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Camillo De Lellis

Abstract

In this article we survey some recent results in the regularity theory of admissible solutions to hyperbolic conservation laws and Hamilton-Jacobi equations.

Functions of bounded variations are L^1 functions with distributional derivatives which are Radon measures. It is well known that the space of functions of bounded variation (shortly BV) plays a prominent role in the well-posedness theory for some important class of hyperbolic equations and hyperbolic systems. In this paper we will consider four different instances of this fact:

(a) **Scalar conservation laws in 1 space dimension**. The unknown is a scalar function u defined on a subset of $\mathbb{R}_t \times \mathbb{R}_x$ which solves the equation

$$u_t + [f(u)]_x \;=\; 0\,. \tag{0.1}$$

We will always assume that f is at least C^2.

(b) **Hamilton-Jacobi equations.** The unknown is a scalar function v defined on a subset of $\mathbb{R}_t \times \mathbb{R}^n_x$ which solves the equation

$$v_t + H(D_x v) \;=\; 0\,. \tag{0.2}$$

We will always assume that

$$H \text{ is } C^2 \text{ and uniformly convex.} \tag{0.3}$$

(c) **Hyperbolic systems of conservation laws in 1 space dimension.** The unknown is a vector-valued function u defined on a subset of $\mathbb{R}_t \times \mathbb{R}_x$ which solves the system

$$u_t + [f(u)]_x \;=\; 0 \tag{0.4}$$

with some restrictions on f. If the target of u is k-dimensional, then $f = (f_1, \ldots, f_k)$ is a map from (a subset of) $\mathbb{R}^k$ into $\mathbb{R}^k$ and (0.4) consists of the k equations

$$(u_i)_t + [f_i(u)]_x \;=\; 0\,.$$

The hyperbolicity of the system requires that the $k \times k$ matrix $Df(\bar{u})$ has real eigenvalues for every $\bar{u}$. The system is *strictly* hyperbolic if the k eigenvalues are distinct for every $\bar{u}$.

(d) **Scalar conservation laws in n space dimension**. The unknown is a scalar function u defined on a subset of $\mathbb{R}_t \times \mathbb{R}^n_x$ which solves the equation

$$u_t + \mathrm{div}_x[f(u)] \;=\; 0\,. \tag{0.5}$$

This paper surveys some recent regularity theorems for admissible solutions for the examples (a), (b) and (c). Often these equations are coupled with an initial condition of the form $u(0,x) = u_0(x)$ and one seeks solutions on the domain $[0,+\infty[\times\mathbb{R}^n$, which are therefore called *global solutions of the Cauchy problem*. It turns out that most of the theorems reported below are valid for general domains. However, in most of our discussion we will restrict our attention to solutions of the Cauchy problem, in order to make our presentation easier.

It is well known that generically solutions of the Cauchy problem lose their regularity after a finite time, even when the initial data u_0 is extremely regular. Consider for instance (0.1) with the choice $f(\bar{u}) := \frac{\bar{u}^2}{2}$. The resulting equation is called *Burgers' equation* and it is a famous prototype for the "finite-time-blowup" of classical solutions, taken as an example in most textbooks on partial differential equations (see for instance Section 3.4 in [18]). A thoroughly studied question is whether one can solve the equations "after" the appearing of singularities and give a well-posedness theory in a suitable class of solutions which allows singularities. It turns out that this is possible in a variety of cases. Briefly, we start with:

- bounded maps u which solve (0.1), (0.5) and (0.4) in the sense of distributions;
- Lipschitz maps v which satisfy (0.2) almost everywhere.

It is well known that such solutions are not unique (again, the most common example is Burgers' equation when it is coupled, for instance, with the initial condition $u(0,\cdot) = \mathbf{1}_{[0,+\infty[}$; cp. with Section 4.2 of [15]). In all the cases we define admissible solutions by requiring that the solutions above satisfy some additional constraints:

- in the case of (0.1) and (0.5) we impose some inequalities in the sense of distributions, commonly called *entropy conditions* (see for instance Section 6.2 of [15]); these solutions are therefore called *entropy solutions*;
- in the case of (0.2) we consider *viscosity solutions* (see for instance Section 10.1 of [18]);
- in the case of (0.4) we consider *BV* solutions subject to several restrictions (see for instance Section 9.3 of [11]); we will call them *semigroup solutions.*

In the cases (0.1)-(0.5) and (0.2) there is global existence and uniqueness of admissible solutions when starting from, respectively, bounded and Lipschitz initial data (cp. with [15] and [18]). Moreover:

- the entropy solutions of (0.1) and (0.5) are *BV* if the initial data are;
- the gradients of viscosity solutions are always locally *BV* (this regularization property holds for (0.1) as well, when the flux f is uniformly convex).

