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ORDINARY DIFFERENTIAL EQUATIONS WITH ROUGH COEFFICIENTS AND THE RENORMALIZATION THEOREM OF AMBROSIO
[d'après Ambrosio, DiPerna, Lions]

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INTRODUCTION

Consider the Cauchy problem for transport equations on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot \nabla_x u(t,x) \;=\; 0 \\ \\ u(0,x) \;=\; \overline{u}(x)\,. \end{cases} \tag{1}$$

Here $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth vector field, $\overline{u}$ a given smooth initial condition and u the unknown function. Smooth solutions of (1) are constant along curves $\phi : [a,b] \to \mathbb{R}^n$ solving the system of ordinary differential equations $\dot{\phi}(t) = b(t,\phi(t))$. Indeed, differentiating $g(t) = u(t,\phi(t))$ we find

$$\frac{dg}{dt} \;=\; \partial_t u(t,\phi(t)) + \dot{\phi}(t)\cdot\nabla_x u(t,\phi(t)) \;=\; \partial_t u(t,\phi(t)) + b(t,\phi(t))\cdot\nabla_x u(t,\phi(t)) \;=\; 0\,.$$

Thus, if $\Phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the one–parameter family of diffeomorphisms solving

$$\begin{cases} \partial_t \Phi(x,t) \;=\; b(t,\Phi(x,t)) \\ \\ \Phi(0,x) \;=\; x \end{cases} \tag{2}$$

and $\Phi^{-1}(t,\cdot)$ denotes the inverse of the diffeomorphism $\Phi(t,\cdot)$, then the unique solution u of (1) is given through the formula $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$. This is the classical method of characteristics for transport equations. Our discussion justifies the name transport equation: the quantity u is simply "transported" along the trajectories of the ODE (2). It is therefore not surprising that these equations appear in the mathematical description of many phenomena in classical and statistical physics.

When b is Lipschitz, existence and uniqueness of solutions to (2) are given by the classical Cauchy–Lipschitz Theorem, but for less regular b this elegant and elementary picture breaks down. On the other hand, many physical phenomena lead naturally to consider transport equations where the coefficients b are discontinuous. The literature related to this kind of problems is huge and I will not try to give an account of it here. Let me just mention that in many of these problems one deals with coefficients which typically have jump discontinuities, take for instance the theory of shock waves.

It is therefore desirable to have a theory of solutions for ODEs and transport equations which allows for non–smooth coefficients. The Sobolev spaces $W^{1,p}$ (given by functions

$u \in L^p$ with distributional derivatives in L^p) are probably the most popular spaces of irregular functions in partial differential equations. In their groundbreaking paper [28], motivated by their celebrated work on Boltzmann equation, DiPerna and Lions introduced a theory of generalized solutions for transport equations and ODEs with Sobolev coefficients. Loosely speaking, this is done at the loss of a "pointwise" point of view into an "almost everywhere" point of view. Though a generic function $u \in W^{1,p}(\Omega)$ might be extremely irregular, its singular set, at least in a suitable measure theoretic sense, has necessarily codimension higher than 1. In particular, functions with jump discontinuities do not belong to $W^{1,p}$. Indeed, if the discontinuities are along nice regular surfaces, the distributional derivatives are nothing more than Radon measures.

A commonly used functional–analytic closure of such "jump functions" is the BV space, i.e. the set of summable functions whose distributional derivatives are Radon measures. The extention of the DiPerna–Lions theory to BV functions has been for a while an important open problem. After some attempts by other authors leading to partial results (see [33], [15], [21]; some of these works were motivated by specific problems in partial differential equations and mathematical physics), Ambrosio solved the problem in its full generality in [4]. This note is an attempt to illustrate the most important ideas of the DiPerna–Lions theory and of Ambrosio's result. In order to focus on the main points, I will not consider the most general results proved so far. Moreover, I will not follow the shortest proofs and often I will consider cases which later on become corollaries of more general theorems.

In the first section I discuss the first key idea of [28]: the notion of renormalized solutions and its link to the uniqueness and stability for (1). In Section 2 I discuss the hard core of the DiPerna–Lions theory for $W^{1,p}$ fields: the so called commutator estimate. In Section 3, following the ideas of Ambrosio, I push gradually the DiPerna–Lions approach towards the BV case. The proof of Ambrosio's Theorem is finally achieved in Section 4 in two different ways, based on observations of Bouchut and Alberti. Section 5 discusses the third key idea of [28], a sort of converse of the classical theory of characteristics: appropriate results on transport equations can be used to infer interesting conclusions on ODEs. Section 6 surveys further results, conjectures and open problems in three different directions of research. Section 7 contains the proof of one technical proposition on BV functions used in Section 3.

1. RENORMALIZED SOLUTIONS

1.1. Distributional solutions

Let us start by rewriting (1) in the following way:

$$\left\{ \begin{array}{rcl} \partial_t u + \mathrm{div}_x(ub) - u\,\mathrm{div}_x b &=& 0 \\ \\ u(0,x) &=& \overline{u}(x)\,. \end{array} \right. \tag{3}$$

Here and in what follows I denote by $\mathrm{div}_x b$ the divergence (in space) of the vector b. Clearly any classical solution of (3) is a solution of (1) and viceversa. However, equation (3) can be understood in the distributional sense under very mild assumptions on u and b. This is stated more precisely in the following definition.

DEFINITION 1.1. — *Let b and $\overline{u}$ be locally summable functions such that the distributional divergence of b is locally summable. We say that $u \in L^\infty_{\mathrm{loc}}$ is a distributional solution of* (3) *if the following identity holds for every test function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$*

$$\int_0^\infty \int_{\mathbb{R}^n} u\,[\partial_t\varphi + b\cdot\nabla_x\varphi + \varphi\,\mathrm{div}_x b]\,dx\,dt \;=\; -\int_{\mathbb{R}^n} \overline{u}(x)\varphi(0,x)\,dx \tag{4}$$

Of course for classical solutions the identity (4) follows from a simple integration by parts. The existence of weak solutions under quite general assumptions is an obvious corollary of the maximum principle for transport equations combined with a standard approximation argument.

LEMMA 1.2 (Maximum Principle). — *Let b be smooth and let u be a smooth solution of* (3). *Then, for every t we have* $\sup_{x\in\mathbb{R}^n} u(t,x) \le \sup_{x\in\mathbb{R}^n}\overline{u}(x)$ *and* $\inf_{x\in\mathbb{R}^n} u(t,x) \ge \inf_{x\in\mathbb{R}^n}\overline{u}(x)$. *Hence* $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|\overline{u}\|_\infty$.

Proof. — The lemma is a trivial consequence of the method of characteristics. Indeed, arguing as in the introduction $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$, where Φ is the solution of (2). From this representation formula the inequalities follow trivially. □

THEOREM 1.3. — *Let $b \in L^p$ with $\mathrm{div}_x b \in L^1_{loc}$ and let $\overline{u}\in L^\infty$. Then there exists a distributional solution of* (3).

Proof. — Consider a standard family of mollifiers ζ_ε and η_ε respectively on $\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{R}^n$. Let $b_\varepsilon = b*\eta_\varepsilon$ and $\overline{u}_\varepsilon = \overline{u}*\zeta_\varepsilon$ be the corresponding regularizations of b and $\overline{u}$. Then $\|\overline{u}_\varepsilon\|_\infty$ is uniformly bounded. Consider the classical solutions u_ε of

$$\begin{cases} \partial_t u_\varepsilon + b_\varepsilon\cdot\nabla_x u_\varepsilon \;=\; 0 \\[2ex] u_\varepsilon(0,\cdot) \;=\; \overline{u}_\varepsilon\,. \end{cases} \tag{5}$$

Note that such solutions exist because we can solve the equation with the method of characteristics: indeed each b_ε is Lipschitz and we can apply the classical Cauchy–Lipschitz theorem to solve (2). By Lemma 1.2 we conclude that $\|u_\varepsilon\|_\infty$ is uniformly bounded. Hence there exists a subsequence converging weakly* to a function $u \in L^\infty(\mathbb{R}^+\times\mathbb{R}^n)$. Let us fix a test function $\varphi\in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$. Since the u_ε are classical solutions of (5), the identity (4) is satisfied if we replace u, b and $\overline{u}$ with u_ε, b_ε and $\overline{u}_\varepsilon$. On the other hand, since $b_\varepsilon\to b$, $\mathrm{div}_x b_\varepsilon \to \mathrm{div}_x b$ and $\overline{u}_\varepsilon\to\overline{u}$ locally strongly in L^1_{loc}, we can pass into the limit in such identities to achieve (4) for u, $\overline{u}$ and b. □

1.2. Renormalized solutions

Of course the next relevant questions are whether such distributional solutions are unique and stable. Under the general assumptions above the answer is negative, as it is for instance witnessed by the elegant example of [27]. However, DiPerna and Lions in [28] proved stability and uniqueness when $b \in W^{1,p} \cap L^\infty$ and $\mathrm{div}_x b \in L^\infty$.

Theorem 1.4. — *Let $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n)) \cap L^\infty$ with bounded divergence. Then for every $\overline{u} \in L^\infty$ there exists a unique distributional solution of (3). Moreover, let b_k and $\overline{u}_k$ be two smooth approximating sequences converging strongly in L^1_{loc} to b and $\overline{u}$ such that $\|\overline{u}_k\|_\infty$ is uniformly bounded. Then the solutions u_k of the corresponding transport equations converge strongly in L^1_{loc} to u.*

In order to understand their proof, we first go back to classical solutions u of (3), and we observe that, whenever $\beta : \mathbb{R} \to \mathbb{R}$ is a C^1 function, $\beta(u)$ solves

$$(6) \qquad \begin{cases} \partial_t[\beta(u)] + \mathrm{div}_x[\beta(u) b] - \beta(u)\, \mathrm{div}_x b \;=\; 0 \\ \\ [\beta(u)] \;=\; \beta(\overline{u})\,. \end{cases}$$

This can be seen, for instance, using the chain rule for differentiable functions, i.e. $\partial_t \beta(u) + b \cdot \nabla_x \beta(u) \;=\; \beta'(u)[\partial_t u + b \cdot \nabla_x u]$. Otherwise, one can observe that, since u must be constant along the trajectories (2), so must be $\beta(u)$. Motivated by this observation, we introduce the following terminology.

