Existence of Optimal Strategies for a Fire Confinement Problem

ALBERTO BRESSAN
Penn State University

CAMILLO DE LELLIS
Institut für Mathematik, Universität Zürich

Abstract

We consider a class of variational problems for differential inclusions related to the control of forest fires. The area burned by the fire at time $t > 0$ is modeled as the reachable set for a differential inclusion $\dot{x} \in F(x)$ starting from an initial set R_0. To block the fire, a barrier can be constructed progressively in time at a given speed. In this paper we prove the existence of an optimal strategy, which minimizes the value of the area destroyed by the fire plus the cost of constructing the barrier. © 2008 Wiley Periodicals, Inc.

1 Introduction

The aim of this paper is to analyze a new class of optimization problems, motivated by the control of forest fires, or the spatial spreading of a contaminating agent. The propagation of wildfires has been studied in various mathematical papers: see, for example, [9, 10], the survey articles [3, 13], and the references therein. A related optimal control problem was considered in [12]. Our present goal is to study the optimal allocation of firefighting resources, depending on the geometric structure of the advancing fire front and on the value of the land at risk.

The basic mathematical framework was introduced in [4]. At each time $t \geq 0$, we denote by $R(t) \subset \mathbb{R}^2$ the *contaminated region*. In the absence of control, the time evolution of the set $R(t)$ will be modeled in terms of a differential inclusion. More precisely, we consider a Lipschitz-continuous multifunction $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with compact, convex values and a bounded, open set $R_0 \subset \mathbb{R}^2$. At any given time $t \geq 0$, the contaminated set $R(t)$ is defined as the reachable set for the differential inclusion

$$\dot{x} \in F(x), \quad x(0) \in R_0, \tag{1.1}$$

where the upper dot denotes a derivative with respect to time. In other words,

$$R(t) = \big\{x(t) : x(\cdot) \text{ absolutely continuous, } x(0) \in R_0, \\ \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t]\big\}.$$

Communications on Pure and Applied Mathematics, Vol. LXII, 0789–0830 (2009)

This is a generalization of the models in [9, 10], where, for computational purposes, all velocity sets $F(x)$ are ellipses. Throughout the following, we assume that

$$0 \in F(x) \quad \forall x \in \mathbb{R}^2. \tag{1.2}$$

Of course, this implies

$$R(t_1) \subseteq R(t_2) \quad \text{whenever } t_1 < t_2. \tag{1.3}$$

We assume that the spreading of the contamination can be controlled by constructing walls. In the case of a forest fire, one may think of a thin strip of land that is either soaked with water poured from above (by airplane or helicopter) or cleared of all vegetation using a bulldozer. This will prevent the fire from crossing that particular strip of land. In mathematical terms, we thus assume that the controller can construct a one-dimensional rectifiable set γ that blocks the spreading of the contamination (see Section 2 for the definition of rectifiable set). To model this, we consider a continuous, strictly positive function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}^+$. Calling $\gamma(t) \subset \mathbb{R}^2$ the portion of the wall constructed within time $t \geq 0$, we make the following assumptions:

(H1) For any $0 \leq t_1 < t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$.
(H2) For every $t \geq 0$, the total length of the wall satisfies

$$\int_{\gamma(t)} \psi \, dm_1 \leq t. \tag{1.4}$$

Here and throughout the paper, m_1 denotes the one-dimensional Hausdorff measure, normalized so that $m_1(\Gamma)$ yields the usual length of a smooth curve Γ. In the above formula, $1/\psi(x)$ is the speed at which the wall can be constructed at the location x. In particular, if $\psi(x) \equiv \psi_0$ is constant, then (1.4) simply says that the length of the curve $\gamma(t)$ is $\leq t/\psi_0$. A strategy γ satisfying (H1) and (H2) will be called an *admissible strategy*.

When a wall is being constructed, the contaminated set is reduced. Indeed, we define

$$\begin{aligned} R^\gamma(t) \doteq \big\{x(t) : \ & x(\cdot) \text{ absolutely continuous, } x(0) \in R_0, \\ & \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0, t], \\ & x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0, t]\big\}. \end{aligned} \tag{1.5}$$

To define an optimization problem, we now introduce a cost functional. In general, this should take into account:

- the value of the area destroyed by the contamination and
- the cost of building the wall.

We thus consider two continuous, nonnegative functions $\alpha, \beta : \mathbb{R}^2 \mapsto \mathbb{R}_+$ and define the following functional:

$$J(\gamma) \doteq \int_{R_\infty^\gamma} \alpha \, dm_2 + \int_{\gamma_\infty} \beta \, dm_1, \tag{1.6}$$

where the sets R_∞^γ and γ_∞ are defined, respectively, as

$$R_\infty^\gamma \doteq \bigcup_{t>0} R^\gamma(t), \qquad \gamma_\infty \doteq \bigcup_{t>0} \gamma(t). \tag{1.7}$$

In (1.6), m_2 denotes the two-dimensional Lebesgue measure. In the case of a fire, $\alpha(x)$ is the value of a unit area of land at the point x, while $\beta(x)$ is the cost of building a unit length of wall at the point x. We remark that the set γ_∞ is a rectifiable set; hence it is measurable. On the other hand, the measurability of the set R_∞^γ is not an obvious fact. This issue will be addressed in Section 3.

It is clear that, in connection with a large advancing fire front, different strategies can be adopted. For example, one could try to stop the fire first at the center, then move toward the sides. Alternatively, one could start by blocking the right and left edges of the fire and then progress toward the center. Different strategies will result in different areas eventually being burned. It is thus natural to seek an optimal strategy that minimizes the total cost (1.6). The main purpose of this paper is to establish the existence of an optimal solution γ^* to the minimization problem

$$\min_{\gamma \in \mathcal{A}} J(\gamma), \tag{1.8}$$

where $\mathcal{A}$ is the set of all admissible strategies $t \mapsto \gamma(t) \subset \mathbb{R}^2$ satisfying (H1) and (H2).

For convenience, we list all the assumptions below:

- (A1) The initial set R_0 is open and bounded. Its boundary satisfies $m_2(\partial R_0) = 0$.
- (A2) The multifunction F is Lipschitz-continuous with respect to the Hausdorff distance. For each $x \in \mathbb{R}^2$ the set $F(x)$ is compact and convex, and contains a ball of radius $\rho_0 > 0$ centered at the origin.
- (A3) For every $x \in \mathbb{R}^2$ one has $\alpha(x) \geq 0$, $\beta(x) \geq 0$, and $\psi(x) \geq \psi_0 > 0$. Moreover, α is locally integrable, while β and ψ are both lower-semi-continuous.

As in [2], the Hausdorff distance between two compact sets X and Y is defined as

$$d_H(X, Y) \doteq \max\{\max_{x \in X} d(x, Y), \max_{y \in Y} d(y, X)\} \tag{1.9}$$

where

$$d(x, Y) \doteq \inf_{y \in Y} d(x, y).$$

The multifunction F is Lipschitz-continuous if there exists a constant L such that

$$d_H(F(x), F(y)) \leq L \cdot d(x, y)$$

for every couple of points x, y. On $\mathbb{R}^2$ we shall always use the Euclidean distance $d(x, y) \doteq |x - y|$.

Our main result can now be stated as follows:

THEOREM 1.1 *Let assumptions* (A1) *through* (A3) *hold. If there exists an admissible strategy such that $J(\gamma) < \infty$, then the minimization problem* (1.8) *admits an optimal solution.*

We remark that, for any admissible strategy $\gamma(\cdot)$, by assumption (A3) the length of walls constructed within a given time t satisfies $m_1(\gamma(t)) \leq t/\psi_0$. In the case where $\beta(x) \geq \beta_0 > 0$, the total length of all walls constructed by an optimal strategy γ^* remains uniformly bounded for all times. Indeed, $m_1(\gamma^*_\infty) \leq J(\gamma^*)/\beta_0 < \infty$. If we also assume that $\alpha(x) \geq \alpha_0 > 0$, then the burned region $R^{\gamma^*}_\infty$ must be bounded. On the other hand, if $\int_{\mathbb{R}^2} \alpha \, dx < \infty$, then the area of the burned region may be infinite. Indeed, one can conceive an optimal strategy that surrounds with walls a few precious enclaves, allowing everything else to be burned by the fire.

Our construction of the optimal strategy γ^* implies directly a mild regularity result:

COROLLARY 1.2 *With the assumptions of Theorem* 1.1*, there exists an optimal strategy $\gamma^*(\cdot)$ having the following property*: *With the exception of at most countably many t's, the set $\gamma^*(t)$ is the union of countably many compact connected components together with a set $\mathcal{N}(t)$ with $m_1(\mathcal{N}(t)) = 0$.*

We remark, however, that this regularity is not good enough, in order to apply the necessary conditions for optimality derived in [4].

The plan of the paper is as follows: In Section 2 we recall some notions and theorems from geometric measure theory, which will be used in the rest of the paper. In Section 3 we show that the reachable set is measurable (and hence that the functional J is well-defined). Section 4 contains the definition of *completion* of a strategy γ and a proof of its existence. Roughly speaking, this completion uses the same amount of walls as γ while making the reachable set as small as possible. Section 5 considers a minimizing sequence of complete strategies and selects a suitable concept of limiting strategy γ.

The following four sections show that γ is a minimizer. In Section 6 the proof is split into the two main lemmas, Lemmas 6.1 and 6.2. The proof of Lemma 6.1 is contained in Sections 7 and 8. The proof of Lemma 6.2 is given in Section 9. In Section 10 we discuss the role of the various assumptions and possible extensions of the main result. Finally, we prove Corollary 1.2.

2 Preliminaries from Geometric Measure Theory

2.1 Rectifiable Sets and Densities

By definition, a one-dimensional rectifiable set $\Gamma \subset \mathbb{R}^k$ is an m_1-measurable set that can be covered by countably many Lipschitz-continuous curves except for a set of m_1-measure zero (cf., for instance, definition 15.3 in [11] and definition 4.1 of [7]). In this paper we will always deal with one-dimensional rectifiable subsets of the Euclidean plane, which we will simply call rectifiable sets. A particularly useful reference for the classical theory of rectifiable one-dimensional sets is the book [8]. However, we warn the reader that in [8] rectifiable sets are called Y-sets; see page 33 therein. The following are the classical notions of densities for one-dimensional sets (see [8, p. 20] and definition 2.14 in [7]).

DEFINITION 2.1 Let E be a m_1-measurable set. Its *lower and upper densities* at a point x are then defined, respectively, as

$$\theta^*(E,x) \doteq \limsup_{r\downarrow 0} \frac{m_1(B(x,r)\cap E)}{2r}, \tag{2.1}$$

$$\theta_*(E,x) \doteq \liminf_{r\downarrow 0} \frac{m_1(B(x,r)\cap E)}{2r}. \tag{2.2}$$

When these two numbers coincide, we simply speak of the *density* of E at x, denoted by $\theta(E,x)$.

We recall a classical result concerning one-dimensional sets, densities, and rectifiable sets (compare with corollary 3.9 and theorem 3.25 of [8]).

PROPOSITION 2.2 *If $m_1(E) < \infty$, then $\theta^*(E,x) = 0$ for m_1-a.e. $x \notin E$ and $\theta^*(E,x) \geq \frac{1}{2}$ for m_1-a.e. $x \in E$. Moreover, E is rectifiable if and only if $\theta(E,x) = 1$ for m_1-a.e. $x \in E$.*

2.2 A Simple Estimate

The following proposition will be used quite often throughout the paper:

PROPOSITION 2.3 *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz map and $E \subset \mathbb{R}^2$ a m_1-measurable set. Let*

$$\mathcal{E} \doteq \{t \in \mathbb{R} : F^{-1}(t) \cap E \neq \varnothing\} = F(E).$$

Then

$$\operatorname{Lip}(F)\cdot m_1(E) \;\geq\; m_1(\mathcal{E}). \tag{2.3}$$

PROOF: The inequality is a straightforward consequence of the definition of Hausdorff measure. $\square$

2.3 Continua

A continuum is a compact connected set. The next lemma contains some elementary properties of continua with finite one-dimensional measure. Throughout the following, by $\overline{E}$ we denote the closure of a set E.

LEMMA 2.4 *If E is a connected set with $m_1(\overline{E}) < \infty$, then*:

(i) *Either E consists of a single point or $\theta_*(E, x) \geq \frac{1}{2}$ for every point $x \in \overline{E}$.*

(ii) $m_1(\overline{E} \setminus E) = 0$.

(iii) *$\overline{E}$ is compact and arcwise connected.*

(iv) *The set $\overline{E}$ is rectifiable.*

COROLLARY 2.5 *Let E be a set with $m_1(E) < \infty$, and let*

$$\tilde{E} \doteq E \cup \{x : \theta^*(E, x) > 0\}. \tag{2.4}$$

If E' is a connected component of $\tilde{E}$, then E' is a continuum.

Finally, the following semicontinuity result will play a central role in this paper.

THEOREM 2.6 *Let $\beta : \mathbb{R}^2 \to [0, \infty[$ be a lower-semicontinuous function. Consider a sequence $\{E_n\}$ of continua converging to a compact set E in the Hausdorff distance and assume that $m_1(E_n) \leq C < \infty$ for some constant C independent of n. Then E is a continuum and*

$$\liminf_{n\uparrow\infty} \int_{E_n} \beta \, dm_1 \geq \int_{E} \beta \, dm_1 . \tag{2.5}$$

We end this subsection by sketching the proofs of the above results for the interested reader. For the statements that are not proved in detail, we refer to chapter 3 of [8].

PROOF OF LEMMA 2.4:

(i) Let $r > 0$ and $x \in \overline{E}$. If $E \setminus B(x, r) \neq \varnothing$, then necessarily $\partial B(x, r) \cap E \neq \varnothing$, otherwise $B(x, r)$ and $\mathbb{R}^2 \setminus B(x, r)$ would disconnect E. Hence either E consists of a single point, or for every $x \in \overline{E}$ there exists a $\rho > 0$ for which $\partial B(x, r) \cap E \neq \varnothing$ for every $r \in]0, \rho]$.

From Proposition 2.3 we conclude $m_1(E \cap B(x, r)) \geq r$ for every $r \leq \rho$. By the definition of lower density, this implies (i).

(ii) By (i), for every $x \in \overline{E}$ we have $\theta^*(E, x) \geq \frac{1}{2}$. Hence (ii) follows from Proposition 2.2.

(iii) The same argument used in the proof of (i) shows that E is bounded; otherwise we would have $m_1(E) = \infty$. Therefore $\overline{E}$ is a compact connected set (i.e., a continuum) with $m_1(E) < \infty$.

Let $x \in \overline{E}$ and $\varepsilon > 0$ be given. We say that $y \in \overline{E}$ is ε-chain connected to x if there exist points $x = x_0, \ldots, x_N = y$ such that $|x_i - x_{i-1}| \leq \varepsilon$ and $x_i \in \overline{E}$. Since the set of points z that are ε-chain connected to x is relatively open and closed, the connectedness of $\overline{E}$ implies that any $y \in \overline{E}$ is ε-chain connected to x.

