A Quantitative Compactness Estimate for Scalar Conservation Laws

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Abstract

In the case of a scalar conservation law with convex flux in space dimension one, P. D. Lax proved [*Comm. Pure and Appl. Math.* **7** (1954)] that the semigroup defining the entropy solution is compact in L^1_{loc} for each positive time. The present note gives an estimate of the ε-entropy in L^1_{loc} of the set of entropy solutions at time $t > 0$ whose initial data run through a bounded set in L^1.

1 Estimates for Entropy Solutions

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that

$$f'' \geq a > 0 \ \text{ on } \mathbb{R} \quad \text{and} \quad f(0) = 0. \tag{1.1}$$

We consider the Cauchy problem for the conservation law with flux f:

$$\begin{cases} u_t + f(u)_x = 0 \\ u\big|_{t=0} = u^{\text{in}}, \end{cases} \qquad x \in \mathbb{R}\,. \tag{1.2}$$

Without loss of generality, we can assume that

$$f'(0) = 0; \tag{1.3}$$

otherwise, changing x into $x + tf'(0)$ and $f(u)$ into $f(u) - uf'(0)$ reduces the general case to this one.

For each $u^{\text{in}} \in L^1(\mathbb{R})$, there exists a unique entropy solution $u \equiv u(t,x)$ to (1.2). This defines a (nonlinear) semigroup $S(t)$ by $u(t,\cdot) = S(t)u^{\text{in}}$. Let us recall some well-known properties of the entropy solution to (1.2). First, it satisfies the Lax-Oleinik bound:

$$\text{for each } t > 0, \quad u_x(t,\cdot) \leq \frac{1}{at} \tag{1.4}$$

in the sense of distributions on $\mathbb{R}$. It also satisfies the following L^∞ bound (see formula (4.9) in chapter 4 of [4]):

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PROPOSITION 1.1 *Assume that* $u^{\text{in}} \in L^1(\mathbb{R})$ *and that* f *satisfies* (1.1). *Then, for each* $t > 0$*, one has*

$$\|u(t,\cdot)\|_{L^\infty} \le \sqrt{\frac{2\|u^{\text{in}}\|_{L^1}}{at}}\,. \tag{1.5}$$

For the convenience of the reader, we recall the short proof of (1.5) below, since the constants in it differ slightly from those in [4].

PROOF: We recall that for each $t > 0$, the entropy solution $u(t,\cdot)$ to (1.2) belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and is expressed by the Lax-Oleinik variational principle

$$u(t, x \pm 0) = (f')^{-1}\left(\frac{x - y_\pm(t,x)}{t}\right) \tag{1.6}$$

where $y_-(t,x)$ (respectively, $y_+(t,x)$) is the smallest (respectively, largest) minimum point y of the action functional

$$L_{t,x}(y) = tf^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} u^{\text{in}}(z)dz. \tag{1.7}$$

As usual, f^* designates the Legendre dual of f, defined by

$$f^*(p) = \sup_{u\in\mathbb{R}}(pu - f(u))$$

or equivalently by

$$f^*(f'(u)) = uf'(u) - f(u)$$

(because, thanks to the first property in (1.1), f' is an increasing, one-to-one mapping of $\mathbb{R}$ onto itself). Therefore

$$\begin{aligned} f^*(f'(u(t,x\pm 0))) &= \frac{1}{t}\left(\inf_{y\in\mathbb{R}} L_{t,x}(y) - \int_{-\infty}^{y_\pm(t,x)} u^{\text{in}}(z)dz\right) \\ &\le \frac{1}{t}\left(L_{t,x}(x) - \int_{-\infty}^{y_\pm(t,x)} u^{\text{in}}(z)dz\right) \\ &= \frac{1}{t}\left(\int_{-\infty}^{x} u^{\text{in}}(z)dz - \int_{-\infty}^{y_\pm(t,x)} u^{\text{in}}(z)dz\right), \end{aligned}$$

where the last equality follows from $f^*(0) = f^*(f'(0)) = -f(0) = 0$ (recall (1.3) and the second property in (1.1)). Hence

$$f^*\big(f'(u(t,x\pm 0))\big) \le \frac{1}{t}\|u^{\text{in}}\|_{L^1}. \tag{1.8}$$