Definition 1.5. — *Let $b \in L^1_{\mathrm{loc}}$ with $\mathrm{div}_x b \in L^1_{\mathrm{loc}}$. A bounded distributional solution of (3) is said* renormalized *if $\beta(u)$ is a solution of (6) for any $\beta \in C^1$. The field b is said to have the* renormalization property *if every bounded distributional solution of (3) is renormalized.*

When b and u are not regular we cannot use the chain rule, neither the theory of characteristics. Therefore, whether a distributional solution is renormalized might be a nontrivial question. Actually, for quite general b, there do exist distributional solutions which are not renormalized (see again [27]). The proof of Theorem 1.4 by DiPerna and Lions consists of two parts, the first one, which is "soft" can be stated as follows.

Proposition 1.6. — *If $b \in L^\infty$ has the renormalization property and its divergence is bounded, then the uniqueness and stability properties of Theorem 1.4 hold.*

The second one, which is the "hard" part of the proof, states essentially that $W^{1,p}$ fields have the renormalization property.

Theorem 1.7. — *Any $b \in L^1([0,\infty[, W^{1,p}(\mathbb{R}^n))$ has the renormalization property.*

We postpone the "hard part" to the next section and come first to Proposition 1.6.

Proof. — **Uniqueness.** Fix a u_0 and let u and v be two distributional solutions of (3). It then follows that $w = u - v$ is a distributional solution of the same transport equation with initial data 0. By the renormalization property so is w^2, i.e.

$$
\left\{\begin{array}{rcl} \partial_t w^2 + \mathrm{div}_x(w^2 b) &=& w^2\, \mathrm{div}_x b \\ \\ w^2(0,\cdot) &=& 0 \end{array}\right. \tag{7}
$$

Integrating (7) "formally" in space we obtain

$$
\partial_t \int_{\mathbb{R}^n} w^2(t,x)\,dx = \int_{\mathbb{R}^n} w^2(t,x)\,\mathrm{div}_x b \;\leq\; \|\mathrm{div}_x b\|_\infty \int_{\mathbb{R}^n} w^2(t,x)\,.
$$

Since $\int_{\mathbb{R}^n} w^2(0,x)\,dx = 0$, by Gronwall's Lemma we would conclude $\int_{\mathbb{R}^n} w^2(t,x)\,dx = 0$ for every t. We sketch how to make rigorous this formal argument. Assume for simplicity $\|b\|_\infty \leq 1$. Let $T, R > 0$ be given and choose a smooth cut–off function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$ such that $\varphi = 1$ on $[0,T]\times B_R(0)$ and $\partial_t\varphi \leq -|\nabla_x\varphi|$ on $[0,2T]\times\mathbb{R}^n$ Now let $\psi \in C^\infty_c(]-2T,2T[)$ be nonnegative and test (7) with $\psi(t)\varphi(t,x)$. Define $f(t) = \int_{\mathbb{R}^n} w^2(t,x)\,\varphi(t,x)\,dx$ and use Fubini's Theorem to get

$$
\begin{aligned}
-\int_0^\infty f(t)\partial_t\psi(t)\,dt \;&=\; \int_0^\infty\!\!\int \psi(t)\varphi(t,x)\,w^2(t,x)\,\mathrm{div}_x b(t,x)\,dx\,dt \\
&\quad + \int_0^\infty\!\!\int \psi(t)w^2(t,x)\left[\partial_t\varphi(t,x) + b(t,x)\cdot\nabla_x\varphi(t,x)\right]dx\,dt\,.
\end{aligned}
$$

Note that the second integral in the right hand side is nonpositive, whereas the first one can be estimated by $\|\mathrm{div}_x b\|_\infty \int f(t)\psi(t)\,dt$. We conclude that f satisfies a "distributional" form of Gronwall's inequality for $t \in [0,2T[$. It can be easily seen that this implies $f = 0$. Thus $w = 0$ a.e. on $[0,T]\times B_R(0)$, and by the arbitrariness of R and T we conclude $w = 0$.

Stability. Arguing as in Theorem 1.3, we easily conclude that, up to subsequences, u_k converges weakly* in L^∞ to a distributional solution u of (3). However, by the uniqueness part of the Theorem, this solution is unique, and hence the whole sequence converges to u. Since the b_k and the u_k are both smooth, u_k^2 solves the corresponding transport equations with initial data $\overline{u}^2$. Arguing as above, u_k^2 must then converge, weakly* in L^∞, to the unique solution of (3) with initial data $\overline{u}^2$. But by the renormalization property this solution is u^2. Summarizing, $u_k \overset{*}{\rightharpoonup} u$ and $u_k^2 \overset{*}{\rightharpoonup} u^2$ in L^∞, which clearly implies the strong convergence in L^1_{loc}. □

2. THE COMMUTATOR ESTIMATE OF DIPERNA AND LIONS

In this section we come to the "hard part", i.e. Theorem 1.7. We first prove a milder conclusion, neglecting the initial conditions, which will be adjusted later.

PROPOSITION 2.1. — *Assume $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n))$ and let $u \in L^\infty$ solve*

$$\partial_t u + \mathrm{div}_x(ub) - u\,\mathrm{div}_x b = 0 \tag{8}$$

distributionally on $\mathbb{R}^+ \times \mathbb{R}^n$. Then, for every $\beta \in C^1$,

$$\partial_t[\beta(u)] + \mathrm{div}_x(\beta(u)b) - \beta(u)\,\mathrm{div}_x b \;=\; 0\,. \tag{9}$$

2.1. Commutators

Let us fix u and b as in Proposition 2.1 and consider a standard smooth and even kernel ρ in $\mathbb{R}^n$. By a slight abuse of notation we denote by $u * \rho_\varepsilon$ the convolution in the x variable, that is $[u * \rho_\varepsilon](t,x) = \int u(t,y)\rho_\varepsilon(x-y)dy$. Mollify (8) to obtain $0 = \partial_t u * \rho_\varepsilon + [\mathrm{div}_x(bu)] * \rho_\varepsilon - [u\,\mathrm{div}_x b] * \rho_\varepsilon$. We rewrite this identity as

$$\partial_t u * \rho_\varepsilon + b\cdot\nabla_x u * \rho_\varepsilon \;=\; -R_\varepsilon + [(u\,\mathrm{div}_x b) * \rho_\varepsilon - u * \rho_\varepsilon\,\mathrm{div}_x b] \tag{10}$$

where R_ε are simply the commutators

$$R_\varepsilon \;=\; [\mathrm{div}_x(bu)] * \rho_\varepsilon - \mathrm{div}_x[b(u * \rho_\varepsilon)]\,. \tag{11}$$

Since R_ε is a locally summable function, the identity (10) implies that $\partial_t u * \rho_\varepsilon$ is also locally summable. Thus, $u * \rho_\varepsilon$ is a Sobolev function in space and time, and we can use the chain rule for Sobolev functions (see for instance Section 4.2.2 of [30]) to compute

$$\partial_t[\beta(u*\rho_\varepsilon)] + b\cdot\nabla_x\big[\beta(u*\rho_\varepsilon)\big] = \beta'(u*\rho_\varepsilon)\big[\partial_t u * \rho_\varepsilon + b\cdot\nabla_x u * \rho_\varepsilon\big]\,.$$

Inserting (10) in this identity we get

$$\partial_t[\beta(u*\rho_\varepsilon)] + b\cdot\nabla_x\big[\beta(u*\rho_\varepsilon)\big] \;=\; \beta'(u*\rho_\varepsilon)\big\{R_\varepsilon + [(u\,\mathrm{div}_x b) * \rho_\varepsilon - u * \rho_\varepsilon\,\mathrm{div}_x b]\big\}\,. \tag{12}$$

Now, the left hand side of (12) converges distributionally to the left hand side of (9). Recall that $\|\beta'(u_\varepsilon)\|_\infty$ and $\|u * \rho_\varepsilon\|_\infty$ are uniformly bounded, whereas

$$[(u\,\mathrm{div}_x b) * \rho_\varepsilon - u * \rho_\varepsilon\,\mathrm{div}_x b] \qquad \longrightarrow \qquad 0$$

strongly in L^1_{loc}. Therefore, in order to prove Proposition 2.1 we just need to show that $\beta'(u*\rho_\varepsilon)R_\varepsilon$ converges to 0. This is implied by the following lemma.

LEMMA 2.2 (Commutator estimate). — *Let $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n))$, $u \in L^\infty$ and R_ε as in* (11)*. Then $R_\varepsilon \to 0$ in L^1_{loc}.*

2.2. The commutator estimate of DiPerna and Lions

Proof of Lemma 2.2. — Without loss of generality we assume that the kernel ρ is supported in $B_1(0)$. First we use the elementary identity

$$R_\varepsilon = -\sum_i (ub_i) * \partial_{x_i}\rho_\varepsilon + \sum_i b_i(u * \partial_{x_i}\rho_\varepsilon) - u * \rho_\varepsilon\,\mathrm{div}_x b$$

and we expand the convolutions to obtain

$$R_\varepsilon(t,x) \;=\; \int u(t,y)(b(t,x)-b(t,y))\cdot\nabla\rho_\varepsilon(x-y)\,dy - \big[u * \rho_\varepsilon\,\mathrm{div}_x b\big](t,x)\,. \tag{13}$$

Since $\nabla\rho_\varepsilon(\xi) = \varepsilon^{-n-1}\nabla\rho(\xi/\varepsilon)$, we perform the change of variables $z = (x-y)/\varepsilon$ to get

$$(14)\quad R_\varepsilon(t,x) \;=\; -\int u(t,x+\varepsilon z)\frac{b(t,x+\varepsilon z)-b(t,x)}{\varepsilon}\cdot\nabla\rho(z)\,dz - \big[u*\rho_\varepsilon\,\mathrm{div}_x b\big](t,x)$$

Next, fix a compact set K. By standard properties of Sobolev functions (see for instance Section 5.8.2 of [29]), the difference quotients

$$(15)\qquad d_{\varepsilon,z}(t,x) \;=\; \frac{b(t,x+\varepsilon z)-b(t,x)}{\varepsilon}$$

are bounded in $L^p(K)$ independently of $z\in B_1(0)$ and $\varepsilon\in]0,1[$. We now let $\varepsilon\downarrow 0$. For each fixed z, $d_{\varepsilon,z}$ converges strongly in $L^p(K)$ to $\partial_z b$. The functions $u_{z,\varepsilon}(t,x) = u(t,x+\varepsilon z)$ are instead uniformly bounded in L^∞, and, by the L^1–continuity of the translation, they converge strongly in $L^1(K)$ to u.

