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Sharp upper bounds for a variational problem with singular perturbation

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Abstract Let Ω be a C^2 bounded open set of $\mathbb{R}^2$ and consider the functionals

$$F_\varepsilon(u) \ := \int\limits_{\Omega} \left\{ \frac{(1-|\nabla u(x)|^2)^2}{\varepsilon} + \varepsilon |D^2 u(x)|^2 \right\} \mathrm{d}x\,.$$

We prove that if $u \in W^{1,\infty}(\Omega)$, $|\nabla u| = 1$ a.e., and $\nabla u \in BV$, then

$$\Gamma - \lim_{\varepsilon \downarrow 0} F_\varepsilon(u) = \frac{1}{3} \int\limits_{J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1\,.$$

The new result is the $\Gamma - \limsup$ inequality.

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1 Introduction

1.1 History of the problem

We consider the energy functionals

$$F_\varepsilon[u,\Omega] := \int_\Omega \left\{ \frac{(1-|\nabla u(x)|^2)^2}{\varepsilon} + \varepsilon |D^2 u(x)|^2 \right\} \mathrm{d}x. \tag{1.1}$$

Here ε is a positive number, Ω is a (sufficiently smooth) bounded open set of $\mathbb{R}^2$, u is an element of $H^2(\Omega)$, and $|D^2u(x)|^2$ denotes the square of the Hilbert–Schmidt norm of the Hessian $D^2u(x)$. These functionals have been proposed as models for different physical problems (see [3], [18], and [13]). In all these cases one seeks minimizers of F_ε among the u's such that

$$u|_{\partial\Omega} = 0 \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = -1. \tag{1.2}$$

In [3] (see also [4]) the following conjectures were made. First of all, if $\limsup_{\varepsilon\downarrow 0} F_\varepsilon(u^\varepsilon) < \infty$, then u^ε converges, up to subsequences, to a Lipschitz function u solving the eikonal equation $|\nabla u| = 1$. Second, if $\{u^\varepsilon\}$ is a family of minimizers, then the limits u must minimize

$$F_0[v,\Omega] := \frac{1}{3} \int_{\Omega\cap J_{\nabla v}} |[\nabla v]|^3 \,\mathrm{d}\mathscr{H}^1, \tag{1.3}$$

among all v solving the eikonal equation. Here $J_{\nabla v}$ is the set where "∇v jumps", and $[\nabla v]$ is the "jump".

The first conjecture has been proved independently in [2] and [14], building on the works [17] and [5]. Concerning the second question, one first has to interpret (1.3). A possible choice is to restrict F_0 to u's with $\nabla u \in BV(\Omega)$. In [5] Aviles and Giga proved that F_0 is lower semicontinuous on the space $BV_e(\Omega) := \{u \in W^{1,\infty} : |\nabla u| = 1, \nabla u \in BV(\Omega)\}$; endowed with the L^1 topology. However an example of [2] shows that $\{F_0 \le c\} \cap BV_e(\Omega)$ is not compact. This, combined with a construction of [10], gives a family $\{u^\varepsilon\}$ bounded in energy which converges to an u such that $\nabla u \notin BV_{loc}$. In [2], inspired by [5], a larger space $AG(\Omega)$ and a functional $\tilde F : AG(\Omega) \to \mathbb{R}$ were proposed. Summarizing the various results available in the literature, we have that:

(a) $BV_e(\Omega) \subset AG(\Omega)$ and $\tilde F = F_0$ on $BV_e(\Omega)$;
(b) $\tilde F$ is lower semicontinuous on $AG(\Omega)$;
(c) The sublevel sets of $\tilde F$ are compact on $AG(\Omega)$;
(d) If $\limsup F_\varepsilon(u^\varepsilon) < \infty$, then u^ε clusters to elements $u \in AG(\Omega)$;

(e) If $\{u^{\varepsilon_k}\} \subset H^2(\Omega)$ converges to $u \in AG(\Omega)$, then

$$\tilde{F}(u) \leq \liminf_{k\uparrow\infty} F_{\varepsilon_k}(u^{\varepsilon_k}).$$

One can also impose the boundary conditions (1.2) and hence prove the existence of a $\tilde{F}$–minimizer. In the language of Γ–convergence (see [9] and the books [8,7,6]), (e) is the *Γ–liminf inequality*. Combined with the *Γ–limsup inequality* of Conjecture 1, (a)–(e) would give a full positive answer to the problem raised by Aviles and Giga.

Conjecture 1 If $u \in AG(\Omega)$ satisfies (1.2), then there exists a family of functions $\{u^\varepsilon\} \subset H^2(\Omega)$ which satisfy (1.2), converge to u, and such that $\tilde{F}(u) \geq \limsup_\varepsilon F_\varepsilon(u^\varepsilon)$.

We finally remark that all these results are restricted to two dimensions because of the structure of F_ε: As it was shown in [11], already in three dimensions the situation is very different.

1.2 Statement of the result

As far as we know, the existence of the optimal family of Conjecture 1 is known only when u is the distance from the boundary and the jump set of ∇u is a finite union of smooth arcs, with a finite number of intersections (see [15]). A milder problem than Conjecture 1 is to exhibit such an optimal family without imposing boundary conditions. For the case that ∇u jumps between two values such a family was constructed in [3]; the construction was extended to piecewise affine u's with finitely many pieces in [10]. Therefore, in order to construct an optimal family for any given u, it would suffice to approximate it with piecewise affine maps u_k such that $\lim \tilde{F}(u_k) = \tilde{F}(u)$. This "approximation in energy" is the standard procedure adopted to tackle Γ–limsup inequalities: First one proves the existence of the optimal family for a suitable class of functions, and then one shows that this class is dense in energy.

In our case an approximation by piecewise affine maps would be delicate if we want to impose the boundary conditions: In particular it would require a Whitney type triangulation of Ω which refines towards the boundary. However, even neglecting the boundary conditions and assuming that $\nabla u \in BV$, it is not clear at all whether an approximation *in energy* by piecewise affine maps is possible. This difficulty is due to the rigidity of the eikonal constraint. Using a completely different approach, in this paper we prove the following.

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a C^2 bounded domain, and $u \in W^{1,\infty}(\Omega,\mathbb{R})$ with $\nabla u \in BV(\Omega,\mathbb{S}^1)$. Then there is a family $\{u^\varepsilon\} \subset C^\infty(\overline{\Omega})$ such that $u^\varepsilon \to u$ in $W^{1,p}(\Omega)$ for every $p < \infty$ and*

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon[u^\varepsilon, \Omega] \leq F_0[u, \Omega].$$

If u additionally satisfies (1.2)*, then the family u^ε can be chosen to also satisfy* (1.2).

Remark 1 We shall actually prove that the more general boundary conditions $u|_{\partial\Omega} = g$ and $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = h$, for $g, h \in C^2(\partial\Omega)$, can be preserved.

Remark 2 A similar result has been independently obtained by A. Poliakovsky [19,20].

Remark 3 Our proofs rely on the fine properties of BV functions and on estimates upon the F_ε energy of convolutions. This approach is remarkably different from the traditional one based on approximation by affine maps, and can be directly extended to more general functionals and higher dimension, provided that the limiting function is in BV.

Concerning the more general case of Conjecture 1, we remark that nothing is known about fine properties of functions in $AG(\Omega)$. Motivated by [13], in [12] a (possibly smaller) function space $A(\Omega) \subset AG(\Omega)$ was proposed. On this set it is possible to define a functional $\bar{F}$ in such a way that the pair $(A(\Omega), \bar{F})$ has the five properties (a)–(e). Moreover in [12] it was shown that $A(\Omega)$ has fine properties very similar to BV. However one can see that there are $u \in A(\Omega)$ such that, for any convolution kernel φ, $F_\varepsilon(u * \varphi_\varepsilon) \uparrow \infty$. One such u can be obtained by suitably modifying the example of [2]. Therefore this particular u has finite energy and can be approximated in energy by piecewise affine maps with finitely many pieces.

1.3 Rough strategy of the proof

We start by mollifying $u\chi_\Omega$ to get an approximating family $\{u_\varepsilon\}$. Standard estimates give $\limsup_\varepsilon F_\varepsilon[u_\varepsilon, \Omega] \le C$ (see Lemma 4). We build our result upon this construction and this estimate. First, we prove a sharper bound on the energy away from the jump set of ∇u, via Lemma 5. Then, around the 'good part' of the jump set we replace u_ε with the appropriate one-dimensional optimal profile, which gives the optimal energy. The remainder has small energy by the fine properties of the BV function ∇u.

More precisely, in Sect. 3 we define the set $J^g(\bar\theta, k, \eta, \varepsilon)$ of points $x \in J_{\nabla u}$ such that:

- The jump of ∇u at x is at least $\bar\theta$;
- At a scale $k\varepsilon$, ∇u is η–close to a pure jump and $\|D^2 u\|$ is at least of order $k\varepsilon\bar\theta$

(see Definition 3). We denote by $\Omega^g(\bar\theta, k, \eta, \varepsilon)$ the $k\varepsilon$–neighborhood of J^g. In Proposition 1 we show that

$$\lim_{\bar\theta \downarrow 0} \limsup_{k \uparrow \infty} \limsup_{\eta \downarrow 0} \limsup_{\varepsilon \downarrow 0} F_\varepsilon[u_\varepsilon, \Omega \setminus \Omega^g(\bar\theta, k, \eta, \varepsilon)] = 0.$$

Here the quadratic estimate of Lemma 5 plays a fundamental role.

In Sect. 4 we cover Ω^g with balls B_i's of size $k\varepsilon$, in such a way that the number of overlaps is controlled. On each ball we inductively substitute the original gradient with a pure jump and glue in coronas of size ε. After we make all these modifications we mollify the final function at a scale ε. We obtain in this way a smooth $v_{\bar\theta,k,\eta,\varepsilon}$. On most of the set Ω^g this function is a ε–mollification of a "roof" function, on most of the complement it equals u_ε. We suitably define a family of disjoint rectangles $\{R^i_{\bar\theta,k,\eta,\varepsilon}\}_i$ which are centered on the jumps of the roof functions and have vertical size $\sqrt{k}\varepsilon$. In each of these rectangles, $v_{\bar\theta,k,\eta,\varepsilon}$ is the ε–mollification of a *single* roof function. In Proposition 2 we show that the F_ε energy of $v_{\bar\theta,k,\eta,\varepsilon}$ lies mostly in $\cup_i R^i$. In Sect. 5 we take $v_{\bar\theta,k,\eta,\varepsilon}$ and in each R^i we substitute it with a suitable one-dimensional profile, glueing the two functions in a layer of size ε. Denote by $w_{\bar\theta,k,\eta,\varepsilon}$ the final result. Then we prove that, for sets $\Omega^{(\varepsilon)} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$,

$$\limsup_{\bar\theta\downarrow 0}\ \limsup_{k\uparrow\infty}\ \limsup_{\eta\downarrow 0}\ \limsup_{\varepsilon\downarrow 0} F_\varepsilon[w_{\bar\theta,k,\eta,\varepsilon}, \Omega^{(\varepsilon)}] \le F_0[u,\Omega],$$

A standard diagonal argument and an additional construction on the boundary layer yields the desired optimal family. Finally, an interpolation argument permits to enforce the boundary conditions (Sect. 6).