Finally, since $(zf'(z) - f(z))' = zf''(z)$, one has

$$f^*(f'(z)) = zf'(z) - f(z) \ge \frac{a}{2}z^2\,. \tag{1.9}$$

The inequalities (1.8) and (1.9) imply (1.5). □

We also recall that the entropy solution satisfies the maximum principle: If $u^{\text{in}} \in L^1 \cap L^\infty(\mathbb{R})$, then

$$\|S(t)u^{\text{in}}\|_{L^\infty} \le \|u^{\text{in}}\|_{L^\infty} \quad \text{for each } t > 0. \tag{1.10}$$

Moreover, the semigroup $S(t)$ is an L^1-contraction (see [2, 7]), that is,

$$\|S(t)u - S(t)v\|_{L^1(\mathbb{R})} \le \|u - v\|_{L^1(\mathbb{R})} \quad \forall t > 0 \text{ and } \forall u, v \in L^1(\mathbb{R}).$$

Thus, since $S(t)0 = 0$, we have

$$\|S(t)u^{\text{in}}\|_{L^1(\mathbb{R})} \le \|u^{\text{in}}\|_{L^1(\mathbb{R})}. \tag{1.11}$$

2 Compactness of the Semigroup $S(t)$

P. D. Lax proved in [3] that, for each $t > 0$, the map $S(t)$ is compact from $L^1(\mathbb{R})$ to $L^1_{\text{loc}}(\mathbb{R})$. In [6], he asked whether it is possible to give a quantitative estimate of the compactness of $S(t)$ and suggested using the notion of ε-entropy to do so. We first recall this notion, introduced by A. Kolmogorov (see [1]).

DEFINITION 2.1 Let (X, d) be a metric space and E a precompact subset of X. Let $N_\varepsilon(E)$ be the minimal number of sets in an ε-covering of E—i.e., a covering of E by subsets of X with diameter no greater than 2ε. The *ε-entropy of E* is defined as

$$H_\varepsilon(E \mid X) = \log_2 N_\varepsilon(E).$$

In the rest of this note, given $A \subset L^1(\mathbb{R})$, we denote by $S(t)A$ the set $\{S(t)u^{\text{in}} \mid u^{\text{in}} \in A\}$. By using the Lax-Oleinik bound (1.4) and the L^∞ bound (1.5), we arrive at the following quantitative variant of the Lax compactness theorem:

THEOREM 2.2 *Assume that f satisfies* (1.1) *and* (1.3). *For $L > 0$ and $M > 0$, set $c_M = \sup_{|z| \le M} |f''(z)|$ and define*

$$\mathcal{C}_{L,m,M} = \left\{u^{\text{in}} \in L^\infty(\mathbb{R}) \mid \operatorname{supp}(u^{\text{in}}) \subset [-L, L],\ \|u^{\text{in}}\|_{L^1} \le m,\ \text{and } \|u^{\text{in}}\|_{L^\infty} \le M\right\}.$$

Then, for ε sufficiently small, the ε-entropy of $S(t)\mathcal{C}_{L,m,M}$ in $L^1(\mathbb{R})$ satisfies

$$H_\varepsilon(S(t)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})) \le \frac{4}{\varepsilon}\left(\frac{4L(t)^2}{at} + 4L(t)\sqrt{\frac{2m}{at}}\right),$$

with $L(t) = L + 2c_M\sqrt{2mt/a}$ for every $t > 0$.

The theorem below is a localized version of Theorem 2.2. If $R > 0$ and $\mathcal{A} \subset L^1(\mathbb{R})$, we denote by $H_\varepsilon(\mathcal{A} \mid L^1([-R, R]))$ the ε-entropy of $\mathcal{A}' = \{f|_{[-R,R]}$ when f runs through $\mathcal{A}\}$ as a subset of $L^1([-R, R])$.

