BLOWUP OF THE BV NORM IN THE MULTIDIMENSIONAL KEYFITZ AND KRANZER SYSTEM

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Abstract
We consider the Cauchy problem for the system $\partial_t u_i + \mathrm{div}_z(g(|u|)u_i) = 0$ $i \in \{1, \dots, k\}$*, in* m *space dimensions and with* $g \in C^3$*. When* $k \geq 2$ *and* $m = 2$*, we show a wide choice of* g*'s for which the bounded variation (BV) norm of admissible solutions can blow up, even when the initial data have arbitrarily small oscillation and arbitrarily small total variation, and are bounded away from the origin. When* $m \geq 3$*, we show that this occurs whenever* g *is not constant, that is, unless the system reduces to* k *decoupled transport equations with constant coefficients.*

1. Introduction

Let us consider the system of conservation laws

$$\partial_t U + \mathrm{div}_z[F(U)] = 0, \qquad U : \Omega \subset \mathbf{R}_t \times \mathbf{R}^m_z \to \mathbf{R}^k_u, \tag{1}$$

where $F : \mathbf{R}^k \to \mathbf{R}^{k\times m}$ is a C^1-function. In what follows we use the notation $F = (F_1, \dots, F_m)$, where each F_i is a map from $\mathbf{R}^k$ to $\mathbf{R}^k$, and we restrict our attention to *symmetric* systems, that is, systems for which $DF_i|_V$ is a symmetric matrix for every i and for every $V \in \mathbf{R}^k$. It is known, by a result of Rauch [14] based on a previous paper of Brenner [6] for linear hyperbolic systems, that certain types of BV estimates (and L^p-estimates for $p \neq 2$) fail for all the systems (1) which do not satisfy the commutator conditions

$$DF_i|_V \cdot DF_j|_V = DF_j|_V \cdot DF_i|_V \quad \text{for every } V \in \mathbf{R}^k. \tag{2}$$

When $n = 2$, it was proved in [8] that (2) is also sufficient to get L^p-estimates for every $p \leq 2$ and, under additional conditions, also for $p = \infty$.

A particular class of maps F which satisfy the commutator condition (2) is given by $F(v) = g(|v|) \otimes v$, where $g \in C^1(\mathbf{R}, \mathbf{R}^m)$. Note that in this case the requirement

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$F \in C^1$ implies $g'(0) = 0$. However, in the rest of the paper we do not impose this condition since it is not needed in any of the proofs.

Under this particular choice of F, the system (1) becomes

$$\begin{cases} \partial_t u + \mathrm{div}_z[g(|u|)u] = 0, \\ u(0, \cdot) = u_0, \end{cases} \qquad u : \mathbf{R}_t^+ \times \mathbf{R}_z^m \to \mathbf{R}^k. \tag{3}$$

A natural *formal* procedure to construct weak solutions to (1) consists in the following two steps.

- First, we impose the fact that $\rho := |u|$ solves, in the sense of Kruzhkov, the Cauchy problem for the scalar law:

$$\begin{cases} \partial_t \rho + \mathrm{div}_z[g(\rho)\rho] = 0, \\ \rho(0, \cdot) = |u_0|. \end{cases} \tag{4}$$

- Second, we impose the fact that $\theta := u/|u|$ solves the transport equation

$$\begin{cases} \partial_t (\rho\theta) + \mathrm{div}_z[g(\rho)\rho\theta] = 0, \\ \theta(0, \cdot) = u_0/|u_0|. \end{cases} \tag{5}$$

Following [12], we call such a solution $u = \rho\theta$ a *renormalized entropy* solution. More precisely, we use the following definition.

Definition 1.1
We say that u is a renormalized entropy solution of (3) if $\rho := |u|$ is a Kruzhkov solution of (4).

Renormalized entropy solutions are entropy (or admissible) solutions in the classical sense of the theory of hyperbolic systems of conservation laws (see Section 2). In [7] Bressan showed a Lipschitz map $F : \mathbf{R}^2 \to \mathbf{R}^{2\times 2}$ of the form $g(|u|) \otimes u$ for which the Cauchy problem for renormalized entropy solutions is ill posed in L^∞. Stimulated by this work, in [2] the authors proved that such a Cauchy problem is well posed in the space $X := \{u \in L^\infty : |u| \in \mathrm{BV}_{\mathrm{loc}}\}$ using the extension of DiPerna-Lions theory to BV fluxes recently achieved in [1].

In one space dimension, the fundamental result of Glimm (see [9]) gives the existence of BV entropy solutions if one starts with initial data that have sufficiently small total variation. Hence it is natural to ask whether renormalized entropy solutions u of (3) enjoy BV regularity when the whole initial datum u_0 (and not only its modulus) belongs to BV. In analogy with the one-dimensional case, one could ask whether such regularity holds at least for small times and when u_0 is close to a constant, uniformly and in the BV norm.

Note that

- in our case, Rauch's result does not apply since F satisfies condition (2);
- in any case, when Rauch's result applies, it does not exclude BV regularity, but it implies that estimates of a certain kind are not available;
- one might wonder whether the notion of renormalized entropy solution is a suitable one, and thus we can ask whether BV regularity holds for some other notion of admissible solutions.

In this paper we prove in Theorem 2.4 that for $k, m \geq 2$ and for a wide choice of g's, there are initial data u_0 (with arbitrarily small total variation and uniformly close to a constant $c \neq 0$) such that the BV norm of admissible solutions u of (3) blows up instantaneously, no matter what criterion of admissibility is chosen among the ones proposed in the literature. When m in (3) is strictly bigger than 2, such initial data can be constructed for every g which is not constant, that is, for every system of the form (3) which does not reduce to k decoupled transport equations with constant coefficients.

2. Statement of the result and preliminaries

2.1. Statement of the result

In order to state Theorem 2.4, we first need some definitions.

Definition 2.1

A pair of functions $\eta \in W^{1,\infty}(\mathbf{R}^k, \mathbf{R})$, $q \in W^{1,\infty}(\mathbf{R}^k, \mathbf{R}^m)$ is called an *entropy–entropy flux pair* for (1) if $Dq = D\eta \cdot DF$.

General hyperbolic systems of conservation laws may not have any entropy–entropy flux pair besides the ones where η is an affine function. However, the systems (3) always have an abundance of entropies (see Remark 2.3).

Definition 2.2

Let $U_0 \in L^\infty$. Then U is called an *entropy solution* of

$$\begin{cases} \partial_t U + \mathrm{div}_z[F(U)] = 0, \\ U(0, \cdot) = U_0 \end{cases} \tag{6}$$

if, for every convex entropy–entropy flux pair η, q and for every smooth test function

$\psi \geq 0$,

$$\int_{t>0}\int_{\mathbf{R}^m} \big[\partial_t \psi(t,z)\, \eta(U(t,z)) + \nabla_z \psi(t,z) \cdot q(U(t,z))\big]\, dt\, dz \\ + \int_{\mathbf{R}^m} \psi(0,z)\eta\big(U_0(z)\big)\, dz \geq 0. \tag{7}$$

The (nonpositive) *entropy production* measure $\partial_t[\eta(U)] + \mathrm{div}_z[q(U)]$ is denoted by μ_η.

Remark 2.3
Consider F of the form $F(u) = g(|u|) \otimes u$. Then there are two natural families of entropies for the system (3).

- The first one is given by entropies of the form $\xi(|u|)$, where $\xi \in W^{1,\infty}(\mathbf{R})$. The related entropy fluxes are of the form $\omega(|u|)$, where $\omega'(u) = \xi'(u)ug'(u) + \xi'(u)g(u)$. Hence (ξ, ω) is an entropy–entropy flux pair for the scalar law (4).
- The second family is given by entropies of the form $|u|G(u/|u|)$, where $G \in W^{1,\infty}(\mathbf{S}^{k-1})$. The related entropy fluxes are of the form $g(|u|)|u|G(u/|u|)$.

Clearly, any linear combination of the entropies above is an entropy as well. Furthermore, [12, Lemma 1.1] shows that every entropy η for the system (3) is of the form $\xi(|u|) + |u|G(u/|u|)$. Note that, if η is convex, then ξ must be also convex. From the proof of [2], it follows that for all entropies of the form $|u|G(u/|u|)$, the entropy production vanishes. Hence, since $|u|$ is an entropy solution of the corresponding scalar law, we conclude that *renormalized entropy* solutions of (3) are indeed *entropy* solutions. However, there are some entropy solutions that are not renormalized. This happens already in one space dimension and is due to the hyperbolic degeneracy of the system (3) at the origin (see, for instance, the example in [7] or [9, p. 188, top]).

The chain rule of Vol'pert for BV functions implies that if U is a BV entropy solution of (6), then $\mu_\eta(\Omega \setminus J_U) = 0$; that is, μ_η "lives" on the jump set J_U (which in this case is called the set of shocks of the solution U). We refer to Section 2.2 for the relevant definitions. For BV weak solutions U of (1), other criteria of admissibility have been studied in the literature. We remark that all of them are based on imposing some conditions on the set of shocks and that these conditions turn out to be stronger than the entropy ones. Hence the notion of entropy solution is the weakest among the ones proposed in the literature. We are now in a position to state the main result of this paper.