Therefore we conclude that R_ε converges strongly in L^1_{loc} to

$$\begin{aligned} R_0(t,x) \;&=\; -u(t,x)\int \partial_z b(t,x)\cdot\nabla\rho(z)\,dz - \big[u\,\mathrm{div}_x b\big](t,x)\\ &=\; -u(t,x)\sum_{i,j}\partial_i b^j(t,x)\int z_i\partial_{z_j}\rho(z)\,dz - u(t,x)\,\mathrm{div}_x b(t,x)\,. \end{aligned}$$

Integrating by parts we have $\int z_i\partial_{z_j}\rho = -\delta_{ij}$. So $R_0 = 0$, which completes the proof. □

2.3. The initial condition

In order to prove Theorem 1.7 we still need to show that $\beta(u)$ takes the initial condition $[\beta(u)](0,\cdot) = \beta(\overline{u})(\cdot)$. This is achieved with a small trick.

Proof of Theorem 1.7. — Consider b and u as in Theorem 1.7 and extend both of them to negative times by setting $b(t,x)=0$ and $u(t,x)=\overline{u}(x)$ for $t<0$. It is then immediate to check that $\partial_t u + \mathrm{div}_x(bu) = u\,\mathrm{div}_x b$ distributionally on the whole space–time $\mathbb{R}\times\mathbb{R}^n$. On the other hand the proof of Proposition 2.1 remains valid if we replace $\mathbb{R}^+$ with $\mathbb{R}$ (actually the proof remains the same on any open set $\Omega\subset\mathbb{R}\times\mathbb{R}^n$). Therefore

$$\partial_t[\beta(u)] + \mathrm{div}_x[b\beta(u)] = \beta(u)\,\mathrm{div}_x b$$

distributionally on $\mathbb{R}\times\mathbb{R}^n$. We test this equation with a $\varphi\in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$, recalling that $[\beta(u)](t,x) = \beta(\overline{u}(x))$ and $b(t,x)=0$ for $t<0$. We then conclude

$$(16)\quad \int_0^\infty\int_{\mathbb{R}^n}\beta(u)\big[\partial_t\varphi + b\cdot\nabla_x\varphi + \mathrm{div}_x b\,\varphi\big]\,dx\,dt \;=\; -\int_{\mathbb{R}^n}\beta(\overline{u}(x))\int_{-\infty}^0\partial_t\varphi(t,x)\,dt\,dx\,.$$

On the other hand, since φ is smooth, we can integrate by parts in t in the right hand side of (16) in order to get $-\int\beta(\overline{u}(x))\varphi(0,x)dx$. This concludes the proof. □

3. THE BV CASE: THE COMMUTATOR ESTIMATE OF AMBROSIO

Let us try to push the proof of DiPerna and Lions to the BV case (we recall here that a function of bounded variation is simply a summable function whose distributional derivatives are Radon measures). Notice however that, in order to make sense of a distributional solution of (3) as in Definition 1.1, we do need the additional assumption $\text{div}_x b \in L^1$, because for a generic BV function the divergence is only a Radon measure.

The only point where the strategy of DiPerna and Lions does not work is in the proof of Lemma 2.2. There we can still conclude that the difference quotients (15) are bounded in L^1_{loc}, but we cannot conclude that they converge strongly in L^1_{loc} to $\partial_z b$. In fact, $\partial_z b$ is now a Radon measure, and $d_{\varepsilon,z}$ converges to it weakly* in the space of Radon measures (this weak* convergence is the one coming from duality with continuous functions through the Riesz Representation Theorem). However, though we cannot conclude that Lemma 2.2 holds, we still get some information: the right hand side of (12) is uniformly bounded in L^1_{loc}, and hence converges (up to subsequences) to a Radon measure μ. We include this statement in a Lemma to which we will refer later.

LEMMA 3.1. — *Let $u \in L^\infty$ and $b \in L^1(\mathbb{R}^+, BV(\mathbb{R}^n))$ with $\text{div}_x b \in L^1$. Assume that $\partial_t u + \text{div}_x(ub) - u\,\text{div}_x b = 0$ distributionally on $\mathbb{R}^+ \times \mathbb{R}^n$. Then, for every $\beta \in C^1$,*

$$\partial_t[\beta(u)] + \text{div}_x(\beta(u)b) - \beta(u)\,\text{div}_x b = \mu \tag{17}$$

for some Radon measure μ.

3.1. Difference quotients of BV functions

In what follows we will denote by $D_x b$ the distributional differential in the space variables of the vector field b. That is, the matrix of distributional partial derivatives. In order to go beyond Lemma 3.1, consider that, by the Radon–Nikodym decomposition, the distributional derivative $D_x b$, which is a measure, can be split into the part which is absolutely continuous with respect to the Lebesgue measure and the singular part. We denote them by $D^a_x b$ and $D^s_x b$. The Sobolev space $W^{1,1}$ is simply given by those BV functions b for which the singular part $D^s_x b$ vanishes. For such functions, according to Proposition 2.1, the measure μ in (17) vanishes. It is therefore natural to conjecture that, in the general BV case, μ is a singular measure.

In order to show this, we need a refined analysis of the difference quotients of BV functions. We start by introducing a bit of terminology. First of all, we can regard $D_x b$ as a matrix of measures or as a matrix–valued measure. Since $D^a_x b$ is an absolutely continuous function, we can write it as $f\mathscr{L}^{n+1}$, where $\mathscr{L}^{n+1}$ denotes the $n+1$–dimensional Lebesgue measure, and f is a matrix–valued function. In this case f is usually denoted by $\nabla_x b$ in the literature (indeed it coincides with an appropriate measure–theoretic notion of pointwise differential of b, see [14]).

Thanks to the Radon–Nikodym decomposition, a similar splitting holds for $D^s_x b$ as well. That is, we might write $D^s_x b = M|D^s_x b|$, where $|D^s_x b|$ is the total variation measure of $D^s_x b$ (and hence a nonnegative measure), and M is a matrix–valued Borel function. We are now ready to state the following

PROPOSITION 3.2. — *Let $b \in BV(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and let $z \in \mathbb{R}^n$. Then the difference quotients*

$$\frac{b(t, x + \delta z) - b(t, x)}{\delta}$$

can be canonically written as $b_{1,\delta}(z)(t, x) + b_{2,\delta}(z)(t, x)$, where

(a) *$b_{1,\delta}(z)$ converges strongly in $L^1_{\rm loc}$ to $\nabla_x b \cdot z$ as $\delta \downarrow 0$.*

(b) *For any compact set $K \subset \mathbb{R} \times \mathbb{R}^n$ we have*

$$\limsup_{\delta \downarrow 0} \int_K \left| b_{2,\delta}(z)(t, x) \right| dx\, dt \;\le\; |D^s_x b \cdot z|(K)\,. \tag{18}$$

(c) *For every compact set $K \subset \mathbb{R} \times \mathbb{R}^n$ we have*

$$\sup_{\delta \in]0,\varepsilon[} \int_K \left(\left| b_{1,\delta}(z)(t, x) \right| + \left| b_{2,\delta}(z)(t, x) \right| \right) dx\, dt \;\le\; |z||D_x b|(K_\varepsilon) \tag{19}$$

where $K_\varepsilon = \{(t, x) : \mathrm{dist}\,((t, x), K) \le \varepsilon\}$.

Loosely speaking, in this canonical splitting $b_{1,\delta}(z)$ is converging towards the absolutely continuous part of $\partial_z b$, whereas $b_{2,\delta}(z)$ is converging towards the singular part. In order to understand why this decomposition is possible, consider the case when b is a function of one real variable, and split its derivative b' into the sum $b'_a + b'_s$ of its absolutely continuous part and its singular part. Let b_1 be a primitive of b'_a and b_2 a primitive of b'_s. For instance we can define $b_1(x) = b'_a([0, \tau[)$ and $b_2(x) = b'_s([0, \tau[)$ for τ positive and $b_1(x) = -b'_a(]\tau, 0])$ and $b_2(x) = b'_s(] - \tau, 0])$ for τ negative. The sum of the difference quotients of b_1 and b_2 give the difference quotients of b, and it is, actually, the splitting of Proposition 3.2. For instance, since b_1 is a $W^{1,1}$ function, its difference quotients converge strongly in L^1 to its derivative, that is b'_a: this gives (a). The remaining points (b) and (c) follow in a similar way. The proof of the proposition in the general case is perhaps the most technical part of this note, but it is based on the 1–dimensional case sketched above through the "slicing theory" of BV functions. The interested reader will find it in the appendix.

