Min-Max Theory for Minimal Hypersurfaces with Boundary

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Jusuf Ramic

aus

Bosnien und Herzegowina

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Zürich, 2016

Abstract

In this thesis we present a result concerning existence and regularity of minimal surfaces with boundary in Riemannian manifolds obtained via a min-max construction, both in fixed and free boundary context.

We start by considering a smooth, compact, oriented Riemmanian manifold $(\mathcal{M}, g)$ of dimension $n+1$ with a uniform convexity property (the principal curvatures of $\partial\mathcal{M}$ with respect to the inner normal have a uniform positive lower bound). In the spirit of an analogous approach presented in Colding-De Lellis (cf. [10]), we then construct generalized families of hypersurfaces in $\mathcal{M}$, with appropriate boundary conditions. To be more precise, in the fixed boundary setting this will basically mean that these hypersurfaces will have a fixed common boundary γ, which is an $(n-1)$-dimensional smooth, closed, oriented submanifold of $\partial\mathcal{M}$, and in the free boundary setting that their boundaries lie in $\partial\mathcal{M}$. Moreover, we will consider more general parameter spaces for these families. After constructing a suitable homotopy class of such families and assuming that it satisfies a certain "energy gap", we can prove the existence of a nontrivial, embedded, minimal hypersurface (with a codimension 7 singular set) and corresponding boundary conditions.

As a corollary, we consider a special case of two smooth, strictly stable surfaces bounding an open domain A and meeting only in the common boundary $\gamma \subset \partial\mathcal{M}$, and show the existence of a homotopy class with the required energy gap. The main theorem then furnishes the existence of a third minimal surface.

Zusammenfassung

In dieser Arbeit stellen wir ein Resultat über die Existenz und Regularität von Minimalflächen mit Randbedingung in Riemann'sche Mannigfaltigkeiten, erhalten durch Min-Max Konstruktion, vor. Wir betrachten die festen und freien Randbedingungen.

Sei $(\mathcal{M}, g)$ eine glatte, kompakte, orientierbare, $(n + 1)$-dimensionale Riemann'sche Mannigfaltigkeit mit einer uniformen Konvexitätseigenschaft (die Hauptkrümmungen von $\partial\mathcal{M}$ bezüglich dem nach innen gerichteten Normalvektor haben eine uniforme, positive, untere Schranke). Analog zum Vorgehen in Colding-De Lellis (cf. [10]) konstruieren wir verallgemeinerte Familien von Hyperflächen in $\mathcal{M}$ mit entsprechenden Randbedingungen. Genauer, bei dem Problem mit festem Rand bedeutet das, dass diese Hyperflächen einen gemeinsamen Rand γ haben, welcher eine $(n - 1)$-dimensionale, glatte, geschlossene, orientierbare Untermannigfaltigkeit von $\partial\mathcal{M}$ ist. Bei dem Problem mit freiem Rand liegen die Ränder von diesen Hyperflächen in $\partial\mathcal{M}$. Ausserdem betrachten wir allgemeinere Parameterräume für diese Familien. Nachdem wir eine geeignete Homotopieklasse solcher Familien konstruiert haben, und angenommen dass diese Klasse eine bestimmte "Energielücke"-Annahme erfüllt, können wir die Existenz einer nicht trivialen, eingebetteten, minimalen Hyperfläche (mit Kodimension 7 Menge der Singularitäten) mit entsprechender Randbedingung zeigen.

Als Korollar betrachten wir den Spezialfall von zwei glatten, strikt stabilen Hyperflächen, die eine offene Menge A begrenzen und nur auf dem gemeinsamen Rand $\gamma \subset \partial\mathcal{M}$ treffen, und zeigen die Existenz von einer Homotopieklasse mit der benötigten "Energielücke". Das Haupttheorem liefert dann die Existenz von einer dritten Minimalfläche.

Acknowledgements

Firstly, I would like to thank my advisor, Prof. Dr. Camillo De Lellis. His exceptionally keen mathematical mind and unrelenting optimism have been an indispensable source of inspiration and encouragement that got me through these past years. I feel extremely lucky, not only to have had him as an advisor, but to consider him as a friend. Secondly, my deep and sincere gratitude goes to Prof. Dr. Guido De Philippis. His suggestions and input, as well as those of Prof. Dr. Brian White by proxy, have been of invaluable help in the process of completing this thesis.

I would like to thank all my friends and colleagues from the university (Andrea, Giacomo, Luca, Salvatore, Antonio, Annalisa, Davide, Andrea, etc.) for the stimulating discussions and the fun times we had during my time here. Although I am sorry for it, I also feel somehow strangely proud of not succumbing to the pressure to learn almost any Italian, thus making you all speak English in my presence.

Last but not least, I wish to thank my close friends and family, and especially my parents, Mustafa and Amira. Their love and unwavering support is something that I will always cherish and never take for granted.

Contents

Chapter 1

Introduction

1.1 Theory of Minimal Surfaces

The foundations of what would later develop into a mathematical theory of minimal surfaces were laid out by Lagrange and Euler in the 1760s. In its inception, the objects of interest were surfaces which minimize area within a given boundary configuration. Lagrange considered these surfaces as graphs over a plane, sometimes referred to as non-parametric surfaces, and through the use of variational calculus derived the minimal surface equation, which gives a necessary condition that they must satisfy. Meusnier later showed a more geometric but equivalent characterization, which states that such surfaces must have a vanishing mean curvature, i.e. $H = 0$. He also found two (non-trivial) examples of such objects, the helicoid and the catenoid. In the modern sense of the name *minimal surfaces*, the condition $H = 0$ is the most widely used as definition. Hence it refers to surfaces which are critical points of the area functional, but not necessarily minimizers.

Usually, one distinguishes between two major types of boundary configurations when treating the problem of finding these surfaces; the first is when the boundary of the surface is fixed *a priori*, and the second is the so called *free boundary* problem, when the boundary must lie on some given supporting surface S. In the second case, the surface must be stationary for the area with respect to deformations which allow its boundary to move freely on S. This will be made more precise in Chapter 3. A comprehensive collection of nearly all of the historical references presented throughout this introduction (and many more) can be found in the book by Dierkes, Hildebrandt and Sauvigny [14].

1.1.1 A Short History of the Plateau Problem

Although the study of minimal surfaces is interesting from a purely theoretical point of view, a connection to real world phenomena was established by the Belgian physicist Plateau, when he conducted experiments with soap films. He conjectured that every closed wire frame, when dipped into a soap solution, should bound a surface of least energy. In mathematical terms, it will correspond to a minimal surface, and the problem of finding one (i.e. showing the existence thereof) was later named as the *Plateau problem.* However, there are many ways to precisely formulate such a problem, depending on the definition of the words *surface* and *boundary.*

When considering the nonparametric definition of surfaces as mentioned above, the first general existence proof was published by Haar [26] in 1927, with important contributions from Rado. The more general (and interesting) formulation of the problem was the parametric one, which was to find a surface parametrized by a two dimensional disc of least area spanning a given rectifiable Jordan curve. The breakthrough came in the early 1930s, when Douglas [15] and Rado [43] independently proved the existence of a solution. A generalization to Riemannian manifolds is due to Morrey [37].

A natural question which arises once existence of a solution is known is its regularity. In the case of the Douglas-Rado solution, the surface is known only to be an immersion, after Osserman, Gulliver and Alt [40],[25],[6] excluded the existence of interior branch points. In particular, it might have self-intersections (and in a lot of situations it will). The question of boundary branch points is still open, unless the boundary curve is real analytic (see [24]). In order to achieve embeddedness of the solution, one has to impose certain restrictions on the boundary curve. For instance, if the boundary curve lies inside the C^2 boundary ∂K of a mean convex set K, the minimal disk is embedded by a result of Meeks and Yau [36] (see also Almgren-Simon [3]).

In view of these drawbacks and in order to attack the Plateau problem in more generality, it was clear that the parametric formulation is not very well suited. Perhaps the greatest disadvantage is that one *a priori* fixes the topology of the minimal surface to be that of a disk. This in particular makes it difficult for minimal surfaces to be embedded (by constructions similar to that of Almgren and Thurston in [4], one can produce unknotted boundary curves for which the only embedded minimal surfaces have an arbitrarily prescribed lower bound on the genus). Moreover, the Douglas-Rado approach consists of minimizing the Dirichlet energy, which has better functional analytic properties than the area functional, in a suitable class of surfaces. It then relies on the facts that for conformally parametrized surfaces the two functionals are equal, and that by the celebrated Lichtenstein theorem, in two dimensions one can always find conformal parametrizations. This theorem is false in general for higher dimensions.

Using tools from geometric measure theory, several different generalized notions of surfaces were introduced to overcome the difficulties presented above, the most notable of which perhaps are currents and varifolds. Loosely speaking, both of these notions represent classes of objects which are regular enough to be considered surfaces in some weak sense, and at the same time general enough to have good compactness properties, ensuring the existence of area minimizers through the use of the direct method of calculus of variations (both of these objects will be introduced in more detail in Chapter 2). In the framework of integer rectifiable currents, one is able to formulate and solve the Plateau problem in general dimension (and codimension), since there is a natural notion of a boundary for such surfaces.

As for the regularity of such objects, in the codimension 1 case of n-dimensional surfaces in $\mathbb{R}^{n+1}$ (which is of most interest to us in this thesis), the regularity theory due to De Giorgi, Fleming, Almgren, Simons and Federer [11],[18],[2],[50],[17] shows that in the interior the minimizing current is smooth and embedded for $n \leq 6$, has isolated singularities for $n = 7$ and has a codimension 7 singular set for $n \geq 8$. Moreover, this regularity is sharp by a paper of Bombieri, De Giorgi and Giusti [9], which shows the (7-dimensional) Simons' cone to be area minimizing in $\mathbb{R}^8$. The question of boundary regularity was settled by Hardt and Simon [27], who show that if the boundary of the minimizing current is an oriented, closed $(n-1)$-dimensional $C^{1,\alpha}$ submanifold, then at every boundary point it is a C^1 manifold with boundary. We briefly mention that this regularity results come at a cost that, due to the weak notion of convergence, one has in general no control over the topological type of the minimal surface.

1.1.2 Min-Max Method

Since minimal surfaces are defined as critical points of the area and not merely as minimizers, another important problem involves finding minimal surfaces of higher index. A natural setting to use in this problem would be the theory of critical points due to M. Morse and to L.A. Lusternik and L. Schnirelmann, which generally relates topological properties of a manifold to the number and type of critical points of certain functionals defined on it. For the purpose of this thesis, we are interested in a special part of that theory, which is a type of statement commonly referred to as the "Mountain Pass Theorem". The name stems from an appropriate topographical analogy; given two points which are separated by a "wall" of high elevation (in particular if they are strict local minima), there exists a path between them with a "mountain pass", which is an unstable critical point of saddle type (i.e. unstable) of the elevation function. To find this point, heuristically speaking, one would find a point of maximum elevation on a path between the two points, and then minimize among all possible paths - hence the label "min-max" is sometimes used for this type of argument.

First results of this kind for parametric surfaces, which show the existence of unstable minimal surfaces spanning a given boundary, were proved simultaneously by Shiffman [48] and by Morse and Tompkins [38]. A nice exposition of Shiffman's approach can be found in the treatise of Nitsche [39]. We will state the theorem here, as an appropriate analogy to the result in this thesis can be drawn later on. We define, for a rectifiable [1] Jordan curve Γ, the space of surfaces

$$\begin{aligned}\mathfrak{H}(\Gamma) = \{X \in H^{1,2}(B,\mathbb{R}^3) \cap C^0(\bar{B},\mathbb{R}^3) \mid X \text{ harmonic in } B,\ X|_{\partial B} : \partial B \to \Gamma \text{ is} \\ \text{weakly monotonic and fixed for 3 points } y_i \in \partial B, i = 0, 1, 2\},\end{aligned}$$

[1] there is an additional condition on this curve which we omit here, see §25 in [39]

where B is the unit 2-dimensional disk, and endow this space with the C^0 topology. Now let $X_1, X_2 \in \mathfrak{H}(\Gamma)$ be two maps with Dirichlet integrals $D(X_i) < \infty$, and let $\mathcal{I}$ be a closed connected subset of $\mathfrak{H}$ containing these maps. We denote $d[X_1, X_2] = \inf_{\mathcal{I}} \sup_{X \in \mathcal{I}} D(X)$. We then say that the two maps X_1 and X_2 are *separated by a wall* if

$$d[X_1, X_2] - \max\{D(X_1), D(X_2)\} > 0. \tag{1.1.1}$$

We can now state the theorem as follows:

Theorem 1. *Let the space $\mathfrak{H}(\Gamma)$ contain two different maps which define generalized*[2] *minimal surfaces (by the isoperimetric inequality, these surfaces have finite Dirichlet integrals). If X_1 and X_2 are separated by a wall of positive elevation, then there exists another map (distinct from X_1 and X_2) which defines a generalized minimal surface.*

Using a degree theory approach to the Plateau problem, Tromba [56, 55] was able to derive a limited Morse theory for disk-type surfaces, generalizing the Morse-Shiffman-Tompkins result. M. Struwe [52, 53, 54] then developed a general Morse theory for minimal surfaces of disk and annulus type, based on the $H^{1,2}$ topology (as opposed to C^0 topology used in Morse-Shiffman-Tompkins approach). These (and other related works) were expanded by Jost and Struwe in [29], where they consider minimal surfaces of arbitrary topological type. Among other things, they succeed in applying saddle-point methods to prove the existence of unstable minimal surfaces of prescribed genus.

Min-max arguments have also been used to show existence of closed minimal submanifolds in compact Riemannian manifolds. This approach is particularly effective in producing closed geodesics. In this setting, the two local minimizers correspond to constant maps γ_0 and γ_1, and the paths between them to 1-parameter families $\{\gamma_t\}$ of maps from $\mathbf{S}^1$ into the manifold. Whereas in the Morse-Tompkins and Shiffman result one had to work with a weaker topology in order to obtain a necessary compactness argument, in the case of curves it is the energy *functional* (defined on an appropriate space) that satisfies the so called Palais-Smale compactness condition (C), which (along with some other topological properties) thus enabled Lyusternik and Fet [32] to prove existence of closed geodesics in arbitrary closed Riemannian manifolds via a mountain pass argument. This had already been done in the case of manifolds homeomorphic to the sphere, by Birkhoff [8] back in 1917. Unfortunately, one cannot apply the same method directly to produce closed minimal submanifolds, due to lack of such compactness. Nevertheless, for 2-spheres, there is a famous work by Sacks and Uhlenbeck [44] where they consider a family of perturbed energy functionals that (unlike the Dirichlet energy) satisfy the condition (C), find their critical points with a mountain pass argument, and finally obtain the minimal surface as a limit for these critical points.

[2] this means isolated branch points are allowed, see definition in §283 of [39]

A renewed effort to produce closed minimal surfaces via the min-max construction using tools from geometric measure theory started with Almgren [5], who showed the existence of stationary varifolds in arbitrary dimension and codimension. His student J. Pitts [41] later expanded on this work and proved regularity in the codimension one case. More precisely, he showed the existence of smooth, closed, embedded minimal hypersurfaces in Riemannian manifolds of dimension less than or equal to 6, since the curvature estimates of Schoen, Simon and Yau [47], valid in those dimensions, were crucially used. These estimates were improved by Schoen and Simon [46], which enabled them to extend Pitts' existence proof to arbitrary dimensions, with the addition of a possible singular set of codimension 7 to the hypersurface.

Currently, interest in min-max constructions is once again starting to gain traction. This is largely due to the celebrated recent work of Marques and Neves [33] which used Almgren-Pitts min-max theory to solve a long-standing conjecture of Willmore in differential geometry.

Unfortunately, due to the lack of Hilbert space structure and the Palais-Smale condition in the approach of Almgren and Pitts, it turns out to be very difficult to extract information on the Morse index, or to establish genus bounds for the min-max minimal surface. The main problem is the very weak notion of convergence (in the sense of varifolds). In fact, no general information on the Morse index in Pitts' construction was known until, very recently, such bounds were established by Marques and Neves [34]. There is however a variant of Almgren-Pitts theory for 3-dimensional ambient manifols, attributed to Simon and Smith, which allows to control the topology of the min-max minimal surface, see [51], [42], [12], [30]. This variant is restricted to three dimensions, as it crucially uses a famous result by Meeks, Simon and Yau [35] that is not available in higher dimensions. We omit further details here, as it goes beyond the scope of this thesis. M. Grüter and J. Jost [23] [28] used the Simon-Smith variant of min-max theory to construct embedded minimal disks with free boundary when the ambient manifold $\mathcal{M}$ is diffeomorphic to the sphere and $\partial\mathcal{M}$ is mean convex. Within the same setting, Martin Man-chun Li [31] proved existence of minimal surfaces with free boundaries in 3-manifolds, without any convexity assumption on $\partial\mathcal{M}$.

1.2 Setting and Main Result

Pitts' groundbreaking monograph introduced new ideas used in many subsequent works, including this thesis, which implements the same general approach in order to prove existence and regularity of minimal hypersurfaces with certain boundary conditions. In fact, Pitts' proof was shortened considerably by De Lellis and Tasnady in [13], and we will develop our min-max theory in the framework introduced in that paper, which essentially is a variant of that developed by Pitts. This allows us to avoid a lot of technical details and yet be sufficiently self-contained. However, several of the tools developed in this thesis can be applied to a suitable modification of Pitts' theory as well and we believe that the same statements can be proved in that framework.

We start by considering a smooth, compact, oriented Riemannian manifold $(\mathcal{M}, g)$ of dimension $n+1$ with boundary $\partial\mathcal{M}$. We will assume that $\partial\mathcal{M}$ is strictly uniformly convex, namely:

Assumption 1.1. *The principal curvatures of $\partial\mathcal{M}$ with respect to the unit normal ν pointing inside $\mathcal{M}$ have a uniform, positive lower bound.*

Sometimes we write the condition above as $A_{\partial\mathcal{M}} \succeq \xi g$, where $\xi > 0$, $A_{\partial\mathcal{M}}$ denotes the second fundamental form of $\partial\mathcal{M}$ (with the choice of inward pointing normal) and g the induced metric as submanifold of $\mathcal{M}$. We also note that we do not really need $\mathcal{M}$ to be C^∞ since a limited amount of regularity (for instance $C^{2,\alpha}$ for some positive α) suffices for all our considerations, although we will not pay any attention to this detail.

We proceed by recalling the continuous families of hypersurfaces used in [13].

Definition 1.2. *We fix a smooth compact k-dimensional manifold $\mathcal{P}$ with boundary $\partial\mathcal{P}$ (possibly empty) and we will call it the* space of parameters.

A smooth family of hypersurfaces in $\mathcal{M}$ parametrized by $\mathcal{P}$ is given by a map $t \mapsto \Gamma_t$ which assigns to each $t \in \mathcal{P}$ a closed subset Γ_t of $\mathcal{M}$ and satisfies the following properties:

- *(s1) For each t there is a finite $S_t \subset \mathcal{M}$ such that Γ_t is a smooth* oriented *hypersurface in $\mathcal{M} \setminus S_t$ with boundary $\partial\Gamma_t \subset \partial\mathcal{M} \setminus S_t$;*
- *(s2) $\mathcal{H}^n(\Gamma_t)$ is continuous in t and $t \mapsto \Gamma_t$ is continuous in the Hausdorff sense;*
- *(s3) on any $U \subset\subset \mathcal{M} \setminus S_{t_0}$, $\Gamma_t \xrightarrow{t \to t_0} \Gamma_{t_0}$ smoothly in U.*

From now on we will simply refer to such objects as *families parametrized by $\mathcal{P}$* and we will omit to mention the space of parameters when this is obvious from the context. Additionally we will distinguish between two classes of smooth families according to their behaviour at the boundary $\partial\mathcal{M}$.

Definition 1.3. *Consider a smooth, closed submanifold $\gamma \subset \partial\mathcal{M}$ of dimension $n-1$. A smooth family of hypersurfaces parametrized by $\mathcal{P}$ is constrained by γ if $\partial\Gamma_t \setminus S_t = \gamma \setminus S_t$ for every $t \in \mathcal{P}$. Otherwise we talk about "uncostrained families".*

Two unconstrained families $\{\Gamma_t\}$ and $\{\Sigma_t\}$ parametrized by $\mathcal{P}$ are homotopic if there is a family $\{\Lambda_{t,s}\}$ parametrized by $\mathcal{P} \times [0,1]$ such that

- $\Lambda_{t,0} = \Sigma_t \ \forall t \in \mathcal{P}$,
- $\Lambda_{t,1} = \Gamma_t \ \forall t \in \mathcal{P}$,
- *and $\Lambda_{t,s} = \Lambda_{t,0} \ \forall t \in \partial\mathcal{P}$ and for all $s \in [0,1]$.*

When the two families are constrained by γ we then additionally require that the family $\{\Lambda_{t,s}\}$ is also constrained by γ.

Finally a set X of constrained (resp. unconstrained) families parametrized by the same $\mathcal{P}$ is called homotopically closed if X includes the homotopy class of each of its elements.

Definition 1.4. *Let X be a homotopically closed set of constrained (resp. unconstrained) families parametrized by the same $\mathcal{P}$. The min-max value of X, denoted by $m_0(X)$ is the number*

$$m_0(X) = \inf \left\{ \max_{t\in\mathcal{P}} \mathcal{H}^n(\Sigma_t) : \{\Sigma_t\} \in X \right\} . \tag{1.2.1}$$

The boundary-max value of X is instead

$$bM_0(X) = \max_{t\in\partial\mathcal{P}} \{\mathcal{H}^n(\Sigma_t) : \{\Sigma_t\} \in X\} . \tag{1.2.2}$$

A minimizing sequence is given by a sequence of elements $\{\{\Sigma_t\}^\ell\} \subset X$ such that

$$\lim_{\ell\uparrow\infty} \max_{t\in\mathcal{P}} \mathcal{H}^n(\Sigma_t^\ell) = m_0(X).$$

A min-max sequence is then obtained from a minimizing sequence by taking the slices $\{\Sigma^\ell_{t_\ell}\}$, for a choice of parameters $t_\ell \in \mathcal{P}$ such that $\mathcal{H}^n(\Sigma^\ell_{t_\ell}) \to m_0(X)$.

As it is well known, even the solutions of the codimension one Plateau problem can exhibit singularities if the dimension $n+1$ of the ambient manifold is strictly larger than 7. If we say that an embedded minimal hypersurface Γ is *smooth* then we understand that it has no singularities. Otherwise we denote by $\mathrm{Sing}\,(\Gamma)$ its closed singular set, i.e. the set of points where Γ cannot be described locally as the graph of a smooth function. Such singular set will always have Hausdorff dimension at most $n-7$ and thus with a slight abuse of terminology we will anyway say that Γ is embedded, although in a neighborhood of the singularities the surface might not be a continuous embedded submanifold. When we write $\dim(\mathrm{Sing}\,(\Gamma)) \leq n-7$ we then understand that the singular set is empty for $n \leq 6$.

Our main theorem is the following.

Theorem 1.5. *Let $\mathcal{M}$ be a smooth Riemannian manifold that satisfies Assumption 1.1 and X be a homotopically closed set of constrained (resp. unconstrained) families parametrized by $\mathcal{P}$ such that*

$$m_0(X) > bM_0(X) . \tag{1.2.3}$$

Then there is a min-max sequence $\{\Sigma^\ell_{t_\ell}\}$, finitely many disjoint embedded and connected minimal hypersurfaces $\{\Gamma_1, \ldots, \Gamma_N\}$ with boundaries $\partial\Gamma_i \subset \partial\mathcal{M}$ (possibly empty) and finitely many positive integers c_i such that

$$\Sigma^\ell_{t_\ell} \quad \rightharpoonup^* \quad \sum_i c_i \Gamma_i$$

in the sense of varifolds and $\dim(\mathrm{Sing}\,(\Gamma_i)) \leq n-7$ *for each i. In addition:*

(a) If X consists of unconstrained families, then each Γ_i meets $\partial\mathcal{M}$ orthogonally;

(b) If X consists of families constrained by γ, then: $\sum \partial\Gamma_i = \gamma$, $\operatorname{Sing}(\Gamma_i) \cap \partial\mathcal{M} = \emptyset$ for each i and $c_i = 1$ whenever $\partial\Gamma_i \neq \emptyset$.

We can immediately draw a parallel with Theorem 1, since the assumption (1.2.3) for $\mathcal{P} = [0,1]$ can be interpreted as surfaces Σ_0 and Σ_1 being "separated by a wall", similar to the assumption (1.1.1) in that theorem. Thus it can be understood in the spirit of a mountain-pass theorem in a somewhat generalized sense.

Our main concern is in fact the case (b) of Theorem 1.5, because the regularity at the boundary requires much more effort. The regularity at the boundary for the case (a) is instead much more similar to the usual interior regularity for minimal surfaces and for this reason we will not spend much time on it but rather sketch the needed changes in the arguments. As an application of the main theorem we give the following two interesting corollaries.

Corollary 1.6. *Under the assumptions above there is always a nontrivial embedded minimal hypersurface Γ in $\mathcal{M}$, meeting the boundary $\partial\mathcal{M}$ orthogonally, with* $\dim(\operatorname{Sing}(\Gamma)) \leq n-7$

Note that the corollary above does not necessarily imply that Γ has nonempty boundary; we do not exclude that Γ might be a closed minimal surface. On the other hand, if Γ has nonempty boundary, then it is contained in $\partial\mathcal{M}$ and any connected component of Γ is thus a nontrivial solution of the free boundary problem. Therefore the existence of such nontrivial solution is guaranteed by the following

Assumption 1.7. *$\mathcal{M}$ does not contain any nontrivial, minimal, closed hypersurface Σ, embedded and smooth except for a singular set* $\operatorname{Sing}(\Sigma)$ *satisfying* $\dim(\operatorname{Sing}(\Sigma)) \leq n-7$.

Note that the property above holds if $\mathcal{M}$ satisfies some stronger convexity condition than Assumption 1.1: for instance if there is a point p such that $\mathcal{M}\setminus\{p\}$ can be foliated with convex hypersurfaces, it follows from the maximum principle. In particular both the Assumptions 1.1 and 1.7 are satisfied by any bounded convex subset of the Euclidean space, or by any ball of a closed Riemannian manifold with radius smaller than the convexity radius.

Likewise, under the very same assumptions we can conclude the following Morse-theoretical result for the Plateau's problem.

Corollary 1.8. *Let $\mathcal{M}$ be a smooth Riemannian manifold satisfying Assumptions 1.1 and 1.7 and let $\gamma \subset \partial\mathcal{M}$ be a smooth, oriented, closed $(n-1)$-dimensional submanifold. Assume further that:*

(i) there are two distinct smooth, oriented, minimal embedded hypersurfaces Σ_0 and Σ_1 with $\partial\Sigma_0 = \partial\Sigma_1 = \gamma$ which are strictly stable, meet only at the boundary and bound some open domain A (in particular Σ_0 and Σ_1 are homologous).

Then there exists a third distinct embedded minimal hypersurface Γ_2 *with* $\partial\Gamma_2 = \gamma$ *such that* $\dim(\operatorname{Sing}(\Gamma_2)) \leq n-7$ *and* $\operatorname{Sing}(\Gamma_2) \cap \partial\mathcal{M} = \emptyset$.

The corollary above asks for two technical assumptions which are not really natural:

- Σ_0 and Σ_1 intersect only at the boundary;
- they are regular *everywhere*.

We use both to give an elementary construction of a 1-dimensional sweepout which "connects" Σ_0 and Σ_1 (i.e. a one-parameter family $\{\Sigma_t\}_{t\in[0,1]}$), but by taking advantage of more avdanced techniques in geometric measure theory and algebraic topology - as for instance Pitts' approach via discretized faimly of currents, it should suffice to assume that Σ_0 and Σ_1 are homologous and that the dimension of their singular set does not exceed $n-7$. The smoothness enters however more crucially in showing that any sweepout (i.e. smooth family of hypersurfaces) connecting Σ_0 and Σ_1 must have a "slice" with n-dimensional volume larger than $\max\{\mathcal{H}^n(\Sigma_0), \mathcal{H}^n(\Sigma_1)\}$. It is needed to take advantage of an argument of White [57], where regularity is a key ingredient. The following local minimality property could replace strict stability and smoothness:

- For each $i \in \{1,2\}$ there is a $\varepsilon > 0$ such that any current Γ with boundary γ which is distinct from Γ_i and at flat distance smaller than ε from Γ_i has mass strictly larger than that of Γ_i.

Chapter 2

Preliminaries

2.1 Notation

Since we are always dealing with manifolds $\mathcal{M}$ which have a nonempty boundary, as it is customary an open subset U of $\mathcal{M}$ can contain a portion of $\partial\mathcal{M}$. For instance, if $\mathcal{M}$ is the closed unit ball in $\mathbb{R}^{n+2}$, P the north pole $(0, 0, \dots, 1)$ and $\tilde{U}$ a neighborhood of P in $\mathbb{R}^{n+2}$, then $\tilde{U} \cap \mathcal{M}$ is, in the relative topology, an open subset of $\mathcal{M}$. Hence, although we will denote by $\mathrm{Int}\,(\mathcal{M})$ the set $\mathcal{M} \setminus \partial\mathcal{M}$, the latter *is not* the topological interior of $\mathcal{M}$ and our notation is slightly abusive. In the following table we present notations, definitions and conventions used consistently throughout the thesis:

$B_\rho(x)$, $\overline{B}_\rho(x)$	open and closed geodesic balll of radius ρ and center x in $\mathcal{M}$;
$\partial B_\rho(x)$	geodesic sphere of radius ρ and center x in $\mathcal{M}$
$\mathrm{Int}\,(U)$	"interior" of the open set U, namely $U \setminus \partial\mathcal{M}$;
$\mathrm{Inj}(\mathcal{M})$	injectivity radius of $\mathcal{M}$;
$\mathrm{An}(x, \tau, t)$	open annulus $B_t(x) \setminus \overline{B}_\tau(x)$;
$\mathcal{AN}_r(x)$	the set $\{\mathrm{An}(x, \tau, t)$ with $0 < \tau < t < r\}$;
$\mathrm{diam}(G)$	diameter of a subset $G \subset \mathcal{M}$;
$\mathcal{H}^k$	k-dim Hausdorff measure in $\mathcal{M}$;
ω_k	volume of the unit ball in $\mathbb{R}^k$;
ν	unit normal to $\partial\mathcal{M}$, pointing *inwards*
spt	support (of a function, vector field, varifold, current, etc.);
$\mathfrak{X}_c(U)$	smooth vector fields χ with $\mathrm{spt}(\chi) \subset U$;
$\mathfrak{X}_c^0(U)$	$\chi \in \mathfrak{X}_c(U)$ which vanish on $\partial\mathcal{M}$;
$\mathfrak{X}_c^t(U)$	$\chi \in \mathfrak{X}_c(U)$ tangent to $\partial\mathcal{M}$, i.e. $\chi(x) \in T_x\partial\mathcal{M}\ \forall x \in \partial\mathcal{M}$
$\mathfrak{X}_c^-(U)$	$\chi \in \mathfrak{X}_c(U)$ pointing inwards at $\partial\mathcal{M}$ (i.e. $\chi \cdot \nu \geq 0$)

$\mathcal{V}^k(U), \mathcal{V}(U)$	vector space of k-varifolds in U;
$G^k(U), G(U)$	Grassmanian bundle of unoriented k-planes on U;
$[\![S]\!]$	(rectifiable) current induced by the k-dimensional submanifold S (taken with multiplicity 1);
$\mathbf{M}(S)$	mass norm of a current S;
$\mathbb{F}(S)$	flat norm of a current S;
$\mathrm{v}(R, \theta)$	varifold induced by the k-rectifiable set R, with multiplicity θ;
W_0	the set $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \lvert x_{n+1}\rvert \leq x_1 \tan\theta\}$ with $\theta \in]0, \frac{\pi}{2}[$, which we refer to as the *canonical wedge with opening angle* θ.