THEOREM 2.4
Let $k \geq 2$, $m \geq 3$, $g \in C^3_{\mathrm{loc}}$, and let $c \in \mathbf{R}^k \setminus \{0\}$ be such that $g'(|c|) \neq 0$. Then there exists a sequence of initial data $u_0^n : \mathbf{R}^m \to \mathbf{R}^k$ such that

- $\|u_0^n - c\|_{\mathrm{BV}(\mathbf{R}^m)} + \|u_0^n - c\|_\infty \to 0$ *for* $n \uparrow \infty$*;*
- $u_0^n = c$ *on* $\mathbf{R}^m \setminus B_R(0)$ *for some* $R > 0$ *independent of* n*;*
- *if* u^n *is* any *bounded entropy solution of (3) with initial data* u_0^n*, then there exists* $r > 0$ *(independent of* n*) such that* $\|u^n\|_{\mathrm{BV}(]0,T[\times B_r(0))} = \infty$ *for every positive* T*.*

When $m = 2$*, the same statement holds if in addition we assume that* $g''(|c|)$ *is parallel to* $g'(|c|)$ *(or vanishes).*

2.2. BV functions and Vol'pert's chain rule

For definitions, statements, and proofs of the claims contained in this section, we refer to the book [5]. In the rest of the paper, $\mathscr{L}^m$ denotes the Lebesgue measure on $\mathbf{R}^m$ and $\mathscr{H}^n$ the n-dimensional Hausdorff measure.

When $u : \mathbf{R}^l \to \mathbf{R}^k$ is a BV function, we denote by $|\partial_{x_i} u|$ the total variation measure of the partial derivative $\partial_{x_i} u$. Moreover, we define S_u as the set of points x where u is not approximately continuous and J_u as the set of points x where u jumps, that is, where it has approximate right and left limits with respect to some fixed direction $\nu(x) \in \mathbf{S}^{l-1}$. The BV structure theorem states that J_u is a codimension 1 rectifiable set and that $\mathscr{H}^{l-1}(S_u \setminus J_u) = 0$, where $\mathscr{H}^{l-1}$ denotes the $(l-1)$-dimensional Hausdorff measure. Moreover, $\nu(x)$ can be chosen so that $\nu : J_u \to \mathbf{S}^{l-1}$ is a Borel measurable function and coincides with the approximate normal to J_u. With this choice of $\nu(x)$, we denote by $u^+(x)$ and $u^-(x)$, respectively, the right and left approximate limits of u at x. For any $x \notin S_u$, we denote by $\tilde{u}(x)$ the approximate limit of u at x. The BV structure theorem implies also that the matrix-valued measure Du can be written as

$$Du = \mu + [(u^+ - u^-) \otimes \nu]\, \mathscr{H}^{l-1} \llcorner J_u, \tag{8}$$

where

- $\mathscr{H}^{l-1} \llcorner J_u$ denotes the Borel measure given by

$$\mathscr{H}^{l-1} \llcorner J_u\,(A) = \mathscr{H}^{l-1}(A \cap J_u) \quad \text{for any Borel set } A;$$

- $\mu(A) = 0$ for any Borel set A such that $\mathscr{H}^{l-1}(A) < \infty$.

We denote μ by $D_d u$, and we call it the *diffused part of the measure* Du. (In the notation of [5] this measure is just the sum of the absolutely continuous part $D_a u$ and the Cantor part $D_c u$.) We can now state the chain rule of Vol'pert for BV functions.

THEOREM 2.5
Let $u \in \mathrm{BV}(\mathbf{R}^l, \mathbf{R}^k)$ *and* $\eta \in C^1(\mathbf{R}^k, \mathbf{R}^h)$ *be globally Lipschitz with* $\eta(0) = 0$*. Then* $\eta \circ u \in \mathrm{BV}$ *and*

$$D[\eta \circ u] = [D\eta \circ \tilde{u}] \cdot D_d u + \big\{[\eta(u^+) - \eta(u^-)] \otimes \nu\big\}\, \mathscr{H}^{l-1} \llcorner J_u. \tag{9}$$

Remark 2.6

In [3] the authors proved a suitable extension of Theorem 2.5 to $\eta \in W^{1,\infty}$. In what follows we sometimes consider the measures $D[\eta \circ u]$ for η which are merely Lipschitz. However, we do not need the general result of [3] since in all the cases considered in this paper we are able to use some ad hoc considerations.

2.3. Riemann problem for scalar laws

Let us consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x [h(\rho)] = 0, \\ \rho(0, \cdot) = \rho^{\mathrm{in}}, \end{cases} \qquad \rho : \mathbf{R}^+_t \times \mathbf{R}^m_x \to \mathbf{R}, \tag{10}$$

where $h : \mathbf{R} \to \mathbf{R}^m$ is of class C^3. Fix $\beta, \gamma, \alpha \in \mathbf{R}$, set $\varepsilon := \max\{|\alpha - \beta|, |\alpha - \gamma|\}$, and define

$$\rho^{\mathrm{in}}(x_1, \dots, x_m) = \begin{cases} \beta & \text{for } x_m < 0, \\ \gamma & \text{for } x_m > 0. \end{cases}$$

Consider the entropy solution ρ of (10). It is easy to see that ρ depends only on t and x_m. For each $T > 0$, define

$$\xi := \max\bigl\{x_m \bigm| \rho(T, \cdot, x_m) = \beta\bigr\},$$
$$\eta := \min\bigl\{x_m \bigm| \rho(T, \cdot, x_m) = \gamma\bigr\}.$$

Then the following lemma holds.

LEMMA 2.7

If we denote by h'_m and h''_m the mth components of the vector-valued functions h' and h'', then there exist constants C and δ (depending only on h) such that

$$\max\bigl\{|\xi - T h'_m(\alpha)|, |\eta - T h'_m(\alpha)|\bigr\} \le 2|h''_m(\alpha)|\varepsilon + C\varepsilon^2 \quad \textit{for } \varepsilon \le \delta. \tag{11}$$

2.4. Regular Lagrangian flows

As explained in the introduction, renormalized entropy solutions can be formally constructed by solving first the scalar conservation law (4) and then the transport equation (5).

Let u be a renormalized entropy solution of (3). Assume that the initial data u_0 is bounded away from the origin, that is, that $|u_0| \ge c > 0$. Then, from the maximum principle for scalar conservation laws, it turns out that the renormalized entropy solution u is bounded away from zero as well, that is, that $|u| \ge c > 0$. Hence we can define the angular parts $\theta_0 := u_0/|u_0|$, $\theta := u/|u|$, and we conclude that θ solves the transport equation (5). In [4] the authors showed the existence of a locally bounded map $\Psi : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ such that

- if $\theta_0 \in L^\infty(\mathbf{R}^m)$, then $\tilde{\theta}(t,x) := \theta_0(\Psi(t,x))$ is a weak solution of (5).

In view of [2, Lemma 4.7], which gives the uniqueness of weak solutions to (5), necessarily $\tilde{\theta} = \theta$. In this paper we use the fact that $\Psi(t,\cdot)$ is invertible (in a suitable sense) for a.e. t and that if we denote by $\Phi(t,\cdot)$ the inverse of $\Psi(t,\cdot)$, then Φ satisfies the ordinary differential equation (ODE)

$$\begin{cases} \frac{d}{dt}\Phi(s,x) = g\big(\rho(s,\Phi(s,x))\big), \\ \Phi(0,x) = x, \end{cases} \tag{12}$$

in the sense of Proposition 2.9. This property is not explicitly stated in either [4] or [2], though it can be derived from the analysis of [2] (which combines the results of [1] with suitable extensions of the ideals of [11]). Such a Φ is called *regular Lagrangian flow*.

First, we need the following stability property.

PROPOSITION 2.8

Assume that $\{f_n\} \subset C^\infty$ is a uniformly bounded sequence and that $f_n \to g(\rho)$ in L^1_{loc}. Let Φ_n be the solutions of the ODEs

$$\begin{cases} \frac{d}{dt}\Phi_n(s,x) = f_n\big(s,\Phi_n(s,x)\big), \\ \Phi_n(0,x) = x. \end{cases} \tag{13}$$

If for some constant C we have $C^{-1} \le \det \nabla_x \Phi_n \le C$, then Φ_n converges to a map Φ strongly in L^1_{loc}. Moreover, $\Phi(s,\Psi(s,x)) = \Psi(s,\Phi(s,x)) = x$ for $\mathscr{L}^{m+1}$-a.e. (s,x).

Proof

From the results of [1], it follows that the renormalization [2, Conjecture 4.3] holds for $(\rho, \rho g(\rho))$. Hence it follows from the proof of [2, Proposition 4.4] that $\{\Phi_n\}$ is strongly precompact in L^1_{loc}. Denote by Φ the limit point of a subsequence of $\{\Phi_n\}$.

Note that from the proof of Proposition 4.4 it follows that

- there exists a $\tilde{\Psi}$ such that $\tilde{\Psi}(s,\Phi(s,x)) = \Phi(s,\tilde{\Psi}(s,x)) = x$ for $\mathscr{L}^{m+1}$-a.e. (s,x);
- for any $\theta_0 \in L^\infty$, the function $\tilde{\theta}(t,x) := \theta_0(\tilde{\Psi}(t,x))$ is a weak solution of (5).