2 Preliminaries and basic estimates

2.1 Preliminaries and notation

We denote by $B_r(x)$ the disk of center $x \in \mathbb{R}^2$ and radius $r > 0$. When $x = 0$ we use B_r in place of $B_r(0)$

We fix a standard mollifier $\varphi \in C^\infty_c(B_1)$ with $\int \varphi = 1$ and for every $\delta > 0$ we denote by φ_δ the function $\varphi_\delta(x) = \frac{1}{\delta^2}\varphi\left(\frac{x}{\delta}\right)$. For any $u \in L^1(\Omega)$ we denote by u_δ the function $(u\chi_\Omega) * \varphi_\delta$. All constants below can depend on the choice of φ.

When μ is a Radon measure and Ω an open set, we denote by $\|\mu\|(\Omega)$ the total variation of μ in Ω. This variant of the Poincaré inequality follows, for instance, from Theorem 4.2.1 of [21].

Lemma 1 (Smooth poincaré inequality) *There exists a constant C, depending only on φ, such that*

$$\left[\int_{B_\delta} \left|v(x) - v_\delta(0)\right|^2 \mathrm{d}x\right]^{1/2} \le C\|Dv\|(B_\delta) \qquad \forall v \in BV(B_\delta). \tag{2.1}$$

For any set $A \subset \mathbb{R}^2$ we denote by A_δ the set $\{x : \operatorname{dist}(x, A) < \delta\}$. The next lemma is a small variant of a well–known covering argument:

Lemma 2 (Covering lemma) *There exists a universal constant N such that the following holds for every $r > 0$. Given a bounded $E \subset \mathbb{R}^2$, there exist N finite collections of balls $\mathcal{F}^j = \{B_r(x_i^j)\}_{i=1,\dots,\ell_j}$ such that*

$$x_i^j \in E \text{ for every } j \text{ and } i\,, \qquad E_{r/2} \subset \bigcup_j \bigcup_i B_r(x_i^j)$$
$$\text{and} \qquad B_{2r}(x_i^j) \cap B_{2r}(x_l^j) = \emptyset \text{ for every } i \neq l.$$

Its proof is based on the geometric observation which is the main ingredient of similar covering theorems:

Lemma 3 *For any $k > 1$ there is $N = N(k)$ such that the following holds. Given any $r > 0$ and any set $A \subset \mathbb{R}^2$ such that the balls $\{B_r(x) : x \in A\}$ are disjoint, one can subdivide A into N sets $\{A_j\}_{j=1,\dots,N}$ such that for each j the balls $\{B_{kr}(x) : x \in A_j\}$ are disjoint.*

Proof Let $N(k)$ be such that a disk of radius $2k + 1$ can contain at most $N(k)$ disjoint balls of radius 1. We claim that this constant satisfies the requirement of the lemma.

Indeed, let $\{B_r(x) : x \in A\}$ be a family of disjoint balls. We construct the sets A_j of the statement by induction. We let A_1 be a maximal subset of A such that the balls $\{B_{kr}(x) : x \in A_1\}$ are disjoint. Analogously, we let A_j be a maximal subset of $A \setminus (A_1 \cup \dots \cup A_{j-1})$ such that $\{B_{kr}(x) : x \in A_j\}$ are disjoint. We claim that after at most $N(k)$ steps the set A is exhausted. Indeed, if after $N(k)$ steps one point $x_0 \in A$ were left, then, by the maximality assumption, for each j there would be $x_j \in A_j$ such that $B_{kr}(x_0) \cap B_{kr}(x_j) \neq \emptyset$. This implies that the $N(k) + 1$ disjoint balls $\{B_r(x_j)\}_{j=0,\dots,N(k)}$ are all contained in $B_{(2k+1)r}(x_0)$, which is a contradiction. □

Proof of Lemma 2 Let $\mathcal{F} = \{\overline{B_{r/2}}(x) : x \in E\}$. By the Besicovitch covering theorem [16, Sect. 1.5.2] the family $\mathcal{F}$ contains N_2^{Bes} disjoint families $\tilde{\mathcal{F}}_j = \{\overline{B_{r/2}}(x) : x \in E_j\}$ which still cover E, where N_2^{Bes} is a geometric constant. This means that

- $E_j \subset E$ for each j;
- $\overline{B_{r/2}}(x) \cap \overline{B_{r/2}}(y) = \emptyset$ for every $x, y \in E_j, x \neq y$;
- $E \subset \cup_j \cup_{x \in E_j} \overline{B_{r/2}}(x)$.

By the triangular inequality, the last condition implies

$$E_{r/2} \subset \bigcup_j \bigcup_{x \in E_j} B_r(x)\,.$$

By Lemma 3, each of the sets E_j can be subdivided into $N(4)$ subsets $\{E_j^h\}_{h=1,\dots,N(4)}$ such that, if $x, y \in E_j^h$ and $x \neq y$, then $B_{2r}(x) \cap B_{2r}(y) = \emptyset$.

This proves the statement of the lemma for $N = N_2^{\mathrm{Bes}} N(4)$. □

2.2 Two estimates for the energy of mollifications

We now provide the two basic estimates on the energy of a mollification of u, that will be used to control the error terms. The first one implies that mollifying the limiting u we obtain a sequence bounded in energy, the second that away from the jump set the energy density converges to zero. For clarity we formulate and prove both lemmas keeping the mollification parameter ε distinct from the parameter δ entering the energy; they will be applied for $\varepsilon = \delta$.

Lemma 4 (Linear estimate) *There exists a universal constant C such that the following holds for every $k \geq 1$. If $u \in W^{1,\infty}(B_{2k\varepsilon})$ with $\nabla u \in BV(B_{2k\varepsilon}, \mathbb{S}^1)$, then*

$$F_\delta[u_\varepsilon, B_{k\varepsilon}] \;\leq\; C\left[\frac{\varepsilon}{\delta} + \frac{\delta}{\varepsilon}\right] \|D^2 u\|(B_{2k\varepsilon}). \tag{2.2}$$

Proof It suffices to prove the Lemma for $k = 1$, and then to cover $B_{k\varepsilon}$ with balls of radius ε. First we estimate

$$\begin{aligned}
\int_{B_\varepsilon} |D^2 u_\varepsilon|^2 &\leq \|D^2 u_\varepsilon\|_{L^\infty(B_\varepsilon)} \|D^2 u_\varepsilon\|_{L^1(B_\varepsilon)} \\
&= \|\nabla u * \nabla(\varphi_\varepsilon)\|_{L^\infty(B_\varepsilon)} \|D^2 u * \varphi_\varepsilon\|_{L^1(B_\varepsilon)} \\
&\leq \|\nabla u\|_{L^\infty(B_{2\varepsilon})} \|\nabla(\varphi_\varepsilon)\|_{L^1} \|\varphi_\varepsilon\|_{L^1} \|D^2 u\|(B_{2\varepsilon}) \\
&\leq \frac{C}{\varepsilon} \|D^2 u\|(B_{2\varepsilon}).
\end{aligned} \tag{2.3}$$

For the second term, since $|\nabla u| = 1$ and $|\nabla u_\varepsilon| \leq 1$, we compute

$$\begin{aligned}
(1 - |\nabla u_\varepsilon|^2)^2 &= \left\langle \nabla u - \nabla u_\varepsilon, \nabla u + \nabla u_\varepsilon \right\rangle^2 \\
&\leq |\nabla u - \nabla u_\varepsilon|^2 |\nabla u + \nabla u_\varepsilon|^2 \;\leq\; 8|\nabla u - \nabla u_\varepsilon|\,.
\end{aligned} \tag{2.4}$$

Therefore

$$\begin{aligned}
\int_{B_\varepsilon} (1 - |\nabla u_\varepsilon|^2)^2 \;&\leq\; 8 \int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(x)|\, \mathrm{d}x \\
&\leq 8 \int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(0)|\, \mathrm{d}x + 8 \int_{B_\varepsilon} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(0)|\, \mathrm{d}x.
\end{aligned} \tag{2.5}$$

Using the inequality (2.1) we conclude

$$\int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(0)|\,\mathrm{d}x \le C\varepsilon \left[\int_{B_\varepsilon} \left|\nabla u(x) - \nabla u_\varepsilon(0)\right|^2 \mathrm{d}x \right]^{1/2}$$
$$\le C\varepsilon \|D^2 u\|(B_\varepsilon). \tag{2.6}$$

Note that

$$\|D^2 u_\varepsilon\|_{L^\infty(B_\varepsilon)} \le \|D^2 u\|(B_{2\varepsilon})\|\varphi_\varepsilon\|_{L^\infty} \le \frac{C}{\varepsilon^2}\|D^2 u\|(B_{2\varepsilon}) \tag{2.7}$$

and therefore

$$\int_{B_\varepsilon} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(0)|\,\mathrm{d}x \le \|D^2 u_\varepsilon\|_\infty \varepsilon \mathscr{L}^2(B_\varepsilon) \le C\varepsilon \|D^2 u\|(B_{2\varepsilon}). \tag{2.8}$$

Putting together (2.3), (2.5), (2.6) and (2.8) we get (2.2). □

Lemma 5 (Quadratic estimate) *There exists a universal constant C such that the following holds for all $k \ge 1$. If $u \in W^{1,\infty}(B_{2k\varepsilon})$ and $\nabla u \in BV(B_{2k\varepsilon}, \mathbb{S}^1)$, then*

$$F_\delta[u_\varepsilon, B_{k\varepsilon}] \le C\left[\frac{1}{\delta} + \frac{\delta}{\varepsilon^2}\right]\left\{\|D^2 u\|(B_{2k\varepsilon})\right\}^2. \tag{2.9}$$