Note that all of the different spaces of vector fields introduced above, namely $\mathfrak{X}_c(U), \mathfrak{X}_c^0(U), \mathfrak{X}_c^t(U)$ and $\mathfrak{X}_c^-(U)$, coincide when $U \cap \partial\mathcal{M} = \emptyset$. Otherwise $\mathfrak{X}_c^0(U)$, $\mathfrak{X}_c^t(U)$ and $\mathfrak{X}_c^-(U)$ are all proper subsets of $\mathfrak{X}_c(U)$ since we could construct a vector field $\chi \in \mathfrak{X}_c(U)$ such that $\chi(x) \cdot \nu < 0$ for some $x \in U \cap \partial\mathcal{M}$. Additional clarification is provided at the beginning of Chapter 3.

2.2 Varifolds and Currents

For the notation and terminology about currents and rectifiable varifolds we will follow [49], and we refer the reader to it for more details. Here we present the definitions and recall briefly some basic facts.

If U is an open subset of $\mathcal{M}$, we define a *k-varifold* ($k \leq n+1$) in U to be a Radon measure on the Grassmanian bundle $G^k(U)$ of unoriented k-planes on U, and we denote the space of k-varifolds by $\mathcal{V}^k(U)$. When it is clear from the context, we will sometimes drop the exponent k in the notations.
The space $\mathcal{V}(U)$ is endowed with the topology of weak convergence of measures, thus a sequence $\{V^k\} \subset \mathcal{V}(U)$ converges to a varifold V if

$$\lim_{k\to\infty} \int \varphi(x,t)\, dV^k(x,\pi) = \int \varphi(x,t)\, dV(x,\pi)$$

for every function $\varphi \in C_c(G^n(U))$, where π is a k-plane in $T_x\mathcal{M}$. To any k-varifold V in U we can associate a unique Radon measure $||V||$ obtained by pushing V forward with the projection p of $G^k(U)$ onto U, so $||V||(A) = V(p^{-1}(A))$, $A \subset U$. The number $||V||(A)$ will be called *the mass of* V *in* U.
If $R \subset U$ is a k-rectifiable set, i.e. a countable union of closed subsets of C^1 k-dimensional submanifolds of $\mathcal{M}$ (modulo sets of $\mathcal{H}^k$-measure 0), and if $h : R \to \mathbb{R}$ is a Borel map, we can define a varifold V by

$$\int \varphi(x,\pi)\, dV(x,\pi) = \int_R h(x)\varphi(x, T_xR)\, d\mathcal{H}^k(x).$$

Here, T_xR is the approximate tangent space to R. We say that such a varifold is *induced* by R and we will sometimes denote it by $\mathrm{v}(R,h)$, or simply R when it is clear from the context. Moreover, if the multiplicity h is integer-valued, we say it is an *integer rectifiable varifold.*
If $V \in \mathcal{V}(U)$, then for any diffeomorphism $\psi : U \to U'$ we can define the push-forward $\psi_\sharp V \in \mathcal{V}(U')$ by

$$\int_{G(U')} \varphi(y,\sigma)\, d(\psi_\sharp V)(y,\sigma) = \int_{G(U)} J\psi(x,\pi)\, \varphi(\psi(x), d\psi_x(\pi))\, dV(x,\pi) \tag{2.2.1}$$

where $(y,\sigma) = (\psi(x), d\psi_x(\pi))$, and $J\psi(x,\pi)$ denotes the Jacobian determinant (i.e. area element) of the differential $d\psi_x$ restricted to the plane π. This allows to define variations with respect to local deformations. If $\psi_t := \psi(x,t)$ represents the one-parameter family of smooth maps generated by vector fields[1] $\chi \in \mathfrak{X}_c$ i.e. $\frac{\partial \psi}{\partial t} = \chi$, then the first variation of a k-varifold V with respect to χ is given as

$$[\delta V](\chi) := \frac{d}{dt}(||(\psi_t)_\sharp V||)\Big|_{t=0} = \int_{G^k(U)} \mathrm{div}_\pi \chi(x)\, dV(x,\pi), \tag{2.2.2}$$

where the second equality is obtained by differentiating under the integral in (2.2.1). A varifold V is said to be *stationary in U* if $[\delta V](\chi) = 0$ for any $\chi \in \mathfrak{X}^0_c(U)$. When the varifold is stationary for every open set U, it is called a *stationary varifold.* In the case when the varifold is induced by a surface, this corresponds to the surface being minimal.

A *k-dimensional current* (or simply a k-current) in some open set $U \subset \mathcal{M}$ is a continuous linear functional on the space $\mathcal{D}^k(U)$ of smooth differential n-forms supported in U. The space of k-currents in U is usually denoted by $\mathcal{D}_k(U)$. If S is a k-dimensional oriented submanifold of $\mathcal{M}$, or more generally a k-rectifiable set, we can associate to it a k-current $[\![S]\!] \in \mathcal{D}_k(\mathcal{M})$ defined by

$$[\![S]\!](\omega) = \int_S \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^k(x), \quad \omega \in \mathcal{D}^k(\mathcal{M})$$

where $\xi : S \to \Lambda_k\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} \Lambda_k(T_p\mathcal{M})$ orients S (in case S is k-rectifiable it is an $\mathcal{H}^k$-measurable function such that for $\mathcal{H}^k$-a.e. $x \in S$, $\xi(x)$ can be expressed as $\tau_1 \wedge \ldots \wedge \tau_k$, where $\tau_1, \ldots, \tau_k$ is an orthonormal basis for T_xS), and $\langle , \rangle$ denotes the dual pairing between $\Lambda^k\mathcal{M}$ and $\Lambda_k\mathcal{M}$. Additionally, one can multiply the above integrand with a locally $\mathcal{H}^k$-integrable positive function θ called the *multiplicity.* Generally, such currents are called *rectifiable*, and in the special case where θ is integer-valued, the current is said to be an *integer rectifiable current.*

Viewed as generalized surfaces, currents allow for a natural notion of a boundary, whose definition is motivated by the classical Stokes' theorem (in case the

[1] technically, if at $\partial\mathcal{M}$ the vector field points outwards, one would use Remark 5.3, and thus generate an isotopy in $\tilde{\mathcal{M}}$ which does not preserve $\mathcal{M}$

current is induced by a smooth submanifold with boundary). Thus we define the boundary $\partial T \in \mathcal{D}_{k-1}(\mathcal{M})$ of a k-current $T \in \mathcal{D}_k(\mathcal{M})$ by

$$\partial T(\omega) = T(d\omega), \quad \omega \in \mathcal{D}^k(\mathcal{M}).$$

Although the boundary of an integer rectifiable current T may not be integer rectifiable in general, if it is, then the current T is called an *integral current.*

In addition to the weak* topology on $\mathcal{D}_k(\mathcal{M})$ (which comes naturally equipped by duality), one usually introduces two other important topologies (induced by seminorms) on the space of k-currents: the mass norm $\mathbf{M}$, and the flat norm $\mathbb{F}$. These are defined in the following way for a k-current T:

$$\begin{aligned}\mathbf{M}(T) &= \sup\{T(\omega) \,:\, \sup \|\omega(x)\|^* \leq 1\},\\ \mathbb{F}(T) &= \inf\{\mathbf{M}(A) + \mathbf{M}(B) \,:\, T = A + \partial B,\ A \text{ and } B \text{ integer rectifiable}\},\end{aligned}$$

where $\|.\|^*$ denotes the comass norm for $\Lambda^k(\mathcal{M})$. Note that in case $T = [\![S]\!]$ we have $\mathbf{M}(T) = \mathcal{H}^k(S)$. Furthermore, by the compactness theorem proved by Federer and Flemming (27.3 in [49]), the weak and the flat topology coincide on the space of integral currents with bounded mass and bounded mass of the boundary.

We remark here that in the following, we will only work with integer multiplicity currents. However we warn the reader that, unless we specify that a given varifold is integer rectifiable, in general it will be not and will be understood as a suitable measure on the space of Grassmanians, according to the definition above.

2.3 Outline of the Thesis

As mentioned before, we will follow the presentation in [10], and hence the general outline will be very similar. First of all in the Section 3 we will introduce two adapted classes of stationary varifolds for the constrained and unconstrained case, which are a simple variants of the usual notion of stationary varifold introduced by Almgren, defined in the preceding section. Then in Proposition 3.2 we prove the existence of a suitable sequence of families $\{\{\Sigma_t\}^\ell\}$ in X with the property that each min-max sequence generated by it converges to a stationary varifold: the argument is a straightforward adaptation of Almgren's pull-tight procedure used in Pitts' book and in several other later references. Conceptually, this step is usually attributed to Birkhoff in the case of geodesics, where it is sometimes known as Birkhoff's "curve shortening map". In this thesis we also need to accommodate for the fact that the surfaces in a smooth family which are parametrized by $t \in \partial\mathcal{P}$ must remain fixed.

In the sections 4.1 and 4.2 we adapt the notion of almost minimizing surfaces used in [13] to the case at the boundary, as well as to the more general parameter space, and we ultimately prove the existence of a min-max sequence which is almost minimizing in any sufficiently small annulus centered at any given point, cf. Proposition 4.3. The arguments follow closely those used by Pitts in [41] and a trick introduced in [13] to avoid Pitts' discretized families. The min-max sequence generated in Proposition 4.3 is the one for which we will conclude the properties claimed in Theorem 1.5. Indeed the interior regularity follows from the arguments of Pitts (with a suitable adapation by Schoen and Simon to the case $n \geq 6$) and we refer to [13] for the details. The remaining sections are thus devoted to the boundary regularity.

First of all in Chapter 5 we collect several tools about the boundary behavior of stationary varifolds (such as the monotonicity formulae in both the constrained and uncostrained case and a useful maximum principle in the constrained one), but more importantly, we will use a very recent argument of White to conclude suitable curvature estimates at the boundary in the constrained case, under the assumption that the minimal surface meets $\partial\mathcal{M}$ transversally in a suitable (quantified) sense, cf. Theorem 5.10.

In Chapter 6 we recall the celebrated Schoen-Simon compactness theorem for stable minimal hypersurfaces in the interior and its variant by Grüter and Jost in the free boundary case. Moreover, we combine the Schoen-Simon theorem with Theorem 5.10 to conclude a version of the Schoen-Simon compactness theorem for stable hypersurfaces up to the boundary, when the latter is a fixed given smooth γ and the surfaces meet $\partial\mathcal{M}$ transversally. In Section 6.5 we will then adapt the maximum principle to show that any stationary varifold produced in the constrained case meet the boundary at an angle which is uniformly positive.

In Chapter 7 we modify the proof in [13] to construct replacements for almost minimizing varifolds. The main difficulty and contribution here is to preserve the boundary conditions for the surfaces in the constrained case, throughout the various steps of the construction. Following the arguments in [13], we analogously define the $(2^{m+2}j)^{-1}$ - homotopic Platau problem for $j \in \mathbb{N}$, and we conclude that in sufficiently small balls, the corresponding minimizers are actually minimizing for the Plateau problem. Hence their regularity (with no singular points!) at the boundary will follow from Allard's boundary regularity in [1] (see also Hardt-Simon [27]). Furthermore, we are in position to apply the tools of Chapter 6 and 7, which are then used in Chapter 8 to conclude the boundary regularity and hence the proof of Theorem 1.5.

Finally, in the last Chapter we provide the proofs to Corollaries 1.6 and 1.8. The main challenge here is to find a homotopically closed set of constrained families and prove that it satisfies Assumption 1.2.3, namely the mountain-pass condition.

Chapter 3

Existence of Stationary Varifolds

The first step in the min-max construction consists of finding a *nice* minimizing sequence having the property that *any* min-max sequence belonging to it converges to a stationary varifold. From now on we will denote the subset of stationary varifolds by $\mathcal{V}_s(\mathcal{M})$ (or simply $\mathcal{V}_s$). We will however consider two slightly smaller subclasses of $\mathcal{V}_s$, depending on whether we are dealing with the constrained or unconstrained problem. To get an intuition consider the one-parameter families of smooth maps Φ_τ generated by vector fields in $\mathfrak{X}_c$ following their flows.

(C) If $\chi \in \mathfrak{X}_c^0(U)$ then, for every τ, Φ_τ is a diffeomorphism of $\mathcal{M}$ onto itself which fixes any point not in $\mathrm{Int}\,(U)$ (and thus it is the identity on $\partial\mathcal{M}$);

(T) If $\chi \in \mathfrak{X}_c^t(U)$ then, for every τ, Φ_τ is a diffeomorphism of $\mathcal{M}$ onto itself which fixes any point not in U and maps $\partial\mathcal{M} \cap U$ *onto* itself;

(I) If $\chi \in \mathfrak{X}_c^-(U)$ then Φ_τ is a well-defined map for $\tau \geq 0$, but not necessarily for $\tau < 0$; moreover, for each $\tau \geq 0$, Φ_τ is a diffeomorphism of $\mathcal{M}$ with $\Phi_\tau(\mathcal{M}) \subset \mathcal{M}$, but in general $\Phi_\tau(\mathcal{M})$ will be a proper subset of $\mathcal{M}$, i.e. Φ_τ rather than mapping $\partial\mathcal{M}$ into itself might "push it inwards".

It is thus clear that $\mathfrak{X}_c^t$ is a natural class of variations for the uncostrained problem, whereas a vector field in $\mathfrak{X}_c^-$ gives a natural (one-sided) variation for the constrained problem if we impose that it vanishes on the fixed boundary γ. This motivates the following

Definition 3.1. *In the "constrained" min-max problem, where the boundary constraint is γ, we introduce the set $\mathcal{V}_s^c(\mathcal{M},\gamma)$ (or shortly $\mathcal{V}_s^c(\gamma)$) which consists of those varifolds satisfying the condition*

$$\delta V(\chi) \geq 0 \qquad \textit{for all } \chi \in \mathfrak{X}_c^-(\mathcal{M} \setminus \gamma). \tag{3.0.1}$$

In the "unconstrained" min-max problem we introduce the set $\mathcal{V}_s^u$ which consists of those varifolds which are stationary for all variations in $\chi \in \mathfrak{X}_c^t(\mathcal{M})$:

$$\delta V(\chi) = 0 \qquad \textit{for all } \chi \in \mathfrak{X}_c^t(\mathcal{M}). \tag{3.0.2}$$

Clearly, since $\mathfrak{X}_c^0(M) \subset \mathfrak{X}_c^t(\mathcal{M})$, $\mathcal{V}_s^u$ is a subset of the stationary varifolds $\mathcal{V}_s$. Note moreover that, if $\chi \in \mathfrak{X}_c^0(\mathcal{M})$, then both χ and $-\chi$ belong to $\mathfrak{X}_c^-(\mathcal{M} \setminus \gamma)$: therefore we again conclude $\mathcal{V}_s^c(\gamma) \subset \mathcal{V}_s$.

For the purpose of this section, we will consider the subset $\mathcal{V}(\mathcal{M}, 4m_0)$ of varifolds with mass bounded by $4m_0$. Recall that the weak* topology on this set is metrizable, and we choose a metric $\mathcal{D}$ which induces it. In addition, we introduce the set

$$\mathcal{V}_\partial := \{\Xi_t \,|\, t \in \partial\mathcal{P}, \{\Xi_t\} \subset X\},$$

which is a closed subset of $\mathcal{V}(\mathcal{M})$. Note that, according to our notion of homotopy in Definition 1.3, this definition is independent of the family $\{\Xi_t\} \subset X$ we choose. We are now ready to state the main technical proposition of this section which, as already mentioned, will be proved using the classical pull-tight procedure of Almgren.

Proposition 3.2. *Let X be a homotopically closed set of smooth families which are parametrized by $\mathcal{P}$, such that* (1.2.3) *is satisfied. Then:*

- *(C) In the problem constrained by γ there exists a minimizing sequence $\{\{\Gamma_t\}^\ell\} \subset X$ such that, if $\{\Gamma^\ell_{t_\ell}\}$ is a min-max sequence, then $\mathcal{D}(\Gamma^\ell_{t_\ell}, \mathcal{V}_s^c(\gamma)) \to 0$;*
- *(U) In the unconstrained problem there exists a minimizing sequence $\{\{\Gamma_t\}^\ell\} \subset X$ such that, if $\{\Gamma^\ell_{t_\ell}\}$ is a min-max sequence, then $\mathcal{D}(\Gamma^\ell_{t_\ell}, \mathcal{V}_s^u) \to 0$.*

Proof. In what follows we will use $\mathcal{V}_s^c$ in place of $\mathcal{V}_s^c(\gamma)$ for the constrained case. The key idea of the proof is to find a continuous map $\Omega : \mathcal{V}(\mathcal{M}) \to \mathfrak{Is}(\mathcal{M})$, where $\mathfrak{Is}$ is the set of smooth isotopies - with a slight abuse of this term, note case (I) mentioned above. For a varifold V, the map $\Omega_V : [0,1] \times \mathcal{M} \to \mathcal{M}$, where the subscript V is used to denote $\Omega(V)$, will be the trivial one for any varifold belonging to $\mathcal{V}_\partial$ or $\mathcal{V}_s^c$ (or $\mathcal{V}_s^u$, depending on the case considered), and decrease the mass of V not belonging to either of these sets (when considering the push-forward of V with respect to it).

These maps will be generated as flows of certain vector fields defined for each of the varifolds in $\mathcal{V}(\mathcal{M})$. The first step is therefore to find a map from the set of varifolds to the set of smooth vector fields in $\mathcal{M}$. Afterwards, we construct the corresponding flows for each of these vector fields, and prove that they possess the properties described above. We will in fact repeat the first two steps in the proof of [10, Proposition 4.1] verbatim (for the sake of completeness), the only exception being that we consider vector fields in $\mathfrak{X}_c^t(\mathcal{M})$ or $\mathfrak{X}_c^-(\mathcal{M} \setminus \gamma)$ and thus we replace $\mathcal{V}_s$ with $\mathcal{V}_s^u$ and $\mathcal{V}_s^c$ in the respective cases. Since both these sets of vector fields are convex subsets of $\mathfrak{X}(\mathcal{M})$, the vector field H_V produced in Step 1 will also belong to the same class. Finally, in the last step we take a minimizing sequence and apply the flows generated by such vector fields to construct a competitor sequence. After some technical adjustments to ensure that it has the necessary properties, we prove that this sequence satisfies the requirements of the proposition.

Step 1: A map from $\mathcal{V}$ to the space of vector fields. For $l \in \mathbb{Z}$ we define the annuli

$$\mathcal{V}_l = \{V \in \mathcal{V}(\mathcal{M}) : 2^{-l+1} \geq \mathcal{D}(V, \mathcal{V}_s^\square) \geq 2^{-l-1}\},$$

where $\square$ is either u or c, depending on the case considered. These $\mathcal{V}_l$ are compact. Therefore there are constants $c(l) > 0$ depending only on l, such that for all $V \in \mathcal{V}_l$ there is a smooth vector field $\chi_V \in \mathfrak{X}_c^-(\mathcal{M} \setminus \gamma)$ (alternatively $\mathfrak{X}_c^t(\mathcal{M})$) with

$$\|\chi_V\|_\infty \leq \frac{1}{l}, \quad \delta V(\chi_V) \leq -c(l).$$

For, if not, then there is a sequence $\{V_j\}$ of varifolds in $\mathcal{V}_l$ such that for all vector fields χ with $\|\chi\|_\infty \leq 1$

$$|\delta V_j(\chi)| \leq \frac{1}{j}.$$

Therefore $\|\delta v_j\| \to 0$. Compactness and lower-semicontinuity of the first variation then yield a subsequence which converges to a stationary varifold. But this is a contradiction to the definition of $\mathcal{V}_l$. This proves the existence of the constants $c(l)$.

Next, we want to show that the associated vector fields χ_V can be chosen in a continuous dependence on V. For this first note that we have

$$\begin{aligned}\delta W(\chi_V) &\leq \delta V(\chi_V) + \delta(W - V)(\chi_V) \\ &\leq -c(l) + \|\delta(W - V)\|.\end{aligned}$$

Thus, by the lower semicontinuity of the first variation there is $r > 0$ such that

$$\delta W(\chi_V) \leq -\frac{c(l)}{2}, \quad W \in U_r(V),$$

where $U_r(V)$ denotes the ball in $\mathcal{V}$. Using again compactness we can find for any $l \in \mathbb{Z}$ balls $\{U_i^l\}_{i=1}^{N(l)}$ and corresponding vector fields χ_i^l such that

- the balls $\tilde{U}_i^l$, concentric to U_i^l with half the radii, cover $\mathcal{V}_l$;
- for all $W \in U_i^l$ we have $\delta W(\chi_i^l) \leq -\frac{c(l)}{2}$;
- the balls U_i^l are disjoint from $\mathcal{V}_j$ for $|j - l| \geq 2$.

The balls $\{U_i^l\}_{i,l}$ form a locally finite covering of $\mathcal{V} \setminus \mathcal{V}_s^\square$. Hence we can pick a continuous partition of unity $\{\varphi_i^l\}$ subordinate to this covering. Then we define the vector fields $H_V := \sum_{i,l} \varphi_i^l(V)\chi_i^l$. The map

$$H : \mathcal{V} \to C^\infty(\mathcal{M}, T\mathcal{M}), \quad V \mapsto H_V$$

is continuous and $\|H_V\|_\infty \leq 1$ for all $V \in \mathcal{V}(\mathcal{M})$.

Step 2: A map from $\mathcal{V}$ to the space of isotopies. Let $V \in \mathcal{V}_l$. Then, by the above covering, V is contained in at least one ball $\tilde{U}_i^l$. We denote by $r(V)$ the radius of the smallest such ball. As there are only finitely many such balls, we can find $r(l)$ depending only on l such that $r(V) \geq r(l) > 0$. Moreover, by the properties of the covering,

$$\delta W(H_V) \leq -\frac{1}{2} \min\{c(l-1), c(l), c(l+1)\}$$

for all $W \in U_{r(V)}(V)$. Thus, we have two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\delta W(H_V) \leq -g(\mathcal{D}(V, \mathcal{V}_s^\square)) \quad \text{if} \quad \mathcal{D}(W, V) \leq r(\mathcal{D}(V, \mathcal{V}_s^\square)). \tag{3.0.3}$$

The function $-g$ for instance can be obtained by dominating the step function depending on the $c(l)$ by a continuous function. By the compactness of $\mathcal{M}$ and the smoothness of each H_V we can construct for all V a 1-parameter family of maps

$$\Phi_V : [0, \infty) \times \mathcal{M} \to \mathcal{M} \quad \text{with} \quad \frac{\partial \Phi_V}{\partial t}(t, x) = H_V(\Phi_V(t, x)).$$

Note that in the unconstrained case $\Psi_V(s, \cdot)$ is a diffeomorphism from $\mathcal{M}$ to itself, whereas in the constrained case it is a diffeomorphism of $\mathcal{M}$ with $\Psi_V(s, \mathcal{M})$ which however keeps γ fixed. The key is now to prove that these diffeomorphisms decrease the mass of a varifold by an amount depending on its distance to the stationary varifolds. More precisely, we claim that there are continuous functions $T : \mathbb{R}^+ \to [0, 1]$ and $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- if $\eta = \mathcal{D}(V, \mathcal{V}_s^\square) > 0$ and V' is obtained from V (pushing forward) by the diffeomorphism $\Phi_V(T(\eta), \cdot)$, then $\|V'\|(\mathcal{M}) \leq \|V\|(\mathcal{M}) - G(\eta)$;
- $G(s)$ and $T(s)$ both converge to 0 as $s \to 0$.

For this we fix $V \in \mathcal{V} \setminus \mathcal{V}_s^\square$. For all $r > 0$ there is $T > 0$ such that the curve

$$\{V(t) = (\Phi_V(t, \cdot))_\sharp V, \ t \in [0, T]\}$$

stays in $U_r(V)$. This implies the inequality

$$\|V(T)\|(\mathcal{M}) - \|V\|(\mathcal{M}) = \|V(T)\|(\mathcal{M}) - \|V(0)\|(\mathcal{M}) \leq \int_0^T \delta V(t)(H_V)\, dt.$$

If we choose $r = r(\mathcal{D}(V, \mathcal{V}_s^\square)$ as in (3.0.3), this yields

$$\|V(T)\|(\mathcal{M}) - \|V\|(\mathcal{M}) \leq -Tg(\mathcal{D}(V, \mathcal{V}_s^\square)),$$

or we can rewrite this as

$$\|V(T)\|(\mathcal{M}) - \|V\|(\mathcal{M}) \leq -G(\mathcal{D}(V, \mathcal{V}_s^\square)).$$

Moreover T and G are continuous. Clearly $T(s) \to 0$ as $s \to 0$. The boundedness of g then gives $G(s) \to 0$ as $s \to 0$. Arguing as in the first step, using a continuous partition of unity, we can find a choice of T that is continuous in V and depends only on $\mathcal{D}(V, \mathcal{V}_s^\square)$.

Step 3: Construction of the competitor. At this point we diverge somewhat from [10]. We define $b(V) := \min\{\mathcal{D}(V, \mathcal{V}_\partial), 1\}$ for $V \in \mathcal{V}(\mathcal{M})$, and remark that $b : \mathcal{V}(\mathcal{M}) \to \mathbb{R}$ is a continuous function. Let $V \in \mathcal{V}$ be such that $\mathcal{D}(V, \mathcal{V}_s^\square) = \eta$. We now renormalize the maps Φ_V by setting

$$\Omega_V(t, \cdot) = \Phi_V(b(V)T(\eta)t, \cdot), \quad t \in [0, 1].$$

By the definition of T the varifolds $(\Omega_V(t, \cdot))_\sharp V$ stay in $U_{r(\eta)}(V)$ for all $t \in [0, 1]$. Moreover, the introduction of the additional scaling parameter $b(V)$ keeps the varifolds in $\mathcal{V}_\partial$ intact by the flow. A quick computation as in Step 2 then yields

$$||\Omega_V(1, \cdot)_\sharp V||(\mathcal{M}) \leq ||V||(\mathcal{M}) - b(V)L(\mathcal{D}\big(V, \mathcal{V}_s^\square)\big). \tag{3.0.4}$$

where $L : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $L(0) = 0$. The function G above is not necessarily strictly increasing, but all the choices can be made in such a way that this goal is achieved.

We now take a sequence of families $\{\{\Sigma_t\}^\ell\} \subset X$ such that $\max_{t \in \mathcal{P}} \mathcal{H}^n(\Sigma_t^\ell) \leq m_0(X) + \frac{1}{\ell}$. Naturally, we would consider the family defined by $\tilde{\Gamma}_t^\ell = \Omega_{\Sigma_t^\ell}(1, \Sigma_t^\ell)$ as the required competitor. But, since the dependence of the vector field (corresponding to) $\Omega_{\Sigma_t^\ell}$ is merely continuous in t, the new family is not necessarily smooth. We overcome this obstacle by smoothing the continuous map

$$h : \mathcal{P} \to \mathfrak{X}_c^-(\mathcal{M} \setminus \gamma) \quad (\text{or } \mathfrak{X}_c^t(\mathcal{M})),$$

where $\mathfrak{X}_c^-(\mathcal{M})$ is endowed with the topology of C^k-seminorms, and the smooth vector field $h_t^\ell = b(\Sigma_t^\ell)T\big(\mathcal{D}(\Sigma_t^\ell, \mathcal{V}_s^\square)\big)H_{\Sigma_t^\ell}$ generates $\Omega_{\Sigma_t^\ell}$. Note that $h_t = 0$ for $t \in \partial\mathcal{P}$, and we smooth h by keeping it 0 on $\partial\mathcal{P}$. We obtain this way a smooth map $\tilde{h}$, and consequently, the 1-parameter family of diffeomorphisms $\tilde{\Omega}_t^\ell$ generated by $\tilde{h}(t)$. We then consider the smooth family

$$\Gamma_t^\ell = \tilde{\Omega}_t^\ell(1, \Sigma_t^\ell).$$

Since $\tilde{\Omega}_t^\ell(s, \cdot)$ is the identity for $t \in \mathcal{P}$, the new family $\{\Gamma_t^\ell\}_t$ is homotopic to $\{\Sigma_t^l\}_t$. Whenever $\sup_t \|h_t - \tilde{h}_t\|_{C^1}$ is small enough, the calculations as before give

$$\mathcal{H}^n(\Gamma_t^\ell) \leq \mathcal{H}^n(\Sigma_t^\ell) - \frac{b(\Sigma_t^\ell)L\big(\mathcal{D}(\Sigma_t^\ell, \mathcal{V}_s^\square)\big)}{2}. \tag{3.0.5}$$

Moreover, there will be an increasing continuous map $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lambda(0) = 0$ and

$$\mathcal{D}(\Sigma_t^\ell, \mathcal{V}_s^\square) \geq \lambda\big(\mathcal{D}(\Gamma_t^\ell, \mathcal{V}_s^\square)\big). \tag{3.0.6}$$

Finally, we claim that for every ϵ, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\text{if } \left\{ \begin{array}{c} k > N \\ \text{and } \mathcal{H}^n(\Gamma^\ell_{t_\ell}) > m_0 - \delta \end{array} \right\}, \quad \text{then } \mathcal{D}(\Gamma^\ell_{t_\ell}, \mathcal{V}^\square_s) < \epsilon. \tag{3.0.7}$$

Let us therefore fix $\epsilon > 0$. Considering that $b(W) = 0 \ \forall W \in \mathcal{V}_\partial$, the continuity of mass of varifolds clearly implies that, if we set $\xi := \frac{m_0(X) - bM_0(X)}{2}$, then for all $V \in \mathcal{V}(\mathcal{M})$ with $\mathcal{H}^n(V) \geq m_0 - \xi = bM_0(X) + \xi$ we have $b(V) \geq c(\xi) > 0$. We will choose $0 < \delta < \xi$ and $N \in \mathbb{N}$ satisfying

$$\frac{c(\xi)L(\lambda(\epsilon))}{2} - \delta > \frac{1}{N}.$$

Assume now, contrary to (3.0.7), there are $k > N$ and $t \in \mathcal{P}$ such that

$$\mathcal{H}^n(\Gamma^\ell_{t_\ell}) > m_0 - \delta \quad \text{and} \quad \mathcal{D}(\Gamma^\ell_{t_\ell}, \mathcal{V}^\square_s) > \epsilon.$$

Then, by (3.0.5), (3.0.6) and the fact that $\mathcal{H}^n(\Sigma^\ell_{t_\ell}) \geq \mathcal{H}^n(\Gamma^\ell_{t_\ell}) > m_0 - \delta > m_0 - \xi$, we get

$$\begin{aligned} \mathcal{H}^n(\Sigma^\ell_{t_\ell}) &\geq \mathcal{H}^n(\Gamma^\ell_{t_\ell}) + \delta + \frac{c(\xi)L(\lambda(\epsilon))}{2} - \delta \\ &\geq m_0 + \frac{1}{N}. \end{aligned}$$

This contradicts $\max_{t \in \mathcal{P}} \mathcal{H}^n(\Sigma^\ell_t) \leq m_0(X) + \frac{1}{\ell}$, and thus completes the proof of claim (3.0.7), which in turn implies the proposition. $\square$

Chapter 4

Existence of Almost Minimizing Varifolds

4.1 Almost Minimizing Property

Following its introduction by Pitts [41], an important concept to achieve regularity for stationary varifolds produced by min-max theory is that of *almost minimizing surfaces.* Roughly speaking, a surface is almost minimizing if any area-decreasing deformation must eventually pass through some surface with sufficiently large area. The precise definition we require here is the following:

Definition 4.1. *Let $\epsilon > 0$, $U \in \mathcal{M}$ be an open subset, and fix $m \in \mathbb{N}$. A surface Σ is called* ϵ-almost minimizing *in U if there is* <u>no</u> *family of surfaces $\{\Sigma_t\}_{t\in[0,1]}$ satisfying the properties:*

(4.1.1) (s1), (s2) *and* (s3) *of Definition 0.1 hold*;

(4.1.2) $\Sigma_0 = \Sigma$ *and* $\Sigma_t \setminus U = \Sigma \setminus U$ *for every* $t \in [0,1]$;

(4.1.3) $\mathcal{H}^n(\Sigma_t) \leq \mathcal{H}^n(\Sigma) + \frac{\epsilon}{2^{m+2}}$ *for all* $t \in [0,1]$;

(4.1.4) $\mathcal{H}^n(\Sigma_1) \leq \mathcal{H}^n(\Sigma) - \epsilon$

A sequence $\{\Omega^i\}$ of surfaces is called almost minimizing (or a.m.) in U if each Ω^i is ϵ_i-almost minimizing in U for some sequence $\epsilon_i \to 0$ (with the same m).