The situation for (0.4) is much more complicated. A well-posedness theory for global semigroup solutions has been achieved only recently by Bressan and his school. In this case one needs suitable smallness assumptions for the initial data (we refer to [11] for further details).

1. SBV functions

In general the BV class is a quite satisfactory functional setting for the equations above. Indeed, easy examples show that jump singularities (respectively in u for (0.1), (0.4) and (0.5) and in ∇v for (0.2)) are necessary. The typical picture that one has in mind for these solutions (resp. for their gradients, in the case of (0.2)) is a piecewise C^1 function which undergoes jump discontinuities along space-time hypersurfaces (and hence along space-time curves if the space dimension is 1). The space of BV functions is perhaps the most used functional-analytic "closure" of classical functions which have jump singularities. However, a typical BV function might have a quite complicated behavior.

Consider for a moment a BV scalar function f in 1 variable. We then know (see for instance Section 3.2 of [4]) that, after possibly redefining f on a set of measure zero, f is continuous except for an at most countable set J. Moreover, for every $x_0 \in J$, the following limits exist:

$$f^+(x_0) := \lim_{x \downarrow x_0} f(x) \qquad \text{and} \qquad f^-(x_0) := \lim_{x \uparrow x_0} f(x)\,.$$

The distributional derivative of f is a measure μ. By the Radon-Nykodim Theorem we can decompose it into $\mu^s + \mu^a$, where μ^a is absolutely continuous with respect to the Lebesgue measure and μ^s is singular. We can further decompose μ^s into its atomic and nonatomic part $\mu^{at} + \mu^{na}$. This means that μ^{at} consists of the sum of (at most countably many) Dirac masses, whereas μ^{na} has the property that $\mu^{na}(\{x\}) = 0$ for every $x \in \mathbb{R}$. The atomic part of the measure can be nicely related to the pointwise properties of f through the formula

$$\mu^{at} \;=\; \sum_{y \in J} (f^+(y) - f^-(y))\delta_y\,. \tag{1.1}$$

For this reason, several authors call μ^{at} the *jump part* of μ and denote it by μ^j. The measure μ^{na} is instead much less understood.

A typical example of μ^{na} is the derivative of the classical ternary Cantor function (see Example 1.67 of [4]). Inspired by this example, μ^{na} has been called *Cantor part* and denoted by μ^c by Ambrosio and De Giorgi in [17], where this decomposition of μ^s was proposed for the first time. This idea can be generalized to n dimensions, as it was first done in [17]. In this general setting the measure μ^j is concentrated on a rectifiable set of codimension 1 (i.e. a countable union of pieces of C^1 hypersurfaces) and one can relate the pointwise behavior of f to μ^j with a suitable generalization of (1.1). The remaining "Cantor part" μ^c of the singular measure μ^s is, roughly speaking, a "fractal measure" of "dimension between $n-1$ and n" (for the precise definitions and a complete account of the theory we refer to Chapter 3 of [4]).

In order to deal with some variational problems, Ambrosio and De Giorgi introduced in [17] the space of *special functions of bounded variations*, briefly SBV, which consists of those BV functions for which the Cantor part of the derivative vanishes identically. This space can be regarded as that "regular part" of BV which retains the typical properties of classical functions with jump discontinuities. The following two questions arise naturally in connection with the hyperbolic equations considered in this note:

(Q1) Do admissible solutions of the Cauchy problems for (0.1), (0.2), (0.4), (0.5) preserve also the SBV regularity?

(Q2) Is there an SBV-regularization under suitable assumptions?

