_2} \chi(t,z,w) dz\, dt \leqslant r \int_{B_r} \chi_2(z,w) d\mu_0\, dt, \tag{9.33}$$

and

$$\begin{aligned}
\int_B \chi(t,z,w) dz\, dt &= \int_{1-r}^1 dt \int_{\frac{t+2r-1}{2}}^r \frac{s}{r}\, ds \int_{\partial B_r} \chi_3(z,w)\, dz \\
&\quad + \int_{1-2r}^{1-r} \int_{\frac{1-t}{2}} r\frac{s}{r}\, ds \int_{\partial B_r} \chi_3(z,w)\, dz \leqslant r^2 \int_{\partial B_r} \chi_3(z,w)\, dz\,.
\end{aligned} \tag{9.34}$$

By direct computations one verifies that the integrands in (9.30) are all bounded from above by functions χ with the structure (9.31): in particular,

(i) $|z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathscr{H}|(t,z,w) \leqslant \chi(t,z,w)$ if we choose

$$\chi_1(z,w) = \chi_3(z,w) = |z|^{\gamma_0-1} |\boldsymbol{\eta} \circ \mathscr{N}|(z,w) \quad \text{and} \quad \chi_2(z,w) = |z|^{\gamma_0-1} |\boldsymbol{\eta} \circ \mathscr{L}|(x,y);$$

(ii) $|D\mathscr{H}|^2(t,z,w) \leqslant \chi(t,z,w)$ if we choose

$$\chi_1(z,w) = \chi_3(z,w) = \frac{C}{r^2} |\mathscr{N}|^2(z,w) + C\, |D\mathscr{N}|^2(z,w)$$

$$\chi_2(z,w) = \frac{C}{r^2} |\mathscr{L}|^2(z,w) + C\, |D\mathscr{L}|^2(z,w).$$

for some dimensional constant $C > 0$.

It then turns out from (9.32), (9.33), (9.34) and (i), (ii), (iii) that

$$\begin{aligned}
6R^{-1} \mathbf{M}(\mathbf{T}_K \llcorner (\mathbf{B}_R \setminus \mathbf{B}_{R(1-r)})) &\leqslant Q \left(1 - (1-2r)^3 \right) \mathcal{H}^2 (\boldsymbol{\Psi}(B_r)) \\
&\quad + \mathrm{Err}_1(\mathscr{N}, B_r) + \mathrm{Err}_2(\mathscr{L}, B_r)\,.
\end{aligned} \tag{9.35}$$

Summing (9.35) and (9.26) we conclude

$$6R^{-1} \mathbf{M}(\mathbf{T}_K) \leqslant 2Q\mathcal{H}^2(\boldsymbol{\Psi}(B_r)) + \int_{B_r} |D\mathscr{L}|^2 + \mathrm{Err}_1(\mathscr{N}, B_r) + \mathrm{Err}_2(\mathscr{L}, B_r)\,.$$

Combining the latter estimate with (9.25) we conclude the proof.

9.2 HARMONIC COMPETITOR AND TWO USEFUL INEQUALITIES

The most natural choice for the competitor $\mathscr{L}$ is a suitable "harmonic" extension of the boundary value $\mathscr{N}|_{\partial B_r}$. Following the ideas of [9] we estimate carefully the energy of such competitor. To this purpose it is useful to introduce "polar" coordinates with center 0 in $\mathfrak{B}$ and split accordingly the Dirichlet integrand in radial and angular parts. More precisely, consider $(z_0, w_0) = ((\xi_0, \zeta_0), w_0) \in \partial B_r$ and take, locally, the standard flat coordinates $z = (x_1, x_2)$ of Definition 2.10. We then denote by ν the exterior unit vector normal to ∂B_r at (z_0, w_0) and by τ the corresponding tangent unit vector obtained by rotating ν of an angle $\pi/2$ in the counterclockwise direction, namely

$$\nu := |z_0|^{-1}\left(\xi_0 \frac{\partial}{\partial x_1} + \zeta_0 \frac{\partial}{\partial x_2}\right) \qquad \text{and} \qquad \tau := |z_0|^{-1}\left(-\zeta_0 \frac{\partial}{\partial x_1} + \xi_0 \frac{\partial}{\partial x_2}\right).$$

The directional derivatives of any (multi)function f on $\mathfrak{B}$ gives then two (multi)functions

$$D_\nu f = \sum_i [\![Df_i \cdot \nu]\!] \qquad \text{and} \qquad D_\tau f = \sum_i [\![Df_i \cdot \tau]\!] .$$

The Dirichlet integrand $|Df|^2$ enjoys then the splitting

$$|Df|^2 = |D_\nu f|^2 + |D_\tau f|^2 .$$

For the rigorous justification of these identities see [17].

Proposition 9.3. *There are constants $C > 0$, $\sigma > 0$ such that, for every $r \in (0,1)$ there exists a competitor $\mathscr{L}\colon B_r \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ for $\mathscr{N}$ with the following additional properties:*

(i) $\mathrm{Lip}(\mathscr{L}) \leqslant C_{9.2}$, $\|\mathscr{L}\|_0 \leqslant Cr$;

(ii) The following estimate hold:

$$\int_{B_r} |D\bar{\mathscr{L}}|^2 \leqslant C\, r \int_{\partial B_r} |D\bar{\mathscr{N}}|^2 \leqslant C\, r\, \mathbf{D}'(r)\,, \tag{9.36}$$

$$\int_{B_r} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \mathscr{L}| \leqslant C\, r^{\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathscr{N}| + \mathbf{H}(r)\,; \tag{9.37}$$

(iii) For every $a > 0$ there exists $b_0 > 0$ such that, for all $b \in (0, b_0)$, the following estimate holds:

$$(2\,a + b) \int_{B_r} |D\bar{\mathscr{L}}|^2 \leqslant r \int_{\partial B_r} |D_\tau \mathscr{N}|^2 + \frac{a\,(a+b)}{r} \int_{\partial B_r} |\mathscr{N}|^2 + C r^{1+\sigma} \mathbf{D}'(r)\,. \tag{9.38}$$

Using this competitor in Proposition 9.2, we then infer the following corollary.

Corollary 9.4. *For every $r \in (0,1)$ the following inequality holds*

$$\mathbf{D}(r) \leqslant C\, r\, \mathbf{D}'(r) + C r^{1-\gamma_0}\, \mathbf{H}(r) + C\, \mathbf{F}(r) + C\, \mathbf{m}_0^{\frac{1}{2}}\, r^{\gamma_0} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathscr{N}|\,. \tag{9.39}$$

For every $a > 0$ *there exists* $b_0 > 0$ *such that, for all* $b \in (0, b_0)$ *and all* $r \in]0,1[$

$$\mathbf{D}(r) \leqslant (1 + C\,r)\,\left[\frac{r}{(2\,a+b)}\int_{\partial B_r} |D_\tau \mathscr{N}|^2 + \frac{a\,(a+b)}{r\,(2\,a+b)}\mathbf{H}(r)\right] + C\,\mathcal{E}_{QM}(r) + C\,r^{1+\sigma}\,\mathbf{D}'(r)\,, \tag{9.40}$$

with

$$\mathcal{E}_{QM}(r) \leqslant \mathbf{\Lambda}(r)^{\eta_0}\mathbf{D}(r) + \mathbf{F}(r) + \mathbf{H}(r) + \mathbf{m}_0^{\frac{1}{2}}\,r^{\gamma_0}\int_{\partial B_r} |\boldsymbol{\eta} \circ \mathscr{N}|\,.$$

Proof of Corollary 9.4. Recalling that $\mathbf{H}(r) \leqslant Cr\|\mathscr{N}\|^2_{\partial B_r} \leqslant Cr^{3+\gamma_0}$ we easily infer that $\mathbf{\Lambda}(r) \leqslant Cr^2$ and thus the inequalities follow readily from Proposition 9.2 and Proposition 9.3. $\square$

9.2.1 *Proof of Proposition 9.3: Step 1*

First of all we observe that it suffices to exhibit $\bar{\mathscr{L}}$, as $\mathscr{L}$ can be recovered from it via the formula (9.1). Moreover, it suffices to show the estimates with $\bar{\mathscr{N}}$ replacing $\mathscr{N}$, because we obviously have $|\bar{\mathscr{N}}| \leqslant |\mathscr{N}|$ and $|D\bar{\mathscr{N}}| \leqslant |D\mathscr{N}|$ so that the corresponding error terms can all be absorbed in $\mathcal{E}_{QM}$ (observe that the definition of $\bar{\mathscr{N}}$ also ensures $|\boldsymbol{\eta} \circ \bar{\mathscr{N}}| \leqslant |\boldsymbol{\eta} \circ N|$). Next we wish to relate $\boldsymbol{\eta} \circ L$ and $\boldsymbol{\eta} \circ \bar{L}$ for two maps satisfying the relation (9.1). Note that by a simple Taylor expansion (cp. (10.84)) we have

$$|\boldsymbol{\eta} \circ \mathscr{L}| \leqslant C|\boldsymbol{\eta} \circ \bar{\mathscr{L}}| + C\mathcal{G}(\bar{\mathscr{L}}, \boldsymbol{\eta} \circ \bar{\mathscr{L}})^2\,,$$

where the constant C depends on the C^2 norm of Ψ_0. In particular we record the following conclusion:

$$\int_{B_r} |z|^{\gamma_0 - 1}|\boldsymbol{\eta} \circ \mathscr{L}| \leqslant C\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta} \circ \bar{\mathscr{L}}| + C\int_{B_r} |z|^{\gamma_0-1}|\bar{\mathscr{L}}|^2\,. \tag{9.41}$$

In this step we exhibit an "harmonic"[1] competitor $\mathscr{H}$ which satisfies all the requirements of the proposition except for the Lipschitz estimate. In fact we will show that there is a $W^{1,2}$ map $\mathscr{H} : B_r \to \mathcal{A}_Q(\mathbb{R}^{2+\bar{n}})$ such that

$$\mathscr{H}|_{\partial B_r} = \bar{\mathscr{N}}|_{\partial B_r} \quad \text{and} \quad \|\mathscr{H}\|_{L^\infty(B_r)} \leqslant \|\bar{\mathscr{N}}\|_{L^\infty(\partial B_r)} \tag{9.42}$$

$$\int_{B_r} |D\mathscr{H}|^2 \leqslant Cr\int_{\partial B_r} |D\bar{\mathscr{N}}|^2 \tag{9.43}$$

$$\int_{B_r} |z|^{\gamma_0-1}|\boldsymbol{\eta} \circ \mathscr{H}| \leqslant Cr^{\gamma_0}\int_{\partial B_r} |\boldsymbol{\eta} \circ \bar{\mathscr{N}}| \tag{9.44}$$

$$\int_{B_r} |z|^{\gamma_0-1}|\mathscr{H}|^2 \leqslant Cr^{\gamma_0}\int_{\partial B_r} |\bar{\mathscr{N}}|^2 \tag{9.45}$$

$$(2\,a+b)\int_{B_r} |D\bar{\mathscr{H}}|^2 \leqslant r\int_{\partial B_r} |D_\tau \bar{\mathscr{N}}|^2 + \frac{a\,(a+b)}{r}\int_{\partial B_r} |\bar{\mathscr{N}}|^2\,. \tag{9.46}$$

1 We remark that the competitor used here does not coincide, in general, with the Dirichlet minimizer with boundary value $\bar{\mathscr{N}}|_{\partial B_r}$.

In these estimates we do not use any of the particular properties of $\bar{\mathscr{N}}$ and indeed for any Lipschitz multivalued map $\bar{\mathscr{N}} : B_r \to \mathcal{A}_Q(\mathbb{R}^{2+\bar{n}})$ there is such an "harmonic" competitor. Therefore, given the scaling invariance of the estimates, we will assume without loss of generality that $r = 1$.

Let $D_r := \{|z| < r\}$ denote the disk of radius r in $\mathbb{R}^2$, which we identiy with the complex plane. We start by defining the "winding map" $\boldsymbol{W} : \mathbb{R}^2 \supset \bar{D}_1 \to \mathfrak{B}$ given (in complex notation) by

$$\boldsymbol{W}(z) := (z^{\bar{Q}}, z)\,.$$

We then consider the multivalued map $\mathscr{U} := \bar{\mathscr{N}} \circ \boldsymbol{W}$. Let $\theta \mapsto u(\theta)$ be its trace on $\partial D_1(0)$, which we parametrize with the angle $\theta \in [0, 2\pi]$. According to Lemma 3.16 we can decompose u in a superposition of simple functions $u(\theta) = \sum_{j=1}^{J} u_j(\theta)$ such that, for every $j = 1, \ldots, J$,

$$u_j(\theta) = \sum_{i=1}^{Q_j} \left[\!\!\left[\gamma_j\left(\frac{\theta + 2\pi i}{Q_j}\right)\right]\!\!\right]\,,$$

where the $\gamma_j : [0, 2\pi] \to \mathbb{R}^{2+\bar{n}}$ are periodic Lipschitz functions. Next consider the Fourier's expansion of each γ_j

$$\gamma_j(\theta) = \frac{a_{j,0}}{2} + \sum_{l=1}^{\infty}\big(a_{j,l}\cos(l\theta) + b_{j,l}\sin(l\theta)\big)\,,$$

and its harmonic extension, which in polar coordinates (ρ, θ) reads as

$$\zeta_j(\rho, \theta) := \frac{a_{j,0}}{2} + \sum_{l=1}^{\infty}\rho^l\big(a_{j,l}\cos(l\theta) + b_{j,l}\sin(l\theta)\big)\,. \tag{9.47}$$

We then can define the "harmonic" competitor for $\mathscr{U}$, which is the Q-valued map

$$\mathscr{V}(\rho, \theta) := \sum_{j=1}^{J}\sum_{i=1}^{Q_j}\left[\!\!\left[\zeta_j\left(\rho^{\frac{1}{Q_j}}, \frac{\theta + 2\pi i}{Q_j}\right)\right]\!\!\right]$$

and the "harmonic" competitor for $\bar{\mathscr{N}}$, which is $\mathscr{H} = \mathscr{V} \circ \boldsymbol{W}$. Observe that the first claim in (9.42) is obvious, whereas the second claim follows from the maximum principle for classical harmonic functions.

Simple computations and the conformality of $\boldsymbol{W}$, see for instance [17, Proof of Proposition 5.2], yield

$$\int_{B_1} |D\mathscr{H}|^2 = \int_{D_1} |D\mathscr{V}|^2 = \pi\sum_{j=1}^{J}\sum_{l=1}^{\infty} l\big(|a_{j,l}|^2 + |b_{j,l}|^2\big)\,, \tag{9.48}$$

$$\int_{\partial B_1} |D_\tau\mathscr{H}|^2 = \frac{\pi}{\bar{Q}}\sum_{j=1}^{J}\sum_{l=1}^{\infty}\frac{l^2}{Q_j}\big(|a_{j,l}|^2 + |b_{j,l}|^2\big)\,, \tag{9.49}$$

$$\int_{\partial B_1} |\mathscr{H}|^2 = \pi\bar{Q}\sum_{j=1}^{J} Q_j\Big(\frac{|a_{j,0}|^2}{2} + \sum_{l=1}^{\infty}\big(|a_{j,l}|^2 + |b_{j,l}|^2\big)\Big). \tag{9.50}$$

Clearly, (9.43) follows from the first and second inequality, with the constant $C = \bar{Q}Q_1 \leqslant \bar{Q}Q$. (9.46) follows from the fact that, for any chosen $a > 0$, if b_0 is sufficiently small and $0 < b < b_0$, then

$$(2a+b)\ell \leqslant \frac{\ell^2}{\bar{Q}Q_j} + \bar{Q}Q_j\ell a(a+b) \qquad \forall \ell \in \mathbb{N}.$$

The latter claim is elementary and the reader can consult, for instance, Step 2 in the proof of [17, Proposition 5.2].

Observe next that $\boldsymbol{\eta}\circ\mathscr{V}$ is the classical harmonic extension of the single-valued function $\boldsymbol{\eta}\circ\mathscr{U}|_{\partial D_1}$. We then have the classical estimates

$$\|\boldsymbol{\eta}\circ\mathscr{V}\|_{L^\infty(D_{\frac{1}{2^{Q}}})} + \|\boldsymbol{\eta}\circ\mathscr{V}\|_{L^1(D_1)} \leqslant C\|\boldsymbol{\eta}\circ\mathscr{U}\|_{L^1(\partial D_1)}.$$

In particular we conclude easily

$$\|\boldsymbol{\eta}\circ\mathscr{H}\|_{L^\infty(B_{1/2})} + \|\boldsymbol{\eta}\circ\mathscr{H}\|_{L^1(B_1\setminus B_{1/2})} \leqslant C\int_{\partial B_1} |\boldsymbol{\eta}\circ\bar{\mathscr{N}}|,$$

because the change of variables $\boldsymbol{W}^{-1}$ is smooth on $B_1 \setminus B_{1/2}$. The integrability of $|z|^{\gamma_0-1}$ on B_1 gives then

$$\int_{B_1} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\mathscr{H}(z,w)|\,dz \leqslant C\|\boldsymbol{\eta}\circ\mathscr{H}\|_{L^\infty(B_{1/2})} + C\|\boldsymbol{\eta}\circ\mathscr{H}\|_{L^1(B_1\setminus B_{1/2})},$$

which in turn completes the proof of (9.44).