Remark 3.3. — The decomposition of the proof is canonical in the sense that we give an explicit way of constructing $b_{1,\delta}$ and $b_{2,\delta}$ from the measures $D^a_x b \cdot z$ and $D^s_x b \cdot z$. One important consequence of this explicit construction is the following linearity property: If $b^1, b^2 \in BV_{\rm loc}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $z \in \mathbb{R}^n$, then

$$(\lambda_1 b^1 + \lambda_2 b^2)_{i,\delta}(z)(t, x) \;=\; \lambda_1 b^1_{i,\delta}(z)(t, x) + \lambda_2 b^2_{i,\delta}(z)(t, x)\,.$$

3.2. The commutator estimate of Ambrosio

We now use the technical Proposition 3.2 to give a more careful estimate on the commutators R_ε. The idea is again to follow the proof of Lemma 2.2, but this time, once arrived to (14), we will substitute the difference quotients of b with the splitting given by Proposition 3.2. We will then show that the "$b_{1,\varepsilon}$" cancels with the divergence, whereas for the singular part "$b_{2,\varepsilon}$" we will use the crudest estimate available. In order to state the final result, we first need some notation.

Definition 3.4. — *For any $\eta \in C^\infty_c(\mathbb{R}^n)$ and any $n \times n$ matrix M we define*

$$\Lambda(M, \eta) \;=\; \int_{\mathbb{R}^n} |\nabla \eta(z) \cdot M \cdot z| \, dz\,.$$

We are now ready to state Ambrosio's Commutators Estimate.

Proposition 3.5 (Commutators estimate). — *Let b, u and β be as in Lemma 3.1. Let ρ be any even convolution kernel and let M be the matrix–valued Borel function such that $D^s_x b = M|D^s_x b|$. Then the measure μ of* (17) *satisfies the inequality*

$$|\mu| \leq C\Lambda(M, \rho)|D^s_x b|\,. \tag{20}$$

Proof. — Consider a continuous compactly supported test function φ and use the computations of Subsection 2.2 in order to conclude

$$\begin{aligned}
-\int \varphi \, d\mu \;&=\; \lim_{\varepsilon\downarrow 0} -\int \varphi \beta'(u * \rho_\varepsilon) R_\varepsilon \;=\; \int \varphi \beta'(u) u \,\mathrm{div}_x b \\
&\quad + \lim_{\varepsilon\downarrow 0} \int \varphi(t,x)[\beta'(u)u](t, x+\varepsilon z) \frac{b(t,x+\varepsilon z) - b(t,x)}{\varepsilon} \cdot \nabla \rho(z) \, dz\, dx\, dt \\
&=\; \int \varphi \beta'(u) u \,\mathrm{div}_x b + \lim_{\varepsilon\downarrow 0} \int \varphi(t,x)[\beta'(u)u](t,x+\varepsilon z) b_{1,\varepsilon}(z)(t,x) \cdot \nabla\rho(z)\, dz\, dx\, dt \qquad (21)\\
&\quad + \lim_{\varepsilon\downarrow 0} \int \varphi(t,x)[\beta'(u)u](t,x+\varepsilon z) b_{2,\varepsilon}(z)(t,x) \cdot \nabla\rho(z)\, dz\, dx\, dt\,. \qquad (22)
\end{aligned}$$

We now use Proposition 3.2 to show that (21) vanishes and to estimate (22) with (a suitable modification of) the right hand side of (20).

Indeed, from Proposition 3.2(a) and (c), and from the strong L^1_{loc} convergence of $u * \rho_\varepsilon$ to u, the second integral in (21) converges to

$$\int \varphi(t,x)\beta'(y(t,x))u(t,x) \sum_{i,j} e^j \cdot \nabla b(t,x) \cdot e_i \int z_j \partial_{z_i} \rho(z)\, dz\, dx\, dt\,. \tag{23}$$

Arguing as in Subsection 2.2, (23) is equal to

$$-\int \varphi(t,x) u(t,x) \beta'(u(t,x)) \,\mathrm{tr}\, \nabla b(t,x)\, dx\, dt\,.$$

On the other hand, $\mathrm{tr}\,\nabla b$ is just the absolutely continuous part of the divergence of b. Since by assumption $\mathrm{div}_x b$ is absolutely continuous, it coincides with its absolutely continuous part. Therefore, (21) vanishes.

We now come to (22). Since β' and u are both bounded, (22) can be estimated by

$$C \limsup_{\varepsilon\downarrow 0} \int |\varphi(t,x)| \int |b_{2,\varepsilon}(z)(t,x)\cdot \nabla\rho(z)|\, dz\, dx\, dt \tag{24}$$

Next, let $S = \|\varphi\|_{C^0}$, let K_σ be the closure of $\{(t,x) : |\varphi(t,x)| > \sigma\}$ and rewrite (24) as

$$C \limsup_{\varepsilon\downarrow 0} \int_0^S \int_{K_\sigma} \int |b_{2,\varepsilon}(z)(t,x)\cdot \nabla\rho(z)|\, dz\, dx\, dt\, d\sigma\,. \tag{25}$$

From Proposition 3.2(c), we know that

$$\limsup_{\varepsilon\downarrow 0} \int_{K_\sigma} |b_{2,\varepsilon}(z)(t,x)\cdot \nabla\rho(z)|\, dt\, dx \;\leq\; |D^s_x b\cdot\nabla\rho(z)|(K_\sigma)\,. \tag{26}$$

Moreover, since for z outside the support of ρ the integral in (26) vanishes, the map

$$(\sigma, z) \to \int_{K_\sigma} |b_{2,\varepsilon}(z)(t,x)\cdot\nabla\rho(z)|\, dt\, dx$$

is bounded. Therefore, we integrate (25) first in (t,x) and use (26) and the dominated convergence theorem to bound (25) with a constant time

$$\int_0^S \int |\nabla\rho(z)\cdot D^s_x b\cdot z|(K_\sigma)\, dz\, d\sigma\,. \tag{27}$$

Let ν_z be the measure $|\nabla\rho(z)\cdot D^s_x b\cdot z| = |\nabla\rho(z)\cdot M\cdot z||D^s_x b|$. Then (27) is simply

$$\begin{aligned}
\int\int |\varphi(t,x)| d\nu_z(t,x)\, dz \;&=\; \int\int |\varphi(t,x)|\, |\nabla\rho(z)\cdot M(t,x)\cdot z|\, d|D^s_x b|(t,x)\, dz\\
&=\; \int |\varphi(t,x)| \left[\int |\nabla\rho(z)\cdot M(t,x)\cdot z|\, dz\right] d|D^s_x b|(t,x)\\
&=\; \int |\varphi(t,x)|\Lambda(M(t,x),\rho)\, d|D^s_x b|(t,x)\,.
\end{aligned}$$

Summarizing, we get

$$\int \varphi d\mu \;\leq\; C\int |\varphi(t,x)|\Lambda(M(t,x),\rho) d|D^s_x b|(t,x)$$

for any continuous compactly supported φ. This is indeed the desired claim (20). □

3.3. Optimizing the choice of the kernel

Let us recollect what proved so far in this section. We started with a BV field b, a distributional solution u of $\partial_t u + \mathrm{div}_x\,(ub) = u\,\mathrm{div}_x b$ and a function $\beta\in C^1(\mathbb{R})$ and we have proved that the distribution $\partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] - \beta(u)\,\mathrm{div}_x b$ is a measure μ satisfying

$$|\mu| \leq C\Lambda(M,\rho)|D^s_x b|\,, \tag{28}$$

for any choice of an even convolution kernel $\rho\in C^\infty_c(\mathbb{R}^n)$.

Clearly our estimate is far from being optimal: the measure μ and the constant C are both independent of the kernel ρ. We can therefore optimize in ρ. Since the estimate

(28) has a local nature, this optimization procedure is, in a certain sense, equivalent to vary the regularizing kernel in t and x. In order to state our optimized estimate, we define the set of kernels

$$\mathcal{K} \;=\; \left\{\eta \in C_c^\infty(B_1(0)) \text{ such that } \eta \geq 0 \text{ is even, and } \textstyle\int_{B_1(0)} \eta = 1\right\}. \tag{29}$$

THEOREM 3.6. — *Let u, b, and β be as in Lemma 3.1. Then $\partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] - \beta(u)\,\mathrm{div}_x b = f|D_x^s b|$ for some Borel function f satisfying*

$$|f(t,x)| \;\leq\; C \inf_{\rho\in\mathcal{K}} \Lambda(M(t,x),\rho) \qquad \textit{for } |D_x^s b|\textit{–a.e. } (t,x). \tag{30}$$

Proof. — Let μ be as in (17). The inequality (28) implies its absolute continuity with respect of $|D_x^s b|$. Therefore there exists a Borel function f such that $\mu = f|D_x^s b|$. There is only one technical subtlety to take into account. From Proposition 3.5 we know that

$$|f(t,x)| \leq \Lambda(M(t,x),\rho) \qquad \text{for } |D_x^s b|\text{–a.e. } (t,x)$$

whenever we fix a convolution kernel ρ. However, the set of measure zero where the inequality fails might in principle depend on ρ. This gives no trouble as soon as we infimize on a countable set of kernels $\mathcal{K}'$ (because a countable union of sets of measure zero has measure zero!):

$$|f(t,x)| \leq \inf_{\rho\in\mathcal{K}'} \Lambda(M(t,x),\rho) \qquad \text{for } |D_x^s b|\text{–a.e. } (t,x).$$

However, for any fixed matrix M, the map $\rho \mapsto \Lambda(M,\rho)$ is continuous for the $W^{1,1}$ topology. Therefore, if we choose $\mathcal{K}'$ to be any countable subset of $\mathcal{K}$ dense in the $W^{1,1}$ topology, then the infimum over $\mathcal{K}'$ coincides with the infimum over $\mathcal{K}$. □

4. THE LEMMAS OF BOUCHUT AND ALBERTI

Our plan so far lead us to the following question: given a matrix M, what is the infimum of the functional $\Lambda(M,\rho)$ over the set of kernels $\mathcal{K}$? One lower bound for this infimum follows from a simple integration by parts:

$$\begin{aligned}\Lambda(M,\rho) \;&\geq\; \left|\int_{B_1(0)} \nabla\rho(y)\cdot M\cdot y\,dy\right| \;=\; \left|\sum_{k,j} M_{jk}\int_{B_1(0)} y_j \frac{\partial\rho}{z_k}(y)\,dy\right| \\ &=\; \left|-\sum_{k,j} M_{jk}\int_{B_1(0)} \delta_{jk}\rho(y)\,dy\right| \;=\; |\mathrm{tr}\,M|\,. \end{aligned}\tag{31}$$

Now, in the case at hand, recall that $M|D_x^s b|$ is the singular part of the derivative $D_x b$. Therefore $\mathrm{tr}\,M|D_x^s b|$ is just the singular part of the divergence, which by our assumptions is zero. The proof that $\partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] = 0$ is therefore completed by the following Lemma, whose proof is due to Alberti:

LEMMA 4.1 (Alberti). — *For any $n \times n$ matrix M we have*

$$\inf_{\eta\in\mathcal{K}} \Lambda(M,\eta) \;=\; \big|\mathrm{tr}\, M\big| \,. \tag{32}$$

However, Ambrosio's original proof was instead based on a special case of Alberti's Lemma, proved by Bouchut in [15]. There the author was interested in a renormalization property for fields with special structure.