These questions can be regarded as "regularity problems" for admissible solutions. There are very interesting results on higher regularity for solutions of (0.2), see for instance the book [14]. However, these theorems require some regularity assumptions on the initial data and it is known that they do not hold if we start from general Lipschitz initial data. Besides their intrinsic interest, the questions (Q1) and (Q2) arise naturally in some problems in the control theory for hyperbolic systems of conservation laws (see for instance [12], [6], [5], [7] and [13]). In this note we will survey some recent results that show that, while the answer to (Q1) is quite easily seen to be negative, an SBV-regularization effect holds for (0.1), (0.2) and (0.4) as soon as the equation is sufficiently nonlinear (note that such a hypothesis is needed since transport equations are special cases of all the equations considered so far).

2. A couple of examples

As already anticipated, it is easy to see that the answer to Question (Q1) is negative. Let us consider the simple case of Burgers' equation

$$u_t + \left[\frac{u^2}{2}\right]_x \;=\; 0\,. \tag{2.1}$$

As it is well known, classical solutions are constant along characteristics, which are straight lines and solve the ode $\dot\gamma = u(\gamma)$. Let g be the Cantor ternary function (see Example 1.67 of [4] for the explicit construction of g). As it is well known g is a continuous strictly increasing map from $[0,1]$ onto $[0,1]$. Define $u_1 : \mathbb{R} \to [0,1]$ as

$$u_1(x) \;=\; \begin{cases} g(-x) & \text{for } x \in [-1,0] \\ 0 & \text{for } x > 0 \\ 1 & \text{for } x < -1\,. \end{cases}$$

For every $y \in \mathbb{R}$ consider the half-line $\ell_y \subset]-\infty,1] \times \mathbb{R}$ given by the equation

$$\ell_y := \{(x-y) = u_1(y)(t-1)\}\,.$$

The monotonicity of u_1 easily implies that the collection $\{\ell_y\}_{y\in\mathbb{R}}$ is a fibration of $]-\infty,1[\times\mathbb{R}$. Define therefore the function $u :]-\infty,1] \times \mathbb{R} \to [0,1]$ by requiring that $u|_{\ell_y} \equiv u_1(y)$. It is easy to check that u is locally Lipschitz on $]-\infty,1[\times\mathbb{R}$, continuous on $]0,+\infty] \times \mathbb{R}$ and satisfies $u(1,x) = u_1(x)$. Moreover, by the theory of characteristics, u solves (2.1) on $]-\infty,1[\times\mathbb{R}$. This solution is also an entropy solution (the entropy conditions can be easily checked because the usual chain rule holds for Lipschitz functions).

Consider next the entropy solution $\bar u$ of (2.1) on $[0,+\infty[\times\mathbb{R}$ with initial data $\bar u(0,x) = u_0(x) := u(0,x)$. u_0 is a Lipschitz function, which is constantly equal to 0 for $x >> 1$ and constantly equal to 1 for $x << -1$. Therefore u_0 is an SBV function. On the other hand, $\bar u$ coincides necessarily with u on $[0,1[\times\mathbb{R}$ and by the continuity in time of entropy solutions, we conclude that $\bar u(1,x) = u(1,x)$. Therefore $\bar u(1,\cdot)$ is the Cantor ternary function, which is not in SBV. We can summarize this construction in the following remark.

Remark 2.1. There exists a bounded Lipschitz initial data u_0 such that, if $\bar u$ denotes the unique entropy solution in $[0,+\infty[\times\mathbb{R}$ of (2.1) coupled with the initial condition

$\bar{u}(0,\cdot) = u_0$, then $\bar{u}(1,\cdot)$ is a "Cantor-type" function which belongs to $L^\infty \cap (BV \setminus SBV)$.

The previous example still leaves open that entropy solutions are, for instance, SBV functions when we consider them as functions of both variables: an SBV function of two variables can in fact have the Cantor ternary function as trace on a given line. To see this, consider the map u constructed above and extend it to the whole space-time by setting $u(t,x) = u(1-t,x)$ for $t > 1$. Then it is easy to see that $u \in W^{1,1}_{loc}(\mathbb{R}_t \times \mathbb{R}_x)$. Obviously, this extended function is not even a distributional solution of (2.1): the example just shows that there is no incompatibility between the SBV regularity in two variables and having "bad" trace at time $t = 1$, but the proof of the SBV regularity in two variables for entropy solutions of Burgers' equation does not have anything to do with a "reflection trick".