Next, let $x, y \in \overline{E}$ be given. For each $\varepsilon > 0$ let $x = x_0, \dots, x_{N_\varepsilon} = y$ be as above. By deleting intermediate points in the chain, we might assume $|x_i - x_j| > \varepsilon$ if $|i - j| \geq 2$. Therefore, if we consider the collection of balls $\{B(x_i, \frac{\varepsilon}{2})\}$, a point p is contained in at most two balls of the collection. This fact, together with the argument in (i), implies the following:

$$4m_1(\overline{E}) \geq 2 \sum_{i=1}^{N_\varepsilon} m_1(\overline{E} \cap B(x_i, \tfrac{\varepsilon}{2})) \geq N_\varepsilon \cdot \varepsilon \geq \sum_{i=1}^{N_\varepsilon} |x_i - x_{i-1}|. \tag{2.6}$$

Let $\gamma_\varepsilon : [0, \ell_\varepsilon] \mapsto \mathbb{R}^2$ be the polygonal curve with vertices at the points $x_0, \dots, x_{N_\varepsilon}$, parametrized by arc length. By (2.6) it follows that, its total length ℓ_ε satisfies $|x - y| \leq \ell_\varepsilon \leq 4m_1(\overline{E})$. We can extend each γ_ε to the interval $[0,\ 4m_1(\overline{E})]$ by setting $\gamma_\varepsilon(s) = y$ for $s \geq \ell_\varepsilon$. Since the maps γ_ε are all Lipschitz-continuous with constant 1 and uniformly bounded by the Ascoli-Arzelà compactness theorem, we can extract a subsequence $(\gamma_{\varepsilon_k})_{k \geq 1}$, with $\varepsilon_k \downarrow 0$, converging uniformly to a map γ. This limit curve γ joins x and y and is contained in $\overline{E}$ because $\overline{E}$ is closed.

(iv) One uses (iii) and an inductive procedure to cover m_1-almost all E by rectifiable curves; see the proof of lemma 3.13 in [8] for the details. □

PROOF OF COROLLARY 2.5: Let E' be a connected component of $\tilde{E}$. Then by Lemma 2.4(i) and (2.4), we have $\overline{E'} \subseteq \tilde{E}$. Therefore E' is itself closed. The claim of the corollary is now clear. □

Theorem 2.6 is a well-known fact. However, [8] gives the proof only in the case $\beta \equiv 1$ (i.e., when $F_\beta(E) = m_1(E)$). We report for the reader's convenience the standard arguments needed to modify the proof of [8] in our setting.

PROOF OF THEOREM 2.6: Assume $\beta_0 > 0$ and let $\beta : \mathbb{R}^2 \to [\beta_0, \infty[$ be a lower-semicontinuous function. Let $\mathcal{C}$ be the set of continua of $\mathbb{R}^2$, with the metric d_H defined in (1.9) and consider the functional

$$F_\beta(E) \doteq \int_E \beta \, dm_1. \tag{2.7}$$

Our goal is to prove that F_β is semicontinuous. First of all, β can be written as the pointwise supremum of a sequence of continuous functions $\beta_n : \mathbb{R}^2 \mapsto [\beta_0, \infty[$. By the monotone convergence theorem, $F_{\beta_n}(E) \uparrow F_\beta(E)$ for every $E \in \mathcal{C}$, and hence it suffices to prove the semicontinuity of F_β under the assumption that β is continuous.

This case is a minor modification of the classical argument giving the semicontinuity for the functional $F_1(E) = m_1(E)$. We give here a brief sketch. Let $\{E_j\}$ be a sequence of continua converging in the Hausdorff metric to a continuum E and consider $L \doteq \liminf_{j \uparrow \infty} F_\beta(E_j) < \infty$ (if $L = \infty$, then the claim of the theorem follows trivially). After extracting a subsequence, not relabeled, we may assume that $L = \lim_{j \uparrow \infty} F_\beta(E_j)$.

First of all, we call a *tree* any continuum E where every pair of points x, y can be joined by a unique injective curve of finite length. Using an elementary approximation procedure, we can find trees $F_j \subset E_j$ such that $d_H(F_j, E) \to 0$.

Consider the set $\mathcal{S}_\delta$ of coverings $\{H_l\}$ of E consisting of closed sets with diameter smaller than δ. Define

$$F^\delta_\beta(E) \doteq \inf_{\{H_l\}\in\mathcal{S}_\delta} \sum_l \mathrm{diam}(H_l) \min_{H_l} \beta.$$

By the definition of Hausdorff measure and the continuity of β, we easily obtain

$$F_\beta(E) = \sup_{\delta>0} F^\delta_\beta(E). \tag{2.8}$$

Recalling that $\sup_j m_1(E_j) \le C$, an elementary argument shows that we can decompose each F_j as a finite union of $N_j \le N(\delta, C)$ connected components F_{ij} with $\mathrm{diam}(F_{ij}) \le \delta$ and such that

$$\sum_i^{N_j} F_\beta(F_{ij}) = F_\beta(F_j) \tag{2.9}$$

(see lemma 3.17 of [8]). Up to extraction of subsequences, we can assume that each sequence $\{F_{ij}\}_i$ converges to a set H_i with diameter at most δ and that N_j is a number N independent of j. $\{H_i\}$ is a covering of E with closed sets of diameter less than δ. Therefore we can estimate

$$\begin{aligned} F^\delta_\beta(E) &\le \sum_{i=1}^N \mathrm{diam}(H_i) \min_{H_i} \beta \\ &\le \liminf_{j\uparrow\infty} \sum_{i=1}^N \mathrm{diam}(F_{ij}) \min_{F_{ij}} \beta \le \liminf_{j\uparrow\infty} \sum_{i=1}^N m_1(F_{ij}) \min_{F_{ij}} \beta \\ &\le \liminf_{j\uparrow\infty} \sum_{i=1}^N \int_{F_{ij}} \beta\, dm_1 = \liminf_{j\uparrow\infty} \int_{F_j} \beta\, dm_1 \le \liminf_{j\uparrow\infty} \int_{E_j} \beta\, dm_1. \end{aligned} \tag{2.10}$$

From (2.8) and (2.10) we conclude

$$F_\beta(E) \le \liminf_{j\uparrow\infty} F_\beta(E_j). \tag{2.11}$$

□

2.4 An Inequality on Continua

We prove here a useful geometric inequality.

DEFINITION 2.7 Given a set Γ and a vector $v \in \mathbb{R}^2$, we define its shifted image as

$$\mathcal{T}^v \Gamma \doteq \{v + q : q \in \Gamma\}.$$

LEMMA 2.8 *Consider two continua* $\Gamma, \Gamma' \subset \mathbb{R}^2$ *with finite length. Then*

$$m_2(\{v \in \mathbb{R}^2 : \mathcal{T}^v\Gamma \cap \Gamma' \neq \varnothing\}) \leq m_1(\Gamma) \cdot m_1(\Gamma'). \tag{2.12}$$

PROOF: It is known (see exercise 3.5 of [8]) that Γ and Γ' can be entirely covered by curves $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ and $\gamma' : [0, \ell'] \mapsto \mathbb{R}^2$ parametrized by arc length, with total length $\ell \leq 2m_1(\Gamma)$ and $\ell' \leq 2m_1(\Gamma')$, respectively. Let us define

$$A_k \doteq \{t \in [0, \ell] : \gamma(t) \text{ has } k \text{ preimages}\},$$
$$A'_k \doteq \{t \in [0, \ell'] : \gamma'(t) \text{ has } k \text{ preimages}\}.$$

Note that

$$m_1(\Gamma) = \sum_{k=1}^{\infty} \frac{m_1(A_k)}{k}, \tag{2.13}$$

$$m_1(\Gamma') = \sum_{k=1}^{\infty} \frac{m_1(A'_k)}{k}. \tag{2.14}$$

Next consider the map $\Phi : [0, \ell] \times [0, \ell'] \to \mathbb{R}^2$ given by $\Phi(t, s) = \gamma'(s) - \gamma(t)$. Since both γ and γ' are Lipschitz-continuous with constant 1, the map Φ is also Lipschitz and

$$\Phi([0, \ell] \times [0, \ell']) = \{v \in \mathbb{R}^2 : \mathcal{T}^v\Gamma \cap \Gamma' \neq \varnothing\}.$$

Note that any point $p \in \Phi(A_k \times A'_{k'})$ has at least kk' preimages. Hence, it follows from the area formula that

$$m_2(\Phi([0, \ell] \times [0, \ell'])) \leq \sum_{k,k'} \Phi(A_k \times A'_{k'}) \leq \sum_k \sum_{k'} \frac{1}{kk'} \int\limits_{A_k \times A'_{k'}} |\det d\Phi|.$$

However, note that $|\det d\Phi| \leq 1$. Therefore

$$\begin{aligned} m_2(\Phi([0, \ell] \times [0, \ell'])) &\leq \sum_k \sum_{k'} \frac{1}{kk'} m_1(A_k) m_1(A'_{k'}) \\ &= \left(\sum_k \frac{m_1(A_k)}{k}\right)\left(\sum_{k'} \frac{m_1(A'_{k'})}{k'}\right) = m_1(\Gamma) m_1(\Gamma'). \end{aligned}$$

This completes the proof. □

2.5 A Geometric Lemma

Let W be a rectifiable set contained inside a square Q. We think of W as a set of walls, obstructing the sight. The next result states that, if the total length of W is sufficiently small, then most of the points on one side of the square Q can still be seen, looking from most of the points on the other sides.

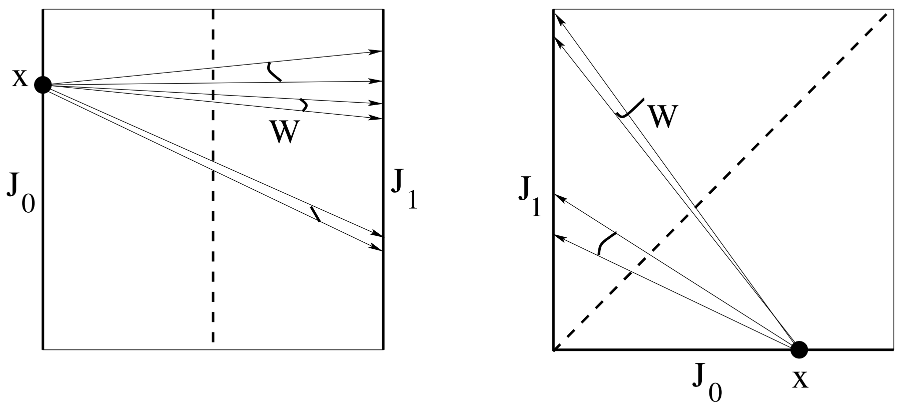

FIGURE 2.1. The projections $\pi_x(W)$ and $\pi_x(W \cap T_x)$: on the left, the case where J_0 and J_1 are opposite sides; on the right, the case where they have a vertex in common.

LEMMA 2.9 *Let $Q = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. Let J_0 and J_1 be any two sides. For any $\mu > 0$ there exists $\kappa > 0$ with the following property: If $W \subset Q$ is a set with $m_1(W) < \kappa$, then there is a set $K_0 \subseteq J_0$ such that:*

(i) $m_1(K_0) \geq 1 - \mu$.

(ii) *For every $x \in K_0$, the set J_x of points $y \in J_1$ for which the segment $[x, y]$ does not intersect W has measure $m_1(J_x) \geq 1 - \mu$.*

PROOF: We parametrize J_0 and J_1 with $[0, 1]$ in the obvious way. Given a set $W \subseteq Q$, consider the set W' of pairs $(x, y) \in J_0 \times J_1$ such that the segment $[x, y]$ intersects W. The lemma can then be reduced to the following claim:

(C) For every $\alpha > 0$ there exists a $\kappa > 0$ such that

$$m_1(W) \leq \kappa \implies m_2(W') \leq \alpha.$$

We divide the proof into two cases.

Case 1. *J_0 and J_1 are opposite sides.* Let $J_0 = \{0\} \times [0, 1]$ and $J_1 = \{1\} \times [0, 1]$. Notice that, by symmetry, it suffices to prove (C) under the assumption that $W \subseteq [\frac{1}{2}, 1] \times [0, 1]$. For each $x = (0, x_2) \in J_0$ consider the projection $\pi_x : [\frac{1}{2}, 1] \times [0, 1] \mapsto J_1$ with base point x, i.e.,

$$\pi_x(z_1, z_2) = \left(1,\ x_2 + \frac{z_2 - x_2}{z_1}\right)$$

(see Figure 2.1, left). It is not difficult to check that each map π_x is Lipschitz-continuous with a Lipschitz constant ≤ 4. We now observe that

$$W' \subseteq \{(x, y) : y \in \pi_x(W)\}.$$

Moreover, $m_1(\pi_x(W)) \leq 4m_1(W)$ by Proposition 2.3. As a result, $m_2(W') \leq 4m_1(W)$. The conclusion thus holds if we choose $\kappa \leq \frac{\alpha}{4}$.

Case 2. *J_0 and J_1 have a vertex in common.* Let $J_0 = [0,1] \times \{0\}$ and $J_1 = \{0\} \times [0,1]$. Again by symmetry, it suffices to prove the claim under the assumption that $W \subseteq T$, where $T = Q \cap \{(z_1, z_2) : z_2 \geq z_1\}$.

For each $x = (x_1, 0) \in J_0$ with $x_1 \geq \frac{\alpha}{2}$, consider the triangle T_x having the segment J_1 as basis and the point x as vertex. Moreover, consider the projection $\pi_x : T_x \mapsto J_1$ defined by

$$\pi_x(z_1, z_2) = \left(0, \frac{z_2\, x_1}{x_1 - z_1}\right)$$

(see Figure 2.1, right). In this case, the Lipschitz constant of π_x can be bounded by a constant C_α depending only on α. Observing that

$$W' \subseteq [0, \tfrac{\alpha}{2}] \times [0,1] \cup \{(x,y) : x \in [\tfrac{\alpha}{2}, 1] \times \{0\},\ y \in \pi_x(T_x \cap W)\},$$

we obtain the estimate

$$m_2(W') \leq \frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right) C_\alpha m_1(W).$$

Therefore, if $m_1(W)$ is sufficiently small, we again conclude that $m_2(W') \leq \alpha$. $\square$

3 Measurability of the Reachable Set

We now address the question of measurability of the set R^γ_∞ reached by the fire, defined at (1.7).

LEMMA 3.1 *Let assumptions* (A1) *through* (A3) *hold and let γ be an admissible strategy. Then the set $R^\gamma_\infty \subseteq \mathbb{R}^2$ is Lebesgue-measurable.*

PROOF: Define the set $\gamma^\dagger(t) \supset \gamma(t)$ as

$$\gamma^\dagger(t) \doteq \gamma(t) \cup \{x \in \mathbb{R}^2 : \theta^*(\gamma(t), x) > \tfrac{1}{6}\}. \tag{3.1}$$

We claim that

$$\overline{R^\gamma(t)} \subseteq R^\gamma(t+1) \cup \gamma^\dagger(t+1) \cup \partial R_0. \tag{3.2}$$

This claim implies the measurability of R^γ_∞. Indeed, on one hand,

$$R^\gamma_\infty = \bigcup_{N \in \mathbb{N}} R^\gamma(N) \subset \bigcup_{N \in \mathbb{N}} \overline{R^\gamma(N)}. \tag{3.3}$$

On the other hand, (3.2) implies

$$\bigcup_{N \in \mathbb{N}} \overline{R^\gamma(N)} \subset R^\gamma_\infty \cup \Big(\partial R_0 \cup \bigcup_{N \in \mathbb{N}} \gamma^\dagger(N)\Big). \tag{3.4}$$

Recalling Proposition 2.2 and assumption (A1), we conclude

$$m_2\Big(\partial R_0 \cup \bigcup_{N \in \mathbb{N}} \gamma^\dagger(N)\Big) = 0. \tag{3.5}$$

Clearly, (3.3), (3.4), and (3.5) imply that R^γ_∞ is measurable.