THEOREM 2.3 *Under the same assumptions as in Theorem* 2.2*,*

$$H_\varepsilon(S(t)\mathcal{C}_{L,m,M} \mid L^1([-R, R])) \le \frac{C_1(t)}{\varepsilon} + 2\log_2\left(\frac{C_2(t)}{\varepsilon} + C_3(t)\right).$$

In the estimate above, one can take

$$C_1(t) = \frac{16R^2}{at} + 16R\sqrt{\frac{2m}{at}}, \qquad C_2(t) = \frac{4RL(t)}{at} + 4R\sqrt{\frac{2m}{at}},$$

$$C_3(t) = \frac{L(t) + \sqrt{2mat}}{R + \sqrt{2mat}} + 2,$$

with $L(t) = L + 2c_M\sqrt{2mt/a}$.

These bounds show that in the ε-entropy of $S(t)\mathcal{C}(L, m, M)$ localized in any segment, the leading-order term vanishes as t tends to $+\infty$.

We end this section with a few remarks on Theorems 2.2 and 2.3 and variants thereof.

Remark 2.4. Define

$$\mathcal{C}_m = \{u^{\text{in}} \mid \|u^{\text{in}}\|_{L^1(\mathbb{R})} \le m\},$$
$$\mathcal{C}_{m,M} = \{u^{\text{in}} \mid \|u^{\text{in}}\|_{L^1(\mathbb{R})} \le m \text{ and } \|u^{\text{in}}\|_{L^\infty} \le M\}.$$

From (1.5), (1.10), and (1.11), it follows that

$$S(t)\mathcal{C}_m \subset S\left(\frac{t}{2}\right)\mathcal{C}_{m,M(t)} \quad \text{where } M(t) = \sqrt{\frac{4m}{at}}.$$

As a result of the finite speed of propagation of equation (1.2), we know that for any $u^{\text{in}} \in \mathcal{C}_{m,M'}$ the values of $S(t)u^{\text{in}}$ on $[-L, L]$ depend only on the values of u^{in} in $[-L - 2c_{M'}\sqrt{2mt/a}, L + 2c_{M'}\sqrt{2mt/a}]$. Hence, if we define $L'(t) = R + 2c_{M(t)}\sqrt{mt/a}$, we conclude that

$$H_\varepsilon(S(t)\mathcal{C}_m \mid L^1([-R, R])) \le H_\varepsilon\left(S\left(\frac{t}{2}\right)\mathcal{C}_{L'(t),m,M(t)} \mid L^1([-R, R])\right).$$

Applying then Theorem 2.3, we eventually arrive at an upper bound of the form

$$H_\varepsilon\big(S(t)\mathcal{C}_m \mid L^1([-R, R])\big) \le \frac{C_4(t)}{\varepsilon} + 2\log_2\left(\frac{C_5(t)}{\varepsilon} + C_6(t)\right), \tag{2.1}$$

where

$$C_4(t) = \frac{32R^2}{at} + 32R\sqrt{\frac{m}{at}},$$
$$C_5(t) = \frac{8R}{at}\left(R + 2c_{M(t)}\sqrt{\frac{mt}{a}}\right) + 8R\sqrt{\frac{m}{at}},$$
$$C_6(t) = \frac{1}{R + \sqrt{mat}}\left(R + 2c_{M(t)}\sqrt{\frac{mt}{a}} + \sqrt{mat}\right) + 2.$$

Remark 2.5. In view of potential applications to the notion of "resolution" of a numerical method, as suggested in [5], it would be interesting to know whether the ε-entropy estimates above are optimal—or to derive lower bounds for those ε-entropies.