Thus, [2, Lemma 4.7] (which gives the uniqueness of weak solutions of (5)) implies that $\tilde{\Psi}$ coincides with Ψ and that Φ is independent of the subsequence. □

PROPOSITION 2.9

For $\mathscr{L}^m$-a.e. x, we have the following:

(a) $\Phi(\cdot,x)$ *is Lipschitz (and hence it is differentiable in t for $\mathscr{L}^1$-a.e. t);*

(b) $(t,\Phi(t,x))$ *is a point of approximate continuity of ρ for $\mathscr{L}^1$-a.e. t;*

(c) $\frac{d}{dt}\Phi(t,x) = g(\rho(t,\Phi(t,x))$ *for* $\mathscr{L}^1$*-a.e.* t.

Proof
By a standard smoothing argument, it is easy to produce a sequence of smooth maps f_n, ρ_n such that

- $C^{-1} \le \rho_n \le C$ for some constant C independent of n;
- f_n is uniformly bounded and converges to $g(\rho)$ strongly in L^1_{loc};
- $\partial_t \rho_n + \text{div}(f_n \rho_n) = 0$ on $\mathbf{R}^+ \times \mathbf{R}^m$.

Denote by Φ_n the solution of (13), and set $J_n := \det(\nabla_x \Phi_n)$. From Liouville's theorem it follows that $\partial_t J_n + \text{div}(f_n J_n) = 0$. Since $J_n(0,\cdot) = 1$, the maximum principle applied to the continuity equation $\partial_t w + \text{div}(f_n w) = 0$ yields the fact that $C^{-1}\rho_n \le J_n \le C\rho_n$, and hence $C^{-2} \le J_n \le C^2$.

Step 1. We can apply Proposition 2.8 and conclude that $\Phi_n \to \Phi$ strongly in L^1_{loc}. Since for every x the curves $\Phi_n(\cdot,x)$ are uniformly Lipschitz, we conclude that $\Phi(\cdot,x)$ is a Lipschitz curve for $\mathscr{L}^m$-a.e. x. This gives (a).

Step 2. Fix a t and a subsequence (not relabeled) of $\Phi_n(t,\cdot)$ which converges to $\Phi(t,\cdot)$ in $L^1_{\text{loc}}(\mathbf{R}^m)$. (Such a subsequence exists for $\mathscr{L}^1$-a.e. t.) Let $E \subset \mathbf{R}^m$ be an open set. It is not difficult to show that

$$\mathscr{L}^m\big(\Phi(t,\cdot)^{-1}(E)\big) \le \limsup_{n\uparrow\infty} \mathscr{L}^m\big(\Phi_n(t,\cdot)^{-1}(E)\big) \le C^2\mathscr{L}^m(E). \tag{14}$$

Hence, for $\mathscr{L}^1$-a.e. t, this bound holds for every open set E. This property gives (b).

Step 3. The strong convergence of Φ_n implies that if $h_n \in C(\mathbf{R}\times\mathbf{R}^m)$ converges locally uniformly to $h \in C(\mathbf{R}\times\mathbf{R}^m)$, then $h_n(\cdot,\Phi_n)$ converges to $h(\cdot,\Phi)$ strongly in L^1_{loc}. If $h_n \to h$ strongly in L^1_{loc} and it is uniformly bounded, applying Egorov's theorem we find a closed set E such that h_n converges locally uniformly to h on E and $\mathscr{L}^{m+1}(\mathbf{R}\times\mathbf{R}^m \setminus E)$ is as small as desired. Recall that Φ_n is locally uniformly bounded. Applying Step 2, it follows that $f_n(\cdot,\Phi_n)$ converges strongly to $f(\cdot,\Phi)$.

Step 4. Since Φ_n solves (13), we have

$$\Phi_n(t,x) = x + \int_0^t f_n\big(\tau,\Phi_n(\tau,x)\big)\,d\tau. \tag{15}$$

From Step 3 we can find a subsequence (not relabeled) of $\{\Phi_n\}$ such that $f_n(\cdot,\Phi_n)$ converges to $g(\rho(\cdot,\Phi))$ pointwise almost everywhere on $\mathbf{R}\times\mathbf{R}^m$. From the dominated convergence theorem we get

$$\Phi(t,x) = x + \int_0^t g\big(\rho(\tau,\Phi(\tau,x))\big)\,d\tau \quad \text{for } \mathscr{L}^{m+1}\text{-a.e. } (t,x).$$

From this identity we easily conclude (c). □

3. Blowup of the BV norm for renormalized entropy solutions

In this section we prove the following proposition.

PROPOSITION 3.1

Let $k \geq 2$, $m \geq 3$, and $g \in C^3_{\text{loc}}$. Then, for every $c \in \mathbf{R}^k \setminus \{0\}$ such that $g'(|c|) \neq 0$, there exists a sequence of initial data $u_0^n : \mathbf{R}^m \to \mathbf{R}^k$ such that

- $\|u_0^n - c\|_{\mathrm{BV}(\mathbf{R}^m)} + \|u_0^n - c\|_\infty \to 0$ *for* $n \uparrow \infty$*;*
- $u_0^n = c$ *on* $\mathbf{R}^m \setminus B_R(0)$ *for some* $R > 0$ *independent of* n*;*
- *if* u^n *denotes the unique renormalized entropy solution of (3) with* $u^n(0, \cdot) = u_0^n$*, then there exists* $r > 0$ *such that* $u^n(t, \cdot) \notin \mathrm{BV}(B_r(0))$ *for every* n *and for every* $t \in]0, 1[$*.*

When $m = 2$*, the same statement holds if, in addition,* $g''(|c|)$ *is parallel to* $g'(|c|)$ *or* $g''(|c|) = 0$*.*

Before giving the proof of Proposition 3.1 we consider the special case of system (3) when $g = (f, 0, \ldots, 0)$, that is,

$$\begin{cases} \partial_t u + \partial_{x_1}[f(|u|)u] = 0, \\ u(0, \cdot) = u_0. \end{cases} \tag{16}$$

The following is a corollary of Proposition 3.1.

PROPOSITION 3.2

Let $k \geq 2$, $m \geq 2$, and $c \in \mathbf{R}^k \setminus \{0\}$ be such that $f'(|c|) \neq 0$. Then there exists a sequence of initial data $u_0^n : \mathbf{R}^m \to \mathbf{R}^k$ such that

- $\|u_0^n - c\|_{\mathrm{BV}(\mathbf{R}^m)} + \|u_0^n - c\|_\infty \to 0$ *for* $n \uparrow \infty$*;*
- $u_0^n = c$ *on* $\mathbf{R}^m \setminus B_R(0)$ *for some* $R > 0$ *independent of* n*;*
- *if* u^n *denotes the unique renormalized entropy solution of (16) with* $u^n(0, \cdot) = u_0^n$*, then there exists* $r > 0$ *such that* $u^n(t, \cdot) \notin \mathrm{BV}_{\text{loc}}(B_r(0))$ *for every* n *and for every* $t \in]0, 1[$*.*

Roughly speaking, the proof of Proposition 3.1 is based on the following remark: When $m = 3$, we can choose initial data, close to a constant, in such a way that the behavior of the renormalized entropy solutions of (3) is close to the behavior of solutions of (16). This seems to be no longer true for $m = 2$ unless $g''(|c|)$ is parallel to $g'(|c|)$ (or $g''(|c|) = 0$). Due to this remark, we choose to give a quick self-contained proof of Proposition 3.2.

Remark 3.3
Concerning the behavior of u^n for large times, in the case of Proposition 3.2 one can construct initial data u_0^n such that $u^n(t, \cdot) \notin \mathrm{BV}_{\mathrm{loc}}$ for any positive time $t > 0$. In the case of Proposition 3.1, it is difficult to track what happens for large times since in order to carry on our proof we need the fact that the rarefaction waves generated by $|u_0|$ do not interact.

3.1. Proof of Proposition 3.2
In the following, for any real number α, we denote by $[\alpha]$ the largest integer that is less than or equal to α.

For the sake of simplicity, we prove the proposition when $m = 2$, $f'(|c|) = 1$, and $f(|c|) = 0$. Only minor adjustments are needed to handle the general case. To simplify the notation, on $\mathbf{R}^2$ we use the coordinates (x, y) in place of (x_1, x_2).

Let $\{m_i\}$ be a sequence of positive even numbers such that

$$\sum_i m_i 2^{-i} < \infty. \tag{17}$$

Let $\delta > 0$ be so small that

- f is injective on $[|c| - 2\delta, |c| + 2\delta]$;
- $[-\delta, \delta] \subset f([|c| - 2\delta, |c| + 2\delta])$.

Then, for i sufficiently large, we define r_i as the unique number in $[-2\delta, 2\delta]$ such that $f(|c| + r_i) = 2^{-i}$. Notice that for i sufficiently large, we have $r_i \leq 2^{-i+1}$. Set $\alpha = c/|c|$, and for every i, choose an $\alpha_i \in \mathbf{S}^{k-1}$ such that $|\alpha_i - \alpha| = i^{-2}$.