Proof Again, it suffices to prove the Lemma for $k = 1$ and then to use a covering argument. First we estimate from (2.7)

$$\int_{B_\varepsilon} |D^2 u_\varepsilon|^2 \le \mathscr{L}^2(B_\varepsilon)\|D^2 u_\varepsilon\|^2_{L^\infty} \le \frac{C}{\varepsilon^2}\left\{\|D^2 u\|(B_{2\varepsilon})\right\}^2. \tag{2.10}$$

For the second term we use again (2.4) to conclude

$$\int_{B_\varepsilon} (1 - |\nabla u_\varepsilon|^2)^2 \le 4 \int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(x)|^2\,\mathrm{d}x$$
$$\le 8 \int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(0)|^2\,\mathrm{d}x + 8 \int_{B_\varepsilon} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(0)|^2\,\mathrm{d}x. \tag{2.11}$$

Using the inequality (2.1) we estimate

$$\int_{B_\varepsilon} |\nabla u(x) - \nabla u_\varepsilon(0)|^2\,\mathrm{d}x \le C\left\{\|D^2 u\|(B_\varepsilon)\right\}^2. \tag{2.12}$$

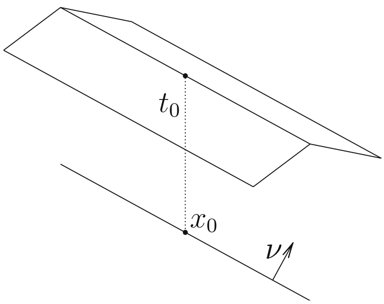

Fig. 1 A roof function t_{θ,ν,x_0,t_0} is continuous, affine on each side of the line through x_0 with normal ν, its gradient jumps by $2\sin\theta$, and its value at x_0 is t_0

From (2.7) we get

$$\int_{B_\varepsilon} |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(0)|^2 \, \mathrm{d}x \leq \|D^2 u_\varepsilon\|_\infty^2 \varepsilon^2 \mathscr{L}^2(B_\varepsilon)$$
$$\leq C\big\{\|D^2 u\|(B_{2\varepsilon})\big\}^2. \tag{2.13}$$

Putting together (2.10), (2.11), (2.12) and (2.13) we get (2.9). □

3 Domain decomposition

Definition 1 (Roof functions) *Given a point $x_0 \in \mathbb{R}^2$, a direction $\nu \in \mathbb{S}^1$, an angle θ and a scalar t_0 we define the roof function*

$$t(x) = t_{\theta,\nu,x_0,t_0}(x) = t_0 + \cos\theta \ (x - x_0) \cdot \nu^\perp + \sin\theta \ |(x - x_0) \cdot \nu|$$

(*see Fig.* 1). *We denote by $\mathcal{J}(x_0)$ the set of all such functions t with a given x_0.*

Definition 2 (Jump points) *Let $u \in W^{1,\infty}(\Omega)$* s.t. *$\nabla u \in BV(\Omega, \mathbb{S}^1)$. Consider the set $J = J_{\nabla u}$ of jump points of ∇u [1, Definition* 3.67*]. Then to each x_0 we associate the roof function $t \in \mathcal{J}(x_0)$ such that $\nu = \nu_u$, $t_0 = u(x_0)$ and $\nabla t^\pm = \nabla u^\pm$ at x_0.*

In our construction we will deal with several positive parameters, involved in the definition of a "good set" (see Definition 3) on which the most complicated part of the construction will later take place:

- ε is the scale of the mollification;
- η denotes the L^1 distance from a single jump;
- $k\varepsilon$ is the scale at which we use the basic estimates of the previous section;
- $\bar{\theta}$ denotes the maximum jump treated as "small jump" via the linear estimate.

Many sets and functions will depend on these parameters, but in order to avoid cumbersome sub and superscripts, we will not make this dependence explicit in our notation. Moreover C will always denote universal constants, which do not depend on any of the parameters but can be different from line to line;

$C_{p_1,\dots,p_j}$. is a constant which depends only on the parameters $p_1, \dots p_j$. Since we will often take the limit

$$\limsup_{\bar\theta\downarrow 0}\ \limsup_{k\uparrow\infty}\ \limsup_{\eta\downarrow 0}\ \limsup_{\varepsilon\downarrow 0},$$

we will denote it by Lim. Further, we define the sets

$$\begin{aligned}\Omega^{(\varepsilon)} &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}\ ,\\ \Omega^* &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 6k\varepsilon\}\ .\end{aligned}$$

The mollification can be used in $\Omega^{(\varepsilon)}$, we shall then perform a more refined construction only in the smaller set Ω^*. The boundary layer $\Omega \setminus \Omega^{(\varepsilon)}$ will be treated in Sect. 6.

Definition 3 *We define J^g as the set of points $x_0 \in J_{\nabla u} \cap \Omega^*$ such that:*

(g1) *The roof function t associated to x_0 satisfies $|\sin\theta| \geq \sin\bar\theta$,*

$$\|D^2u\|(B_{2k\varepsilon}(x_0)) \geq k\varepsilon \sin\bar\theta\,, \tag{3.1}$$

and

$$\frac{1}{|B_{2k\varepsilon}|}\int\limits_{B_{2k\varepsilon}(x_0)} |\nabla u - \nabla t| + \frac{|u-t|}{\varepsilon} \leq \eta. \tag{3.2}$$

(g2) *For the finitely many balls $B_\varepsilon(y) \subset B_{2k\varepsilon}(x_0)$ with $y\cdot\nu = x_0\cdot\nu$ and $(y-x_0)\cdot\nu^\perp \in 2\varepsilon\mathbb{Z}$ (see Fig. 2), one has*

$$\int\limits_{B_\varepsilon(y)\cap J_{\nabla u}} |[\nabla u]|^3\, d\mathscr{H}^1 \geq |2\sin\theta|^3 2\varepsilon - \eta\varepsilon\,. \tag{3.3}$$

We set $\Omega^g := J^g_{k\varepsilon/2} = \{x \in \Omega : \operatorname{dist}(x, J^g) < k\varepsilon/2\}$.

Proposition 1 (Domain decomposition) *Let $\Omega \subset\subset \mathbb{R}^2$ and let $u \in W^{1,\infty}(\Omega)$ be such that $\nabla u \in BV(\Omega, \mathbb{S}^1)$. Then*

$$\operatorname{Lim} F_\varepsilon[u_\varepsilon, \Omega^{(\varepsilon)} \setminus \Omega^g] = 0\,.$$

Proof We cover the domain with four sets, which are treated separately. Fix some $\alpha < \bar\theta$, to be chosen later. The first set is given by the points where D^2u scales less than linearly with the diameter. Therefore it contains most of the measure of Ω, but almost no energy:

$$\Omega^{1,\alpha} := \left\{x \in \Omega^* \ :\ \|D^2u\|(B_{2k\varepsilon}(x)) \leq 4\alpha k\varepsilon\right\}.$$

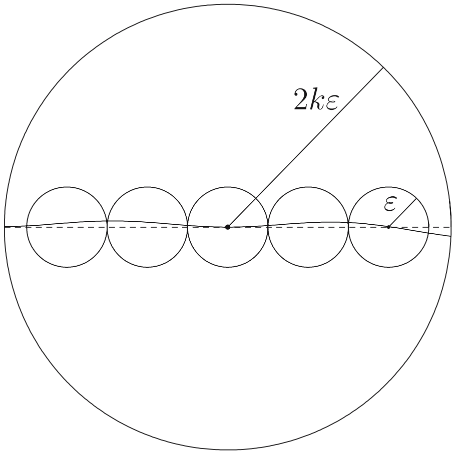

Fig. 2 Sketch of the geometry in Definition 3. The estimate (3.2) states that on the large ball $B_{2k\varepsilon}$ the function u is close to a roof function. The condition (3.3) states that for all balls of radius ε centered on the jump set of the roof function (*dashed line*), the amount of energy contained in the jump set of u (*full curve*) is not significantly smaller than the one of the corresponding roof function

In this set we shall use the quadratic estimate to show that

$$F_\varepsilon[u_\varepsilon, \Omega^{1,\alpha}] \le C\alpha k \|D^2u\|(\Omega). \tag{3.4}$$

The second set contains the points where D^2u scales linearly but with a small coefficient,

$$\Omega^{2,\alpha} := \big\{x \in \Omega^* \ : \ \|D^2u\|(B_{2k\varepsilon}(x)) \ge 4\alpha k\varepsilon \quad \text{and} \quad \|D^2u\|(B_{4k\varepsilon}(x)) \le 32k\varepsilon \sin\bar\theta\big\}.$$

Here we shall use the linear estimate to prove the existence of a function h such that

$$\lim_{\bar\theta\downarrow 0} h(\bar\theta) = 0 \tag{3.5}$$

and

$$\limsup_{\alpha\downarrow 0}\ \limsup_{\varepsilon\downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^{2,\alpha}] \le h(\bar\theta). \tag{3.6}$$

The third set, given by

$$\Omega^3 := \big\{x \in \Omega \setminus \Omega^g : \ \mathrm{dist}\,(x, \partial\Omega) > 7k\varepsilon\,, \|D^2u\|(B_{4k\varepsilon}(x)) > 32k\varepsilon \sin\bar\theta\big\},$$

contains the points where the energy concentrates, but which are sufficiently distant from J^g. Using the fine properties of BV functions we shall show that this set has small energy. More precisely we will prove that

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^3] = 0. \tag{3.7}$$

Finally, the set $\Omega^4 := \{x \in \Omega : \varepsilon < \operatorname{dist}(x, \partial\Omega) < 7k\varepsilon\}$ has small energy, in the sense that

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^4] = 0. \tag{3.8}$$

Note that $\Omega^{(\varepsilon)} \setminus \Omega^g \subset \Omega^{1,\alpha} \cup \Omega^{2,\alpha} \cup \Omega^3 \cup \Omega^4$. Therefore from (3.4), (3.7) and (3.8) we get

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^{(\varepsilon)} \setminus \Omega^g] \le C\alpha k\|D^2u\|(\Omega) + \limsup_{\varepsilon\downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^{2,\alpha}].$$

By (3.6) we can pick $\alpha = \alpha(\bar\theta, k, \eta)$ so that this expression is bounded by $\bar\theta + h(\bar\theta)$. We conclude that

$$\operatorname{Lim} F_\varepsilon[u_\varepsilon, \Omega^{(\varepsilon)} \setminus \Omega^g] \le \lim_{\bar\theta\downarrow 0} h(\bar\theta) \overset{(3.5)}{=} 0.$$

We now proceed to prove the four claims.