Consider next the easiest example of (0.1), i.e. let f be a linear function. For some constant c_0 we then have

$$u_t + c_0 u_x \; = \; 0\,. \tag{2.2}$$

The only distributional solution of (2.2) on $[0,+\infty[$ with the initial condition $u(0,\cdot) = u_0$ is obviously $u(t,x) = u_0(x - c_0 t)$. Therefore, if u_0 is BV function which does not belong to SBV, the solution u is obviously not in SBV. This shows that, in order to hope for an SBV regularization effect, we need some assumption on the flux function f. Indeed, the key property which induces the SBV regularization effect is a "sufficient nonlinearity".

Let us come back to the example of Remark 2.1. In this explicit case, it is possible and instructive to see how the entropy solution behaves at times larger than 1. If we fix $\varepsilon > 0$, no matter how small it is, $u(1+\varepsilon,\cdot)$ is a piecewise constant function taking finitely many values. Therefore, the distributional derivative $u_x(1+\varepsilon,\cdot)$ is a finite sum of Dirac deltas, located on the finitely many points where the function u jumps. The number of these points converge to ∞ as ε approaches 0. Essentially, for $\varepsilon \downarrow 0$, the jump sets $J_{1+\varepsilon}$ of $u(1+\varepsilon,\cdot)$ cluster towards the Cantor ternary set (which is the set where the fractal measure $u_x(1,\cdot)$ is concentrated). It is also not difficult to see that the total mass of the singular measure $u_x(1+\varepsilon,\cdot)$ equals the mass of the singular measure $u_x(1,\cdot)$.

Inspired by this example, one could formulate the following conjecture (which was pointed out to the author by Bressan during a conversation on the problem): if u is an entropy solution of Burgers' and, for a certain positive time T, $u(T,\cdot)$ is not in SBV, at future times $T+\varepsilon$ the "Cantor part" of $u(T,\cdot)$ gets transformed into jump singularities. From this rough picture one should be able to conclude that $u(T,\cdot)$ is "almost always" in SBV, and hence that $u \in SBV$ as a function of *two* variables. We will see in the next section that this picture is correct for general nonlinear scalar laws in 1 space dimension. A suitable version is also true for Hamilton Jacobi equations (under the assumption (0.3)) and for hyperbolic systems of conservation laws which are genuinely nonlinear.

3. The SBV-regularization effect

Following the discussion of the previous section, we list here several theorems which have been proved on the SBV regularity of admissible solutions. The first result has been proved in [2]:

Theorem 3.1. *Assume $f \in C^2(\mathbb{R})$ and $f'' > 0$. Let u be a bounded entropy solution of* (0.1) *in a domain $\Omega \subset \mathbb{R}_t \times \mathbb{R}_x$. Then there exists a set $S \subset \mathbb{R}$ at most countable such that the following holds $\forall \tau \notin S$:*

$$u(\tau, \cdot) \in SBV(I) \qquad \text{for every open interval } I \subset\subset \Omega \cap \{t = \tau\}.$$

In addition, $u \in SBV(\Gamma)$ for every domain $\Gamma \subset\subset \Omega$.

This theorem has been later extended in [19] in two directions:

(i) the assumption $f'' > 0$ can be replaced by the discreteness of the set $\{f'' = 0\}$ (in this case we must however assume that the solution is BV, since there are bounded entropy solutions with unbounded variation);

(ii) we can allow for sufficiently regular source terms and for flux functions depending also on u and (x, t); in other words the result can be extended to balance laws of type $u_t + [f(u, x, t)]_x + g(u, x, t) = 0$.