Let I_i be the interval $[2^{-i}, 2^{-i+1}[$, and subdivide it into m_i equal subintervals

$$I_i^j := \Big[2^{-i} + \frac{(j-1)2^{-i}}{m_i}, 2^{-i} + \frac{j2^{-i}}{m_i}\Big[, \quad j \in \{1, \dots, m_i\}.$$

Next, define the functions $\psi_i : \mathbf{R}^2 \to \mathbf{S}^{k-1}$ as

$$\psi_i(x, y) := \begin{cases} \alpha_i & \text{if } y \in I_i \text{ and } [x2^i] \text{ is odd,} \\ \alpha & \text{otherwise,} \end{cases}$$

and define the functions $\chi_i : \mathbf{R}^2 \to \mathbf{R}$ as

$$\chi_i(x, y) := \begin{cases} r_i & \text{if } y \in I_i^j \text{ for } j \text{ even and } x \in [-M, M], \\ r_{i+1} & \text{if } y \in I_i^j \text{ for } j \text{ odd and } x \in [-M, M], \\ 0 & \text{otherwise.} \end{cases}$$

Here M is a positive real number that is chosen later. Finally, we define

$$\begin{cases} \rho_0^n & := |c| + \sum_{i=n}^{\infty} \chi_i, \\ \theta_0^n(x,y) & := \begin{cases} \psi_i(x,y) & \text{if } y \in I_i \text{ for some } i \geq n \text{ and } x \in [-M, M], \\ \alpha & \text{otherwise,} \end{cases} \\ u_0^n & := \rho_0^n \theta_0^n. \end{cases}$$

Figure 1 gives a picture of the partition of $\mathbf{R}^2$ on which we based the definition of u_0^n.

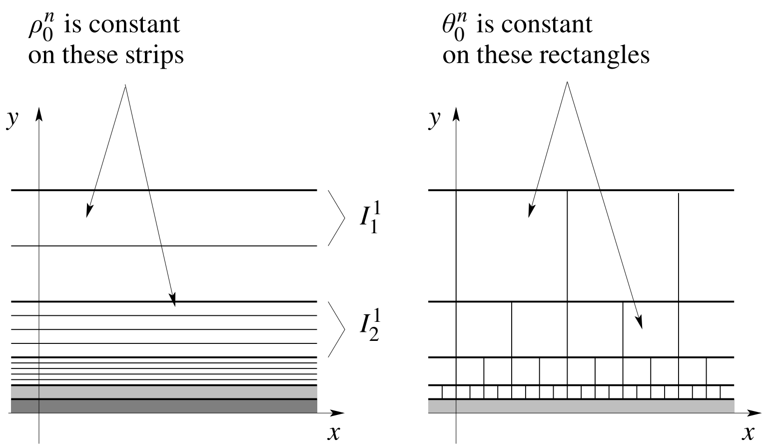

Figure 1. Decomposition of the plane in open sets where ρ_0^n (resp., θ_0^n) is constant

Clearly, $\|u_0^n - c\|_\infty \leq |c||\alpha_n - \alpha| + r_n$. Hence, as $n \uparrow \infty$, we have $\|u_0^n - c\|_\infty \to 0$. Moreover, notice that $u_0^n - c$ is supported on $[-M, M] \times [0, 1]$. From now on we assume that M is chosen larger than 1.

In order to show that

$$\|u_0^n - c\|_{\mathrm{BV}(\mathbf{R}^2)} \to 0,$$

it is sufficient to show

$$\|\rho_0^n - |c|\|_{\mathrm{BV}([-2M,2M]^2)} \to 0, \tag{18}$$

$$\|\theta_0^n - \alpha\|_{\mathrm{BV}([-2M,2M]^2)} \to 0. \tag{19}$$

Note that

$$\begin{aligned} \|\rho_0^n - |c|\|_{\mathrm{BV}([-2M,2M]^2)} &\leq 4\|u_0^n - c\|_\infty M^2 + 2M \sum_{i \geq n} m_i r_i + (4M+2) r_n \\ &\leq 4\|u_0^n - c\|_\infty M^2 + 4M \sum_{i \geq n} m_i 2^{-i} + (4M+2) r_n, \end{aligned}$$

and, since $\sum 2^{-i} m_i$ is summable, we get (18). Moreover,

$$\|\theta_0^n - \alpha\|_{\mathrm{BV}([-2M,2M]^2)} \le 4\|\theta_0^n - \alpha\|_\infty M^2 + 2M \sum_{i \ge n} 2^{-i} i^{-2} 2^i \\ + 2M \sum_{i \ge n} [i^{-2} + (i+1)^{-2}] + (4M+2)n^{-2},$$

and the summability of $\sum i^{-2}$ gives (19).

Now we let u^n be the unique renormalized solution of (16). Recall that $\rho^n := |u^n|$ is the unique entropy solution of (4) with initial data ρ_0^n, which in our case is given by

$$\begin{cases} \partial_t \rho^n + \partial_x \big(f(\rho^n)\rho^n \big) = 0, \\ \rho^n(0, \cdot) = \rho_0^n. \end{cases}$$

Hence, if ρ_0^n did not depend on x, we would have $\rho^n(t, y, x) = \rho_0^n(x, y)$. Since ρ_0^n is "truncated," this is not true. However, $\rho_0^n(\cdot, y)$ is constant on $[-M, M]$, and by the finite speed of propagation of scalar laws, it follows that $\rho^n(t, x, y) = \rho_0^n(x, y)$ if (t, x, y) belongs to the cone

$$\big\{ \sqrt{y^2 + x^2} \le c(M - t) \big\},$$

where c is a constant that depends only on $\|\rho_0^n\|_\infty$. Thus for every $\lambda > 1$, we can choose M large enough (but independent of n) so that

$$\rho^n(t, x, y) = \rho_0^n(x, y) \quad \text{for } t \in [0, 1] \text{ and } (x, y) \in [-\lambda, \lambda] \times [0, 1].$$

To find the angular part $\theta^n(t, x, y) := u^n/|u^n|(t, x, y)$, we use the fact that θ^n is constant on the curves $\Phi_n(\cdot, x)$, where Φ_n solves the ODEs

$$\begin{cases} \frac{d}{dt} \Phi_n(s, x, y) = g\big(\rho^n(s, \Phi_n(s, x, y)) \big), \\ \Phi_n(0, x, y) = (x, y) \end{cases} \tag{20}$$

in the sense of Propositions 2.8 and 2.9. Hence it follows that for $\mathscr{L}^3$-a.e. (τ, x_1, y_1), there is $(x_0, y_0) \in \mathbf{R}^2$ such that

- the curve $\Phi(\cdot, x_0, y_0)$ is Lipschitz;
- $\Phi(\tau, x_0, y_0) = (x_1, y_1)$;
- $\Phi(\cdot, x_0, y_0)$ solves (20) in the sense of Proposition 2.9.

Therefore every connected component of the intersection of the curve $\Phi(\cdot, x_0, y_0)$ with $[0, 1] \times [-\lambda, \lambda] \times [0, 1]$ is a straight segment lying on a plane $\{y = \text{const}\}$. If $(\tau, x_1, y_1) \in [0, 1]^3 \subset [0, 1] \times [-\lambda, \lambda] \times [0, 1]$, one of these segments contains (τ, x_1, y_1), and hence its slope is given by $f(\rho^n(\tau, x_1, y_1))$. If we choose λ large enough, the curve $\Phi(\cdot, x_0, y_0)$ remains "trapped" on the plane $\{y = y_1\}$ for the whole time interval $]0, \tau[$. Note that this choice of λ depends only on f and on the L^∞-norm of ρ^n, which is uniformly bounded.

From now on, we assume that λ (and hence M) have been chosen so as to satisfy the requirement above. Recall that for $\mathscr{L}^3$-a.e. $(t, x, y) \in [0, 1]^3$, we have $\rho^n(t, x, y) = |c| + r_i$ for some i, and hence $f(\rho^n(t, x, y))) = 2^{-i}$. From the previous discussion we derive the following formulas, valid for $\mathscr{L}^3$-a.e. $(t, x, y) \in [0, 1]^3$:

- if $\rho_0^n(x, y) = |c|$, then $\theta^n(t, x, y) = \theta_0^n(x, y)$;
- if $\rho_0^n(x, y) = |c| + r_i$, then $\theta^n(t, x, y) = \theta_0^n(x - t2^{-i}, y)$.

Hence, for $j \in \{1, m_i - 1\}$, $i \geq n$, and $l \in \{1, \ldots, 2^i - 1\}$, the function $\theta^n(t, \cdot)$ jumps on the segments

$$S_{j,i,l} := \left\{ y = 2^{-i} + \frac{j2^{-i}}{m_i} \quad x \in [l2^{-i}, (l + t)2^{-i}] \right\}$$

(see Figure 2).

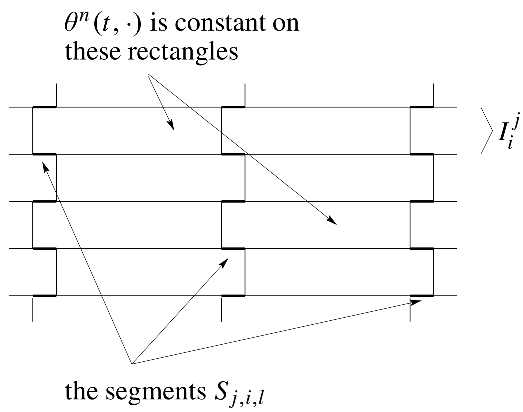

Figure 2. The function $\theta^n(t, \cdot)$ and the segments $S_{j,i,l}$

The total amount of this jump is given by

$$J_i := \int_{S_{j,i,l}} \left| (\theta^n)^+(t, x, y) - (\theta^n)^-(t, x, y) \right| d\mathscr{H}^1(x) = t2^{-i} |\alpha_i - \alpha| = t2^{-i} i^{-2}.$$

Thus

$$\|\theta^n(t, \cdot)\|_{\mathrm{BV}([0,1]^2)} \geq \sum_{i \geq n} \sum_{j=1}^{m_i - 1} \sum_{l=1}^{2^i - 1} J_i = \sum_{i \geq n} (2^i - 1)(m_i - 1) J_i \geq \frac{t}{2} \sum_{i \geq n} (m_i - 1) i^{-2}. \tag{21}$$

Clearly, since $|u^n|(t, \cdot) \in \mathrm{BV} \cap L^\infty$ for every t, and it is bounded away from zero, it is sufficient to show that $\theta^n(t, \cdot) \notin \mathrm{BV}([0, 1]^2)$ for any $t \in]0, 1[$.