Remark 4.2. *The definition above is practically the same as the one given in [13] when $m = 1$. The generalization is due to the more general parameter space $\mathcal{P}$. To be precise, we will henceforth fix $m \in \mathbb{N}$ such that $\mathcal{P}$ can be smoothly embedded into $\mathbb{R}^m$.*

The main goal of this section is to prove an existence result regarding almost minimizing property in annuli:

Proposition 4.3. *Le X be a homotopically closed set of (constrained or unconstrained) families in $\mathcal{M}^{n+1}$, parameterized by a smooth, compact k-dimensional manifold $\mathcal{P}$ (with or without boundary), and satisfying the condition* (1.2.3)*. Then there is a function $r : \mathcal{M} \to \mathbb{R}_+$ and a min-max sequence $\{\Gamma^k\} = \{\Gamma^k_{t_k}\}$ such that*

- $\{\Gamma^k\}$ *is a.m. in every* $An \in \mathcal{AN}_{r(x)}(x)$ *with* $x \in \mathcal{M}$
- $\{\Gamma^k\}$ *converges to a stationary varifold* V *as* $k \to \infty$.

An important corollary of the above proposition is the interior regularity, for which we refer to [13]. We record the consequence here

Proposition 4.4. *The varifold* V *of Proposition 4.3 is a regular embedded minimal surface in* $\operatorname{Int}(\mathcal{M})$, *except for a set of Hausdorff dimension at most* $n-7$.

In order to prove Proposition 4.3 we will be following the strategy laid out in Section 5 of [10] (see also Section 3 of [13]), which contains a similar statement. In fact, the main difference is the significant generalization of the parameter space $\mathcal{P}$. The case of higher dimensional cubes was covered in the master thesis of Fuchs [19], and in this paper, some necessary modifications were made. The key ingredient of the proof is a combinatorial covering argument, a variant of the original one by Almgren and Pitts (see [41]), and which we therefore refer to as the Almgren-Pitts combinatorial lemma. We will use it to prove Proposition 4.3 at the end of this section, and its proof will be provided in the next one.

Definition 4.5. *Let* $d \in \mathbb{N}$ *and* $U^1, \dots U^d$ *be open sets in* $\mathcal{M}$. *A surface* Σ *is said to be* ϵ-almost minimizing in $(U^1, \dots, U^d)$ *if it is* ϵ*-a.m. in at least one of the open sets* $U^1, \dots U^d$.
Furthermore, we define

$$\operatorname{dist}(U, V) := \inf_{u \in U, v \in V} d_g(u, v)$$

as the distance between the two sets U *and* V *(*d_g *being the Riemannian distance). Finally, for any* $d \in \mathbb{N}$ *we denote by* $\mathcal{CO}_d$ *the set of* d*-tuples* $(U^1, \dots, U^d)$, *where* $U^1, \dots, U^d$ *are open sets with the property that*

$$\operatorname{dist}(U^i, U^j) \geq 4 \cdot \min\{\operatorname{diam}(U^i), \operatorname{diam}(U^j)\}$$

for all $i, j \in \{1, \dots, d\}$ *with* $i \neq j$.

We require also the following lemma as preparation:

Lemma 4.6. *Let* $p \in \mathbb{N}$. *Then there exists* $\omega_p \in \mathbb{N}$ *with the following property:*

(CA) Assume $\mathcal{F}_1 = (U_1^1, \dots, U_1^{\omega_p}), \dots, \mathcal{F}_{2^p} = (U_{2^p}^1, \dots, U_{2^p}^{\omega_p})$ *are* 2^p *families of open sets with the property that*

$$\operatorname{dist}(U_i^j, U_i^{j'}) \geq 2 \cdot \min\{\operatorname{diam}(U_i^j), \operatorname{diam}(U_i^{j'})\} \tag{4.1.5}$$

for all $i \in \{1, \dots, 2^p\}$ *and for all* $j, j' \in \{1, \dots, \omega_p\}$ *with* $j \neq j'$.
Then we can extract 2^p *subfamilies* $\mathcal{F}_1^{sub} \subset \mathcal{F}_1, \dots, \mathcal{F}_{2^p}^{sub} \subset \mathcal{F}_{2^p}$ *such that*

- $\operatorname{dist}(U, V) > 0$ *for all* $U \in \mathcal{F}_i^{sub}, V \in \mathcal{F}_j^{sub}$ *with* $i, j \in \{1, \ldots, 2^p\}$ *and* $i \neq j$;

- $\mathcal{F}_i^{sub}$ *contains at least* 2^p *open sets for every* $i \in \{1, \ldots, 2^p\}$.

Proof. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{2^p}$ be as in the assumption (CA), with ω_p some (natural) number, to be fixed later. First note that, if $U \in \mathcal{F}_i$ and $V^1, \ldots V^l \in \mathcal{F}_s$ with $i \neq s$ and $\operatorname{diam}(U) \leq \operatorname{diam}(V^j)$, $j \in \{1, \ldots, l\}$, then there is at most one $j \in \{1, \ldots, l\}$ with $\operatorname{dist}(U, V^j) = 0$. Otherwise, assuming there are two such sets V^{j_1}, V^{j_2} with $\operatorname{dist}(V^{j_1}, U) = 0$, $\operatorname{dist}(V^{j_2}, U) = 0$ and w.l.o.g. $\operatorname{diam}(V^{j_1}) \leq \operatorname{diam}(V^{j_2})$, we would get

$$\operatorname{dist}(V^{j_1}, V^{j_2}) \leq \operatorname{diam}(U) \leq \operatorname{diam}(V^{j_1}),$$

which contradicts the assumption (4.1.5). Now, in order to produce the subfamilies, one can employ the following algorithm:

– take all the sets in all the families and arrange them in an ascending order with respect to their diameters, left to right (from smallest to largest). In the first step, fix the leftmost set;

– at each step of the process, remove all the sets to the right of the fixed set which are at distance zero with respect to it. Furthermore, if to the left of the currently fixed set there are $2^p - 1$ remaining sets from the same family, remove all the sets to the right which belong to the same family;

– move on to the first (remaining) set to the right of the previously fixed set, fix it, and repeat the step above.

We claim that the remaining sets belong to the required subfamilies. Firstly, it is obvious from the construction that for any two remaining sets U, V we have $\operatorname{dist}(U, V) > 0$. Secondly, we see from the consideration at the beginning of the proof, that at each step, at most one set from each family the fixed set does not belong to, is removed (and none from the same family, due to (4.1.5)). Finally, since any family that reaches 2^p remaining elements is removed from the process, it can account for no more than 2^p removed elements from any other family. Hence we can remove no more than $2^p(2^p - 1)$ from each family, so if we choose any $\omega_p \geq 4^p$ we can ensure that at least 2^p elements remain. □

Proposition 4.7. (Almgren-Pitts combinatorial lemma) *Let X be a homotopically closed set of families as in Proposition 4.3. Assume $\mathcal{P}$ is smoothly embedded into $\mathbb{R}^m$, and let ω_m be as in Lemma 4.6. Then there exists a min-max sequence* $\{\Gamma^N\} = \{\Gamma^N_{t_N}\}$ *such that*

- $\{\Gamma^N\}$ *converges to a stationary varifold;*

- *for any* $(U^1, \ldots, U^{\omega_m}) \in \mathcal{CO}_{\omega_m}$, Γ^N *is* $\frac{1}{N}$*-a.m. in* $(U^1, \ldots, U^{\omega_m})$, *for N large enough.*

We can now prove the main proposition as a corollary of the above.

Proof of Proposition 4.3. We will show that a subsequence of $\{\Gamma^j\}$ in Proposition 4.7 satisfies the requirements. Let $r_1 > 0$ be such that $r_1 < \mathrm{Inj}(\mathcal{M})$, and for each $x \in \mathcal{M}$ define the tuple $(U^1_{r_1}(x), \ldots, U^{\omega_m}_{r_{\omega_m}}(x))$ by

(4.1.6)
$$U^1_{r_1}(x) := \mathcal{M} \setminus \overline{B}_{r_1}(x);$$
(4.1.7)
$$U^l_{r_l}(x) := B_{\tilde{r}_l}(x) \setminus \overline{B}_{r_l}(x) \text{ where } \tilde{r}_l := \frac{1}{9} r_{l-1} \text{ and } r_l < \tilde{r}_l \text{ for } 2 \le l \le \omega_m - 1;$$
(4.1.8)
$$U^{\omega_m}_{r_{\omega_m}} := B_{r_{\omega_m}}(x) \text{ where } r_{\omega_m} \le \frac{1}{9} r_{\omega_m - 1}.$$

Then, by definition, $(U^1_{r_1}(x), \ldots, U^{\omega_m}_{r_{\omega_m}}(x)) \in \mathcal{CO}_{\omega_m}$ and Γ^j is therefore (for j large enough) $\frac{1}{j}$-a.m. in at least one $U^l_{r_l}(x)$, $1 \le l \le \omega_m$. Having fixed $r_1 > 0$, one of the following options holds:

(a) either $\{\Gamma^j\}$ is (for j large) $\frac{1}{j}$-a.m. in $(U^2_{r_2}(y), \ldots, U^{\omega_m}_{r_{\omega_m}}(y))$ for every $y \in \mathcal{M}$;

(b) or, for each $K \in \mathbb{N}$, there exists some $s_K \ge K$ and a point $x^{s_K}_{r_1} \subset \mathcal{M}$ such that Γ^{s_K} is $\frac{1}{s_K}$-a.m. in $\mathcal{M} \setminus \overline{B}_{r_1}(x^{s_K}_{r_1})$.

Assume there is no $r_1 > 0$ such that (a) holds. Thus, choosing option (b) with $r_1 = \frac{1}{j}$ and $K = j$ for each $j \in \mathbb{N}$, we obtain a subsequence $\{\Gamma^{s_j}\}_{j \in \mathbb{N}}$, and a sequence of points $\{x^{s_j}_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ such that Γ^{s_j} is $\frac{1}{s_j}$-a.m. in $\mathcal{M} \setminus \overline{B}_{\frac{1}{j}}(x^{s_j}_j)$. Since $\mathcal{M}$ is compact, there exists some $x \in \mathcal{M}$ such that $x^{s_j}_j \to x$. We conclude that, for any $N \in \mathbb{N}$, Γ^{s_j} is $\frac{1}{s_j}$-a.m. in $\mathcal{M} \setminus \overline{B}_{\frac{1}{N}}(x)$ for j large enough. Consequently, if $y \in \mathcal{M} \setminus \{x\}$, we can choose $r(y)$ such that $B_{r(y)}(y) \subset\subset \mathcal{M} \setminus \{x\}$, whereas $r(x)$ can be chosen arbitrarily.
Assume now that there is some fixed $r_1 > 0$ such that (a) holds. Note that, in this case, it is implied that Γ^j is <u>not</u> $\frac{1}{j}$-a.m. in $U^1_{r_1}$ for all j large enough. Due to compactness, we can divide the manifold $\mathcal{M}$ into finitely many, nonempty, closed subsets $\mathcal{M}_1, \ldots, \mathcal{M}_N \subset \mathcal{M}$ such that

- $0 < \mathrm{diam}(\mathcal{M}_i) < \tilde{r}_2 = \frac{1}{9} r_1$ for every $i \in \{1, \ldots, N\}$;
- $\mathcal{M} = \cup \mathcal{M}_i$.

Similar to the reasoning above, for each $\mathcal{M}_i$, starting with $\mathcal{M}_1$, we consider two mutually exclusive cases:

(a) either there exists some fixed $r_{2,i} > 0$ such that $\{\Gamma^j\}$ must be (for j large) $\frac{1}{j}$-a.m. in $(U^3_{r_3}(y), \ldots, U^{\omega_m}_{r_{\omega_m}}(y))$ for every $y \in \mathcal{M}_i$, where of course, $\tilde{r}_3 \leq \frac{1}{9} r_{2,i}$;

(b) or we can extract a subsequence $\{\Gamma^j\}$, not relabeled, and a sequence of points $x_{i,j} \subset \mathcal{M}_i$ such that Γ^j is $\frac{1}{j}$-a.m. in $B_{\tilde{r}_2}(x_{i,j}) \setminus \overline{B}_{\frac{1}{j}}(x_{i,j})$

Again, if (b) holds, we know $x_{i,j} \to x_i \in \mathcal{M}_i$, and we can choose $r(x_i) \in (\text{diam}(\mathcal{M}_i), \tilde{r}_2)$. Accordingly, for any other $y \in \mathcal{M}_i$, we can choose $r(y)$ such that $B_{r(y)}(y) \subset\subset B_{r(x_i)}(x_i) \setminus \{x_i\}$. We proceed onto $\mathcal{M}_{i+1}$, where either (a) gets chosen, or we possibly extract a futher subsequence, and define further values of the function r. For the subsets $\mathcal{M}_{i_1}, \ldots, \mathcal{M}_{i_l}$ where option (a) holds, we define $r_2 := \min\{r_{2,i_1}, \ldots, r_{2,i_l}\}$, and then continue iteratively, by first subdividing the sets and then considering the relevant cases. Finally, note that if in the last instance of the iteration we choose option (a) for certain subsets, it means that, in those sets, Γ^j must be (for j large) $\frac{1}{j}$-a.m. in $B_{r_{\omega_m}}(y)$ for some $r_{\omega_m} > 0$ and all y, hence we can choose $r(y) = r_{\omega_m}$, and we are done. □

4.2 Almgren-Pitts Combinatorial Lemma

In this section, we turn to proving Proposition 4.7, which will be done by contradiction. Assuming no min-max sequence (extracted from an apropriate minimizing sequence) with the required property exists, we are able to construct a competitor minimizing sequence $\{\Sigma_t\}^N$ with energy (i.e. $\max_{t \in \mathcal{P}}\{\Sigma_t\}^N$), lowered by a fixed amount, thus reaching a contradiction to the minimality of the original sequence. This will be done using two main ingredients. The first is a technical lemma which enables us to use the "static" variational principle in Definition 4.1 for a single, fixed time slice to construct a "dynamic" competitor family of surfaces. This is achieved by using a tool called "freezing", introduced in [13] (see Lemma 3.1). The statement and proof we present here are slightly different. They go as follows:

Lemma 4.8. *Let $U \subset\subset U' \subset \mathcal{M}$ be two open sets, and $\{\Xi_t\}_{t \in [0,1]^p}$ be a smooth family parameterized by $[0,1]^p$, with $p \in \mathbb{N}$ fixed. Given an $\epsilon > 0$ and $t_0 \in (0,1)^p$, suppose $\{\Sigma_s\}_{s \in [0,1]}$ is a 1-parameter family of surfaces satisfying properties (4.1.1)-(4.1.4), with $\Sigma_0 = \Xi_{t_0}$ and $m = p$. Then there is an $\eta > 0$ such that the following holds for every a', a with $0 < a' < a < \eta$:*

There is a competitor (smooth) family $\{\Xi'_t\}_{t\in[0,1]^p}$ *such that*

(4.2.1)
$$\Xi_t = \Xi'_t \text{ for } t \in [0,1]^p \setminus Q(t_0,a), \text{ and } \Xi_t \setminus U' = \Xi'_t \setminus U' \text{ for } t \in Q(t_0,a);$$
(4.2.2)
$$\mathcal{H}^n(\Xi'_t) \le \mathcal{H}^n(\Xi_t) + \frac{\epsilon}{2^{p+1}} \text{ for every } t \in [0,1]^p;$$
(4.2.3)
$$\mathcal{H}^n(\Xi'_t) \le \mathcal{H}^n(\Xi_t) - \frac{\epsilon}{2} \text{ for every } t \in Q(t_0,a'),$$

where

$$Q(t_0,r) := \{t = (t^1,\dots,t^p) \in [0,1]^p \mid \forall i \in \{1,\dots,p\} : t_0^i - r < t^i < t_0^i + r\}$$

Moreover, $\{\Xi'_t\}$ *is homotopic to* $\{\Xi_t\}$.

Proof. **Step 1: Freezing**. First we will choose open sets A_1, A_2 and B_1, B_2 satisfying
$$U \subset\subset A_1 \subset\subset A_2 \subset\subset B_1 \subset\subset B_2 \subset\subset U',$$
and such that $\Xi_{t_0} \cap \tilde{C}$ is a smooth surface, where $\tilde{C} := B_2 \setminus \bar{A}_1$, which is possible since Ξ_{t_0} contains only finitely many singularities. In a tubular δ-neighborhood (w.r.t the normal bundle) of $\Xi_{t_0} \cap \tilde{C}$ we fix normal coordinates $(z,\sigma) \in \Xi_{t_0} \cap \tilde{C} \times (-\delta,\delta)$. By choosing δ small enough and/or redefining A_i-s and B_i-s, we can ensure that $\Xi_{t_0} \cap A_2 \times (-\delta,\delta) \subset\subset B_1$ and $\Xi_{t_0} \cap B_1 \times (-\delta,\delta) \subset\subset B_2$. Now, after defining the open sets $A := A_1 \cup \big(\Xi_{t_0} \cap A_2 \times (-\delta,\delta)\big)$, and $B := \big(B_1 \cap \Xi_{t_0} \times (-\delta,\delta)\big) \cup \big(B_2 \setminus (\bar{B}_2 \cap \Xi_{t_0} \times [-\delta,\delta])\big)$, we set $C := B \setminus \bar{A}$ and deduce the following properties:

(a) $U \subset\subset A \subset\subset B \subset\subset U'$;

(b) $\Xi_{t_0} \cap C$ is a smooth surface;

(c) we can fix $\eta > 0$ such that $\Xi_t \cap C$ is the graph of a function g_t over $\Xi_{t_0} \cap C$ for $t \in Q(t_0,\eta)$.

Note that the slightly complicated definitons above are only to ensure the property (c), or in other words, that the set C is "cylindrical" near Ξ_{t_0} so that $\Xi_t \cap C$ can in fact be entirely represented as a graph over $\Xi_{t_0} \cap C$, i.e. $\Xi_t \cap C = \{(z,\sigma) | \sigma = g_t(z),\ z \in \Xi_{t_0} \cap C\}\ \forall t \in Q(t_0,\eta)$.
Next, we fix two smooth functions φ_A and φ_B such that

- $\varphi_A + \varphi_B = 1$;

- $\varphi_A \in C_c^\infty(B)$, $\varphi_B \in C_c^\infty(\mathcal{M} \setminus \bar{A})$

We then introduce the functions

$$g_{t,s,\tau} := \varphi_B g_t + \varphi_A((1-s)g_t + s g_\tau), \quad t, \tau \in Q(t_0, \eta), s \in [0,1]$$

Since g_t converges smoothly to $g_{t_0}(=0)$ as $t \to t_0$, we can make $\sup_{s,\tau} \|g_{t,s,\tau} - g_t\|_{C^1}$ arbitrarily small by choosing η small. Moreover, if we express the area of the graph of a function g over $\Xi_{t_0} \cap C$ as an integral functional of g, we know that it only depends on g and its first derivatives. Thus, if $\Gamma_{t,s,\tau}$ is the graph of $g_{t,s,\tau}$, we can find η small enough such that

$$\max_{s\in[0,1]} \mathcal{H}^n(\Gamma_{t,s,\tau}) \leq \mathcal{H}^n(\Xi_t \cap C) + \frac{\epsilon}{2^{p+3}}. \tag{4.2.4}$$

Now, given $0 < a' < a < \eta$, we choose $a'' \in (a', a)$ and fix:

- a smooth function $\psi : Q(t_0, a) \to [0,1]$ which is identically equal to 0 in a neighborhood of $\partial Q(t_0, a)$ and equal to 1 on $Q(t_0, a'')$;

- a smooth function $\gamma : Q(t_0, a) \to Q(t_0, \eta)$ which is equal to the identity in a neighborhood of $\partial Q(t_0, a)$ and equal to t_0 in $Q(t_0, a'')$.

We now define a new family $\{\Delta_t\}$ as follows:

- $\Delta_t = \Xi_t$ for $t \notin Q(t_0, a)$;

- $\Delta_t \setminus \bar{B} = \Xi_t \setminus \bar{B}$ for all t;

- $\Delta_t \cap A = \Xi_{\gamma(t)} \cap A$ for $t \in Q(t_0, a)$;

- $\Delta_t \cap C = \{(z, \sigma) \,|\, \sigma = g_{t,\psi(t),\gamma(t)},\ z \in \Xi_{t_0} \cap C\}$ for $t \in Q(t_0, a)$.

Note that $\{\Delta_t\}$ is a smooth family homotopic to $\{\Xi_t\}$, they both coincide outside of B (and hence outside of U') for *every* t, and that in A (and hence in U) we have $\Delta_t = \Xi_{\gamma(t)}$ for $t \in Q(t_0, a)$. Since $\gamma(t)$ is equal to t_0 for $t \in Q(t, a'')$, it follows that $\Delta_t \cap U = \Xi_{t_0} \cap U$ for $t \in Q(t_0, a'')$, or in other words, $\Delta \cap U$ is *frozen* in $Q(t_0, a'')$. Furthermore, because of (4.2.4),

$$\mathcal{H}^n(\Delta_t \cap C) \leq \mathcal{H}^n(\Xi_t \cap C) + \frac{\epsilon}{2^{p+3}} \quad \text{for } t \in Q(t_0, a). \tag{4.2.5}$$

Step 2: Dynamic competitor. We fix a smooth function $\chi : Q(t_0, a'') \to [0,1]$ which is identically 0 in a neighborhood of $\partial Q(t_0, a'')$, and identically 1 on $Q(t_0, a')$. We then define a competitor family $\{\Xi'_t\}$ in the following way:

- $\Xi'_t = \Delta_t$ for $t \notin Q(t_0, a'')$;

- $\Xi'_t \setminus A = \Delta_t \setminus A$ for $t \in Q(t_0, a'')$;

- $\Xi'_t \cap A = \Sigma_{\chi(t)} \cap A$ for $t \in Q(t_0, a'')$.

The new family $\{\Xi'_t\}$ is also a smooth family, which is obviously homotopic in the sense of Definition 1.3 to $\{\Delta_t\}$ and hence to $\{\Xi_t\}$, so long as we ensure a is small enough that $Q(t_0, a) \subseteq (0,1)^p$. We can now start estimating $\mathcal{H}^n(\Xi'_t)$.
For $t \notin Q(t_0, a)$, we have $\Xi'_t = \Delta_t = \Xi_t$, so

$$\mathcal{H}^n(\Xi'_t) = \mathcal{H}^n(\Xi_t) \quad \text{for } t \notin Q(t_0, a). \tag{4.2.6}$$

For $t \in Q(t_0, a)$, we have $\Xi_t \setminus \bar{B} = \Xi'_t \setminus \bar{B}$ and hence $\Xi'_t \setminus U' = \Xi_t \setminus U'$. This shows property (4.2.1) of the lemma.
In the set C it holds $\Xi'_t = \Delta_t$ for $t \in Q(t_0, a)$, thus owing to (4.2.5),

$$\begin{aligned} \mathcal{H}^n(\Xi'_t) - \mathcal{H}^n(\Xi_t) &= [\mathcal{H}^n(\Delta_t \cap C) - \mathcal{H}^n(\Xi_t \cap C)] + [\mathcal{H}^n(\Xi'_t \cap A) - \mathcal{H}^n(\Xi_t \cap A)] \\ &\overset{(4.2.5)}{\leq} \frac{\epsilon}{2^{p+3}} + [\mathcal{H}^n(\Xi'_t \cap A) - \mathcal{H}^n(\Xi_t \cap A)]. \end{aligned} \tag{4.2.7}$$

Next, we want to estimate the area in A for $t \in Q(t_0, a)$. To do so, we consider several cases separately:

(i) Let $t \in Q(t_0, a) \setminus Q(t_0, a'')$. Then $\Xi'_t \cap A = \Delta_t \cap A = \Xi_{\gamma(t)} \cap A$. However, $t, \gamma(t) \in Q(t_0, \eta)$ and, having chosen η small enough, we can assume that

$$|\mathcal{H}^n(\Xi_s \cap A) - \mathcal{H}^n(\Xi_\sigma \cap A)| \leq \frac{\epsilon}{2^{p+3}} \quad \text{for every } \sigma, s \in Q(t_0, \eta). \tag{4.2.8}$$

Hence, we deduce with (4.2.7) that

$$\mathcal{H}^n(\Xi'_t) \leq \mathcal{H}^n(\Xi_t) + \frac{\epsilon}{2^{p+2}}. \tag{4.2.9}$$

(ii) Let $t \in Q(t_0, a'') \setminus Q(t_0, a')$. Then $\Xi'_t \cap A = \Sigma_{\chi(t)} \cap A$. Therefore, with (4.2.7) it follows

$$\begin{aligned} \mathcal{H}^n(\Xi'_t) - \mathcal{H}^n(\Xi_t) &\leq \frac{\epsilon}{2^{p+3}} + [\mathcal{H}^n(\Xi_{t_0} \cap A) - \mathcal{H}^n(\Xi_t \cap A)] \\ &\quad + [\mathcal{H}^n(\Sigma_{\chi(t)} \cap A) - \mathcal{H}^n(\Xi_{t_0} \cap A)] \\ &\overset{(4.1.3),(4.2.8)}{\leq} \frac{\epsilon}{2^{p+3}} + \frac{\epsilon}{2^{p+3}} + \frac{\epsilon}{2^{p+2}} = \frac{\epsilon}{2^{p+1}}. \end{aligned} \tag{4.2.10}$$

(iii) Let $t \in Q(t_0, a')$. Then we have $\Xi'_t \cap A = \Sigma_1 \cap A$. Using (4.2.7) again, we have

$$\begin{aligned}\mathcal{H}^n(\Xi_t') - \mathcal{H}^n(\Xi_t) &\leq \frac{\epsilon}{2^{p+3}} + [\mathcal{H}^n(\Sigma_1 \cap A) - \mathcal{H}^n(\Xi_{t_0} \cap A)] \\ &\quad + [\mathcal{H}^n(\Xi_{t_0} \cap A) - \mathcal{H}^n(\Xi_t \cap A)] \\ &\overset{(4.1.4),(4.2.8)}{\leq} \frac{\epsilon}{2^{p+3}} - \epsilon + \frac{\epsilon}{2^{p+3}} < -\frac{\epsilon}{2}. \end{aligned} \tag{4.2.11}$$

Gathering the estimates (4.2.6), (4.2.9), (4.2.10) and (4.2.11), we finally obtain the properties (4.2.2) and (4.2.3) of the lemma, which concludes the proof. □

By retracing the steps of the previous proof, we can see that it allows for (at least) two generalizations, which will be useful.

Remark 4.9. (i) *Note that the choice to have the cubes $Q(t_0, a')$ and $Q(t_0, a)$ centered at t_0 is unnecessary and only for the sake of notational simplicity. Indeed, with the appropriate choice of cut-off functions ψ, γ and χ, the proof is almost identical if we replace them with cubes $Q(t_1, a')$ and $Q(t_2, a)$ (or even more general sets) that are nested inside each other, i.e. $Q(t_1, a') \subset\subset Q(t_2, a) \subset\subset Q(t_o, \eta)$.*
(ii) *The lemma also works with minimal modifications if the family $\{\Xi_t\}_{t\in[0,1]^p}$ is parameterized by a k-dimensional smooth submanifold $\mathcal{P}$ of $[0,1]^p$, with $\partial\mathcal{P} \cap (0,1)^p = \emptyset$ in case it has a boundary. One can simply take restrictions of the relevant subsets of $[0,1]^p$ to their intersection with $\mathcal{P}$, both in the statement and the proof.*

In order to use the previous lemma to construct the aforementioned competitor minimizing sequence and prove the Almgren-Pitts lemma, we will require a combinatorial covering argument, which is the second main ingredient. The idea is to decrease areas of certain "large-area" slices by a definite amount, while simultaneously keeping the potential area increase for other slices under control.

Proof of Proposition 4.7. Let $\{\{\Gamma_t^\ell\}\}^\ell \subset X$ be a minimizing sequence which satisfies Proposition 3.2, and such that $\mathcal{F}(\{\Gamma_t^\ell\}) := \max_{t\in\mathcal{P}} \mathcal{H}^n(\Gamma_t^\ell) < m_0(X) + \frac{1}{2^{m+2}\ell}$. The following claim clearly implies the propositon:

Claim: *For every N large enough there exists $t_N \in \mathcal{P}$ such that $\Gamma^N := \Gamma^N_{t_N}$ is $\frac{1}{N}$-a.m. in every $(U^1, \ldots, U^{\omega_m}) \in \mathcal{CO}_{\omega_m}$ and $\mathcal{H}^n(\Gamma^N) \geq m_0(X) - \frac{1}{N}$.*

We define

$$K_N := \left\{ t \in \mathcal{P} \mid \mathcal{H}^n(\Gamma_t^N) \geq m_0(X) - \frac{1}{N} \right\} \tag{4.2.12}$$

and suppose, contrary to the claim, that there is some subsequence $\{N_j\}_j$ such that for every $t \in K_{N_j}$ there exists an ω_m-tuple $(U^1, \ldots, U^{\omega_m})$ such that $\Gamma_t^{N_j}$ is not $\frac{1}{N_j}$-a.m. in it. After a translation and/or dilation, we can assume, without loss of generality, that $\mathcal{P} \subset [0,1]^m$ (in the embedding). Note that, if we assume

N to be large enough that $m_0(X) - 1/N > bM_0(X)$, the set K_N will surely lie in the interior of $\mathcal{P}$. In fact, in everything that follows, it is tacitly assumed that the subsets of $\mathcal{P}$ we choose stay away from $\partial\mathcal{P}$, in order to comply with our definition of homotopic families.