Proof of (3.4) In order to simplify the notation we write Ω^1 for $\Omega^{1,\alpha}$ and we set $\mu := \|D^2u\|$. From Lemma 2 we get N families of disjoint balls $\mathcal{F}^j = \{B_{2k\varepsilon}(x_i^j)\}_i$ such that $x_i^j \in \Omega^1$ and $\Omega^1 \subset \bigcup_{i,j} B_{k\varepsilon}(x_i^j)$. Using (2.9) we can write

$$\begin{aligned} F_\varepsilon[u_\varepsilon, \Omega^1] &\le \sum_{i,j} F_\varepsilon[u_\varepsilon, B_{k\varepsilon}(x_i^j)] \le \sum_{i,j} C\frac{1}{\varepsilon}\left[\mu(B_{2k\varepsilon}(x_i^j)\right]^2 \\ &\le C\alpha k \sum_{i,j} \mu(B_{2k\varepsilon}(x_i^j)) \le C\alpha k \sum_{j=1}^N \left[\sum_i \mu(B_{2k\varepsilon}(x_i^j))\right] \le CN\alpha k\mu(\Omega). \end{aligned}$$

Proof of (3.5) *and* (3.6) To simplify the notation we write Ω^2 for $\Omega^{2,\alpha}$. Arguing as in the previous step, via the linear estimate (2.2) we get

$$F_\varepsilon[u_\varepsilon, \Omega^2] \le C\mu(\Omega^2_{2k\varepsilon}).$$

Recall that for μ–a.e. $x \in \Omega$ the limit

$$\phi(x) := \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{2r}$$

exists and is finite. Indeed, $\mu = \mu^1 + |\nabla u^+ - \nabla u^-|\mathcal{H}^1 \llcorner J_{\nabla u}$, where $\mu^1(E) = 0$ for any set E such that $\mathcal{H}^1(E) < \infty$. For μ-a.e. $x \notin J_{\nabla u}$, one has $\lim \mu(B_r(x))/r = \lim \mu^1(B_r(x))/r = 0$. For μ-a.e. $x \in J_{\nabla u}$, one has $\lim \mu(B_r(x))/2r = |\nabla u^+ - \nabla u^-|$ because $J_{\nabla u}$ is rectifiable [1, Sect. 3.9].

Denote by A^α the set $\{x \in \Omega^* : \alpha/4 \le \phi(x) \le 16 \sin\bar\theta\}$. We claim that

$$\limsup_{\varepsilon\downarrow 0} \mu(\Omega^2_{2k\varepsilon}) \ \le\ \limsup_{\varepsilon\downarrow 0} \mu(\Omega^2_{2k\varepsilon} \cap A^\alpha)\,. \tag{3.9}$$

Then, since obviously $\mu(\Omega^2_{2k\varepsilon} \cap A^\alpha) \le \mu(A^\alpha)$ and, by the properties of μ,

$$\lim_{\bar\theta\downarrow 0}\ \limsup_{\alpha\downarrow 0}\ \mu(A^\alpha) \ =\ 0\,,$$

the claim (3.9) would conclude the proof of (3.6).

To prove (3.9), fix any $\gamma > 0$. Then there exist $r_0 > 0$ and a set K such that $\mu(\Omega \setminus (A^\alpha \cup K)) < \gamma$ and for any $x \in K$ we have either

$$\frac{\mu(B_r(x))}{2r} \ \le\ \frac{\alpha}{3} \quad \text{for every } r < r_0$$

or

$$\frac{\mu(B_r(x))}{2r} \ \ge\ 12\sin\bar\theta \quad \text{for every } r < r_0.$$

We claim that

$$K \cap \Omega^2_{2k\varepsilon} \ =\ \emptyset \quad \text{for every } \varepsilon < r_0/4k. \tag{3.10}$$

This would imply

$$\limsup_{\varepsilon\downarrow 0} \mu(\Omega^2_{2k\varepsilon} \setminus A^\alpha) \ \le\ \gamma$$

and the arbitrariness of γ would give (3.9).

To prove (3.10) we argue by contradiction. If it were false, there would be $x \in K$ and $y \in \Omega^2$ such that $|x - y| < 2\varepsilon k$. Then either

$$4\alpha k\varepsilon \ \le\ \mu(B_{2k\varepsilon}(y)) \ \le\ \mu(B_{4k\varepsilon}(x)) \ \le\ \frac{8}{3}\alpha k\varepsilon$$

or

$$48k\varepsilon \sin\bar\theta \ \le\ \mu(B_{2k\varepsilon})(x) \ \le\ \mu(B_{4k\varepsilon}(y)) \ \le\ 32k\varepsilon\sin\bar\theta.$$

Both cases would lead to a contradiction. This proves (3.10).

Proof of (3.7) We repeat the covering argument of the first estimate, using the linear estimate (2.2) and covering with balls of radius $B_{k\varepsilon/4}$. We conclude

$$F_\varepsilon[u_\varepsilon, \Omega^3] \ \le\ C\mu(\Omega^3_{k\varepsilon/2}). \tag{3.11}$$

For any point x_0 consider the family of blow-ups $v_{r,x_0}(x) = r^{-1}u(x_0 + rx)$: $B_1 \to \mathbb{R}$.

Let A be the set of $x_0 \in \Omega$ with the following properties:

- There are $\nu \in \mathbb{S}^1$ and $\theta \in \mathbb{R}$ such that

$$\lim_{r\downarrow 0} \|D^2 v_r\|(B_1) = 4|\sin\theta|\,, \quad \lim_{r\downarrow 0} \int_{B_1} |\nabla v_r - \nabla t| + |v_r - t| = 0$$

  where $t = t_{\theta,\nu,x_0,t_0}$ is a fixed roof function, with $t_0 = u(x_0)$.
- Further,

$$\lim_{r\downarrow 0} \int_{B_1\cap J_{\nabla v_r}} |[\nabla v_r] - 2\sin\theta\,\nu|\, d\mathscr{H}^1 = 0$$

  and, for every fixed open set $\omega \subset B_1$,

$$\lim_{r\downarrow 0} \left| \mathscr{H}^1(\omega \cap J_{\nabla v_r}) - \mathscr{H}^1(\omega \cap J_{\nabla t}) \right| = 0\,.$$

From the fine properties of BV functions, we know that $\mu(\Omega \setminus A) = 0$ (for the first property, see [1, Proposition 3.69], for the second it follows from the rectifiability of $J_{\nabla u}$). Moreover, $x_0 \in A$ is in $J_{\nabla u}$ if and only if $\sin\theta \neq 0$; in this case the roof function above corresponds precisely to the roof function associated to x_0 in Definition 2 (although θ and ν are not unique).

A standard measure theoretic argument shows that those properties hold uniformly, up to an arbitrarily small error set. More precisely, let the three parameters $\bar{\theta}$, k and η be fixed. For any positive γ and $\tilde{\eta}$ there exist $\varepsilon_0 > 0$ and $K \subset A$ such that:

- $\mu(\Omega \setminus K) \leq \gamma$;
- For every $x \in K$ and for every $\varepsilon < \varepsilon_0$ the following estimates hold, with the parameters corresponding to x:

$$\|D^2u\|(B_{5k\varepsilon}(x)) \leq 20k\varepsilon|\sin\theta| + \tilde{\eta}\varepsilon\,, \tag{3.12}$$

$$\|D^2u\|(B_{2k\varepsilon}(x)) \geq 8k\varepsilon|\sin\theta| - \tilde{\eta}\varepsilon\,, \tag{3.13}$$

$$\frac{1}{|B_{2k\varepsilon}|} \int_{B_{2k\varepsilon}(x)} |\nabla u - \nabla t| + \frac{|u-t|}{\varepsilon} \leq \tilde{\eta}\,, \tag{3.14}$$

$$\int_{B_\varepsilon(y)\cap J_{\nabla u}} |[\nabla u]|^3 d\mathscr{H}^1 \geq |2\sin\theta|^3 2\varepsilon - \tilde{\eta}\varepsilon\,, \tag{3.15}$$

  where the balls $B_\varepsilon(y)$ are the ones from Condition (g2) in Definition 3.

We now claim that

$$\text{If we choose } \tilde{\eta} \text{ sufficiently small then } K \cap \Omega^3_{k\varepsilon/2} = \emptyset. \tag{3.16}$$

The choice of $\tilde{\eta}$ will depend on $\bar{\theta}$, k and η. With these kept fixed, (3.16) and (3.11) give

$$\limsup_{\varepsilon \downarrow 0} F_\varepsilon[u_\varepsilon, \Omega^3] \leq C\mu(\Omega^3_{k\varepsilon/2}) \leq C\mu(\Omega \setminus K) \leq C\gamma.$$

Since γ is arbitrary and C is a universal constant, this would conclude the proof.

We now prove (3.16). Assume $x \in K \cap \Omega^3_{k\varepsilon/2}$. Then there is $y \in \Omega^3$ such that $|y-x| \leq k\varepsilon/2$. Therefore the ball $B_{5k\varepsilon}(x)$ contains the ball $B_{4k\varepsilon}(y)$, and $x \in \Omega^*$. From the definition of Ω^3 and (3.12) we conclude

$$20k\varepsilon \,|\sin\theta| \geq 32k\varepsilon \sin\bar{\theta} - \tilde{\eta}\varepsilon.$$

Clearly, if $\tilde{\eta}$ is sufficiently small, then $|\sin\theta| \geq \sin\bar{\theta} > 0$. Therefore $x \in J_{\nabla u}$. Moreover, if $\tilde{\eta}$ is small compared to η and $\bar{\theta}$, (3.13) and (3.14) imply (g1) and (3.15) imply (g2) in Definition 3. Hence we conclude that $x \in J^g$. But this is not possible because $\Omega^3_{k\varepsilon/2} \cap J^g = \emptyset$.

Proof of (3.8) We cover Ω^4 with N disjoint families of balls of radius ε. Applying the linear estimate (2.2) (with $k = 1$) to each of them we have

$$F_\varepsilon[u_\varepsilon, \Omega^4] \leq CN\mu\left(\{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) < (7k+1)\varepsilon\}\right).$$

Taking the limit $\varepsilon \to 0$ we obtain (3.8). □

4 Interior construction: the intermediate family

In this section we construct a function which coincides with a smoothed roof function around most of J^g, and with u_ε away from it. We first briefly describe the geometry, sketched in Fig. 3. We cover Ω^g with balls of radius $k\varepsilon$, and have good estimates on the larger balls of radius $2k\varepsilon$. We first interpolate between the original function u and a roof function on the coronas $B_{k\varepsilon}(x_i^j) \setminus B_{(k-1)\varepsilon}(x_i^j)$, and – after *all* interpolations have been performed – mollify on a scale ε (Definition 4). Along the interface we obtain a finite number of rectangles R_i^j such that the function coincides with a mollified roof on each of them (Definition 5). We then prove (Proposition 2) that the energy outside the rectangles is negligible; in the next section we shall then produce an appropriate modification of the construction inside the rectangles.