The Theorem 3.1 implies a similar regularization effect for Hamilton-Jacobi equations (0.2) in 1 space dimension. Indeed, it is well known that, for convex fluxes f, a Lipschitz function v is a viscosity solution of $v_t + f(v_x) = 0$ if and only if $u := v_x$ is an entropy solution of $u_t + [f(u)]_x = 0$. However, this nice equivalence does not have a counterpart when the space dimension is higher than 1. For (0.2), a generalization of Theorem 3.1 to any dimension has been given in [10]:

Theorem 3.2. *Assume v is a Lipschitz viscosity solution of* (0.2) *on a domain $\Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$, under the assumption* (0.3)*. Then there exists a set $S \subset \mathbb{R}_t$ at most countable such that the following holds for every $\tau \notin S$:*

$$\nabla v(\tau, \cdot) \in SBV(\Gamma) \qquad \text{for every domain } \Gamma \subset\subset \Omega \times \{t = \tau\}.$$

Moreover, $v_t, \nabla v \in SBV(\Sigma)$ for every $\Sigma \subset\subset \Omega$.

Tonon [20] has extended this theorem to Hamiltonians depending also on t and x (that is to solutions of $v_t + H(t, x, \nabla v) = 0$).

Finally, very recently Theorem 3.1 has been extended to the much more interesting and difficult case of hyperbolic systems in [9]. First steps in this direction had been taken in [16] (for self-similar solutions, i.e. solutions of the Riemann problem) and in [8] (for genuinely nonlinear Temple systems).

Theorem 3.3. *Consider a strictly hyperbolic system* (0.4) *coupled with an initial data $u(0, \cdot) = \bar{u}$. Assume that each characteristic field is genuinely nonlinear (see Definition 5.2 of* [11]*). If the BV norm of $\bar{u}$ is sufficiently small, then the same conclusions of Theorem 3.1 hold for the unique* semigroup solution *of the corresponding Cauchy problem in* $[0, +\infty[\times\mathbb{R}$.

The investigation of which conditions ensure the SBV regularity of solutions to multidimensional scalar conservation laws has not yet been explored and remains, therefore, an open question. Another interesting question is whether the entropy condition is really needed. Indeed, one could for instance ask whether BV distributional solutions of Burgers' equation are necessarily SBV (as functions of 2 variables). This last question is linked to some problems in the theory of transport equations with rough coefficients (we refer the reader to [3], in particular to Remark 8.1 therein) and might be quite subtle. Indeed, it is possible to show that there are BV functions $u \notin SBV$ such that $u_t + \left(\frac{u^2}{2}\right)_x = \mu$ and μ is a purely jump measure (see the proof of Proposition 1.3 in [3]).

4. Why the SBV regularization holds

A common point in the proofs of all the theorems mentioned above is the following observation. Assume that we are looking at the solution u of a Cauchy problem in $[0, +\infty[\times\mathbb{R}^n$ (the case of a general domain is reduced to this one by the finite speed of propagation). It is then possible to construct a functional $\mathcal{F}$ which we evaluate on the solution $u(t, \cdot)$ and enjoys the following properties:

- $t \mapsto f(t) := \mathcal{F}(u(t, \cdot))$ has some monotonicity properties;
- If $u(\tau, \cdot)$ (resp. $\nabla v(\tau, \cdot)$ for (0.2)) does not belong to SBV, then the function f has a jump discontinuity at τ; indeed it is possible to estimate $|f(\tau^+) - f(\tau^-)|$ from below with (a suitable expression involving) the total variation of the Cantor part of the measure $u_x(t, \cdot)$.

The functionals $\mathcal{F}$ used in the papers cited above are all somewhat similar. However, the various situations pose different levels of difficulty. The proof of [2] is the most elementary and we will give a sketchy explanation of it in this section. The proof of [10] is more difficult to visualize and, though the key estimates have a quite elementary nature, their proof needs some sophisticated technical devices (most notably a reformulation of the theory of monotone functions with tools from geometric measure theory, due to [1]). The proof in [9] is surely the most demanding, due to the complicated interactions which arise when dealing with hyperbolic systems of conservation laws and to the technical difficulties in justifying the computations (in other words, the estimates are achieved through subtle approximation techniques).

To illustrate some of the basic ideas which are common to all the proofs, we choose the most simple situation, i.e. that of Theorem 3.1. In this case the functional $\mathcal{F}$ has an immediate interpretation in terms of Dafermos' generalized characteristics (see [15]). Fix the time t and consider the set of jump discontinuities J_t of the scalar BV function $x \mapsto u(t, x)$. For each $x_i \in J_t$ there are a maximal and minimal backward characteristics ℓ_i^+ and ℓ_i^- which are straight segments in space-time. One endpoint of these segments is (t, x_i), whereas the two other extrema are distinct and lie on the line $\{t = 0\}$ (see Figure 4.1 below). These segments and the line $\{t = 0\}$ bound a nontrivial triangle.