Recall that the bound (17) is the only condition required on the sequence of even numbers $\{m_i\}$. If we set $m_i = 2i^2$, then (17) is clearly satisfied, whereas (21) is infinite. $\square$

3.2. Proof of Proposition 3.1

As in the proof of Proposition 3.2, for $\beta \in \mathbf{R}$ we denote by $[\beta]$ the largest integer that is less than or equal to β.

The idea is to mimic the construction of Proposition 3.2. Hence we start with piecewise constant initial moduli ρ_0^n which are constant along $m-1$ orthogonal directions $e_1, \ldots, e_{m-1}$ and oscillate along the direction ω orthogonal to each e_i. The solution ρ^n of the scalar law (4) is then constant along the directions $e_1, \ldots, e_{m-1}$. Moreover, for small times, this solution consists of shocks and rarefaction waves that do not interact. We impose two requirements on this construction.

- We choose ω and the sizes and heights of the oscillations in such a way that the distinct shocks and rarefaction waves do not interact for times less than 1. Hence in this range of times, between each couple of nearby shock and rarefaction wave there is a space-time strip on which ρ is constant (see Figure 3).
- We choose ω in such a way that the trajectories of solutions of (12) are "trapped" in the strips for a sufficiently long time.

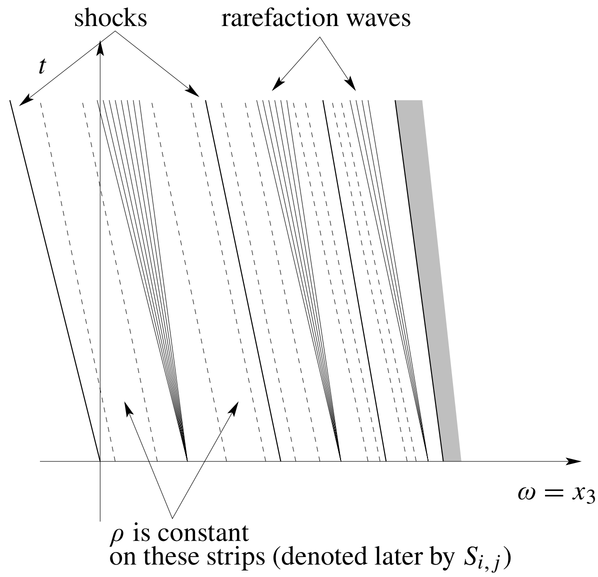

Figure 3. A (t, ω)-slice of the evolution of ρ^n

Finally, we choose initial data θ_0^n which oscillate along a direction perpendicular to ω in such a way that in the strip mentioned above, θ^n reproduce the behavior of the construction of Proposition 3.2.

These requirements translate into geometric conditions on ω and into analytical ones on the various parameters that govern the oscillations. When $m \geq 3$ and g is not constant, we can always satisfy these conditions. When $m = 2$, we are able to do it only in some cases.

Since the construction is the same, we present the proof only when $m \geq 3$, and without losing generality, we assume $m = 3$. We denote by h the function given by $h(\rho) = \rho g(\rho)$ and by β the positive real number $|c|$. Clearly, there exists a unit vector $\omega \in \mathbf{R}^3$ such that

$$\omega \cdot g(\beta) = \omega \cdot h'(\beta), \tag{22}$$

$$\omega \cdot g'(\beta) = 0, \tag{23}$$

$$\omega \cdot h''(\beta) = 0. \tag{24}$$

Indeed, since $h'(\beta) = g(\beta) + \beta g'(\beta)$, (22) reduces to (23). Thus the conditions above reduce to finding a unit vector $\omega \in \mathbf{R}^3$ which is perpendicular to both the vectors $g'(\beta)$ and $h''(\beta)$. We fix an orthonormal system of coordinates in $\mathbf{R}^3$ in such a way that $\omega = (0, 0, 1)$.

Step 1: Construction of the modulus. Let $\{\sigma_l\}$ be a sequence of vanishing positive real numbers such that $\sum \sigma_l < \infty$, and let $I_l \subset \mathbf{R}$ be the intervals

$$I_1 := [0, \sigma_1[, \qquad I_l := \Big[\sum_{i \leq l-1} \sigma_i, \sum_{i \leq l} \sigma_i \Big].$$

Let m_l be a strictly increasing sequence of even integers, and divide every I_l into m_l equal subintervals I_l^j for $j \in \{1, \dots, m_l\}$. Finally, let $\{a_l\}$ be a vanishing sequence of real numbers, and set

$$\rho^{\text{in}}(x_1, x_2, x_3) := \begin{cases} \beta + a_l & \text{if } x_3 \in I_l^j \text{ for some even } j, \\ \beta & \text{otherwise.} \end{cases}$$

Then, let ρ be the entropy solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x[h(\rho)] = 0, \\ \rho(0, \cdot) = \rho^{\text{in}}. \end{cases} \tag{25}$$

Clearly, ρ is a function of t and x_3 only. Moreover, recalling that $h_3''(\beta) = 0$, we can apply Lemma 2.7 in order to get the following property.

(T) For every $C_1 > 0$, there exists a $C_2 > 0$ such that if

$$\frac{\sigma_l}{m_l} \geq C_2 a_l^2, \tag{26}$$

then every I_l^j contains a subinterval J_l^j such that

- the length of J_l^j is bigger than $C_1 a_l^2$;
- for every $(t, \xi_1, \xi_2, \xi_3) \in [0, 1] \times \mathbf{R}^2 \times J_l^j$, we have

$$\rho\big(t, \xi_1, \xi_2, \xi_3 + t h_3'(\beta)\big) = \rho\big(0, \xi_1, \xi_2, \xi_3\big). \tag{27}$$

For each couple j, l, we let $S_{l,j}$ be the strip

$$S_{l,j} := \big\{(t, x_1, x_2, x_3) \,\big|\, 0 \leq t \leq 1 \text{ and } (x_3 - t h_3'(\beta)) \in J_l^j\big\}.$$

Step 2: The flux generated by ρ. Denote by $B_R \subset \mathbf{R}^3$ the ball of radius R centered at the origin. It is easy to check that there exists a constant C_3 such that

$$\|\rho^{\text{in}}\|_{\text{BV}(B_R)} \leq C_3 R^3 + C_3 R^2 \Big(\sum_l (m_l + 1)|a_l|\Big). \tag{28}$$

Hence, to ensure that $\rho^{\text{in}} \in \text{BV}_{\text{loc}}$, it is sufficient to assume

$$\sum_l (m_l + 1)|a_l| < \infty. \tag{29}$$

Assuming that this condition is fulfilled, from the classical result of Kruzhkov we get the existence of a constant M such that $\|\rho\|_{\text{BV}(]0,1[\times B_R)} \leq M\|\rho^{\text{in}}\|_{\text{BV}(B_{R+Mt})}$. Thus we can consider the regular Lagrangian flow Φ for the ODE

$$\begin{cases} \frac{d}{dt}\Phi(s, \cdot) = g\big(\rho(s, \Phi(s, \cdot))\big), \\ \Phi(0, x) = x \end{cases}$$

(see Propositions 2.8 and 2.9). Fix any strip $S_{l,j}$ as defined in Step 1. Clearly, for a.e. x, every connected component of the intersection of the trajectory curve $\gamma_x := \{\Phi(t, x) \mid t \in \mathbf{R}\}$ with the strip $S_{l,j}$ is a straight segment. If j is even, then this segment is parallel to $(1, g(\beta))$; otherwise, it is parallel to $(1, g(\beta + a_l))$. Thus, if j is even and $(t, x) \in S_{l,j}$, the portion of trajectory

$$T_{t,x} := \big\{\Phi(s, \xi) \text{ for } \xi \text{ such that } \Phi(t, \xi) = x \text{ and for } s \in [0, t]\big\}$$

is a straight segment contained in $S_{l,j}$.

Let us now turn to the case where j is odd. Note that

$$g(\beta + a_l) = g(\beta) + g'(\beta) a_l + O(a_l^2). \tag{30}$$

Thanks to the properties of $\omega = (0, 0, 1)$, we have the fact that segments of the form

$$\left\{\left(t, \xi + t(g(\beta) + a_l g'(\beta))\right) \,\middle|\, 0 \le t \le 1 \text{ and } (0, \xi) \in S_{l,j}\right\} \tag{31}$$

are subsets of $S_{l,j}$. Recall (T) of Step 1. From (30) and (31) it follows that for C_1 in (T) sufficiently large, there exists a subinterval $K_{l,j}$ such that

- the length of $K_{l,j}$ is bigger than a_l^2;
- if $t \in [0, 1]$ and $x_3 - tg'(\beta) \in K_{l,j}$, then the set

$$T_{t,x} = \left\{\Phi(s, \xi) \,\middle|\, s \in [0, t] \text{ and } \Phi(t, \xi) = x\right\}$$

  is a straight segment contained in $S_{l,j}$.