LEMMA 4.2 (Bouchut). — *For any pair of vectors $\xi, \chi \in \mathbb{R}^n$ we have*

$$\inf_{\eta\in\mathcal{K}} \Lambda(\chi\otimes\xi,\eta) \;=\; |\xi\cdot\chi| \;=\; \big|\mathrm{tr}\,(\chi\otimes\xi)\big| \,. \tag{33}$$

Indeed, when $M|D^s_x b|$ is the singular part of the distributional derivative of a BV function, $M(t,x)$ is a rank–one matrix for $|D^s_x b|$–a.e. (t,x). This result, which is probably the deepest one in the theory of BV functions, is also due to Alberti (see [2]; for a recent brief, but nonetheless complete, account of the proof, see [25]). In order to understand its statement, the reader might check it on the easiest examples, i.e. functions which are piecewise constants. In this case the result is a trivial fact: the hard core of Alberti's result is that the same property holds also when (part of) the distributional derivative of b is a fractal–type measure.

In any case, by Alberti's Rank–one Theorem, Bouchut's Lemma is already sufficient to prove the renormalization theorem of Ambrosio.

THEOREM 4.3. — *Let u, b, and β be as in Lemma 3.1. Then $\partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] - \beta(u)\,\mathrm{div}_x b = 0$*

Moreover, arguing exactly as in Subsection 2.3, we can adjust the initial condition to conclude

THEOREM 4.4. — *Let $b \in L^1(\mathbb{R}^+, BV(\mathbb{R}^n))$ with absolutely continuous divergence. Then b has the renormalization property.*

Before coming to the proof of these lemmas, we want to point out an important fact. As already said, we can regard the optimization procedure of Theorem 3.6 as an implementation of the idea "a varying regularizing kernel approximates better than a fixed one". Then both Bouchut's and Alberti's Lemmas tell us that, close to points where the singular part of $D^s_x b$ is large, the optimal choice is a very anisotropic kernel. This intuition originated in Bouchut's paper [15].

4.1. Bouchut's Lemma

The proof of Bouchut's Lemma is very elementary and it exploits convolution kernels which have a very simple structure i.e. they are close to the indicator function of a very thin rectangle, whose long sides are parallel to χ.

Proof of Lemma 4.2. — If $d=2$ we can fix an orthonormal basis of coordinates z_1, z_2 in such a way that $\xi=(a,b)$ and $\chi=(0,c)$. Consider the rectangle $r_\varepsilon=[-\varepsilon/2,\varepsilon/2]\times[-1/2,1/2]$ and consider the kernel $\eta_\varepsilon=\frac{1}{\varepsilon}\mathbf{1}_{r_\varepsilon}$. Let $\zeta\in\mathcal{K}$ and denote by ζ_δ the family of mollifiers generated by ζ. Clearly $\eta_\varepsilon*\zeta_\delta\in\mathcal{K}$ for $\varepsilon+\delta$ small enough.

Denote by $\nu=(\nu_1,\nu_2)$ the unit normal to ∂r_ε and recall that

$$\lim_{\delta\downarrow 0}\left|\frac{\partial(\eta_\varepsilon*\zeta_\delta)}{\partial z_i}\right| \quad \stackrel{*}{\rightharpoonup} \quad \frac{|\nu_i|}{\varepsilon}\mathscr{H}^1 \llcorner \partial r_\varepsilon\,. \tag{34}$$

in the sense of measures (here $\mathscr{H}^1\llcorner\partial r_\varepsilon$ denotes the usual 1–dimensional measure on the boundary of r_ε).

Thus, we can compute

$$\begin{aligned}\limsup_{\delta\downarrow 0}\Lambda(M,\eta_\varepsilon*\zeta_\delta) \;&\le\; \limsup_{\delta\downarrow 0}\int_{\mathbb{R}^2}\big(|az_1|+|bz_2|\big)|c|\left|\frac{\partial(\eta_\varepsilon*\zeta_\delta)}{\partial z_2}\right| dz_1dz_2\\ &=\; \frac{2|c|}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2}\left(|az_1|+\frac{|b|}{2}\right)dz_1 \;=\; |ac|\frac{\varepsilon}{2}+|bc|\,.\end{aligned}$$

Note that $bc=\operatorname{tr}M$. Thus, if we define the convolution kernels $\lambda_{\varepsilon,\delta}=\eta_\varepsilon*\zeta_\delta$ we get:

$$\limsup_{\varepsilon\downarrow 0}\;\limsup_{\delta\downarrow 0}\Lambda(M,\eta_\varepsilon*\zeta_\delta)\;\le\;|\operatorname{tr}M|\,. \tag{35}$$

For $n\ge 2$ we consider a system of coordinates $x_1,x_2,\dots,x_n$ such that $\eta=(a,b,0,\dots,0)$, $\xi=(0,c,0,\dots,0)$ and we define the convolution kernels

$$\lambda_{\varepsilon,\delta}(x)\;=\;[\eta_\varepsilon*\zeta_\delta](x_1,x_2)\cdot\zeta(x_3)\cdot\ldots\cdot\zeta(x_n)\,.$$

The conclusion of the Lemma follows easily. □

4.2. Alberti's Lemma

The proof of Alberti's Lemma is in a certain sense a generalization of Bouchut's proof. The basic idea is to take a convolution kernel which is concentrated on a very long tube made of trajectories of the ODE $\dot\gamma=M\cdot\gamma$.

Proof of Lemma 4.1. — By the identity $\nabla\eta(z)\cdot M\cdot z=\operatorname{div}\,(M\cdot z\eta(z))-\operatorname{tr}M\eta(z)$, it suffices to show that for every $T>0$ there exists $\eta\in\mathcal{K}$ such that

$$\int_{\mathbb{R}^n}|\operatorname{div}(M\cdot z\eta(z))|\,dz\;\le\;\frac{2}{T}\,. \tag{36}$$

Given a smooth nonnegative convolution kernel θ with compact support, we claim that the function

$$\eta(z)\;=\;\frac{1}{T}\int_0^T\theta(e^{-tM}\cdot z)\,e^{-t\operatorname{tr}M}\,dt$$

has the required properties. Here e^{tM} is the matrix $\sum_i \frac{t^i M^i}{i!}$. That is, $e^{tM} \cdot z$ is just the solution of the ODE $\dot\gamma = M \cdot \gamma$ with initial condition $\gamma(0) = z$, and $e^{-t \operatorname{tr} M}$ is the determinant of e^{-tM}. The usual change of variables yields

$$\begin{aligned} \int \eta(z)\varphi(z)\, dz &= \frac{1}{T}\int_0^T \int \varphi(z)\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\, dz\, dt \\ &= \frac{1}{T}\int_0^T \int \varphi(e^{tM}\cdot \zeta)\theta(\zeta)\, d\zeta\, dt\,, \end{aligned} \tag{37}$$

for any integrable bounded φ. Hence $\eta\mathscr{L}^d$ is the time average of the push-forward of the measure $\theta\mathscr{L}^d$ along the trajectories of $\dot\gamma = M\cdot\gamma$. This is the point of view taken in [5] to prove (36), for which we argue with the direct computations shown below.

Note that

$$\operatorname{div}(M\cdot z\eta(z)) = \frac{1}{T}\int_0^T \operatorname{div}(M\cdot z\,\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M}\, dt\,.$$

A tedious but straightforward computation (see [26]) shows

$$\operatorname{div}(M\cdot z\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M} = -\frac{d}{dt}\left(\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\right)\,.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |\operatorname{div}(M\cdot z\eta(z))|\, dz &= \int_{\mathbb{R}^n} \frac{1}{T}\left|\int_0^T \operatorname{div}(M\cdot z\theta(e^{-tM}\cdot z))e^{-t\operatorname{tr} M}\, dt\right|\, dz \\ &= \int_{\mathbb{R}^n} \frac{1}{T}\left|\int_0^T \frac{d}{dt}\left(\theta(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\right)\, dt\right|\, dz \\ &= \int_{\mathbb{R}^n} \frac{1}{T}\left|\theta(e^{-TM}\cdot z)e^{-T\operatorname{tr} M} - \theta(z)\right|\, dz \\ &\le \frac{1}{T}\left(\int_{\mathbb{R}^n} \theta(e^{-TM}\cdot z)e^{-T\operatorname{tr} M}\, dz + \int_{\mathbb{R}^n} \theta(z)\, dz\right) \\ &= \frac{1}{T}\left(\int_{\mathbb{R}^n} \theta(\zeta)\, d\zeta + \int_{\mathbb{R}^n}\theta(z)\, dz\right) = \frac{2}{T}\,. \end{aligned}$$

This shows (36) and concludes the proof. □

5. THE CONTINUITY EQUATION AND REGULAR LAGRANGIAN FLOWS

Another major point of the DiPerna–Lions theory is that the classical road from characteristics to transport equations can be reversed: the renormalization property and the induced uniqueness and stability of weak solutions to transport equations can be used to infer existence, uniqueness and stability of a suitable generalized notion of flow for the ODEs (2). In his paper [4], Ambrosio has proposed a new way of looking at this side of the DiPerna–Lions theory, based on the analysis of probability measures on the space of paths. In the present note we follow yet another presentation, given in [26].