Let $\mathcal{F}^j = \{B_{2k\varepsilon}(x_i^j)\}_i$ $(1 \leq j \leq N)$ be N families of disjoint balls such that $x_i^j \in J^g$ and the $B_{k\varepsilon}(x_i^j)$ cover Ω^g. These families exist by Lemma 2. For every

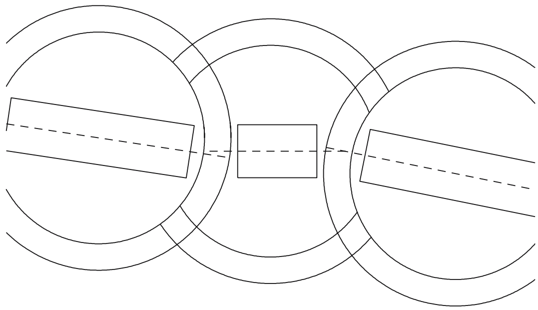

Fig. 3 Sketch of the construction in Definition 4, for a case with $N = 2$. The central ball belongs to the family $\mathcal{F}_1$, the two outer ones to $\mathcal{F}_2$. The interpolation is done in the coronas, the *rectangles* are disjoint and contained in the part where v coincides with a mollified roof function. The *dashed lines* represent the interfaces of the roof functions

k fix a cutoff function $\psi \in C_c^\infty(B_k)$ such that $\psi = 1$ on B_{k-1}. Then for every $\varepsilon > 0$ we define $\psi^\varepsilon \in C_c^\infty(B_{k\varepsilon})$ as $\psi^\varepsilon(x) = \psi(\frac{x}{\varepsilon})$.

Definition 4 (Intermediate family of functions) *We set $v^0 = u$ and inductively define $\{v^j\}_{j=1,\dots,N}$ as follows. At the j–th step we consider the family of balls $\mathcal{F}^j$, and set*

$$v^j(x) := \begin{cases} (1 - \psi^\varepsilon(x - x_i^j))v^{j-1}(x) + \psi^\varepsilon(x - x_i^j)t_i^j(x) & \text{if } x \in B_{k\varepsilon}(x_i^j) \\ & \text{for some } i, \\ v^{j-1}(x) & \text{otherwise.} \end{cases}$$

Here t_i^j is the roof function associated to x_i^j.

Finally we set $v := v^N * \varphi_\varepsilon$.

Note that v depends on the covering, hence on J^g, and therefore on $\varepsilon, \eta, \bar{\theta}$ and k.

Definition 5 *For each i and j, consider the family of rectangles of the form*

$$\left\{x : |(x - x_i^j) \cdot \nu| < \sqrt{k}\varepsilon\,, \quad a < (x - x_i^j) \cdot \nu^\perp < b\right\} \quad \text{where } a, b \in \mathbb{R},$$

and $\nu = \nu_j^i$ is the normal associated to x_i^j.

Let R_i^j be the largest of these rectangles among the ones contained in the ball $B_{(k-2)\varepsilon}(x_i^j)$ and which do not intersect any ball $B_{(k+1)\varepsilon}(x_{i'}^{j'})$ with $j' > j$.

Uniqueness of R_i^j follows from the argument at the end of the proof of Proposition 2. Further, by definition $\operatorname{dist}(R_i^j, R_{i'}^{j'}) > 3\varepsilon$.

Proposition 2 (Energy concentration) *Let v be as in Definition 4 and R_i^j as in Definition 5. Then*

$$\operatorname{Lim} F_\varepsilon\left[v, \Omega^{(\varepsilon)} \setminus \cup_{ij} R_i^j\right] = 0. \tag{4.1}$$

Moreover on each R_i^j one has $v = \varphi_\varepsilon * t_i^j$ and

$$\operatorname{Lim} \frac{1}{3} \sum_{ij} \int_{R_i^j \cap J_{\nabla t_i^j}} |[\nabla t_i^j]|^3 \mathrm{d}\mathscr{H}^1 \leq F_0[u, \Omega]\,. \tag{4.2}$$

Proof We first prove (4.2). Fix an R_i^j and denote by $\{B_{ijl}\}_{l=1,\ldots,N_{ij}}$ the balls of the type considered in point (g2) of Definition 3 which intersect R_i^j. Let ℓ_{ij} denote the length of R_i^j in direction $\nu^\perp$. Then

$$\int_{R_i^j \cap J_{\nabla t_i^j}} |[\nabla t_i^j]|^3 \mathrm{d}\mathscr{H}^1 \;=\; \ell_{ij} |2 \sin\theta|^3 \;\leq\; N_{ij} 2\varepsilon |2\sin\theta|^3.$$

From (g2) in Definition 3 we have

$$2\varepsilon |2\sin\theta|^3 \;\leq\; \int_{B_{ijl} \cap J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1 + \eta\varepsilon.$$

Using first (3.1) in Definition 3, then that the balls B_{ijl} are disjoint, we have

$$\begin{aligned}
\sum_{ij} \int_{R_i^j \cap J_{\nabla t_i^j}} |[\nabla t_i^j]|^3 \mathrm{d}\mathscr{H}^1 \;&\leq\; \sum_{ijl} \int_{B_{ijl} \cap J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1 + 2k\eta\varepsilon \\
&\leq \sum_{ijl} \int_{B_{ijl} \cap J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1 + \sum_{ij} \frac{2\eta}{\sin\bar\theta} \|D^2 u\|(B_{2k\varepsilon}(x_i^j)) \\
&\leq \int_{\Omega \cap J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1 + \frac{2N\eta}{\sin\bar\theta} \|D^2 u\|(\Omega).
\end{aligned}$$

We conclude that

$$\operatorname{Lim} \sum_{ij} \int_{R_i^j \cap J_{\nabla t_i^j}} |[\nabla t_i^j]|^3 \mathrm{d}\mathscr{H}^1 \;\leq\; \int_{\Omega \cap J_{\nabla u}} |[\nabla u]|^3 \mathrm{d}\mathscr{H}^1\,,$$

which gives (4.2).

We now pass to (4.1) and we first note that $\Omega^g \subset \bigcup_{ij} B_{(k+1)\varepsilon}(x_i^j)$ and $v = u_\varepsilon$ on $\Omega^{(\varepsilon)} \setminus \bigcup_{ij} B_{(k+1)\varepsilon}(x_i^j)$. Therefore Proposition 1 implies

$$\operatorname{Lim} F_\varepsilon \left[v, \Omega^{(\varepsilon)} \setminus \bigcup_{ij} B_{(k+1)\varepsilon}(x_i^j) \right] \;\le\; \operatorname{Lim} F_\varepsilon \left[u_\varepsilon, \Omega^{(\varepsilon)} \setminus \Omega^g \right] \;=\; 0.$$

This estimate allows to reduce the proof of (4.1) to the following two identities:

$$\operatorname{Lim} F_\varepsilon \left[v, \bigcup_{i,j} B_{(k+1)\varepsilon}(x_i^j) \setminus B_{(k-2)\varepsilon}(x_i^j) \right] = 0\,, \tag{4.3}$$

and

$$\operatorname{Lim} F_\varepsilon \left[v, \bigcup_{i,j} B_{(k-2)\varepsilon}(x_i^j) \setminus R_i^j \right] = 0\,. \tag{4.4}$$

Proof of (4.3) We first claim that for $J = 0, 1, ...N$, the function v^J obeys

$$\frac{1}{|B_{2k\varepsilon}|} \int\limits_{B_{2k\varepsilon}(x_i^j)} |\nabla v^J - \nabla t_i^j| + \frac{|v^J - t_i^j|}{\varepsilon} \le C_J \eta \quad \forall i, j. \tag{4.5}$$

Notice that we require the control on all balls, not only on those of the J-th family, and that we allow the constant to depend on J. This can be proved by induction. For $J = 0$, (4.5) follows from (3.2) and the fact that $v^0 = u$. At step J, for each $x \in B_{k\varepsilon}(x_i^J)$ we have

$$|v^J - v^{J-1}|(x) \le |v^{J-1} - t_i^J|(x)$$

and

$$|\nabla v^J - \nabla v^{J-1}|(x) \le |\nabla v^{J-1} - \nabla t_i^J|(x) + \frac{C}{\varepsilon} |v^{J-1} - t_i^J|(x)\,.$$

The balls of the J-th family are disjoint, hence for those the result is clear. Consider now a generic ball $B_{2k\varepsilon}(x_i^j)$. Since all balls have the same size, and each family is disjoint, it can intersect at most M (which is a universal constant) of the balls of the family J. Therefore,

$$\begin{aligned} \frac{1}{|B_{2k\varepsilon}|} \int\limits_{B_{2k\varepsilon}(x_i^j)} |v^J - v^{J-1}| &= \sum_{i'} \frac{1}{|B_{2k\varepsilon}|} \int\limits_{B_{2k\varepsilon}(x_i^j) \cap B_{k\varepsilon}(x_{i'}^J)} |v^J - v^{J-1}| \\ &\le M C_{J-1} \eta\,, \end{aligned}$$

and the same for the gradient. The triangular inequality concludes the proof of (4.5).