We now come to the proof of (3.2). Let $y \in R^{\overline{\gamma}}(t) \setminus (\gamma^{\dagger}(t+1) \cup \partial R_0)$. Then by assumption there exists $r_0 > 0$ such that, for all $0 < r \le r_0$, we have

$$\frac{m_1(B(y,r) \cap \gamma(t+1))}{2r} < \frac{1}{5}. \tag{3.6}$$

By further reducing the value of r_0, we can assume $B(y, r_0) \cap R_0 = \varnothing$. In this case, there exists a trajectory $t \mapsto x(t)$ of the differential inclusion (1.1) that reaches a point $x_0 \in B(y, r_0/2)$ at some positive time $t_0 \le t$.

For every $j \ge 1$, let $r_j \doteq 2^{-j} r_0$ and observe that, by (3.6) and Proposition 2.3, the following two properties hold:

- There exists $s_j \in \,]r_j/2, r_j[$ such that $\partial B(y, s_j) \cap \gamma(t+1) = \varnothing$.
- There exists an angle θ_j such that, in a system of polar coordinates centered at y, the segment $S_j \doteq \{(r,\theta) : r \in [r_j/4, r_j]\}$ does not intersect $\gamma(t+1)$.

Let a_j and b_j denote the intersection of S_j with $\partial B(y, s_j)$ and $\partial B(y, s_{j+1})$, respectively. Moreover, denote by $b_{-1} = x(\overline{t})$ the first intersection of the trajectory x with $\partial B(y, s_0)$. Finally, denote by $\widehat{b_{j-1}a_j}$ the shortest arc on $\partial B(y, s_j)$ joining b_{j-1} and a_j.

We are now ready to construct a new admissible trajectory $x^{\sharp}(\cdot)$ for the fire, reaching y and avoiding γ. We first follow the trajectory $x(\cdot)$ to get from a point of R_0 to b_{-1}. Then we travel along the curve

$$\beta \doteq \bigcup_{j \ge 0} \left(\widehat{b_{j-1}a_j} \cup S_j\right)$$

with constant speed $\rho_0 > 0$ (see Figure 3.1). The length of β can be estimated by

$$m_1(\beta) \le \sum_{i=0}^{\infty} (\pi r_j + r_j) = (\pi + 1) r_0 \sum_{i=0}^{\infty} 2^{-i} = 2(\pi + 1) r_0.$$

Thanks to (A2), the new trajectory $x^{\sharp}$ solves the differential inclusion (1.1). Moreover, by construction, it does not cross the walls $\gamma(t+1)$. Now, since β is covered at speed ρ_0, the trajectory $x^{\sharp}(\cdot)$ reaches y at a time $T \le t + 2(\pi+1) r_0/\rho_0$. On the other hand, by choosing r_0 sufficiently small, we achieve $T < t + 1$. This shows that y belongs to $R^{\gamma}(t+1)$. □

4 Complete Strategies

In the previous section we did not impose any regularity assumption on the admissible strategies. We now show that each strategy can be replaced by a slightly larger one, where the walls contain all of their density points.

DEFINITION 4.1 Given a strategy $t \to \gamma(t)$ we define its *completion* $t \to \gamma^c(t)$ in the following way: First, we construct the sets

$$\Gamma(t) \doteq \gamma(t) \cup \{x \in \mathbb{R}^2 : \theta^*(\gamma(t), x) > 0\}.$$

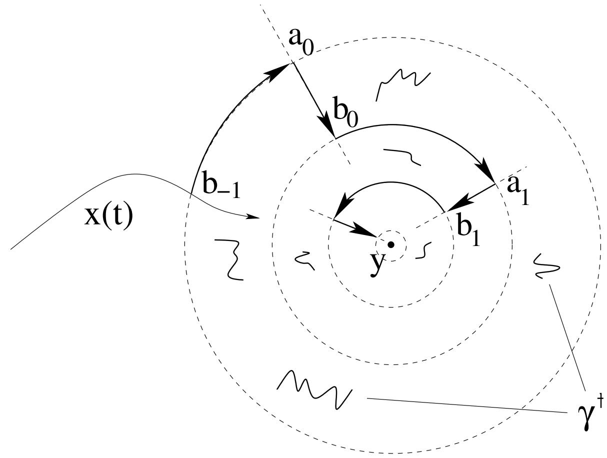

FIGURE 3.1. Constructing an admissible trajectory that reaches the point y avoiding the wall γ.

Then we define $\gamma^c(t) \doteq \bigcap_{s>t} \Gamma(s)$.

LEMMA 4.2 *The completion* $t \mapsto \gamma^c(t)$ *is an admissible strategy with the following properties*:

(i) $\gamma^c(t) = \bigcap_{s>t} \gamma^c(s)$;
(ii) $\gamma(t) \subseteq \gamma^c(t)$;
(iii) $m_1(\gamma^c(t) \setminus \gamma(t)) = 0$ *for all times* $t \geq 0$ *except at most countably many*;
(iv) $\gamma^c(t)$ *contains all its points of positive upper density.*

The lemma motivates the following:

DEFINITION 4.3 A strategy $\gamma(\cdot)$ is *complete* if it coincides with its completion.

We record here the following simple but important consequence of Lemma 4.2.

COROLLARY 4.4 *There exists a sequence of complete strategies* $\{\gamma_n\} \subset \mathcal{A}$ *such that*

$$\lim_{n\uparrow\infty} J(\gamma_n) = \inf_{\gamma\in\mathcal{A}} J(\gamma).$$

PROOF OF LEMMA 4.2:

(i) We simply observe that

$$\bigcap_{s>t} \gamma^c(s) = \bigcap_{s>t} \Big(\bigcap_{\tau>s} \Gamma(\tau) \Big) = \bigcap_{\tau>t} \Gamma(\tau) = \gamma^c(t).$$

(ii) For $t < s$ we have $\gamma(t) \subseteq \gamma(s) \subseteq \Gamma(s)$. Therefore

$$\gamma(t) \subseteq \bigcap_{s>t} \Gamma(s) = \gamma^c(t).$$

(iii) By Proposition 2.2, we have $m_1(\Gamma(t)) = m_1(\gamma(t))$ for every t. On the other hand, since $\gamma(s) \subseteq \gamma(t)$ for every $t > s$, we conclude that the map $t \mapsto m_1(\gamma(t))$ is nondecreasing. Hence this map is continuous everywhere except possibly at countably many points. Let t be a point of continuity. Then

$$m_1(\gamma^c(t)) = \lim_{s \downarrow t} m_1(\Gamma(s)) = \lim_{s \downarrow t} m_1(\gamma(s)) = m_1(\gamma(t)). \tag{4.1}$$

Together, (ii) and (4.1) imply (iii).

(iv) By Proposition 2.2, $m_1(\Gamma(t) \setminus \gamma(t)) = 0$ for every t. Hence, x has positive upper density for $\Gamma(t)$ if and only if it has positive upper density for $\gamma(t)$. So $\Gamma(t)$ contains all the points where its upper density is positive. Let now x be a point where $\gamma^c(t)$ has positive upper density. Since $\gamma^c(t) \subset \Gamma(s)$ for every $s > t$, we conclude that $\Gamma(s)$ has positive upper density at x for every $s > t$. Thus, $x \in \Gamma(s)$ for every $s > t$ and hence

$$x \in \bigcap_{s>t} \Gamma(s) = \gamma^c(t).$$

In order to complete the proof, it remains to show that the strategy $\gamma^c(\cdot)$ is admissible. Clearly γ^c satisfies requirement (H1) because of (i). Next, let S be the (at most) countable set of times $t \geq 0$ where (iii) fails. Then

$$\int_{\gamma^c(t)} \psi \, dm_1 = \int_{\gamma(t)} \psi \, dm_1 \leq t \quad \text{for every } t \notin S.$$

On the other hand, if $s \in S$, since $\gamma^c(s) \subseteq \gamma^c(t)$ for $t > s$, we can write

$$\int_{\gamma^c(s)} \psi \, dm_1 \leq \liminf_{\substack{t \notin S \\ t \downarrow s}} \int_{\gamma^c(t)} \psi \, dm_1 \leq s.$$

Hence γ^c also satisfies requirement (H2). This completes the proof that γ^c is admissible. $\square$

5 Limit of a Minimizing Sequence of Complete Strategies

The proof of Theorem 1.1 follows the direct method of the calculus of variations. By Corollary 4.4, there exists a sequence of complete strategies $\{\gamma_n\} \subset \mathcal{A}$ such that

$$\lim_{n\to\infty} J(\gamma_n) = J^* \doteq \inf_{\gamma\in\mathcal{A}} J(\gamma).$$

In this section we will construct a suitable limiting strategy γ and, in Proposition 5.2, we formulate the main semicontinuity property which will imply that γ is actually a minimizer.

5.1 Canonical Decomposition of γ_n

Note that each $\gamma_n(t)$ is a rectifiable set. Moreover, since $\psi(x) \geq \psi_0 > 0$, by (1.4) its total length is bounded by

$$m_1(\gamma_n(t)) \leq \frac{t}{\psi_0}. \tag{5.1}$$

For each positive rational time $\tau \in \mathbb{Q}^+$, we decompose the set $\gamma_n(\tau)$ into its connected components.

Recall that γ_n is complete. Therefore each set $\gamma_n(\tau)$ contains all its points of positive upper density. By Corollary 2.5, each connected component of $\gamma_n(\tau)$ is a compact connected set, either consisting of a single point or having a strictly positive length. Moreover, by Lemma 2.4 each such component is arcwise connected.

We denote by $\gamma^\tau_{n,i}$, $i = 1, 2, \ldots$, the components of $\gamma_n(\tau)$ having strictly positive length, and we denote by $\ell^\tau_{n,i}$ their lengths. Moreover, without loss of generality, we assume that they are ordered so that

$$\ell^\tau_{n,1} \geq \ell^\tau_{n,2} \geq \cdots > 0. \tag{5.2}$$

By possibly taking a subsequence, we can assume that, for each $\tau \in \mathbb{Q}^+$ and each i, the following limit exists and is finite:

$$\lim_{n\to\infty} \ell^\tau_{n,i} = \ell^\tau_i < \infty. \tag{5.3}$$

We now recall that the metric space of compact subsets $K \subset \mathbb{R}^2$ with the Hausdorff distance d_H is locally compact. Therefore, for each (τ, i), by possibly taking a further subsequence we can assume that either there exists a limit set γ^τ_i with

$$\lim_{n\to\infty} d_H\big(\gamma^\tau_{n,i},\ \gamma^\tau_i\big) = 0, \tag{5.4}$$

or else, by the connectedness of $\gamma^\tau_{n,i}$ and (5.3),

$$\lim_{n\to\infty} [\inf\{|x| : x \in \gamma^\tau_{n,i}\}] = \infty. \tag{5.5}$$

In the latter case, we define $\gamma^\tau_i \doteq \varnothing$. Notice that, by the semicontinuity theorem 2.6,

$$m_1(\gamma^\tau_i) \leq \ell^\tau_i. \tag{5.6}$$

Remark 5.1 (Canonical Decomposition of $\gamma_n(\tau)$). For every rational time τ, there is a smallest index $i_0(\tau)$ such that $\ell_i^\tau = 0$ for all $i \geq i_0$. Possibly $i_0 = \infty$. Hence we can decompose $\gamma_n(\tau)$ as follows:

$$\gamma_n(\tau) = \Big(\bigcup_{i<i_0} \gamma_{n,i}^\tau\Big) \cup \Big(\bigcup_{i\geq i_0} \gamma_{n,i}^\tau\Big) \cup \gamma_n^0(\tau), \tag{5.7}$$

where $\gamma_n^0(\tau)$ is the union of those connected components that consist of one point. Of course, we may well have $m_1\big(\gamma_n^0(\tau)\big) > 0$. Observe that the maximum length of all connected components in the last two terms of (5.7) approaches 0 as $n \to \infty$.

5.2 The Limiting Strategy γ

We now define

$$\gamma^\flat(\tau) = \bigcup_{i<i_0(\tau)} \gamma_i^\tau \quad \text{if } \tau \in \mathbb{Q}_+. \tag{5.8}$$

Then, for every $t \geq 0$ we define

$$\gamma^\sharp(t) \doteq \bigcap_{\substack{\tau>t\\ \tau\in\mathbb{Q}}} \gamma^\flat(\tau) \quad \forall t \geq 0. \tag{5.9}$$

Finally, we use Lemma 4.2 and define γ to be the completion of the strategy $\gamma^\sharp$.

We claim that $\gamma^\sharp$ is an admissible strategy, hence its completion γ is admissible as well.

Indeed, observe that the inclusion $\gamma^\sharp(t) \subseteq \gamma^\sharp(s)$ for $t < s$ is an obvious consequence of the definition. Next, note that each γ_i^τ is a compact connected set. Moreover, by Theorem 2.6 and by (5.1), each γ_i^τ has finite length. Therefore, by Lemma 2.4, γ_i^τ is rectifiable and arcwise connected, which implies that $\gamma^\flat(\tau)$ is a rectifiable set for every $\tau \in \mathbb{Q}^+$. Therefore, by (5.9), $\gamma^\sharp(t)$ is contained in a rectifiable set and hence is rectifiable.

Furthermore, we have

$$\begin{aligned}\int_{\gamma^\flat(\tau)} \psi\, dm_1 &= \sum_{i<i_0} \int_{\gamma_i^\tau} \psi\, dm_1 \leq \sum_{i<i_0} \liminf_{n\uparrow\infty} \int_{\gamma_{n,i}^\tau} \psi\, dm_1 \\ &\leq \liminf_{n\uparrow\infty} \sum_{i<i_0} \int_{\gamma_{n,i}^\tau} \psi\, dm_1 \leq \liminf_{n\uparrow\infty} \int_{\gamma_n(\tau)} \psi\, dm_1,\end{aligned} \tag{5.10}$$

where the inequality in the first line follows from Theorem 2.6. By the admissibility of each strategy γ_n, we conclude that

$$\int_{\gamma^\flat(\tau)} \psi\, dm_1 \leq \tau \tag{5.11}$$

and

$$\int_{\gamma^\sharp(t)} \psi \, dm_1 \le \liminf_{\substack{\tau \in \mathbb{Q} \\ \tau \downarrow t}} \int_{\gamma^\flat(\tau)} \psi \, dm_1 \overset{(5.11)}{\le} t. \tag{5.12}$$

This shows the admissibility of the strategy $\gamma^\sharp$ and hence of its completion γ.

5.3 Semicontinuity

At this point, Theorem 1.1 follows once we prove that

$$J(\gamma) \le \liminf_{n\uparrow\infty} J(\gamma_n). \tag{5.13}$$

Recall that

$$J(\gamma) = \int_{R_\infty^\gamma} \alpha \, dm_2 + \int_{\gamma_\infty} \beta \, dm_1 \doteq J_1(\gamma) + J_2(\gamma). \tag{5.14}$$

Note that

$$\begin{aligned} J_2(\gamma) \le \sup_{\tau \in \mathbb{Q}^+} \int_{\gamma^\flat(\tau)} \beta \, dm_1 &\le \sup_{\tau \in \mathbb{Q}^+} \liminf_{n\uparrow\infty} \int_{\gamma_n(\tau)} \beta \, dm_1 \\ &\le \liminf_{n\uparrow\infty} \sup_{\tau \in \mathbb{Q}^+} \int_{\gamma_n(\tau)} \beta \, dm_1 \le \liminf_{n\uparrow\infty} J_2(\gamma_n), \end{aligned}$$

where the second inequality follows from the same argument used to prove (5.10).