From now on, we fix C_1 (and hence C_2) in such a way as to ensure the existence of the segments $K_{l,j}$.

Step 3: Construction of the angular part. We recall that $g_3'(\beta) = g'(\beta) \cdot \omega = 0$ and that $g_3'(\beta) \neq 0$. Since the construction of Step 2 is independent of the choice of the coordinates x_1 and x_2, we can choose them so that $g'(\beta) = (0, C_4, 0)$, with $C_4 > 0$. Choose the a_l's in such a way that

$$g_2(\beta + a_l) - g_2(\beta) = 2^{-l}.$$

Then, clearly, there exists a constant C_5 such that

$$\frac{2^{-l}}{C_5} \le a_l \le C_5 2^{-l}. \tag{32}$$

Set $\eta = c/|c|$, and let $\eta_l \in \mathbf{S}^{k-1}$ be such that $|\eta_l - \eta| = l^{-2}$. Then define

$$\theta^{\text{in}}(x_1, x_2, x_3) := \begin{cases} \eta_l & \text{if } x_3 \in I_l \text{ and } [2^l x_2] \text{ is even,} \\ \eta & \text{otherwise.} \end{cases}$$

Set $u^{\text{in}} := \rho^{\text{in}}\theta^{\text{in}}$. Let u be the renormalized entropy solution of

$$\begin{cases} \partial_t u + \operatorname{div}_z[g(|u|)u] = 0, \\ u(0, \cdot) = u^{\text{in}}. \end{cases} \tag{33}$$

We denote by θ the angular part $u/|u|$. According to Propositions 2.8 and 2.9, θ is given by the formula

$$\theta(t, x) = \theta^{\text{in}}\big(\Psi(t, x)\big),$$

where Ψ is a map such that $\Phi(t, \Psi(t, x)) = \Psi(t, \Phi(t, x)) = x$ for $\mathscr{L}^4$-a.e. (t, x). In what follows, we denote by Φ_t^{-1} the map $\Psi(t, \cdot)$.

Step 4: Choice of parameters. We prove that for an appropriate choice of the various parameters, $u^{\mathrm{in}} \in \mathrm{BV}_{\mathrm{loc}}$, whereas $u(t,\cdot)$ is not in $\mathrm{BV}_{\mathrm{loc}}$ for any $t \in]0,1]$. Recall that $\rho^{\mathrm{in}} = |u^{\mathrm{in}}|$ and $\rho(t,\cdot) = |u|(t,\cdot)$ are both in $\mathrm{BV}_{\mathrm{loc}}$ and that $C_6^{-1} \le \rho \le C_6$ for some positive constant C_6. Thus our goal is to choose the parameters σ_l and m_l in such a way that $\theta^{\mathrm{in}} \in \mathrm{BV}_{\mathrm{loc}}$ and $\theta(t,\cdot) \notin \mathrm{BV}_{\mathrm{loc}}$ for every $t \in]0,1]$. Note that for some constant C_7,

$$\|\theta^{\mathrm{in}}\|_{\mathrm{BV}(B_R)} \le C_7 R^3 + C_7 R^2 \Big(\sum_l \frac{2^l}{l^2} \sigma_l + \sum_l l^{-2} \Big). \tag{34}$$

Hence, choosing $\sigma_l = 2^{-l}$, we conclude that $\theta^{\mathrm{in}} \in \mathrm{BV}(B_R)$ for every $R > 0$.

Now we choose $m_l = 2l^2$, and since from (32) we have $a_l \le C_5^2 2^{-l}$, we clearly fulfill the condition (29), which is the only one we required on the sequence $\{m_l\}$. Thus we get

$$\frac{\sigma_l}{m_l} = l^{-2} 2^{-l+1}.$$

Since from (32) we have $a_l^2 \le C_5 2^{-2l}$, clearly (26) is fulfilled for any constant C_2, provided that l is large enough. Thus we get the existence of a constant C_8 such that the segments $K_{l,j}$ of Step 2 exist for any $l \ge C_8$.

Fix $t \in]0,1]$ and $l \ge C_8$. Recalling that $\theta(t,x) = \theta^{\mathrm{in}}(\Phi_t^{-1}(x))$ and taking into account the properties of Φ proved in Step 2, we conclude the following.

- If $j \in [1, m_l]$ is even and $\xi_{l,j}$ belongs to the segment $J_{l,j}$, then

$$\theta\big(t, x_1, x_2, \xi_{l,j} + t g_3(\beta)\big) = \begin{cases} \eta_l & \text{if } [2^l(x_2 - t g_2(\beta))] \text{ is even,} \\ \eta & \text{otherwise.} \end{cases}$$

- If $j \in [1, m_l]$ is odd and $\xi_{l,j}$ belongs to the segment $K_{l,j}$, then

$$\theta\big(t, x_1, x_2, \xi_{l,j} + t g_3(\beta)\big) = \begin{cases} \eta_l & \text{if } [2^l(x_2 - t g_2(\beta + a_l))] \text{ is even,} \\ \eta & \text{otherwise.} \end{cases}$$

Recall that $g_2(\beta + a_l) - g_2(\beta) = 2^{-l}$. Thus for any $j \in [1, m_l - 1]$, we have

$$\begin{aligned} A_{l,j} &:= \int_{[0,1]^2} \big|\theta\big(t, x_1, x_2, \xi_{l,j} + t g_3(\beta)\big) - \theta\big(t, x_1, x_2, \xi_{l,j+1} + t g_3(\beta)\big)\big| \, dx_1 \, dx_2 \\ &= t|\eta_l - \eta| = t l^{-2}. \end{aligned}$$

Thus

$$\sum_{l \ge C_8} \sum_{1 \le j \le m_l - 1} A_{l,j} = t \sum_{l \ge C_8} \frac{m_l - 1}{l^2} = t \sum_{l \ge C_8} \frac{2l^2 - 1}{l^2} = \infty. \tag{35}$$

Note that if $\theta(t,\cdot)$ were locally in BV, then $\partial_{x_3}\theta(t,\cdot)$ would be a Radon measure. Denote by μ the total variation measure of $\partial_{x_3}\theta(t,\cdot)$, and denote by $\mathscr{S}_{l,j}$ the stripes

$$\mathscr{S}_{l,j} := \bigl\{(x_1,x_2,x_3) \,\big|\, (x_1,x_2)\in[0,1]^2 \text{ and } \bigl(x_3 - tg_3(\beta)\bigr)\in[\xi_{l,j},\xi_{l,j+1}]\bigr\}.$$

Then $A_{l,j} \le \mu(\mathscr{S}_{l,j})$. The $\mathscr{S}_{l,j}$ are pairwise disjoint, and for R' sufficiently large, they are all contained in the ball $B_{R'}$. Thus we would get

$$\sum_{l\ge C_8}\sum_{1\le j\le m_l-1} A_{l,j} \le \sum_{l\ge C_8}\sum_{1\le j\le m_l-1} \mu(\mathscr{S}_{l,j}) \le \mu(B_{R'}) < \infty,$$

which contradicts (35). Hence we conclude that $\theta(t,\cdot)$ is not in $\mathrm{BV}(B_{R'})$ for any $t\in\,]0,1]$.

Step: Truncation of the construction and conclusion. Next, define $\tilde{u}_0^n : \mathbf{R}^3 \to \mathbf{R}^2$ as

$$\tilde{u}_0^n(x_1,x_2,x_3) := \begin{cases} u^{\mathrm{in}}(x_1,x_2,x_3) & \text{if } x_3\in I_l \text{ for some } l\ge n,\\ c & \text{otherwise.}\end{cases}$$

Clearly, $\|\tilde{u}_0^n - c\|_\infty + \|\tilde{u}_0^n - c\|_{\mathrm{BV}(U)} \to 0$ for every bounded open set $U\subset\mathbf{R}^3$. Moreover, if we denote by $\tilde{u}^n$ the renormalized entropy solution of

$$\begin{cases} \partial_t u + \mathrm{div}_z[g(|u|)u] = 0,\\ u(0,\cdot) = \tilde{u}_0^n,\end{cases} \tag{36}$$

then $\tilde{u}^n(t,\cdot)\notin \mathrm{BV}(B_{R'})$ for any $t\in\,]0,1]$. Finally, let $M>0$, and define

$$u_0^n(x_1,x_2,x_3) := \begin{cases} \tilde{u}_0^n(x_1,x_2,x_3) & \text{if } x_1^2+x_2^2+x_3^2\le M,\\ c & \text{otherwise.}\end{cases}$$

Let u^n be the renormalized entropy solution of

$$\begin{cases} \partial_t u + \mathrm{div}_x[g(|u|)u] = 0,\\ u(0,\cdot) = u_0^n.\end{cases} \tag{37}$$

The proof of [2, Proposition 2.11] gives the existence of an M sufficiently large such that $u^n = \tilde{u}^n$ on $[0,1]\times B_{R'}$ for any n. Hence the sequence $\{u_0^n\}$ has all the properties required by the proposition. □

3.3. Bounds in different spaces

Varying the parameters involved in the proof of Proposition 3.2, one can construct initial data u_0^n with renormalized solutions u^n which have the following properties:

- the sequence $\{u_0^n\}$ satisfies the BV- and L^∞-bounds of Proposition 3.2;
- the restriction of u^n to $\{t\} \times [0,1]^m$ is piecewise constant on the elements U_i^n of a countable partition by open sets;
- $J_{u^n} := \big(\{t\} \times [0,1]^m\big) \setminus \bigcup U_i^n$ is the union of countably many Lipschitz hypersurfaces;
- if we denote by $(u^n)^+$ and $(u_n)^-$ the right and left traces of u^n on J_{u^n}, then

$$\int_{J_{u^n}} |(u^n)^+ - (u^n)^-|^j \, d\mathscr{H}^{m-1} = \infty \quad \text{for every positive } j. \tag{38}$$

Similar examples are possible in the more general case, but the maximal j for which (38) holds seems to depend on the number of space variables and on the regularity of g. Building on them, it would be interesting to understand if $W^{\alpha,p}$-norms (for $\alpha < 1$) also blow up in finite time. However, this requires more than (38); in particular, it needs more subtle computations involving the dislocation of the jump set (see, for instance, [10]).