A similar argument proves (9.45). Using the classical theory of single valued harmonic functions we see indeed that $\|\zeta_j\|_{L^2(B_1)} + \|\zeta_j\|_{L^\infty(B_{1/2})} \leqslant C\|\gamma_j\|_{L^2(\partial B_1)}$ and thus, using the fact that $\boldsymbol{W}$ is smooth on $B_1 \setminus B_{1/2}$, we conclude that

$$\|\mathscr{H}\|^2_{L^\infty(B_{1/2})} + \|\mathscr{H}\|^2_{L^2(B_1\setminus B_{1/2})} \leqslant C\int_{\partial B_1} |\bar{\mathscr{N}}|^2.$$

From this we easily conclude (9.45).

9.2.2 *Proof of Proposition 9.3: Step 2*

We keep the notation of the previous paragraphs and assume that $\bar{\mathscr{N}}$ is defined in B_1, after scaling. The specific scaling that we are using is the one which preserves the Lipschitz constant and is given by

$$\bar{\mathscr{N}}(z,w) \mapsto r^{-1}\mathscr{N}\big(rz, r^{\frac{1}{Q}}w\big).$$

Under this scaling we then have the estimates $\|\bar{\mathscr{N}}\|_{L^\infty} \leqslant C\mathbf{m}_0^{\frac{1}{4}} r^{\gamma_0/2}$ and $\mathrm{Lip}(\bar{\mathscr{N}}) \leqslant \boldsymbol{\Lambda}(r)^\eta$ and we want to show that we can modify $\mathscr{H}$ to a competitor $\bar{\mathscr{L}}$ with $\mathrm{Lip}(\bar{\mathscr{L}}) \leqslant C_{9.2}$, satisfying

$$\bar{\mathscr{L}}|_{\partial B_1} = \bar{\mathscr{N}}|_{\partial B_1} \qquad \text{and} \qquad \|\bar{\mathscr{L}}\|_{L^\infty(B_1)} \leqslant \|\bar{\mathscr{N}}\|_{L^\infty(\partial B_1)} \tag{9.51}$$

$$\int_{B_1} |D\bar{\mathscr{L}}|^2 \leqslant \int_{B_1} |D\mathscr{H}|^2 + Cr^\sigma \int_{\partial B_1} |D\bar{\mathscr{N}}|^2 \tag{9.52}$$

$$\int_{B_r} |z|^{\gamma_0 - 1} |\bar{\mathscr{L}}|^2 \leqslant C \int_{\partial B_1} |\bar{\mathscr{N}}|^2 \tag{9.53}$$

$$\int_{B_1} |z|^{\gamma_0 - 1} |\boldsymbol{\eta} \circ \bar{\mathscr{L}}| \leqslant C \int_{\partial B_1} |\boldsymbol{\eta} \circ \bar{\mathscr{N}}| \,. \tag{9.54}$$

Observe that the harmonic functions ζ_j defined in (9.47) are Lipschitz in every ball D_{1-t} for $0 < t < 1$ with an estimate of the form

$$\|D\zeta_j\|_{L^\infty(D_{1-t})} \leqslant \frac{C}{t}\mathrm{Lip}(\gamma_j) \leqslant \frac{C}{t}\mathrm{Lip}(\bar{\mathscr{N}}) \leqslant \frac{C\boldsymbol{\Lambda}(r)^{\eta_0}}{t} \,. \tag{9.55}$$

They are not Lipschitz up to the boundary ∂D_1 because the Dirichlet to Neumann map $\gamma_j \to \frac{\partial \zeta_j}{\rho}(1, \cdot)$ does not map L^∞ into L^∞. However we have the Schauder estimate

$$\|D\zeta_j\|_{L^p(D_1)} \leqslant C_p \|\gamma_j\|_{W^{1,p}(\partial D_1)} \leqslant C_p \boldsymbol{\Lambda}(r)^{\eta_0}$$

for every $p < \infty$. In particular, we can bound

$$\|\zeta_j(1-t, \cdot) - \gamma_j\|_{W^{1,1}(\partial D_1)} \leqslant C_2 t^{\frac{1}{2}} \boldsymbol{\Lambda}(r)^{\eta_0} \,,$$

which in turn implies

$$\max_\theta |\zeta_j(1-t, \theta) - \gamma_j(\theta)| \leqslant C_2 t^{\frac{1}{2}} \boldsymbol{\Lambda}(r)^{\eta_0} \,. \tag{9.56}$$

Choose $t := \boldsymbol{\Lambda}(r)^{\frac{\eta_0}{2}}$ and define a new map ξ_j as

$$\xi_j(\rho, \theta) := \begin{cases} \zeta_j(\rho, \theta) & \text{for } \rho \leqslant 1 - t \\ \\ \frac{1-\rho}{t}\zeta_j(1-t, \theta) + \frac{\rho - (1-t)}{t}\gamma_j(\theta) & \text{for } 1 - t \leqslant \rho \leqslant 1. \end{cases}$$

Now, (9.55) and (9.56) imply that $\|D\zeta_j\| \leqslant C\boldsymbol{\Lambda}(r)^{\frac{\eta_0}{2}}$. Moreover we obviously have

$$\begin{aligned} \int_{D_1} |D\xi_j|^2 &\leqslant \int_{D_1} |D\zeta_j|^2 + C\boldsymbol{\Lambda}(r)^{\frac{\eta_0}{2}} \Big(\int_{\partial D_{1-t}} |D\zeta_j|^2 + \int_{\partial D_1} |D\gamma_j|^2 \Big) \\ &\leqslant \int_{D_1} |D\zeta_j|^2 + C\boldsymbol{\Lambda}(r)^{\frac{\eta_0}{2}} \int_{\partial B_1} |D\gamma_j|^2 \,. \end{aligned} \tag{9.57}$$

We can now define two "intermediate" maps

$$\mathscr{V}^0(\rho, \theta) := \sum_{j=1}^{J} \sum_{i=1}^{Q_j} \left[\!\left[\xi_j \left(\rho^{\frac{1}{Q_j}}, \frac{\theta + 2\pi i}{Q_j} \right) \right]\!\right]$$

and $\mathscr{L}^0 := \mathscr{V}^0 \circ \mathbf{W}^{-1}$. It is then immediate to see that $\mathscr{L}^0$ enjoys the bound $\mathrm{Lip}(\mathscr{L}^0) \leqslant C\boldsymbol{\Lambda}(r)^{\frac{\eta}{2}}$ on the domain $B_1 \setminus B_{1/4}$ and that all the estimates (9.51), (9.52) and (9.54). On the other hand the differential $D\mathscr{L}^0$ is singular in the origin and in fact it is rather easy to see that we have the bound

$$|D\mathscr{L}^0(z,w)|^2 \leqslant C|z|^{2-\frac{2}{(Q\bar{Q})}} \int_{B_1} |D\mathscr{L}^0|^2 . \tag{9.58}$$

In order to produce $\bar{\mathscr{L}}$ we need to smooth the singularity of $\mathscr{L}^0$ at the origin. There are several ways to do this and we present here one possibility. First of all we fix $2 < p < 2Q\bar{Q}/(2Q\bar{Q}-2)$ and observe that (9.58) yields the estimate

$$\int_{B_{3/4}} |D\mathscr{L}^0(z,w)|^p \leqslant C\Big(\int_{B_1} |D\mathscr{L}^0|^2\Big)^{\frac{p}{2}} . \tag{9.59}$$

Next we define

$$M|D\mathscr{L}^0(z,w)| := \sup_{\rho<1/4} \frac{1}{\rho^2} \int_{B_\rho(z,w)} |D\mathscr{L}^0(z,w)|$$

and let

$$A := \{(z,w) : M|D\mathscr{L}^0(z,w)| \geqslant c_0\}$$

where c_0 is a constant to be chosen later. Observe that, given the Lipschitz bound for $\mathscr{L}^0$ outside the origin, for r sufficiently small the set A is contained in $B_{1/2}$. Arguing as in the proof of [17, Proposition 4.4] we have the Lipschitz estimate $\mathrm{Lip}(\mathscr{L}^0) \leqslant Cc_0$ on $B_1 \setminus A$, where C is a dimensional constant. We can then use the Lipschitz extension of Proposition 3.4 to extend $\mathscr{L}^0$ to $\bar{\mathscr{L}}$ on A so that $\mathrm{Lip}(\mathscr{L}) \leqslant Cc_0$. Choosing c_0 accordingly we achieve the desired Lipschitz bound on B_1. As for (9.51) and (9.53) observe that the extension satisfies

$$\|\bar{\mathscr{L}}\|^2_{L^\infty(B_{1/2})} \leqslant C\|\mathscr{H}\|^2_{L^\infty(B_{3/4})}$$

and coincides with $\mathscr{L}_0$ on $B_1 \setminus B_{1/2}$. As for (9.54), it would suffice to show that $|\boldsymbol{\eta} \circ \bar{\mathscr{L}}| \leqslant C|\boldsymbol{\eta} \circ \bar{\mathscr{N}}|$. This can be easily achieved in the following way: we make a Lipschitz extension of $\mathscr{L}^0$, subtract from each sheet the average and then sum back to each sheet a Lipschitz extension of $\boldsymbol{\eta} \circ \mathscr{L}^0$.

As for (9.52) we compute

$$\begin{aligned}\int |D\bar{\mathscr{L}}|^2 &\leqslant \int |D\mathscr{L}^0|^2 + Cc_0^2|A| \leqslant \int |D\mathscr{L}^0|^2 + Cc_0^{2-p} \int_{B_{3/4}} |D\mathscr{L}^0|^p \\ &\leqslant \int |D\mathscr{L}^0|^2\Big(1 + Cc_0^{2-p}\Big(\int |D\mathscr{L}^0|^2\Big)^{\frac{p}{2}-1}\Big) .\end{aligned} \tag{9.60}$$

Observe that $p/2-1 > 0$ and that by (9.57) and (9.43)

$$\int |D\mathscr{L}^0|^2 \leqslant \int |D\mathscr{H}|^2 + C\boldsymbol{\Lambda}(r)^{\frac{\sigma}{2}} \int_{\partial B_1} |D\bar{\mathscr{N}}|^2 \leqslant C \int_{\partial B_1} |D\bar{\mathscr{N}}|^2 \leqslant Cr^\sigma ,$$

so that

$$\int_{B_1} |D\bar{\mathscr{L}}|^2 \leqslant (1 + C\,r^\sigma) \int_{B_1} |D\mathscr{H}|^2 + Cr^\sigma \int_{\partial B_1} |D\bar{\mathscr{N}}|^2 \overset{(9.43)}{\leqslant} \int_{B_1} |D\mathscr{H}|^2 + Cr^\sigma \int_{\partial B_1} |D\bar{\mathscr{N}}|^2 .$$

10 BLOW UP ANALYSIS

This chapter is dedicated to the proof of Theorem 2.20, which we recall for the reader convenience.

Theorem 10.1 (Blowup Analysis)**.** *Under the assumptions of Theorem 2.18, the following dichotomy holds:*

(i) either there exists $s > 0$ *such that* $\mathcal{N}|_{B_s} \equiv Q\,[\![0]\!]$*,*

(ii) or there exist constants $I_0 > 1$*,* $a_0, \bar{r}, C > 0$ *and an* I_0*-homogeneous nontrivial Dir-minimizing function* $g : \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *such that* $\boldsymbol{\eta} \circ g \equiv 0$*,* $g = \sum_i [\![(0, \bar{g}_i, 0)]\!]$*, where* $\bar{g}_i(x) \in \mathbb{R}^{\bar{n}}$*, and*

$$\mathcal{G}\big(\mathcal{N}(z,w), g(z,w)\big) \leqslant C|z|^{I_0+a_0} \quad \forall\, (z,w) \in \mathfrak{B}_Q,\ |z| < \bar{r}, \tag{10.1}$$

and moreover the following estimates hold

$$\int_{B_{r+2\rho}\setminus B_{r-2\rho}} |D\mathcal{N}|^2 \leqslant C\, r^{2I_0+a_0} + C\, r^{2I_0-1}\,\rho \quad \forall\, 4\,\rho \leqslant r < 1, \tag{10.2}$$

$$\mathbf{H}(r) \leqslant C\, r\, \mathbf{D}(r) \quad \forall\, r < 1. \tag{10.3}$$

10.1 OUTER VARIATIONS AND THE POINCARÉ INEQUALITY

In this section we begin to exploit the variations of the area functional on T in conjunction with the estimates of the previous section. The main conclusion will be the following Poincaré inequality:

Theorem 10.2 (Poincaré inequality)**.** *There exists a constant* $C_{10.2} > 0$ *such that if* r *is sufficiently small, then*

$$\mathbf{H}(r) \leqslant C_{10.2}\, r\, \mathbf{D}(r)\,. \tag{10.4}$$

We record however the two main tools used to prove Theorem 10.2, since they will be useful in the future. The first one is an elementary computation. In order to state it we introduce the quantity

$$\mathbf{E}(r) := \int_{\partial B_r} \sum_{j=1}^{Q} \langle \mathcal{N}_j, D_\nu \mathcal{N}_j \rangle\,. \tag{10.5}$$

Lemma 10.3. $\mathbf{H}$ *is a Lipschitz function and the following identity holds for a.e.* $r \in (0,1)$

$$\mathbf{H}'(r) = \frac{\mathbf{H}(r)}{r} + 2\,\mathbf{E}(r)\,. \tag{10.6}$$

The second identity is a consequence of the first variation of T under certain specific vector fields, which we call "outer variations": such variations "stretch" the normal bundle of $\mathcal{M}$ suitably and they are defined using the map $\mathscr{N}$. In the case of semicalibrated currents it is convenient to modify the Dirichlet energy suitably to gain a new quantity which enjoys better estimates. Thus, from now on $\mathbf{\Omega}$ will denote $\mathbf{D}$ in the cases (a) and (c) of Definition 1.1, whereas in the case (b) it will be given by

$$\mathbf{\Omega}(r) := \mathbf{D}(r) + \mathbf{L}(r) := \mathbf{D}(r) + \sum_{i=1}^{Q} \int_{\mathbf{\Psi}(B_r)} \langle \xi_1 \wedge D_{\xi_2} N_i \wedge N_i + D_{\xi_1} N_i \wedge \xi_2 \wedge N_i, d\omega \rangle\,.$$

Proposition 10.4 (Outer variations). *There exist constants $C_{10.4} > 0$ and $\kappa > 0$ such that, if $r > 0$ is small enough, then the inequality*

$$|\mathbf{\Omega}(r) - \mathbf{E}(r)| \leqslant C_{10.4}\, \mathcal{E}_{OV}(r) \tag{10.7}$$

holds with

$$\mathcal{E}_{OV}(r) = \mathbf{\Lambda}(r)^{\kappa}\Big(\mathbf{D}(r) + \frac{\mathbf{H}(r)}{r} + r\mathbf{D}'(r)\Big) + \mathbf{F}(r) + r^{1+\gamma_0}\frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))\,. \tag{10.8}$$

Moreover

$$|\mathbf{L}(r)| \leqslant C\, \mathbf{m}_0^{\frac{1}{2}}\, r^{2-\gamma_0}\mathbf{D}(r) + C\, \mathbf{m}_0^{\frac{1}{2}}\, \mathbf{F}(r). \tag{10.9}$$

10.1.1 *Proof of Lemma 10.3*

The Lipschitz regularity of $\mathbf{H}$ follows from the Lipschitz regularity of $\mathscr{N}$. Consider next the map $i_r : \mathfrak{B} \to \mathfrak{B}$ given by $i_r(z, w) = \left(rz, r^{\frac{1}{Q}} w\right)$. By a simple change of variables we compute

$$\mathbf{H}(r) = \int_{\partial B_1} |\mathscr{N}|^2(i_r(z', w'))\, r\,.$$

The formula (10.6) is then an elementary computation using the chain rule for multifunctions, cf. Proposition 3.6.