Hence, in order to show (5.13), the heart of the matter is to prove the following proposition:

PROPOSITION 5.2 *Let $(\gamma_n)_{n\ge 1}$ be a minimizing sequence of complete strategies and consider the strategy γ constructed in Section* 5.2*. Then*

$$\int_{R_\infty^\gamma} \alpha \, dm_2 \le \liminf_{n\uparrow\infty} \int_{R_\infty^{\gamma_n}} \alpha \, dm_2. \tag{5.15}$$

The proof of Proposition 5.2 will be developed in Sections 6 through 8.

5.4 Monotonicity of $\gamma^\flat$

We end the present section with a simple lemma that plays a crucial role in the proof of Proposition 5.2.

LEMMA 5.3 *Consider $\tau, \sigma \in \mathbb{Q}^+$ such that $\tau < \sigma$. Then*

$$\gamma^\flat(\tau) \subseteq \gamma^\flat(\sigma). \tag{5.16}$$

PROOF: Let $\tau, \sigma \in \mathbb{Q}^+$ be given, with $\tau < \sigma$. Consider any $i < i_0(\tau)$. Since $\gamma_n(\tau) \subseteq \gamma_n(\sigma)$, each connected component $\gamma^\tau_{n,i}$ of $\gamma_n(\tau)$ is contained in some component $\gamma^\sigma_{n,j}$ of $\gamma_n(\sigma)$. Here $j = j(i,n)$ possibly depends on n. For all n sufficiently large we have

$$m_1(\gamma^\sigma_{n,j}) \geq m_1(\gamma^\tau_{n,i}) \geq \frac{\ell^\tau_i}{2}. \tag{5.17}$$

Hence

$$\frac{\sigma}{\psi_0} \geq j \cdot m_1(\gamma^\sigma_{n,j}) \geq \frac{j\,\ell^\tau_i}{2}. \tag{5.18}$$

This shows that the index $j = j(i,n)$ can be bounded by an integer N independent of n. Set $N_0 \doteq \min\{N, i_0(\sigma) - 1\}$ (or simply $N_0 \doteq N$ in the case $i_0(\sigma) = \infty$). Then

$$\gamma^\tau_{n,i} \subseteq \bigcup_{j=1}^{N_0} \gamma^\sigma_{n,j}. \tag{5.19}$$

By letting $n \uparrow \infty$ we conclude $\gamma^\tau_i \subseteq \gamma^\flat(\sigma)$. This implies that

$$\gamma^\flat(\tau) = \bigcup_{i < i_0(\tau)} \gamma^\tau_i \subseteq \gamma^\flat(\sigma). \tag{5.20}$$

This completes the proof of the lemma. □

6 Proof of Proposition 5.2

The proof of Proposition 5.2 relies on the following lemmas.

LEMMA 6.1 *Let $y \in R^\gamma_\infty$. Then, for every radius r, there exists N large enough, possibly depending on y and r, such that $R^{\gamma_n}_\infty \cap B(y,r) \neq \emptyset$ for every $n \geq N$.*

LEMMA 6.2 *For m_2-a.e. $y \in R^\gamma_\infty$ the following holds: For any $\varepsilon > 0$ there exists $r > 0$ such that*

$$\liminf_{n\uparrow\infty} \frac{m_2(B(y,\rho) \cap R^{\gamma_n}_\infty)}{\pi\rho^2} \geq 1 - \varepsilon \quad \forall \rho < r. \tag{6.1}$$

These lemmas will be proved in subsequent sections. We show here that Proposition 5.2 follows from Lemma 6.2 by a Besicovitch-Vitali covering argument.

PROOF OF PROPOSITION 5.2: Recall that a.e. point $y \in \mathbb{R}^2$ is a Lebesgue point for α (see [1, cor. 2.23]) and that α is nonnegative. Therefore,

$$\lim_{r\downarrow 0} \frac{\int_{B(y,r)} |\alpha(x) - \alpha(y)|dx}{\int_{B(y,r)} \alpha(x)\,dx} = 0 \quad \text{for a.e. } y \in R^\gamma_\infty \cap \{\alpha > 0\}. \tag{6.2}$$

Let $M, \varepsilon > 0$ be given and consider the measurable set

$$S_M \doteq \{x \in R^\gamma_\infty : |x| \leq M,\ \alpha(x) > 0\}.$$

For m_2-a.e. $y \in S_M$ that is a Lebesgue point for α, choose a radius $r(y) > 0$ such that, for every $\rho < r(y)$, the inequality (6.1) holds together with

$$\int_{B(y,\rho)} |\alpha(x) - \alpha(y)| dx \leq \varepsilon \int_{B(y,\rho)} \alpha(x) dx. \tag{6.3}$$

Note that this choice of $r(y)$ is possible because of Lemma 6.2.

Using the Besicovitch-Vitali covering argument (see [1, theorem 2.19]), we select a sequence of pairwise disjoint balls $\{B(y_i, r_i)\}_{i \geq 1}$ such that

- $y_i \in R^\gamma_\infty$,
- $r_i < r(y_i)$, and
- the union $\bigcup_{i \geq 1} B(y_i, r_i)$ of all these balls covers almost all the set S_M.

Since α is locally integrable, we can select a finite subcollection $\{B(y_i, r_i)\}_{i=1,\dots,k}$ such that

$$\sum_{i=1}^{k} \int_{B(y_i,r_i)} \alpha(x) dx \geq \int_{S_M} \alpha(x) dx - \varepsilon. \tag{6.4}$$

Combining (6.3) and (6.4) we then get

$$\sum_{i=1}^{k} \alpha(y_i) \pi r_i^2 \geq (1 - \varepsilon) \int_{S_M} \alpha(x) dx - \varepsilon. \tag{6.5}$$

By our choice of r_i, there is N sufficiently large such that

$$m_2(R^{\gamma_n}_\infty \cap B_{r_i}(y_i)) \geq (1 - \varepsilon) \pi r_i^2 \quad \forall n > N. \tag{6.6}$$

Moreover, combining (6.6) with (6.3), we obtain

$$\begin{aligned} \int_{R^{\gamma_n}_\infty \cap B(y_i,r_i)} \alpha(x) dx &\geq \alpha(y_i) m_2(R^{\gamma_n}_\infty \cap B(y_i, r_i)) - \varepsilon \int_{B(y_i,r_i)} \alpha(x) dx \\ &\geq (1 - \varepsilon) \alpha(y_i) \pi r_i^2 - 2\varepsilon \alpha(y_i) \pi r_i^2 = (1 - 3\varepsilon) \alpha(y_i) \pi r_i^2 \end{aligned} \tag{6.7}$$

for all $n > N$. Finally, by (6.5) and (6.7),

$$\begin{aligned} \int_{R^{\gamma_n}_\infty} \alpha(x) dx &\geq \sum_{i=1}^{k} \int_{R^{\gamma_n}_\infty \cap B(y_i,r_i)} \alpha(x) dx \geq (1 - 3\varepsilon) \sum_{i=1}^{k} \alpha(y_i) \pi r_i^2 \\ &\geq (1 - 4\varepsilon) \int_{S_M} \alpha(x) dx - \varepsilon \quad \text{for all } n > N. \end{aligned} \tag{6.8}$$

Hence

$$\liminf_{n \uparrow \infty} \int_{R^{\gamma_n}_\infty} \alpha(x) dx \geq (1 - 4\varepsilon) \int_{S_M} \alpha(x) dx - \varepsilon. \tag{6.9}$$

Since

$$\lim_{M\uparrow\infty} \int_{S_M} \alpha(x)dx = \int_{R^\gamma_\infty} \alpha(x)dx,$$

and in (6.9) the constants $M, \varepsilon > 0$ are arbitrary, we have established Proposition 5.2. □

7 Proof of Lemma 6.1: Part I

7.1 Strategy of the Proof

To help the reader, we first give an intuitive sketch of the main ideas in the proof (see Figure 7.1). Then we work out the details. Let $t \mapsto x(t)$ be an admissible trajectory of (1.1) reaching the point y at time T, avoiding all the walls constructed by the limiting strategy γ. Given $r > 0$, we need to show that for every n sufficiently large there exists a trajectory $x_n(\cdot)$ of (1.1), which at time T reaches a point $x_n(T)$ at a distance $< r$ from y and avoids the walls γ_n.

Step 1. Relying on the fact that the initial set R_0 is open, we construct a trajectory $x^\sharp$ whose speed $\dot{x}^\sharp(t)$ lies strictly in the interior of the velocity set $F(x^\sharp(t))$ and reaches the terminal point y at an earlier time: $x^\sharp(T-\delta) = y$. The new trajectory satisfies the inclusion $\{x^\sharp(t) : t \in [0, T-\delta]\} \subset \{x(t) : t \in [0,T]\}$. This first step is accomplished in Section 7.2.

Step 2. The walls in γ_n are decomposed into three parts; see (7.4). Namely, (i) a finite number of large connected components, (ii) a countable number of small connected components, whose total length is $\varepsilon \ll \delta$, and (iii) an additional set, possibly with big total length, but with the property that the size of its largest connected component approaches 0 as $n \to \infty$. This last portion of the walls we regard as "debris": in the end it has no effect on confining the spread of the fire. The decomposition is described in detail in Section 7.3, whereas Section 7.4 contains a technical covering lemma for the debris.

Step 3. By the definition of the limit strategy γ, it follows that, for all n large enough, the trajectory $t \mapsto x^\sharp(t)$ remains at a distance $> r_1 > 0$ from the finitely many large components of the wall γ_n. Here $r_1 < r$ is a sufficiently small radius. Relying on Lemma 2.8, we then show that, by considering the trajectory $x^v(t) = x^\sharp(t) + v$ for some vector $v \in B(0, r_1)$, this new trajectory intersects a minimal amount of debris. Indeed, by a suitable choice of v, the total length of all these connected components of the set in (iii) crossed by the shifted trajectory can be rendered $< \varepsilon$ for all n large enough. This is accomplished in Section 7.5.

For notational simplicity, the shifted trajectory found in step 3 will be denoted by Y.

Step 4. To complete the proof, we will construct a new trajectory $t \mapsto x_n(\cdot)$, which stays close to Y during most of the time but avoids the small walls (ii) and the debris (iii) constructed by the strategy γ_n. By the previous step, this requires

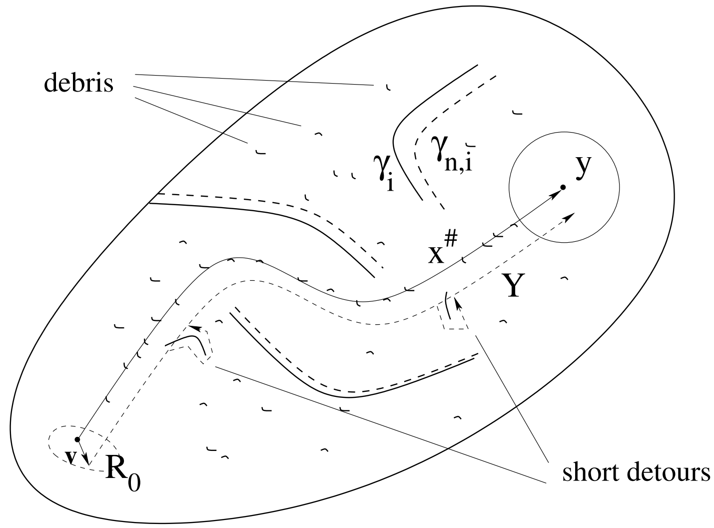

FIGURE 7.1. Constructing a trajectory $x_n(\cdot)$ that reaches a point close to y and avoids the walls in γ_n.

taking short "detours," whose total length is $O(1) \cdot \varepsilon \ll r$. Since the velocity sets $F(x)$ contain a neighborhood of the origin, the additional time required to make these detours is again $O(1) \cdot \varepsilon \ll \delta$. Hence the modified trajectory x_n reaches a point close to y within time T.

Though this last step is intuitively quite clear, its rigorous proof requires a quite elaborate procedure, which is the longest and most technical part of this paper. The entire Section 8 is devoted to the proof of this last step.

In the actual proof, one should also keep in mind that the sets $\gamma_n(t)$ are time dependent. An additional argument will show that it suffices to consider the walls constructed by the strategies γ_n at a finite set of times $0 < t_1 < \cdots < t_n < T$. We will actually show the existence of a subsequence of γ_n, chosen independently of y, for which the statement of the lemma holds (compare with Remark 7.2). The argument shows the existence of such a subsequence for each minimizing sequence of complete strategies. We deduce therefore that the conclusion of the lemma holds without extraction of a subsequence.

7.2 Faster Trajectories

We start by fixing $y \in R_\infty^\gamma$ and showing that this point can be reached by a trajectory with speed lying strictly in the interior of the sets $F(x)$ of admissible velocities. More precisely, we have the following:

LEMMA 7.1 *Consider an admissible strategy γ, a point $y \in R_\gamma^\infty$, and a Lipschitz trajectory $x : [0, T] \mapsto \mathbb{R}^2$ reaching y. That is,*

(i) $x(0) \in R_0$, $x(T) = y$,

(ii) $\dot{x}(t) \in F(x(t))$ *for a.e.* $t \in [0, T]$, *and*
(iii) $x(t) \notin \gamma(t)$ *for every* $t \in [0, T]$.

Then there are $\delta > 0$, $\eta > 0$, *and a trajectory* $x^\sharp : [0, T-\delta] \mapsto \mathbb{R}^2$ *such that*

(iv) $x^\sharp(0) \in R_0$, $x^\sharp(T-\delta) = y$,
(v) $x^\sharp(t) \notin \gamma(t+\delta)$ *for every* $t \in [0, T-\delta]$, *and*
(vi) $B(\dot{x}^\sharp(t), 4\eta) \subseteq F(x^\sharp(t))$ *for a.e.* $t \in [0, T-\delta]$.

PROOF: Since $x(0) \in R_0$ and R_0 is open, there is an interval of time $[0, 2\delta]$ such that $x(\tau) \in R_0$ for every $\tau \in [0, 2\delta]$. Consider the trajectory $x^\sharp : [0, T-\delta] \to \mathbb{R}^2$ defined by

$$x^\sharp(t) \doteq x\left(\frac{T}{T+\delta}(t+2\delta)\right).$$

Clearly $x^\sharp(0) = x(2\delta T/(T+\delta)) \in R_0$ and $x^\sharp(T-\delta) = x(T) = y$, which gives (iv). Moreover, for every $t \in [0, T-\delta]$ we have

$$x^\sharp(t) = x\left(\frac{T}{T+\delta}(t+2\delta)\right) \notin \gamma\left(\frac{T}{T+\delta}(t+2\delta)\right) \supseteq \gamma(t+\delta),$$

where the last inequality follows because, for $t \in [0, T-\delta]$, we have $(t+\delta)(T+\delta) \le T(t+2\delta)$. This proves (v).