4. Proof of Theorem 2.4

Theorem 2.4 is a consequence of Proposition 3.1 and of the following.

PROPOSITION 4.1
Let $u_0 \in L^\infty(\mathbf{R}^m, \mathbf{R}^k)$, and let C be the closure of the convex hull of its essential image. Assume that
(a) *either $0 \notin C$ or it is an extremal point of C;*
(b) *u is a bounded entropy solution of (3);*
(c) *$u \in \mathrm{BV}(]0, T[\times U)$ for some $T > 0$ and for some bounded open $U \subset \mathbf{R}^m$.*
Then u is a renormalized entropy solution of (3) on $]0, T[\times U$.

Proof
Define $\rho := |u|$ and $\rho_0 := |u_0|$. The goal is to show that ρ is an entropy solution of the scalar law

$$\begin{cases} \partial_t \rho + \mathrm{div}_x[g(\rho)\rho] = 0, \\ \rho(0, \cdot) = |u_0| \end{cases} \tag{39}$$

in $]0, T[\times U$.

Actually, it is sufficient to show that ρ is a *weak* solution of (39) in $]0, T[\times U$. Indeed, note that for every $\eta : \mathbf{R}^+ \to \mathbf{R}$ which is convex and increasing, $\eta(|u|)$ is a convex entropy for the system (3). (The entropy flux is of the form $q(|u|)$ for q such that $q' = \eta' g'$.) Thus we have

$$\int_{t>0}\int_{\mathbf{R}^m}\big[\partial_t\psi(t,z)\,\eta\big(\rho(t,z)\big)+\nabla_x\psi(t,z)\cdot q\big(\rho(t,z)\big)\big]\,dt\,dz + \int_{\mathbf{R}^m}\psi(0,z)\eta\big(\rho_0(z)\big)\,dz \ge 0 \tag{40}$$

for every nonnegative smooth test function ψ. Moreover, if ρ is a weak solution of (39) in $]0,T[\times U$, then for every linear function $L:\mathbf{R}\to\mathbf{R}$ we get the following equality for every $\psi \in C_c^\infty(]-T,T[\times U)$:

$$\int_{t>0}\int_{\mathbf{R}^m}\big[\partial_t\psi(t,z)\,L\big(\rho(t,z)\big)+\nabla_x\psi(t,z)\cdot Q\big(\rho(t,z)\big)\big]\,dt\,dz + \int_{\mathbf{R}^m}\psi(0,z)L\big(\rho_0(z)\big)\,dz = 0, \tag{41}$$

where $Q:\mathbf{R}\to\mathbf{R}^m$ is given by $Q=(L(g_1),\dots,L(g_m))$. Given any convex function ξ, we can write it as $L+\eta$, where L is an appropriate linear function and η is increasing on the half-line $\mathbf{R}^+$. Thus, summing (40) and (41), we conclude that ρ satisfies the entropy inequality for ξ and for every nonnegative $\psi \in C_c^\infty(]-T,T[\times U)$, and hence that ρ is an entropy solution of (39) in $]0,T[\times U$.

We now come to the proof that ρ is a weak solution of (39), which we split into several steps.

Step 1. Recall that ρ is a weak solution of (39) in $]0,T[\times U$ if it satisfies the identity

$$\int_{t>0}\int_{\mathbf{R}^m}\rho(t,z)\big[\partial_t\psi(t,z)+g\big(\rho(t,z)\big)\cdot\nabla_x\psi(t,z)\big]\,dt\,dz+\int_{\mathbf{R}^m}\psi(0,z)\rho_0(z)\,dz = 0 \tag{42}$$

for every $\psi \in C_c^\infty(]-T,T[\times U)$.

Recall that $\|u\|_{\mathrm{BV}(U\times]0,T[)}$ is finite. Hence we claim that, thanks to the trace properties of BV functions, in order to prove (42) it suffices to check that

$$\text{the Radon measure } \mu = \partial_t\rho + \operatorname{div}\big(\rho g(\rho)\big) \text{ vanishes on }]0,T[\times U. \tag{43}$$

Indeed, by a standard approximation argument, we get the following estimate for every $t < T$:

$$\int_0^t\int_U |u(\tau,z)-u_0(z)|\,dz\,d\tau \le \int_0^t |\partial_t u|(]0,\tau[\times U)\,d\tau \le t|\partial_t u|(]0,t[\times U).$$

From this we conclude

$$\int_0^t\int_U |\rho(\tau,z)-\rho_0(z)|\,dz\,d\tau \le t|\partial_t u|(]0,t[\times U). \tag{44}$$

Fix $\psi \in C_c^\infty(]-T,T[\times U)$, and let $\{\chi_i\}\subset C^\infty([0,T])$ be such that

- $\chi_i = 1$ for $t \geq 2/i$;
- $\chi_i = 0$ for $t \leq 1/i$;
- $0 \leq \chi_i' \leq 4i$.

Then $\psi\chi_i$ is compactly supported in $]0, T[\times U$, and from (43) we get

$$\int_0^T \int_{\mathbf{R}^m} \chi_i(\tau)\rho(\tau,z)\big[\partial_t\psi(\tau,z) + g\big(\rho(\tau,z)\big)\cdot\nabla_x\psi(\tau,z)\big]\,dz\,d\tau + \int_0^{2/k}\int_{\mathbf{R}^m} \chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau = 0. \tag{45}$$

As $i \uparrow \infty$, the first integral in (45) converges to

$$\int_0^T \int_{\mathbf{R}^m} \rho(\tau,z)\big[\partial_t\psi(\tau,z) + g\big(\rho(\tau,z)\big)\cdot\nabla_x\psi(\tau,z)\big]\,dz\,d\tau.$$

Concerning the second integral, we recall that $\int_0^{2/i}\chi_i' = 1$ and we write

$$\begin{aligned}
\Big|\int_0^{2/i}&\int_{\mathbf{R}^m}\chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau - \int_{\mathbf{R}^m}\rho_0(z)\psi(0,z)\,dz\Big| \\
&= \Big|\int_0^{2/i}\int_{\mathbf{R}^m}\chi_i'(\tau)[\rho(\tau,z)\psi(\tau,z) - \rho_0(z)\psi(0,z)]\,dz\,d\tau\Big| \\
&\leq 4i\int_0^{2/i}\int_{\mathbf{R}^m}|\rho(\tau,z)\psi(\tau,z) - \rho_0(z)\psi(0,z)|\,d\tau\,dz \\
&\leq 4i\|\rho\|_\infty\int_0^{2/i}\int_{\mathbf{R}^m}|\psi(\tau,z)-\psi(0,z)|\,d\tau\,dz \\
&\quad + 4i\|\psi\|_\infty\int_0^{2/i}\int_{\mathbf{R}^m}|\rho(\tau,z)-\rho(0,z)|\,d\tau\,dz.
\end{aligned}$$

Note that, for $i \uparrow \infty$ the first term tends to 0 because ψ is smooth. Thanks to (44), the second term is bounded by

$$C|\partial_t u|(]0, 2/i[\times U), \tag{46}$$

where C is a constant independent of t and U is a bounded set. Since $|\partial_t u|$ is a Radon measure, we conclude that the expression (46) tends to zero for $i \uparrow \infty$. Thus we conclude that

$$\lim_{i\uparrow\infty}\int_0^{2/i}\int_{\mathbf{R}^m}\chi_i'(\tau)\rho(\tau,z)\psi(\tau,z)\,dz\,d\tau = \int_{\mathbf{R}^m}\rho_0(z)\psi(0,z)\,dz.$$

Hence, passing into the limit in (45), we get (42). Therefore we are left with the task of proving (43).

Step 2. We wish to use the entropy inequalities and apply Theorem 2.5 to conclude that μ is supported on the jump set (or shock set) J_u. However, this is not possible since the function $|u|$ is not C^1 in the origin (cf. Remark 2.6). We approximate this function uniformly with smooth C^1 convex functions of the form $\eta_n(|u|)$. Clearly, these functions are also entropies for the system of Keyfitz and Kranzer, and their entropy fluxes are of the form $q_n(|u|)$ for some functions $q_n(t)$ which converge uniformly to $tg(t)$.