We start by defining our generalized notion of flow.

Definition 5.1. — *Let* $b \in L^\infty([0,\infty[\times\mathbb{R}^n,\mathbb{R}^n)$. *A map* $\Phi : [0,\infty[\times\mathbb{R}^n \to \mathbb{R}^n$ *is a regular Lagrangian flow for* b *if*

(a) *For* $\mathscr{L}^1$*–a.e.* t *we have* $|\{x : \Phi(t,x) \in A\}| = 0$ *for every Borel set* A *with* $|A| = 0$*;*
(b) *The following identity is valid in the sense of distributions*

$$\left\{\begin{array}{lll} \partial_t \Phi(t,x) & = & b(t,\Phi(t,x)) \\ \\ \Phi(0,x) = x\,. & & \end{array}\right. \tag{38}$$

Note that assumption (a) guarantees that $b(t,\Phi(t,x))$ is well defined. Indeed, if $\hat{b} = b$ $\mathscr{L}^{n+1}$–a.e., then $\hat{b}(t,\Phi(t,x)) = b(t,\Phi(t,x))$ for $\mathscr{L}^{n+1}$–a.e. (t,x).

Ideally one could divide the DiPerna–Lions theory into two separate parts: how to prove "renormalization–type" properties and which kind of "renormalization–type" properties imply existence, uniqueness and stability of regular Lagrangian flows. An example of this approach is given by the notes [26], where the two parts are presented in completely independent ways. Instead, here we focus on the specific theorem below, with the hope to keep the notation and details to a minimum and highlight the mechanisms which link renormalized solutions to regular Lagrangian flows.

Theorem 5.2. — *Let* $b \in L^1(\mathbb{R}^+, BV(\mathbb{R}^n)) \cap L^\infty$ *with bounded divergence. Then there exists a unique regular Lagrangian flow* Φ *for* b*. Moreover, if* b_k *is a sequence of smooth vector fields converging strongly in* $L^1_{\rm loc}$ *to* b *such that* $\|{\rm div}_x b\|_\infty$ *is uniformly bounded, then the flows of* b_k *converge strongly in* $L^1_{\rm loc}$ *to* Φ*.*

During the proof of this theorem we will recover an important fact: the regular Lagrangian flow is a suitable weak notion of characteristics for the transport equation.

5.1. The density of a regular Lagrangian flow and the continuity equation

Denote by μ_Φ the measure $({\rm id}\,,\Phi)_\# \mathscr{L}^{n+1} \llcorner ([0,\infty[\times\mathbb{R}^n)$, i.e. the push–forward via the map $(t,x) \mapsto (t,\Phi(t,x))$ of the Lebesgue $n+1$–dimensional measure on $[0,\infty[\times\mathbb{R}^n$. Such push–forward is simply defined by the property

$$\int_{[0,\infty[\times\mathbb{R}^n} \psi(t,x)\,d\mu_\Phi(t,x) \;=\; \int_{[0,\infty[\times\mathbb{R}^n} \psi(t,\Phi(t,x))\,d\mathscr{L}^{n+1}(t,x)$$

valid for every $\psi \in C_c(\mathbb{R}\times\mathbb{R}^m)$. Observe that (a) is equivalent to the absolute continuity of μ_Φ with respect to the Lebesgue measure, and hence to the existence of a $\rho \in L^1_{\rm loc}([0,\infty[\times\mathbb{R}^n)$ such that $\mu_\Phi = \rho\mathscr{L}^{n+1}$.

Definition 5.3. — *The* ρ *defined above will be called the* density *of the regular Lagrangian flow* Φ*.*

When b is smooth and Φ is the classical solution of (38), $t \mapsto \Phi(t,\cdot)$ is a one–parameter family of diffeomorphisms. For each t let us denote by $\Phi^{-1}(t,\cdot)$ the inverse of $\Phi(t,\cdot)$. Then ρ can be explicitly computed as $\rho(t,x) = \det \nabla_x \Phi(t, \Phi^{-1}(t,x))$ and the classical Liouville Theorem states that ρ solves the continuity equation $\partial_t \rho + \mathrm{div}_x(\rho b) = 0$. Moreover, since $\Phi(0,x) = x$, the initial condition for ρ is $\rho(0,x) = 1$. This property remains true for regular Lagrangian flows and it is simply the special case $\overline{\zeta} = 1$ in the following Proposition.

Proposition 5.4. — *Let Φ be a regular Lagrangian flow for a field b. Let $\overline{\zeta} \in L^\infty(\mathbb{R}^n)$ set $\mu = (\mathrm{id}\,,\Phi)_\#(\overline{\zeta}\mathscr{L}^{n+1})$. Then there exists $\zeta \in L^1_{\mathrm{loc}}([0,\infty[\times\mathbb{R}^n)$ such that $\mu = \zeta\mathscr{L}^{n+1}$. This ζ solves (distributionally)*

$$\left\{\begin{array}{rcl} \partial_t \zeta + \mathrm{div}_x(\zeta b) &=& 0 \\ \\ \zeta(0,\cdot) &=& \overline{\zeta}\,. \end{array}\right. \tag{39}$$

Proof. — First of all, notice that $\mu \leq \|\overline{\zeta}\|_\infty \mu_\Phi$. So μ is absolutely continuous and hence there exists a $\zeta \in L^1_{\mathrm{loc}}$ such that $\mu = \zeta \mathscr{L}^{n+1}$. Now, let $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$ be any given test function. Our goal is to show that

$$-\int_{[0,\infty[\times\mathbb{R}^n} \zeta(t,x)\big(\partial_t\psi(t,x) + b(t,x)\cdot\nabla_x\psi(t,x)\big)\,dx\,dt \;=\; \int_{\mathbb{R}^n} \overline{\zeta}(x)\psi(0,x)\,dx\,. \tag{40}$$

By definition, the left hand side of (40) is equal to

$$-\int_{\mathbb{R}^n} \overline{\zeta}(x)\left[\int_0^\infty \big(\partial_t\psi(t,\Phi(t,x)) + \nabla_x\psi(t,\Phi(t,x))\cdot b(t,\Phi(t,x))\big)\,dt\right]dx\,. \tag{41}$$

The proof would follow if we could integrate by parts in t, since $\psi(0,\Phi_x(0)) = \psi(0,x)$ and $\psi(T,\Phi_x(T)) = 0$ for any T large enough (because ψ is compactly supported). On the other hand this integration by parts is easy to justify for a.e. x, since (38) implies that the curve $t \mapsto \Phi(t,x)$ is Lipschitz for a.e. x. □

5.2. Uniqueness of solutions to the continuity equation

Next, let us assume that $\mathrm{div}_x b$ is bounded in L^∞. Then we would expect, formally, that the density of Φ is bounded away from 0 and $+\infty$. Indeed, assume that b and Φ are both smooth and rewrite the continuity equation as $\partial_t\rho + b\cdot\nabla_x\rho + \rho\,\mathrm{div}_x b = 0$. Fix x and differentiate the function $\omega(t) = \rho(t,\Phi(t,x))$ to get

$$\begin{aligned} \frac{d\omega}{dt}(t) &= \partial_t\rho(t,\Phi(t,x)) + \partial_t\Phi(t,x)\cdot\nabla_x\rho(t,\Phi(t,x)) \\ &= \partial_t\rho(t,\Phi t,x)) + b(t,\Phi(t,x))\cdot\nabla_x\rho(t,\Phi(t,x)) = -\mathrm{div}_x b(t,\Phi(t,x))\rho(t,\Phi(t,x)) \\ &= -\mathrm{div}_x b(t,\Phi(t,x))\omega(t)\,. \end{aligned} \tag{42}$$

Since $-\|\mathrm{div}_x b\|_\infty \leq -\mathrm{div}_x b(t,\Phi(t,x)) \leq \|\mathrm{div}_x b\|_\infty$ and $\omega(0) = 1$, we can use Gronwall's Lemma to conclude $\exp(-T\|\mathrm{div}_x b\|_\infty) \leq \omega(T) \leq \exp(T\|\mathrm{div}_x b\|_\infty)$. But $\Phi(T,\cdot)$ is surjective, because it is a diffeomorphism. Therefore we conclude

$$\exp(-T\|\mathrm{div}_x b\|_\infty) \le \rho \le \exp(T\|\mathrm{div}_x b\|_\infty). \tag{43}$$

We cannot use this formal argument on the density of a general regular Lagrangian flow. On the other hand, by a standard approximation procedure, we can show the following Lemma.