Finally, note that

$$\frac{T+\delta}{T}\dot{x}^\sharp(t) = \dot{x}\left(\frac{T}{T+\delta}(t+2\delta)\right) \in F\left(\frac{T}{T+\delta}(t+2\delta)\right) = F(x^\sharp(t)). \tag{7.1}$$

We now recall that, by assumption (A1), each set $F(x)$ contains the ball $B(0, \rho_0)$ centered at the origin with radius $\rho_0 > 0$. In turn, the convexity of $F(x)$ yields

$$z \in F(x) \text{ and } 0 \le \lambda < 1 \quad \Longrightarrow \quad B(\lambda z, (1-\lambda)\rho_0) \subseteq F(x). \tag{7.2}$$

We apply (7.2) with

$$\lambda = \frac{T}{T+\delta}, \quad x = x^\sharp(t), \quad z = \dot{x}\left(\frac{T(t+2\delta)}{T+\delta}\right).$$

Thus, if we choose $\eta = \delta\rho_0/[4(T+\delta)]$, (vi) follows from (7.1). □

7.3 Decomposition of γ_n

From now on, we consider a given point $y \in R_\infty^\gamma$ and a radius $r > 0$. We let $x(\cdot)$ be a trajectory reaching y, as in Lemma 7.1, and select $\delta, \eta > 0$ and a corresponding trajectory $x^\sharp(\cdot)$ satisfying conditions (iv) through (vi) in the lemma.

Next, for any rational time $\tau \in \mathbb{Q}^+$, we use the decomposition of Remark 5.1. According to (5.7) we can divide $\gamma_n(\tau)$ into two parts:

$$\gamma_n(\tau) = \Big(\bigcup_{i < i_0(\tau)} \gamma_{n,i}^\tau\Big) \cup \gamma_n''(\tau),$$

where

(P1) The set $\gamma_n''(\tau)$ has the property that, if $\ell_n''(\tau)$ denotes the maximum length of all connected components of $\gamma_n''(\tau)$, then $\lim_{n\to\infty} \ell_n''(\tau) = 0$. By a diagonalization argument, we can extract a subsequence such that $\sum \ell_n''(\tau) < \infty$ for every $\tau \in \mathbb{Q}^+$.

(P2) The lengths $\ell_{n,i}^\tau$ of the curves $\{\gamma_{n,i}(\tau)\}_{i<i_0}$ are nonincreasing in i.

Remark 7.2. The only step where we need to extract a subsequence is to achieve property (P1). Notice that the choice of this subsequence is independent of the point y.

Let $\varepsilon > 0$ be a positive constant whose precise value will be chosen later. Recall that, by (5.3), each sequence $\ell_{n,i}^\tau$ converges to a number ℓ_i^τ. We let $i_1(\tau)$ be the smallest integer such that

$$\sum_{i\geq i_1(\tau)} \ell_i^\tau \leq \varepsilon. \tag{7.3}$$

We then further subdivide

$$\gamma_n(\tau) = \Big(\bigcup_{i\leq i_1(\tau)} \gamma_{n,i}^\tau \Big) \cup \gamma_n'(\tau) \cup \gamma_n''(\tau). \tag{7.4}$$

With this new subdvision, in addition to (P1) and (P2) we achieve the further property

(P3) The total length of $\gamma_n'(\tau)$ is smaller than 2ε for all n large enough.

We next choose $N = N(T,\delta)$ rational times $t_1 < t_2 < \cdots < t_N$ such that, for every $t \in [0, T-\delta]$, there is a $t_j \in [t+\frac{\delta}{2}, t+\delta]$.

Now, consider the limits $\gamma_i^{t_j}$ of $\gamma_{n,i}^{t_j}$ as $n \to \infty$. By the construction of γ and thanks to Lemma 5.3, $\gamma_i^{t_j} \subseteq \gamma(t_j)$. Fix $t \in [0, T-\delta]$ and let $t_j \in [t+\frac{\delta}{2}, t+\delta]$. We then know that $x^\sharp(t) \notin \bigcup_{i\leq i_1(t_j)} \gamma_i^{t_j}$. Since i_1 is finite, and the convergence of compact, connected components $\gamma_{n,i}^{t_j} \to \gamma_i^{t_j}$ takes place with respect to the Hausdorff metric, we can find a positive radius $r_1 \in \,]0, \frac{r}{2}[$ and an integer N_1 with the following property:

(P4) If $n > N_1$, $t \in [0, T-\delta]$, and $t_j \in [t+\frac{\delta}{2}, t+\delta]$, then

$$B(x^\sharp(t), 2r_1) \cap \Big(\bigcup_{i\leq i_1(t_j)} \gamma_{n,i}^{t_j} \Big) = \varnothing.$$

7.4 Covering with Lipschitz Arcs

Let us define

$$\ell_n \doteq \max_j \ell_n''(t_j), \quad L_n \doteq \sum_j m_1(\gamma_n''(t_j)).$$

Note that $L_n \leq L$ for some constant $L(N,T)$. Recalling the decomposition (7.4), we observe that, in general, the set $\Gamma_{n,j} \doteq \gamma_n'(t_j) \cup \gamma_n''(t_j)$ is the union of (at most) countably many continua with positive length and a totally disconnected set, possibly with positive measure. For future use, in this section we want to cover the set $\Gamma_{n,j}$ with countably many continua in an efficient way.

We first note that $\{x : \theta^*(\Gamma_{n,j}, x) > 0\} \subseteq \Gamma_{n,j}$. For n, j fixed, choose countably many Lipschitz curves $\{\Xi_i\}_{i\geq 1}$ covering m_1-a.a. $\Gamma_{n,j}$. For every i and every $k \in \mathbb{N}$, let $A_{i,k} \subseteq \Xi_i$ be a relatively open set such that

$$\Xi_i \cap \Gamma_{n,j} \subseteq A_{i,k}, \quad m_1(A_{i,k} \setminus \Gamma_{n,j}) \leq 2^{-i-k} \quad \text{and} \quad A_{i,k} \supseteq A_{i,k+1}.$$

Let then $G_k \doteq \bigcup_i A_{i,k}$ and let $E_k \doteq \{x : \theta^*(G_k, x) > 0\}$. Clearly $G_k \subseteq E_k$. Moreover, if $x \in G_k$, then $x \in A_{i,k}$ for some i and hence x is contained in a connected component of E_k with positive length. Recall also that, by Corollary 2.5, every connected component of E_k is a continuum. Therefore E_k can be decomposed into

- (at most) countably many connected components that are continua of positive length, together with
- the union of the connected components that are single points.

By the discussion above, this last set has zero length. Note also that we have $\{x : \theta^*(\Gamma_{n,j}, x) > 0\} \subset E_k$.

For a fixed k, let $B_{k,1}, \ldots, B_{k,M(k)}$ be the finitely many connected components of E_k with length larger than $2\ell_n$. Recalling (7.3)–(7.4), we claim that, if k is large enough, the sum of the lengths of these bigger components satisfies

$$\sum_{i=1}^{M(k)} m_1(B_{k,i}) < 2\varepsilon. \tag{7.5}$$

Note first that $M(k)$ is bounded by a constant depending only on $\ell_n > 0$. By construction, we know that every $B_{k+1,i}$ is contained in some $B_{k,j}$ for a suitable j. Consider the k's for which there is a $B_{k,i}$ containing two distinct connected components $B_{k+1,j}$ and $B_{k+1,j'}$. It is easy to see that there are only a finite number of such k's, otherwise the union $\bigcup_j B_{1,j}$ would have infinite length. Next, call $M \doteq \limsup_k M(k)$. The previous considerations show that, for all k large enough, $M(k) = M$ and each $B_{k,i}$ contains one and only one $B_{k+1,j}$. After a relabeling, we can thus assume that $B_{k,i} \supseteq B_{k+1,i}$ for all $i \in \{1, \ldots, M\}$ and every k large enough.

Let $B_i \doteq \cap_k B_{k,i}$. Note that, by construction, $m_1(B_i \setminus \Gamma_{n,j}) = 0$. Therefore

$$B_i \subseteq \{x : \theta^*(\Gamma_{n,j}, x) > 0\} \subseteq \Gamma_{n,j}.$$

Moreover, B_i is a continuum, and hence a connected subset of $\gamma_n'(t_j) \cup \gamma_n''(t_j)$ with length larger than $2\ell_n$. It follows that it must be a subset of $\gamma_n'(t_j)$. Therefore $\sum_i m_1(B_i) \leq m_1(\gamma_n'(t_j)) \leq \varepsilon$. We conclude that, for k large enough, the bound (7.5) must hold.

Based on the previous analysis, for each $\Gamma_{n,j}$ we choose k large enough and, by a slight abuse of notation, we now denote by $E_{n,j}$ the covering set denoted above by E_k. Summarizing, for each n and j, we have the following properties:

(P5) Each connected component of $E_{n,j}$ is a continuum.

(P6) $E_{n,j} \supseteq \{x : \theta^*(\gamma_n'(t_j) \cup \gamma_n''(t_j), x) > 0\}$ and $m_1(E_{n,j}) \leq 2L$.

(P7) Call $\{E_{n,j,i}\}_{i \geq 1}$ the connected components of $E_{n,j}$ with positive length. Then

$$m_1\Big(E_{n,j} \ \setminus \bigcup_i E_{n,j,i}\Big) = 0.$$

(P8) Consider the connected components of $E_{n,j}$ with length larger than $2\ell_n$. The sum of their lengths is smaller than 2ε.

7.5 A Shifted Trajectory

Given the trajectory $t \mapsto x^\sharp(t)$ considered in Lemma 7.1, for every $v \in \mathbb{R}^2$ consider the shifted trajectory

$$x^v(t) = x^\sharp(t) + v. \tag{7.6}$$

If Γ is the image of $x^\sharp(\cdot)$, then the image of $x^v(\cdot)$ will be a curve denoted by $\mathcal{T}^v\Gamma$.

We next divide $E_{n,j}$ into the union $E'_{n,j}$ of the connected components with length larger than $2\ell_n$ and the remaining part $E''_{n,j}$. For every $v \in B(0,r)$, let $\varphi_{n,j}(v)$ be the sum of the lengths of all connected components of $E''_{n,j}$ that intersect $\mathcal{T}^v\Gamma$. Then define $\varphi_n(v) \doteq \sum_j \varphi_{n,j}(v)$.

Let $r_1 > 0$ be a radius small enough so that (P4) holds. Applying Lemma 2.8, we conclude

$$\int_{B(0,r_1)} \varphi_n(v)dv \leq 2LN\ell_n m_1(\Gamma). \tag{7.7}$$

Note also that, by property (P1), one has $\sum_n \ell_n < \infty$. Therefore, $\sum_n \varphi_n$ is an integrable function. It follows that $\sum_n \varphi_n(v)$ has a finite value for a.e. $v \in B(0,r_1)$, which in turn implies

$$\lim_{n\downarrow 0} \varphi_n(v) = 0 \quad \text{for a.e. } v \in B(0,r_1). \tag{7.8}$$

We can thus select a $v \in B(0,r_1)$ for which (7.8) holds. For notational convenience, from now on we denote by $Y(\cdot) \doteq x^v(\cdot)$ this particular trajectory.

Since the multifunction F is Lipschitz-continuous, by choosing $r_1, \eta > 0$ sufficiently small, we can assume that the map $Y : [0, T-\delta] \mapsto \mathbb{R}^2$ has the following properties:

(P9) $Y(0) \in R_0$, $|Y(t) - x^\sharp(t)| < r_1$ for all $t \in [0, T-\delta]$.

(P10) $B(\dot{Y}(t), 2\eta) \subset F(Y(t))$ for a.e. $t \in [0, T-\delta]$.

Next, recalling the decomposition of (7.4), for all n sufficiently large we can also achieve

(P11) For every $t \in [0, T-\delta]$, if $t_j \in [t+\frac{\delta}{2}, t+\delta]$, then the following hold:

$$B(Y(t), r_1) \cap \Big(\bigcup_{i \le i_1(t_j)} \gamma_{n,i}^{t_j} \Big) = \varnothing. \tag{7.9}$$

(P12) The sets $E_{n,j}$ enjoy all the properties (P5)–(P8) listed in Section 7.4.

(P13) Let $\mathcal{I}_n$ be the set of all couples of indices (j,i) such that the set $E_{n,j,i}$ (defined in Section 7.4) intersects the trajectory $\{Y(t) : t \in [0, T-\delta]\}$. Then

$$\sum_{(j,i)\in\mathcal{I}_n} m_1(E_{n,j,i}) \le 3\varepsilon. \tag{7.10}$$

Recall that, at this stage, $r, \delta > 0$ are already assigned but we have not yet chosen $\varepsilon > 0$. Notice that r_1 depends indeed on ε, whereas η does not. In the next section we will choose $\varepsilon \ll r, \delta$ and show that, when n is so large that (P11)–(P13) hold, there is an admissible trajectory $x_n : [0,T] \to \mathbb{R}^2$ that starts at $Y(0)$, avoids all the walls of γ_n, and terminates at a distance $O(\varepsilon)$ from the desired target point $y = Y(T-\delta)$. This will achieve the proof of the lemma.

8 A Discretization Procedure to Get around Small Walls

8.1 Strategy of the Proof

In this section we conclude the proof of Lemma 6.1 by constructing a trajectory that travels most of the time close to Y and takes short detours to avoid efficiently the small connected components and the debris crossed by Y. Namely, we show here step 4 of Section 7.1. We start by outlining the strategy of our proof.

Step 4.1. We first choose a standard square grid with mesh size σ, and we change the trajectory Y to a new trajectory Y^σ such that $Y^\sigma \cap Q$ is a segment for every square Q in the grid, with endpoints belonging to Y. Y^σ starts in R_0 and reaches a point close to $Y(t-\delta)$. The discretization procedure is described in detail in Section 8.2.

Step 4.2. We further divide the squares of the grid in black squares and white squares: black squares are those that contain a (sufficiently large) amount of the walls γ_n. Most of the squares touched by Y^σ must therefore be white. If we meet a black square, we wish to get around it by a chain of white squares, which we call "detour" (compare with Definitions 8.1, 8.2, and 8.4, given in Section 8.3). The total length of the detours is proportional to the total length of chains of black squares met by Y^σ. However, since Y^σ meets either debris or a short connected component of γ_n with small total length, for σ sufficiently small the total length of the detours is estimated by $O(1)\varepsilon$. This is accomplished in Section 8.4.

Step 4.3. We next construct a piecewise linear trajectory Z that stays close to Y^σ when it crosses white squares and runs through the detours otherwise (reaching at the end a point close to $Y(T-\delta)$). This new trajectory runs through white squares only. Since white squares contain a small portion of walls γ_n, we use here Lemma 2.9 to achieve the following:

- Z does not cross the walls.
- On the white squares crossed by Y^σ, the segments of Z stay very close to those of Y^σ.

The construction of Z is given in Section 8.5.

Step 4.4. We finally specify the choice of the various parameters involved in the construction and show that Z is an admissible trajectory, which starts from R_0, avoids the walls of γ_n, and reaches a point close to $Y(T-\delta)$ in a time not larger than T (see Section 8.6).

Before coming to the rigorous proofs of these steps, let us remark that, without loss of generality, we can assume that:

(P14) The walls of γ_n do not intersect the set R_0.

In fact, if we replace $\gamma_n(t)$ with the strategies $t \mapsto \gamma_n(t) \setminus R_0$, we obtain new complete strategies with fewer walls but with the same contaminated region.