Next we claim that (4.5) implies

$$F_\varepsilon[v_\varepsilon^j, B_{(k+1)\varepsilon}(x_i^j) \setminus B_{(k-2)\varepsilon}(x_i^j)] \le C\varepsilon + C_k\eta\varepsilon\,, \tag{4.6}$$

where as usual $v_\varepsilon^j = \varphi_\varepsilon * v^j$, and C_k can only depend on k. For the moment let us assume the claim, which will be proved below. Attributing each x in the union of the $B_{(k+1)\varepsilon} \setminus B_{(k-2)\varepsilon}$ to the level j where $v(x)$ was last modified, i.e. to the largest j such that $x \in B_{(k+1)\varepsilon}(x_i^j)$, we get

$$\begin{aligned} &F_\varepsilon\left[v, \bigcup_{i,j} B_{(k+1)\varepsilon} \setminus B_{(k-2)\varepsilon}\right] \\ &\quad \le \sum_j F_\varepsilon\left[v_\varepsilon^j, \bigcup_i B_{(k+1)\varepsilon}(x_i^j) \setminus B_{(k-2)\varepsilon}(x_i^j)\right] \le \sum_{ij} C\varepsilon + C_k\eta\varepsilon\,. \end{aligned}$$

Using (3.1) we obtain, for $C_{\bar\theta} = C/\sin\bar\theta$,

$$\begin{aligned} F_\varepsilon\left[v, \bigcup_{i,j} B_{(k+1)\varepsilon} \setminus B_{(k-2)\varepsilon}\right] &\le \sum_{ij}\left(\frac{C_{\bar\theta}}{k} + C_{\bar\theta}C_k\eta\right)\|D^2u\|(B_{2k\varepsilon}(x_i^j)) \\ &\le N\left(\frac{C_{\bar\theta}}{k} + C_{\bar\theta}C_k\eta\right)\|D^2u\|(\Omega)\,. \end{aligned}$$

This bound is uniform in ε. Therefore

$$\lim_{k\uparrow\infty}\ \limsup_{\eta\downarrow 0}\ \limsup_{\varepsilon\downarrow 0} F_\varepsilon\left[v, \bigcup_{i,j} B_{(k+1)\varepsilon} \setminus B_{(k-2)\varepsilon}\right] \le N\lim_{k\uparrow\infty}\frac{C_{\bar\theta}}{k}\|D^2u\|(\Omega) \ = \ 0.$$

To complete the proof of (4.3) it remains to prove the claim (4.6). After scaling and translating we conclude that it suffices to prove that (4.5) implies (4.6) when $\varepsilon = 1$ and $x_i^j = 0$. Then it is clear that (4.5) gives

$$\|\nabla v_1^j - \nabla(t_i^j)_1\|_{L^1(B_{k+1})} + \|\nabla^2 v_1^j - \nabla^2(t_i^j)_1\|_{L^2(B_{k+1})} \le C_k\eta\,,$$

where the constant can depend on k (the dependence on j can be removed, taking the maximum between the finitely many C_j's). Finally,

$$F_1[v_1^j, B_{k+1} \setminus B_{k-2}] \le F_1[(t_i^j)_1, B_{k+1} \setminus B_{k-2}] + C_k\eta \le C + C_k\eta\,,$$

which is equivalent to (4.6).

Proof of (4.4) Roughly speaking (4.4) follows from the fact that the rectangles cover most of the interface, i.e. that at most a length of order ε is lost in any ball. In turn, this follows from the fact that overlapping balls have roof functions which are η-close.

In each ball $B_{(k-2)\varepsilon}(x_i^j)$ we consider the set

$$E_i^j = B_{(k-2)\varepsilon}(x_i^j) \setminus R_i^j \setminus \bigcup_{j'>j} B_{(k+1)\varepsilon}\left(x_{i'}^{j'}\right).$$

Note that in each $(E_i^j)_\varepsilon$ we have $v^N = t_i^j$, and correspondingly in each E_i^j we have $v = \varphi_\varepsilon * t_i^j$. By (4.3) it suffices to show that

$$\operatorname{Lim} F_\varepsilon[v, \cup_{i,j} E_i^j] = 0\,.$$

In order to achieve our goal we claim that

(Cl) For any fixed $\bar\theta > 0$ and $k > 2$, there exist positive η_0 and ε_0 such that

$$\left\| E_i^j \cap \left\{ |(x - x_i^j) \cdot \nu_i^j| \le \varepsilon \right\} \right\| \le C\varepsilon^2 \quad \text{for all } i \text{ and } j, \tag{4.7}$$

whenever $\varepsilon < \varepsilon_0$ and $\eta < \eta_0$.

For the moment we assume (Cl), which will be proved later. Since $\|\nabla(\varphi_\varepsilon * t_i^j)\| \le 1$ and $\|D^2(\varphi_\varepsilon * t_i^j)\| \le C/\varepsilon$, from (4.7) and (3.1) we would get

$$F_\varepsilon[v, E_i^j] \le C\varepsilon \le \frac{C}{k \sin\bar\theta} \|D^2 u\|(B_{2k\varepsilon}(x_i^j)) \quad \forall i, j, \forall \varepsilon < \varepsilon_0, \forall \eta < \eta_0. \tag{4.8}$$

Summing all the contributions and taking into account that the families $\mathcal{F}^j$ are formed by disjoint balls, from (4.8) we conclude

$$\limsup_{\eta\downarrow 0}\ \limsup_{\varepsilon\downarrow 0}\ F_\varepsilon[v, \cup_{ij} E_i^j] \le \frac{CN}{k\sin\bar\theta}\|D^2 u\|(\Omega)$$

and hence we get (4.4).

It remains to prove (Cl). Without loss of generality we can assume $x = x_i^j = 0$ and $\nu = e_2$. Moreover for simplicity we drop the indices i and j.

The maximality of R implies that there is a point on the left side (and a point on the right side) which is either in $\partial B_{(k-2)\varepsilon}$ or in $\partial B_{(k+1)\varepsilon}(x_{i'}^{j'})$, for some $j' > j$. In the first case the result is obvious (see Fig. 4). In the second case, it follows from the fact that the center $x' = x_{i'}^{j'}$ of the other ball is close to the horizontal axis. Indeed, if we denote by t and t' the two roof functions corresponding to

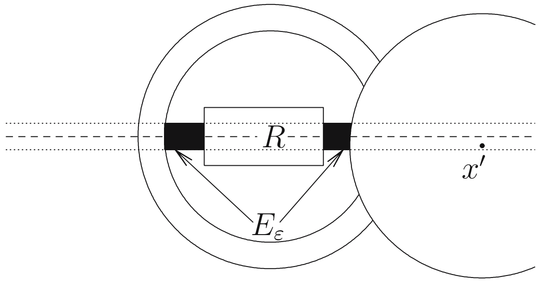

Fig. 4 Set $E_\varepsilon = E \cap \{|x_2| \le \varepsilon\}$ (*black*) entering (4.7) in the proof of Proposition 2. The *dashed line* is the jump set of ∇t, the *dotted lines* delimit the region where x' needs to lie. It follows that the part of $\partial B'$ delimiting R has to be approximately vertical. For clarity one rectangle smaller than R is illustrated, the optimal one touches both left and right boundaries

$x = 0$ and x', by (3.2) we have

$$\frac{1}{|B_{2k\varepsilon}|} \int_{B_{2k\varepsilon} \cap B_{2k\varepsilon}(x')} |\nabla t - \nabla t'| + \frac{|t - t'|}{\varepsilon} \le 2\eta \,.$$

At the same time, $|B_{2k\varepsilon} \cap B_{2k\varepsilon}(x')| \ge C|B_{2k\varepsilon}|$, for some universal constant C, because $B_{(k+1)\varepsilon} \cap B_{(k+1)\varepsilon}(x') \ne \emptyset$ and $k \ge 2$. Since both $|\sin\theta|$ and $|\sin\theta'|$ are bounded from below by $\sin\bar\theta$, the distance of x' from the horizontal axis (which is the jump set of ∇t) is controlled by $C_{k,\bar\theta}\eta\varepsilon$. Choosing η such that $C_{k,\bar\theta}\eta < 1$ we get (Cl). $\square$

5 Interior construction: the optimal family

We complete the construction of Sect. 4 by modifying it inside the "good" rectangles, and obtain the recovery sequence (in a subdomain). We prove

Proposition 3 *Let $\Omega \subset \mathbb{R}^2$ be a C^2 bounded domain, and $u \in W^{1,\infty}(\Omega)$ with $\nabla u \in BV(\Omega, \mathbb{S}^1)$. Then there is a family $\{u^\varepsilon\} \subset C^\infty(\overline{\Omega})$ such that $u^\varepsilon \to u$ in $W^{1,p}(\Omega)$ for every $p < \infty$, $u^\varepsilon = u_\varepsilon$ in a neighbourhood of $\partial\Omega^{(\varepsilon)}$, and*

$$\limsup_{\varepsilon \downarrow 0} F_\varepsilon[u^\varepsilon, \Omega^{(\varepsilon)}] \le F_0[u, \Omega] \,.$$

Proof Consider the function v of Definition 4 and the decomposition of the domain obtained in Definition 5. We claim that for each rectangle R_i^j we can find a smooth w such that $v = w$ outside R_i^j and

$$F_\varepsilon[w, R_i^j] \le \ell_{ij} \frac{1}{3} |2\sin\theta|^3 + C\varepsilon \left(\frac{1}{\bar\theta} + k e^{-\bar\theta\sqrt{k}} \right) , \tag{5.1}$$

where θ is the angle of the roof function t_i^j, and $\ell_{ij} = \mathscr{H}^1(J_{\nabla v^N} \cap R_i^j)$ the length of the rectangle. We assume for the moment the claim, which will be proved below.

We repeat the construction in each of the (disjoint) rectangles; let w be the result. Then

$$F_\varepsilon\left[w, R_i^j\right] \leq \frac{1}{3}|2\sin\theta|^3\mathscr{H}^1(J_{\nabla v^N}\cap R_i^j) + \frac{C\varepsilon}{\bar\theta} + C\varepsilon k e^{-\bar\theta\sqrt{k}}$$
$$\leq \frac{1}{3}\int\limits_{J_{\nabla u}\cap R_i^j} |[\nabla u]|^3 d\mathscr{H}^1 + C\varepsilon + k\varepsilon\eta + \frac{C\varepsilon}{\bar\theta} + C\varepsilon k e^{-\bar\theta\sqrt{k}}\,,$$

where we estimated the first term via the integral of the jump of ∇u using (3.3). (The rectangle contains up to k disjoint balls of the type appearing in (3.3), the sum of their diameters is at least $\mathscr{H}^1(J_{\nabla v_N}\cap R_i^j) - 4\varepsilon$.)