By a slight abuse of notation, from now on we do not rename the subsequence, and also drop the super- and subscript N from Γ_t^N and K_N. Thus for every $t \in K$ there is a ω_m-tuple of open sets $(U_{1,t}, \ldots, U_{\omega_m,t}) \in \mathcal{CO}_{\omega_m}$ and ω_m families $\{\Sigma_{i,t,\tau}\}_{\tau\in[0,1]}$ such that the following properties hold for every $i \in \{1, \ldots, \omega_m\}$:

- $\Sigma_{i,t,0} = \Gamma_t$;
- $\Sigma_{i,t,\tau} \setminus U_{i,t} = \Gamma_t \setminus U_{i,t}$;
- $\mathcal{H}^n(\Sigma_{i,t,\tau}) \leq \mathcal{H}^n(\Gamma_t) + \frac{1}{2^{m+2}N}$;
- $\mathcal{H}^n(\Sigma_{i,t,1}) \leq \mathcal{H}^n(\Gamma_t) - \frac{1}{N}$.

By recalling the definition of $\mathcal{CO}_{\omega_m}$, for every $t \in K$ and every $i \in \{1, \ldots, \omega_m\}$ we can choose an open set $U'_{i,t}$ such that $U_{i,t} \subset\subset U'_{i,t}$ and

$$\operatorname{dist}(U'_{i,t}, U_{i,t}) \geq 2 \cdot \min\{\operatorname{diam}(U'_{i,t}), \operatorname{diam}(U'_{j,t})\} \tag{4.2.13}$$

for all $i, j \in \{1, \ldots, \omega_m\}$ with $i \neq j$. Next, we apply Lemma 4.8 with $\Xi_t = \Gamma_t$, $U = U_{i,t}$, $U' = U'_{i,t}$ and $\Sigma_\tau = \Sigma_{i,t,\tau}$. Hence, for every $t \in K$ and $i \in \{1, \ldots, \omega_m\}$ we get a corresponding constant $\eta_{i,t}$ given by the statement of the lemma.

Step 1: Initial covering. We first assign to each $t \in K$ exactly one constant η_t, by setting $\eta_t := \min_{i\in\{1,\ldots,\omega_m\}} \eta_{i,t}$. We would like to initially decompose the cube $[0, 1]^m$ into a grid of small, slightly overlapping cubes, such that we might be able to apply the constructions in Lemma 4.8 to each of those (after discarding the ones which have empty intersection with K). For this, we would like their size to be smaller than the size of the cube given by the lemma for any point lying in the center of one of these cubes. Therefore, we choose a covering of K:

$$\left\{ Q\big(((2r_1 + 1)\tilde{\eta}, \ldots, (2r_m + 1)\tilde{\eta}), \eta\big) \;\middle|\; \begin{array}{l} r_1, \ldots, r_m \in \{1, \ldots, \xi\} \\ Q\big(((2r_1 + 1)\tilde{\eta}, \ldots, (2r_m + 1)\tilde{\eta}), \eta\big) \cap K \neq \emptyset \end{array} \right\},$$

where $\tilde{\eta} = \frac{9}{10}\eta$, $\xi = \min\{n \in \mathbb{N}_0 \mid (2n+1)\tilde{\eta} > 1-\eta\}$, and η is yet to be determined. Ideally, we would like η to be smaller than any η_t. The problem, however, is that for each $t \in K$, the constant η_t (which is determined by the proof of Lemma 4.8) depends also on the sets $U_{i,t}$, so one might not in general expect to prove lower boundedness. Nevertheless, using Remark 4.9(i), we deduce that if $t_0 \in K$, then for any $t \in Q(t_0, \frac{\eta_{i,t_0}}{2})$, the conclusions of the lemma hold with $\eta = \frac{\eta_{i,t_0}}{2}$ (t being the center of the cubes now), and $U = U_{i,t_0}$. Therefore, for $t\,(\in K)$ close enough to t_0, we can replace $(U_{1,t},\dots,U_{\omega_m,t})$ by $(U_{1,t_0},\dots,U_{\omega_m,t_0})$ if necessary. Now, we can start by covering K with $Q(t,\frac{\eta_t}{2})$, $t\in K$. Since K is compact, it suffices to pick finitely many $t_0,\dots,t_l$ with $K\subset\bigcup Q(t_i,\frac{\eta_{t_i}}{2})$. We then set:

$$\eta' := \min_{j\in\{0,\dots,l\}} \frac{\eta_{t_j}}{2} \tag{4.2.14}$$

Also note that for N large enough, because of condition (1.2.3), the set K lies in the interior of $\mathcal{P}$ (in case it has a boundary). That means there exists some $\eta''>0$ such that for any cube $Q(t,\eta'')$ intersecting K we have $\partial\mathcal{P}\cap Q(t,\eta'')=\emptyset$. We define $\eta:=\min\{\frac{\eta'}{4},\eta''\}$, which determines the size of the cubes in the covering. Furthermore, we set

$$\begin{aligned}
\mathfrak{r} &:= (r_1,\dots,r_m);\\
t_{\mathfrak{r}} &:= ((2r_1+1)\tilde\eta,\dots,(2r_m+1)\tilde\eta)\\
\mathfrak{Q}_{\mathfrak{r}} &:= Q\big(((2r_1+1)\tilde\eta,\dots,(2r_m+1)\tilde\eta),\eta\big)
\end{aligned}$$

To each $\mathfrak{Q}_{\mathfrak{r}}$ with $t_{\mathfrak{r}}\in K$ we can assign a corresponding ω_m-tuple $(U_{1,t_{\mathfrak{r}}},\dots,U_{\omega_m,t_{\mathfrak{r}}})\in\mathcal{CO}_{\omega_m}$ by assumption. On the other hand, to any cube $\mathfrak{Q}_{\mathfrak{r}}$ in the covering (i.e. $\mathfrak{Q}_{\mathfrak{r}}\cap K\neq\emptyset$) where the center $t_{\mathfrak{r}}\notin K$, owing to Remark 4.9(i) and the choice of η above, we can also assign $(U_{1,\tilde t},\dots,U_{\omega_m,\tilde t})$ belonging to some $\tilde t\in K$, where we are able to apply Lemma 4.8. With a slight abuse of notation, we will denote this tuple by $(U_{1,t_{\mathfrak{r}}},\dots,U_{\omega_m,t_{\mathfrak{r}}})$.

Step 2: Refinement of the covering. Our aim is to find a refinement $\{\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})\}$, $\mathfrak{a}\in\{-\frac{2}{5},\frac{2}{5}\}^m$ of the initial covering, such that

(i) $\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})\subset\mathfrak{Q}_{\mathfrak{r}}$ for any $\mathfrak{a}$;

(ii) for every $\mathfrak{r}$ and every $\mathfrak{a}$ there is a choice of $U_{\mathfrak{a},t_{\mathfrak{r}}}$ such that

- $U'_{\mathfrak{a},t_{\mathfrak{r}}}\in\{U'_{1,t_{\mathfrak{r}}},\dots,U'_{\omega_m,t_{\mathfrak{r}}}\}$,
- $\operatorname{dist}(U'_{\mathfrak{a},t_{\mathfrak{r}}},U'_{\mathfrak{a}',t'_{\mathfrak{r}}})>0$ if $\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})\cap\mathfrak{Q}_{\mathfrak{r}'}(\mathfrak{a}')\neq\emptyset$;

(iii) every point $t\in[0,1]^m$ is contained in at most 2^m cubes $\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})$.

To do this, we cover each cube $\mathfrak{Q}_{\mathfrak{r}}$ with 2^m smaller cubes in the following way:

$$\left\{Q\Big(((2r_1+1)\tilde\eta+a_1\eta,\dots,(2r_m+1)\tilde\eta+a_m\eta),\frac{3}{5}\eta\Big)\mid a_1,\dots,a_m\in\left\{-\frac{2}{5},\frac{2}{5}\right\}\right\} \tag{4.2.15}$$

We simplify the notation by setting

$$\mathfrak{a} := (a_1, \dots, a_m) \in \left\{-\frac{2}{5}, \frac{2}{5}\right\};$$
$$\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a}) := Q\Big(\big((2r_1+1)\tilde{\eta} + a_1\eta, \dots, (2r_m+1)\tilde{\eta} + a_m\eta\big), \frac{3}{5}\eta\Big).$$

Note that this choice of the refinement, as well as that of the initial covering, immediately guarantees properties (i) and (iii).

After assigning a family of open sets to each cube of the initial covering in the previous step, we now want to assign a subfamily to every cube of the refined covering. Consider a cube $\mathfrak{Q}_{\mathfrak{r}_1}(\mathfrak{a}) \subset \mathfrak{Q}_{\mathfrak{r}_1}$ of the refinement. Assume that $\mathfrak{Q}_{\mathfrak{r}_1}(\mathfrak{a})$ intersects $1 \le j \le 2^m - 1$ different cubes of the initial covering, say $\mathfrak{Q}_{\mathfrak{r}_2}, \dots, \mathfrak{Q}_{\mathfrak{r}_j}$, and let

$$\mathcal{F}_{\mathfrak{r}_1} := (U'_{1,t_{\mathfrak{r}_1}}, \dots, U'_{\omega_m,t_{\mathfrak{r}_1}}), \dots, \mathcal{F}_{\mathfrak{r}_j} := (U'_{1,t_{\mathfrak{r}_j}} \dots, U'_{\omega_m,t_{\mathfrak{r}_j}})$$

be the corresponding tuples of open sets. Applying Lemma 4.6, we extract subfamilies $\mathcal{F}^{sub}_{\mathfrak{r}_i} \subset \mathcal{F}_{\mathfrak{r}_i}$ for every $i \in \{1, \dots, j\}$, each containing at least 2^m open sets such that

$$\text{dist}(U, V) > 0 \quad \forall\, U \in \mathcal{F}^{sub}_{\mathfrak{r}_a},\ V \in \mathcal{F}^{sub}_{\mathfrak{r}_b}. \tag{4.2.16}$$

We then assign to $\mathfrak{Q}_{\mathfrak{r}_1}(\mathfrak{a})$ the subfamily $\mathcal{F}^{sub}_{\mathfrak{r}_1}$, which we now denote by $\mathcal{F}_{\mathfrak{r}}(\mathfrak{a})$. We can do this for every cube in the refinement. By construction, the property (4.2.16) surely holds for each two subfamilies $\mathcal{F}_{\mathfrak{r}_i}(\mathfrak{a}), \mathcal{F}_{\mathfrak{r}_j}(\mathfrak{a}')$ assigned to cubes $\mathfrak{Q}_{\mathfrak{r}_i}(a), \mathfrak{Q}_{\mathfrak{r}_j}(a')$, such that $\mathfrak{Q}_{\mathfrak{r}_i}(a) \cap \mathfrak{Q}_{\mathfrak{r}_j}(a') \neq \emptyset$ and $\mathfrak{Q}_{\mathfrak{r}_i} \neq \mathfrak{Q}_{\mathfrak{r}_j}$. On the other hand, the subfamilies assigned to two cubes belonging to the same cube of the initial covering (i.e. $\mathfrak{Q}_{\mathfrak{r}_i} = \mathfrak{Q}_{\mathfrak{r}_j}$), are not necessarily different. Note however, that each subfamily contains at least 2^m open sets, and every cube $\mathfrak{Q}_{\mathfrak{r}}$ of the initial covering is covered by exactly 2^m cubes of the refinement. Hence we can assign to each of those a distinct open set $U'_{\mathfrak{a},t_{\mathfrak{r}}} \in \mathcal{F}_{\mathfrak{r}}(\mathfrak{a})$.
Thus we have a refinement of the covering $\{\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})\}$ and corresponding open sets $U'_{\mathfrak{a},t_{\mathfrak{r}}}$ which have all the three properties listed in the beginning of Step 2. Moreover, since K is compact, and $\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a}) = Q(t_{\mathfrak{r}} + \mathfrak{a}\eta, \frac{3}{5}\eta)$ according to (4.2.15), we can choose a $\delta > 0$ such that every $t \in K$ is contained in at least one of the cubes $Q(t_{\mathfrak{r}} + \mathfrak{a}\eta, \frac{3}{5}\eta - \delta)$.

For the sake of simplicity, let us now rename the refinement $\{\mathfrak{Q}_{\mathfrak{r}}(\mathfrak{a})\}$ and call it $\{P_i\}$, the corresponding smaller cubes $Q(t_{\mathfrak{r}} + \mathfrak{a}\eta, \frac{3}{5}\eta - \delta)$ we call P_i^δ, and the associated open sets we now denote by U_i and U'_i.

Step 3: Conclusion. In order to deduce the existence of a family $\{\Gamma_{i,t}\}$ with the properties

- $\Gamma_{i,t} = \Gamma_t$ if $t \notin P_i$ and $\Gamma_{i,t} \setminus U'_i = \Gamma_t \setminus U'_i$ if $t \in P_i$;

- $\mathcal{H}^n(\Gamma_{i,t}) \leq \mathcal{H}^n(\Gamma_t) + \frac{1}{2^{m+1}N}$ for every t;

- $\mathcal{H}^n(\Gamma_{i,t}) \leq \mathcal{H}^n(\Gamma_t) - \frac{1}{2N}$ if $t \in P_i^\delta$,

we apply Lemma 4.8 for $\Xi_t = \Gamma_t, U = U_i, U' = U_i'$ and $\Sigma_\tau = \Sigma_{i,t,\tau}$.
Recall that from the construction of the refined covering $\{P_i\}$ and the choice of U_i' it follows that, if $P_i \cap P_j \neq \emptyset$ for $i \neq j$, then $\mathrm{dist}(U_i', U_j') > 0$. We can therefore define a new family $\{\Gamma_t'\}_{t \in \mathcal{P}}$ with

- $\Gamma_t' = \Gamma_t$ if $t \notin \cup P_i$;

- $\Gamma_t' = \Gamma_{i,t}$ if t is contained in in a single P_i;

- $\Gamma_t' = \left[\Gamma_t \setminus (U_{i_1}' \cup \ldots \cup U_{i_s}')\right] \cup \left[\Gamma_{i_1,t} \cap U_{i_1}'\right] \cup \ldots \cup \left[\Gamma_{i_s,t} \cap U_{i_s}'\right]$ if $t \in P_{i_1} \cap \ldots \cap P_{i_s}$, $s \geq 2$.

This family is clearly homotopic to $\{\Gamma_t\}$ and hence belongs to X.
We now want to estimate $\mathcal{F}(\{\Gamma_t'\})$. If $t \notin K$, then t is contained in at most 2^m P_i's and Γ_t' can therefore gain at most $2^m \cdot \frac{1}{2^{m+1}N}$ in area:

$$t \notin K \implies \mathcal{H}^n(\Gamma_t') \leq \mathcal{H}^n(\Gamma_t) + 2^m \cdot \frac{1}{2^{m+1}N} \leq m_0(X) - \frac{1}{2N}. \tag{4.2.17}$$

Note that the last inequality is due to the definition of K. If $t \in K$, then t is contained in at least one cube P_i^δ and at most $2^m - 1$ other cubes $P_{i_1}, \ldots, P_{i_{2^m-1}}$. Hence the area of Γ_t' looses at least $\frac{1}{2N}$ and gains at most $(2^m - 1) \cdot \frac{1}{2^{m+1}N}$ in area. Thus,

$$t \in K \implies \mathcal{H}^n(\Gamma_t') \leq \mathcal{H}^n(\Gamma_t) + (2^m - 1) \cdot \frac{1}{2^{m+1}N} - \frac{1}{2N} \leq m_0(X) - \frac{1}{2^{m+2}N}, \tag{4.2.18}$$

where the last inequality holds since $\mathcal{H}^n(\Gamma_t) \leq \mathcal{F}(\{\Gamma_s^N\}_{s \in \mathcal{P}}) \leq m_0(X) + \frac{1}{2^{m+2}N}$ by assumption.
From the preceding inequalities we conclude

$$\mathcal{F}(\{\Gamma_t'\}) \leq m_0(X) - \frac{1}{2^{m+1}N},$$

which is a contradiction to $m_0(X) = \inf_X \mathcal{F}$. This finishes the proof. □

Chapter 5

Boundary behavior of stationary varifolds

In this chapter, we start by recollecting some of the basic tools and concepts in regularity theory for minimal surfaces. In the last section we introduce a crucial ingredient, in the form of a bound for the second fundamental form of a minimal surface at the boundary, necessary to conclude regularity in the constrained case.

5.1 Maximum Principle

The first important tool which we recall is the following classical maximum principle for the constrained case.

Proposition 5.1 (Maximum principle). *Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional submanifold satisfying Assumption 1.1 and $U \subset \mathcal{M}$ an open set. If $V \in \mathcal{V}_s^c(U, \gamma)$ for some $C^{2,\alpha}$ $(n-1)$-dimensional submanifold of γ (namely $\delta V(\chi) \geq 0$ for every $\chi \in \mathfrak{X}_c^-(U \setminus \gamma)$), then* $\mathrm{spt}(V) \cap \partial\mathcal{M} \subset \gamma$.

The above proposition is classical if we were to consider $\mathcal{M}$ as a subset of a larger manifold $\tilde{\mathcal{M}}$ without boundary and we had a varifold V which were stationary in $\mathcal{M}' \setminus \gamma$. For a proof we refer the reader to White's paper [59]. However it is straightforward to check that the proof in [59] works in our setting, since the condition $\delta V(\chi) \geq 0$ for the class of vector fields $\mathfrak{X}_c^-(U \setminus \gamma)$ pointing "inwards" is what White really uses in his proof.

As a corollary to the above proposition we obtain the following

Corollary 5.2. *Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional Riemannian manifold isometrically embedded in a Euclidean space $\mathbb{R}^N$ and satisfying Assumption 1.1. If U' is an open subset of $\mathbb{R}^N$ and V a varifold in $\mathcal{V}_s^c(U' \cap \mathcal{M}, \gamma)$ for some $n-1$-dimensional $C^{2,\alpha}$ submanifold γ of ∂M, then V has, as a varifold in $\mathbb{R}^N$, bounded generalized mean curvature in the sense of Allard away from γ: in particular all the conclusions of Allard's boundary regularity theory in [1] are applicable.*

The proof is straightforward: after viewing $\mathcal{M}$ as a subset of a closed submanifold $\tilde{\mathcal{M}}$, Proposition 5.1 implies the stationarity of V in $\tilde{\mathcal{M}} \setminus \gamma$ and reduces the statement to a classical computation (see for instance [49, Remark 16.6(2)]).

Remark 5.3. *While one can in principle work with objects defined intrinsically on $\mathcal{M}$, it is often more convenient to embed $\mathcal{M}$ (smoothly) isometrically into some Euclidean space $\mathbb{R}^N$. In fact, by possibly choosing a larger N, one can do this so that $\mathcal{M}$ is a compact subset of a closed $(n+1)$-dimensional manifold $\tilde{\mathcal{M}}$.*

5.2 Monotonicity Formulae

Perhaps the most important tool in regularity theory for stationary varifolds is the monotonicity formula. For $x \in \mathrm{Int}(\mathcal{M})$ it says that there exists a constant Λ (depending on the ambient Riemannian manifold $\mathcal{M}$, and which is 0 if the metric is flat, see [49]) such that the function

$$f(\rho) := e^{\Lambda\rho}\frac{||V||(B_\rho(x))}{\omega_n\rho^n} \tag{5.2.1}$$

is non-decreasing for every $x \in \mathcal{M}$ and $\rho < \min\{\mathrm{Inj}(\mathcal{M}), \mathrm{dist}(x, \partial\mathcal{M})\}$. A similar conclusion assuming the existence of a "boundary" was reached by Allard [1]. Combined with Corollary 5.2 the results in [1] give the following

Proposition 5.4. *Consider an open subset $U \subset \mathcal{M}$ and a varifold $V \in \mathcal{V}_s^c(U, \gamma)$ for some $C^{2,\alpha}$ submanifold γ of $\mathcal{M}$. Then for every $x \in \gamma$, there exists a $\rho_0 > 0$ and a (smooth) function $\Phi(\rho)$ with $\Phi(\rho) \to 0$ as $\rho \to 0$, such that the quantity*

$$f(\rho) = e^{\Phi(\rho)}\frac{||V||(B_\rho(x))}{\omega_n\rho^n} \tag{5.2.2}$$

is a monotone non-decreasing function of ρ as long as $0 < \rho < \rho_0$.

In particular, we conclude that the limit $\frac{||V||(B_\rho(x))}{\omega_n\rho^n}$ exists and it is finite at any point $x \in \gamma$, which in turn, by a standard covering argument implies the following important fact:

Corollary 5.5. *Let V and γ be as in Proposition 5.4. Then $\|V\|(\gamma) = 0$. In particular the varifold V of Proposition 4.3 is integer rectifiable in the whole $\mathcal{M}$ in the constrained case.*

The case with free boundaries has been addressed by Grüter and Jost in [23, 21, 22], who proved a suitable version of the monotonicity formula. The results in these papers were proved in the Euclidean space, but they are easily extendable to the case of stationary varifolds in compact Riemannian manifolds using the embedding trick of Remark 5.3. We summarize the conclusion in the following

Proposition 5.6. *Assume $\mathcal{M} \subset \tilde{\mathcal{M}} \subset \mathbb{R}^N$, where $\tilde{\mathcal{M}}$ is a closed manifold, let $U \subset \mathcal{M}$ be an open set and V a varifold in $\mathcal{V}_s^u(U)$. Then for each $x \in U$, there* *exists an* $r < dist(x, \partial U)$*, and a constant* $c(x,r)$*, with* $c(x,r) \to 1$ *as* $r \to 0$*, such that*

$$\frac{||V||(B_\sigma(x)) + ||V||(\tilde{B}_\sigma(x))}{\omega_k \sigma^k} \leq c(x,r) \frac{||V||(B_\rho(x)) + ||V||(\tilde{B}_\rho(x))}{\omega_k \rho^k} \tag{5.2.3}$$

for all $0 < \sigma < \rho < r$*. Here,* $\tilde{B}_\sigma(x)$ *denotes the reflection of the ball* $B_\sigma(x)$ *across the boundary* ∂M*.*

Note that, for points in Int (U) and $r < \mathrm{dist}(x, \partial\mathcal{M})$, the monotonicity formula of Grüter and Jost reduces to (5.2.1). An important consequence of the monotonicity in all of the above cases is the existence of the *density function* of the varifold under consideration:

$$\Theta(V, x) = \lim_{r \to 0} \frac{||V||(B_r(x))}{\omega_n r^n} \tag{5.2.4}$$

is well defined at all points $x \in U$. Moreover, in the case $V \in \mathcal{V}_s^u$, one can conclude that the function

$$\tilde{\Theta}(V, x) := \begin{cases} \Theta(V, x) & x \in \mathrm{Int}(\mathcal{M}) \cap U \\ 2\Theta(V, x) & x \in \partial\mathcal{M} \cap U \end{cases}$$

is upper semicontinuous in U. In the constrained case we conclude instead that the density function is upper semicontinuous in Int (U) and in $\partial\mathcal{M} \cap U$.

5.3 Blow-up and Tangent Cones

In this section we recall the usual "rescaling" procedure which allows to blow-up minimal surfaces at a given point. Following Remark 5.3, we adhere to the standard procedure of first embedding the Riemannian manifold $\mathcal{M}$ into $\mathbb{R}^N$. We will use the term $(n+1)$-dimensional wedge of opening angle $\theta \in]0, \frac{\pi}{2}[$ for any closed subset W of the form $R(W_0)$, where $R \in SO(n+1)$ is an orientation-preserving isometry of $\mathbb{R}^{n+1}$ and we recall that that W_0 is the canonical wedge with opening angle θ, namely the set

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : |x_{n+1}| \leq x_1 \tan\theta\}.$$

The half-hyperplane $R(\{x_{n+1} = 0, x_1 > 0\})$ will be called the *axis* of the wedge and the $(n-1)$-dimensional plane $\ell := R(\{x_{n+1} = x_1 = 0\})$ will be called the *tip of the wedge*. As stated above, when $W = W_0$, we call it the *canonical wedge with opening angle* θ.

Definition 5.7. *Let* $\mathcal{M}$ *be a smooth* $(n+1)$*-dimensional manifold with boundary satisfying Assumption 1.1 and* γ *a* $C^{2,\alpha}$ $(n-1)$*-dimensional submanifold of* $\partial\mathcal{M}$*. We say that a closed set* $K \subset \mathcal{M}$ *meets* $\partial\mathcal{M}$ *in* γ *with opening angle at most* θ *if the following holds:*

- $\gamma = K \cap \partial\mathcal{M}$;
- *for any* $x \in \gamma$, *let* $\tau \in T_x\partial\mathcal{M}$ *be a unit vector orthogonal to* $T_x\gamma$ *and* $\nu \in T_x\mathcal{M}$ *be the unit vector orthogonal to* $T_x\partial\mathcal{M}$ *and pointing inwards; then for every* C^1 *curve* $\sigma : [0,1] \to K$, *with* $\sigma(0) = x$ *and parameterized by arc length, we have*

$$|\langle \dot\sigma(0), \tau\rangle| \leq \langle \dot\sigma(0), \nu\rangle \tan\theta\,. \tag{5.3.1}$$

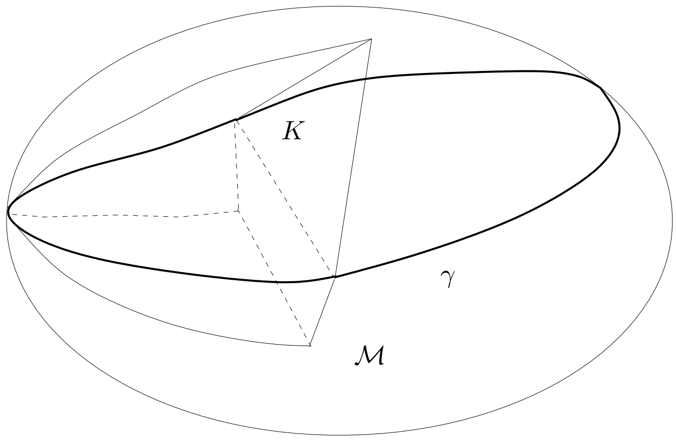

Figure 5.1: A set K meeting γ at some angle at most $\theta < \frac{\pi}{2}$.

We are now ready to state the blow-up procedure which we will use in the rest of the thesis, especially at boundary points. Recall that, for $x \in \partial\mathcal{M}$, ν is the unit vector of $T_x\mathcal{M}$ orthogonal to $T_x\partial\mathcal{M}$ and pointing inwards.

Lemma 5.8. *Let* $\mathcal{M} \subset \mathbb{R}^N$ *be a smooth Riemannian manifold satisfying Assumption 1.1,* $U \subset \mathcal{M}$ *an open set and* V *a varifold which is stationary in* $\mathrm{Int}\,(U)$. *Given a point* $x \in \mathrm{spt}\,(V) \subset \mathcal{M}$ *we introduce the map* $\iota_{x,r} : \mathbb{R}^N \to \mathbb{R}^N$ *defined by* $\iota_{x,r}(y) := (y-x)/r$ *and let* $\mathcal{M}_{x,r} := \iota_{x,r}(\mathcal{M})$ *and* $V_{x,r} := (\iota_{x,r})_\sharp V$.

- *(I) If* $x \in \mathrm{Int}\,(U)$, *then* $\mathcal{M}_{x,r}$ *converges as* $r \to 0$, *locally in the Hausdorff sense, to* $T_x\mathcal{M}$ *(which is identified with the corresponding linear subspace of* $\mathbb{R}^N$*). Up to subsequences* $V_{x,r}$ *converges, in the sense of varifolds, to a varifold* S *which is a cone and it is stationary in* $T_x\mathcal{M}$.
- *(B) If* $x \in \partial\mathcal{M}$, *then* $\mathcal{M}_{x,r}$ *converges, locally in the Hausdorff sense, to* $T_x^+\mathcal{M} := T_x\mathcal{M} \cap \{y : \nu\cdot y \geq 0\}$. $V_{x,r}$ *converges, in the sense of varifolds, to an integer varifold* S *which is a cone, it is supported in* $T_x^+\mathcal{M}$ *and it is stationary in* $T_x\mathcal{M} \cap \{y : y\cdot\nu > 0\}$.
- *(W) If* $x \in \partial\mathcal{M}$ *and we assume in addition that* $\mathrm{spt}\,(V)$ *is contained in a closed* K *which meets* $\partial\mathcal{M}$ *at a* $C^{2,\alpha}$ *submanifold* γ *with opening angle at most* θ, *then each such* S *(as in statement (B)) is supported in the wedge* $W \subset T_x\mathcal{M}$ *of opening angle* θ *with tip* $T_x\gamma$ *and axis orthogonal to* $T_x\partial M$.

In the cases (I) and (W) the limit varifold W is integer rectifiable if V is integer rectifiable. From now on any such varifold will be called a tangent varifold to V ay the point x.

Definition 5.9. *At every point x we denote by* $\mathrm{Tan}\,(x, V)$ *the set of varifolds W which are limits of subsequences (with $r_k \downarrow 0$) of $\{V_{x,r}\}_r$ and which will be called* tangent cones to V at x. *We observe moreover that*

$$\Theta(V, x) = \Theta(W, 0) = \frac{\|W\|(B_r(0))}{\omega_n r^n} \qquad \forall W \in \mathrm{Tan}\,(x, V), \forall r > 0\,. \tag{5.3.2}$$

5.4 White's curvature estimate at the boundary

In this section we introduce the most important tool in the boundary regularity theory which we will develop in the sequel. The tool is a suitable curvature estimate at the boundary, suggested by Brian White, which is valid for stationary smooth hypersurfaces constrained in a wedge.

Theorem 5.10. *Let $\mathcal{M}$ be an $(n+1)$-dimensional smooth Riemannian manifold satisfying Assumption 1.1, $\gamma \subset \partial\mathcal{M}$ a $C^{2,\alpha}$ submanifold of $\partial\mathcal{M}$ and $r \in]0, 1[$. Denote by D the inverse of the distance between the closest pair of points in γ which belong to distinct connected components; if there is a single connected component, set $D = 0$. For every $M > 0$ and $\theta \in [0, \pi[$ there are positive constants $C(D, M, \mathcal{M}, \gamma, n, \eta)$ and $\delta(D, M, \mathcal{M}, \gamma, n, \eta)$ with the following property: Assume that*

(CE1) $x_0 \in \gamma$ and Σ is a stable, minimal hypersurface in $B_{2r}(x_0)$ such that:

- *$\mathcal{H}^n(\Sigma) \leq Mr^n$, $\partial\Sigma \subset \partial B_{2r}(x_0) \cup \partial\mathcal{M}$ and $\partial\Sigma \cap \partial\mathcal{M} = \gamma$;*
- *Σ is C^1 apart from a closed set* $\mathrm{sing}\,(\Sigma)$ *with* $\mathcal{H}^{n-2}(\mathrm{sing}\,(\Sigma)) = 0$ *and* $\gamma \cap \mathrm{sing}\,(\Sigma) = \emptyset$;
- *Σ is contained in a closed set K meeting $\partial\mathcal{M}$ in γ with opening angle at most θ.*

Then Σ is $C^{2,\alpha}$ in $B_{\delta r}(x_0)$ and

$$|A| \leq Cr^{-1} \qquad \text{in } B_{\delta r}(x_0). \tag{5.4.1}$$

Furthermore, $\Sigma \cap B_{\delta r}(x_0)$ consists of a single connected component.