8.2 Choice of a Grid and Discretization of *Y*

Let $Y : [0, T-\delta] \mapsto \mathbb{R}^2$ be the shifted trajectory constructed in the previous section. As a first step, we replace Y by a polygonal trajectory Y^σ.

Choose coordinates (w_1, w_2) on $\mathbb{R}^2$. By a suitable rotation, we can assume that every vertical and horizontal line intersects every set $\gamma_n(t)$ in a set of zero length. This leaves free the choice of the origin of our system of coordinates, which for convenience we shall fix to be at $Y(0)$.

Given a mesh size $\sigma > 0$, we consider the grid formed by the lines $\{w_i = j\sigma\}$ for $i = 1, 2$ and $j \in \mathbb{Z}$. Points of the form $(w_1, w_2) = (j_1\sigma, j_2\sigma)$ with $j_1, j_2 \in \mathbb{Z}$ are the *vertices* of the grid, while horizontal or vertical segments of length σ connecting two vertices will be called *edges* of the grid. Finally, the sets of type $[j_1\sigma, (j_1+k)\sigma] \times [j_2\sigma, (j_2+k)\sigma]$ will be called *squares of size* $k\sigma$ of the grid. When not specified, we understand that the size of the square is σ.

We now choose times $\tau_0 < \tau_1 < \cdots < \tau_M \le T-\delta$ and values $Y^\sigma(\tau_i) \in \mathbb{R}^2$ with the following inductive procedure:

- $\tau_0 = 0$.
- Assume that τ_i has been chosen, together with a closed square Q_i of the grid containing $Y^\sigma(\tau_i)$. Consider the eight squares surrounding Q_i, which together with Q_i form a square $\tilde{Q}_i$ of size 3σ. The procedure stops if in the interval $[\tau_i, T-\delta]$ the trajectory Y^σ remains in $\tilde{Q}_i$. Otherwise we let τ^* be the first time larger than τ_i such that $Y(\tau^*) \in \partial\tilde{Q}_i$. If $\tau^* - t_i < \frac{\delta}{4}$, we define $\tau_{i+1} = \tau^*$ and $Y^\sigma(\tau_{i+1}) = Y(\tau_{i+1})$; otherwise we set $\tau_{i+1} = \tau_i + \frac{\delta}{8}$ and $Y^\sigma(\tau_{i+1}) = Y^\sigma(\tau_i)$.

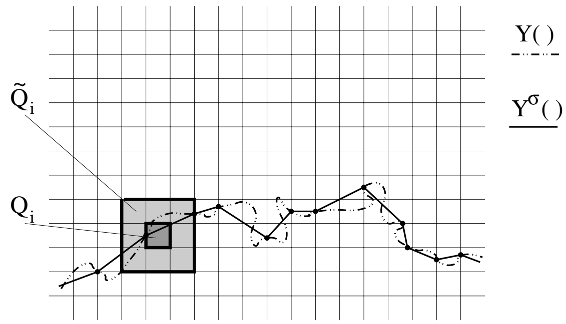

FIGURE 8.1. The grid of mesh σ and the new trajectory Y^σ.

Note that, at the time τ_M when the inductive procedure stops, we necessarily have $|Y^\sigma(\tau_M) - Y(T-\delta)| \le 2\sqrt{2}\,\sigma$.

We then define the piecewise affine map $Y^\sigma : [0, \tau_M] \to \mathbb{R}^2$ by setting

$$Y^\sigma(t) = \frac{t-\tau_i}{\tau_{i+1}-\tau_i} Y^\sigma(\tau_i) + \frac{\tau_{i+1}-t}{\tau_{i+1}-\tau_i} Y^\sigma(\tau_{i+1}) \quad \text{for } t \in [\tau_i, \tau_{i+1}]. \tag{8.1}$$

This new polygonal trajectory (see Figure 8.1) consists of the segments s_i joining the points $Y^\sigma(\tau_i)$ and $Y^\sigma(\tau_{i+1})$. We now recall that the multifunction F is Lipschitz-continuous, convex valued, and satisfies (P10) in Section 7.5. By choosing $\sigma > 0$ sufficiently small, we thus achieve

$$B(\dot{Y}^\sigma(t), \eta) \subseteq F(Y^\sigma(t)) \quad \forall t \in [0, \tau_M]. \tag{8.2}$$

8.3 Black Squares and White Squares

Let $t_1 < \cdots < t_N$ be the finitely many rational times chosen in Section 7.3. At a given time t_j, squares on the σ-grid will be classified as j-white (containing a minimum amount of walls in $\gamma_n(t_j)$) or j-black (containing a larger amount of walls in $\gamma_n(t_j)$, relative to their side length σ). This is reminiscent of a strategy proposed in the last section of [6] for the angel problem. More precisely, we fix a constant $\kappa > 0$ whose precise value will be specified later, and consider a $j \in \{1, \dots, N\}$. All the closed squares Q of the grid for which

$$m_1(\gamma_n(t_j) \cap Q) \ge \kappa\sigma \tag{8.3}$$

will be called j-black. The other closed squares will be called j-white. It is useful here to recall that, by the choice of the coordinates (w_1, w_2), we always have $m_1(\gamma_n(t_j) \cap \partial Q) = 0$.

Next, for each τ_i we choose $j(i)$ such that $t_{j(i)} \in [\tau_i + \frac{\delta}{2}, \tau_i + \delta]$. Note that the map j is not injective: for small σ, the number of times τ_i is of order $O(\sigma^{-1})$, while N is a fixed integer.

DEFINITION 8.1 A *path* of squares is an ordered collection of squares $Q_1, \dots, Q_L$ with the property that every two consecutive squares Q_i and Q_{i+1} have an edge in

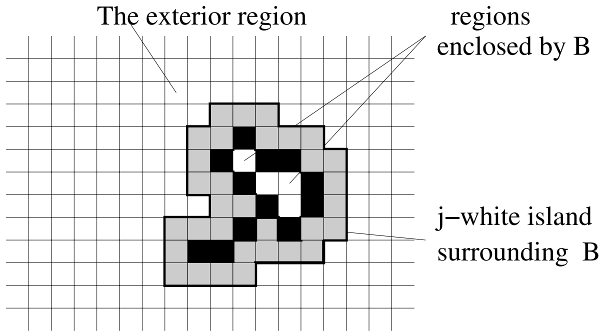

FIGURE 8.2. A j-black island B, the regions enclosed by the island, the exterior region, and the j-white island surrounding B.

common. Moreover, a set B of j-black squares $Q_1, \dots, Q_L$ form a *j-black island* if the following conditions hold:

(i) B is *connected*. That is, for every $Q_k, Q_l \in I$ there is a sequence $Q_{i_0}, \dots, Q_{i_R} \in B$ with $i_0 = k$, $i_R = l$, and the property that any two consecutive squares $Q_{i_m}, Q_{i_{m+1}}$ have at least one vertex in common.

(ii) B is *maximal*. That is, if a j-black square Q has a point in common with an element of B, then it belongs to B.

(iii) *B gets close to Y^σ*. That is, there is a square Q of B and an index i such that $j(i) = j$ and $\operatorname{dist}(Q, Y^\sigma(\tau_i)) \le 2\sqrt{2}\,\sigma$.

The length of a black island and the length of a path are both defined as σ multiplied by the number of squares forming the collection.

Next, given a j-black island B, we can subdivide the set of squares not belonging to B (and hence denoted by B^c) into connected components: two squares will belong to the same connected component if and only if there is a path connecting them that is completely contained in B^c. Of course, among these connected components there is one and only one consisting of infinitely many squares. We will call it the *region exterior to the island*. The remaining components will be called the *regions enclosed by the island*.

DEFINITION 8.2 Now, for every j-black island, consider the collection I of all squares that lie on the exterior region and have at least one vertex in common with a square of the black island. By (i) and (ii), all these squares are j-white. We will call I the *j-white island surrounding B* (see Figure 8.2).

It is straightforward to check that any two squares of I can be connected by a path of squares contained in I.

The following are obvious elementary observations:

- The j-white island enclosing a j-black island of length ℓ consists of at most $8\ell/\sigma$ squares.

- The diameter of a region enclosed by a j-black island of length ℓ is at most ℓ.
- If a point is at distance $2\sqrt{2}(\ell+\sigma)$ from a square of a j-black island with length ℓ, then it lies necessarily in the region exterior to the island.

We denote by $\mathcal{L}$ the sum of the lengths of all the j-black islands, also summed over all $j = 1, \ldots, N$. Notice that, by properties (i)–(iii) of black islands, $\mathcal{L}$ is closely related to the total length of all connected components of the walls constructed by γ_n that cross the polygonal Y^σ. We will prove below that, if σ is chosen sufficiently small, then

$$\mathcal{L} \leq \frac{4\varepsilon}{\kappa} \tag{8.4}$$

(see property (P22) in the next section). By choosing $\varepsilon \ll \kappa$, we can thus make $\mathcal{L}$ smaller than any given constant.

Remark 8.3. In fact, using more refined tools from geometric measure theory, it is possible to show that, for σ sufficiently small,

$$\mathcal{L} \leq 4\varepsilon. \tag{8.5}$$

However, the estimate (8.4) is easier to show and suffices to our purposes because we are free to choose ε much smaller than κ (compare with the final choice of the parameters, described in Section 8.6).

Note that, by (P14), R_0 does not contain any wall of the strategy γ, but it contains $Y(0) = 0$. This can be used to show that, for σ and ε sufficiently small, $Y(0)$ lies in the region exterior to any j-black island.

Indeed, consider $R_0^\sigma \doteq \{z \in R_0 : d(z, \partial R_0) \leq 2\sqrt{2}\,\sigma\}$. All squares at distance $\sqrt{2}\,\sigma$ from R_0^σ are white. Moreover, for σ sufficiently small R_0^σ contains 0. Let Δ^σ be the diameter of the connected component $\tilde{R}$ of R_0^σ containing 0. If B is any j-black island, by our choice, $\tilde{R}$ must be contained in a connected component of the complement $B^c = \mathbb{R}^2 \setminus B$. We can assume that Δ^σ is larger than a fixed constant (Δ^σ can only get larger when σ becomes smaller). By the discussion above, if ε is sufficiently small, we conclude that $\tilde{R}$ belongs necessarily to the region exterior to B. In conclusion, by choosing the mesh σ of the grid small enough, we can assume that:

(P15) For every j-black island B, the point $0 = Y(0)$ belongs to the region exterior to B.

We will now prove that, for σ and ε sufficiently small, we can find indices $0 = i_0 < i_i < \cdots < i_R$ with the following properties:

(P16) $|Y^\sigma(\tau_{i_R}) - Y^\sigma(\tau_M)| \leq 2\sqrt{2}\,\mathcal{L} + 2\sqrt{2}\,\sigma$.

(P17) For any index i, one of the following possibilities hold:
- $i_{k+1} = i_k + 1$ and the segment $s_{i_k} = [Y^\sigma(\tau_{i_k}), Y^\sigma(\tau_{i_{k+1}})]$ is either trivial or contained in a $j(i_k)$-white square or in a square of size 2σ formed by four $j(i_k)$-white squares;

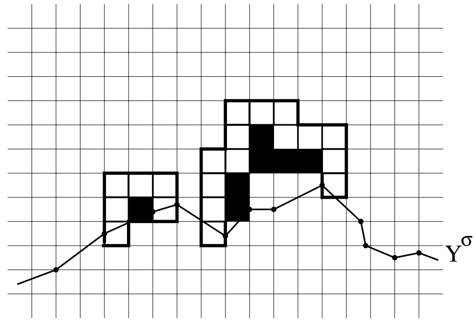

FIGURE 8.3. Two j-black islands and the surrounding paths of j-white squares, which provide the detours.

- there exists a path of $j(i_k)$-white squares $\mathcal{D}^k = \{Q_1^k, \dots Q_{L_k}^k\}$, where Q_1^k contains $Y^\sigma(\tau_{i_k})$, $Q_{L_k}^k$ contains $Y^\sigma(\tau_{i_{k+1}})$, and each square Q_l^k is contained in some j'-white island enclosing a j'-black island.

Observe that in this second case, the index j' might be distinct from $j(i_k)$.

DEFINITION 8.4 If k is an index such that the second case occurs, the corresponding path of white squares $\mathcal{D}_k$ will be called a *detour* and $L_k\sigma$ will be called the *length of the detour* (see Figure 8.3).

Note finally that there is a wide choice of detours: $\mathcal{D}_k$ is not uniquely determined.

In order to show the existence of a chain of indices satisfying (P16) and (P17) we proceed as follows: Consider the family $\mathcal{F}$ of chains of indices $0 = i_0 < i_1 < \cdots < i_R$ such that (P17) holds. Note that $Y(\tau_0)$ is contained in R_0, which is an open set. Therefore, for σ sufficiently small, the chain $\{i_0, i_1\}$ given by $i_0 = 0$ and $i_1 = 1$ satisfies (P17). Consider now a chain $\mathcal{C} \in \mathcal{F}$ such that i_R is as large as possible. We claim that, for such a maximal chain, (P16) holds.

We argue by contradiction and assume that (P16) does not hold. Consider the segment $s_{i_R} = [Y^\sigma(i_R), Y^\sigma(i_R+1)]$. This segment cannot be trivial, it cannot be contained in a $j(i_R)$-white square, and it cannot be contained in a square of size 2σ formed by four $j(i_R)$-white squares. Indeed, in any of these three cases, we could add $i_{R+1} = i_R + 1$ to our chain $\mathcal{C}$, contradicting its maximality.

It follows that $Y^\sigma(\tau_{i_R})$ belongs either to a $j(i_R)$-black island B or to a white square touching a $j(i_R)$-black island B. In both cases, since $|Y^\sigma(\tau_{i_R}) - Y^\sigma(\tau_M)| \geq 2\sqrt{2}\,\mathcal{L} + 2\sqrt{2}\,\sigma$ and the length of B is at most $\mathcal{L}$, $Y^\sigma(\tau_M)$ belongs necessarily to the region that is exterior to the black island B (compare with (P15)). Therefore, there is an $m > i_R$ such that $Y^\sigma(m)$ is contained in the $j(i_R)$-white island I surrounding B.

On the other hand, by (P15), $0 = Y^\sigma(0) = Y^\sigma(i_0)$ also belongs to the region exterior to B. So, either the j-white island I contains a point $Y^\sigma(i_k)$ for some index i_k, or it has a square in common with some detour $\mathcal{D}_k$. In the first case, we find a path $\mathcal{D}'_k = \{Q_1, \dots, Q_S\}$ of squares contained in the island I and connecting i_k and m. The island I is a $j(i_R)$-island surrounding a $j(i_R)$-black island and $i_k \le i_R$; each Q_l is therefore $j(i_k)$-white. Hence the chain $0 = i_0 < i_1 < \cdots < i_k < m$ contradicts the maximality of $\mathcal{C}$.

In the second case, a detour $\mathcal{D}_k$ has a square Q^k_l that also belongs to I. Therefore we can construct a path of squares $\mathcal{D}'_k = \{Q^k_1, \dots, Q^k_l, Q_{l+1}, \dots, Q_S\}$ where

- for $s \le l$, each Q^k_s is $j(i_k)$-white and is contained in a j'-white island surrounding a j'-black island;
- for $s \ge l$, Q_s is contained in I, and hence it is $j(i_R)$-white.

Even in this case the chain $0 = i_0 < i_1 < \cdots < i_k < m$ contradicts the maximality of $\mathcal{C}$.