Let $u^{\text{in}} \in L^1(\mathbb{R})$; under assumptions (1.1) through (1.3) the entropic solution of (1.2) converges to the N-wave

$$N_{p,q}(t,x) = \begin{cases} \frac{x}{f''(0)t} & \text{if } -\sqrt{pt} < x < \sqrt{qt} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$p = -2f''(0) \inf_y \int_{-\infty}^{y} u^{\text{in}}(z)dz,$$
$$q = 2f''(0) \sup_y \int_{y}^{+\infty} u^{\text{in}}(z)dz\,,$$

in the sense that $\|u(t,\cdot) - N_{p,q}(t,\cdot)\|_{L^1(\mathbb{R})} \to 0$ as $t \to +\infty$. This result was proven by Lax (see [4], theorem 4.1 on p. 19). Since the family of N-waves is completely determined by the two independent parameters p and q, one has

$$\overline{\lim_{t\to+\infty}} \text{(resp. } \underline{\lim}_{t\to+\infty}\text{)}\ H_\varepsilon(S(t)\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})) \sim 2|\log_2 \varepsilon|$$

as $\varepsilon \to 0$.

The upper bounds in Theorems 2.2 and 2.3 and in (2.1) do not capture the above asymptotic behavior; however, applying (2.1) with $R = o(\sqrt{t})$ shows that

$$\overline{\lim_{t\to+\infty}}\ H_\varepsilon(S(t)\mathcal{C}_m \mid L^1([-R,R])) = O(1)$$

as $\varepsilon \to 0$. This is compatible with the convergence to the N-wave; indeed, the dependence of the N-wave in the parameters p and q appears on centered intervals with length $O(\sqrt{t})$ only.

Remark 2.6. We conclude this section with a few words on the periodic case. Consider the Cauchy problem (1.2) with f as in (1.1) and where $u^{\text{in}} \in L^1_{\text{loc}}(\mathbb{R})$ satisfies

$$u^{\text{in}} \text{ is periodic with period } L \quad \text{and} \quad \int_0^L u^{\text{in}}(z)dz = 0;$$

call $\mathcal{C}_{\text{per}}$ this class of functions. For each $u^{\text{in}} \in \mathcal{C}_{\text{per}}$ and each $t > 0$,

$$\|S(t)u^{\text{in}}\|_{L^\infty} \le \frac{2L}{at} \quad \text{and} \quad \text{TV}(S(t)u^{\text{in}} \mid [0,L]) \le \frac{2L}{at}.$$

With the methods described below, one can show that

$$H_\varepsilon(S(t)\mathcal{C}_{\text{per}} \mid L^1([0,L])) \le 8\frac{L^2}{at\varepsilon} + 4\log_2\left(\frac{2L^2}{at\varepsilon} + 4\right).$$

3 ε-Entropy in L^1 for the Class of Nondecreasing Functions with Prescribed Total Variation

For $L > 0$ and $V > 0$, set

$$\mathcal{I}_{L,V} = \{w : [0, L] \to [0, V] \mid w \text{ is nondecreasing}\}.$$

In what follows we use the notation $[x] = \max\{z \in \mathbb{Z} \mid z \leq x\}$.

LEMMA 3.1 *For* $0 < \varepsilon \leq \frac{LV}{6}$ *the following holds*:

$$H_\varepsilon(\mathcal{I}_{L,V} \mid L^1([0, L])) \leq 4\left[\frac{LV}{\varepsilon}\right].$$

PROOF: Let N be a positive integer and set $\Delta x = \frac{L}{N}$ and $\Delta y = \frac{V}{N}$. To each $w \in \mathcal{I}_{L,V}$ we associate the pair of functions $(\chi^+[w], \chi^-[w])$ defined by

$$\chi^\pm[w] = \sum_{k=0}^{N-1} \chi_k^\pm \Delta y \mathbf{1}_{[k\Delta x,(k+1)\Delta x[} \tag{3.1}$$

where the notation $\mathbf{1}_S$ designates the indicator function of the set S, and where

$$\chi_k^- = \left[\frac{w(k\Delta x + 0)}{\Delta y}\right], \quad \chi_k^+ = \left[\frac{w((k+1)\Delta x - 0)}{\Delta y}\right] + 1.$$