The proof requires two elementary but important lemmas, which we state immediately.

Lemma 5.11. *Let V be an integer n-dimensional rectifiable varifold in $\mathbb{R}^{n+1}$ such that*

(a) V is stationary in a wedge W_0 of opening angle θ;

(b) $\delta V = (w_1, w_2, 0, \dots, 0)\mathcal{H}^{n-1} \llcorner \ell$ *for some Borel vector field* $w = (w_1, w_2) \in L^1_{loc}(\mathcal{H}^{n-1} \llcorner \ell; \mathbb{R}^2)$.

Then for $\mathcal{H}^{n-1}$*-a.e.* $x \in \ell$ *we have the representation*

$$w(x) = \sum_{i=1}^{m} v_i(x)$$

where

- $m = 2\Theta(x, V)$;
- *each* v_i *is of the form* $(-\cos\theta_i, -\sin\theta_i)$ *for some* $\theta_i \in [-\theta, \theta]$.

Lemma 5.12. *Let* $k \in \mathbb{N} \setminus \{0\}$ *and* $v_i = (-\cos\theta_i, -\sin\theta_i)$ $2k+1$ *unit vectors in the plane with* $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$. *Then the sum* $v_1 + \ldots + v_{2k+1}$ *has length strictly larger than* 1.

The simple proofs of the lemmata will be postponed to the end of the section, while we first deal with the proof of the main Theorem (given the two lemmata).

Proof of Theorem 5.10. We will in fact prove that the constants δ and C depend on the $C^{2,\alpha}$ regularity of γ, $\mathcal{M}$ and $\partial\mathcal{M}$. First of all we focus on the curvature estimate.

Without loss of generality, we again assume that $\mathcal{M}$ is isometrically embedded in a euclidean space $\mathbb{R}^N$. Observe that the dimension N can be estimated by n and thus we can assume that N is some fixed number, depending only on n. Upon rescaling we can also assume that $r = 1$: the rescaling would just lower the $C^{2,\alpha}$ norm of $\mathcal{M}$, $\partial\mathcal{M}$ and γ and increase the distance D between different connected components of γ.

Assuming by contradiction that the statement does not hold, we would find a sequence of manifolds $\mathcal{M}_k$, boundaries γ_k, minimal surfaces Σ_k and points $p_k \in \Sigma$ with the properties that:

- $|A_{\Sigma_k}|(p_k) \uparrow \infty$, or p_k is a singular point, and the distance between p_k and γ_k converges to 0
- $\mathcal{M}_k, \Sigma_k$ and γ_k satisfy the assumptions of the Theorem with $r = 1$, with a uniform bound on the $C^{2,\alpha}$ regularities of both γ_k and $\mathcal{M}_k$ and with a uniform bound on M and θ.

We let $q_k \in \gamma_k$ be the closest point to p_k and, w.l.o.g. we translate the surfaces so that $q_k = 0$. We next rescale them by a factor ρ_k^{-1} where ρ_k is the maximum between $|p_k|$ and $|A_{\Sigma_k}(p_k)|^{-1}$ (where we understand the latter quantity to be 0 if p_k is a singular point). We denote by $\bar{\gamma}_k$, $\bar{\mathcal{M}}_k$, $\bar{\Sigma}_k$ and $\bar{p}_k$ the corresponding rescaled objects. It turns out that, up to subsequences,

(a) the rescaled manifolds $\bar{\mathcal{M}}_k$ are converging, locally in $C^{2,\alpha}$, to a half $(n+1)$-dimensional plane, that w.l.o.g. we can assume to $\mathbb{R}^{n+1}_+ = \{x : x_{n+2} = \ldots = x_N = 0,\ x_1 \geq 0\}$;

(b) the rescaled manifolds $\partial\bar{\mathcal{M}}_k$ are converging, locally in $C^{2,\alpha}$, to an n-dimensional plane, namely $\{x : x_1 = x_{n+2} = \ldots = x_N = 0\}$;

(c) the rescaled surfaces $\bar{\gamma}_k$ are converging, locally in $C^{2,\alpha}$, to an $n-1$-dimensional plane, that w.l.o.g. we can assume to be $\ell := \{x_1 = x_{n+1} = x_{n+2} = \ldots = x_N = 0\}$;

(d) the points $\bar{p}_k$ are converging to some point $\bar{p}$ and $\liminf_k |A_{\bar{\Sigma}_k}| > 0$;

(e) the surfaces $\bar{\Sigma}_k$ are converging, in the sense of varifolds, to an integral varifold V, which is supported in the standard wedge W contained in $\mathbb{R}^{n+1}_+$ with tip ℓ, axis $\pi + = \{x_1 > 0, x_{n+1} = x_{n+2} = \ldots = x_N = 0\}$;

(f) the integral varifold V is stationary inside $W \setminus \ell$ and in fact $|\delta V| \leq \mathcal{H}^{n-1} \llcorner \ell$.

All these statements are simple consequences of elementary considerations and of the theory of varifolds. For (f), observe that $|\delta\bar{\Sigma}_k| \leq \mathcal{H}^{n-1} \llcorner \gamma_k + \|A_{\mathcal{M}_k}\|_{C^0} \mathcal{H}^n \llcorner \Sigma_k$ and use the semicontinuity of the total variation of the first variations under varifold convergence.

We next show that the varifold V is necessarily half of an n-dimensional plane τ bounded by ℓ and lying in W. This would imply, by Allard's regularity theorem, that the surfaces Σ_k are in fact converging in $C^{2,\alpha}$ to τ, contradicting (d).

We first start to show that the density $2\Theta(V, x)$ is odd at $\mathcal{H}^{n-1}$ a.e. $p \in \ell$. By White's stratification theorem, see Theorem 5 of White [58], with the exception of a closed set of dimension at most $n-2$, for any point $x \in \ell$ there is a tangent cone V_∞ to V which is invariant under translations along ℓ. This implies that V_∞ is necessarily given by

$$\sum_{i=1}^{m} [\![\pi_i \cap W_0]\!]$$

for some family of n-dimensional planes (possibly with repetitions) containing ℓ, where $m = m(x) = 2\Theta(V_\infty, 0) = 2\Theta(V, x)$. Observe that any such plane is contained in the wedge W. Consider the first of them, π_1 and let $B \subset \pi_1$ be a compact connected set not intersecting ℓ. By a simple diagonal argument, V_∞ is also the limit of an appropriate sequence of rescalings of the surfaces $\bar{\Sigma}_k$, namely $(\bar{\Sigma}_{k(j)})_{0,r_j}$. If $k(j)$ converges to infinity sufficiently fast, we keep the convergence conclusions in (a), (b), (c), (e) and (f) even when we replace $\bar{\Sigma}_k$, $\bar{\mathcal{M}}_k$, $\bar{\partial}\mathcal{M}_k$ and $\bar{\gamma}_k$ with the corresponding rescalings $(\bar{\Sigma}_{k(j)})_{0,r_j}$, $(\bar{\mathcal{M}}_{k(j)})_{0,r_j}$, $(\bar{\partial}\mathcal{M}_{k(j)})_{0,r_j}$ and $(\bar{\gamma}_{k(j)})_{0,r_j}$. For notational simplicity, let us keep the label $\bar{\Sigma}_k$ even for the rescaled surfaces.

Note that, since the varifold V_∞ is a finite number of affine graphs over the set B (and m is the sum of the multiplicities, including the ones of π_1), the Schoen-Simon theorem implies smooth convergence of the $\bar{\Sigma}_k$, so the $\bar{\Sigma}_k$ will also be a union of m graphs over B (distinct, because the Σ_k are surfaces with multiplicity 1). Let $\varkappa = \pi_1^\perp$ and, after giving compatible orientations to π_1 and $\varkappa$, for every $x \in B$ where $\varkappa + x$ intersects $\bar\Sigma_k$ transversally, we define the degree

$$d(x) := \sum_{y \in \varkappa \cap \bar\Sigma_k} \varepsilon(T_y \bar\Sigma_k, \varkappa),$$

where $\varepsilon(T_y\bar\Sigma_k, \varkappa)$ takes, respectively, the value 1 or -1 according to whether the two transversal planes have compatible or non-compatible orientation. For k large enough γ_k does not intersect $B + \varkappa$ and thus d is constant on B. Moreover, it turns out that d is either 1 or -1. To see this, one can for instance consider $\bar\Sigma_k$ as integral currents and project them onto π_1. Due to (c), for k large enough (and inside some large ball around the origin), the projection of $\bar\gamma_k$'s will have multiplicity one, and since the projection and the boundary operator commute, the projection of $\bar\Sigma_k$'s onto π_1 inside B will be simply $\pm[\![\pi_1]\!] \llcorner B$. Thus the number of intersections of $y + \varkappa$ with $\bar\Sigma_k$ must be odd for a.e. $y \in B$. This obviously implies that m is odd.

We next infer that $2\Theta(V, x)$ must be 1 at $\mathcal{H}^{n-1}$-a.e. $x \in \ell$. Apply indeed Lemma 5.11 and, using the Borel maps w and v_i defined in there, consider the Borel function

$$f(x) := |w(x)| = \left| \sum v_i(x) \right| .$$

We then have $|\delta V| = f\mathcal{H}^{n-1} \llcorner \ell$ and from Lemma 5.12, we conclude that $f > 1$ at every point where $2\Theta(V, x)$ is an odd number larger than 1. Since by the previous step such number is odd a.e., we infer our claim by using item (f) from above. By Allard's regularity theorem, any point x as above (i.e. where there is at least one tangent cone invariant under translations along ℓ) is then a regular point.

Hence, it turns out that

- the set of interior singular points of V has Hausdorff dimension at most $n - 7$, by the Schoen-Simon compactness theorem;
- the set of boundary singular points has Hausdorff dimension at most $n - 2$.

Consequently, there is only one connected component of the regular set of V whose closure contains ℓ. Thus there cannot be any other connected component, because its closure would not touch ℓ and would give a stationary varifold contained in the wedge W, violating the maximum principle. Hence we infer that any interior regular point of V can be connected with a curve of regular points to a regular boundary point. In turn this implies that the varifold V has density 1 at every regular point. So V can be given the structure of a current and in particular we conclude that the Σ_k's are converging to V as a current.

Consider next that,

$$\lim_{R \uparrow \infty} \frac{\|V\|(B_R(0))}{R^n}$$

is bounded uniformly, depending only on the constant M. Thus, by the usual monotonicity formula, there is a sequence $R_k \to \infty$ such that V_{0,R_K} converges to a cone V_∞ stationary in W. Again, by a diagonal argument, V_∞ is also the limit of a sequence of rescalings $(\bar\Sigma_{k(j)})_{0,R_j}$, and if $k(j)$ converges to infinity sufficiently fast, we retain the conclusions in (a), (b), (c), (e) and (f) when we replace $\bar\Sigma_k$, $\bar{\mathcal{M}}_k$, $\bar\partial\mathcal{M}_k$ and $\bar\gamma_k$ with the corresponding rescalings $(\bar\Sigma_{k(j)})_{0,R_j}$, $(\bar{\mathcal{M}}_{k(j)})_{0,R_j}$, $(\bar\partial\mathcal{M}_{k(j)})_{0,R_j}$ and $(\bar\gamma_{k(j)})_{0,R_j}$.

All the conclusions inferred above for V are then valid for V_∞ as well, namely: V_∞ has multiplicity 1 a.e., it can be given the structure of a current and the surfaces $(\bar\Sigma_{k(j)})_{0,R_j}$ are converning to it in the sense of currents. In particular the boundary of V_∞ (as a current) is given by ℓ (with the appropriate orientation). We can then argue as in [1, Lemma 5.2] to conclude that the current V_∞ is in fact the union of finitely many half-hyperplanes meeting at ℓ. But since ℓ has many regular points, where the multiplicity must be $\frac{1}{2}$, we conclude that indeed V_∞ consists of a single plane.

In particular we infer from the argument above that $\Theta(V_\infty, 0) = \frac{1}{2}$. This in turn implies

$$\lim_{R\uparrow\infty} \frac{\|V\|(B_R(0))}{R^n} = \frac{\omega_n}{2}\,.$$

On the other hand

$$\lim_{r\downarrow 0} \frac{\|V\|(B_r(0))}{r^n} = \omega_n \Theta(V,0)\,.$$

But the upper semicontinuity of the density implies implies that $\Theta(V,0) \geq \frac{1}{2}$.

Since ℓ is flat, Allard's monotonicity formula implies that

$$r \mapsto \frac{\|V\|(B_r(0))}{r^n}$$

is monotone and thus constant. Again the monotonicity formula implies that such function is constant if and only if V is itself a cone. This means that V coincides with V_∞ and is half of a hyperplane, as desired.

We now come to the claim that, choosing δ possibly smaller, the surface Σ has a single connected component in $B_{\delta r}(x_0)$. Again this is achieved by a blow-up argument. Given the estimate on the curvature, for every sufficiently small η we have that x_0 belongs to a connected component of Σ which is the graph of a function f for some given system of coordinates in $B_{2\eta}(x_0)$. Let us denote by Γ such a connected component. For η small we can assume that the tangent to Γ is as close to $T_{x_0}\Gamma$ as we desire and thus we can assume that the connected component is actually a graph of a function $f : T_{x_0}\Gamma \to T_{x_0}\Gamma^\perp$, with gradient smaller than some $\varepsilon > 0$, whose choice we specify in a moment. From now on in all our discussion we assume to work in normal coordinates based at x_0. In fact it is convenient to consider a closed manifold $\tilde{\mathcal{M}}$ which contains $\partial\mathcal{M}$ and from now on we let $\tilde B_r(x_0)$ be the corresponding geodesic balls.

Assume now, by contradiction, that $B_{\delta\eta}(x_0)$ contains another point y_0 which does not belong to Γ, where δ is a small parameter, depending on the maximal opening angle θ with which the set K can meet $\partial\mathcal{M}$. Thus y_0 belongs to a second connected component Γ'. By the curvature estimates we can assume that Γ' as well is graphical and more precisely it is a graph over some plane π of a function g with gradient smaller than ε and height smaller than $\varepsilon\eta$. Moreover, without loss of generality, we can assume that π passes through the point y_0.

Observe that by assumption (CE1) Γ' cannot intersect $\partial\mathcal{M}$, hence any point in $\partial\Gamma'$ is at distance 2η from x_0. Since $\|g\|_0 \leq \varepsilon\eta$, it turns out that any point $z_0 \in \pi \cap \tilde{B}_{(2-2\varepsilon)\eta}(x_0)$ must be in the domain of g, which we denote by $\mathrm{Dom}\,(g)$. To see this observe first that, since $\pi \cap \tilde{B}_{(2-2\varepsilon)\eta}(x_0)$ is convex, we can join y_0 and z_0 with a path γ lying in $\pi \cap \tilde{B}_{(2-2\varepsilon)\eta}(x_0)$. Assume that γ is parametrized over $[0,1]$ and that $\gamma(0) = y_0$. For a small ε we know that $\gamma([0,\varepsilon]) \subset \mathrm{Dom}\,(g)$. If $\gamma(1) \in \mathrm{Dom}\,(g)$ we are finished. Otherwise we let τ be the infimum of $\{t : \gamma(t) \notin \mathrm{Dom}\,(g)\}$. Obviously the point p in the closure of the graph of g lying over $\gamma(\tau)$ is a boundary point for Γ'. On the other hand, since $\gamma(\tau) \in \tilde{B}_{(2-2\varepsilon)\eta}(x_0)$ and $\|g\| \leq \varepsilon\eta$, clearly p cannot be at distance 2η from x_0. This is a contradiction and thus we have proved the conclusion

$$\pi \cap \tilde{B}_{(2-2\varepsilon)\eta}(x_0) \subset \mathrm{Dom}\,(g)\,.$$

In particular we conclude that $\pi\cap\tilde{B}_{(2-4\varepsilon)\eta}(x_0)$ cannot meet $\partial\mathcal{M}$: if the intersection were not empty, then there would be a point q contained in $\pi \cap (\tilde{B}_{(2-2\varepsilon)\eta}(x_0) \setminus \mathcal{M})$ which lies at distance at least $\frac{3}{2}\varepsilon\eta$ from $\partial\mathcal{M}$. In particular the point of the graph of g lying on top of q could not belong to $\mathcal{M}$, although it would be a point of Γ'.

By a similar argument, we conclude that $\pi \cap \tilde{B}_{(2-6\varepsilon)\eta}(x_0)$ cannot intersect $T_{x_0}\Sigma$, otherwise we would have nonempty intersection between the graphs of f and g, i.e. a point belonging to $\Gamma \cap \Gamma'$, which we know to be different connected components of $\Sigma \cap B_{2\eta}(x_0)$, hence disjoint.

At this point we choose $\varepsilon = \frac{1}{12}$. Summarizing, the plane π has the following properties:

(a) π contains a point $y_0 \in B_{\delta\eta}(x_0)$;

(b) π does not intersect $\partial\mathcal{M} \cap \tilde{B}_{3\eta/2}$;

(c) π does not intersect $T_{x_0}\Sigma \cap \tilde{B}_{3\eta/2}$;

(d) $T_{x_0}\Sigma$ meets $T_{x_0}\partial\mathcal{M}$ at an opening angle at most θ.

It is now a simple geometric property that, if δ is chosen sufficiently small compared to θ, then the plane π cannot exist, cf. Figure 5.2.

□

Proof of Lemma 5.11. By White's stratification theorem, see Theorem 5 of White [58], at $\mathcal{H}^{n-1}$-a.e. point $x \in \ell$ there is a tangent cone V_∞ to V which is

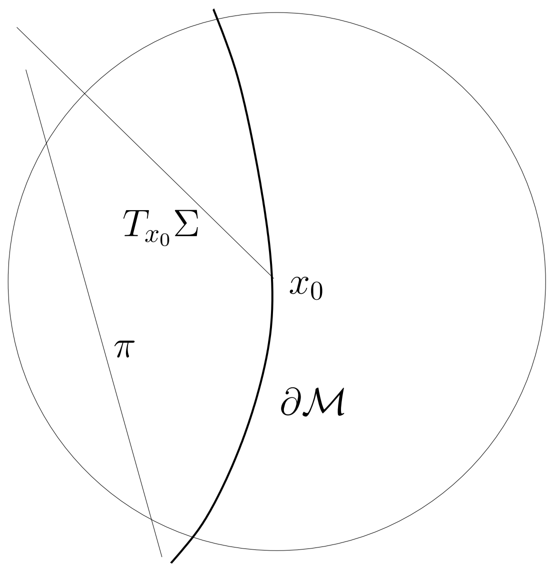

Figure 5.2: If two planes π and $T_{x_0}\Sigma$ satisfy the assumption (b), (c) and (d), then π cannot contain a point which is $\delta\eta$ close to x_0.

invariant under translations along ℓ. This implies that V_∞ is necessarily given by

$$\sum_{i=1}^{m} [\![\pi_i \cap W_0]\!]$$

for some family of n-dimensional planes containing ℓ, where $m = m(x) = 2\Theta(V_\infty, 0) = 2\Theta(V, x)$. It is therefore obvious that

$$\delta V_\infty = \sum_{i=1}^{m} v_i \mathcal{H}^{n-1} \llcorner \ell$$

where each $v_i = v_i(x)$ is the unit vector contained in $\pi_i \setminus W_0$ and orthogonal to ℓ: therefore for each i we have $v_i = (-\cos\theta_i, -\sin\theta_i)$ for some $\theta_i \in [-\theta, \theta]$.

Let $r_k \downarrow 0$ be a sequence such that the rescaled varifolds V_{x,r_k} converge weakly to V_∞. Then $\delta V_{x,r}$ converges to δV_∞ in the sense that $\delta V_{x,r_k}(\varphi) \to \delta V_\infty(\varphi)$ for any smooth compactly supported vector field on $\mathbb{R}^{n+1}$. On the other hand for a.e. x we have $\delta V_{x,r} \rightharpoonup^* w(x)\mathcal{H}^{n-1} \llcorner \ell$. This completes the proof since for a.e. x we must have $w(x) = \sum_i^{m(x)} v_i(x)$. □

Proof of Lemma 5.12. We order the vectors so that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{2k+1}$. For each $i \leq k$, the sum w_i of the pair $v_i + v_{2k+2-i}$ is a positive multiple of

$$\left(-\cos\tfrac{\theta_i+\theta_{2k+2-i}}{2}, -\sin\tfrac{\theta_i+\theta_{2k+2-i}}{2}\right).$$

Since $\theta_i \leq \theta_{k+1} \leq \theta_{2k+2-i}$, it is easy to see that the vectors w_i and v_{k+1} form an angle strictly smaller than $\frac{\pi}{2}$. We therefore have $\langle w_i, v_{k+1}\rangle > 0$ and we can estimate

$$|v_1 + \ldots + v_{2k+1}|^2 \geq \sum_{j=1}^{2k+1} \langle v_j, v_{k+1} \rangle = 1 + \sum_{i=1}^{k} \langle w_i, v_{k+1} \rangle > 1\,.$$

□

Chapter 6

Stability and compactness

A varifold will be called *stable* (in an open set U) if the second variation $\delta^2 V$ is nonnegative when evaluated at every vector field compactly supported in $\mathrm{Int}\,(U)$. Strict stability will mean that the second variation is actually strictly positive, except for the trivial situation where the vector field vanishes everywhere on the support of the varifold. Since the ground-breaking works of Schoen [45], Schoen-Simon-Yau [47] and Schoen-Simon [46], it is known that, roughly speaking, all the smoothness and compactness results which are valid for hypersurfaces (resp. integer rectifiable hypercurrents) which minimize the area are also valid (in the form of suitable *a priori* estimates) for stable hypersurfaces.

6.1 Interior compactness and regularity

We recall here the fundamental compactness/regularity theorem of Schoen and Simon (cf. [46]) for stable minimal surfaces.

Theorem 6.1. *Let $\{\Sigma^k\}$ be a sequence of stable minimal hypersurfaces in some open subset $U \subset \mathcal{M} \setminus \partial\mathcal{M}$ and assume that*

(i) each Σ^k is smooth except for a closed set of vanishing $\mathcal{H}^{n-2}$-measure;

(ii) Σ^k has no boundary in U;

(iii) $\sup_k \mathcal{H}^n(\Sigma^k) < \infty$.

Then a subsequence of $\{\Sigma^k\}$ (not relabeled) converges, in the sense of varifolds, to an integer rectifiable varifold V such that

(a) V is, up to multiplicity, a stable minimal hypersurface Γ with $\dim(\mathrm{Sing}\,(\Gamma)) \leq n-7$;

(b) at any point $p \notin \mathrm{Sing}\,(\Gamma)$ the convergence is smooth, namely there is a neighborhood of V such that, for k large enough, $\Sigma^k \cap V$ can be written as the union of N distinct smooth graphs over (the normal bundle of) $\Gamma \cap U$,

converging smoothly (where the number N is uniformly controlled by virtue of (iii)).

In fact the Theorem of Schoen and Simon gives a more quantitative version of the smooth convergence, since for every point $p \notin \mathrm{Sing}\,(\Gamma)$ the second fundamental form of Σ^k at p can be bounded, for k large enough, by $C\mathrm{dist}(p, \mathrm{Sing}\,(\Gamma))^{-1}$, where the constant C is independent of k.

6.2 Boundary version for free boundary surfaces

In [23] the fundamental result of Schoen and Simon has been extended to the case of free boundary minimal surfaces, under a suitable convexity assumption in the case of $n = 2$ in the Euclidean case. However, it can be readily checked that the arguments presented in [23] to adapt the proof of Schoen and Simon in [46] to the free boundary case are independent both of the dimensional assumption $n = 2$ and of the assumption that $\mathcal{M}$ is a convex subset of the Euclidean space. We state the resulting theorem below, where we need the following stronger stability condition, which we will call *stability for the free boundary problem.*

Definition 6.2. *Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional Riemannian manifold and $U \subset \mathcal{M}$ an open set. A vaifold $V \in \mathcal{V}_s^u(U)$ is said to be stable for the free boundary problem if $\delta^2 V(\chi) \geq 0$ for every $\chi \in \mathfrak{X}_c^t(U)$.*

Theorem 6.3. *Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional Riemannian manifold which satisfies Assumption 1.1. Let Σ^k be a sequence of stable minimal hypersurfaces in some open subset $U \subset \mathcal{M}$ and assume that*

(i) each Σ^k is smooth except for a closed set of vanishing $\mathcal{H}^{n-2}$-measure;

(ii) $\partial\Sigma^k \cap U$ is contained in $\partial\mathcal{M}$ and Σ^k meets $\partial\mathcal{M}$ orthogonally (thus, Σ^k is stationary for the free boundary problem);

(iii) Σ^k is stable for the free boundary problem;

(iv) $\sup_k \mathcal{H}^n(\Sigma^k) < \infty$.

Then a subsequence of Σ^k (not relabeled) converges, in the sense of varifolds, to an integer rectifiable varifold V such that

(a) V is, up to multiplicity, a stable minimal hypersurface Γ with $\dim(\mathrm{Sing}\,(\Gamma)) \leq n-7$;

(b) at any point $p \notin \mathrm{Sing}\,(\Gamma)$ the convergence is smooth;

(c) Γ meets $\partial\mathcal{M}$ orthogonally, thus $V \in \mathcal{V}_s^u(U)$;

(d) V is stable for the free boundary problem.

6.3 Boundary version for the constrained case

We shall now combine Theorem 5.10 with the interior estimates of Schoen and Simon to get a compactness theorem for stable minimal hypersurfaces which have a fixed given boundary γ and meet $\partial\mathcal{M}$ transversally in a suitable quantified way.

Theorem 6.4. *Let $\mathcal{M}$ be an $(n+1)$-dimensional smooth Riemannian manifold which satisfies Assumption 1.1, $\gamma \subset \partial\mathcal{M}$ a $C^{2,\alpha}$ submanifold of $\partial\mathcal{M}$, U an open subset of $\mathcal{M}$ and $K \subset U$ a set which meets $\partial\mathcal{M}$ in γ at an opening angle smaller than $\frac{\pi}{2}$. Let Σ^k be a sequence of stable minimal hypersurfaces in $U \subset \mathcal{M}$ and assume that*

(i) *each Σ^k is smooth except for a closed set of vanishing $\mathcal{H}^{n-2}$-measure and $\gamma \cap \operatorname{sing}(\Sigma) = \emptyset$;*

(ii) $\partial\Sigma^k \cap U = \gamma \cap U$;

(iii) $\sup_k \mathcal{H}^n(\Sigma^k) < \infty$;

(iv) $\Sigma^k \subset K$.

Then a subsequence of Σ^k, not relabeled, converges, in the sense of varifolds, to an integer rectifiable varifold V such that

(a) *V is, up to multiplicity, a stable minimal hypersurface Γ with $\dim(\operatorname{Sing}(\Gamma)) \leq n-7$;*

(b) *at any point $p \notin \operatorname{Sing}(\Gamma)$ the convergence is smooth;*

(c) *$\operatorname{Sing}(\Gamma) \cap \partial\mathcal{M} = \emptyset$ and $\partial\Gamma = \gamma$ (in particular, the multiplicity of any connected component of Γ which intersects $\partial\mathcal{M}$ must be 1).*

Proof. First of all, after extraction of a subsequence we can assume that Σ^k converges to a varifold V. Observe that V is stationary in $\operatorname{Int}(U)$ and thus it is integer rectifiable in there, by Allard's compactness theorem. Note also that each Σ^k belongs to $\mathcal{V}^c_s(U,\gamma)$ and thus, by continuity of the first variations, V belongs as well to $\mathcal{V}^c_s(U,\gamma)$. Thus, by the maximum principle of Proposition 5.1 we conclude that $\|V\|(\partial\mathcal{M}) = \|V\|(\gamma)$. In particular, as argued for Corollary 5.5, Allard's monotonicity formula (cf. Proposition 5.4) implies that $\|V\|(\partial\mathcal{M}) = 0$ and that V is integer rectifiable in U.

Next observe that in $U \setminus \partial\mathcal{M}$ we can apply the Schoen-Simon compactness theorem: thus, except for a set K' in $U \setminus \partial\mathcal{M}$, the smooth convergence holds at every point $x_0 \in U \setminus (\gamma \cup K')$ and $\dim(K') \leq n-7$. As for the points $x \in \gamma$, consider first an open subset U' which has positive distance from $\partial U \setminus \partial\mathcal{M}$. By the boundary curvature estimates of Theorem 5.10, there is an $r_0 > 0$ and a constant C_0, both independent of k, such that $|A_{\Sigma_k}| \leq C_0$ in any ball $B_{r_0}(x)$ with center $x \in \gamma \cap U'$. This implies that, in a fixed neighborhood U'' of γ, Σ^k consists of a single smooth component which is a graph at a fixed scale, independent of

k. The estimate on the curvature in Theorem 5.10 gives then the convergence of these graphs in $C^{1,\alpha}$ for every $\alpha < 1$. Since the limit turns out to be (locally) graphical and a solution of an elliptic PDE, classical Schauder estimates implies its smoothness and the smooth convergence. □

6.4 Varying the ambient manifolds

In all the situations above, we can allow also for the manifolds $\mathcal{M}$ to vary in a controlled way, namely to change as $\mathcal{M}_k$ along the sequence. One version which is particularly useful is when the $\mathcal{M}_k$ are embedded in a given, fixed, Euclidean space and they are converging smoothly to a $\mathcal{M}$. All the compactness statements above still hold in this case and in particular the corresponding obvious modifications (left to the reader) will be used at one occasion in the very simple situation where the $\mathcal{M}_k$ are rescalings of the same $\mathcal{M}$ at a given point, thus converging to the tangent space at that point, cf. Section 8.1 and Section 8.3.2 below.

6.5 Wedge property

In this section we use the maximum principle to prove that, given a smooth γ any stationary varifold $V \in \mathcal{V}_s^c(U, \gamma)$ meets γ "transversally" in a quantified way, namely it lies in suitable wedges that have a controlled angle. This property is necessary to apply to the compactness Theorem 6.4. The precise formulation is the following

Lemma 6.5. *Let $\mathcal{M}$ be a smooth $(n+1)$-dimensional submanifold satisfying Assumption 1.1, γ be a C^2 $(n-1)$-dimensional submanifold of $\mathcal{M}$ and $U' \subset\subset U$ two open subsets of $\mathcal{M}$. Then there is a constant $\theta_0(U, U', \gamma) < \frac{\pi}{2}$ and a compact set $K \subset \overline{U'}$ with the following properties:*

(a) K meets γ at an opening angle at most θ_0;

(b) $\mathrm{spt}(V) \cap U' \subset K$ for every varifold $V \in \mathcal{V}_s^c(U, \gamma)$.