Fix now a chain of indices $0 = i_0 < i_1 < \cdots < i_R$ satisfying (P16) and (P17) and fix a choice for the corresponding detours $\mathcal{D}_k = \{Q^k_1, \dots, Q^k_{L_k}\}$. The following procedure shortens the total length $L\sigma = (L_1 + \cdots + L_k)\sigma$ of the detours:

- If $Q^k_i = Q^k_j$ for some $i < j$, then we replace the detour $\mathcal{D}_k$ with the new detour
$$\mathcal{D}'_k = \{Q^1_k, \dots, Q^k_i, Q^k_{j+1}, \dots Q^k_{L_k}\}.$$
- If $Q^k_i = Q^{k'}_j$ for some $k < k'$, then we consider the new chain $0 = i_0 < i_1 < \cdots < i_k < i_{k'} < \cdots < i_R$, where i_k and $i_{k'}$ are connected by the detour $\mathcal{D}' = \{Q^k_1, \dots, Q^k_i, Q^{k'}_{j+1}, \dots Q^{k'}_{L_{k'}}\}$.

Each time that we can apply this procedure, the total length of the detours is decreased by at least σ. It turns out that, after a finite number of steps, the procedure cannot be applied any more. This means that we reach a chain of indices $0 = i_0 < i_1 < \cdots < i_R$ (and a choice of detours $\mathcal{D}_k = \{Q^k_1, \dots, Q^k_{L_k}\}$) satisfying (P16), (P17), and the following:

(P18) $Q^k_i \neq Q^{k'}_{i'}$ for any distinct pairs of indices (k, i) and (k', i'); namely, any square of the detours gets covered only once.

Property (P18) implies, in particular, that the total length of the detours can be bounded by the sum of the lengths of the j-white islands surrounding the j-black islands. This length, in turn, is bounded by 8 times the sum of the lengths of the black islands, which we denoted by $\mathcal{L}$. Summarizing, we have achieved that

(P19) The sum of the lengths of all detours is estimated by $8\mathcal{L}$.

8.4 Estimating the Total Length of the Black Islands

Fix any integer $n \ge 1$ large enough so that property (P13) holds. We claim that, if the grid size σ is sufficiently small, then none of the squares of the j-black islands intersect the large connected components $\{\gamma^{t_j}_{n,i}\}_{i \le i_1(t_j)}$ considered at (7.9).

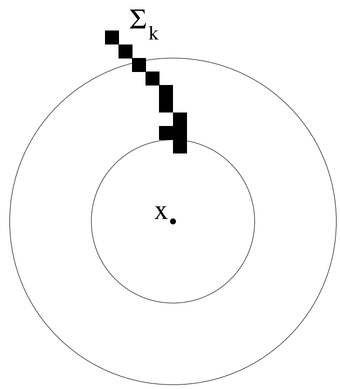

FIGURE 8.4. A chain of black squares joining a point in $B(x,r)$ to a point outside $B(x,2r)$ must contain at least $r/(\sqrt{2}\,\sigma_k)$ squares.

Since these large components are finitely many, it suffices to consider one of them, say $\beta = \gamma_{n,i}^{t_j}$, and show that for σ sufficiently small, it does not intersect any j-black island. We argue by contradiction and assume the claim to be false. Then there is a sequence of σ_k's converging to 0 and of black islands I_k intersecting β. Let Σ_k be the union of the closed squares forming the island I_k. Each Σ_k is a compact connected set. Moreover, it is easy to see that the diameter of Σ_k is uniformly bounded. Hence, up to extraction of a subsequence, we can assume that Σ_k converges to some compact set Σ in the Hausdorff metric. This limit set Σ is clearly connected and intersects β somewhere. Moreover, as $\sigma_k \downarrow 0$, the corresponding discretized trajectories Y^{σ_k} converge to Y. Therefore, Σ intersects the trajectory Y at some point $Y(t)$ with the property that $t_j \in [t + \frac{\delta}{2}, t + \delta]$.

We claim that $\Sigma \subseteq \gamma_n(t_j)$. This would imply that $\beta \cup \Sigma$ is a connected subset of $\gamma_n(t_j)$. But since β is a connected component of $\gamma_n(t_j)$, we would conclude that $\Sigma \subseteq \beta$. In turn, this implies that β contains the point $Y(t)$ above, thus contradicting (P11). Fix therefore a point $x \in \Sigma \setminus \beta$. Our goal is to show that $x \in \gamma_n(t_j)$. Consider any $r < d(x,\beta)/2$. For k sufficiently large, the disk $B(x,r)$ contains a square of I_k. On the other hand, I_k contains a point of β, which lies outside $B(x,2r)$. Since I_k is connected, $B(x,2r)$ must contain at least $r/(\sqrt{2}\,\sigma)$ squares of I_k (see Figure 8.4). Since each such square is j-black, we can estimate

$$m_1(B(x,2r) \cap \gamma_n(t_j)) \geq \frac{r}{\sqrt{2}\,\sigma}\,\kappa\sigma = \frac{\kappa}{\sqrt{2}}\,r.$$

By the arbitrariness of r, we conclude $\theta^*(\gamma_n(t_j), x) > 0$, and hence $x \in \gamma_n(t_j)$, because γ_n is a complete strategy. This proves our claim.

The above argument shows that, if $\sigma > 0$ is sufficiently small, for any j-black island I and any square $Q \in I$,

$$Q \cap \gamma_n(t_j) \subset \gamma_n'(t_j) \cup \gamma_n''(t_j). \tag{8.6}$$

In other words, the j-black islands can intersect the small connected components or the debris produced by γ_n, but they do not intersect any of the large connected components. It follows therefore that $Q \cap \gamma_n(t_j) \subset E_{n,j}$, where $E_{n,j}$ is the set enjoying properties (P5)–(P8) of Section 7.4 and (P13) of Section 7.5.

Using (P13), we now choose a finite set $\mathcal{P}$ of pairs (j,i) satisfying

(P20) for every $(j,i) \in \mathcal{P}$, the set $E_{n,j,i}$ does not intersect the trajectory Y,

and so that, defining

$$H_n \doteq \bigcup_{(j,i)\notin\mathcal{P}} E_{n,j,i}, \tag{8.7}$$

the following estimate holds:

$$m_1(H_n) \le 4\varepsilon. \tag{8.8}$$

Repeating the previous arguments, we conclude that, for σ sufficiently small,

(P21) $Q \cap \gamma_n(t_j) \subseteq H_n$ for every square Q in a j-black island.

Since every black island contains an amount $\ge \kappa\sigma$ of walls, by (P21) and (8.8) it follows that, for any n large enough, by choosing the grid size σ sufficiently small we can achieve the following:

(P22) The total length $\mathcal{L}$ of all black islands is $\le 4\frac{\varepsilon}{\kappa}$.

8.5 The Admissible Trajectory

At this stage, we assume that n is so large and σ is so small that the choice of indices $i_1 < \cdots < i_R$ in Section 8.3 is possible and satisfies all properties (P16)–(P19). We still have not chosen the parameters κ and ε. For technical reasons we also assume that none of the points $Y^\sigma(\tau_{i_k})$ lies exactly on a vertex of the grid. This can easily be achieved with an arbitrarily small perturbation, which does not affect any of the previous estimates. Therefore, only two possibilities can occur:

- $i_{k+1} = i_k + 1$ and the segment $s_{i_k} = [Y^\sigma(\tau_{i_k}), Y^\sigma(\tau_{i_{k+1}})]$ is either trivial or contained in a $j(i_k)$-white square or in a square of size 2σ consisting of $j(i_k)$-squares.
- Or else the points $Y^\sigma(\tau_{i_k})$ and $Y^\sigma(\tau_{i_{k+1}})$ can be joined by a polygonal curve α_k entirely contained inside the path of $j(i_k)$-white squares, say of length $L\sigma$. The polygonal α_k will consist of N_k segments, each contained in a distinct square of a $j(i_k)$-white path of length $N_k\sigma$, and with vertices lying on different sides of the square (see Figure 8.5). As before, we assume that none of these vertices lies exactly at the intersection between a horizontal and a vertical grid line.

We think of the curves α_k as "detours," which are needed to bypass the black islands. The union of the segments s_{i_k} and of the curves α_k gives us a new polygonal curve, which we denote by Y^*. A time parametrization of the curve Y^* is obtained as follows: Let $p_0, p_1, \dots, p_K$ be the vertices of this polygonal curve. If

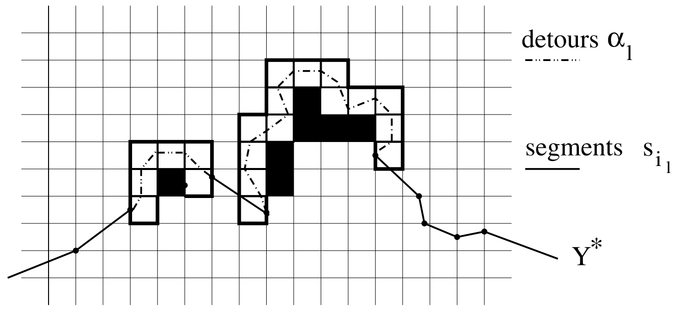

FIGURE 8.5. The curve Y^*. Compared with Y^σ in Figure 8.3, some segments s_i have been removed and replaced by the detours α_ℓ to get around black islands.

the segment joining p_j and p_{j+1} is one of the segments s_{i_k} of the original polygonal curve Y^σ, then we cover it at the same constant speed as Y^σ, in a time equal to $\tau_{i_k+1} - \tau_{i_k}$. Notice that, by construction, this segment is either trivial (i.e., a point) or it has length

$$\sigma \le |p_{j+1} - p_j| \le 2\sqrt{2}\,\sigma. \tag{8.9}$$

On the other hand, if the segment joining p_j and p_{j+1} is part of some detour α_k, we cover it at constant speed ρ_0, where $\rho_0 > 0$ is the constant in the assumption (A2) in the introduction. By estimates (P19) and (P22), the sum of the lengths of all detours α_k is $\le 40\sqrt{2}\,\varepsilon/\kappa$. Choosing $0 < \varepsilon < \kappa\rho_0/(320\sqrt{2})$, we have the inequality

$$\frac{40\sqrt{2}\,\varepsilon}{\kappa\rho_0} \le \frac{\delta}{8}. \tag{8.10}$$

We thus achieve a parametrization $t \mapsto Y^*(t)$ of the entire polygonal curve Y^*, defined for $t \in [0, T^*]$, with $T^* \le T - \delta + 40\sqrt{2}\,\varepsilon/\kappa\rho_0 \le T$, such that:

(P23) $B(\dot{Y}^*(t), \eta) \subseteq F(Y^*(t))$ on the segments of the original polygonal curve Y^σ.

(P24) $|\dot{Y}^*(t)| = \rho_0$ along the detours α_ℓ.

From now on, we denote by ξ_i the time such that $Y^*(\xi_i) = p_i$. At this stage, one should notice that $Y^*(\cdot)$ is a solution of the differential inclusion (1.1), but it may not be admissible. Indeed, it may still hit some of the walls of γ_n. By our construction, however, the trajectory Y^* crosses only white squares, containing a very small amount of walls.

More precisely, consider two consecutive times ξ_i and ξ_{i+1} and the corresponding segment $[p_i, p_{i+1}]$. Two cases might occur:

Case 1. $[p_i, p_{i+1}] = [Y^\sigma(\tau_{l_k}), Y^\sigma(\tau_{l_{k+1}})]$ where $l_{k+1} = l_k + 1$. The segment $[p_i, p_{i+1}]$ might be trivial. Otherwise it is contained in a square $\mathcal{Q}_i$ that is either of size σ or of size 2σ and consists of $j(l_k)$-white squares. In both cases

$$m_1(\mathcal{Q}_i \cap \gamma(t_{j(l_k)})) \leq 4\kappa\sigma. \tag{8.11}$$

On the other hand:

- $\xi_{i+1} - \xi_i = \tau_{l_k} - \tau_{l_k+1} < \frac{\delta}{8}$.
- $\xi_i \leq \tau_{l_k} + \frac{\delta}{8}$, because in covering the detours of Y^*, we get delayed by at most $\frac{\delta}{8}$,
- $t_{j(l_k)} \geq \tau_{l_k} + \frac{\delta}{2}$, by the choice of $j(l_k)$ in Section 8.3.

We conclude that $\xi_{i+1} \leq t_{j(l_k)} - \frac{\delta}{4}$ and hence, by (8.11),

$$m_1(\mathcal{Q}_i \cap \gamma(\xi_{i+1} + \tfrac{\delta}{4})) \leq 4\kappa\sigma. \tag{8.12}$$

Case 2. $[p_i, p_{i+1}]$ is contained in a detour, connecting an index l_k with an index l_{k+1} in the chain selected to define Y^*. Consider then the time ξ_m when $Y^*(\xi_m) = Y^\sigma(\tau_{l_k})$. Once again we know that $\xi_m \leq \tau_{l_k} + \frac{\delta}{8}$. Moreover, $\xi_{i+1} - \xi_m$ is smaller than the total time taken to cover all the detours, which is $\leq \frac{\delta}{8}$. Hence, also in this case we have $\xi_{i+1} \leq \tau_{l_k} + \frac{\delta}{4} \leq t_{j(l_k)} - \frac{\delta}{4}$. Recall that $[p_i, p_i + 1]$ is contained in a $j(l_k)$-white square $\mathcal{Q}_i$. Therefore, we deduce that for this square (8.12) holds.

Next, to simplify our discussion, we eliminate from Y^* the interval of times when the velocity is 0. The only difference is that we may cover Y^* in less time. In any case, all the previous conclusions continue to hold. By a slight abuse of notation, we still write

- Y^* for the trajectory,
- p_i for the endpoints of the segments, and
- ξ_i for the times such that $Y^*(\xi_i) = p_i$.

Therefore, we now have that $p_{i+1} \neq p_i$ for every i and that there always exists a square $\mathcal{Q}_i$, of size σ or 2σ, that contains $[p_{i+1}, p_i]$ and such that

(P25) $m_1(\mathcal{Q}_i \cap \gamma(\xi_{i+1} + \frac{\delta}{4})) \leq 4\kappa\sigma$.

The final step in the proof will show that, by a small perturbation of the vertices $p_0, \dots, p_K$, we obtain another solution of (1.1) that does not intersect any of the walls.

Towards this goal, given a constant $0 < \mu < 1$, we consider a second polygonal curve Z with vertices at points $p'_0, \dots, p'_K$ (see Figure 8.6).

DEFINITION 8.5 We say that the polygonal curve Z is *μ-close* to Y^* if, for every $i = 0, 1, \dots, K$, the point p'_i lies on the same edge of the grid as p_i, and moreover $|p'_i - p_i| \leq \mu\sigma$.

Starting from the parametrization of Y^*, a μ-close polygonal curve Z can be parametrized as follows:

- If the segment joining p_j and p_{j+1} is one of the segments of the original polygonal curve Y^σ, then the corresponding segment $[p'_j, p'_{j+1}]$ in the polygonal curve Z will be covered with constant speed
$$\dot{Z} = \frac{p'_{j+1} - p'_j}{\xi_{j+1} - \xi_j}.$$
Recall that, by construction, the segments $[p_j, p_{j+1}]$ have length at least σ. This fact, together with (P23), implies that, if we choose $\mu \ll \eta$, then $\dot{Z}(t) \in F(Z(t))$ during these time intervals.
- If the segment joining p_j and p_{j+1} is part of a detour, then the corresponding segment $[p'_j, p'_{j+1}]$ will be covered with constant speed
$$\dot{Z} = \rho_0 \cdot \frac{p'_{j+1} - p'_j}{|p'_{j+1} - p'_j|}.$$
By assumption (A2), this again implies $\dot{Z}(t) \in F(Z(t))$.