10.1.2 *Proof of Proposition 10.4*

The inequality (10.9) is a simple consequence of

$$|\mathbf{L}(r)| \leqslant C\mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |D\mathscr{N}||\mathscr{N}| \leqslant C\mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |z|^{2-\gamma_0}|D\mathscr{N}|^2 + C\mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |z|^{\gamma_0 - 2}|\mathscr{N}|^2\,.$$

In order to show (10.7) we fix a test function $\phi \in C_c^\infty(\mathbb{R})$, nonnegative, symmetric, with support in $]-1, 1[$ and monotone decreasing on $[0, 1]$. We then follow [21, Section 3.3] and, having fixed r, we define the vector field X_o on $\mathbf{V}_{u,a}$ via

$$X_o(p) := \varphi(\mathbf{p}(p))(p - \mathbf{p}(p)) \qquad \text{where} \qquad \varphi(\mathbf{\Psi}(z, w)) = \phi\left(\tfrac{|z|}{r}\right)\,.$$

For r small enough, by (8.2) we can use Theorem 3.53 and deduce via the change of coordinates given by $\mathbf{\Psi}$, that

$$\delta \mathbf{T}_F(X) = \int_{\mathfrak{B}} \phi\big(\tfrac{|z|}{r}\big)\, |D\mathscr{N}|^2 + r^{-1} \textstyle\int_{\mathfrak{B}} \phi'\big(\tfrac{|z|}{r}\big)\, \sum_{j=1}^{Q} \langle \mathscr{N}_j, D_\nu \mathscr{N}_j\rangle + \sum_{i=1}^{3} \mathrm{Err}_i^o, \tag{10.10}$$

with

$$\mathrm{Err}_1^o = \Big| \int_{\mathcal{M}} \varphi\, \langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle \Big| \leqslant C\, \mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |z|^{\gamma_0 - 1}\, |\boldsymbol{\eta}\circ\mathscr{N}| \overset{(8.3)}{\leqslant} C\, \boldsymbol{\Lambda}^{\eta}(r)\, \mathbf{D}(r) + C\, \mathbf{F}(r)\,, \tag{10.11}$$

$$\mathrm{Err}_2^o \leqslant C \int_{\mathcal{M}} |\varphi|\, |A_{\mathcal{M}}|^2\, |N|^2 \leqslant C\, \mathbf{F}(r)\,, \tag{10.12}$$

$$\begin{aligned}
\mathrm{Err}_3^o \leqslant & C \int_{\mathcal{M}} \Big(|\varphi| \big(|DN|^2\, |N|\, |A_{\mathcal{M}}| + |DN|^4 \big) + |D\varphi|\, \big(|DN|^3\, |N| + |DN|\, |N|^2\, |A_{\mathcal{M}}| \big) \Big) \\
\leqslant & C \int_{B_r} \Big[\Big(\frac{|\mathscr{N}|^2}{|z|^{2-2\gamma_0}} + |D\mathscr{N}|^4 \Big) - r^{-1} \phi'\big(\tfrac{|z|}{r}\big)\, r^{1+\gamma_0}\, |D\mathscr{N}|^3 - r^{-1} \phi'\big(\tfrac{|z|}{r}\big)\, |D\mathscr{N}|\, \tfrac{|\mathscr{N}|^2}{|z|^{1-\gamma_0}} \Big] \\
\overset{(8.2)\&(8.1)}{\leqslant} & C\, \boldsymbol{\Lambda}^{\eta}(r)\, \mathbf{D}(r) + C\mathbf{F}(r) - C\, \boldsymbol{\Lambda}(r)^{\eta} \int_{B_r} r^{-1} \phi'\big(\tfrac{|z|}{r}\big)\, \tfrac{|\mathscr{N}|^2}{|z|^{1-\gamma_0}} \\
& - C\, r^{1+\gamma_0}\, \boldsymbol{\Lambda}^{\eta} \int_{B_r} r^{-1} \phi'\big(\tfrac{|z|}{r}\big)\, |D\mathscr{N}|^2\,.
\end{aligned} \tag{10.13}$$

(We recall that $\phi' \leqslant 0$ on $[0,1]$)).

We next distinguish two situations:

- in the cases (a) and (c) of Definition 1.1, we denote by $X^\perp$ and X^T the projections of X on the normal and the tangential bundle of Σ, respectively. Then $\delta T(X^T) = 0$ and therefore
$$|\delta \mathbf{T}_F(X)| \leqslant \underbrace{|\delta \mathbf{T}_F(X) - \delta T(X)|}_{\mathrm{Err}_o^4} + \underbrace{|\delta T(X^\perp)|}_{\mathrm{Err}_o^5};$$
- in case (b), since $\delta T(X) = T(dw \lrcorner X)$, we estimate
$$\big| \delta \mathbf{T}_F(X) - \mathbf{T}_F(d\omega \lrcorner X) \big| \leqslant \underbrace{|\delta \mathbf{T}_F(X) - \delta T(X)| + |T(d\omega \lrcorner X) - \mathbf{T}_F(d\omega \lrcorner X)|}_{\mathrm{Err}_o^4}\,.$$

In both cases we have

$$\begin{aligned}
\mathrm{Err}_4^o \leqslant & Q \int_{\mathrm{spt}(T)\setminus \mathrm{Im}(F)} \big| \mathrm{div}_{\vec{T}} X \big|\, d\|T\| + Q \int_{\mathrm{Im}(F)\setminus \mathrm{spt}(T)} \big| \mathrm{div}_{\vec{\mathbf{T}}_F} X \big|\, d\|\mathbf{T}_F\| \\
& + Q \|d\omega\|_\infty \int |X| d\|T - \mathbf{T}_F\|\,,
\end{aligned}$$

where we use the convention that $\omega = 0$ in the cases (a) and (c). We then can estimate

$$\begin{aligned}
\mathrm{Err}_4^o &\leqslant C \int (\varphi'(\mathbf{p}(p))\,|p-\mathbf{p}(p)| + \varphi(\mathbf{p}(p)))\; d\|T-\mathbf{T}_F\| \\
&\overset{(8.1)\&(8.4)}{\leqslant} C\,\boldsymbol{\Lambda}^\eta(r)\,\mathbf{D}(r) + C\,\mathbf{F}(r) + C\,r^{1+\gamma_0} \underbrace{\int |\nabla\varphi(\mathbf{p}(p))|\,|p-\mathbf{p}(p)|\,d\|T-\mathbf{T}_F\|}_{S(\varphi)}\,.
\end{aligned} \tag{10.14}$$

In case (b) we have that

$$\mathbf{T}_F(d\omega \lrcorner X) = \sum_{i=1}^{Q} \int_{\mathcal{M}} \varphi\,\langle(\xi_1 + D_{\xi_1}N_i)\wedge(\xi_2 + D_{\xi_2}N_i\cdot\xi_2)\wedge N_i\,,\,d\omega(p+N_i(p)),$$

and therefore

$$\big|\mathbf{D}(r) + \mathbf{L}(r) - \mathbf{E}(r)\big| \leqslant \sum_{j=1}^{4} \mathrm{Err}_j^o\,.$$

Letting ϕ converge to the characteristic function of the interval $[-1,1]$, we reach the conclusion. The only term which needs some care is the term $S(\varphi)$ in (10.14). Note that we can approximate the characteristic function of $[-1,1]$ with an increasing sequence of functions ϕ_j with the property that $|\phi_j'| \leqslant Cj$, $0 \leqslant \phi_j \leqslant 1$ and $\phi_j \equiv 1$ on $[-1+1/j, 1-1/]$. Then we would have

$$\limsup_j S(\varphi_j) \leqslant C \limsup_j \frac{j}{r}\|T-\mathbf{T}_F\|(\boldsymbol{\Psi}(B_r\setminus B_{r(1-1/j)})) \leqslant C\frac{d}{dr}\|T-\mathbf{T}_F\|(\boldsymbol{\Psi}(B_r))\,,$$

by the monotonicity of the function $r \mapsto \|T-\mathbf{T}_F\|(\boldsymbol{\Psi}(B_r))$.

In the cases (a) and (c) we follow the same argument, but we need to bound the additional term Err_o^5. In order to deal with the latter term we argue as in [21, Section 4.1]. In particular we bound

$$\begin{aligned}
\mathrm{Err}_5^o &\leqslant \left|\int \mathrm{div}_{\vec{T}} X^\perp\; d\|T\|\right| \\
&\leqslant \underbrace{\int_{\mathrm{spt}(T)\setminus\mathrm{Im}(F)} \left|\mathrm{div}_{\vec{T}} X\right|\; d\|T\| + \int_{\mathrm{Im}(F)\setminus\mathrm{spt}(T)} \left|\mathrm{div}_{\vec{\mathbf{T}}_F} X\right|\; d\|\mathbf{T}_F\|}_{I_1} \\
&\quad + \underbrace{\left|\int \langle X^\perp, h(\vec{\mathbf{T}}_F(p))\rangle\, d\|\mathbf{T}_F\|\right|}_{I_2}\,,
\end{aligned} \tag{10.15}$$

where $h(v_1\wedge v_2) := \sum_{i=1}^2 A_\Sigma(v_i,v_i)$. Since the projection on the normal to Σ is a C^{1,ε_0} map, $X^\perp$ enjoys the same C^1 bounds as X and I_1 can be controlled as Err_o^4. The term I_2 can be estimated using

$$|X^{o\perp}(p)| = \varphi\,|\mathbf{p}_{T_p\Sigma^\perp}(p-\mathbf{p}(p))| \leqslant C\mathbf{c}(\Sigma)\,\varphi\,|p-\mathbf{p}(p)|^2 \leqslant C\mathbf{m}_0^{\frac{1}{2}}\,\varphi\,|p-\mathbf{p}(p)|^2 \quad \forall\, p\in\Sigma.$$

In particular we achieve $I_2 \leqslant C\mathbf{H}(r)$, which concludes the proof.

10.1.3 *Proof of Theorem 10.2*

In order to prove the theorem we start estimating the error term $\mathbf{F}$.

Lemma 10.5. *There exist a constant* $C_{10.5} > 0$ *(depending on* γ_0*) such that*

$$\mathbf{F}(r) \leqslant C_{10.5}\, r^{\gamma_0 - 1}\, \mathbf{H}(r) + C_{10.5}\, r^{\gamma_0}\, \mathbf{D}(r) \quad \forall\, r \in (0,1). \tag{10.16}$$

Proof. Using (10.6) and an integration by parts we infer that

$$\gamma_0 \int_0^r \frac{\mathbf{H}(\rho)}{\rho^{2-\gamma_0}}\, d\rho = \frac{\mathbf{H}(\rho)}{\rho^{1-\gamma_0}}\Big|_0^r - \int_0^r \frac{d}{d\rho}\left(\frac{\mathbf{H}(\rho)}{\rho}\right) \rho^{\gamma_0}\, d\rho = \frac{\mathbf{H}(r)}{r^{1-\gamma_0}} - \int_0^r \frac{2\,\mathbf{E}(\rho)}{\rho^{1-\gamma_0}}\, d\rho. \tag{10.17}$$

The Cauchy–Schwarz inequality yields then the following bound for every ε:

$$|\mathbf{E}(r)| \leqslant \frac{\varepsilon}{r}\int_{\partial B_r} |\mathscr{N}|^2 + \frac{r}{4\varepsilon}\int_{\partial B_r} |D\mathscr{N}|^2 = \varepsilon\,\frac{\mathbf{H}(r)}{r} + \frac{r\,\mathbf{D}'(r)}{4\varepsilon}. \tag{10.18}$$

Therefore, by choosing $\varepsilon = \frac{\gamma_0}{2}$, we deduce (10.16) from (10.17) and (10.18). □

Proof of Theorem 10.2. In view of Lemma 10.5, for r sufficiently small, the almost minimizing condition (9.39) reads as

$$\mathbf{D}(r) \leqslant C\, r\, \mathbf{D}'(r) + C\,\frac{\mathbf{H}(r)}{r^{1-\gamma_0}} + C\,\mathbf{m}_0^{\frac{1}{2}}\, r^{\gamma_0}\int_{\partial B_r} |\boldsymbol{\eta}\circ\mathscr{N}|\,.$$

Dividing by the radius and integrating we get

$$\begin{aligned}
\int_0^r \frac{\mathbf{D}(s)}{s}\, ds &\leqslant C\int_0^r \left(\mathbf{D}'(\rho) + \frac{\mathbf{H}(\rho)}{\rho^{2-\gamma_0}} + \rho^{\gamma_0 - 1}\int_{\partial B_\rho} |\boldsymbol{\eta}\circ\mathscr{N}|\right) d\rho \\
&\leqslant C\,\mathbf{D}(r) + C\,\mathbf{F}(r) + C\,\mathbf{m}_0^{\frac{1}{2}}\int_{B_r} \frac{|\boldsymbol{\eta}\circ\mathscr{N}|}{|z|^{1-\gamma_0}} \\
&\overset{(8.3)}{\leqslant} C\,\mathbf{D}(r) + C\left(\boldsymbol{\Lambda}^{\eta}(r)\,\mathbf{D}(r) + \mathbf{F}(r)\right) \overset{(10.16)}{\leqslant} C\,\mathbf{D}(r) + C\, r^{\gamma_0 - 1}\,\mathbf{H}(r)
\end{aligned} \tag{10.19}$$

Therefore, using Lemma 10.3 we deduce that

$$\begin{aligned}
\frac{\mathbf{H}(r)}{r} &= \int_0^r \frac{2\,\mathbf{E}(\rho)}{\rho}\, dt \overset{(10.7)}{\leqslant} C\int_0^r \frac{\mathbf{D}(\rho)}{\rho}\, d\rho \\
&\quad + C\int_0^r \left(\frac{\mathbf{H}(\rho)}{\rho^{2-2\gamma_0}} + \rho^{\gamma_0}\,\mathbf{D}'(\rho) + \rho^{\gamma_0}\frac{d}{d\rho}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_\rho)))\right) d\rho \\
&\overset{(10.19)}{\leqslant} C\,\mathbf{D}(r) + C\,\frac{\mathbf{H}(r)}{r^{1-\gamma_0}} + C\, r^{\gamma_0}\mathbf{D}(r) + C\,\mathbf{F}(r) + C\, r^{\gamma_0}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r))) \\
&\overset{(8.4)\&(10.16)}{\leqslant} C\,\mathbf{D}(r) + C\,\frac{\mathbf{H}(r)}{r^{1-\gamma_0}},
\end{aligned}$$

For r sufficiently small this concludes the proof. □

10.2 INNER VARIATIONS AND KEY ESTIMATES

Using the Poincaré inequality in Theorem 10.2, we can give a very simple estimates of the error terms in the "inner variations" of the current T. The latter corresponds to deformations of T along appropriate vector fields which are tangent to $\mathcal{M}$. In order to state out main conclusion we need to introduce yet another quantity

$$\mathbf{G}(r) := \int_{\partial B_r} |D_\nu \mathscr{N}|^2 \,. \tag{10.20}$$

Proposition 10.6 (Inner Variations). *There exist constants $C_{10.6} > 0$ and $\eta > 0$ such that, if $r > 0$ is small enough, than the following holds*

$$\left|\mathbf{D}'(r) - 2\,\mathbf{G}(r)\right| \leqslant C\,\mathcal{E}_{IV}(r)\,, \tag{10.21}$$

where

$$\begin{aligned}\mathcal{E}_{IV}(r) &= r^{2\eta-1}\mathbf{D}(r) + \mathbf{D}(r)^\eta\,\mathbf{D}'(r) + \frac{\mathbf{m}_0^{\frac{1}{2}}}{r^{1-\gamma_0}}\int_{\partial B_r} |\boldsymbol{\eta}\circ\mathscr{N}(z,w)| \\ &\quad + \frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))\,.\end{aligned} \tag{10.22}$$

For further use we summarize in the next lemma a set of inequalities which will be used in the next sections and which are direct consequences of all the conclusions derived so far

Lemma 10.7. *There exist constant $C_{10.7} > 0$ and $\eta > 0$ such that for every r sufficiently small the following holds:*

$$\mathbf{F}(r) + r\mathbf{F}'(r) \leqslant C_{10.7}\, r^{\gamma_0}\mathbf{D}(r) \tag{10.23}$$

$$|\mathbf{L}(r)| \leqslant C_{10.7}\, r\,\mathbf{D}(r) \tag{10.24}$$

$$|\mathbf{L}'(r)| \leqslant C_{10.7}\,\left(\mathbf{H}(r)\,\mathbf{D}'(r)\right)^{\frac{1}{2}} \tag{10.25}$$

$$\mathcal{E}_{OV} \leqslant C_{10.7}\,\mathbf{D}^{1+\eta}(r) + C_{10.7}\,\mathbf{F}(r) + C_{10.7} r\mathbf{D}^\eta(r)\mathbf{D}'(r) + C\,r\,\mathcal{E}_{BP}(r), \tag{10.26}$$

$$\mathcal{E}_{IV}(r) \leqslant C_{10.7}\, r^{2\eta-1}\mathbf{D}(r) + C_{10.7}\,\mathbf{D}(r)^\eta\,\mathbf{D}'(r) + C\,\mathcal{E}_{BP}(r), \tag{10.27}$$

where

$$\mathcal{E}_{BP}(r) := \frac{\mathbf{m}_0^{\frac{1}{2}}}{r^{1-\gamma_0}}\int_{\partial B_r} |\boldsymbol{\eta}\circ\mathscr{N}| + \frac{d}{dr}\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r)))$$

Moreover, for every $a > 0$ there exist constants $b_0(a), C(a) > 0$ such that

$$\mathbf{D}(r) \leqslant \frac{r\,\mathbf{D}'(r)}{2(2\,a+b)} + \frac{a(a+b)\,\mathbf{H}(r)}{r(2\,a+b)} + C(a)\,r\,\mathcal{E}_{IV}(r) \quad \forall\, b < b_0(a). \tag{10.28}$$