LEMMA 5.5. — *Let $b \in L^\infty$ with bounded divergence. Then there exists a $\tilde\rho \in L^\infty_{\mathrm{loc}}$ satisfying the bounds* (43) *and solving*

$$\begin{cases} \partial_t \tilde\rho + \mathrm{div}_x(\tilde\rho b) \;=\; 0 \\ \\ \tilde\rho(0,\cdot) \;=\; 1\,. \end{cases} \tag{44}$$

Proof. — Let φ be a standard convolution kernel, and consider $b_k = b * \varphi_{k^{-1}}$. Consider the densities ρ_k of the classical flows of b_k. Equation (39) holds with b and $\tilde\rho$ replaced by b_k and ρ_k. On the other hand, for ρ_k we can argue as above and get the bounds $\exp(-\|\mathrm{div}_x b_k\|_\infty) \le \rho_k(t,x) \le \exp(-\|\mathrm{div}_x b_k\|_\infty)$. Since $\|\mathrm{div}_x b_k\|_\infty \le \|\mathrm{div}_x b\|_\infty$, there exists a subsequence of ρ_k which converges weakly* in L^∞ to a $\tilde\rho$ satisfying (43). Arguing as in Theorem 1.3 we obtain (44) by passing into the limit in the continuity equations for $\tilde\rho_k$. □

If we knew the uniqueness of solutions to the continuity equation, this existence result would become a proof of the formal bound (43) for the density of any regular Lagrangian flow. As usual, we consider the case of b smooth in order to get some insight. Let ρ and $\tilde\rho$ be two smooth solutions of (44), with $\tilde\rho > 0$, and define $u = \rho/\tilde\rho$. Then we could use the chain rule to compute

$$\partial_t u + b\cdot\nabla_x u \;=\; \tilde\rho^{-2}\big\{\tilde\rho\big[\partial_t\rho + b\cdot\nabla_x\rho\big] - \rho\big[\partial_t\tilde\rho + b\cdot\nabla_x\tilde\rho\big]\big\}\,.$$

Adding and subtracting $\tilde\rho^{-2}(\rho\tilde\rho\,\mathrm{div}_x b)$, we achieve

$$\partial_t u + b\cdot\nabla_x u \;=\; \tilde\rho^{-2}\big\{\tilde\rho\big[\partial_t\rho + \mathrm{div}_x(\rho b)\big] - \rho\big[\partial_t\tilde\rho + \mathrm{div}_x(\tilde\rho b)\big]\big\} \;=\; 0\,.$$

But since $u(0,x) = \rho(0,x)/\tilde\rho(0,x) = 1$, we conclude $u(t,x) = 1$ for every t and x.

The computations above are very similar, in spirit, to the renormalization property. It is therefore not a surprise that the theorem below follows from suitable modifications of the proof of Theorem 4.4.

THEOREM 5.6. — *Let $b \in L^1(\mathbb{R}^+, BV(\mathbb{R}^n)) \cap L^\infty$ with bounded divergence and let $\tilde\rho$ and ζ be L^1_{loc} functions solving respectively* (44) *and* (39)*. If $\tilde\rho \ge C > 0$, then $u = \zeta/\tilde\rho$ is a distributional solution of*

$$\begin{cases} \partial_t u + \mathrm{div}_x(ub) - u\,\mathrm{div}_x b \;=\; 0 \\ \\ u(0,\cdot) \;=\; \overline{\zeta}\,. \end{cases} \tag{45}$$

By minor modifications of the ideas of Section 1, Lemma 5.5 and Theorem 5.6 yield the desired uniqueness for solutions of the continuity equations.

Corollary 5.7. — *Let b be as in Theorem 5.6. Then there exists a unique $\zeta \in L^1_{\rm loc}$ solving* (39)*. Therefore, if Φ is a regular Lagrangian flow for b, the density of Φ coincides with the density $\tilde\rho$ of Lemma 5.5 and hence satisfies the bounds* (43)*.*

5.3. Uniqueness and stability of regular Lagrangian flows

The uniqueness of solutions of the continuity equations yields easily the uniqueness and stability of regular Lagrangian flows.

Proof of the uniqueness and stability parts in Theorem 5.2. — **Uniqueness.** Let Φ and Ψ be two regular Lagrangian flows for b. Fix a $\overline{\zeta} \in C_c(\mathbb{R}^n)$ and consider the unique solution ζ of (39). According to Proposition 5.4 we have $({\rm id}\,,\Phi)_\#(\overline{\zeta}\mathscr{L}^{n+1}) = \zeta\mathscr{L}^{n+1} = ({\rm id}\,,\Psi)_\#(\overline{\zeta}\mathscr{L}^{n+1})$. This identity means that

$$\int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx \;=\; \int \varphi(t,\Psi(t,x))\overline{\zeta}(x)\,dt\,dx$$

for every test function $\varphi \in C_c(\mathbb{R}\times\mathbb{R}^n)$. But since $\overline{\zeta}$ has compact support, one can infer the equality even when $\varphi(t,y) = \chi(t)y_i$ for $\chi \in C_c(\mathbb{R})$. So

$$\int \Phi_i(t,x))\chi(t)\overline{\zeta}(x)\,dt\,dx \;=\; \int \Psi_i(t,x))\chi(t)\overline{\zeta}(x)\,dt\,dx$$

for any pair of functions $\chi \in C_c(\mathbb{R})$ and $\overline{\zeta} \in C_c(\mathbb{R}^n)$. This easily implies $\Phi_i = \Psi_i$ a.e..

Stability. Consider a sequence $\{b_k\}$ as in the statement of the Theorem and let Φ^k be the corresponding classical flows. Fix a $\overline{\zeta} \in C_c(\mathbb{R}^n)$ and consider the ζ_k and u_k solving, respectively, the continuity equations and the transport equations with coefficients b_k and initial data $\overline{\zeta}$. Recall that, if ρ_k are the densities of Φ^k, then $\zeta_k = u_k\rho_k$. The u_k are essentially bounded functions, and by the bounds in Subsection 5.2, the ρ_k are locally uniformly bounded. Therefore the ζ_k are locally uniformly bounded and, up to subsequences, they converge, weakly* in $L^\infty_{\rm loc}$, to some ζ. Arguing as in Theorem 1.3, this ζ must be the unique distributional solution of (39). So, fixing a test function $\varphi \in C_c(\mathbb{R}\times\mathbb{R}^n)$ and arguing as in the uniqueness part, we get

$$\lim_{k\uparrow\infty}\int \varphi(t,\Phi^k(t,x))\overline{\zeta}(x)\,dt\,dx \;=\; \int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx\,,$$

where we are allowed to test with $\varphi(t,y) = \chi(t)y_i$: this gives the weak* convergence of Φ^k to Φ in $L^\infty_{\rm loc}$. Testing with $\varphi(t,y) = \chi(t)|y|^2$, we conclude as well the weak* convergence of $|\Phi^k|^2$ to $|\Phi|^2$. This implies of course the strong $L^1_{\rm loc}$ convergence. □

5.4. Existence of regular Lagrangian flows

The proof of existence of regular Lagrangian flows follows from an approximation argument. Indeed, let b_k be a standard regularization of b, with $\|b_k\|_\infty + \|{\rm div}_x b_k\|_\infty$ bounded by a constant C and $b_k \to b$ strongly in $L^1_{\rm loc}$. Consider the flows Φ^k of b_k. By the bounds of Subsection 5.2, $\exp(-Ct) \le \det\nabla_x\Phi^k(t,x) \le \exp(Ct)$, which translates into the bounds $\exp(-Ct)|A| \le |\Phi^k(t,A)| \le \exp(Ct)|A|$ for every Borel set A. Assume for the moment that we could prove the strong convergence of Φ^k to a map Φ. Then, clearly $\exp(-Ct)|A| \leq |\Phi(t,A)| \leq \exp(Ct)|A|$, and hence Φ satisfies condition (a) in Definition 5.1. It is then an exercise in elementary measure theory to show that $b_k(t,\Phi^k(t,x))$ converges to $b(t,\Phi(t,x))$ strongly in $L^1_{\rm loc}$. Since Φ^k solves

$$\begin{cases} \partial_t \Phi^k(t,x) &= b_k(t,\Phi^k(t,x)) \\ \Phi^k(0,x) &= x \end{cases}$$

it is straightforward to conclude that Φ solves (38) distributionally.

The main point is therefore to show the strong convergence of Φ_k. This follows from the stability of the corresponding transport equations.

Proof of the strong convergence of Φ^k. — Consider, backward in time, the ODE

$$\begin{cases} \partial_t \Lambda^k(t,x) &= b_k(t,\Lambda^k(t,x)) \\ \Lambda^k(T,x) &= x \end{cases} \tag{46}$$

Let $\Gamma^k(t,\cdot)$ be the inverse of the diffeomorphism $\Lambda^k(t,\cdot)$. If $\overline{u} \in L^\infty(\mathbb{R}^n)$, then $u_k(t,x) = \overline{u}(\Gamma^k(t,x))$ is the unique (backward) solution of the transport equation

$$\begin{cases} \partial_t u_k + {\rm div}_x(b_k u_k) &= u_k\, {\rm div}_x b_k \\ u_k(T,\cdot) &= \overline{u}(\cdot)\,. \end{cases}$$

By Theorem 4.4 and Proposition 1.6, u_k converges strongly in $L^1_{\rm loc}$ to the unique (backward) solution u of

$$\begin{cases} \partial_t u + {\rm div}_x(bu) &= u\, {\rm div}_x b \\ u(T,\cdot) &= \overline{u}(\cdot)\,. \end{cases}$$

Choose $\overline{u}(x) = \chi(x)x_i$, where χ is a smooth cutoff functions. Since $u^k(t,x) = \chi(\Gamma^k(t,x))\Gamma^k_i(t,x)$ we infer easily the strong $L^1_{\rm loc}$ convergence of the components Γ^k_i. This implies that Γ^k converges to a map Γ strongly in $L^1_{\rm loc}([0,T]\times\mathbb{R}^n)$. On the other hand, for any given x, $\Gamma^k(\cdot,x)$ is a Lipschitz curve with Lipschitz constant bounded independently of k. It is then easy to see that $\Gamma^k(t,\cdot)$ is a Cauchy sequence in $L^1(A)$ for every bounded A and every $t\in[0,T]$. In particular, $\Gamma^k(0,\cdot)$ converges to some map strongly in $L^1_{\rm loc}$.