Since the segments of the polygonal curve Z cover the same white squares as Y^*, recalling (8.10) we conclude that the total time spent to cover all the detours in Z is still $\leq 40\sqrt{2}\,\varepsilon/(\kappa\rho_0) \leq \frac{\delta}{8}$. Hence Z can also be parametrized by a solution of (1.1) on some time interval $[0, T_Z] \subseteq [0, T]$. In the end, by possibly taking $\dot{Z}(t) \equiv 0$ for $t \in [T_Z, T]$, it is not restrictive to assume $T_Z = T$. Moreover, if we denote by ζ_i the times such that $Z(\zeta_i) = p'_i$, we get

$$\zeta_i \leq \xi_i + \frac{\delta}{8}. \tag{8.13}$$

To conclude the proof of Lemma 6.1, it suffices to show that, given $\mu > 0$, if κ is chosen sufficiently small, then we can find a polygonal curve Z that is μ-close to Y^* and does not hit any of the walls constructed by the strategy γ_n. In other words, we need to find the points p'_k in such a way that:

(P26) The segments $[p'_k, p'_{k+1}]$ do not intersect the walls of $\gamma_n(\xi_{k+1} + \frac{\delta}{8})$.

By (8.13), it follows that $[p'_k, p'_{k+1}]$ does not cross the walls $\gamma_n(\zeta_{k+1})$, which in turn implies $Z(t) \notin \gamma_n(t)$ for every t.

In order to find the new points satisfying (P26), we will use (P25).

For each point p_k choose a segment J_k of size $\mu\sigma$ that lies on the side of the corresponding square of the grid and contains p_k. For each point $p \in J_k$, we let $G(p)$ be the set of points $q \in J_{k+1}$ for which the segment $[q, p]$ does not intersect $\gamma_n(\xi_{k+1} + \frac{\delta}{4})$.

Using Lemma 2.9 and a simple rescaling argument we obtain the following:

(P27) If the constant κ is sufficiently small, then there is a set $J'_k \subseteq J_k$ with $m_1(J'_k) \geq 2\mu\sigma/3$ such that $m_1(G(p)) \geq 2\mu\sigma/3$ for every $p \in J'_k$.

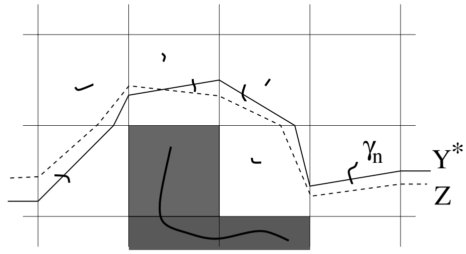

FIGURE 8.6. The curve Y^* (solid line) and a μ-close polygonal curve Z that avoids the walls in γ_n.

By (P27), we can now choose inductively the points p'_k so that $p'_{k+1} \in G(p'_k) \cap J'_{k+1}$. We start by taking any $p'_1 \in J'_1$. Having chosen p'_k, we pick $p'_{k+1} \in G(p'_k) \cap J'_{k+1}$. Indeed, since

- J'_{k+1} and $G(p'_k)$ are contained in J_{k+1},
- $m_1(J_{k+1} \setminus J'_{k+1}) \le \mu\sigma/3$,
- $m_1(J_{k+1} \setminus G(p'_k)) \le \mu\sigma/3$ (because $p'_k \in J'_k$), and
- $m_1(J_{k+1}) = \mu\sigma$,

we conclude that $m_1(G(p'_k) \cap J'_{k+1}) \ge \mu\sigma/3 > 0$ and hence that a point $p'_{k+1} \in G(p'_k) \cap J'_{k+1}$ exists. This achieves the construction of the modified polygonal curve Z.

8.6 Choice of Constants and Conclusion of Proof

We summarize here the main steps of the proof, clarifying the order in which the various constants are chosen.

(1) The radius $\rho_0 > 0$ is given in assumption (A2).

(2) A minimizing sequence γ_n is given. The limiting strategy γ is defined by ordering the connected components of $\gamma_n(t)$ in decreasing length and taking limits componentwise.

(3) By possibly extracting a subsequence, we can assume that $\sum_n \ell''_n < \infty$, where ℓ''_n is the maximum length of connected components of the debris produced by γ_n.

(4) A point $y \in R^\gamma(T)$ is given, together with a radius $r > 0$ and an admissible trajectory $x(\cdot)$, which starts from r_0 and reaches y at time T.

(5) We choose $\eta, \delta > 0$ and construct a faster trajectory $x^\sharp(\cdot)$ reaching y at time $T - \delta$ and satisfying the conclusions of Lemma 7.1. In turn, the size of δ determines the choice of the times $t_1 < \cdots < t_N$ in (P4).

(6) Depending on η and δ, we choose the constant $\mu > 0$ in 8.5, determining by how much a polygonal curve can be perturbed. Using Lemma 2.9, given μ we choose the corresponding constant $\kappa > 0$, specifying the maximum size of walls allowed inside a white square.

(7) Depending on η, δ, and κ, we choose the constant $\varepsilon > 0$ measuring the total length of small connected components in the strategies γ_n. This determines the finitely many integers $i_1(t_j)$, specifying which connected components are considered "large" and which are "small" in the decomposition (7.4).

(8) Depending on ε, we choose a radius $0 < r_1 < \frac{r}{2}$ such that, for $|v| \leq r_1$, the shifted trajectory x^v in (7.6) does not intersect the large connected components of the walls in γ_n for all n sufficiently large.

(9) We choose a particular shifted trajectory $Y = x^v$ satisfying (P9) and (P10).

(10) For each n sufficiently large, we construct a grid with mesh σ (possibly depending on n and on every other previous constant). The trajectory Y is approximated by a polygonal curve Y^σ, then replaced by a second polygonal curve Y^* that passes entirely over white squares.

(11) Finally, we determine a new polygonal curve Z that is μ-close to Y^* and does not hit any of the walls constructed by γ_n.

With a suitable parametrization, the polygonal curve $t \mapsto Z(t)$ satisfies the differential inclusion (1.1), starts inside R_0, and reaches a point $Z(T) \in B(y, r)$ at time T. This achieves the proof of Lemma 6.1.

9 Proof of Lemma 6.2

For any rational time $t \geq 0$ consider the measures μ_n^t defined by

$$\mu_n^t(E) \doteq m_1(E \cap \gamma_n(t)).$$

By possibly extracting a subsequence, we can assume the weak-star convergence $\mu_n^t \rightharpoonup^* \mu^t$ for some limit measure μ^t. For each $t \in \mathbb{Q}^+$, consider the set

$$E_t \doteq \left\{ x \in \mathbb{R}^2 : \omega(t,x) \doteq \liminf_{r \downarrow 0} \frac{\mu^t(B(x,r))}{2r} > 0 \right\}.$$

Clearly, if we set $E_{t,i} \doteq \{x : \omega(t,x) > i^{-1}\}$, then $E_t = \bigcup_i E_{t,i}$. By a well-known theorem in measure theory (see, for instance, proposition 2.56 of [1]), $m_1(E_{t,i}) \leq i\mu^t(\mathbb{R}^2) < \infty$. Therefore, setting $E \doteq \bigcup_{t \in \mathbb{Q}} E_t$, we have $m_2(E) = 0$.

We will prove the claim of the lemma for every $x \in R_\infty^\gamma \setminus E$. From now on we assume that $x \in R_\infty^\gamma$ and $\varepsilon > 0$ are given. Consider a trajectory of (1.1) that reaches the point x at a time earlier than some $T \in \mathbb{Q}$ and avoids the walls constructed by the strategy γ.

We choose r_0 so that

$$0 < r_0 < d(x, R_0), \quad 2(\pi + 1)\frac{r_0}{\rho_0} < 1.$$

We also fix a small constant $\kappa > 0$, whose specific value will be determined later. Since $x \notin E_{T+1}$, there exists $0 < r < r_0/2$ such that

$$\mu^{T+1}(B(x, 4r)) \leq \kappa r.$$

Because of Lemma 6.1 and the weak convergence of μ_n^{T+1} to μ^{T+1}, we can find an integer N so large that, for every $n > N$, the following two conditions hold:

- There exists a trajectory $x_n(\cdot)$ of (1.1) avoiding the walls of γ_n and reaching a point of $B(x,r)$ at a time $T_0 < T$.
- $m_1(\gamma_n(T+1) \cap B(x,2r)) \leq 2\kappa r$.

Next, for each direction $\theta \in [0, 2\pi]$ we consider the radial segment

$$\sigma_\theta \doteq \{x + \rho(\cos\theta, \sin\theta) : \rho \in [\varepsilon, 2r]\}.$$

Let G be the union of all segments σ_θ that do not intersect the set of walls $\gamma_n(T+1)$. If κ is chosen sufficiently small, then $m_2(G \cap B(x,r)) \geq (1-\varepsilon)\pi r^2$. Moreover, if $\kappa < \frac{1}{4}$, then there exists a circle $\partial B(x,\rho)$ with radius $\rho \in \,]r, 2r[$ that does not intersect $\gamma_n(T+1)$.

Since $\rho < 2r < r_0 < d(x, R_0)$, the trajectory x_n must cross the circle $\partial B(x,\rho)$ at some point $x_n(\tau)$. Consider now a point $z \in G$ and the segment σ_θ containing z. We then can construct a trajectory $x^\sharp$ as follows (see Figure 9.1).

(1) We follow x_n until we reach a point $x_n(\tau)$ on the circumference $\partial B(x,\rho)$.
(2) We move along $\partial B(x,\rho)$ until we reach a point y on the radial segment σ_θ.
(3) Starting from y, we move along the segment σ_θ until we reach z.

We cover parts (2) and (3) at the constant speed $\rho_0 > 0$, considered in assumption (A2). In this way, the trajectory reaches y before the time

$$T + 2(\pi+1)\frac{r_0}{\rho_0} \leq T + 1.$$

By the previous arguments, the trajectory $x^\sharp$ avoids the walls $\gamma_n(T+1)$. Therefore, $G \subseteq R_\infty^{\gamma_n}$. We thus conclude that

$$m_2\big(B(x,r) \cap R_\infty^{\gamma_n}\big) \geq (1-\varepsilon)\pi r^2$$

for every $n > N$. This concludes the proof of the lemma.

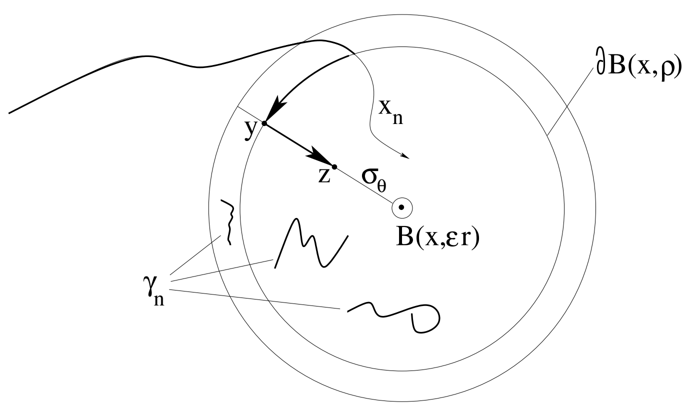

FIGURE 9.1. The trajectory $x^\sharp$, reaching z and avoiding the walls γ_n.

10 Concluding Remarks

10.1 Relevance of Assumptions

In this paper we proved the existence of optimal strategies under hypotheses (A1) through (A3). We discuss here the relevance of these assumptions, and hint at possible extensions.

The assumption that the initial set R_0 is open is used in an essential way. It guarantees that, given any admissible trajectory $x : [0, T] \mapsto \mathbb{R}^2$, there exists a second trajectory $x^\sharp$ reaching the same endpoint at an earlier time $T - \delta$. This assumption is also natural in the sense that, in a realistic situation, we expect that for any $t > 0$ the set of walls $\gamma(t)$ will be a closed, piecewise smooth curve, while the set $R^\gamma(t)$ is an open set, having $\gamma(t)$ as part of its boundary.

The lower semicontinuity of the constraint ψ and of the cost function β are obvious assumptions for a minimization problem. Another reasonable restriction is $\alpha(x) \geq \alpha_0 > 0$. Indeed, this was the basic setting considered in [4, 5].

Assuming that the multifunction F is Lipschitz-continuous with convex values is entirely appropriate for applications. The assumption $F(x) \supseteq B(0, \rho_0)$ is used in an essential way in our proof. Indeed, it guarantees that the fire can move in all directions, and go through all detours, as described in Section 8, at some uniformly positive speed. Using the continuity of F, one may somewhat weaken this assumption by only requiring that $F(x)$ be a neighborhood of the origin for every $x \in \mathbb{R}^2$.

10.2 Proof of Corollary 1.2

The regularity of the optimal strategy, stated in the corollary, is a simple consequence of the construction given in Section 5. We recall that:

- For each $\tau \in \mathbb{Q}$, the set $\gamma^\flat(\tau)$ is the union of (at most) countably many continua.
- By definition, $\gamma^\sharp(t) \doteq \bigcap_{\tau \in \mathbb{Q}, \tau > t} \gamma^\flat(\tau)$.
- The optimal strategy $\gamma(\cdot)$ is then obtained as the completion of $\gamma^\sharp(\cdot)$.

By Lemma 4.2, $\gamma(t) \supset \gamma^\sharp(t)$, and, except for at most countably many points, $m_1(\gamma(t) \setminus \gamma^\sharp(t)) = 0$. Therefore, it suffices to prove that, with the exception of countably many times t, $\gamma^\sharp(t)$ consists of countably many continua with positive length plus a set of m_1-measure zero.

Note that, by construction, $\gamma^\sharp(t) \subset \gamma^\sharp(s)$ for $t < s$. Therefore, the points of discontinuity of $t \mapsto m_1(\gamma^\sharp(t))$ are at most countable. Consider now a t_0 such that $t \mapsto m_1(\gamma^\sharp(t))$ is continuous at t_0. For any $t < \sigma < t_0$, with $\sigma \in \mathbb{Q}$, we have $\gamma^\sharp(t) \subset \gamma^\flat(\sigma) \subset \gamma^\sharp(t_0)$, where the second inclusion is a consequence of Lemma 5.3. Therefore, looking at the set $\tilde{\gamma}(t_0) \doteq \bigcup_{\tau < t_0} \gamma^\flat(\tau)$, we conclude that $\tilde{\gamma}(t_0) \subset \gamma^\sharp(t_0)$ and $m_1(\gamma^\sharp(t_0) \setminus \tilde{\gamma}(t_0)) = 0$. Since $\tilde{\gamma}(t_0)$ is the union of (at most) countably many continua, this concludes the proof.

Acknowledgments. The work of the first author was supported by National Science Foundation Grant DMS-0505430. The work of the second author was supported by the Swiss National Foundation.

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ALBERTO BRESSAN
Penn State University
Department of Mathematics
201 McAllister Building
University Park, PA 16802
E-mail: `bressan@math.psu.edu`

CAMILLO DE LELLIS
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
SWITZERLAND
E-mail: `camillo.delellis@math.unizh.ch`

Received March 2008.
Revised September 2008.