An important corollary of the previous lemma is the following

Corollary 10.8. *There exists a constant $C_{10.8} > 0$ such that, if η is the constant of Lemma 10.7, then for every $\gamma < \eta$ and r sufficiently small, the function*

$$\mathbf{s}(r) := \mathbf{\Sigma}_{OV}(r) + \mathbf{\Sigma}_{IV}(r) := \frac{\mathcal{E}_{IV}(r)}{r^\gamma\,\mathbf{D}(r)} + \frac{\mathcal{E}_{OV}(r)}{r^{1+\gamma}\mathbf{D}(r)}$$

is integrable and, setting $\mathbf{\Sigma}(r) := \int_0^r \mathbf{s}(t)\,dt$,

$$\mathbf{\Sigma}(r) \leqslant C_{10.8}\, r^{\eta-\gamma}\,. \tag{10.29}$$

10.2.1 *Proof of Proposition 10.6*

We evaluate the first variation of T along a suitably defined vector field X. To this aim we fix a function $\phi \in C_c^\infty(]-1,1[)$, symmetric, nonnegative and identically one on $]-1+1/j, 1-1/j[$ and with the property that $|\phi'| \leqslant Cj$. Then we introduce the vector field $Y\colon \mathcal{M} \to \mathbb{R}^{n+2}$ defined, for every $(z,w) \in \mathfrak{B}$, by

$$Y(\mathbf{\Psi}(z,w)) := \tfrac{|z|}{r}\,\phi\big(\tfrac{|z|}{r}\big)\, D_\nu \mathbf{\Psi}(z,w) \in T_{\mathbf{\Psi}(z,w)}\mathcal{M}\,,$$

Next we define the vector field $X_i\colon \mathbf{V}_{a,u} \to \mathbb{R}^{n+2}$ by $X_i(p) := Y(\mathbf{p}(p))$. Note that X_i is the infinitesimal generator of a one parameter family of diffeomorphisms Φ_ε defined as $\Phi_\varepsilon(p) := \Gamma_\varepsilon(\mathbf{p}(p)) + p - \mathbf{p}(p)$, where Γ_ε is the one-parameter family of biLipschitz homeomorphisms of $\mathcal{M}$ generated by Y. In fact, since Γ_ε fixes the origin, we can consider it as a C^{2,γ_0} map of $\mathcal{M}\setminus\{0\}$ onto itself. Note moreover that X_i is Lipschitz on the entire $\mathfrak{B}$.

Observe that, by Lemma 10.5 and the Poincaré inequality, $\mathbf{F}(r) \leqslant C\, r^{\gamma_0}\, \mathbf{D}(r)$, so that $\mathbf{\Lambda}(r) \leqslant C\, \mathbf{D}(r)$. Moreover,

$$|D_{\mathcal{M}} Y|(\mathbf{\Psi}(z,w)) + |\mathrm{div}_{\mathcal{M}}\, Y|(\mathbf{\Psi}(z,w)) \leqslant -Cr^{-2}|z|\,\phi'\big(\tfrac{|z|}{r}\big) + Cr^{-1}\,\phi\big(\tfrac{|z|}{r}\big)\,, \tag{10.30}$$

where we recall that $\phi' \leqslant 0$ on $[0,1]$.

If r is small enough, by (8.2) we can apply Theorem 3.54 and, proceeding as in the proof of Proposition 10.4, deduce that

$$\frac{1}{2}\left|\int_{\mathcal{M}} \left(|DN|^2\, \mathrm{div}_{\mathcal{M}} Y - 2\sum_{i=1}^{Q} \langle DN_i \colon\, (DN_i \cdot D_{\mathcal{M}} Y)\rangle\right)\right| \leqslant \sum_{k=1}^{5} \mathrm{Err}_k^i\,,$$

where the error terms can be bounded in the following manner.

First of all,

$$\begin{aligned}
\mathrm{Err}_1^i &= Q\left|\int_{\mathcal{M}} \big(\langle H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\, \mathrm{div}_{\mathcal{M}} Y + \langle D_Y H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle\big)\right| \\
&\leqslant Cr^{-1}\,\mathbf{m}_0^{\frac12}\int_{\mathfrak{B}} \Big(\phi\big(\tfrac{|z|}{r}\big)\, |z|^{\gamma_0 - 1}\, |\boldsymbol{\eta}\circ \mathscr{N}(z,w)| - \phi'\big(\tfrac{|z|}{r}\big)\, |z|^{\gamma_0-1}\, |\boldsymbol{\eta}\circ\mathscr{N}(z,w)|\Big) \\
&\overset{(8.3)}{\leqslant} Cr^{-1}\,\mathbf{D}^{1+\eta}(r) - C\,\mathbf{m}_0^{\frac12}\, r^{\gamma_0-1}\int_{B_r} r^{-1}\phi'\big(\tfrac{|z|}{r}\big)\, |\boldsymbol{\eta}\circ\mathscr{N}(z,w)|\,,
\end{aligned}$$

where in the first inequality we used (10.30) and the fact that

$$\langle D_Y H_{\mathcal{M}}, \boldsymbol{\eta}\circ N\rangle \leqslant |Y|\,|DH_{\mathcal{M}}|\,|\boldsymbol{\eta}\circ N| \leqslant C\,\tfrac{|z|}{r}\phi\big(\tfrac{|z|}{r}\big)\,|z|^{\gamma_0-2}\,|\boldsymbol{\eta}\circ\mathscr{N}|\,.$$

As for Err_2^i and Err_i^3 we have similarly

$$\begin{aligned}
\mathrm{Err}_2^i &= C\int_{\mathcal{M}} |A_{\mathcal{M}}|^2\big(|DY|\,|N|^2 + |Y|\,|N|\,|DN|\big) \\
&\leqslant C\,\mathbf{m}_0\int_{\mathfrak{B}}\left[r^{-1}\left(-\tfrac{|z|}{r}\phi'\big(\big(\tfrac{|z|}{r}\big)\big) + \phi\big(\tfrac{|z|}{r}\big)\right)\frac{|\mathscr{N}|^2}{|z|^{2-2\gamma_0}} + \tfrac{|z|}{r}\,\phi\big(\tfrac{|z|}{r^2}\big)\frac{|\mathscr{N}|\,|D\mathscr{N}|}{|z|^{2-2\gamma_0}}\right] \\
&\leqslant C\,\mathbf{m}_0\, r^{\gamma_0-1}\,\mathbf{D}(r) - Cr^{-1}\int_{B_r} r^{-1}\phi'\big(\tfrac{|z|}{r}\big)\,\frac{|\mathscr{N}|^2}{|z|^{1-\gamma_0}}\,,
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{Err}_3^i &\leqslant C \int_{\mathcal{M}} \Big(|Y|\,|A_{\mathcal{M}}|\,|DN|^2 \big(|N| + |DN|\big) + |DY|\big(|A_{\mathcal{M}}|\,|DN|\,|N|^2 + |DN|^4\big) \Big) \\
&\leqslant C r^{\gamma_0 - 1} \mathbf{D}(r) - C\,\mathbf{D}(r)^{\eta} \int_{\mathfrak{B}} r^{-1} \phi'\big(\tfrac{|z|}{r}\big)\, |D\mathscr{N}|^2 + C\, r^{-1}\, \mathbf{D}(r)^{\eta} \int_{\mathfrak{B}} r^{-1} \phi\big(\tfrac{|z|}{r}\big)\, \frac{|\mathscr{N}|^2}{|z|^{2-\gamma_0}}\,.
\end{aligned}
$$

The errors Err_4^i and Err_5^i are the same as Err_4^o and Err_5^o respectively, in Section 10.1.2, evaluated along a different vector field. Proceeding in the same way as in the estimate of Err_4^o, we deduce

$$
\begin{aligned}
\mathrm{Err}_4^i &= \int_{\mathrm{spt}(T)\setminus \mathrm{Im}(F)} \left|\mathrm{div}_{\vec{T}} X_i\right|\, d\|T\| + \int_{\mathrm{Im}(F)\setminus \mathrm{spt}(T)} \left|\mathrm{div}_{\vec{\mathbf{T}}_F} X_i\right|\, d\|\mathbf{T}_F\| \\
&\leqslant C\, r^{\gamma_0 - 1}\, \mathbf{D}(r) + C \underbrace{\int \alpha\, d\|T - \mathbf{T}_F\|}_{S(\phi)}\,.
\end{aligned}
$$

where $\alpha(p) = \varphi(\mathbf{p}(p))$ and $\varphi(\boldsymbol{\Psi}(z,w)) = r^{-2}|z|\phi\big(r^{-1}|z|\big) - r^{-1}\phi'(r^{-1}|z|)$. In particular using (8.4) and the fact that $-\phi' \leqslant Cj$ on $[0,1]$, we infer

$$
S(\phi) \leqslant C r^{\gamma_0 - 1} \mathbf{D}(r) + C \frac{j}{r} \|T - \mathbf{T}_F\| (\mathbf{p}^{-1}(\boldsymbol{\Psi}(B_r \setminus B_{r(1-1/j)})))\,.
$$

As for Err_i^5, we observe that it only appears in the cases (a) and (c) and arguing as in Section 10.1.2 we can bound it as

$$
\mathrm{Err}_i^5 \leqslant I_1 + \underbrace{\left| \int \langle X_i^{\perp}, h(\vec{\mathbf{T}}_F(p)) \rangle\, d\|\mathbf{T}_F\| \right|}_{I_2},
$$

where $h(\nu_1 \wedge \nu_2) := \sum_{i=1}^2 A_\Sigma(\nu_i, \nu_i)$ and I_1 enjoys the same bounds as Err_i^4. Denote by $\nu_1, \ldots, \nu_l$ an orthonormal frame for $T_p\Sigma^{\perp}$ of class C^{2,ε_0} (cf. [18, Appendix A]) and set $h_p^j(\vec{\lambda}) := -\sum_{k=1}^m \langle D_{\nu_k}\nu_j(p), \nu_k\rangle$ whenever $\nu_1 \wedge \ldots \wedge \nu_m = \vec{\lambda}$ is an m-vector of $T_p\Sigma$ (with $\nu_1, \ldots, \nu_m$ orthonormal). For the sake of simplicity, we write

$$
h_p^j := h_p^j(\vec{\mathbf{T}}_F(p)) \quad \text{and} \quad h_p = \sum_{j=1}^{l} h_p^j \nu_j(p),
$$

$$
h_{\mathbf{p}(p)}^j := h_{\mathbf{p}(p)}^j(\vec{\mathcal{M}}(\mathbf{p}(p))) \quad \text{and} \quad h_{\mathbf{p}(p)} = \sum_{j=1}^{l} h_{\mathbf{p}(p)}^j \nu_j(\mathbf{p}(p)).
$$

Consider the exponential map $\mathbf{ex}_{\mathbf{p}(p)} : T_{\mathbf{p}(p)}\Sigma \to \Sigma$ and its inverse $\mathbf{ex}_{\mathbf{p}(p)}^{-1}$. Recall that:

- the geodesic distance $d_\Sigma(p,q)$ is comparable to $|p-q|$ up to a constant factor;
- ν_j is C^{2,ε_0} and $\|D\nu_j\|_{C^{1,\varepsilon_0}} \leqslant C\mathbf{m}_0^{\frac{1}{2}}$;
- $\mathbf{ex}_{\mathbf{p}(p)}$ and $\mathbf{ex}_{\mathbf{p}(p)}^{-1}$ are both C^{2,ε_0} and $\|d\,\mathbf{ex}_{\mathbf{p}(p)}\|_{C^{1,\varepsilon_0}} + \|d\,\mathbf{ex}_{\mathbf{p}(p)}^{-1}\|_{C^{1,\varepsilon_0}} \leqslant \mathbf{m}_0^{\frac{1}{2}}$;

- $|h^j_p| \leqslant C\|A_\Sigma\|_{C^0} \leqslant C\mathbf{m}_0^{\frac{1}{2}}$;

where all the constants involved are just geometric. We then conclude that

$$\begin{aligned}
h_p - h_{\mathbf{p}(p)} &= \sum_j \nu_j(p)(h^j_p - h^j_{\mathbf{p}(p)}) + \sum_j \big(\nu_j(p) - \nu_j(\mathbf{p}(p))\big)h^j_{\mathbf{p}(p)} \\
&= \sum_j \nu_j(p)(h^j_p - h^j_{\mathbf{p}(p)}) + \sum_j D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\, h^j_{\mathbf{p}(p)} + O(|p - \mathbf{p}(p)|^2). \qquad (10.31)
\end{aligned}$$

On the other hand, $X_i(p) = Y(\mathbf{p}(p))$ is tangent to $\mathcal{M}$ in $\mathbf{p}(p)$ and hence orthogonal to $h_{\mathbf{p}(p)}$. Thus

$$\begin{aligned}
\langle X_i(p), h_p\rangle = \langle X^i(p), (h_p - h_{\mathbf{p}(p)})\rangle &= \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} \\
&\quad + \sum_j \langle \nu_j(p), X_i(p)\rangle \big(h^j_p - h^j_{\mathbf{p}(p)}\big) + O\left(|p - \mathbf{p}(p)|^2\right) \\
= \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} & \\
+ O\left(|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p))||p - \mathbf{p}(p)| + |p - \mathbf{p}(p)|^2\right), & \qquad (10.32)
\end{aligned}$$

where we used elementary calculus to infer that $|\langle X^i(p), \nu_j(p)\rangle| \leqslant C|p - \mathbf{p}(p)|$ and

$$|h^j_p - h^j_{\mathbf{p}(p)}| \leqslant C\left(|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p)| + |p - \mathbf{p}(p)|\right).$$

We only need that the constants C appearing in the above inequalities are bounded by a geometric factor: in fact they enjoy explicit bounds in terms of $\mathbf{m}_0^{\frac{1}{2}}$ which are at least linear, but such degree of precision is not needed. Finally recalling that $p \in \mathrm{spt}(\mathbf{T}_F)$, we can bound $|p - \mathbf{p}(p)| \leqslant |N(p)|$ and $|\vec{\mathbf{T}}_F(p) - \vec{\mathcal{M}}(\mathbf{p}(p))| \leqslant C|DN(\mathbf{p}(p))|$. We therefore conclude the estimate

$$\langle X_i(p), h_p\rangle = \sum_j \langle X_i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p)) \cdot \mathbf{ex}^{-1}_{\mathbf{p}(p)}(p)\rangle h^j_{\mathbf{p}(p)} + O\big(|N|^2(\mathbf{p}(p)) + |DN|^2(\mathbf{p}(p))\big).$$

We combine it with the expansion of the area functional in [18, Theorem 3.2] to conclude the estimate on I^i_2. Recalling that $\mathbf{p}(F_i(x)) = x$ we get

$$\begin{aligned}
I_2 &= \left|\int \langle X_i, h_p\rangle d\|\mathbf{T}_F\|\right| = \left|\sum_{i=1}^{Q} \int_{\mathcal{M}} \langle Y, h_{F_i}\rangle \mathbf{J}F_i\right| \\
&\overset{(10.32)}{\leqslant} \left|\int_{\mathcal{M}} \sum_{j=1}^{l} \sum_{i=1}^{Q} \langle Y(x), D\nu_j(x) \cdot \mathbf{ex}^{-1}_x(F_i(x))\rangle h^j_x\, d\mathcal{H}^m(x)\right| + C\int_{\mathcal{M}} \varphi_r(|N|^2 + |DN|^2)
\end{aligned}$$

Using the Taylor expansion for $\mathbf{ex}^{-1}_x$ at x (and recalling that $F_i(x) - x = N_i(x)$) we conclude

$$\left|\sum_{i=1}^{Q} \mathbf{ex}^{-1}_x(F_i(x))\right| \leqslant \left|d\,\mathbf{ex}^{-1}_x(\boldsymbol{\eta} \circ N(x))\right| + O(|N|^2) \leqslant C|\boldsymbol{\eta} \circ N(x)| + C|N|^2.$$

Next consider that $|\langle Y, D\nu_j \cdot v\rangle| \leqslant C\varphi \|A_\Sigma\|_{C^0}|v| \leqslant C\varphi\, \mathbf{m}_0^{\frac{1}{2}}|v|$ for every tangent vector v and $|h^j_x| \leqslant C\|A_\Sigma\|_{C^0} \leqslant \mathbf{m}_0^{\frac{1}{2}}$. We thus conclude with the estimate

$$I_2 \leqslant C\,\mathbf{m}_0 \int_{\mathcal{M}} \varphi\,|\boldsymbol{\eta}\circ N| + C\int_{\mathcal{M}} \varphi(|N|^2 + |DN|^2) =: J_1 + J_2\,.$$

Clearly J_1 can be estimated as Err^i_1 and J_2 as Err^i_2.