If at each time we consider the shadowed region R_t in Figure 4.1, it is clear from the geometric properties of generalized characteristics that this region can only increase (in other words, $R_s \subset R_t$ for every $s < t$). Obviously this suggests several possible candidates for the functional $\mathcal{F}$. Observe that the region R_t might be unbounded. To overcome this technical problem we fix $M > 0$ and define the function $f_M(t)$ as the sum of the lengths of the triangles belonging to R_t and having tips lying in $[-M-Ct, M+Ct]$. Note that, by the geometric properties of generalized characteristics, the triangles do not overlap. If C is sufficiently large, then f_M is a nondecreasing function of time. The choice of C depends only on f and $\|u\|_\infty$ (and is in fact related to the finite speed of propagation: roughly speaking it ensures that, if a triangle Γ_s has tip lying in $[-M - Cs, M + Cs]$, at later time $t > s$ there is a triangle Γ_t of R_t which contains Γ_s and has tip in $[-M - Ct, M + Ct]$)).

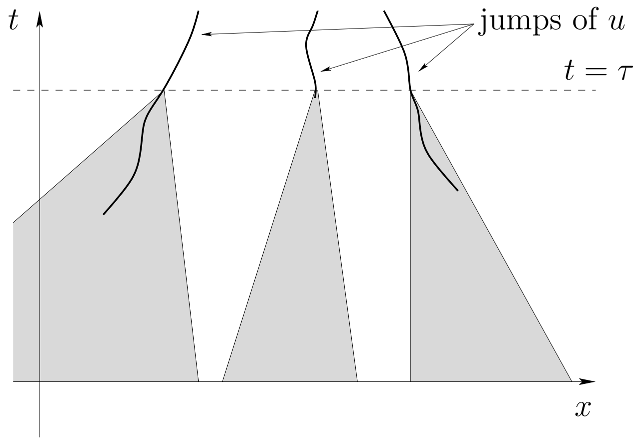

Figure 4.1: The shadowed region R_τ.

It remains to check that, if the restriction of the measure $u_x(t, \cdot)$ to the segment $[-M - Ct, M + Ct]$ has non-vanishing Cantor part, then the function f_M "jumps" up at t. The set of Theorem 3.1 is then the union of the jump discontinuities of all f_M, with M ranging among all natural numbers. To check that f_M jumps when "Cantor-type behaviors' appear is obviously the delicate part of the proof. One thing is, however, quite clear.

Consider a point x_0 where the Cantor part of $u_x(t, \cdot)$ "prevails" (more precisely, if we denote by μ the Cantor part of u_x, we then require that

$$\left. \lim_{r\downarrow 0} \frac{|\mu|([x_0 - r, x_0 + r])}{|u_x|([x_0 - r, x_0 + r])} \;=\; 1 \right) .$$

Then, for most points x_0 of this type, we have

$$\lim_{y\to x_0} \frac{u(t, y) - u(t, x_0)}{y - x_0} = -\infty .$$

Indeed, the difference quotient diverges because of the prevalence of the Cantor part, which is a singular measure; the minus sign is instead an effect of Oleinik's estimate $u_x \leq \frac{C}{t}$. Consider now two characteristics passing through points (t, y) and (t, z) with $y < x_0 < z$. If the points y and z are very close, these characteristics must "collide" shortly after the time t.

At time $t + \varepsilon$ we will therefore see plenty of newly formed triangles in the region $R_{t+\varepsilon}$. In particular, these triangles contain a set of the form $\{t\} \times A$, where A is an open subset of $\mathbb{R}$ covering the "region where the Cantor part of $u_x(t, \cdot)$ prevails". At this point one needs to estimate the total number of new segments that must be counted when computing $f_M(t+\varepsilon) - f_M(t)$. A careful, yet not long, computation, succeeds in estimating this number from below. The important point is obviously that the estimate is independent of ε and gives, therefore, the inequality $f_M(t^+) - f_M(t) > 0$.

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