Summing over all rectangles and using (3.1) we get

$$F_\varepsilon\left[w, \cup_{ij}R_i^j\right] \leq \frac{1}{3}\int\limits_{J_{\nabla u}} |[\nabla u]|^3\, \mathrm{d}\mathscr{H}^1 + \sum_{i,j}\left(\frac{C_{\bar\theta}}{k} + C_{\bar\theta}\eta + C\mathrm{e}^{-\bar\theta\sqrt{k}}\right)\|D^2u\|(B_{2k\varepsilon}(x_i^j))$$
$$\leq F_0[u,\Omega] + \frac{C_{\bar\theta}}{k}\|D^2u\|(\Omega) + C_{\bar\theta}\eta\|D^2u\|(\Omega) + C\mathrm{e}^{-\bar\theta\sqrt{k}}\|D^2u\|(\Omega). \tag{5.2}$$

Taking the Lim, we send first $\varepsilon\to 0$, then $\eta\to 0$, then $k\to\infty$. Combining (5.2) with Proposition 2 we get

$$\mathrm{Lim}\, F_\varepsilon[w, \Omega^{(\varepsilon)}] \leq F_0[u,\Omega]\,.$$

A standard diagonal argument concludes the proof.

It remains to prove (5.1). By scaling and translating it suffices to consider the case $\varepsilon = 1$, $t(x) = \cos\theta\, x_1 + \sin\theta\, |x_2|$, and

$$R = (-a, a)\times(-\sqrt{k}, \sqrt{k})\,.$$

If $a \leq 2$, simply $w = v$ will do, since $F_\varepsilon[t_1, R] \leq Ca$. We can therefore assume $2 \leq a \leq k$.

The ideal profile for a transition with jump $2\sin\theta$ across an horizontal interface is

$$w^\theta = \cos\theta\, x_1 + \ln\left[2\cosh(x_2\sin\theta)\right]\,.$$

Its gradient takes the form

$$\nabla w^\theta = \begin{pmatrix}\cos\theta\\ \sin\theta\tanh(x_2\sin\theta)\end{pmatrix}$$

hence

$$|\nabla w^\theta|^2 - 1 = \sin^2\theta(\tanh^2 s - 1)^2\,, \quad \nabla^2 w^\theta = \frac{\sin^2\theta}{\cosh^2 s} e_1 \otimes e_1\,,$$

where $s = x_2 \sin\theta$. The energy across the interface is

$$\int_{\mathbb{R}} (|\nabla w^\theta|^2 - 1)^2 + |\nabla^2 w^\theta|^2 \mathrm{d}x_2$$
$$= \int_{\mathbb{R}} \left[\sin^4\theta(\tanh^2 s - 1)^2 + \frac{\sin^4\theta}{\cosh^4 s}\right] \frac{1}{|\sin\theta|} \mathrm{d}s \;=\; \frac{8}{3}|\sin\theta|^3\,.$$

Let now ψ be a cutoff function in $C_c^\infty(R,[0,1])$, with $\psi = 1$ on the smaller rectangle

$$R' = (-(a-1), a-1) \times \big(-\sqrt{k}+1, \sqrt{k}-1\big)$$

and $|\psi| + |\nabla\psi| + |\nabla^2\psi| \le C$ for some universal constant C (independent of a and k).

We define $w = \psi w^\theta + (1-\psi)v$. To prove the claim it suffices to estimate the contribution from the boundary layer $R \setminus R'$. Recall that on R we have $v = t_1 = t * \phi_1$. We get

$$\nabla w = \nabla t_1 + \psi(\nabla w^\theta - \nabla t_1) + \nabla\psi(w^\theta - t_1)\,.$$

Now we estimate the energy. The explicit expressions above give

$$|w^\theta - t| + |\nabla w^\theta - \nabla t| \le C\mathrm{e}^{-|x_2 \sin\theta|}\,.$$

Analogously one gets $|\nabla^2 w| \le Ce^{-|x_2 \sin\theta|}$. If $|x_2| > 1$, then $t_1 = t$, and in particular $|\nabla t_1| = 1$. Therefore,

$$F_1[w, R\setminus R'] \le 4k \int_{\sqrt{k}-1}^{\sqrt{k}} Ce^{-2|x_2 \sin\theta|}\mathrm{d}x_2 + 2\int_{-\sqrt{k}}^{\sqrt{k}} Ce^{-2|x_2\sin\theta|}\mathrm{d}x_2$$
$$\le Ck\mathrm{e}^{-\bar\theta\sqrt{k}} + \frac{C}{|\sin\theta|}\,.$$

Since $|\sin\theta| \ge \sin\bar\theta \ge \bar\theta/2$, this proves the claim (5.1). □

6 Construction up to the boundary

We show how to extend the construction up to the boundary, and how to enforce boundary values for u_ε and the normal derivative. We start with an estimate on the mollification of smooth functions, that will be crucial in the estimate of term (VI) below (through (6.10)).

Lemma 6 *Let φ be an* even *mollifier (that is $\varphi_\delta(x) = \varphi_\delta(-x)$) supported in B_1, and $\varphi_\delta(x) = \delta^{-2}\varphi(x/\delta)$. Let $A \subset \mathbb{R}^2$ be open and $u \in C^2(\overline{A})$. Then there exists a constant C_u, depending on u and φ, such that*

$$|u(x) - u * \varphi_\delta(x)| \le C_u\delta^2 \quad \textit{for every } x \in A \textit{ and } \delta < \mathrm{dist}\,(x, \partial A).$$

Proof When w is affine and $\mathrm{dist}\,(x, \partial A) < \delta$, then $w(x) = w * \varphi_\delta(x)$. Therefore it suffices to prove the lemma when $x = 0$, $u(0) = 0$, and $\nabla u(0) = 0$. In this case we can write

$$|u(0) - u * \varphi_\delta(0)| \;\le\; C_u \int_{B_\delta} |x|^2 |\varphi_\delta(x)| \;\le\; C_u\delta^2$$

where C_u depends only on $\|u\|_{C^2}$. □

Proof of Theorem 1 In this proof we do not explicitly indicate the dependence of constants on the domain Ω, as well as on the boundary data g and h.

For every $\zeta > 0$ we consider the tubular neighborhood of the boundary $T_\zeta := (\partial\Omega)_\zeta$ and the open set $\Omega_\zeta := \{x : \mathrm{dist}\,(x, \Omega) < \zeta\}$. Let $\nu : \partial\Omega \to \mathbb{S}^1$ be the outer normal to $\partial\Omega$, and $\eta > 0$ be such that $(x, t) \to x + t\nu(x)$ is a diffeomorphismus between $\partial\Omega \times (-3\eta, 3\eta)$ and $T_{3\eta}$. We define $w : \Omega_\eta \to \mathbb{R}$ by setting $w(x) = u(x)$ on Ω, and

$$w(y + t\nu(y)) = 3u(y - t\nu(y)) - 2u(y - 2t\nu(y)) \quad \text{for } y \in \partial\Omega\,,\ \ t \in (0, \eta)$$

(this is the standard extension procedure for Sobolev functions). Then

$$\nabla w \in BV(\Omega_\eta, \mathbb{R}^2)\,, \quad \|\nabla w\|_{L^\infty(\Omega_\eta)} \le C\,, \tag{6.1}$$

$$\|D^2 w\|(\partial\Omega) = 0\,, \quad \text{and} \quad w\Big|_\Omega = u\,. \tag{6.2}$$

Let u^ε be the result of Proposition 3, and set $w^\varepsilon = u^\varepsilon$ on $\Omega^{(\varepsilon)}$, and $w^\varepsilon = w_\varepsilon$ on $\Omega \setminus \Omega^{(\varepsilon)}$. These match smoothly since both equal u_ε around $\partial\Omega^{(\varepsilon)}$. We claim that the family w^ε satisfies,

$$\limsup_{\varepsilon \downarrow 0} F_\varepsilon[w^\varepsilon, \Omega] \le F_0[u, \Omega]\,, \tag{6.3}$$

$$w^\varepsilon \to u \quad \text{in } W^{1,p}(\Omega), \forall p < \infty\,, \quad w^\varepsilon = w_\varepsilon \ \text{ on } T_\varepsilon \cap \Omega\,. \tag{6.4}$$

The last condition is by definition; the middle one follows from Proposition 3 and the facts $|w_\varepsilon| \leq C$, $|T_\varepsilon| \to 0$. To prove the first one, by Proposition 3 it suffices to show that

$$\lim_{\varepsilon \downarrow 0} F_\varepsilon[w_\varepsilon, \Omega \setminus \Omega^{(\varepsilon)}] = 0\,. \tag{6.5}$$

The latter is proved by a variant of the argument leading to Lemma 4. Indeed, let $x \in \Omega$. Following (2.3), (2.4), and (2.5) we get

$$\begin{aligned} F_\varepsilon[w_\varepsilon, B_\varepsilon(x) \cap \Omega] &\leq C\|D^2 w\|(B_{2\varepsilon}(x)) \\ &\quad + \frac{8}{\varepsilon} \int_{B_\varepsilon(x)} |\nabla w(y) - \nabla w_\varepsilon(x)| + |\nabla w_\varepsilon(y) - \nabla w_\varepsilon(x)|\,. \end{aligned}$$

The rest of the argument is unchanged. We conclude that

$$F_\varepsilon[w_\varepsilon, B_\varepsilon(x) \cap \Omega] \leq C\|D^2 w\|(B_{2\varepsilon}(x))\,, \quad \text{for all } x \in \Omega\,. \tag{6.6}$$

Let now $\mathcal{F}^j = \{B_{2\varepsilon}(x_i^j)\}_i$ $(1 \leq j \leq N)$ be N families of disjoint balls such that the union of the $B_\varepsilon(x_i^j)$ covers $\Omega \cap T_\varepsilon$ and $x_i^j \in \Omega \cap T_\varepsilon$. Then (6.6) gives

$$\begin{aligned} F_\varepsilon[w_\varepsilon, T_\varepsilon \cap \Omega] &\leq \sum_{ij} F_\varepsilon[w_\varepsilon, B_\varepsilon(x_i^j) \cap \Omega] \\ &\leq C \sum_{ij} \|D^2 w\|(B_{2\varepsilon}(x_i^j)) \leq C\|D^2 w\|(T_{3\varepsilon})\,, \end{aligned}$$

and (6.5) follows since $\lim_{\varepsilon\downarrow 0} \|D^2 w\|(T_{3\varepsilon}) = \|D^2 w\|(\partial\Omega) = 0$. This concludes the proof of (6.3) and (6.4) and of the first part of the Theorem.