To conclude the proof notice that, with analogous computation as in [17, Proposition 3.1],

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathcal{M}} |D(N\circ\Gamma_\varepsilon)|^2 = \int_{\mathcal{M}} \left(2\sum_{i=1}^{Q} \langle DN_i \colon (DN_i \cdot D_{\mathcal{M}} Y)\rangle - |DN|^2 \operatorname{div}_{\mathcal{M}} Y\right)\,. \tag{10.33}$$

However, by the conformal invariance of the Dirichlet energy, we have

$$\int_{\mathcal{M}} |D(N\circ\Gamma_\varepsilon)|^2 = \int_{\mathfrak{B}} |D(\mathcal{N}\circ\hat\Gamma_\varepsilon)|^2\,,$$

where $\hat\Gamma_\varepsilon$ is the one parameter family of diffeomorphisms generated by the vector field $\hat Y\colon \mathfrak{B}\to\mathfrak{B}$ defined by

$$\hat Y(z,w) := \frac{|z|}{r}\,\phi\left(\frac{|z|}{r}\right)\nu\,.$$

Hence

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathcal{M}} |D(N\circ\Gamma_\varepsilon)|^2 = \int_{\mathfrak{B}} \left(2\sum_{i=1}^{Q} \langle D\mathcal{N}_i \colon \big(D\mathcal{N}_i \cdot D\hat Y\big)\rangle - |D\mathcal{N}|^2 \operatorname{div}\hat Y\right)\,, \tag{10.34}$$

where the differentiation is taken with respect to the (local) flat structure of $\mathfrak{B}$.

In particular we conclude

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathcal{M}} |D(N\circ\Gamma_\varepsilon)|^2 = \int_{B_r} \frac{|z|}{r^2}\,\phi'\left(\frac{|z|}{r}\right)(2|D_\nu\mathcal{N}|^2 - |D\mathcal{N}|^2)\,. \tag{10.35}$$

Collecting together (10.33), (10.35) and the error estimates, and letting ϕ converge to the to the indicator function of $[-1,1]$ (namely letting $j\uparrow\infty$) we conclude the proof.

10.2.2 *Proof of Lemma 10.7*

The lemma is a very simple corollary of the estimates proven so far. (10.23) is a simple consequence of the Poincaré inequality (10.4) and of (10.16). Similarly, by Lemma 10.5, we have that $\boldsymbol{\Lambda}(r) \leqslant C\,\mathbf{D}(r)$, and therefore (10.26) follows in view of (10.23). The same arguments hold for (10.27). Next for (10.24) we can estimate as follows:

$$\begin{aligned} |\mathbf{L}(r)| &\leqslant C\,\mathbf{m}_0^{\frac{1}{2}} \int_{B_r} |\mathcal{N}|\,|D\mathcal{N}| \leqslant C\,\mathbf{m}_0^{\frac{1}{2}} \left(\int_0^r \mathbf{H}(t)\,dt\right)^{\frac{1}{2}} \mathbf{D}^{\frac{1}{2}}(r) \\ &\overset{(10.2)}{\leqslant} C\,\mathbf{m}_0^{\frac{1}{2}} \left(C_{10.2}\int_0^r t\,\mathbf{D}(t)\,dt\right)^{\frac{1}{2}} \mathbf{D}^{\frac{1}{2}}(r) \leqslant C\,\mathbf{m}_0^{\frac{1}{2}}\, r\,\mathbf{D}(r)\,. \end{aligned} \tag{10.36}$$

Similarly

$$|\mathbf{L}'(r)| \leqslant C\, \mathbf{m}_0^{\frac{1}{2}} \int_{\partial B_r} |\mathscr{N}|\,|D\mathscr{N}| \leqslant C\, \mathbf{m}_0^{\frac{1}{2}} \left(\mathbf{D}'(r)\, \mathbf{H}(r)\right)^{\frac{1}{2}} . \tag{10.37}$$

Finally, we notice that by Proposition 10.6 implies

$$\left|\frac{\mathbf{D}'(r)}{2} - \int_{\partial B_r} |D_\tau \mathscr{N}|^2\right| \leqslant C\, \mathcal{E}_{IV}(r).$$

Therefore, using the almost minimizing property in (9.40) and the Poincaré inequality we infer that

$$\mathbf{D}(r) \leqslant (1 + C\, r) \left[\frac{r\, \mathbf{D}'(r)}{2(2\,a+b)} + \frac{a(a+b)\,\mathbf{H}(r)}{r(2\,a+b)}\right] + C(a)\, r\, \mathcal{E}_{IV}(r) + \mathcal{E}_{QM}(r) + C\, r^{1+\sigma}\, \mathbf{D}'(r)\,.$$

Absorbing the error term $r^{1+\sigma}\, \mathbf{D}'(r)$ and dividing by $(1 + C\, r)$ we get

$$\mathbf{D}(r) \leqslant \left[\frac{r\, \mathbf{D}'(r)}{2(2\,a+b)} + \frac{a(a+b)\,\mathbf{H}(r)}{r(2\,a+b)}\right] + C(a)\, r\, \mathcal{E}_{IV}(r) + \mathcal{E}_{QM}(r) + C\, r\mathbf{D}(r)\,.$$

from which (10.28) follows straightforwardly by noticing that $\mathcal{E}_{QM}(r) + r\, \mathbf{D}(r) \leqslant C\, r\, \mathcal{E}_{IV}(r)$.

10.2.3 *Proof of Corollary 10.8*

Recall first thet $\eta < \gamma_0$. We start with $\mathcal{E}_{BP}(r)$. Notice that, using $\mathbf{H}(t) \leqslant C\, t\, \mathbf{D}(t)$ together with the definition of $\mathbf{F}(r)$, we have

$$\int_0^r \left(\frac{1}{t^\gamma\, \mathbf{D}(t)}\right)' \mathbf{F}(t)\, dt \leqslant C\, \frac{\mathbf{F}(r)}{r^\gamma\, \mathbf{D}(r)} + C \int_0^r \frac{1}{t^\gamma\, \mathbf{D}(t)}\, \frac{\mathbf{H}(t)}{t^{2-\gamma_0}}\, dt \leqslant C r^{\gamma_0-\gamma}$$

Next, by a simple integration by parts and the fact that $\mathbf{D}(r) \leqslant C r^2$, we deduce

$$\begin{aligned}
\int_0^r \frac{1}{t^\gamma \mathbf{D}(t)} \frac{d}{dt} \|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_t)))\, dt &= \frac{1}{r^\gamma \mathbf{D}(r)} \|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_r))) \\
&\quad + \int_0^r \left(\frac{1}{t^\gamma \mathbf{D}(t)}\right)' \|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_t)))\, dt \\
&\overset{(8.4)}{\leqslant} C\, \frac{\mathbf{D}^{1+\eta}(r) + \mathbf{F}(r)}{r^\gamma\, \mathbf{D}(r)} + \int_0^r \left(\frac{1}{t^\gamma \mathbf{D}(t)}\right)' \left(\mathbf{D}(t)^{1+\eta} + \mathbf{F}(t)\right)\, dt \leqslant C\, r^{\eta-\gamma}\,.
\end{aligned} \tag{10.38}$$

In a similar fashion we have

$$\begin{aligned}
\int_0^r \frac{\mathbf{m}_0^{\frac{1}{2}}}{t^\gamma\, \mathbf{D}(t)} \int_{\partial B_t} \frac{|\boldsymbol{\eta} \circ \mathscr{N}(z,w)|}{t^{1-\gamma_0}}\, dt &\leqslant \frac{\mathbf{m}_0^{\frac{1}{2}}}{r^\gamma\, \mathbf{D}(r)} \int_{B_r} \frac{|\boldsymbol{\eta} \circ \mathscr{N}(z,w)|}{|z|^{1-\gamma_0}} \\
&\quad + \int_0^r \left(\frac{1}{t^\gamma \mathbf{D}(t)}\right)' \mathbf{m}_0^{\frac{1}{2}} \int_{B_t} \frac{|\boldsymbol{\eta} \circ \mathscr{N}(z,w)|}{|z|^{1-\gamma_0}} \\
&\overset{(8.3)}{\leqslant} C\, \frac{\mathbf{D}^{1+\eta}(r) + \mathbf{F}(r)}{r^\gamma\, \mathbf{D}(r)} + \int_0^r \left(\frac{1}{t^\gamma \mathbf{D}(t)}\right)' \left(\mathbf{D}(t)^{1+\eta} + \mathbf{F}(t)\right)\, dt \leqslant C\, r^{\eta-\gamma}\,.
\end{aligned} \tag{10.39}$$

so that

$$\int_0^r \frac{\mathcal{E}_{BP}(t)}{t^\gamma\,\mathbf{D}(t)}\,dt \leqslant C\,r^{\eta-\gamma}$$

To conclude, we compute separately the integral of each addendum of $\mathbf{s}$.

$$\begin{aligned}\int_0^r \frac{\mathcal{E}_{IV}(t)}{t^\gamma\,\mathbf{D}(t)}\,dt &\overset{(10.27)}{\leqslant} 2\,C_{10.7}\int_0^r \left(t^{\gamma_0-\gamma-1} + t^{-\gamma}\mathbf{D}(t)^{\eta-1}\,\mathbf{D}'(t) + \frac{\mathcal{E}_{BP}(t)}{t^\gamma\,\mathbf{D}(t)}\right)dt\\ &\leqslant C\,r^{\eta-\gamma}\left(1+\mathbf{D}(t)^{\frac{\eta}{2}}\right) \leqslant C\,r^{\eta-\gamma}\,, \end{aligned}\tag{10.40}$$

where in the second inequality we used $\mathbf{D}(t) \leqslant C\,t^2$, and

$$\begin{aligned}\int_0^r \frac{\mathcal{E}_{OV}(t)}{t^{1+\gamma}\mathbf{D}(t)}\,dt \overset{(10.26)}{\leqslant} C_{10.7}\int_0^r \bigg(&\frac{\mathbf{D}^\eta(t)}{t^{1+\gamma}} + \frac{\mathbf{F}(t)}{t^{1+\gamma}\mathbf{D}(t)} + t^{-\gamma}\mathbf{D}^{\eta-1}(t)\mathbf{D}'(t)\\ &+ \frac{\mathcal{E}_{BP}(t)}{t^\gamma\,\mathbf{D}(t)}\bigg)\,dt \leqslant C\,r^{\eta-\gamma}\,. \end{aligned}\tag{10.41}$$

10.3 ALMOST MONOTONICITY AND DECAY OF THE FREQUENCY FUNCTION

In this section we study the asymptotic behaviour of the normal approximation $\mathscr{N}$. The first step consists in proving approximate monotonicity and decay estimates for the frequency function.

For every $r \in (0,1)$ such that $\mathbf{H}(r) > 0$, we set $\bar{\mathbf{I}}(r) := \frac{r\,\mathbf{\Omega}(r)}{\mathbf{H}(r)}$ where we recall that

$$\mathbf{\Omega}(r) := \begin{cases} \mathbf{D}(r) & \text{in the cases (a) and (b) of Definition 1.1;}\\ \mathbf{D}(r)+\mathbf{L}(r) & \text{in case (c).}\end{cases}$$

Furthermore we define $\bar{\mathbf{K}}(r) := \bar{\mathbf{I}}(r)^{-1}$ whenever $\mathbf{\Omega}(r) \neq 0$. By (10.24) there exists $r_0 > 0$ such that

$$\frac{1}{2}\mathbf{D}(r) \leqslant (1-C\,r)\mathbf{D}(r) \leqslant \mathbf{\Omega}(r) \leqslant (1+C\,r)\mathbf{D}(r) \leqslant 2\,\mathbf{D}(r) \qquad \forall r \leqslant r_0\,. \tag{10.42}$$

Having fixed r_0, $\bar{\mathbf{K}}(r)$ is well defined whenever $\mathbf{D}(r) > 0$ and hence, by the Poincaré inequality, whenever $\bar{\mathbf{I}}(r)$ is defined. Moreover, if for some $\rho \leqslant r_0$, $\bar{\mathbf{K}}(\rho)$ is not well defined, that is $\mathbf{\Omega}(\rho) = 0$, then obviously $\mathbf{\Omega}(r) = \mathbf{D}(r) = 0$ for every $r \leqslant \rho$.

We are now ready to state the first important monotonicity estimate.

Theorem 10.9. *There exists a constant $C_{10.9} > 0$ with the following property: if $\mathbf{D}(r) > 0$ for some $r \leqslant r_0$, then the function*

$$\bar{\mathbf{K}}(r)\,\exp(-4\mathbf{\Sigma}_{IV}(r)) - 4\,\mathbf{\Sigma}_{OV}(r) \tag{10.43}$$

is monotone non-increasing on any interval $[a,b]$ where $\mathbf{D}$ is nowhere 0. In particular, either there is $\bar{r} > 0$ such that $\mathbf{D}(\bar{r}) = 0$ or $\bar{\mathbf{K}}$ is well-defined on $]0, r_0[$ and the limit $K_0 := \lim_{r\to 0}\bar{\mathbf{K}}(r)$ exists.

A fundamental consequence of Theorem 10.9 is the following dichotomy.

Corollary 10.10. *There exists* $\bar{r} > 0$ *such that*

(A) *either* $\bar{\mathbf{K}}(r)$ *is well-defined for every* $r \in]0, r_0[$*, the limit*

$$K_0 := \lim_{r \downarrow 0} \bar{\mathbf{K}}(r) \tag{10.44}$$

is positive and thus there is a constant C *and a radius* $\bar{r}$ *such that*

$$C^{-1}\, r\, \mathbf{D}(r) \leqslant \mathbf{H}(r) \leqslant C\, r\, \mathbf{D}(r) \qquad \forall\, r \in]0, \bar{r}[\,; \tag{10.45}$$

(B) *or* $T \llcorner \mathbf{p}^{-1}(\mathbf{\Psi}(B_{\bar{r}})) = Q\, [\![\mathbf{\Psi}(B_{\bar{r}})]\!]$ *for some positive* $\bar{r}$*.*

In turn, using the above dichotomy we will show

Theorem 10.11. *Assume that condition (i) in Theorem 10.1 fails. Then the frequency* $\bar{\mathbf{I}}(r)$ *is well-defined for every sufficiently small* r *and its limit* $I_0 = \lim_{r \to 0} \bar{\mathbf{I}}(r) = K_0^{-1}$ *exists and it is finite and positive. Moreover there exist constants* $\lambda, C_{10.11}, H_0, D_0 > 0$ *such that, for every* r *sufficiently small the following holds:*

$$\left|\mathbf{I}(r) - I_0\right| + \left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| + \left|\frac{\mathbf{D}(r)}{r^{2I_0}} - D_0\right| \leqslant C_{10.11}\, r^{\lambda}\,. \tag{10.46}$$

10.3.1 *Proof of Theorem 10.9*

In the first step we claim the monotonicity of the function $\bar{\mathbf{K}}(r)\, \exp(-\mathbf{\Sigma}_{IV}(r)) - 2\,\mathbf{\Sigma}_{OV}(r)$ on any interval contained in $[a, b]$ on which $\mathbf{D}$ is everywhere positive. Recalling that $\mathbf{\Omega}$ and $\mathbf{H}$ are absolutely continuous functions, we can compute the following derivative: for every $r \in [a, b]$

$$\begin{aligned} \bar{\mathbf{K}}'(r) &= \left(\frac{\mathbf{H}(r)}{r}\right)' \frac{1}{\mathbf{\Omega}(r)} - \frac{\mathbf{H}(r)}{r}\, \frac{\mathbf{\Omega}'(r)}{\mathbf{\Omega}^2(r)} \\ &\overset{(10.6)}{\leqslant} \frac{1}{r\mathbf{\Omega}^2(r)} \big(2\, \mathbf{E}(r)\, \mathbf{\Omega}(r) - \mathbf{D}'(r)\, \mathbf{H}(r) + |\mathbf{L}'(r)|\, \mathbf{H}(r)\big)\,. \end{aligned} \tag{10.47}$$

Then, either $\bar{\mathbf{K}}' \leqslant 0$, or the RHS of the inequality above is positive, that is

$$\mathbf{D}'(r)\, \mathbf{H}(r) \leqslant 2\mathbf{E}(r)\, \mathbf{\Omega}(r) + |\mathbf{L}'(r)|\, \mathbf{H}(r) \overset{(10.37)}{\leqslant} 2\mathbf{E}(r)\, \mathbf{\Omega}(r) + r\, \mathbf{D}'(r)\, \mathbf{H}(r) + \frac{\mathbf{H}^2(r)}{r}\,.$$

In turn, using $\mathbf{H}(r) \leqslant C\, r\, \mathbf{D}(r) \leqslant C\, r\, \mathbf{\Omega}(r)$, the latter inequality implies

$$\mathbf{D}'(r)\, \mathbf{H}(r) \leqslant C\, \mathbf{E}(r)\, \mathbf{\Omega}(r) + C\, r\, \mathbf{\Omega}^2(r)\,.$$