Notice that, since w is nondecreasing,

$$\chi_k^+ - \chi_{k+1}^- \leq 1, \quad k = 0, 1, \dots, N-2. \tag{3.2}$$

Hence

$$\begin{aligned}
\|\chi^+[w] - \chi^-[w]\|_{L^1} &= \sum_{k=0}^{N-1}(\chi_k^+ - \chi_k^-)\Delta y \Delta x \\
&= (\chi_{N-1}^+ - \chi_0^-)\Delta y \Delta x + \sum_{k=0}^{N-2}(\chi_k^+ - \chi_{k+1}^-)\Delta y \Delta x \\
&\leq N\Delta y \Delta x + (N-1)\Delta y \Delta x = (2N-1)\Delta y \Delta x. \qquad (3.3)
\end{aligned}$$

For $\zeta^\pm \in \mathcal{I}_{L,V}$, define

$$U(\zeta^-, \zeta^+) = \{v \in \mathcal{I}_{L,V} \mid \zeta^- \leq v \leq \zeta^+\}.$$

For each $w \in \mathcal{I}_{L,V}$, $w \in U(\chi^-[w], \chi^+[w])$, so that the set

$$\mathcal{U} = \{U(\chi^-[v], \chi^+[v]) \mid v \in \mathcal{I}_{L,V}\}$$

is a covering of $\mathcal{I}_{L,V}$. On the other hand, by (3.3)

$$\operatorname{diam} U(\chi^-[w], \chi^+[w]) = \|\chi^+[w] - \chi^-[w]\|_{L^1} \leq 2N\Delta y \Delta x\,. \tag{3.4}$$

Notice that $\{\chi_k^-\}$ and $\{\chi_k^+ - 1\}$ are nondecreasing sequences of nonnegative integers smaller than $N+1$. Thus

$$\#\mathcal{U} \le (\#\{0 \le a_0 \le a_1 \le \cdots \le a_{N-1} \le N \mid a_k \in \mathbb{N}\})^2. \tag{3.5}$$

Define

$$\pi(N,k) := \#\{(p_1,\ldots,p_N) \in \mathbb{N}^N \mid p_1 + \cdots + p_N = k\}.$$

We recall the elementary method for computing $\pi(N,k)$: by definition of $\pi(N,k)$,

$$\begin{aligned}\sum_{k\ge 0} \pi(N,k) X^k = (1-X)^{-N} &= \frac{1}{(N-1)!}\,\frac{d^{N-1}}{dx^{N-1}}\,(1-X)^{-1} \\ &= \frac{1}{(N-1)!}\sum_{k\ge 0}(k+1)\cdots(k+N-1)X^k\end{aligned}$$

and therefore

$$\pi(N,k) = \binom{N-1+k}{k}.$$

Clearly

$$\begin{aligned}&\#\{0 \le a_0 \le a_1 \le \cdots \le a_{N-1} \le N \mid a_k \in \mathbb{N}\} \\ &\quad = \#\{(p_1,\ldots,p_{N+1}) \in \mathbb{N}^{N+1} \mid p_1 + \cdots + p_{N+1} = N\} \\ &\quad = \pi(N+1,N) = \binom{2N}{N}.\end{aligned} \tag{3.6}$$

Therefore, if $N \ge 6$ we have

$$\#\mathcal{U} \le \left(\binom{12}{6}\prod_{i=7}^{N}\frac{2i(2i-1)}{i^2}\right)^2 = \left(924 \cdot 2^{2(N-6)}\right)^2 \le 2^{4N-4}.$$

For $\varepsilon > 0$, set

$$N = \left[\frac{LV}{\varepsilon}\right] + 1;$$

the set $\mathcal{U}$ is a covering of $\mathcal{I}_{L,V}$ with at most $2^{4[LV/\varepsilon]}$ sets of diameter not exceeding

$$2N\Delta x \Delta y = 2\frac{LV}{N} \le 2\varepsilon.$$

Hence

$$H_\varepsilon(\mathcal{I}_{L,V} \mid L^1[0,L])) \le 4\left[\frac{LV}{\varepsilon}\right].$$

□

Next, consider the class of functions

$$\mathcal{I}_{L,M,V} = \{w \in \mathcal{I}_{L,M} \mid w(L-0) - w(0^+) \le V\}.$$