Note that in the special case of $U = \mathcal{M}$, we are allowed to choose $U' = \mathcal{M}$ and thus we conclude a uniform transversality property for any varifold in $\mathcal{V}_s^c(\mathcal{M}, \gamma)$, in particular for the varifold V of Proposition 4.3. On the other hand we do need the local version above for several considerations leading to the regularity of V at the boundary. When $\mathcal{M}$ is a subset of the Euclidean space, the lemma above follows easily from the following two considerations:

(i) By the classical maximum principle, $\mathrm{spt}(V)$ is contained in the compact subset K which is the convex hull of $(\gamma \cap \overline{U}) \cup (\partial U \setminus \partial \mathcal{M})$, see for instance the reference [49];

(ii) Such convex hull K meets γ at an opening angle which is strictly less than $\frac{\pi}{2}$ at every point $x \in \gamma \cap U$ (here the C^2 regularity of γ is crucially used, cf. the elementary Lemma 6.6 below).

The uniform (upper) bound on the angle is then obtained in $U' \subset\subset U$ simply by compactness.

Unfortunately, although the extension of (i) above to general Riemannian manifolds is folklore among the experts, we do not know of a reference that we could invoke for Lemma 6.5 without some additional technical work. This essentially amounts to reducing to the Euclidean situation by a suitable choice of coordinates.

6.5.1 Wedge property and convex hull

We start by recording the following elementary fact, which in fact proves claim (ii) above.

Lemma 6.6. *Consider a bounded, open, smooth, uniformly convex set $\mathcal{M} \subset \mathbb{R}^{n+1}$ and a C^2 $(n-1)$-dimensional connected submanifold $\gamma \subset B_r(0) \cap \partial\mathcal{M}$ passing through the origin. Then there is a wedge W contaning γ such that:*

(a) The axis of W is orthogonal to $T_0\partial\mathcal{M}$;

(b) The tip of W is $T_0\gamma$;

(c) The opening angle is bounded away from $\frac{\pi}{2}$ in terms of the principal curvatures of $\partial\mathcal{M}$ and of $\|A_\gamma\|_\infty$.

Proof. For simplicity fix coordinates so that $T_0\gamma = \{x_1 = x_{n+1} = 0\}$, $T_0\partial\mathcal{M} = \{x_1 = 0\}$ and $\mathcal{M}$ is lying in $\{x_1 > 0\}$. For every $\theta < \frac{\pi}{2}$ let $\mathcal{M}_\theta$ be the portion of $\mathcal{M}$ lying in $\{x_{n+1} > x_1 \tan\theta\}$, and consider $r > 0$ such that the open ball

$$B_r((0, 0, \dots, 0, r))$$

contains $\mathcal{M}_\theta$. Let $\rho(\theta)$ be the smallest such radius. $\rho(\theta)$ is a non-increasing function of θ and by the uniform convexity of $\mathcal{M}$, $\rho(\theta) \to 0$ as $\theta \uparrow \frac{\pi}{2}$. On the other hand we know that if $\rho(\theta) < \|A_\gamma\|_\infty^{-1}$, then $B_\rho((0, 0, \dots, 0, \rho))$ is an osculating ball for γ at 0 and cannot contain any point of γ. This shows that for all θ sufficiently close to $\frac{\pi}{2}$, γ is contained in $\{x_{n+1} \le x_1 \tan\theta\}$. By a simple reflection argument we obtain the same property with $\{-x_{n+1} \le x_1 \tan\theta\}$, which completes the proof of the lemma. □

6.5.2 Proof of Lemma 6.5

First of all, we observe that by a simple covering argument it suffices to show the lemma in a sufficiently small neighborhood U of any point $p \in \partial\mathcal{M}$, since we already know by the maximum principle in Proposition 5.1 that $\mathrm{spt}(V) \cap \partial M \subset \gamma$.

Recall that we can assume that $\mathcal{M}$ is a subset of a closed Riemannian manifold $\tilde{\mathcal{M}}$, cf. Remark 5.3. Let $p \in \gamma$, and $\tilde{U}$ a normal neighborhood of p in $\tilde{\mathcal{M}}$. We then consider normal coordinates on $\tilde{\mathcal{M}}$ centered at p, given by the chart $\varphi := E \circ \exp_p^{-1} : U \to \mathbb{R}^{n+1}$, where the isomorphism $E : T_p\tilde{\mathcal{M}} \to \mathbb{R}^{n+1}$ is chosen so that $E(T_p\partial\mathcal{M}) = \{x \in \mathbb{R}^{n+1} : x_1 = 0\}$, and $E(T_p\gamma) = \{x \in \mathbb{R}^{n+1} : x_1 = x_{n+1} = 0\}$.

Now, if we let A denote the second fundamental form of $\partial\mathcal{M}$ in $\mathcal{M}$ with respect to the unit normal ν pointing inside $\mathcal{M}$, B the second fundamental form of $\varphi(\partial\mathcal{M})$ in $\mathbb{R}^{n+1}$ with respect to the unit normal n pointing inside $\varphi(\mathcal{M})$, and ∇, $\bar{\nabla}$ the ambient Riemannian and Euclidean connection respectively, we immediately see that

$$
\begin{aligned}
A(X,Y)\big|_p = -g(\bar{\nabla}_X \nu, Y)\big|_p = g(\nu, \bar{\nabla}_X Y)\big|_p = \langle n, \nabla_X Y\rangle\big|_0 \\
= -\langle \nabla_X n, Y\rangle\big|_0 = B(X,Y)\big|_0,
\end{aligned}
$$

since $\nu(p) = n(0)$, $g(.,.)\big|_p = \langle .,. \rangle$ and $\nabla\big|_p = \bar{\nabla}\big|_0$ by the properties of the exponential map. Hence, it follows from Assumption 1.1 that $B \succeq \xi\,\mathrm{Id}$ at 0. Thus, if we represent $\varphi(\partial\mathcal{M})$ as a graph of a function f over its tangent plane $\{x \in \mathbb{R}^{n+1} | \, x_1 = 0\}$ at 0, the Hessian of f is equal to B at 0, and hence there are some Cartesian coordinates $(y_2, \ldots, y_{n+1})$ on this plane such that f has the form

$$
f(y_2, \ldots, y_{n+1}) = \frac{1}{2}(\kappa_2 y_2^2 + \ldots + \kappa_{n+1} y_{n+1}^2) + O(|y|^3), \tag{6.5.1}
$$

where $\kappa_2, \ldots, \kappa_{n+1} > \xi > 0$ are principal curvatures (w.r.t. inward pointing normal at 0).

In particular we can assume that U is chosen so small that f is uniformly convex in the Euclidean sense, namely that $D^2 f > 0$ everywhere on $\varphi(\tilde{U})$. By abuse of notation we keep using $\tilde{U}$ for $\varphi(\tilde{U})$, $\mathcal{M}$ for $\varphi(\mathcal{M})$ and thus V for the varifold $\varphi_\sharp V$. Since we can now regard $\mathcal{M}$ as a convex subset of the euclidean space, we can apply Lemma 6.6 and conclude that γ is contained in a wedge W of the form $\{|x_{n+1}| \le \tan\theta x_1\}$. However we cannot apply the maximum principle to conclude that $\mathrm{spt}(V) \subset W$ because V is not *stationary in the Euclidean metric*. Our aim is however to show that, if we enlarge slightly θ, but still keep it smaller than $\frac{\pi}{2}$, then $\mathrm{spt}(V) \subset W$. The resulting θ will depend on the manifold $\mathcal{M}$, the submanifold γ and the size of $\tilde{U}$, but not on the point p. Thus this argument completes the proof, since the set K can be taken to be, in a neighborhood of $\gamma \cap U'$, the union of the corresponding wedges for p (intersected with the corresponding neighborhoods $\tilde{U}$) as p varies in $\gamma \cap U'$.

Recall that, in our notation, $\mathcal{M}$ is in fact the set $\{y_1 \ge f(y_2, \ldots, y_{n+1})\}$. For each $\lambda \ge 0$ consider now the function

$$
f_\lambda(y_2, \ldots, y_{n+1}) = (1-\lambda) f(y_2, \ldots, y_{n+1}) + \lambda \frac{y_{n+1}}{\tan\theta}\,.
$$

For $\lambda \downarrow 0$, the function f_λ converges in C^2 to the function f. Thus the set

$\mathcal{M}_\lambda = \{y_1 \geq f_{c,\lambda}(y_2, \ldots, y_{n+1})\}$ is uniformly convex in the Riemannian manifold $\tilde{\mathcal{M}}$ as soon as $\lambda \leq \varepsilon$.

Observe next that all the graphs of all the functions f_λ intersect in an $n-1$-dimensional submanifold, which is indeed the intersection of the graphs of f_1 and $f_0 = f$. Consider now the region

$$R = \{f_0(y_2, \ldots, y_{n+1}) \leq y_1 \leq f_\varepsilon(y_2, \ldots, y_{n+1})\},$$

cf. Fig. 6.1. Since the graph of f_0 is in fact $\partial\mathcal{M}$, we know from Proposition 5.1 that $\mathrm{spt}(V)\cap\partial\mathcal{M}\cap R \subset \gamma$ and from the choice of the wedge W we thus know that $\mathrm{spt}(V) \cap \partial\mathcal{M} \cap R = \{0\}$. Assume now by contradiction that R contains another point $p \in \mathrm{spt}(V)$. Then this point does not belong to γ. On the other hand there must be a minimum δ such that the graph of f_δ contains this point. But then, by the fact that $\mathcal{M}_\delta$ is uniformly convex in $\tilde{\mathcal{M}}$, this would be a contradiction to the maximum principle of Proposition 5.1.

We thus conclude that the region R intersects the support of V only in the origin. On the other hand recall that f_δ is convex also in the Euclidean sense. Thus its graph lies above its tangent at 0, which is given by $\{y_{n+1} = y_1\delta^{-1}\tan\theta\}$. This implies that the support of V intersected with $\tilde{U}$ is in fact contained in

$$\{y_{n+1} \leq y_1\delta^{-1}\tan\theta\}.$$

Symmetrizing the argument we find the new desired wedge in which the support of our varifold is contained. □

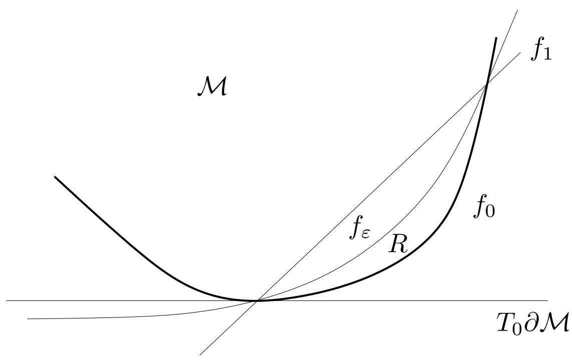

Figure 6.1: The region R foliated by the graps of f_λ.

Chapter 7

Replacements at the Boundary

We have now all the tools for proving the boundary regularity of the varifold V in Proposition 4.3 and we can start with the argument leading to

Theorem 7.1. *The varifold V of Proposition 4.3 has all the properties claimed in Theorem 1.5.*

The argument is indeed split into two main steps. In the first one we employ another important concept first developed by Pitts, called a *replacement*.

Definition 7.2. *Let $V \in \mathcal{V}(\mathcal{M})$ be a stationary varifold in $\mathcal{M}$ and $U \subset \overline{\mathcal{M}}$ an open set. A stationary varifold $V' \in \mathcal{V}(\mathcal{M})$ (in the appropriate sense) is called a replacement for V in U if $V = V'$ on $G(\mathcal{M} \setminus U)$, $\|V\|(\mathcal{M}) = \|V'\|(\mathcal{M})$, and $V \llcorner U$ is a stable minimal hypersurface Γ. In the constrained case we require that $\partial\Gamma \cap U = \gamma \cap U$ (in particular the connected components of Γ that intersect γ will arise with multiplicity 1 in the varifold V). In the unconstrained case the surface $\Gamma \cap U$ meets $\partial\mathcal{M}$ orthogonally.*

Our goal now is to show that the almost minimizing property of the sequence $\{\Gamma^j\}$ from Proposition 4.3 is sufficient to prove the existence of a replacement for the varifold V. More precisely, we prove:

Proposition 7.3. *Let $\{\Gamma^j\}, V$ and r be as in Proposition 4.3. Fix $x \in \mathcal{M}$ and consider an annulus $An \in \mathcal{AN}_{r(x)}(x)$. Then there exist a varifold $\tilde{V}$, a sequence $\{\tilde{\Gamma}^j\}$ and a function $r' : \mathcal{M} \to \mathbb{R}^+$ such that*

- *$\tilde{V}$ is a replacement for V in An and $\tilde{\Gamma}^j$ converges to $\tilde{V}$ in the sense of varifolds;*
- *$\tilde{\Gamma}^j$ is almost minimizing in every $An' \in \mathcal{AN}_{r'(y)}(y)$ with $y \in \mathcal{M}$;*
- *$r(x) = r'(x)$.*

7.1 Homotopic Plateau's Problem

Let us fix a point $x \in \mathcal{M}$ and $An \in \mathcal{AN}_{r(x)}(x)$ from now on. If $x \in \operatorname{Int}\mathcal{M}$, then the statement above is indeed proved in [13]. We fix therefore $x \in \partial M$. The strategy of the proof will be analogous to the one in [13] and follows anyway the pioneering ideas of Pitts: in An we will indeed replace the a.m. sequence Γ^j with a suitable $\tilde{\Gamma}^j$, which is a minimizing sequence for a suitable (homotopic) variational problem.

As a starting point for the proof we consider for each $j \in \mathbb{N}$ the following class and the corresponding variational problem:

Definition 7.4. *Let $U \subset \mathcal{M}$ be an open set and for each $j \in \mathbb{N}$ consider the class $\mathcal{H}_c(\Gamma^j, U)$ (resp. $\mathcal{H}_s(\Gamma^j, U)$) of surfaces Ξ such that there is a constrained (resp. unconstrained) family of surfaces $\{\Gamma_t\}$ satisfying $\Gamma_0 = \Gamma^j$, $\Gamma_1 = \Xi$, (4.1.1), (4.1.2), and (4.1.3) for $\epsilon = 1/j$ (recall that m is fixed by Remark 4.2). The subscript c (resp. u) will dropped when clear from the context. A minimizing sequence in $\mathcal{H}(\Gamma^j, U)$ is a sequence $\Gamma^{j,k}$ for which the volume of $\Gamma^{j,k}$ converges towards the infimum.*

We will call the variational problem above the $(2^{m+2}j)^{-1}$ - *homotopic Plateau* problem. Next, we take a minimizing sequence $\{\Gamma^{j,k}\}_{k\in\mathbb{N}} \subset \mathcal{H}(\Gamma^j, An)$. Up to subsequences, we have that

- as integral currents, $[\![\Gamma^{j,k}]\!]$ converge weakly to an integral current Z^j;
- as varifolds, $\Gamma^{j,k}$ converge to a varifold V^j;
- V^j, along with a suitable diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$ converges to a varifold $\tilde{V}$.

The rest of the section will then be devoted to prove that the varifold $\tilde{V}$ is in fact the replacement of Proposition 7.3 and that the sequence $\tilde{\Gamma}^j$ satisfies the requirements of the same proposition.

The proof is split into two steps. In the first one we will show that, at all sufficiently small scales, the current Z^j is indeed a minimizer of the area in the corresponding variational problem (constrained and unconstrained) without *any restriction* on the competitors. More precisely we show that

Lemma 7.5. *Let $j \in \mathbb{N}$ and $y \in An$. Then there are a ball $B = B_\rho(y) \subset An$ and a $k_0 \in \mathbb{N}$ such that every set Ξ with the following properties (satisfied for some $k \geq k_0$) belongs to the class $\mathcal{H}(\Gamma^j, An)$:*

- *Ξ is a smooth hypersurface away from a finite set;*
- *$\partial\Xi \cap \partial\mathcal{M} \cap B = \gamma \cap B$ in the constrained problem, whereas $\partial\Xi \cap \operatorname{Int}(B) = \emptyset$ in the unconstrained problem;*
- *$\Xi \setminus B = \Gamma^{j,k} \setminus B$;*

- $\mathcal{H}^n(\Xi) < \mathcal{H}^n(\Gamma^{j,k})$.

As a simple corollary, using the regularity theory for area minimizing currents for a given prescribed boundary (and the corresponding regularity theory for the minimizers in the free boundary case, as developed by Grüter in [21]) we then get the following

Corollary 7.6. *Let $\tilde{B}$ be the ball concentric to the ball B in Lemma 7.5 with half the radius. In the constrained case the current Z^j has boundary γ in B and any competitor $Z^j + \partial S$, where S is an integer rectifiable current supported in $\tilde{B}$, cannot have mass smaller than that of Z^j. In the unconstrained case Z^j is a minimizer with respect to free boundary perturbations, namely any current $Z^j + T$ with $\mathrm{spt}(\partial T) \subset B \cap \partial\mathcal{M}$ and $\mathrm{spt}(T) \subset \tilde{B}$, cannot have mass smaller than that of Z^j.*

Thus, $Z^j \llcorner An = V^j \llcorner An = \bar{\Gamma}^j$ is a regular, minimal, embedded hypersurface except for a closed set $\mathrm{Sing}\,(\bar{\Gamma}^j)$ of dimension at most $n-7$. In the unconstrained case it meets the boundary $\partial\mathcal{M}$ orthogonally and it is stable for the free boundary problem. In the constrained case $\mathrm{Sing}\,(\bar{\Gamma}^j)$ does not intersect $\partial\mathcal{M}$ and $\partial\bar{\Gamma}^j = \gamma$ (in An; in particular any connected component of Γ^j that intersects $\partial\mathcal{M}$ must have multiplicity 1).

The second step in the proof of Proposition 7.3 takes advantage of the compactness theorems in Section 6 to pass into the limits in j and conclude that $\tilde{V}$ has the desired regularity properties.

7.2 Proof of Lemma 7.5

We focus on the constrained case, since the proof in the unconstrained case follows the same line and it is indeed easier.

We will exhibit a suitable homotopy between $\Gamma^{j,k}$ and Ξ by first deforming $\Gamma^{j,k}$ inside B to a cone with vertex y and base $\Gamma^{j,k} \cap \partial B$, and then deforming this cone back to Ξ, without increasing the area by more than $(2^{m+2}j)^{-1}$, which will prove the claim. To this end, we borrow the "blow down -blow up" procedure from [13], which in turn is borrowed from Smith [51] (see also Section 7 of [10]) and we only need to modify the idea because $x \in \partial\mathcal{M}$.

Fix $y \in An \cap \partial\mathcal{M}$, and $j \in \mathbb{N}$. If $y \notin \gamma$, by considering $\mathcal{M}$ as a subset of $\tilde{\mathcal{M}}$ as in Remark 5.3, and simply making sure to choose ρ small enough that $B_\rho(y) \subset\subset \tilde{\mathcal{M}} \setminus \gamma$, we can reduce to the interior case. Note that we also make use of the convexity assumption on $\mathcal{M}$ to make sure all the surfaces in the homotopy stay inside $\mathcal{M}$. Therefore, we are left to prove the case $y \in An \cap \partial\mathcal{M} \cap \gamma$.
First, in a small neighborhood around y, we can find (smooth) diffeomorphisms

$$\Psi_1 : \mathbb{R}^{n-1} \times \mathbb{R} \to \partial\mathcal{M}, \quad \Psi_1^{-1}(\gamma) \subset \mathbb{R}^{n-1} \times \{0\}, \quad \Psi_1(0) = \iota(y);$$
$$\Psi_2 : \partial\mathcal{M} \times \mathbb{R} \to \mathcal{M}, \quad \Psi_2(x,t) = \exp_x(t\nu(x)),$$

with $\iota : \partial\mathcal{M} \to \mathcal{M}$ the inclusion map, and $\nu(x)$ unit normal to $\partial\mathcal{M}$. By taking $\Psi_2(\Psi_1(x), t)$ and composing it with a linear map if necessary, we get a (smooth) local coordinate chart $\Psi : U \subseteq \mathbb{R}^n \to V$ in a neighborhood $V \subset An \subset \mathcal{M}$ of y, with $\Psi(0) = y$, $\partial\mathcal{M} \cong \mathbb{R}^n \times \{0\}, \gamma \cong \mathbb{R}^{n-1} \times \{0\} \times \{0\}$, and $D\Psi_0 = \mathrm{Id}$. In the following, $B^e_r(0)$ and $\mathcal{H}^{n,e}$ are used to denote the ball of radius r and the Hausdorff measure w.r.t the euclidean metric in the given coordinates. We will choose $\tau > 0$ small enough, that $B^e_{2\tau}(0) \subseteq U$. The required radius ρ of the geodesic ball $B = B_\rho(y)$ will be fixed later, but chosen small enough that $\Psi^{-1}(B_\rho(y)) \subset\subset B^e_\tau(0)$ (and, of course, smaller than the injectivity radius). Furthermore, by choosing U (and consequently τ) small enough, we can ensure for any surface $\Sigma \subset B^e_{2\tau}(0)$ that

$$\frac{1}{c}\mathcal{H}^{n,e}(\Sigma) \le \mathcal{H}^n(\Sigma) \le c\mathcal{H}^{n,e}(\Sigma), \tag{7.2.1}$$

where c depends on the metric, and $c \to 1$ for $\tau \to 0$. From now on, we will use the same symbols to denote sets and their representations in the coordinates given by Ψ.

Step 1: Stretching $\Gamma^{j,k} \cap \partial B^e_r(0)$. First of all , we will choose $r \in (\tau, 2\tau)$ such that, for every k,

$$\begin{aligned}\Gamma^{j,k} \text{ is regular in a neighborhood of } \partial B^e_r(0)\\ \text{and intersects it transversally}\end{aligned} \tag{7.2.2}$$

This is implied by Sard's lemma, since each $\Gamma^{j,k}$ has only finitely many singularities. We let K be the cone

$$K = \{\lambda z \,|\, 0 \le \lambda < 1, z \in \partial B^e_r(0) \cap \Gamma^{j,k}\}$$

We now show that $\Gamma^{j,k}$ can be homotopized through a family $\tilde{\Omega}_t$ to a surface $\tilde{\Omega}_1$ in such a way that

- $\max_t \mathcal{H}^{n,e}(\tilde{\Omega}_t) - \mathcal{H}^{n,e}(\Gamma^{j,k})$ can be made arbitrarily small;
- $\tilde{\Omega}_1$ coincides with K in a neighborhood of $\partial B^e_r(0)$

To this end, we consider a smooth function $\varphi : [0, 2\tau] \to [0, 2\tau]$ with

- $|\varphi(s) - s| \le \epsilon$ and $0 \le \varphi' \le 2$;
- $\varphi(s) = s$ if $|s - r| > \epsilon$ and $\varphi \equiv r$ in a neighborhood of r.

Set $\Phi(t, s) := (1 - t)s + t\varphi(s)$. If A is any set, we use λA as usual to denote the set $\{\lambda x \,|\, x \in A\}$. We can now define $\tilde{\Omega}_t$ in the following way:

- $\tilde{\Omega}_t \setminus An^e(0, r - \epsilon, r + \epsilon) = \Gamma^{j,k} \setminus An^e(0, r - \epsilon, r + \epsilon)$;

- $\tilde{\Omega}_t \cap \partial B^e_s(0) = \frac{s}{\Phi(t,s)}\big(\Gamma^{j,k} \cap \partial B^e_{\Phi(t,s)}\big)$ for every $s \in (r-\epsilon, r+\epsilon)$,

where the annuli (with the superscript e) are with respect to the euclidean metric. Note that our choice of coordinates ensures that γ is preserved as the boundary. Furthermore, the surfaces are smooth (with the exception of a finite number of singularities), since the only possible irregularity of the slice $\Gamma^{j,k} \cap \partial B^e_r(0)$, which is at $\partial B^e_r(0) \cap \partial\mathcal{M}$, gets propagated along the boundary γ as we "stretch" it to a cone, and hence we are within the parameters of a constrained family. Moreover, owing to (7.2.1) and (7.2.2), and for ϵ sufficiently small, $\tilde{\Omega}_t$ will have the desired properties. Finally, since Ξ coincides with $\Gamma^{j,k}$ on $\mathcal{M} \setminus B_\rho(y)$ (and in particular, outside $B^e_\tau(0)$), the same argument can be applied to Ξ. This shows that

$$\text{w.l.o.g. we can assume } K = \Xi = \Gamma^{j,k} \text{ in a neighborhood of } B^e_r(0) \tag{7.2.3}$$

Step 2: The homotopy. We now construct the required homotopy mentioned in the beginning of the proof, as the family $\{\Omega_t\}_{t\in[0,1]}$ of hypersurfaces which satisfy:

- $\Omega_t \setminus \bar{B}^e_r(y) = \Gamma^{j,k} \setminus \bar{B}^e_r(y)$ for every t;
- $\Omega_t \cap An^e(0, |1-2t|r, r) = K \cap An(y, |1-2t|r, r)$ for every t;
- $\Omega_t \cap \bar{B}^e_{(1-2t)r}(0) = (1-2t)(\Gamma^{j,k} \cap \bar{B}^e_r(0))$ for $t \in [0, \frac{1}{2}]$;
- $\Omega_t \cap \bar{B}^e_{(2t-1)r}(0) = (2t-1)(\Xi \cap \bar{B}^e_r(0))$ for $t \in [\frac{1}{2}, 1]$.

Note again that, because of the way we chose our coordinates and deformations, and consequently (7.2.3), this satisfies the properties of a smooth constrained family. The only property left to check is that

$$\max_t \mathcal{H}^n(\Omega_t) \le \mathcal{H}^n(\Gamma^{j,k}) + \frac{1}{2^{m+2}j} \quad \forall k \ge k_0 \tag{7.2.4}$$

holds for a suitable choice ρ, r and k_0.
First we observe the following standard facts, for every $r < 2\tau$ and $\lambda \in [0,1]$:

$$\mathcal{H}^{n,e}(K) = \frac{r}{n}\mathcal{H}^{n-1,e}\big(\Gamma^{j,k} \cap \partial B^e_r(0)\big); \tag{7.2.5}$$

$$\mathcal{H}^{n,e}\big(\lambda(\Gamma^{j,k} \cap \bar{B}^e_r(0)\big) = \mathcal{H}^{n,e}\big(\lambda(\Gamma^{j,k} \cap B^e_r(0)\big) \le \mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_r(0)\big); \tag{7.2.6}$$

$$\mathcal{H}^{n,e}\big(\lambda(\Xi \cap \bar{B}^e_r(0)\big) = \mathcal{H}^{n,e}\big(\lambda(\Xi \cap B^e_r(0)\big) \le \mathcal{H}^{n,e}\big(\Xi \cap B^e_r(0)\big); \tag{7.2.7}$$

$$\int_0^{2\tau} \mathcal{H}^{n-1,e}\big(\Gamma^{j,k} \cap \partial B^e_s(0)\big)\, ds \le \mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_{2\tau}(0)\big), \tag{7.2.8}$$

where the equalities in (7.2.6) and (7.2.7) are due to (7.2.2). From (7.2.1) and the assumption on Ξ we conclude $\mathcal{H}^{n,e}(\Xi \cap B^e_{2\tau}(0)) \le c^2\mathcal{H}^{n,e}(\Gamma^{j,k} \cap B^e_{2\tau}(0))$, which together with (7.2.5), (7.2.6) and (7.2.7) gives us the estimate

$$\begin{aligned}\max_t \mathcal{H}^n(\Omega_t) - \mathcal{H}^n(\Gamma^{j,k}) &\le c\mathcal{H}^{n,e}\big(\Omega_t \cap B^e_{2\tau}(0)\big)\\ &\le c^2\mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_{2\tau}(0)\big) + r\mathcal{H}^{n-1,e}\big(\Gamma^{j,k} \cap \partial B^e_r(0)\big) \qquad (7.2.9)\end{aligned}$$

By (7.2.8) we can find $r \in (\tau, 2\tau)$ which, in addition to (7.2.2) (and consequently (7.2.3)), satisfies

$$\mathcal{H}^{n-1,e}\big(\Gamma^{j,k} \cap B^e_r(0)\big) \le \frac{2}{\tau}\mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_{2\tau}(0)\big). \qquad (7.2.10)$$

Hence,

$$\max_t \mathcal{H}^n(\Omega_t) \le \mathcal{H}^n(\Gamma^{j,k}) + (4+c^2)\mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_{2\tau}(0)\big). \qquad (7.2.11)$$

By a metric comparison argument similar to (7.2.1) relating the lenghts of curves inside $B^e_{2\tau}(0)$, we can obtain the inclusions $B_\rho(y) \subset\subset B^e_\tau(0) \subset B^e_{2\tau}(0) \subset\subset B_{\bar{c}\rho}(y)$, where the constant $\bar{c}$ depends on the metric, assuming of course that τ is initially chosen small enough. Next, by the convergence of $\Gamma^{j,k}$ to the stationary varifold V^j, we can choose k_0 such that

$$\mathcal{H}^{n,e}\big(\Gamma^{j,k} \cap B^e_{2\tau}(0)\big) \le 2||V^j||(B_{\bar{c}\rho}(y)) \quad \text{for } k \ge k_0. \qquad (7.2.12)$$

Finally, by the monotonicity formula (see Theorem 3.4.(2) in [1]),

$$||V^j||(B_{\bar{c}\rho}(y) \le C_{\mathcal{M}}||V^j||(\mathcal{M})\rho^n. \qquad (7.2.13)$$

By gathering the estimates (7.2.11), (7.2.12), and (7.2.13) (and having chosen τ small enough as instructed, depending only on $\mathcal{M}$), we deduce that if ρ is chosen small enough that

$$2(4+c^2)C_{\mathcal{M}}||V^j||(\mathcal{M})\rho^n < \frac{1}{2^{m+2}j},$$

holds, k_0 large enough that (7.2.12) holds, and finally fixing $r \in (\tau, 2\tau)$ satisfying (7.2.2) and (7.2.10), we can construct $\{\Omega_t\}$ as above, concluding the proof. □

7.3 Proof of Corollary 7.6

Step 1. Minimality in the interior. Again, we focus on the constrained problem, since the unconstrained problem is exactly the same. Strictly speaking, the conclusion of the corollary is new even in the interior, because in [13] the homotopic Plateau's problem was stated in the framework of Caccioppoli sets, i.e. not allowing multiplicities for our currents. We thus first show how to remove this technical point in the interior.