From this we deduce

$$\mathbf{E}^2(r) \leqslant \mathbf{H}(r)\, \mathbf{G}(r) \leqslant \mathbf{H}(r)\, \mathbf{D}'(r) \leqslant C\frac{\mathbf{\Omega}^2(r)}{2} + \frac{\mathbf{E}^2(r)}{2}$$

which implies that $\mathbf{E}(r) \leqslant C\mathbf{\Omega}(r)$ and so, by (10.24),

$$|\mathbf{L}'(r)| \leqslant C\, \mathbf{m}_0^{\frac{1}{2}}\, (\mathbf{D}'(r)\, \mathbf{H}(r))^{\frac{1}{2}} \leqslant C\, \mathbf{m}_0^{\frac{1}{2}}\, \mathbf{\Omega}(r)\,. \tag{10.48}$$

Next using again the Cauchy-Schwarz inequality and (10.26), we have

$$\begin{aligned}\mathbf{\Omega}(r)\,\mathbf{E}(r) &\leqslant \mathbf{\Omega}(r)\,\mathbf{H}(r)^{\frac{1}{2}}\,\mathbf{G}(r)^{\frac{1}{2}} \leqslant \frac{\mathbf{\Omega}(r)^2}{2} + \frac{\mathbf{H}(r)\,\mathbf{G}(r)}{2}\\ &\leqslant \frac{\mathbf{\Omega}(r)\mathbf{E}(r)}{2} + \frac{\mathbf{\Omega}(r)\,\mathcal{E}_{OV}(r)}{2} + \frac{\mathbf{H}(r)\,\mathbf{G}(r)}{2},\end{aligned}$$

which implies

$$\mathbf{\Omega}(r)\,\mathbf{E}(r) \leqslant \mathbf{H}(r)\,\mathbf{G}(r) + \mathbf{\Omega}(r)\,\mathcal{E}_{OV}(r)\,. \tag{10.49}$$

Collecting all these estimates together and using (10.21), we conclude that, if $\bar{\mathbf{K}}'(r) \geqslant 0$, then

$$\begin{aligned}\bar{\mathbf{K}}'(r) &\overset{(10.49)}{\leqslant} \frac{1}{r\mathbf{\Omega}^2(r)}\big(2\,\mathbf{H}(r)\,\mathbf{G}(r) - \mathbf{D}'(r)\,\mathbf{H}(r) + |\mathbf{L}'(r)|\,\mathbf{H}(r) + 2\,\mathbf{\Omega}(r)\,\mathcal{E}_{OV}(r)\big)\\ &\overset{(10.21)\&(10.48)}{\leqslant} \frac{1}{r\mathbf{\Omega}^2(r)}\big(2\,\mathbf{H}(r)\,\mathbf{G}(r) - 2\,\mathbf{H}(r)\,\mathbf{G}(r) + \mathbf{\Omega}(r)\,\mathbf{H}(r) + \mathbf{H}(r)\,\mathcal{E}_{IV}(r) + 2\,\mathbf{\Omega}(r)\,\mathcal{E}_{OV}(r)\big)\\ &\leqslant 2\frac{\mathcal{E}_{OV}(r)}{r\mathbf{\Omega}(r)} + \bar{\mathbf{K}}(r)\,\frac{\mathcal{E}_{IV}(r)}{\mathbf{\Omega}(r)} \leqslant 4\,\frac{\mathcal{E}_{OV}(r)}{r\mathbf{D}(r)} + 4\,\bar{\mathbf{K}}(r)\,\frac{\mathcal{E}_{IV}(r)}{\mathbf{D}(r)}\,.\end{aligned} \tag{10.50}$$

On the other hand the final inequality

$$\mathbf{K}'(r) \leqslant 4\frac{\mathcal{E}_{OV}(r)}{r\mathbf{D}(r)} + 4\,\bar{\mathbf{K}}(r)\,\frac{\mathcal{E}_{IV}(r)}{\mathbf{D}(r)}$$

is certainly correct when $\mathbf{K}'(r) \leqslant 0$, because the right hand side is positive. The monotonicity of the function in (10.43) is then obvious.

Next, as already observed, either $\mathbf{D}$ is always positive, or it vanishes on some interval $]0, \bar{r}[$. If $\mathbf{D}$ is always positive, then $\bar{\mathbf{K}}$ is well defined on $]0, r_0[$ and the existence of the limit $K_0 := \lim_{r\downarrow 0} \bar{\mathbf{K}}(r)$ is a direct consequence of (10.43) and Corollary 10.8.

10.3.2 *Proof of Corollary 10.10*

First of all observe that, if $\mathbf{D}(\bar{r})$ vanishes, then $\mathscr{N} \equiv Q\,[\![0]\!]$ on $B_{\bar{r}}$. In particular by (8.4) we conclude that we are in the alternative (B). We can thus assume, without loss of generality, that $\mathbf{D}$ is positive on $]0, r_0[$. Assuming that K_0 vanishes we will then reach a contradiction.

Under the assumption $K_0 = 0$, consider the monotonicity of $\bar{\mathbf{K}}(r)\exp(-4\mathbf{\Sigma}_{IV}(r)) - 4\,\mathbf{\Sigma}_{OV}(r)$ between two radii $0 < s < r$ and let $s \to 0$ to get

$$\bar{\mathbf{K}}(r) \leqslant 4\,e^{4\mathbf{\Sigma}_{IV}(r)}\,\mathbf{\Sigma}_{OV}(r) \leqslant C\,\mathbf{\Sigma}_{OV}(r)\,,$$

where the last inequality holds for r sufficiently small, since $\mathbf{\Sigma}_{IV}(r) \leqslant Cr^{\eta-\gamma}$. Next observe that, since the function $\mathbf{\Sigma}_{OV}(r)$ is non-decreasing (it is the primitive of a positive function),

$$\frac{\mathbf{F}(r)}{\mathbf{D}(r)} \leqslant \frac{1}{\mathbf{D}(r)}\int_0^r \frac{\mathbf{H}(s)}{s^{2-\gamma_0}}\frac{\mathbf{D}(s)}{\mathbf{D}(s)}\,ds \leqslant C\int_0^r \frac{\bar{\mathbf{K}}(s)}{s^{1-\gamma_0}}\,ds \leqslant C\,r^{\gamma_0}\,\mathbf{\Sigma}_{OV}(r)\,, \tag{10.51}$$

and that

$$\begin{aligned}
&\int_0^r \frac{1}{\mathbf{D}(s)}\frac{d}{ds}\|T-\mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_s)))\,ds\\
&\overset{(8.4)}{\leqslant} C\frac{\mathbf{D}^{1+\eta}(r)+\mathbf{F}(r)}{\mathbf{D}(r)}+C\int_0^r\left(\frac{1}{\mathbf{D}(s)}\right)'\left(\mathbf{D}^{1+\eta}(s)+\mathbf{F}(s)\right)\,ds\\
&\leqslant C\,\mathbf{D}^\eta(r)+C\,r^{\gamma_0}\mathbf{\Sigma}_{OV}(r)+C\,\frac{\mathbf{F}(r)}{\mathbf{D}(r)}+C\int_0^r\frac{\mathbf{F}'(s)}{\mathbf{D}(s)}\,ds\\
&\leqslant C\,\mathbf{D}^\eta(r)+C\,r^{\gamma_0}\mathbf{\Sigma}_{OV}(r)+C\int_0^r\frac{\bar{\mathbf{K}}(s)}{s^{1-\gamma_0}}\,ds\leqslant C\,\mathbf{D}^\eta(r)+C\,r^{\gamma_0}\mathbf{\Sigma}_{OV}(r)\,. \qquad (10.52)
\end{aligned}$$

Using these two estimates and $2^{-1}\mathbf{D}(r)\leqslant\mathbf{\Omega}(r)\leqslant 2\mathbf{D}(r)$ in the formula for $\mathcal{E}_{OV}$, we have

$$\begin{aligned}
&\mathbf{\Sigma}_{OV}(r)\\
&\leqslant C\int_0^r\frac{1}{s\mathbf{D}(s)}\left(\mathbf{D}(s)^{1+\eta}+s\mathbf{D}^\eta(s)\mathbf{D}'(s)+\mathbf{F}(s)+s\frac{d}{ds}\|T-\mathbf{T}_F\|(\mathbf{p}^{-1}(\mathbf{\Psi}(B_s)))\right)\,ds\\
&\leqslant C\,r^\eta\mathbf{D}(r)^{\frac{\eta}{2}}+Cr^{\gamma_0}\,\mathbf{\Sigma}_{OV}(r)\,.
\end{aligned}$$

Hence, for r sufficiently small,

$$\bar{\mathbf{K}}(r)\leqslant C\mathbf{\Sigma}_{OV}(r)\leqslant C\mathbf{D}(r)^{\frac{\eta}{2}}\,. \qquad (10.53)$$

In particular this implies that

$$\mathbf{H}(r)\leqslant C\,r\,\mathbf{D}(r)^{1+\frac{\eta}{2}}\,. \qquad (10.54)$$

Combining this with (10.7) and the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
\frac{1}{2}\mathbf{D}(r)\leqslant\mathbf{\Omega}(r)&\leqslant\frac{\mathbf{E}(r)}{r}+\mathcal{E}_{OV}(r)\leqslant\left(\frac{\mathbf{H}(r)}{r\,\mathbf{D}(r)^{\frac{\eta}{4}}}\right)^{\frac{1}{2}}\left(r\,\mathbf{D}'(r)\,\mathbf{D}(r)^{\frac{\eta}{4}}\right)^{\frac{1}{2}}+\mathcal{E}_{OV}(r)\\
&\overset{(10.54)}{\leqslant} C\,\mathbf{D}(r)^{1+\frac{\eta}{4}}+C\,r\,\mathbf{D}(r)^{\frac{\eta}{4}}\mathbf{D}'(r)+\mathcal{E}_{OV}(r)\,.
\end{aligned}$$

Dividing the expression above by $r\mathbf{D}(r)$, integrating between two radii $0<s<r$ and using the bound $\mathbf{D}(r)\leqslant C\,r^2$ we obtain

$$\log\left(\frac{r}{s}\right)\leqslant C\int_s^r\left(\frac{\mathbf{D}(\rho)^{\frac{\eta}{4}}}{\rho}+\mathbf{D}(\rho)^{\frac{\eta}{4}-1}\,\mathbf{D}'(\rho)+\frac{\mathcal{E}_{OV}(\rho)}{\rho\,\mathbf{D}(\rho)}\right)\,d\rho\leqslant C\,(r^{\frac{\eta}{2}}-s^{\frac{\eta}{2}})\,.$$

Sending $s\to 0$ we get a contradiction.

10.3.3 *Proof of Theorem 10.11*

Clearly, if (i) in Theorem 10.1 does not hold, then $\mathbf{D}$ is always positive and we are in alternative (A) of Corollary 10.10. Thus K_0 is positive and the first statement is obvious.

Let $\mathbf{K}(r):=\mathbf{I}(r)^{-1}$ and observe that by (10.42) we have

$$(1-C\,r)\mathbf{I}(r)\leqslant\bar{\mathbf{I}}(r)\leqslant(1+C\,r)\mathbf{I}(r)\,,\quad\forall 0\leqslant r\leqslant r_0\,,$$

which implies

$$(1 - C\,r)\bar{\mathbf{K}}(r) \leqslant \mathbf{K}(r) \leqslant (1 + C\,r)\bar{\mathbf{K}}(r) \quad \forall\, 0 \leqslant r \leqslant r_0\,,$$

so that in particular $\mathbf{K}(r) \leqslant C\,\bar{\mathbf{K}}(r) < \infty$ for every $0 < r < r_0$ and $\mathbf{K}(r) \to K_0$ as $r \to 0$. Using the monotonicity formula of Theorem 10.9 together with Corollary 10.8 we have

$$\bar{\mathbf{K}}(r) - K_0 \leqslant C\,\mathbf{s}(r) \leqslant C\,r^{\eta}\,.$$

and therefore

$$\mathbf{K}(r) - K_0 \leqslant C\,r^{\eta} + C\,\mathbf{K}(r)\,r \leqslant C\,r^{\eta}\,. \tag{10.55}$$

To control $\mathbf{K}(r) - K_0$ from below we apply (10.28) with $a = I_0 = \frac{1}{K_0}$ and $b = \lambda \leqslant \min\{\frac{\eta}{2}, b_0(I_0)\}$ to infer, after dividing by $r\mathbf{D}(r)$, that

$$-\frac{\mathbf{D}'(r)}{\mathbf{D}(r)} \leqslant \frac{2}{r}\left(I_0(I_0+\lambda)\mathbf{K}(r) - (2I_0+\lambda)\right)\,.$$

Multiplying this expression by $\mathbf{K}(r) > 0$ and adding $\frac{2}{r}$, we get

$$\begin{aligned}\frac{2}{r} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)}\mathbf{K}(r) &\leqslant \frac{2}{r}\left[1 + I_0(I_0+\lambda)\mathbf{K}^2(r) - (2I_0+\lambda)\mathbf{K}(r)\right] + \frac{C\,\mathcal{E}_{IV}(r)}{\mathbf{D}(r)}\\ &\leqslant \frac{2}{r}\,I_0\left(\mathbf{K}(r) - \frac{1}{I_0}\right)\left((I_0+\lambda)\mathbf{K}(r) - 1\right) + \frac{C\,\mathcal{E}_{IV}(r)}{\mathbf{D}(r)}\end{aligned} \tag{10.56}$$

Since $(I_0+\lambda)\mathbf{K}(r)$ converges to $1 + \lambda K_0$, we easily deduce that for r small enough $(I_0 + \lambda)\bar{\mathbf{K}}(r) - 1 \geqslant \frac{\lambda}{2}K_0$. Using this together with (10.55), we deduce from (10.56) that

$$\frac{2}{r} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} \leqslant \frac{\lambda}{r}\left(\mathbf{K}(r) - \frac{1}{I_0}\right) + \frac{C\,\mathcal{E}_{IV}(r)}{\mathbf{D}(r)} + C\frac{r^{\eta}}{r}. \tag{10.57}$$

A simple application of the usual variational formulas leads to

$$\begin{aligned}\mathbf{K}'(r) = \left(\frac{\mathbf{H}(r)}{r}\right)'\frac{1}{\mathbf{D}(r)} - \frac{\mathbf{H}(r)}{r\mathbf{D}(r)}\frac{\mathbf{D}'(r)}{\mathbf{D}(r)} &\overset{(10.6)}{\leqslant} \frac{2\,\mathbf{E}(r)}{r\,\mathbf{D}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)}\mathbf{K}(r)\\ &\overset{(10.7)}{\leqslant} \frac{2}{r} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)}\mathbf{K}(r) + C\,\frac{\mathcal{E}_{OV}(r)}{r\mathbf{D}(r)}\\ &\overset{(10.57)}{\leqslant} \frac{\lambda}{r}\left(\mathbf{K}(r) - \frac{1}{I_0}\right) + \frac{C\,\mathcal{E}_{IV}(r)}{\mathbf{D}(r)} + C\,\frac{\mathcal{E}_{OV}(r)}{r\mathbf{D}(r)} + C\frac{r^{\eta}}{r}\,.\end{aligned} \tag{10.58}$$

Recalling that $\mathbf{K}(r) \leqslant C$, we deduce

$$\frac{d}{dr}\left[\frac{\mathbf{K}(r) - K_0}{r^{\lambda}}\right] \leqslant C\,\frac{\mathcal{E}_{OV}(r)}{r^{1+\lambda}\mathbf{D}(r)} + C\,\frac{\mathcal{E}_{IV}(r)}{r^{\lambda}\,\mathbf{D}(r)} + C\frac{1}{r^{1+\lambda-\eta}} \tag{10.59}$$

Integrating (10.59) on the interval $]s, r[$ and using (10.29), we get

$$\mathbf{K}(r) - \mathbf{K}_0 \leqslant \frac{r^{\lambda}}{s^{\lambda}}\left(\mathbf{K}(s) - K_0\right) + C\,r^{\eta-\lambda}$$

that is $\mathbf{K}(s) - K_0 \geqslant Cs^\lambda$. The inequality $|\mathbf{K}(r) - K_0| \leqslant C\, r^\lambda$ easily implies $|\mathbf{I}(r) - I_0| \leqslant C\, r^\lambda$.