Let $\nu : J_u \to \mathbf{R}^m$ be a Borel vector field, and let $\zeta : J_u \to \mathbf{R}$ be a nonnegative Borel function such that $(\zeta, \nu)/\sqrt{\zeta^2 + |\nu|^2}$ is normal to J_u $\mathscr{H}^m$-a.e. Then the chain rule of Vol'pert gives

$$\begin{aligned}\partial_t[\eta_n(\rho)] + \mathrm{div}_x[q_n(\rho)] = (\zeta^2 + |\nu|^2)^{-1/2}\big[\big(\eta_n(|u^+|) - \eta_n(|u^-|)\big)\zeta \\ + \big(q_n(|u^+|) - q_n(|u^-|)\big)\cdot\nu\big]\,\mathscr{H}^m \llcorner J_u.\end{aligned}$$

Passing to the limit in n, we get

$$\mu = (\zeta^2 + |\nu|^2)^{-1/2}\big[(|u^+| - |u^-|)\zeta + \big(|u^+|g(|u^+|) - |u^-|g(|u^-|)\big)\cdot\nu\big]\,\mathscr{H}^m \llcorner J_u. \tag{47}$$

Thus we must prove that

$$\big(\zeta + g(|u^+|)\cdot\nu\big)|u^+| = \big(\zeta + g(|u^-|)\cdot\nu\big)|u^-|, \quad \mathscr{H}^m\text{-a.e. on } J_u. \tag{48}$$

In what follows, for the sake of simplicity we drop the $\mathscr{H}^m$-a.e.

Since u is a weak solution of (3), when $F(v) := g(|v|)\otimes v$ is C^1 we can apply Theorem 2.5 to get

$$\big(g(|u^+|)\cdot\nu + \zeta\big)u^+ = \big(g(|u^-|)\cdot\nu + \zeta\big)u^-. \tag{49}$$

In order to derive (49) when 0 is a singularity for DF, we approximate F with $F_n := g(\eta_n(u))\otimes u$. Then we get

$$\begin{aligned}\partial_t u + \mathrm{div}_x\big(F_n(u)\big) = D_d u + DF_n(\tilde{u})\cdot D_d u \\ + \big[(u^+ - u^-)\zeta + \big(F(u^+) - F(u^-)\big)\cdot\nu\big]\,\mathscr{H}^m \llcorner J_u.\end{aligned} \tag{50}$$

Clearly, the left-hand side converges to $0 = \partial_t u + \mathrm{div}_x(F(u))$. Moreover, the second term of the right-hand side converges to

$$\big[\big(g(|u^+|)\cdot\nu + \zeta\big)u^+ - \big(g(|u^-|)\cdot\nu + \zeta\big)u^-\big]\,\mathscr{H}^m \llcorner J_u$$

in the sense of measures. If we choose the approximations F_n in such a way that the DF_n are locally uniformly bounded, the measures $DF_n(\tilde{u})\cdot D_d u$ converge to $\gamma\,|D_d u|$ for some bounded Borel function γ. Since $|D_d u|(J_u) = 0$, we conclude that (49) holds.

From (49) we get

$$\left|g(|u^+|)\cdot\nu+\zeta\right||u^+|=\left|g(|u^-|)\cdot\nu+\zeta\right||u^-|. \tag{51}$$

If $|u^+|$ (or $|u^-|$) vanishes, (48) follows trivially. Hence, after setting $\rho^\pm := |u^\pm|$, we restrict our attention to the subset of J_u given by $G := \{\rho^+ \neq 0 \neq \rho^-\}$. On this set we define $\theta^\pm := u^\pm/\rho^\pm$, and we note that (49) becomes

$$\left[\left(g(\rho^+)\cdot\nu+\zeta\right)\right]\rho^+\theta^+=\left[\left(g(\rho^-)\cdot\nu+\zeta\right)\right].\rho^-\theta^-. \tag{52}$$

Since $\theta^\pm \in \mathbf{S}^{k-1}$, we conclude that either $\theta^+ = \theta^-$ or $\theta^+ = -\theta^-$. In the next step we prove that if D is the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbf{R}^m}$, then either $0 \notin D$ or 0 is an extremal point of D. This rules out the alternative $\theta^+ = -\theta^-$. Therefore we conclude that $\theta^+ = \theta^-$ on G, from which (48) easily follows.

Step 3. In order to complete the proof, it remains to show that if D denotes the closure of the convex hull of the essential image of $u|_{]0,T[\times\mathbf{R}^m}$, then either the origin is not contained in D or it is an extremal point of D. Recalling (a), this property is true for the closure C of the convex hull of the essential image of u_0. Choose $\xi_1,\dots,\xi_k$ unit vectors of $\mathbf{R}^k$ such that

$$C \;\subset\; \left\{x \,\middle|\, x\cdot\xi_i \le 0 \text{ for every } i\right\}$$

and 0 is an extremal point of $\{x \mid x\cdot\xi_i \le 0 \text{ for every } i\}$. We show that the essential image of u is contained in $\{x \mid x\cdot\xi_i \le 0\}$ for every i.

Fix i, and denote by $\eta : \mathbf{R}^k \to \mathbf{R}$, $q : \mathbf{R}^k \to \mathbf{R}^m$ the functions

$$\eta(U) := \begin{cases} 0 & \text{if } \xi_i\cdot U \le 0,\\ \xi_i\cdot U & \text{otherwise,}\end{cases} \qquad q(U) := g(|U|)\eta(U).$$

Note that (η, q) is a convex entropy–entropy flux pair. Clearly, $\eta(u_0) = 0$, and a standard argument borrowed from the theory of Kruzhkov for scalar laws implies that $\eta(u) = 0$. We include this argument for the convenience of the reader.

Since $\eta(u_0) = 0$, the boundary term in the entropy inequality (7) disappears. Fix a nonnegative test function $\psi \in C^\infty_c(\mathbf{R}\times\mathbf{R}^m)$, and for every $\tau < T$, define a sequence $\chi_{\tau,i} \subset C^\infty([0,\tau])$ such that

- $\chi_{\tau,i}(s) = 0$ for $s > \tau$;
- $\chi_{\tau,i}(s) = 1$ for $s < \tau - 1/i$;
- $0 \ge \chi'_{\tau,i} \ge -2/i$.

We test (7) with $\psi(s,x)\chi_{\tau,i}(s)$, and we let $i \uparrow \infty$. Using a Fubini-type argument, we conclude that the following inequality is valid for a.e. τ:

$$\int_0^\tau \int_{\mathbf{R}^m} \eta(u)\partial_t\psi(s,x) + q(u)\cdot\nabla_x\psi(s,x)\,ds\,dx - \int_{\mathbf{R}^m} \eta\big(u(\tau,x)\big)\psi(\tau,x)\,dx \geq 0. \tag{53}$$

Using a countable dense set of test functions, we conclude that for a.e. τ, (53) holds for any test function $\psi \in C_c^\infty(\mathbf{R}\times\mathbf{R}^m)$.

Since u is bounded, there is a constant C such that $|g(|U|)| \leq C$, and hence $C\eta(u) \geq |q(u)|$. It is not difficult to show that for every $R > 0$, there exists a non-negative test function ψ such that

$$-\partial_t\psi \geq C|\nabla_x\psi| \quad \text{on } [0,\tau]\times\mathbf{R}^m,$$
$$\psi(\tau,x) = 1 \quad \text{for every } x \in B_R.$$

For such a test function, the first term on the left-hand side of (53) is nonpositive. Thus we get

$$\int_{B_R} \eta\big(u(\tau,x)\big)\,dx \leq \int_{\mathbf{R}^m} \eta\big(u(\tau,x)\big)\psi(\tau,x)\,dx \leq 0.$$

Since this inequality holds for every $R > 0$ and for a.e. $\tau \in]0,T[$, we get $\eta(u) \leq 0$. This completes the proof. □

In view of this proof, the following seems likely.

CONJECTURE 4.2
Let u be a bounded entropy solution of (3), and denote by C the closure of the convex hull of its essential image. If $0 \notin C$ or if it is an extremal point of C, then u is a renormalized entropy solution.

We conclude the section with the following.

Proof of Theorem 2.4
Let u_0^n be the initial data of Proposition 3.1, and let $\lambda > 0$ be such that the corresponding renormalized entropy solutions $u^n(t,\cdot)$ are not in $\mathrm{BV}(B_\lambda(0))$ for any $t \in]0,1[$. Let $\tilde{u}^n$ be any other entropy solution of (3) with initial data u_0^n. For any $c > \|u_0^n\|_\infty$, we apply the argument of Step 3 of the proof of Proposition 4.1 to the entropy $\eta(|u|) := (|u|-c)\mathbf{1}_{|u|\geq c}$. It turns out that $\eta(|u|) = 0$, from which we conclude $\|\tilde{u}^n\|_\infty \leq \|u_0^n\|_\infty$. Hence $\tilde{u}^n$ is uniformly bounded.

Fix $T \in]0,1[$, and let $\gamma \geq 0$ be the supremum of the nonnegative R's such that $\tilde{u}^n \in \mathrm{BV}(]0,T[\times B_R(0))$. From Proposition 4.1, we get the fact that $\tilde{u}^n$ is a renormalized entropy solution on $]0,T[\times B_\gamma(0)$. The proof of [2, Proposition 2.11] gives that there exists a constant $C > 1$ depending only on f and $\|u^n\|_\infty$ such that if $\gamma \geq C\lambda$, then $\tilde{u}^n = u^n$ on $]0, T[\times B_\lambda(0)$. But in this case we have $u^n \in \mathrm{BV}(]0, T[\times B_\lambda)$. Hence we conclude that $\gamma < C\lambda$. Therefore $\tilde{u}^n \notin \mathrm{BV}(]0, T[\times B_{C\lambda}(0))$ for every $T \in]0, 1[$, and hence $\tilde{u}^n \notin \mathrm{BV}(]0, T[\times B_{C\lambda}(0))$ for any positive T. □

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