COROLLARY 3.2 *For* $0 < \varepsilon \le \frac{LV}{6}$ *the following holds*:

$$H_\varepsilon(\mathcal{I}_{L,M,V} \mid L^1([0,L])) \le 4\left[\frac{LV}{\varepsilon}\right] + 2\log_2\left[\frac{ML}{\varepsilon} + \frac{M}{V} + 2\right].$$

PROOF: Using the same notation as in the proof of Lemma 3.1—and especially with the same definitions of Δx and Δy—we introduce

$$\mathcal{U}' = \{U(\chi^-[v], \chi^+[v]) \mid v \in \mathcal{I}_{L,M,V}\}.$$

Let

$$N = \left[\frac{LV}{\varepsilon}\right] + 1.$$

As before, $\mathcal{U}'$ is a covering of $\mathcal{I}_{L,M,V}$ by sets of diameter at most

$$2N\Delta x \Delta y = 2\frac{LV}{N} \le 2\varepsilon.$$

On the other hand $\#\mathcal{U}' \le (\#A)^2$, where A is the set

$$\left\{0 \le a_0 \le \cdots \le a_{N-1} \le \left[\tfrac{M}{V}N\right] + 1 \mid a_k \in \mathbb{N} \text{ and } a_{N-1} - a_0 \le N\right\}.$$

To any such sequence $a_0, \dots, a_{N-1}$ we associate $(a_0, p_1, \dots, p_{N-1})$ defined by $p_k = a_k - a_{k-1}$ for $k = 1, \dots, N-1$. Define

$$B = \left\{a_0 \in \mathbb{N} \mid 0 \le a_0 \le \left[\tfrac{M}{V}N\right] + 1\right\}$$
$$C = \left\{(p_1, \dots, p_{N-1}) \in \mathbb{N}^{N-1} \mid p_1 + \cdots + p_{N-1} \le N\right\}.$$

By using the estimate (3.6) for $\#C$, we obtain

$$\#A \le \#B \cdot \#C \le \left(\left[\tfrac{M}{V}N\right] + 2\right)2^{2N-2},$$

which implies

$$\#\mathcal{U}' \le 2^{4(N-1)}\left(\left[\tfrac{M}{V}N\right] + 2\right)^2.$$

Hence

$$H_\varepsilon(\mathcal{I}_{L,M,V} \mid L^1([0,L])) \le 4(N-1) + 2\log_2\left[\tfrac{M}{V}N + 2\right].$$

With $N = [LV/\varepsilon] + 1$, this leads to the desired estimate. □

4 Proofs of Theorem 2.2 and 2.3

PROOF OF THEOREM 2.2: First, we use the finite speed of propagation for the hyperbolic equation (1.2). Let $c_M = \sup_{|z|\le M} |f''(z)|$. For every $u^{\text{in}} \in \mathcal{C}_{L,m,M}$ and $t > 0$ we have (see [4], p. 19)

$$\text{(4.1)} \qquad \operatorname{supp}(S(t)u^{\text{in}}) \subset [-L(t), L(t)] \quad \text{where} \quad L(t) = L + 2c_M\sqrt{\frac{2mt}{a}}.$$

(Indeed, by the maximum principle, $\|u(t,\cdot)\|_{L^\infty} \le M$, and the maximum speed of propagation at each time $t > 0$ is

$$\sup_{z\in\mathbb{R}} |f'(u(t,z))| \le c_M\|u(t,\cdot)\|_{L^\infty} \le c_M\sqrt{\frac{2m}{at}}$$

by Proposition 1.1).