We finally enforce the boundary conditions. Let $g, h \in C^2(\partial\Omega)$, as in Remark 1. Then

$$v(y + t\nu(y)) = g(y) + h(y)t \quad y \in \partial\Omega\,, \quad t \in (-\eta, \eta)$$

defines a map $v \in C^2(T_\eta)$, and $v = u$ up to the gradient on $\partial\Omega$. Let $w : \Omega_\eta \to \mathbb{R}$ be given by

$$w(x) \;=\; \begin{cases} u(x) \text{ for } x \in \Omega \\ v(x) \text{ for } x \in T_\eta \setminus \Omega\,. \end{cases}$$

By the trace properties of BV functions, the new w still satisfies (6.1) and (6.2). We repeat the above construction using the new definition of w, and obtain a family $\{w^\varepsilon\}_\varepsilon \subset C^\infty(\bar{\Omega})$ which obeys (6.3). We extend each w^ε to $\Omega_{3\varepsilon}$ by setting

$w^\varepsilon = w_\varepsilon$. Next, fix a cutoff $\psi \in C^\infty([0,\infty),[0,1])$ with $\psi = 0$ on $[3/4,\infty)$ and $\psi = 1$ on $[0,1/4]$. We define $\Psi^\varepsilon : \mathbb{R}^2 \to [0,1]$ as

$$\Psi^\varepsilon(x) := \begin{cases} \psi\left(\operatorname{dist}(x,\partial\Omega)/\varepsilon\right), & \text{for } x \in \Omega, \\ 0 & \text{else.} \end{cases}$$

We set $z^\varepsilon := (1-\Psi^\varepsilon)w^\varepsilon + \Psi^\varepsilon v$. For small ε, $\{z^\varepsilon\} \subset C^2(\bar{\Omega})$. Moreover $z^\varepsilon|_{\partial\Omega} = g$ and $\left.\frac{\partial z^\varepsilon}{\partial \nu}\right|_{\partial\Omega} = h$. We claim that $\{z^\varepsilon\}$ is the desired optimal sequence. Since $z^\varepsilon|_{\Omega\setminus T_\varepsilon} = w^\varepsilon$ and $w^\varepsilon \to w$ in $W^{1,p}(\Omega)$ for all p, it suffices to prove

$$\lim_{\varepsilon\downarrow 0} F_\varepsilon[z^\varepsilon, \Omega\cap T_\varepsilon] = 0, \tag{6.7}$$

$$\lim_{\varepsilon\downarrow 0} \int_{T_\varepsilon} |\nabla z^\varepsilon|^p = 0 \quad \text{for all } p < \infty. \tag{6.8}$$

First step By (6.1) we have $\|\nabla w_\varepsilon\|_{C^0} \le C$ and $\|D^2 w_\varepsilon\| \le C\varepsilon^{-1}$ on T_ε. Moreover

$$\|\nabla z^\varepsilon\|_{C^0(T_\varepsilon)} \le \|\nabla w_\varepsilon\|_{C^0(T_\varepsilon)} + \|\nabla v\|_{C^0(T_\varepsilon)} + \frac{C}{\varepsilon}\|w_\varepsilon - v\|_{C^0(T_\varepsilon)}. \tag{6.9}$$

We claim that $\|w_\varepsilon - v\|_{C^0(T_\varepsilon)} \le C\varepsilon$. Indeed, notice that on $\Omega_{3\varepsilon}\setminus\Omega_\varepsilon$ we have $w_\varepsilon = v_\varepsilon$. By the smoothness of v, from Lemma 6 we get $\|v_\varepsilon - v\|_{C^0(\Omega_{3\varepsilon}\setminus\Omega_\varepsilon)} \le C\varepsilon^2$, and hence

$$\|w_\varepsilon - v\|_{C^0(\Omega_{3\varepsilon}\setminus\Omega_\varepsilon)} \le C\varepsilon^2. \tag{6.10}$$

Recall that $\|\nabla w_\varepsilon\| + \|\nabla v\| \le C$. Therefore using (6.10) and integrating over the segments perpendicular to $\partial\Omega$, we easily get $\|w_\varepsilon - v\|_{C^0(T_\varepsilon)} \le C\varepsilon$. Plugging this into (6.9) we conclude $\|Dz^\varepsilon\|_{C^0(T_\varepsilon)} \le C$ and hence (6.8) follows easily. In a similar way we get $\|D^2 z^\varepsilon\|_{C^0(T_\varepsilon)} \le C\varepsilon^{-1}$. Summarizing,

$$\|\nabla z^\varepsilon\|_{C^0(T_\varepsilon)} \le C \quad \text{and} \quad \|D^2 z^\varepsilon\|_{C^0(T_\varepsilon)} \le \frac{C}{\varepsilon} \quad \text{on } T_\varepsilon. \tag{6.11}$$

Second step Using (6.11) we can write

$$F_\varepsilon[z^\varepsilon, \Omega\cap T_\varepsilon] \le F_\varepsilon[w_\varepsilon, \Omega\cap T_\varepsilon] + C \int_{\Omega\cap T_\varepsilon} \frac{|\nabla z^\varepsilon - \nabla w_\varepsilon|}{\varepsilon} + |D^2 z^\varepsilon|. \tag{6.12}$$

The first term is infinitesimal by (6.5). To estimate the second one, we compute

$$\begin{aligned} |\nabla z^\varepsilon - \nabla w_\varepsilon| &\le |\nabla\Psi^\varepsilon|\,|w_\varepsilon - v| + |\nabla w_\varepsilon - \nabla v| \\ &\le C\varepsilon^{-1}|w_\varepsilon - v| + |\nabla w_\varepsilon - \nabla w| + |\nabla w - \nabla v|. \end{aligned}$$

Similarly

$$
\begin{aligned}
|D^2 z^\varepsilon| &\le |D^2 w_\varepsilon| + |D^2 v| + C\varepsilon^{-1}|\nabla w_\varepsilon - \nabla v| + C\varepsilon^{-2}|w_\varepsilon - v| \\
&\le |D^2 w_\varepsilon| + |D^2 v| + C\varepsilon^{-1}|\nabla w_\varepsilon - \nabla w| \\
&\quad + C\varepsilon^{-1}|\nabla w - \nabla v| + C\varepsilon^{-2}|w_\varepsilon - v|.
\end{aligned}
$$

Therefore the integral in (6.12) is bounded by a universal constant times

$$
\int_{T_\varepsilon} |D^2 v| + \int_{T_\varepsilon} |D^2 w_\varepsilon| + \int_{T_\varepsilon} \frac{|\nabla w_\varepsilon - \nabla w|}{\varepsilon} + \int_{T_\varepsilon} \frac{|\nabla w - \nabla v|}{\varepsilon} + \int_{T_\varepsilon} \frac{|w_\varepsilon - v|}{\varepsilon^2}.
$$

We denote these integrals respectively by (I), (II), (III), (IV), and (V) and we will prove that they all vanish as $\varepsilon \downarrow 0$.

Third step The limit of (I) vanishes because $|D^2 v|$ is bounded. In the proof of (6.5) we have already shown that the limits of (II) and (III) also vanish.

Next, note that the BV function $\nabla w - \nabla v$ has trace 0 on $\partial\Omega$. Therefore (IV) vanishes thanks to the trace properties of BV functions.

Finally, integrating over segments perpendicular to $\partial\Omega$ we get

$$
(V) \;\le\; C \int_{\Omega_{3\varepsilon}\setminus\Omega_\varepsilon} \frac{|w_\varepsilon - v|}{\varepsilon^2} + \frac{C}{\varepsilon}\int_{T_{3\varepsilon}} |\nabla w_\varepsilon - \nabla v| \;=:\; (VI) + (VII).
$$

The limit of (VI) vanishes by (6.10); and (VII) is treated as (III) above. □

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References

1. Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. Mathematical Monographs. Oxford University Press, Oxford (2000)
2. Ambrosio, L., De Lellis, C., Mantegazza, C.: Line energies for gradient vector fields in the plane. Calc. Var. Part. Diff. Eq. **9**, 327–355 (1999)
3. Aviles, P., Giga, Y.: A mathematical problem related to the physical theory of liquid crystal configurations. Proc. Centre Math. Anal. Austr. Nat. Univ. **12**, 1–16 (1987)
4. Aviles, P., Giga, Y.: The distance function and defect energy. Proc. R. Soc. Edin. Sect. A **126**, 923–938 (1996)
5. Aviles, P., Giga, Y.: On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields. Proc. R. Soc. Edin. Sect. A **129**, 1–17 (1999)
6. Braides, A.: Γ-convergence for beginners. Oxford University Press, Oxford (2002)
7. Braides, A., Defranceschi, A.: Homogenization of multiple integrals. Claredon Press, Oxford (1998)
8. Dal Maso, G.: An introduction to Γ-convergence, Birkhäuser, Boston (1993)
9. De Giorgi, E., Franzoni, T.: Su un tipo di convergenza variazionale. Atti Accad Naz. Lincei Rend. Cl. Sci. Mat. **8**, 842–850 (1975)

10. De Lellis, C.: Energie di linea per campi di gradienti. Tesi di Laurea. Università di Pisa, Pisa (1999)
11. De Lellis, C.: An example in the gradient theory of phase transitions. ESAIM Control Optim. Calc. Var. **7**, 285–289 (2002)
12. De Lellis, C., Otto, F.: Structure of entropy solutions to the eikonal equation. J. Eur. Math. Soc. **5**, 107–145 (2003)
13. DeSimone, A., Kohn, R.V., Müller, S., Otto, F.: Magnetic microstructures—a paradigm of multiscale problems. ICIAM 1999, vol. 131, pp. 833–844 Oxford University Press, Oxford (1999)
14. DeSimone, A., Kohn, R.V., Müller, S., Otto, F.: A compactness result in the gradient theory of phase transitions. Proc. R. Soc. Edin. A **131**, 833–844 (2001)
15. Ercolani, N.M., Indik, R., Newell, A.C., Passot, T.: The geometry of the phase diffusion equation. J. Nonlinear Sci. **10**, 223–274 (2000)
16. Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. studies in advanced mathematics, CRC Press, Boca Raton (1992)
17. Jin, W., Kohn, R.V.: Singular perturbation and the energy of folds. J. Nonlinear Sci. **10**, 355–390 (2000)
18. Ortiz, M., Gioia, G.: The morphology and folding patterns of buckling-driven thin-film blisters. J. Mech. Phys. Solids **42**, 531–559 (1994)
19. Poliakovsky, A.: A method for establishing upper bounds for singular perturbation problems. C. R. Math. Acad. Sci. Paris **341**, 97–102 (2005)
20. Poliakovsky, A.: Upper bounds for singular perturbation problems involving gradient fields, preprint (2005)
21. Ziemer, W.P.: Weakly differentiable functions, Sobolev spaces and functions of bounded variation. Graduate Texts in Mathematics, vol. 120. Springer, Berlin Heidelberg New York (1989)