For what concerns the other inequalities we compute

$$\left[\log\left(\frac{\mathbf{H}(r)}{r^{2I_0+1}}\right)\right]' = \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{2\,I_0+1}{r} = \frac{2\,\mathbf{E}(r)}{r\,\mathbf{H}(r)} - \frac{2\,I_0}{r} \leqslant \frac{2\,\mathbf{D}(r)}{\mathbf{H}(r)} - \frac{2\,I_0}{r} + C\,\frac{\mathcal{E}_{OV}(r)}{\mathbf{H}(r)}$$
$$\leqslant \frac{2}{r}\left(\mathbf{I}(r) - I_0\right) + C\,\frac{\mathcal{E}_{OV}(r)}{\mathbf{H}(r)} \tag{10.60}$$

and similarly

$$\left[\log\left(\frac{\mathbf{H}(r)}{r^{2I_0+1}}\right)\right]' \geqslant \frac{2}{r}\left(\mathbf{I}(r) - I_0\right) - C\,\frac{\mathcal{E}_{OV}(r)}{\mathbf{H}(r)}\,. \tag{10.61}$$

Integrating (10.60) and (10.61) and using (10.29), we deduce that there exists the limit

$$H_0 := \lim_{s\downarrow 0} \frac{\mathbf{H}(s)}{s^{2I_0+1}}, \quad \text{with} \quad \left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| \leqslant C\, r^\lambda.$$

Moreover, from (10.60) we also infer that for r sufficiently small

$$H_0 \geqslant \frac{\mathbf{H}(r)}{r^{2I_0+1}} e^{-C\,r^\lambda} > 0.$$

Finally the last assertion follows simply setting $D_0 := I_0 \cdot H_0$ and from

$$\left|\frac{\mathbf{D}(r)}{r^{2\,I_0}} - D_0\right| = \left|\mathbf{I}(r)\,\frac{\mathbf{H}(r)}{r^{2\,I_0+1}} - I_0\,H_0\right|$$
$$\leqslant |\mathbf{I}(r) - I_0|\,\frac{\mathbf{H}(r)}{r^{2\,I_0+1}} + I_0\,\left|\frac{\mathbf{H}(r)}{r^{2\,I_0+1}} - H_0\right| \leqslant C\, r^\lambda.$$

10.4 PROOF OF THE BLOW-UP THEOREM

As a consequence of the decay estimate in Theorem 10.11 we can show that suitable rescaling of the normal approximation N converge to a unique limiting profile. To this aim we consider for every $r \in (0,1)$ the functions $f_r : \partial B_1 \to \mathcal{A}_{Q_1}(\mathbb{R}^{2+n})$ given by

$$f_r(z,w) := \frac{\mathscr{N}(\mathbf{i}_r(z,w))}{r^{I_0}}.$$

Recall that $T_0\mathcal{M} = \mathbb{R}^2 \times \{0\}$, and $T_0\Sigma = \mathbb{R}^2 \times \mathbb{R}^{\bar n} \times \{0\}$. In the following, with a slight abuse of notation, we write $\mathbb{R}^{\bar n}$ for the subspace $\{0\} \times \mathbb{R}^{\bar n} \times \{0\}$.

The final step in the proof of Theorem 10.1 is then the following proposition.

Proposition 10.12. *Assume alternative (i) in Theorem 10.1 fails and let I_0 and λ be the positive numbers of Theorem 10.11. Then $I_0 > 1$ and there exists a function $f_0 : \partial B_1 \to \mathcal{A}_Q(\mathbb{R}^{\bar n})$ such that*

(i) $\boldsymbol{\eta} \circ f_0 = 0$ *and* $f_0 \not\equiv Q_1\,[\![0]\!]$*;*

(ii) for every r sufficiently small

$$\mathcal{G}(f_r(z,w), f_0(z,w)) \leqslant C\, r^{\frac{\lambda}{16}} \quad \forall\,(z,w) \in \partial B_1\,; \tag{10.62}$$

(iii) the I_0*-homogeneous extension* $g(z,w) := |z|^{I_0} f_0\left(\frac{z}{|z|}, \frac{w}{|w|}\right)$ *is nontrivial and* Dir*-minimizing.*

In particular, by (iii) $\mathrm{Im}(g_0)\setminus\{0\}\subset\mathbb{R}^{2+n}$ *is a real analytic submanifold.*

Theorem 10.1 follows immediately from Proposition 10.12 and Theorem 10.11.

Proof of Theorem 10.1. Since we have identitied $\mathbb{R}^{\bar n}$ with $\{0\}\times\mathbb{R}^{\bar n}\times\{0\}$, it is obvious that the map g has all the properties claimed in (ii), namely it is Dir-minimizing, $\boldsymbol{\eta}\circ g\equiv 0$ and it is nontrivial. (10.1) is a corollary of (10.62) provided $a_0\leqslant\frac{\lambda}{16}$. Next note that (10.3) has been shown in Theorem 10.2. As for (10.2) observe that, if $4\rho\leqslant r<1$, then, by Theorem 10.11,

$$D_0(r-2\rho)^{2I_0}-C(r-2\rho)^{2I_0+\lambda}\leqslant \mathbf{D}(r-2\rho)\leqslant\mathbf{D}(r+2\rho)\leqslant D_0(r+2\rho)^{2I_0}+C(r+2\rho)^{2I_0+\lambda}\,.$$

Since $2I_0>2$, (10.2) follows easily from

$$\int_{B_{r+2\rho}\setminus B_{r-2\rho}}|D\mathscr{N}|^2=\mathbf{D}(r+2\rho)-\mathbf{D}(r-2\rho)\,,$$

provided $a_0\leqslant\lambda$. □

The rest of this final section of the note is devoted to the proof Proposition 10.12, which is split in several steps. Before starting with it, let us however observe that the conclusion $I_0>1$ is an obvious consequence of the decay estimates of Theorem 10.11 and the fact that $\mathbf{D}(r)\leqslant Cr^{2+2\gamma_0}$.

10.4.1 *Step 1: uniqueness of the limit* f_0

For r sufficienly small and $s\in[\frac{r}{2},r]$, we start estimating the following quantity:

$$\int_{\partial B_1}\mathcal{G}(f_r,f_s)^2\leqslant(r-s)\int_{\partial B_1}\int_s^r\left|\frac{d}{dt}f_t(z,w)\right|^2dt\,. \tag{10.63}$$

Using the differentiability properties of Lipschitz multiple valued functions and the 1-dimensional theory in Chapter 3 (note that $t\mapsto\mathscr{N}(\iota_t(z,w))$ is a Lipschitz map), we easily infer that

$$\begin{aligned}\left|\frac{d}{dt}f_t(z,w)\right|^2&=\sum_{j=1}^Q\left|\frac{D\mathscr{N}_j(\iota_t(z,w))\cdot z}{t^{I_0}}-I_0\frac{\mathscr{N}_j(\iota_t(z,w))}{t^{I_0+1}}\right|^2\\&=\frac{|z|^2|\partial_{\hat r}\mathscr{N}|^2(\iota_t(z,w))}{t^{2I_0}}-2\,I_0\frac{|z|}{t^{2I_0+1}}\sum_{j=1}^Q\langle\partial_{\hat r}\mathscr{N}_j,\mathscr{N}_j\rangle(\iota_t(z,w))+\frac{|\mathscr{N}|^2(\iota_t(z,w))}{t^{2I_0+2}}.\end{aligned}$$

Therefore, by the change of variable $(z',w')=\iota_t(z,w)$ in (10.63) we infer that

$$\begin{aligned}\int_{\partial B_1}\mathcal{G}(f_r,f_s)^2&\leqslant\frac{r}{2}\int_{\frac{r}{2}}^r\left(\frac{\mathbf{G}(t)}{t^{2I_0+1}}-2\,I_0\frac{\mathbf{E}(t)}{t^{2I_0+2}}+I_0^2\frac{\mathbf{H}(t)}{t^{2I_0+3}}\right)dt\\&\leqslant\frac{r}{2}\int_{\frac{r}{2}}^r\left(\frac{\mathbf{D}'(t)}{2t^{2I_0+1}}-2\,I_0\frac{\mathbf{D}(t)}{t^{2I_0+2}}+I_0^2\frac{\mathbf{H}(t)}{t^{2I_0+3}}+C\frac{\mathcal{E}_{IV}(t)}{t^{2I_0+1}}+C\frac{\mathcal{E}_{OV}(t)}{t^{2I_0+2}}\right)dt\\&=\frac{r}{2}\int_{\frac{r}{2}}^r\left[\frac{1}{2t}\left(\frac{\mathbf{D}(t)}{t^{2I_0}}\right)'+I_0\frac{\mathbf{H}(t)}{t^{2I_0+3}}\,(I_0-\mathbf{I}(t))+C\frac{\mathcal{E}_{IV}(t)}{t^{2I_0+1}}+C\frac{\mathcal{E}_{OV}(t)}{t^{2I_0+2}}\right]dt.\end{aligned}$$

Using Theorem 10.11, we can then conclude that

$$\begin{aligned}\int_{\partial B_1} \mathcal{G}(f_r, f_s)^2 &\leqslant C \left|\frac{\mathbf{D}(r)}{r^{2I_0}} - \frac{\mathbf{D}\left(\frac{r}{2}\right)}{\left(\frac{r}{2}\right)^{2I_0}}\right| + C \int_{\frac{r}{2}}^{r} \left[\frac{|I_0 - \mathbf{I}(t)|}{t} + C\,\frac{\mathcal{E}_{IV}(t)}{\mathbf{D}(t)} + C\,\frac{\mathcal{E}_{OV}(t)}{t\,\mathbf{D}(t)}\right] dt \\ &\leqslant C\, r^{\lambda}. \end{aligned} \tag{10.64}$$

By an elementary dyadic argument analogous to that of [17, Theorem 5.3], we then infer the existence of $f_0 : \partial B_1 \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ such that, for r sufficiently small,

$$\|\mathcal{G}(f_r, f_0)\|^2_{L^2(\partial B_1)} \leqslant C\, r^{\lambda}. \tag{10.65}$$

10.4.2 *Step 2: uniform convergence*

Set next $h(z,w) := \mathcal{G}\left(\frac{\mathcal{N}(z,w)}{|z|^{I_0}}, \frac{\mathcal{N}(i_{1/2}(z,w))}{|\frac{z}{2}|^{I_0}}\right)$. It follows from (10.64) that for r sufficiently small

$$\int_{B_r} h^2 \leqslant \int_0^r \int_{\partial B_1} \mathcal{G}(f_t, f_{\frac{t}{2}})^2\, t\, dt \overset{(10.64)}{\leqslant} C\, r^{2+\lambda}, \tag{10.66}$$

and from (8.1) and (8.2)

$$\mathrm{Lip}(h|_{B_1 \setminus B_s}) \leqslant C\, s^{-I_0}. \tag{10.67}$$

Moreover, for every $\rho < \frac{|z|}{4}$ we claim the estimate

$$\int_{B_\rho(z,w)} |Dh|^2 \leqslant C\,\rho + C\,|z|^{\lambda}. \tag{10.68}$$

Indeed $|Dh| \leqslant C \left|D\left(\frac{\mathcal{N}}{|z|^{I_0}}\right)\right|$ and by Theorem 10.11

$$\begin{aligned}\int_{B_\rho(z,w)} \left|D\left(\frac{\mathcal{N}}{|z|^{I_0}}\right)\right|^2 &\leqslant 2 \int_{|z|-\rho}^{|z|+\rho} \int_{\partial B_t} \left(\frac{|D\mathcal{N}|^2}{t^{2I_0}} + I_0^2\, \frac{|\mathcal{N}|^2}{t^{2I_0+2}}\right) dt \\ &\leqslant \int_{|z|-\rho}^{|z|+\rho} \left(\left(\frac{\mathbf{D}(t)}{t^{2I_0}}\right)' + 2\, I_0 \frac{\mathbf{D}(t)}{t^{2I_0+1}} + I_0^2\, \frac{\mathbf{H}(t)}{t^{2I_0+2}}\right) dt \\ &\leqslant C\ (|z|+\rho)^{\lambda} + C \log\left(\frac{|z|+\rho}{|z|-\rho}\right) \leqslant C\,|z|^{\lambda} + C\,\frac{\rho}{|z|}. \end{aligned}$$

In particular, applying (10.66), (10.67) and (10.68) with $\rho = |z|^{1+\frac{\lambda}{4}}$, we infer that for every point $p = (z, w) \in \mathfrak{B}_{\bar Q}$ with $|z|$ sufficiently small

$$\begin{aligned}
h(p) &\leqslant \Big|h(p) - \fint_{B_{\frac{|z|^{1+\frac{\lambda}{4}}}{2^k}}(p)} h\Big| + \sum_{i=0}^{k-1}\Big|\fint_{B_{\frac{|z|^{1+\frac{\lambda}{4}}}{2^i}}(p)} h - \fint_{B_{\frac{|z|^{1+\frac{\lambda}{4}}}{2^{i+1}}}(p)} h\Big| + \fint_{B_{|z|^{1+\frac{\lambda}{4}}}(p)} h \\
&\leqslant \mathrm{Lip}(h|_{B_1(p)\setminus B_{\frac{|z|}{2}}(p)})\frac{|z|^{1+\frac{\lambda}{4}}}{2^k} + C\sum_{i=0}^{k-1}\frac{|z|^{1+\frac{\lambda}{4}}}{2^i}\fint_{B_{\frac{|z|^{1+\frac{\lambda}{4}}}{2^i}}(p)} |Dh| + \fint_{B_{|z|^{1+\frac{\lambda}{4}}}(p)} h \\
&\overset{(10.67)}{\leqslant} C\,|z|^{1+\frac{\lambda}{4}} + C\sum_{i=0}^{k-1}\left(\int_{B_{|z|^{1+\frac{\lambda}{4}}}} |Dh|^2\right)^{\frac{1}{2}} + \frac{C}{|z|^{1+\frac{\lambda}{4}}}\left(\int_{B_{2|z|}} |h|^2\right)^{\frac{1}{2}}, \qquad (10.69)
\end{aligned}$$

where we have used the standard Poincaré inequality

$$\Big|\fint_{B_r} f - \fint_{B_{\frac{r}{2}}} f\Big| \leqslant C\,r\fint_{B_r}|Df| \qquad f \in W^{1,2}.$$

Now choose $k \in \mathbb{N}$ such that $\frac{|z|^{1+\frac{\lambda}{4}}}{2^k} < |z|^{1+\frac{\lambda}{4}+I_0} \leqslant \frac{|z|^{1+\frac{\lambda}{4}}}{2^{k-1}}$ (in particular $k \leqslant |\log|z||$) and use (10.66) together with (10.68) to bound

$$h(z,w) \leqslant C\,|z|^{1+\frac{\lambda}{4}} + C\,|\log|z||\,|z|^{\frac{\lambda}{8}} + C\,|z|^{\frac{\lambda}{4}} \leqslant C\,|z|^{\frac{\lambda}{16}}, \qquad (10.70)$$

This gives that, for a sufficiently small r,

$$\max_{\partial B_1} \mathcal{G}(f_r, f_{r/2}) \leqslant C r^{\frac{\lambda}{16}}.$$

Thus

$$\max_{\partial B_1} \mathcal{G}(f_r, f_0) \leqslant \sum_{k=0}^{\infty} \mathcal{G}(f_{r2^{-k}}, f_{r2^{-k-1}}) \leqslant C r^{\frac{\lambda}{16}}.$$

10.4.3 *Step 3: nontriviality of the limit and other properties*

To show that $f_0 \neq Q\,[\![0]\!]$ it is enough to observe that, by Theorem 10.11,

$$\int_{\partial B_1} |f_0|^2 = \lim_{r\to 0}\int_{\partial B_1}|f_r|^2 = \lim_{r\to 0}\frac{\mathbf{H}(r)}{r^{2I_0+1}} = H_0 > 0.$$

In order to show that $\boldsymbol{\eta}\circ f_0 \equiv 0$, we notice that by a simple slicing argument combined with (8.3) there exists a sequence of radii $r_k \in [2^{-k-1}, 2^{-k}]$ such that

$$\begin{aligned}
\int_{\partial B_{r_k}} |\boldsymbol{\eta}\circ\mathscr{N}| &\leqslant 2^{k+1}\int_{B_{2^{-k}}\setminus B_{2^{-k-1}}} |\boldsymbol{\eta}\circ\mathscr{N}| \leqslant C\,r_k^{\gamma_0}\int_{B_{2^{-k}}} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\mathscr{N}| \\
&\leqslant C\,r_k^{\gamma_0+2\eta}\mathbf{D}(2r_k) \leqslant C\,r_k^{\gamma_0+2\eta+2I_0}, \qquad (10.71)
\end{aligned}$$

from which

$$\int_{\partial B_1} |\boldsymbol{\eta} \circ f_0| = \lim_{r_k \to 0} \int_{\partial B_1} |\boldsymbol{\eta} \circ f_{r_k}| = \lim_{r_k \to 0} r_k^{-I_0 - 1} \int_{\partial B_r} |\boldsymbol{\eta} \circ \mathscr{N}|$$
$$\leqslant C \lim_{r_k \to 0} r_k^{\gamma_0 + 2\eta + I_0 - 1} = 0.$$

Next we show that f_0 takes values in $\mathbb{R}^{\bar{n}}$. We start by showing that f_0 must take values in $T_0\Sigma = \mathbb{R}^{2+\bar{n}} \times \{0\}$. Indeed, if we set $f_r(z,w) := \bar{\mathscr{N}}(i_r(z,w))$, using (10.84) and $|\mathscr{N}|(i_r(z,w)) \leqslant C\, r^{1+\frac{\gamma_0}{2}}$ we conclude

$$\int_{\partial B_1} \mathcal{G}(f_r, \bar{f}_r)^2 \leqslant \frac{Cr^2}{r^{2I_0+1}} \int_{\partial B_r} |\mathscr{N}|^2 \leqslant Cr^2\,,$$

which implies that $f_0(z,w) \in \mathcal{A}_Q(T_0\Sigma)$.