Next, for each $t > 0$, $u(t,\cdot) = S(t)u^{\text{in}}$ is a function of bounded variation. Specifically, by the Lax-Oleinik bound (1.4),

$$\mu_t = \frac{1}{at} - u_x(t,\cdot)$$

is a nonnegative distribution—and therefore a nonnegative Radon measure—for each $t > 0$. Thus, $u(t,\cdot)$ is decomposed into the difference of two nondecreasing functions in the following way:

$$u(t, x \pm 0) = \int_{-L(t)}^{x} \frac{dz}{at} - \int_{-L(t)-0}^{x\pm 0} d\mu_t(z) = u_1(t,x) - u_2(t, x \pm 0)\,. \tag{4.2}$$

Below, we use the notation

$$S_2(t)u^{\text{in}} = u_2(t,\cdot) \quad \text{for each } t > 0.$$

Notice that the function u_1 in the decomposition (4.2) is independent of u^{in}. Hence, for each $t > 0$ and $\varepsilon \in\,]0,1[$, we have

$$H_\varepsilon\big(S(t)\,\mathcal{C}_{L,m,M} \mid L^1(\mathbb{R})\big) = H_\varepsilon\big(S_2(t)\,\mathcal{C}_{L,m,M} \mid L^1([-L(t), L(t)])\big)\,. \tag{4.3}$$

Next we discuss the properties of $u_2(t,\cdot)$. Clearly, $u_2(t,\cdot)$ is a nondecreasing function on $[-L(t), L(t)]$ that satisfies $u_2(-L(t)-0) = 0$ and

$$\begin{aligned} u_2(L(t)+0) &= \int_{-L(t)-0}^{L(t)+0} d\mu_t(z) \\ &= \frac{2L(t)}{at} - \big(u(t, L(t)+0) - u(t, -L(t)-0)\big) \\ &\le \frac{2L(t)}{at} + 2\sqrt{\frac{2m}{at}} \end{aligned}$$

with $L(t) = L + 2c_M\sqrt{2mt/a}$. Thus, for each $t > 0$, the set of functions

$$x \mapsto S_2(t)u^{\text{in}}(x + L(t))$$

is included in the class $\mathcal{I}_{L',V'}$ with

$$L' = 2L(t) \quad \text{and} \quad V' = \frac{2L(t)}{at} + 2\sqrt{\frac{2m}{at}}\,.$$

Theorem 2.2 follows from this observation after applying Lemma 3.1 and (4.3). □

PROOF OF THEOREM 2.3: Observe that $S_2(t)u^{\text{in}}$ is a nondecreasing function such that

$$S_2(t)u^{\text{in}}(R-0) - S_2(t)u^{\text{in}}(-R+0) \le \frac{2R}{at} + 2\sqrt{\frac{2m}{at}}$$

and

$$0 \le S_2(t)u^{\text{in}} \le \frac{2L(t)}{at} + 2\sqrt{\frac{2m}{at}} \quad \text{on } [-R, R].$$

In other words, the set of functions defined on $[0, 2R]$ by

$$x \mapsto S_2(t)u^{\text{in}}(x+R)$$

belongs to the class $\mathcal{I}_{2R,M',V'}$ with

$$M' = \frac{2L(t)}{at} + 2\sqrt{\frac{2m}{at}} \quad \text{and} \quad V' = \frac{2R}{at} + 2\sqrt{\frac{2m}{at}}.$$

Then we conclude as in the proof of Theorem 2.2 by applying this time Corollary 3.2. □

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Bibliography

[1] Kolmogorov, A. N.; Tikhomirov, V. M. ε-Entropy and ε-capacity of sets in functional spaces. *Uspekhi Mat. Nauk* **14** (1959), 3–86.

[2] Kružkov, S. N. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)* **81** (1970), 228–255.

[3] Lax, P. D. Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Math.* **7** (1954), 159–193.

[4] Lax, P. D. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. SIAM, Philadelphia, 1973.

[5] Lax, P. D. Accuracy and resolution in the computation of solutions of linear and nonlinear equations. *Recent advances in numerical analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1978)*. Publ. Math. Res. Center Univ. Wisconsin, pp. 107–117. Academic Press, New York, 1978.

[6] Lax, P. D. Course on hyperbolic systems of conservation laws. XXVII Scuola Estiva di Fis. Mat., Ravello, 2002.

[7] Quinn-Keyfitz, B. Solutions with shocks: an example of an L_1-contractive semigroup. *Comm. Pure Appl. Math.* **24** (1971), 125–132.

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