Fix $j \in \mathbb{N}, y \in \operatorname{Int}(An)$, and let $B_\rho(y) \subset An$ be the ball given by Lemma 7.5 where we assume in addition $\rho < \operatorname{dist}(y, \partial\mathcal{M})$. We will prove, by contradiction, that the integral current Z^j (obtained as the weak limit of currents $[\![\Gamma^{j,k}]\!]$) is area minimizing in $B_{\rho/2}(y)$. Assume, therefore, it is not, and there exists and integral current S, with $\partial S = \gamma$, $S = Z^j$ on $\mathcal{M} \setminus B_{\rho/2}(y)$ and

$$\mathbf{M}(S) < \mathbf{M}(Z^j) - \eta \tag{7.3.1}$$

Since $\sup_k(\mathbf{M}(\Gamma^{j,k}) + \mathbf{M}(\gamma)) < \infty$, and therefore the weak and flat convergence are equivalent, we have

$$\Gamma^{j,k} - Z^j = \partial A_j + B_j, \quad \mathbf{M}(A_j) + \mathbf{M}(B_j) \to 0 \tag{7.3.2}$$

In fact, considering that $\partial(\Gamma^{j,k} - Z^j) = 0$, we can assume w.l.o.g. that $B_j = 0$. By slicing theory, we can choose $\rho/2 < \tau < \rho$ and a subsequence (not relabeled) such that

$$\partial(A_j \llcorner B_\tau(y)) = (\partial A_j) \llcorner B_\tau(y) + R_j, \quad \mathbf{M}(R_j) \to 0 \tag{7.3.3}$$

where $\operatorname{spt} R_j \subset \partial B_\tau(y)$, and R_j is integer multiplicity (cf. Fig. 7.1). Now, define the integer n-rectifiable current

$$S^{j,k} := S \llcorner B_\tau(y) - R_j + \Gamma^{j,k} \llcorner (\mathcal{M} \setminus B_\tau(y)).$$

It is easy to check from the above that $\partial S^{j,k} = \gamma$. Moreover, from the weak convergence $\Gamma^{j,k} \rightharpoonup Z^j$ we get $\mathbf{M}(Z^j \llcorner B_\tau) \leq \liminf_{k\to\infty} \mathbf{M}(\Gamma^{j,k} \llcorner B_\tau)$, and together with (7.3.1), (7.3.3), this implies

$$\limsup_{j\to\infty} \left(\mathbf{M}(S^{j,k}) - \mathbf{M}(\Gamma^{j,k})\right) \leq -\eta. \tag{7.3.4}$$

We now proceed to approximate $S^{j,k}$ with smooth surfaces, which would by construction exhibit a similar gap in area (mass) with respect to $\Gamma^{j,k}$. The idea is to then apply Lemma 7.5, thereby showing that these smooth surfaces belong to the class $\mathcal{H}(\Gamma^j, An)$, and thus contradicting the minimality of the original sequence $\Gamma^{j,k}$.

Let us first fix $(a, b) \subset\subset (\tau, \rho)$ with the property that $\Gamma^{j,k} \cap An(y, a, b)$ is a smooth surface. Since $\partial[S^{j,k} \llcorner B_b(y)] \subset \partial B_b(y)$, we can find an n-rectifiable current Ξ with $\operatorname{spt}(\Xi) \subset \partial B_b(y)$ and $\partial\Xi = \partial[S^{j,k} \llcorner B_b(y)]$. Taking $R = S^{j,k} \llcorner B_b(y) - \Xi$ we apply 4.5.17 of [16] to find a decreasing sequence of $\mathcal{H}^{n+1}$-measurable sets $\{U_i\}_{i=-\infty}^{\infty}$ (of finite perimeter in B_b) and use them to construct rectifiable currents

$$\begin{aligned} &S_i^{j,k} = \partial[\![U_i]\!] \llcorner B_b(y) \quad \text{with} \quad \operatorname{spt} \partial S_i^{j,k} \subset \partial B_b(y), \quad \text{and} \\ &S^{j,k} \llcorner B_b(y) = \sum_{i\in\mathbf{Z}} S_i^{j,k}, \qquad \mathbf{M}(S^{j,k} \llcorner B_b(y)) = \sum_{i\in\mathbf{Z}} \mathbf{M}(S_i^{j,k}). \end{aligned} \tag{7.3.5}$$

In fact, $R = \partial T$ where $T = \sum_{i=1}^{\infty}[\![U_i]\!] - \sum_{i=-\infty}^{0}[\![B_b(y) \setminus U_i]\!]$. Let us therefore

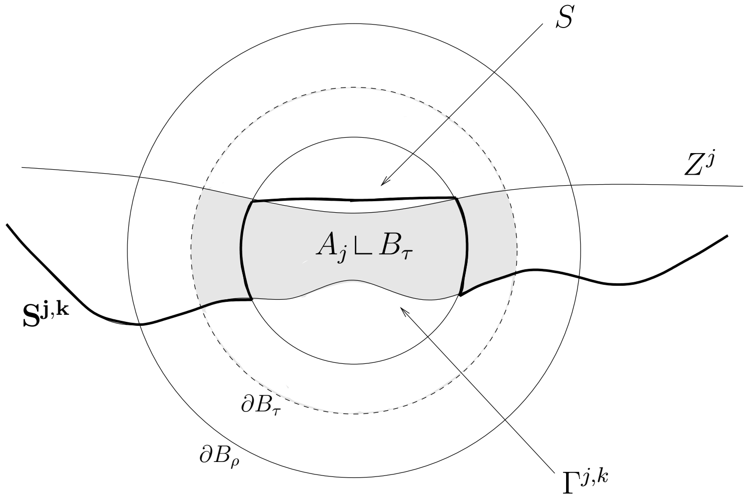

Figure 7.1: The cut-and-paste procedure to produce a suitable competitor.

define the integer valued function $f : B_b(y) \to \mathbf{Z}$ by

$$f := \sum_{i=1}^{\infty} \chi_{U_i} - \sum_{i=-\infty}^{0} \chi_{B_b \setminus U_i},$$

where χ_A denotes the characteristic function of a set A. Because the sequence $\{U_i\}_{i=-\infty}^{\infty}$ is decreasing, we see immediately that $U_i = \{x : f(x) \geq i\}$. In fact, f is of bounded variation inside $B_b(y)$, which follows from (7.3.5) and the fact that (see Remark 6.3.7 in [49])

$$\mathbf{M}(\partial [\![U_i]\!] \llcorner B_b(y)) = \int_{B_b(y)} |D\chi_{U_i}|. \tag{7.3.6}$$

By recalling the standard way of approximating functions of bounded variation by smooth functions, we take a compactly supported convolution kernel φ and consider the functions $f_\epsilon = f * \varphi_\epsilon$, for $\epsilon < \rho - b$ (hence spt $f_\epsilon \subset B_\rho(y)$). Of course, $\int_{B_\rho} |Df_\epsilon| \to \int_{B_\rho} |Df|$ for $\epsilon \to 0$. If we define $U_{t,\epsilon} := \{x : f_\epsilon(x) \geq t\}$, then by coarea formula

$$\int |Df_\epsilon| = \int_{-\infty}^{\infty} dt \int |D\chi_{U_{t,\epsilon}}|.$$

By a simple argument, which essentially follows from Chebyshev's inequality applied to the function $f_\epsilon(x) - f(x)$ (see Lemma 1.25 in [20]), we get $\chi_{U_{t,\epsilon}} \to \chi_{U_i}$ in L^1 for every $t \in (i-1, i)$, $i \in \mathbf{Z}$. Taking a sequence $\epsilon_l \to 0$, and using the lower semicontinuity of the perimeter w.r.t L^1 convergence, we deduce

$$\begin{aligned}\int_{B_b} |Df| &= \lim_{j\to\infty} \int_{B_b} |Df_{\epsilon_l}| \geq \int_{-\infty}^{\infty} dt \liminf_{l\to\infty} \int_{B_b} |D\chi_{U_{t,\epsilon_l}}| \\ &\geq \sum_{i=-\infty}^{\infty} \int_{i-1}^{i} dt \int_{B_b} |D\chi_{U_i}| = \int_{B_b} |Df|. \qquad (7.3.7)\end{aligned}$$

Hence, for all $i \in \mathbf{Z}$ and almost all $t \in (i-1, i)$, $\liminf_{l\to\infty} \int_{B_b} |D\chi_{U_{t,\epsilon_l}}| \to \int_{B_b} |D\chi_{U_i}|$. Moreover, since almost all level sets are smooth by Sard's lemma, for all $i \in \mathbf{Z}$ we may choose a $t_i \in (i-1, i)$ such that:

- $\partial U_{t_i,\epsilon_l}$ is smooth;

- $\liminf_{j\to\infty} \int_{B_b} |D\chi_{U_{t_i,\epsilon_l}}| \to \int_{B_b} |D\chi_{U_i}|$.

By choosing a diagonal subsequence (without relabeling), we can ensure that the $\liminf_{j\to\infty}$ is replaced by a $\lim_{j\to\infty}$. We now define a current

$$\Delta^{j,k,l} = \sum_{i=-\infty}^{\infty} \partial [\![U_{t_i,\epsilon_l}]\!] \llcorner B_b(y),$$

and note that it is induced by a smooth surface (for each $l \in \mathbb{N}$), since it is composed of smooth level sets of a smooth function. Furthermore, the properties above together with (7.3.5) and (7.3.6) imply that $\mathbf{M}(\Delta^{j,k,l}) \to \mathbf{M}(S^{j,k} \llcorner B_b(y))$ as $l \to \infty$.

We would now like to patch $\Delta^{j,k,l}$ with $\Gamma^{j,k}$ outside $B_b(y)$. For this, recall that $S^{j,k} \cap An(y,a,b) = \Gamma^{j,k} \cap An(y,a,b)$ is also a smooth surface. Therefore, fixing a regular tubular neighborhood T of $S^{j,k}$ inside $An(y,a,b)$ and the corresponding normal coordinates (ξ,σ) on it, we conclude that for l sufficiently large (consequently ϵ_l sufficiently small), $T \cap \Delta^{j,k,l}$ is the set $\{\sigma = g_{\epsilon_l}(\xi)\}$ for some function g_{ϵ_l}. Moreover, $g_{\epsilon_l} \to 0$ smoothly, as $l \to \infty$. Now, using a patching argument entirely analogous to the one of the freezing construction in Lemma 4.8 (one dimensional version) allows us to modify $\Delta^{j,k,l}$ to coincide with $S^{j,k}$ (and therefore $\Gamma^{j,k}$) in some smaller annulus $An(y,b',b) \subset An(y,a,b)$, without increasing the area too much. Thus, observing the definition of $S^{j,k}$ and (7.3.4), we are able to construct currents $\Delta^{j,k}$ with the following properties:

- $\Delta^{j,k}$ is smooth outside of a finite set;
- $\Delta^{j,k} \llcorner (\mathcal{M} \setminus B_\rho(y)) = \Gamma^{j,k} \llcorner (\mathcal{M} \setminus B_\rho(y))$;
- $\limsup_k \left(\mathbf{M}(\Delta^{j,k}) - \mathbf{M}(\Gamma^{j,k})\right) \leq -\eta < 0$.

For k large enough, Lemma 7.5 tells us that $\Delta^{j,k} \in \mathcal{H}(\Gamma^j), An)$, which would in turn imply that $\Gamma^{j,k}$ is not a minimizing sequence, thus closing the contradiction argument.

Step 2. Minimality at the boundary. We are still left with proving the statement in case $y \in \gamma \subset \partial\mathcal{M}$. As before, we start with a competitor current S and the assumption (7.3.1). As a matter of fact, we will reduce this to the previous case by constructing the current $S^{j,k}$, "pushing" it slightly towards the interior of $\mathcal{M}$, and then "attaching" to it a smooth segment which connects it to γ. If the mass of the resulting current is very close to the mass of $S^{j,k}$, we retain (7.3.1) with a smaller constant, and proceed with smoothing as before. First, analogously to the above, we obtain the currents $S^{j,k}$ and (7.3.4). Choose $(a, b) \subset\subset (\tau, \rho)$ such that $\Gamma^{j,k} \cap An(y, a, b)$ (and hence also $S^{j,k}$) is a smooth surface with boundary $\gamma \cap An(y, a, b)$. Parametrize a tubular neighborhood $U_\delta(\partial\mathcal{M}) = \{x \in \mathcal{M} : |\mathrm{dist}(x, \partial M)| < \delta\}$ of $\partial\mathcal{M}$ with the usual smooth diffeomorphism

$$\Phi : \partial\mathcal{M} \times [0, \delta) \to U_\delta(\partial\mathcal{M}), \quad (t, s) \mapsto \Phi(t, s) = \exp_t(s\nu(t)),$$

where $\nu(t)$ is the inward pointing normal of $\partial\mathcal{M}$ at t. Let us denote by $N := \gamma \times [0, \delta)$ the smooth hypersurface which meets $\partial\mathcal{M}$ orthogonally in γ. Next, we pick $a < a' < b' < b$ and slightly deform $S^{j,k}$ to make it coincide with N in $An(y, a', b') \cap U_\xi(\partial\mathcal{M})$ for some ξ small enough. To do this, note for example that near γ, $S^{j,k} \cap An(y, a', b')$ is a graph of a function g over N, due to the convexity assumption on $\mathcal{M}$. By considering $g(1 - \psi)$, where ψ is a suitable cutoff function supported in $An(y, a, b) \cap U_{2\xi}(\partial\mathcal{M})$ and equal to 1 in $An(y, a', b') \cap U_\xi(\partial\mathcal{M})$, we obtain the desired surface. Furthermore, its area will be arbitrarily close to the area of $S^{j,k}$, provided ξ is chosen small enough. Thus, w.l.o.g. we can assume

$$S^{j,k} = N \text{ in } An(y, a', b') \cap U_\xi(\partial\mathcal{M}), \text{ for some } \xi \text{ small enough.} \tag{7.3.8}$$

We fix:

- a smooth function $\varphi : [0, \infty) \to [0, \epsilon]$ such that $\varphi(0) = \epsilon$, $\varphi(x) = 0$ for $x \geq \sqrt{\epsilon}$, and $|\varphi'(x)| \leq C\sqrt{\epsilon}$ (where ϵ will be fixed later);
- a smooth function $\eta : \partial\mathcal{M} \to [0, 1]$ such that $\eta(t) = 1$ for $t \in \partial\mathcal{M} \cap B_{a'}(y)$ and $\eta(t) = 0$ for $t \in \partial\mathcal{M} \setminus B_{b'}(y)$.

Consider now the map

$$\Psi(x) := \begin{cases} (t, s) \mapsto (t, s + \varphi(s)\eta(t)) & \text{for } x = (t, s) \in U_{\sqrt{\epsilon}}(\partial\mathcal{M}); \\ \mathrm{Id} & \text{for } x \in \mathcal{M} \setminus U_{\sqrt{\epsilon}}(\partial\mathcal{M}). \end{cases} \tag{7.3.9}$$

If $\epsilon < \delta^2$ is small enough that $|\varphi'(x)| < 1$, we ensure that $s \mapsto s + \varphi(s)\eta(t)$ is monotone increasing, and $\Psi : \mathcal{M} \to \mathcal{M}$ is a well defined, smooth, proper map, with a Lipschitz constant $1 + O(\sqrt{\epsilon})$. This means that we can push forward the current $S^{j,k}$ to obtain $\Psi_\#(S^{j,k})$ with a (possibly) small gain in mass, and with $\partial(\Psi_\#(S^{j,k})) = \Psi_\#(\partial S^{j,k}) = \Psi_\#(\gamma)$ being a smooth submanifold of N. It is now

obvious that, by attaching to it a smooth surface $\gamma|_{\mathrm{spt}\,(\eta)} \times [0, \epsilon\eta(t))$ with mass $O(\epsilon)$ (and the proper orientation assigned), we are able to construct a current $\tilde{S}^{j,k}$ with $\partial\tilde{S}^{j,k} = \gamma$, $\tilde{S}^{j,k} \setminus B_\rho(y) = \Gamma^{j,k} \setminus B_\rho(y)$ and with $\mathbf{M}(\tilde{S}^{j,k})$ arbitrarily close to $\mathbf{M}(S^{j,k})$. Moreover, it follows from the construction and (7.3.8) that $\tilde{S}^{j,k}$ is smooth in $U_\epsilon(\partial\mathcal{M}) \cap B_b(y)$ (in fact, it coincides with N in $U_\epsilon(\partial\mathcal{M}) \cap B_{b'}(y)$), cf. Figure 7.2.
We can now repeat the smoothing procedure from the previous case, centered around the point $y' = \Psi(y) \in \tilde{S}^{j,k}$, with one modification; we may not be able to actually choose (metric) balls around y' with some radii $\tilde{a}, \tilde{b}$, contained in $\mathrm{Int}(\mathcal{M})$ such that $\tilde{S}^{j,k} \cap An(y', \tilde{a}, \tilde{b})$ is smooth, as before. Nevertheless, it follows from the above that we may choose some open neighborhoods $V_a(y') \subset\subset V_b(y') \subset\subset B_b(y)$ diffeomorphic to balls, such that this is true. All the arguments can be easily modified for this case, and we reach a contradiction once again.

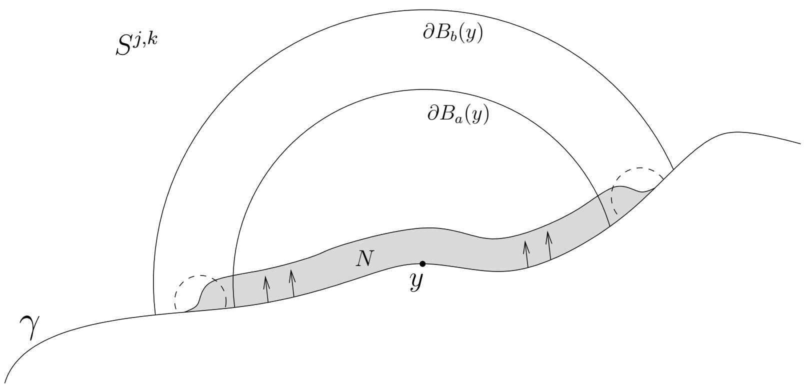

Figure 7.2: The currrent $S^{j,k}$ is pulled away from γ by the map Ψ, and the surface N (in the shaded region) is attached to it. In the regions bounded by the dotted line, $S^{j,k}$ and N coincide.

Step 3. $\boldsymbol{Z^j = V^j}$**.** We first show that $\mathbf{M}(\Gamma^{j,k})$ converges to $\mathbf{M}(Z^j)$. Indeed, if this were not the case, we would have

$$\mathbf{M}(Z^j \cap B_{\rho/2})(y) < \limsup_{k\to\infty} \mathbf{M}(\Gamma^{j,k} \cap B_{\rho/2})(y)$$

for some $y \in An$ and some ρ to which we can apply the conclusion of Lemma 7.5. We can then use Z^j instead of S in the beginning of this proof to once again contradict the minimality of the sequence $\{\Gamma^{j,k}\}_{k\in\mathbb{N}}$. The convergence of the mass is then a simple consequence of the following well known fact

Lemma 7.7. *Let V^j be a sequence of rectifiable currents in $\mathcal{M}$ such that*

(i) $V^j \rightharpoonup V$ *in the flat norm;*

(ii) $\mathbf{M}(V^j) \to \mathbf{M}(V)$.

Let the rectifiable varifolds W^j associated do V^j converge to a (rectifiable) varifold W. Then W is the varifold associated to the rectifiable current V.

In the codimension 1 case (which we need), the proof of the lemma follows from Reshetnyak continuity theorem (cf. [7, Theorem 2.39]), whereas the general case can be proved by constructions analogous to those of [16, 5.1.5] (cf. [41, page 63]).

Step 4. Regularity and stability. In the constrained case the regularity in the interior follows from the standard theory for area-minimizing currents, see for instance [49]. The regularity at the boundary follows instead from [1] because $\partial\mathcal{M}$ is uniformly convex. Indeed, one could apply Lemma 5.2 in that paper, which admittedly deals with the case where $\mathcal{M}$ is a subset of the Euclidean space, but the modifications to handle the case of a general Riemannian manifold are just routine ones. In the unconstrained case, the regularity is proved in Grüter's work [21] (here again, the arguments, given in the euclidean setting, can be easily adapted to deal with the general Riemannian one).

We finally turn to the proof of Proposition 7.3, which mimics precisely the one of Proposition 4.6 in [13]. We write it here for completeness.

Proof of Proposition 7.3. Consider the varifolds V^j, and the diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$ in the beginning of this chapter. Observe that $\tilde{\Gamma}^j$ is obtained from Γ^j through a suitable homotopy which leaves everything fixed outside An. It follows then from the a.m. property of $\{\Gamma^j\}$ that $\{\tilde{\Gamma}^j\}$ is also a.m. in any $An(x, \epsilon, r(x)-\epsilon)$ containing An. Note next that if a sequence is a.m. in an open set U, then it is also a.m. in an open set U' contained in U. This trivial observation and the one preceding it imply that $\{\tilde{\Gamma}^j\}$ is a.m. in any $An \in \mathcal{AN}_{r(x)}(x)$.

Fix now an annulus $An' = An(x, \epsilon, r(x)-\epsilon) \supset\supset An$. Then $\mathcal{M} = An' \cup (\mathcal{M} \setminus An)$. For any $y \in \mathcal{M} \setminus An$ (and $y \neq x$) set $r'(y) := \min\{r(y), \mathrm{dist}(y, An)\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $\tilde{\Gamma}^j \cap An'' = \Gamma^j \cap An''$, and hence $\{\tilde{\Gamma}^j\}$ is a.m. in An''. For $y \in An'$ we set $r'(y) := \min\{r(y), \mathrm{dist}(y, \mathcal{M} \setminus An')\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $An'' \subset An'$ and, since $\{\tilde{\Gamma}^j\}$ is a.m. in An' by the argument above, it must also be a.m. in An''.

We next show that $\tilde{V}$ is a replacement for V. By the compactness theorems in Chapter 6, $\tilde{V}$ is a stable minimal hypersurface in An, with the regularity properties and appropriate boundary conditions required by Definition 7.2. In the unconstrained case we use Theorem 6.3, whereas in the constrained case we use Theorem 6.4. Note that we can apply the latter theorem thanks to Lemma 6.5. It remains to show that $\tilde{V}$ is stationary. It is obviously stationary in $\mathcal{M}\setminus An$, because it coincides with V there. Let now $An' \supset\supset An$. Since $\{An', \mathcal{M}\setminus An\}$ is a covering of $\mathcal{M}$, we can subordinate a partition of unity $\{\varphi_1, \varphi_2\}$ to it. By the linearity of the first variation, we get $[\delta\tilde{V}](\chi) = [\delta\tilde{V}](\varphi_1\chi) + [\delta\tilde{V}](\varphi_2\chi) = [\delta\tilde{V}](\varphi_1\chi)$, so it suffices to show stationarity in An'. Suppose to the contrary that there is some $\chi \in \mathfrak{X}_c^-(An')$ or $\chi \in \mathfrak{X}_c^t(An')$, depending on whether we are considering the fixed boundary or the free boundary case, such that $[\delta\tilde{V}](\chi) \leq -C < 0$. Let ψ denote the flow generated by χ, i.e. $\frac{\partial\psi(x,t)}{\partial t} = \chi(\psi(x,t))$. We set

$$\tilde{V}(t) := \psi(t)_\sharp \tilde{V}, \qquad \Sigma^j(t) = \psi(t, \tilde{\Gamma}^j). \tag{7.3.10}$$

By continuity of first variation there is $\epsilon > 0$ such that $\delta\tilde{V}(t)(\chi) \leq -C/2$ for all $t \leq \epsilon$. Moreover, since $\Sigma^j(t) \to \tilde{V}(t)$ in the sense of varifolds, there is J such that

$$[\delta\Sigma^j(t)](\chi) \leq -\frac{C}{4} \quad \text{for } j > J \text{ and } t \leq \epsilon. \tag{7.3.11}$$

Integrating (7.3.11) we conclude $\mathcal{H}^n(\Sigma^j(t)) \leq \mathcal{H}^n(\tilde{\Gamma}^j) - Ct/8$ for every $t \in [0, \epsilon]$ and $j \geq J$. This contradicts the a.m. property of $\tilde{\Gamma}^j$ in An' for j large enough.

Finally, observe that $\mathcal{H}^n(\tilde{\Gamma}^j) \leq \mathcal{H}^n(\Gamma^j)$ and

$$\liminf_n (\mathcal{H}^n(\tilde{\Gamma}^j) - \mathcal{H}^n(\Gamma^j)) \geq 0$$

because otherwise we would contradict the a.m. property of $\{\Gamma^j\}$ in An. We thus conclude $||V||(\mathcal{M}) = ||\tilde{V}||(\mathcal{M})$. □

Chapter 8

Proof of Theorem 7.1 and Theorem 1.5

Clearly Theorem 1.5 is a direct consequence of Theorem 7.1 and Proposition 4.3. Thus from now on we focus on Theorem 7.1: we fix a varifold V as in there and we want to prove that it is regular. In particular, we already know that V is regular in the interior. Moreover,

(a) In the constrained case we know that $\mathrm{spt}(V) \cap \partial\mathcal{M} \subset \gamma$. We thus need to show the regularity of V at *any* point $p \in \gamma$, and more precisely that for every $p \in \gamma$ there is a neighborhood U such that V is a regular minimal surface Γ in U counted with multiplicity 1, such that $\partial\Gamma = \gamma$ (in U).

(b) In the unconstrained case we need to show that, with the exception of a closed set of dimension at most $n-7$, for any $p \in \partial\mathcal{M}$ there is a neighborhood U such that V is a regular minimal surface Γ in U (counted with integer multiplicity, not necessarily 1) which meets $\partial\mathcal{M}$ orthogonally.

8.1 Tangent Varifolds and Integrality

We already know, in the constrained case, that V is an integer rectifiable varifold and that $\|V\|(\partial\mathcal{M}) = 0$. In the unconstrained case we know the integrality of V in $\mathrm{Int}\,(\mathcal{M})$. We now wish to show that $\|V\|(\partial\mathcal{M}) = 0$ even in this case. Fix a point $p \in \partial\mathcal{M}$ and consider the blow-up procedure of Lemma 5.8. Denote by $\mathrm{Tan}\,(V,p)$ the set of tangent varifolds and fix a $W \in \mathrm{Tan}\,(V,p)$. Let V_{p,r_k} be a corresponding sequence of rescaled varifolds which is converging to W. Thanks to Proposition 7.3, there exists a varifold $\tilde{V}_k$ which is a replacement for V in the annulus $An(p, r_k, 2r_k)$. Rescaling such a replacement suitably we get a second varifold $\bar{V}_k$ which is a replacement for V_{p,r_k} in $\iota_{p,r_k}(An(p, r_k, 2r_k))$. In particular, by the compactness Theorem 6.3 (in the appropriately modified version discussed in Section 6.4) we obtain the convergence of $\bar{V}_k$ to a replacement $\bar{W}$ for W. Now, this replacement has the property that it is regular in $B_2(0) \setminus B_1(0) \subset T_p\mathcal{M}$ and meets $T_p\partial\mathcal{M}$ orthogonally. However, since W is a cone, it turns out that such property is in fact valid in the punctured plane $T_p\mathcal{M} \setminus \{0\}$. Moreover, by the considerations in [21] and [23], the reflection of W along $T_p\partial\mathcal{M}$ gives a stable minimal hypercone in $T_p\mathcal{M} \setminus \{0\}$, regular up to a set of codimension at least 7. Finally, see for instance [46], since the origin has zero 2-capacity, such a cone turns out to be stable on the whole $T_p\mathcal{M}$. In particular, by the classical result of Simons, the cone is in fact a hyperplane if $n \leq 6$.

Before going on, we observe that the argument above applies literally in the same way to the constrained case as well. We conclude that W is a cone C in $T_p\mathcal{M} \setminus \{0\}$ with the property that $\partial C = T_p\gamma$. In particular we conclude that C is a multiplicity 1 half-hyperplane and indeed, by the Wedge property of Lemma 6.5, C meets $T_p\partial\mathcal{M}$ transversally.

We summarize our conclusions in the following

Lemma 8.1. *Let V be as in Theorem 7.1, p a point in $\partial\mathcal{M}$ and $W \in \mathrm{Tan}\,(V, p)$ a tangent varifold.*

(i) In the constrained case $W = 0$ unless $p \in \gamma$ and if $p \in \gamma$ then W is a half hyperplane of $T_p\mathcal{M}$, counted with multiplicity 1, which meets $T_p\partial\mathcal{M}$ transversally at $T_p\gamma$.

(ii) In the unconstrained case W is a minimal hypersurface Ξ meeting $T_p\partial\mathcal{M}$ orthogonally, which is half of a stable minimal cone in $T_p\mathcal{M}$ (counted with multiplicity), regular up to a set of dimension at most $n-7$. When $n \leq 6$, Ξ is half of a hyperplane meeting $T_p\partial\mathcal{M}$ orthogonally.

Next, in the unconstrained case the lemma above implies that $\|W\|(\partial\mathcal{M}) = 0$. In particular, since $\partial\mathcal{M}$ is a closed subset of $\mathcal{M}$, we easily conclude that

$$\begin{aligned}\lim_{k\to\infty} r_k^{-n}\|V\|(\partial\mathcal{M} \cap \overline{B}_{r_k}(p)) &= \lim_{r\downarrow 0} \|V_{p,r_k}\|(\iota_{p,r_k}(\partial\mathcal{M} \cap \overline{B}_{r_k}(p)) \\ &\leq \|W\|(T_p\partial\mathcal{M} \cap \overline{B}_1) = 0\,.\end{aligned}$$

Therefore,

$$\lim_{r\downarrow 0} r^{-n}\|V\|(\partial\mathcal{M} \cap B_r(p)) = 0 \qquad \text{for every } p \in \partial M,$$

which in turn implies easily $\|V\|(\partial\mathcal{M}) = 0$.

8.2 Constrained Case

In the constrained case, Lemma 8.1 implies that we fall under the assumptions of Allard's boundary regularity theorem for stationary varifolds: V is therefore regular at every point $p \in \gamma$, which completes the proof.

8.3 Unconstrained Case

Note that the modification of the arguments from [10] to conclude regularity at the free boundary has already been covered in [31]. We present the main arguments in the remaining sections, but refer the reader to that paper for additional details, if necessary.

8.3.1 Regularity in the Punctured Ball

Our first goal is to show that, if $p \in \partial\mathcal{M}$, then there is a radius r such that V is regular *up to the boundary* in the punctured ball $B_r(p) \setminus \{p\}$. First of all, we observe the following simple consequence of the maximum principle:

Lemma 8.2. *Assume ρ is smaller than the convexity radius of $\mathcal{M}$. Let V be an integer rectifiable varifold in $B_\rho(p)$, which is stationary for the free boundary problem. Then* $\mathrm{spt}(V)$ *must contain at least a point q at distance ρ from p.*

Proof. Assume the opposite and let $q \in \mathrm{spt}(V)$ be a point at maximum distance from p. By integer rectifiability, the support of V cannot be contained in $\{p\}$ and thus the distance between p and q is a positive number $\bar{\rho}$ smaller than ρ. But then V touches the convex set $B_{\bar{\rho}}(p)$ from the interior and it can be seen from the argument in [59] that there is a vector field orthogonal to $\partial B_{\bar{\rho}}(p) \setminus \partial\mathcal{M}$ which violates the stationarity condition for V: to get a contradiction we only need to show that such a vector field can be taken to be tangent to $\partial\mathcal{M}$, which however is a simple consequence of the construction in [59]. □

Fix now a point $p \in \partial\mathcal{M}$ and a radius r which is smaller than the convexity radius of $\mathcal{M}$. Recall that in $\mathrm{Int}\,(B_r(p))$, Γ is given by a stable minimal hyprersurface which is regular up to a set $\mathrm{Sing}\,(\Gamma)$ of dimension at most $n-7$. Let $\tilde{\Gamma}$ be any connected component of $\Gamma \cap B_r(p)$. We want to argue that the closure C of $\tilde{\Gamma}$ must contain at least one point of $\partial B_r(p) \setminus \partial\mathcal{M}$. If this were not the case the closure of $C \setminus \mathrm{Reg}\,(\tilde{\Gamma})$ would then be contained in the union of $\partial B_r(p) \cap \partial\mathcal{M}$ and $\mathrm{Sing}\,(\Gamma)$, which is a set of dimension at most $n-2$. We can then apply [13, Lemma 5.5] to conclude that the closure of $\tilde{\Gamma}$ induces a varifold which is stationary in the whole $\mathcal{M}$, and hence in particular in any $B_\rho(p)$ with $\rho > r$ ([13, Lemma 5.5] is not stated in the context of the free boundary problem, but the proof given in [13, Appendix A] easily transposes to cover this case as well). Choosing ρ smaller than the convexity radius and applying Lemma 8.2 above we then get a contradiction.