Next observe that $f_r(z,w) = \sum_i [\![\mathscr{N}_i(i_r(z,w))]\!]$ has the property that each $\mathscr{N}_i(i_r(z,w))$ is orthogonal to $T_{\boldsymbol{\Psi}(i_r(z,w))}\mathcal{M}$. In particular, if $|z| = 1$ and $r \downarrow 0$, the tangent planes $T_{\boldsymbol{\Psi}(i_r(z,w))}\mathcal{M}$ converge to $\mathbb{R}^2 \times \{0\}$: it follows, by the uniform convergence of f_r to f_0, that $f_0(z,w) = \sum_i [\![(f_0)_i(z,w)]\!]$ for some $(f_0)_i(z,w)$ which are orthogonal to $\mathbb{R}^2 \times \{0\}$. We thus conclude that each $(f_0)_i(z,w)$ belongs to $\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}$.

10.4.4 *Step 4: Minimality of* g

In order to complete the proof of Proposition 10.12 we need to show that g is Dir-minimizing. Given the homogeneity of g in the radial direction, it suffices to show that there is no $W^{1,2}$ multifunction $h : B_1 \to \mathcal{A}_Q(\mathbb{R}^{\bar{n}})$ which has the same trace of g on ∂B_1 and less energy on B_1. Assume thus by contradiction that there is an $h \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^{\bar{n}}))$ such that $h|_{\partial B_1}$ and

$$\int |Dh|^2 \leqslant \int |Dg|^2 - \delta \tag{10.72}$$

for some positive $\delta > 0$. Recall the definition of $W^{1,2}$ according to Remark 2.19: using the map $\boldsymbol{W}$ in there and the functions $h \circ \boldsymbol{W}$ and $g \circ \boldsymbol{W}$ we can use the theory of Chapter 3 and assume that $h \circ \boldsymbol{W}$ is a Dir-minimizer on the euclidean disk $D_1 \subset \mathbb{R}^2$. Observe also that, since $\boldsymbol{\eta} \circ g \equiv 0$, we must have $\boldsymbol{\eta} \circ h \equiv 0$ as well. Indeed since $h \circ \boldsymbol{W} = g \circ \boldsymbol{W}$ on ∂D_1, we have $\boldsymbol{\eta} \circ h \circ \boldsymbol{W} = \boldsymbol{\eta} \circ g \circ \boldsymbol{W} = 0$ on the boundary and considering that

$$\int_{D_1} \sum_i |D(h_i \circ \boldsymbol{W} - \boldsymbol{\eta} \circ h \circ \boldsymbol{W})|^2 \leqslant \int_{D_1} |D(h \circ \boldsymbol{W})|^2 - Q \int_{D_1} |D(\boldsymbol{\eta} \circ h \circ \boldsymbol{W})|^2\,,$$

the minimality of $h \circ \boldsymbol{W}$ forces the Dirichlet energy of $\boldsymbol{\eta} \circ h \circ \boldsymbol{W}$ to vanish identically.

Using (8.3), the decay $\mathbf{D}(r) \leqslant C\, r^{2I_0}$ and a Fubini-type argument we can find a sequence of radii $s_j \to 0$ such that

$$\int_{\partial B_1} |Df_0|^2 \leqslant \limsup_j \int_{\partial B_1} |Df_{s_j}|^2 \leqslant \limsup_j \frac{\mathbf{D}'(s_j)}{s_j^{2I_0-1}} \leqslant C\,. \tag{10.73}$$

We now wish to "smooth" h, i.e. to approximate it with a sequence of Lipschitz maps h_ε such that $\boldsymbol{\eta}\circ h_\varepsilon \equiv 0$,

$$\int_{B_1} |Dh_\varepsilon|^2 - |Dh|^2 \leqslant \varepsilon^2 \tag{10.74}$$

$$\int_{\partial B_1} \mathcal{G}(f_0, h_\varepsilon)^2 + \left|\int_{\partial B_1} |Df_0|^2 - |Dh_\varepsilon|^2\right| \leqslant \varepsilon^2 \,. \tag{10.75}$$

We would like to appeal to Lemma 3.11, but there is the slight technical complication that $\mathfrak{B}$ is not regular. We postpone this technical step and continue with the argument assuming the existence of the approximations h_ε.

Next we would like to apply Lemma 3.15 to h_ε and $\mathbf{p}_{T_0\Sigma}(f_{s_j}) =: \bar f_{s_j}$, to get a family of competitor functions $(\hat f_{s_j}) \subset W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^{2+\bar n}))$, such that $\hat f_{s_j}|_{\partial B_1} = \bar f_{s_j}|_{\partial B_1})$ and

$$\int_{B_1} |D\hat f_{s_j}|^2 \leqslant \int_{B_1} |Dh_\varepsilon|^2 + \varepsilon \int_{\partial B_1} \left(|D_\tau h_\varepsilon|^2 + |D_\tau \bar f_{s_j}|^2\right) + \frac{C}{\varepsilon}\int_{\partial B_1} \mathcal{G}(h_\varepsilon, \bar f_{s_j})^2 \,, \tag{10.76}$$

$$\mathrm{Lip}(\hat f_{s_j}) \leqslant C\left(\mathrm{Lip}(h_\varepsilon) + \mathrm{Lip}(\bar f_{s_j}) + \frac{1}{\varepsilon}\sup_{\partial B_1} \mathcal{G}(\bar f_{s_j}, h_\varepsilon)\right) \tag{10.77}$$

$$\boldsymbol{\eta}\circ \hat f_{s_j} = \boldsymbol{\eta}\circ \bar f_{s_j} \,. \tag{10.78}$$

Again, this is not straightforward because Lemma 3.15 is stated for euclidean domains. We postpone this second technical problem and continue with our argument assuming the existence of $\hat f_{s_j}$.

We are now ready to define our comptetior function. We set $\bar{\mathscr{L}}_{s_j}(z,w) := s_j^{I_0} \hat f_{s_j}(\iota_{\frac{1}{s_j}}(z,w))$ and, observing that $\bar{\mathscr{L}}_{s_j}$ takes value in $\mathcal{A}_Q(T_0\Sigma)$, we use (9.1) to define a corresponding $\mathscr{L}_{s_j}$, which clearly is a competitor $\mathscr{N}$ in B_{s_j} according to Definition 9.1. Moreover

$$\mathrm{Lip}(\mathscr{L}_{s_j}) \leqslant C\, s_j^{I_0+1} \mathrm{Lip}(\hat f_{s_j}|_{B_1}) \overset{(10.67)}{\leqslant} C\, s_j{}^{\eta} \,.$$

Therefore we can apply Proposition 9.2 with $\bar{\mathscr{L}} = \bar{\mathscr{L}}_{s_j}$. In particular, taking into account Theorem 10.11 and (10.73), we conclude that

$$\int_{B_{s_j}} |D\bar{\mathscr{N}}|^2 \leqslant (1 + Cs_j)\int_{B_{s_j}} |D\bar{\mathscr{L}}_{s_j}|^2 + C\mathbf{m}_0^{\frac12} \int_{B_{s_j}} |z|^{\gamma_0 - 1} |\boldsymbol{\eta}\circ \mathscr{L}_{s_j}| + Cs_j^{2I_0+\eta} \,.$$

Next, recall the inequality (9.41):

$$\int_{B_{s_j}} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\mathscr{L}_{s_j}| \leqslant C\int_{B_{s_j}} |z|^{\gamma_0-1} |\boldsymbol{\eta}\circ\bar{\mathscr{L}}_{s_j}| + C\int_{B_{s_j}} |z|^{\gamma_0-1}|\bar{\mathscr{L}}_{s_j}|^2 \,.$$

By (10.78) the first term in the right hand side equals indeed

$$C\int_{B_{s_j}} |z|^{\gamma_0-1}|\boldsymbol{\eta}\circ\bar{\mathscr{N}}| \leqslant Cs_j^{\eta}\,\mathbf{D}(s_j) \leqslant Cs_j^{2I_0+\eta} \,.$$

For the second term we use the Poincaré inequality

$$\int_{B_{s_j}} |z|^{\gamma_0-1}|\bar{\mathscr{L}}_{s_j}|^2 \leqslant Cs_j^{1+\gamma_0}\int_{B_{s_j}} |D\bar{\mathscr{L}}_{s_j}|^2 + Cs_j^{\gamma_0}\int_{\partial B_{s_j}} |\bar{\mathscr{L}}_{s_j}|^2 \,, \tag{10.79}$$

whose proof is given in Lemma 10.13.

Using that

$$\int_{\partial B_{s_j}} |\bar{\mathscr{L}}_{s_j}|^2 = \int_{\partial B_{s_j}} |\bar{\mathscr{N}}|^2 = \mathbf{H}(s_j) \leqslant C s_j^{2I_0+1}\,,$$

we easily conclude that

$$\int_{B_{s_j}} |D\bar{\mathscr{N}}|^2 \leqslant (1 + C s_j) \int_{B_{s_j}} |D\bar{\mathscr{L}}_{s_j}|^2 + C s_j^{2I_0+\eta}\,. \tag{10.80}$$

Changing variables and dividing by $s_j^{2I_0}$ we infer that

$$\int_{B_1} |D\bar{f}_{s_j}|^2 \leqslant \int_{B_1} |D\hat{f}_{s_j}|^2 + C s_j^{\eta}\,. \tag{10.81}$$

Using (10.74), (10.75) and (10.76), we conclude

$$\begin{aligned}\int_{B_1} |D\bar{f}_{s_j}|^2 &\leqslant \int_{B_1} |Dh|^2 + C s_j^{\eta} + C\varepsilon + \frac{C}{\varepsilon}\int_{\partial B_1} \mathcal{G}(f_0, \bar{f}_{s_j})^2 \\ &\leqslant \int_{B_1} |Dg|^2 - \delta + C s_j^{\eta} + C\varepsilon + \frac{C}{\varepsilon}\int_{\partial B_1} \mathcal{G}(f_0, \bar{f}_{s_j})^2\,,\end{aligned}$$

where the constant C is independent of ε. In particular, if we fix ε sufficiently small and we then let $s_j \downarrow 0$, by the uniform convergence of f_{s_j} to f_0 on ∂B_1 we conclude

$$\limsup_{j\to\infty} \int_{B_1} |D\bar{f}_{s_j}|^2 \leqslant \int_{B_1} |Dg|^2 - \frac{\delta}{2}\,.$$

Since however $f_{s_j} \to g$ in B_1, the latter inequality contradicts the semicontinuity of the Dirichlet energy.

10.4.5 *Step 5: Technical leftovers*

First of all we show the existence of the map h_ε as in (10.74) and (10.75). We consider $h \circ \boldsymbol{W}$, which is defined on the closed unit disk $\bar{D}_1 \subset \mathbb{R}^2$. We then can apply Lemma 3.11 to the latter map and generate approximations $\hat{h}_\varepsilon$ which satisfy the bounds (10.74) and (10.75) with D_1 in place of B_1 and $h \circ \boldsymbol{W}$ in place of h. The maps $h_\varepsilon := \hat{h}_\varepsilon \circ \boldsymbol{W}$ would then satisfy the desired estimates because of the conformality of $\boldsymbol{W}^{-1}$ (which keeps the Dirichlet energy invariant) and its regularity in $B_1 \setminus \{0\}$ (which results into the loss of a constant factor in (10.75)). However the resulting map would not be Lipschitz because of the singularity of $\boldsymbol{W}^{-1}$ in the origin. To overcome this difficulty it suffices to perturb slightly $\hat{h}_\varepsilon$ so that it is constant in a small neighborhood of the origin. As for the condition $\boldsymbol{\eta} \circ h_\varepsilon \equiv 0$, this can easily be achieved subtracting the average to whichever extension satisfies (10.74) and (10.75).

Secondly we show the existence of $\hat{f}_{s_j}$. First of all we observe that the condition (10.78) can be easily achieved after we prove the existence of a map which satisfies the other two conditions: as above it suffices to subtract the average of this map and add back $\boldsymbol{\eta} \circ \bar{f}_{s_j}$. At this point we observe that it suffices, as above, to compose with the map $\boldsymbol{W}$, apply [17, Lemma 2.14] and Lemma 3.15 and compose the resulting map with $\boldsymbol{W}^{-1}$: indeed the latter would coincide with $h_\varepsilon \circ \boldsymbol{W}$ on $D_{1-\varepsilon}$ and on the complement $\boldsymbol{W}^{-1}$ is regular.

10.5 APPENDIX A: SOME USEFUL LEMMAS.

The first lemma is a simple version of the Poincaré inequality for $W^{1,2}$ functions.

Lemma 10.13. *There exists a universal constant* $C > 0$ *such that for every* $f \in W^{1,2}(B_r, \mathcal{A}_Q)$*, where* $B_r \subset \mathfrak{B}_Q$*, the following two inequalities hold*

$$\int_{B_r} |f|^2 \leqslant Cr^2 \int_{B_r} |Df|^2 + Cr \int_{\partial B_r} |f|^2 \tag{10.82}$$

$$\int_{B_r} |z|^{\gamma_0 - 1} |f|^2 \leqslant C\, r^{1+\gamma_0} \int_{B_r} |Df|^2 + C\, r^{\gamma_0} \int_{\partial B_1} |f|^2 \,. \tag{10.83}$$

Proof. By approximation we can assume, without loss of generality, that f is Lipschitz and, by scaling, it suffices to show the inequalities (10.82) and (10.83) on the ball B_1. Fixing $|z| = 1$ and integrating along rays

$$|f(rz, r^{1/Q}w)|^2 \leqslant 2|f(z,w)|^2 + 2\int_r^1 |Df(tz, t^{1/Q}w)|^2\, dt\,.$$

Using radial coordinates we then conclude

$$\int_{B_1} |z|^{\gamma_0 - 1}|f|^2 \leqslant C \int_{\partial B_1} |f|^2 + \int_{\partial B_1}\int_0^1 r_0^{\gamma} \int_r^1 |Df(tz, t^{\frac{1}{Q}}w)|^2\, dt\, dr\, dz\,.$$

Using Fubini the latter integral can be rewritten as

$$\int_0^1 \int_{\partial B_1} |Df(tz, t^{\frac{1}{Q}}w)|^2 \int_0^t r^{\gamma_0}\, dt\, dz\, dr \leqslant \int_0^1 t \int_{\partial B_1} |Df(tz, t^{\frac{1}{Q}}w)|^2\, dz\, dr\,.$$

This completes the proof of (10.83). The proof of (10.82) is a simple variation of this one and is left to the reader. □

Lemma 10.14. *Let* $\bar{\mathscr{L}} \colon \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbb{R}^{2+\bar{n}})$ *be Lipschitz and consider the map* $\mathscr{L} \colon \mathfrak{B}_{\bar{Q}} \to \mathcal{A}_Q(\mathbb{R}^{2+n})$ *defined by* (9.1)*. Then there exists a constant* $C := C(\|\Psi_0\|_{C^3}) > 0$ *such that*

$$\mathcal{G}(\mathscr{L}, \bar{\mathscr{L}})(z,w) \leqslant C\, r\, |\bar{\mathscr{L}}|(z,w) + C\, |\bar{\mathscr{L}}|^2(z,w)\,, \quad \forall (z,w) \in B_r \tag{10.84}$$

$$\int_{B_r} |D\mathscr{L}|^2 \leqslant (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + C\, r \int_{\partial B_r} |\bar{\mathscr{L}}|^2\,. \tag{10.85}$$

Proof. For what concerns (10.84), observe that $D\Psi(0) = 0$ implies $\|D\Psi_0\|_{L^\infty(B_r))} \leqslant Cr$. Therefore, by the C^3 regularity of Ψ_0, we get

$$\begin{aligned}
\mathcal{G}(\mathscr{L}, \bar{\mathscr{L}})(z,w) &= \sum_{j=1}^{Q} |\Psi(\mathbf{p}_0(\mathbf{\Psi}) + \bar{\mathscr{L}}_j) - \Psi_0(\mathbf{p}_0(\mathbf{\Psi}))|(z,w) \\
&\leqslant \|D\psi\|(\mathbf{\Psi}(z,w))\, |\bar{\mathscr{L}}|(z,w) + \|A_\Sigma\|\, |\bar{\mathscr{L}}|^2(z,w) \\
&\leqslant C\, r\, |\bar{\mathscr{L}}|(z,w) + C\, |\bar{\mathscr{L}}|^2\,.
\end{aligned}$$

An analogous computation gives

$$\int_{B_r} |D\mathscr{L}|^2 \leqslant (1 + Cr) \int_{B_r} |D\bar{\mathscr{L}}|^2 + C \int_{B_r} |\bar{\mathscr{L}}|^2$$

and we conclude (10.85) using Lemma 10.13. □

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