Fix now an $r > 0$ which is smaller than the convexity radius and smaller than the number $r(p)$ of Proposition 4.3. Moreover, by Sard's Lemma, we can assume that $\partial B_r(p)$ does not intersect $\mathrm{Sing}(\Gamma)$ and that at any point of $\partial B_r(p) \cap \Gamma$ the tangent plane to Γ is transversal to $\partial B_r(p)$. Consider a replacement V' for V in any $An(p, \sigma, r)$ with $\sigma < r$. Let Ξ be the corresponding stable minimal surface which induces V' in the annulus: for such a hypersurface we know the regularity *up to the boundary* $\partial\mathcal{M}$, except for a closed singular set of dimension at most

$n-7$. The arguments of [13, Section 5.4] apply at any point $y \in \partial B_r(p) \setminus \partial\mathcal{M}$, since such a point is in $\mathrm{Int}\,(\mathcal{M})$, and show that the varifold V' is indeed regular in a neighborhood of any such point. Thus $\Gamma \setminus B_r(p)$ and $\tilde{\Gamma}$ join smoothly across $\partial B_r(p) \setminus \partial\mathcal{M}$. In particular, if $\tilde{\Gamma}$ is a connected component of $\Gamma \cap B_\rho(p)$, knowing that there is a point $q \in \partial B_\rho \setminus \partial\mathcal{M}$ which belongs to the closure of $\tilde{\Gamma}$, we conclude that $\tilde{\Gamma}$ and Ξ intersect on a set of positive $\mathcal{H}^n$ measure. But then a classical unique continuation argument implies actually that $\tilde{\Gamma} \subset \Xi$. Since Ξ is regular up to the boundary and meets it orthogonally, we conclude that the same property holds for $\tilde{\Gamma}$.

This gives the desired regularity of $\Gamma \cap An(p, \sigma, r)$ for any $\sigma < r$ and thus, letting $\sigma \downarrow 0$, we conclude the desired regularity in the punctured ball $B_r(p) \setminus \{p\}$.

8.3.2 Removing Singular Points for $n \leq 6$

From the previous step and by a simple covering argument, we conclude that the set of singular points at the boundary is at most finite when $n \leq 6$. We now wish to remove said points. Again the argument is a suitable variant of the argument which deals with the same issue in the interior. Consider the smooth surface Γ (counted with multiplicity) which gives the varifold V in $B_r(p) \setminus \{p\}$. If we choose r sufficiently small, by Lemma 8.1, for every $\rho < r$ we know that the rescalings $\iota_{p,\rho}(\Gamma)$ are ε close, in the varifold sense and in the annulus $\iota_{p,\rho}(An(p, \rho/8, 4\rho))$, to a varifold of the form $\Theta(V, p)\pi(\rho)$ where $\pi(\rho) \subset T_p\mathcal{M}$ is a half-hyperplane meeting $T_p\partial\mathcal{M}$ orthogonally. We can also assume that the tilt between $\pi(\rho)$ and $\pi(2\rho)$ is smaller than ε, provided r is chosen even smaller.

By the compactness Theorem 6.3 (again, in the more general version where the ambient manifolds can change, cf. Section 6.4), if r is sufficiently small and $\rho < r$, then $V_{p,\rho} \llcorner \iota_{p,\rho}(An(p, \rho/4, 2\rho))$ consists of finitely many Lipschitz graphs $\Gamma_1(\rho), \ldots, \Gamma_k(\rho)$ over $\pi(\rho)$, with controlled Lipschitz constant (say, at most 1), each counted with multiplicity m_i. The same then holds for $V \llcorner An(p, \rho/4, 2\rho)$. Moreover since the tilt between $\pi(\rho)$ and $\pi(\rho/2)$ is small, we easily conclude that each of the numbers of connected components in $An(p, \rho/8, \rho)$ is the same and that they can be ordered so that $\Gamma_i(\rho)$ and $\Gamma_i(\rho/2)$ overlap smoothly.

We can repeat the above argument over dyadic radii $\rho 2^{-j}$ and we conclude that $V \llcorner (B_\rho(p) \setminus \{p\}$ consists of finitely many connected components Γ_i counted with multiplicity m_i, which are topologically punctured n-dimensional balls, smooth up to $\partial\mathcal{M}$. Taking one such connected component and removing the multiplicity, we get a multiplicity 1 varifold in $B_\rho(p)$ which is stationary for the free boundary problem and has flat tangent cones at p, with multiplicity 1. This falls therefore under the assumptions of the Allard's type theorem proved by Grüter and Jost in the paper [22], from which we conclude that p is a regular point. Hence each Γ_i continues smoothly across p. The classical maximum principle now implies that the Γ_i cannot actually touch at the point p, implying in fact that the number of connected components of Γ in any ball B_ρ is 1.

Chapter 9

Competitors: proofs of Corollary 1.6 and 1.8

We start with Corollary 1.6, as the proof is a straightforward adaptation of the sweepout construction in [10].

Proof of Corollary 1.6. Without loss of generality we can assume that $\mathcal{M}$ is connected.

First of all we show that there is a generalized family $\{\Sigma_t\}_{t\in[0,1]}$ where Σ_0 and Σ_1 are trivial (namely as closed sets which consist of a collection of finitely many points). Indeed it suffices to take the level sets of a Morse function f whose range is $[0, 1]$, with the additional requirement that the restriction of f to $\partial\mathcal{M}$ is also a Morse function. Since Morse functions are generic on smooth manifolds, the existence of such an f is guaranteed. We now construct a homotopically closed family X by taking the smallest such family which contains Σ_t.

Take now any $\{\Sigma'_t\}_t \in X$. Away from the singularities S_t the family $\{\Sigma'_t\}$ can be given locally and for t in an interval $[a, b]$ as the image of a smooth map $\Phi : U \times [a, b]$. Thus the family $\{\Sigma'_\tau\}_{\tau\in[0,1]}$ induces canonically a current Ω'_t such that $\partial\Omega'_t = \Sigma'_t$. If $\{\Gamma_{t,s}\}_{(t,s)\in[0,1]^2}$ is a homotopy between $\{\Sigma_t\}_{t\in[0,1]}$ and $\{\Sigma'_t\}_{t\in[0,1]}$, it is easy to check that the corresponding currents $\Omega_{t,s}$ such that $\partial\Omega_{t,s} = \Gamma_{t,s}$ also vary continuosly. Observe however that:

- $\Omega_{t,0} = [\![\{f < t\}]\!]$ and thus $\Omega_{1,0} = [\![\mathcal{M}]\!]$, whereas $\Omega_{0,0} = 0$;
- Since $\Sigma_{1,s}$, resp. $\Sigma_{0,s}$ are all trivial currents, each $\Omega_{1,s}$, resp. $\Omega_{0,s}$, is either 0 or $\mathcal{M}$ (because we are assuming that $\mathcal{M}$ is connected);
- The continuity of $\Omega_{1,t}$ and $\Omega_{0,t}$ ensures then that $\Omega'_1 = \Omega_{1,1} = [\![\mathcal{M}]\!]$ and $\Omega'_0 = \Omega'_{0,1} = 0$.

We thus conclude that there must be one Ω'_t such that $\mathbf{M}(\Omega'_t) = \frac{1}{2}\mathrm{Vol}(\mathcal{M})$. Now the isoperimetric inequality implies that $\mathcal{H}^n(\Sigma'_t) \geq c_0(\mathcal{M}) > 0$, where the constant c_0 depends only upon the ambient manifold.

The above argument shows that $m_0(X) > bM_0(X) = 0$ and thus we can apply Theorem 1.5 to find a free boundary minimal hypersurface with total area equal to $m_0(X)$. This completes the proof. □

Similarly, Corollary 1.8 will be an immediate consequence of Theorem 1.5 applied to constrained families, once we are able to show the existence of two strictly stable minimal surface gives a homotopically closed set X of constrained families parametrized by $\mathcal{P} = [0,1]$ which satisfies the condition (1.2.3). The proof will be divided into two lemmas. In the first one we show the existence of a particular smooth family of hypersurfaces $\{\Sigma\}_{t\in[0,1]}$, starting from Σ_0 and ending in Σ_1. In the second lemma we show that any integer rectifiable current with sufficiently small flat distance to Σ_0 or Σ_1 must have mass which is strictly greater, with a uniform lower bound depending on the distance. More precisely our two lemmas are

Lemma 9.1. *Assume Σ_0 and Σ_1 are as in Corollary 1.8. Then there exists a smooth family of hypersurfaces $\{\Sigma_t\}$ parametrized by $[0,1]$ which is constrained by γ.*

Lemma 9.2. *Let Σ_0, Σ_1 be as above. There exists an $\epsilon_0 > 0$ and $f : (0,\epsilon_0] \to \mathbb{R}^+$ such that*

(S) *If Γ is an integer rectifiable current with $\mathbb{F}([\![\Sigma_i]\!] - \Gamma) = \epsilon$, $i \in \{0,1\}$, and $\partial\Gamma = \partial[\![\Sigma_i]\!] = \gamma$, then $\mathbf{M}(\Gamma) \geq \mathbf{M}([\![\Sigma_i]\!]) + f(\epsilon)$*

The two lemmas above easily imply our corollary.

Proof of Corollary 1.8. Obviously, by taking the homotopy class of the family in Lemma 9.1 we construct a homotopically closed set X. The second lemma then clearly implies that any smooth family $\{\Gamma_t\}$ with $\Gamma_0 = \Sigma_0$ and $\Gamma_1 = \Sigma_1$ must satisfy (1.2.3), since $\mathbb{F}(\Gamma_t, \Gamma_s)$ is a continuous function of t and s and $\mathbb{F}(\Gamma_0, \Gamma_1) > 0$. In particular there is a smooth minimal surface Γ with volume equal to $m_0(X) > \max\{\mathcal{H}^n(\Sigma_0), \mathcal{H}^n(\Sigma_1)\}$ which bounds γ.

Now, by Assumption 1.7 the surface Γ cannot be given by Σ_0 (or Σ_1) plus a closed minimal hypersurface, since the latter cannot exist. Moreover, since the volume of Γ must be strictly larger than Σ_0 (resp. Σ_1) and the multiplicity of Γ must be everywhere 1 thanks to part (b) of Theorem 1.5, we conclude that Γ is distinct from Σ_0 (resp. Σ_1). □

9.1 Proof of Lemma 9.1

Step 1: Let us first extend $\mathcal{M}$ slightly across $\partial\mathcal{M}$, in order to make the following arguments more elegant (as per Remark 5.3 we can even do this such that $\mathcal{M} \subset \tilde{\mathcal{M}}$, for some closed manifold $\tilde{\mathcal{M}}$, if necessary). Consider the normal tubular neighborhood of γ in $\mathcal{M}$, which is realized by an embedding $\iota : U \to \mathcal{M}$, where $U \subset N\gamma$ is a neighborhood of the zero section of the normal bundle $N\gamma$, such that $\iota|_\gamma = 1_\gamma$ and $\iota(U)$ is open in $\mathcal{M}$. Take a (smooth) vector field $e_1(x)$ along γ, which is the normal to $\partial\mathcal{M}$ pointing inwards.

For each point x of γ, consider the sets $\iota^{-1}(\Sigma_i \cap \iota(U)) \cap U_x,\ i \in \{1,2\}$, where $U_x \cong \mathbb{R}^2$ is the fiber of the normal bundle at x. Since Σ_0 and Σ_1 are smooth and minimal, we can use the same arguments as in the proof of Lemma 6.5 to conclude that, if U is small enough, these are smooth, non-intersecting curves (starting at the origin) which are contained inside a 2-dimensional wedge of opening angle at most $\theta < \frac{\pi}{2}$, with $e_1(x)$ lying on its axis. Hence, choosing U even smaller than necessary, we can make sure that they are graphs over $e_1(x)$. That is, for each U_x there exist (smooth) functions ϕ^0_x, ϕ^1_x such that:

$$\phi^0_x, \phi^1_x : W_x \to \mathbb{R}, \qquad \phi^i_x(W_x) = \iota^{-1}(\Sigma_i \cap \iota(U)) \cap U_x \text{ for } i = 0, 1$$

where $W_x := e_1(x) \cap U_x$.

Consider a point $y \in \bar{A} \cap \iota(U_x)$ (recall A from the statement of the lemma), for some $x \in \gamma$. Note that the orthogonal projection of $\iota^{-1}(y)$ on $e_1(x)$, which we denote by $\bar{y}$, lies on the line segment W_x. We define:

$$u_x(y) := t, \qquad \text{where} \quad \iota^{-1}(y) = t\phi^1_x(\bar{y}) + (1-t)\phi^0_x(\bar{y}),\ t \in [0,1]. \tag{9.1.1}$$

Now, by the properties of the tubular neighborhood, to each $y \in A \cap \iota(U)$ is associated a unique fiber U_x, hence there exists an $\eta > 0$, such that when we are at most η-away from $\partial\mathcal{M}$, i.e. on some open set $E_0 = A \cap \iota(U) \cap (\mathcal{M}\backslash\mathcal{M}_\eta)$ with $\mathcal{M}_\eta := \{x \in \mathcal{M} : \mathrm{dist}(x, \partial\mathcal{M}) \geq \eta\}$, these fiber-wise constructions yield a well defined function $f_0 : E_0 \to \mathbb{R}$ such that,

$$f_0(y) := u_x(y), \quad \text{where} \quad y \in \iota(U_x).$$

Furthermore, this function is smooth (by smoothness of Σ_0, Σ_1 and ι), and it has no critical points, provided we choose η small enough, since obviously the derivative in the direction orthogonal to e_1 (and γ) will be different from 0.

Now, we construct a covering of the rest of $\bar{A}$ with balls, satisfying the following two properties:

a) Each ball has a radius less or equal than $\frac{\eta}{2}$;

b) Each ball can only contain points from one of the surfaces Σ_0 and Σ_1, and if it does, its center must lie on the surface.

Through compactness, we obtain a finite subcover, consisting of balls centered at the points $x_1, x_2, \ldots, x_N$. We will denote these balls by $E_1, \ldots, E_N$.
Around each of these points x_k lying on one of the Σ_i-s, we can characterize the submanifold through a local trivialization, i.e. there exists a neighborhood $W \subset \mathcal{M}$ of the point (which we can w.l.o.g. assume to be bigger than the ball E_k), an open set $W' \subset \mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}^1$, and a diffeomorphism

$$\Psi_{x_k} : W \to W', \quad \Psi_{x_k}(\Sigma_i \cap U) = W' \cap (\mathbb{R}^n \times \{0\}).$$

We assume in these cases that the points lying inside the set A are mapped into the positive half-space $\mathbb{R}^{n+1}_+ := \{(y_1, \dots, y_{n+1}),\ y_{n+1} > 0\}$. We now define the functions f_i defined on the balls E_i, $i \in \{1, \dots, N\}$, the following way:

$$f_i(y) := \begin{cases} \frac{1}{2} & \text{if } x_i \text{ lies in the interior of } A; \\ (y_{n+1} \circ \Psi_{x_i})(y) & \text{if } x_i \text{ lies on } \Sigma_0; \\ 1 - (y_{n+1} \circ \Psi_{x_i})(y) & \text{if } x_i \text{ lies on } \Sigma_1. \end{cases} \tag{9.1.2}$$

Here, $y_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}$ is just a function which evaluates the corresponding coordinate. Functions f_i defined in this way are obviously smooth.

Finally, take a partition of unity $\{\varphi_j\}_{0 \le j \le N}$ of $\bar{A}$, subordinate to the covering $E_0, \dots, E_N$. This allows us to define a function $h : \bar{A} \to \mathbb{R}$ via:

$$h(x) := \sum_{i=0}^{N} \varphi_i(x) f_i(x), \tag{9.1.3}$$

which is smooth up to the boundary of A, excluding γ of course.

Step 2: *The function h defined in* (9.1.3) *has no critical points near Σ_0 and Σ_1, as well as in a small neighborhood of γ. Moreover, it holds $h(\Sigma_0 \setminus \gamma) = 0$ and $h(\Sigma_1 \setminus \gamma) = 1$.*

It is obvious from (9.1.1), (9.1.2) and (9.1.3) that $h(\Sigma_0 \setminus \gamma) = 0$ and $h(\Sigma_1 \setminus \gamma) = 1$. Note also that, by the definitions of E_i, when we are at most $\frac{\eta}{2}$ away from $\partial\mathcal{M}$, only φ_0 is supported in this region, hence here it must hold $h(x) = f_0(x)$, and we already know that f_0 has no critical points in it. In points $q \in \Sigma_0$, we have

$$\left.\frac{\partial h}{\partial y_{n+1}}\right|_q = \sum_i \frac{\partial \varphi_i}{\partial y_{n+1}} \cdot f_i + \sum_i \varphi_i \cdot \frac{\partial f_i}{\partial y_{n+1}}$$

where y_{n+1} again denotes the "height" with respect to some fixed local chart E_j. It holds $f_i(q) = 0\ \forall i$ according to (9.1.1) and (9.1.2), so the first sum vanishes. We also see that $\frac{\partial f_j}{\partial y_{n+1}} = 1$ and $\frac{\partial f_i}{\partial y_n} > 0$ for $i \neq j$, $i > 0$ due to the compatibility of charts. so since φ_i-s are nonnegative and $\sum_i \varphi_i = 1$, it follows that the second sum is positive. Hence q cannot be a critical point of h. With similar arguments, we deduce this also for points lying on Σ_1. It can be seen from the construction, however, that the function h will be mostly constant inside the open set A away from Σ_0 and Σ_1. So in this region we will use the fact that Morse functions form a dense, open subset in the C^2 topology, and define one such function g, say on the open set $B := A \cap \operatorname{Int}(\mathcal{M}_{\eta/4})$ (recall the definition above), such that

$$\|h - g\|_{C^2(B)} < \epsilon \tag{9.1.4}$$

for some small $\epsilon > 0$, which will be fixed later. Next, we define a cut-off function

$\psi : \mathcal{M} \to \mathbb{R}$, such that $\psi = 0$ on $\mathcal{M} \setminus \mathcal{M}_{\eta/4}$ and $\psi = 1$ on $\mathcal{M}_{\eta/2}$, and finally we define:

$$f : \bar{A} \to \mathbb{R}, \qquad f(x) := \psi(x)g(x) + (1 - \psi(x))h(x) \tag{9.1.5}$$

Step 3: *For ϵ small enough, the function f is Morse inside A, and its level sets provide a smooth family parametrized by $[0, 1]$, where $f^{-1}(1) = \Sigma_1$ and $f^{-1}(0) = \Sigma_0$.*

It follows from the construction that f does not have any degenerate critical points in the regions where $\psi = 0$ or $\psi = 1$. In the intermediate region, due to (9.1.4) we have:

$$Df = Dh + D\psi(g - h) + \psi(Dg - Dh)$$

Due to the previous steps, we know $h = f_0$ when at most $\frac{\eta}{2}$ away from $\partial\mathcal{M}$, and thus not near critical points, so $|Dh| > \delta$ for some $\delta > 0$. So we have

$$|Df| \geq |Dh| - (|D\psi| + |\psi|)\|h - g\|_{C^2} \geq \delta - C\epsilon,$$

for some constant C depending on ψ (which in turn depends only on η). Now, we can fix ϵ small enough so that $|Df| > 0$. It is clear from the construction that the level sets of f will be smooth hypersurfaces near γ and will in fact have γ as boundary. □

9.2 Proof of Lemma 9.2

W.l.o.g. we assume $i = 0$. By Theorem 2 of White [57], there exists an open set U containing Σ_0 such that

$$\mathbf{M}(\Gamma) > \mathbf{M}([\![\Sigma_0]\!]) \quad \forall \Gamma \text{ with } \partial\Gamma = \partial[\![\Sigma_0]\!], \text{ and } \operatorname{spt}(\Gamma) \subset U. \tag{9.2.1}$$

Observe also that, if the neighborhood U is sufficiently small, having the same boundary is equivalent to be in the same homology class.

We define

$$m_0(\epsilon) := \inf\{\mathbf{M}(\Gamma) : \partial\Gamma = \partial[\![\Sigma_0]\!] \text{ and } \mathbb{F}(\Gamma - [\![\Sigma_0]\!]) = \epsilon\}. \tag{9.2.2}$$

Our aim is to show that $m_0(\epsilon) > \mathbf{M}([\![\Sigma_0]\!])$ $\forall \epsilon \in (0, \epsilon_0]$, which clearly implies the statement (S), by setting $f(\epsilon) = m_0(\epsilon) - \mathbf{M}([\![\Sigma_0]\!])$. Note that the infimum in (9.2.2) is actually a minimum. Observe that, by setting ϵ_0 small enough, we ensure that any current, which is in flat distance at most ϵ_0 away from Σ_0 and with the same boundary as Σ_0, must be homologous to Σ_0 (see proof of Theorem 5.7 in [**?**]). We would like to show that, if ε sufficiently small, a minimizer Γ_ε must be contained in the tubular neighborhood U of Σ_0: this would then conclude the proof because by (9.2.1) the mass of Γ_ε would be strictly larger than that of $[\![\Sigma_0]\!]$. In fact, what we will really show is that there is certainly a Z which has at most the same mass as Γ_ε, has boundary γ and is contained in U, which still suffices to reach the desired conclusion.

Step 1: Let us denote by $U_\delta(\Sigma_0)$ the δ-tubular neighborhood of Σ_0. We will choose δ small enough so that $U_{2\delta}(\Sigma_0) \subset U$. Note that $\partial U_\tau(\Sigma_0) \setminus \partial\mathcal{M}$ is smooth for all $\tau \in (\delta, 2\delta)$. Hence, by the isoperimetric inequality, we can choose some constant $C > 0$ (independent of τ) such that for every $(n-1)$-dimensional integer rectifiable current α homologous to 0 in $\partial U_\tau(\Sigma_0)$, there exists an n-dimensional i.r. current S in $\partial U_\tau(\Sigma_0)$ with

$$\partial S = \alpha \quad \text{and} \quad \mathbf{M}(S) \leq C\mathbf{M}(\alpha)^{\frac{n}{n-1}}. \tag{9.2.3}$$

Take Γ_ϵ to be the minimizer in (9.2.2). For every $\tau \in (\delta, 2\delta)$ we define:

- $A(\tau) := \mathbf{M}\big(\Gamma_\epsilon \llcorner (U_\tau(\Sigma_0))^c\big)$;

- $L(\tau) := \mathbf{M}\big(\partial\big(\Gamma_\epsilon \llcorner (U_\tau(\Sigma_0))^c\big)\big) = \mathbf{M}\big(\partial(\Gamma_\epsilon \llcorner U_\tau(\Sigma_0)) - \gamma\big)$.

A standard inequality using coarea formula yields

$$L(\tau) \leq -A'(\tau). \tag{9.2.4}$$

Let us now fix $\tau \in (\delta, 2\delta)$. One of the following alternatives must hold:

(A1) $L(\tau) = 0$. This means that $\partial\big(\Gamma_\epsilon \llcorner U_\tau(\Sigma_0)\big) = \gamma$, and hence $\Gamma_\epsilon \llcorner U_\tau(\Sigma_0)$ is homologous to Σ_0 in U. Consequently, by (9.2.1),

$$m_0(\epsilon) = \mathbf{M}(\Gamma_\epsilon) \geq \mathbf{M}(\Gamma_\epsilon \llcorner U_\tau(\Sigma_0)) > \mathbf{M}([\![\Sigma_0]\!]),$$

hence we are finished.

(A2) $L(\tau) > 0$. Since $\mathbb{F}(\Gamma_\epsilon - [\![\Sigma_0]\!])$ is sufficiently small, then $\mathbb{F}(\Gamma_\epsilon - [\![\Sigma_0]\!]) = \mathbf{M}(T)$ with $\partial T = \Gamma_\epsilon - [\![\Sigma_0]\!]$. Note that the slice of the $(n+1)$-current T which is supported in $\partial U_\tau(\Sigma_0)$ has the slice of n-current Γ_ϵ as its boundary. Let us denote this slice by S. This means that S lies in $\partial U_\tau(\Sigma_0)$ with $\partial S = \gamma - \partial(\Gamma_\epsilon \llcorner U_\tau(\Sigma_0))$, and by (9.2.3),

$$\mathbf{M}(S) \leq CL(\tau)^{\frac{n}{n-1}}$$

Let us set $Z = \Gamma_\epsilon \llcorner U_\tau(\Sigma_0) + S$. At this point, we make a further distinction between two cases:

(A2.1) $\mathbf{M}(Z) \leq \mathbf{M}(\Gamma_\epsilon)$. By construction, Z is homologous to Σ_0 in U; thus by (9.2.1),

$$m_0(\epsilon) = \mathbf{M}(\Gamma_\epsilon) \geq \mathbf{M}(Z) > \mathbf{M}([\![\Sigma_0]\!]),$$

and the claim follows.

(A2.2) $\mathbf{M}(Z) \geq \mathbf{M}(\Gamma_\epsilon)$. By the above, this implies

$$\mathbf{M}\big(\Gamma_\epsilon \llcorner (U_\tau(\Sigma_0))^c\big) \leq \mathbf{M}(S) \leq CL(\tau)^{\frac{n}{n-1}}.$$

In summary, it follows from the considerations above that for the rest of the proof we may assume w.l.o.g. $\forall \tau \in (\delta, 2\delta)$:

- $L(\tau) > 0$
- $A(\tau) \leq CL(\tau)^{\frac{n}{n-1}}$.

Step 2: We claim that the minimizers Γ_ϵ satisfy

$$\mathbf{M}(\Gamma_\epsilon) \to \mathbf{M}([\![\Sigma_0]\!]) \quad \text{as } \epsilon \to 0. \tag{9.2.5}$$

By the lower semicontinuity of mass with respect to flat convergence, we immediately get

$$\liminf_{\epsilon\to 0} \mathbf{M}(\Gamma_\epsilon) \geq \mathbf{M}([\![\Sigma_0]\!]).$$

The other inequality needed to prove the claim follows by constructing suitable competitors. Consider the currents $\tilde{\Gamma}_r := [\![\Sigma_0]\!] + \partial[\![B_r(p)]\!]$, where $B_r(p) \subset U$, $B_r(p) \cap \Sigma_0 = \emptyset$. Clearly, $\mathbb{F}(\tilde{\Gamma}_r - [\![\Sigma_0]\!]) \to 0$ as $r \to 0$, hence (for ϵ small enough) there exists some $r(\epsilon)$ such that $\mathbb{F}(\tilde{\Gamma}_{r(\epsilon)} - [\![\Sigma_0]\!]) = \epsilon$. Moreover, $\mathbf{M}(\tilde{\Gamma}_{r(\epsilon)}) \to \mathbf{M}([\![\Sigma_0]\!])$ as $\epsilon \to 0$. This shows that

$$\limsup_{\epsilon\to 0} \mathbf{M}(\Gamma_\epsilon) \leq \mathbf{M}([\![\Sigma_0]\!]),$$

and the claim follows.

Step 3: We next prove that

$$\lim_{\epsilon\to 0} \mathbf{M}\big(\Gamma_\epsilon \llcorner (U_{3\delta/2}(\Sigma_0))^c\big) = 0. \tag{9.2.6}$$

As before, we can assume $\Gamma_\epsilon - [\![\Sigma_0]\!] = \partial T_\epsilon$, with $\mathbf{M}(T_\epsilon) \to 0$ as $\epsilon \to 0$. If (9.2.6) were wrong, there would exist a sequence $\epsilon_k \downarrow 0$ and an $\alpha > 0$ such that

$$\mathbf{M}\big(\Gamma_{\epsilon_k} \llcorner (U_{3\delta/2}(\Sigma_0))^c\big) \geq \alpha. \tag{9.2.7}$$

If we let $\langle T_{\epsilon_k}, \tau\rangle = \partial(T_{\epsilon_k} \llcorner U_\tau(\Sigma_0)) - (\partial T_{\epsilon_k}) \llcorner U_\tau(\Sigma_0)$ denote the slices of T_{ϵ_k} (w.r.t the distance from Σ_0), then by coarea formula

$$\int_\delta^{\frac{3}{2}\delta} \mathbf{M}(\langle T_{\epsilon_k}, \tau\rangle)\, d\tau \leq \mathbf{M}(T_{\epsilon_k}) \to 0$$

as $k \to 0$. Since L^1 convergence implies a.e. pointwise convergence, we are able to extract a subsequence (not relabeled) and a $\tau \in (\delta, \frac{3}{2}\delta)$ such that $\mathbf{M}(\langle T_{\epsilon_k}, \tau\rangle) \to 0$.

On the other hand, we can apply (9.2.1) to the current $\langle T_{\epsilon_k}, \tau\rangle + \Gamma_{\epsilon_k} \llcorner U_\tau(\Sigma_0)$ as we did in Step 1, which gives us

$$\mathbf{M}\big(\langle T_{\epsilon_k}, \tau\rangle + \Gamma_{\epsilon_k} \llcorner U_\tau(\Sigma_0)\big) > \mathbf{M}([\![\Sigma_0]\!]).$$

Using these two facts together with (9.2.5), one easily concludes

$$\lim_{k\to\infty} \mathbf{M}\big(\Gamma_{\epsilon_k} \llcorner (U_\tau(\Sigma_0))^c\big) \to 0,$$

which is a contradiction to (9.2.7).

Step 4: Note that the previous step tells us that $A\left(\frac{3}{2}\delta\right) \to 0$ as $\epsilon \to 0$. Recall that we assume $L(\tau) > 0$ $\forall \tau \in (\delta, 2\delta)$ since Step 1, which immediately implies that also $A(\tau) > 0$. However, from (9.2.4) and the other conclusion of Step 1 we deduce

$$A(\tau) \leq CL(\tau)^{\frac{n}{n-1}} \leq C(-A'(\tau))^{\frac{n}{n-1}},$$

giving (by a slight abuse of notation regarding the constants involved)

$$-\frac{A'(\tau)}{A(\tau)^{\frac{n-1}{n}}} \geq \frac{1}{C} \quad \forall \tau \in (\delta, 2\delta).$$

Integrating the above inequality between $\frac{3}{2}\delta$ and 2δ, we get

$$A\left(\tfrac{3}{2}\delta\right)^{\frac{1}{n}} \geq A\left(\tfrac{3}{2}\delta\right)^{\frac{1}{n}} - A(2\delta)^{\frac{1}{n}} \geq \frac{\delta}{2nC}$$

which gives a contradiction for ϵ small enough. □

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