Now, $\Gamma^k(0,\cdot)$ is the inverse of $\Lambda^k(0,\cdot)$, which in view of (46) is the inverse of $\Phi^k(T,\cdot)$. Therefore we conclude that for each T there exists a map $\Phi(T,\cdot)$ such that $\Phi^k(T,\cdot)\to\Phi(T,\cdot)$ strongly in $L^1_{\rm loc}$. Again, using the fact that, for each x, $\Phi^k(\cdot,x)$ is a Lipschitz curve with Lipschitz constant bounded independently of k, it is not difficult to see that Φ^k is a Cauchy sequence in $L^1(A)$ for any bounded $A\subset\mathbb{R}^+\times\mathbb{R}^n$. This concludes the proof. □

6. BEYOND BV AND BEYOND RENORMALIZED SOLUTIONS: FURTHER RESULTS, CONJECTURES AND OPEN PROBLEMS

6.1. Nearly incompressible BV fields

By nearly incompressible fields b we understand those fields for which there exists a regular Lagrangian flow Φ satisfying the bounds $c(t)|A| \leq |\Phi(t,A)| \leq C(t)|A|$, for some continuous and nonvanishing functions c and C. At a first glance there are at least two obstructions to build a theory of renormalized solutions for nearly incompressible flows. On the one hand, it seems necessary to give a meaning to $u\,\mathrm{div}_x b$ in order to define distributional solutions u of (3). On the other hand, it is not clear how to define nearly incompressible fields without referring to some flow.

Both these issues can be naturally solved by using the continuity equation. Indeed, we can define nearly incompressible fields as those b for which there exists a distributional solution $\tilde{\rho}$ of (44). Moreover, there are appropriate versions of the renormalization property which use only the continuity equation and hence can be stated without assumptions on the divergence of b. This point of view was first taken in [7] and it has been systematically explored in [26]. The "soft" part of the DiPerna–Lions theory can be extended naturally to this setting. Concerning the "hard" part, i.e. the proof of the corresponding renormalization properties, the $W^{1,p}$ case of this theory follows from the DiPerna–Lions estimate for the commutators. The BV case is instead still open. Indeed, the motivation in [7] was the following conjecture raised by Bressan in [17].

Conjecture 6.1 (Bressan's compactness conjecture). — *Let $b_k : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a sequence of smooth vector fields and denote by Φ^k the corresponding flows. Assume that $\|b_k\|_\infty + \|\nabla b_k\|_{L^1}$ is uniformly bounded and that $C^{-1} \leq \det(\nabla_x \Phi^k(t,x)) \leq C$ for some constant $C>0$. Then the sequence $\{\Phi^k\}$ is strongly precompact in L^1_{loc}.*

Bressan's conjecture was initially motivated by a problem in the theory of hyperbolic systems of conservation laws. However, in order to solve this problem one does not need to tackle Conjecture 6.1: a milder statement, which is a corollary of Ambrosio's result, suffices (see [10] and [7]). At present, the best result available in the direction of Conjecture 6.1 is contained in [11] and goes towards a theory of renormalized solutions for nearly incompressible BV fields. This paper makes strong use of a refined theory of traces for transport equations, developed in [9].

6.2. Beyond BV fields

Can one hope for the renormalization property when b is in a space larger than BV? The counterexamples available in the literature show fields which are quite close to be BV and do not have the renormalization property (see [27] and [22], both inspired by an older construction of Aizenmann [1]). Moreover, these examples have severe consequences on the possibility of building a general theory of existence for hyperbolic systems of conservation laws on transport equations (see [23]).

Nonetheless there are still many interesting open problems in this direction. For instance, in two dimensions and for divergence–free autonomous fields, renormalization theorems are available even under very mild assumptions, because of the underlying Hamiltonian structure (see [16], [31], [20]). In the recent paper [3] the authors have given a necessary and sufficient condition for the renormalization property when b is divergence–free, planar, autonomous and bounded. In particular, they produce a striking example of such a b which does not have the renormalization property.

A very interesting open question, naturally linked to Euler's equations, is whether the renormalization property holds for divergence–free fields $b \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^2))$ when the vorticity of b is a measure. Another open question is whether the renormalization property holds for fields b with absolutely continuous divergence when the symmetric part of the gradient is a measure. The property indeed holds when the symmetric part of the gradient is in L^1, see [19]. For a more general result in this direction, see [9].

6.3. A direct Lagrangian approach

In the DiPerna–Lions theory, conclusions on the "Lagrangian point of view" are recovered from theorems on the "Eulerian point of view". A natural question is whether one could get the same results directly, for instance proving a–priori estimates on the solutions of the ODEs. Indeed, the whole theory of regular Lagrangian flows for $W^{1,p}$ fields with $p > 1$ can be recovered by proving appropriate estimates in the Lagrangian formulation, as it has been recently shown in [24]. These estimates also provide mild regularity properties for regular Lagrangian flows and distributional solutions to transport equations. In a nutshell, if $b \in W^{1,p}$ and Φ is the corresponding flow, the L^p norm of the difference of $\Phi(t,\cdot) - \Phi(t,\cdot+v)$ can be estimated by a constant (depending on the compressibility of b, and the L^p norm of ∇b) times $|\log(|v|)|^{-1}$.

The estimates of [24] were inspired by some computations of [12], where the authors proved the approximate differentiability of regular Lagrangian flows. In turn, [12] was inspired by another result of [32] on weak differentiability properties for regular Lagrangian flows. See also [13] for a comparison among the various weak notions of differentiability used in these papers.

The estimates of [23] quantify the compactifying properties of transport equations with Sobolev coefficients. In particular they imply the L^p version of a second conjecture of Bressan on the mixing of flows (see [18]), which we state below.

Fix coordinates $x = (x_1, x_2) \in [0,1[\times[0,1[$ on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ and consider the set $A = \big\{(x_1,x_2) : 0 \leq x_2 \leq 1/2\big\} \subset \mathbb{T}$. Given a smooth divergence–free field $b : [0,1]\times\mathbb{T} \to \mathbb{R}^2$ denote by Φ its flow. For a fixed $\kappa \in]0,1/2[$, we say that Φ *mixes the set A up to scale ε* if for every ball $B_\varepsilon(x)$ we have

$$\kappa|B_\varepsilon(x)| \leq |B_\varepsilon(x) \cap \Phi(1,A)| \leq (1-\kappa)|B_\varepsilon(x)|\,.$$

Conjecture 6.2 (Bressan's mixing conjecture). — *Under these assumptions, there exists a constant C depending only on κ s.t., if Φ mixes the set A up to scale ε, then*

$$\int_0^1 \int_{\mathbb{T}} |D_x b| \, dxdt \geq C|\log \varepsilon| \qquad \textit{for every } 0 < \varepsilon < 1/4.$$

7. APPENDIX: PROOF OF PROPOSITION 3.2

Proof. — Let $e_1, \ldots, e_n$ be orthonormal vectors in $\mathbb{R}^n$. In the corresponding system of coordinates we use the notation $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$. Without loss of generality we can assume that $z = e_n$. Recall the following elementary fact: if μ is a Radon measure on $\mathbb{R}$, then the functions

$$\hat{\mu}_\delta(\tau) \;=\; \frac{\mu([\tau, \tau+\delta])}{\delta} \;=\; \mu * \frac{\mathbf{1}_{[-\delta,0]}}{\delta}(\tau) \qquad \tau \in \mathbb{R}$$

satisfy

$$\int_K |\hat{\mu}_\delta| \, d\tau \;\leq\; \mu(K_\delta) \tag{47}$$

for every compact set $K \subset \mathbb{R}$, where K_δ denotes the δ–neighborhood of K.

Consider the measure $D_{e_n} b = D_x b \cdot e_n$, and the vector–valued function $\nabla_x b \cdot e_n$. Clearly this function is the Radon–Nikodym derivative of $D_{e_n} b$ with respect to $\mathscr{L}^{n+1}$ and we denote by $D^s_{e_n} b$ the singular measure $D^s_x b \cdot e_n = D_{e_n} b - \nabla_x b \cdot e_n \mathscr{L}^{n+1}$.

We define

$$b_{1,\delta}(t, x', x_d) \;=\; \frac{1}{\delta} \int_{x_n}^{x_n+\delta} \nabla_x b \cdot e_n(t, x', s) \, ds \,.$$

By Fubini's Theorem and standard arguments on convolutions, we get that $b_{1,\delta} \to \nabla_x b \cdot e_n$ strongly in L^1_{loc}. Next set

$$b_{2,\delta}(t, x', x_n) \;=\; \frac{b(t, x', x_n + \delta) - b(t, x', x_n)}{\delta} - b_{1,\delta}(t, x', x_n)\,,$$

and, for $\mathscr{L}^n$–a.e. $(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, define $b_{t,y} : \mathbb{R} \to \mathbb{R}$ by $b_{t,y}(s) = b(t, y, s)$.

We recall the following slicing properties of BV functions (see Theorem 3.103, Theorem 3.107, and Theorem 3.108 of [14]):

(a) $b_{t,y} \in BV_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ for $\mathscr{L}^n$–a.e. (t, y);

(b) If we let $D^s b_{t,y} + b'_{t,y}\mathscr{L}^1$ be the Radon–Nikodym decomposition of $Db_{t,y}$, then we have

$$\nabla_x b(t, y, s) \cdot e_n \;=\; b'_{t,y}(s) \qquad \text{for } \mathscr{L}^{n+1}\text{–a.e. } (t, y, s)$$

and

$$|D^s_{e_n}|(A) \;=\; \int_{\mathbb{R}^n} |D^s b_{t,y}|(A \cap \{(t, y, s) : s \in \mathbb{R}\}) \, dt \, dy\,;$$

(c) $b_{t,y}(s + \delta) - b_{t,y}(s) = Db_{t,y}([s, s + \delta])$.

Therefore, for any $\delta > 0$ and for $\mathscr{L}^n$–a.e. (t,y) we have

$$\begin{aligned}
\frac{b(t,y,x_n+\delta)-b(t,y,x_n)}{\delta} &= \frac{b_{t,y}(x_n+\delta)-b_{t,y}(x_n)}{\delta} = \frac{Db_{t,y}([x_n,x_n+\delta])}{\delta} \\
&= (\widehat{b'_{t,y}\mathscr{L}^1})_\delta(x_n) + (\widehat{D^s b_{t,y}})_\delta(x_n) \\
&= b_{1,\delta}(t,y,x_n) + (\widehat{D^s b_{t,y}})_\delta(x_n) \qquad \text{for } \mathscr{L}^1\text{–a.e. } x_n.
\end{aligned}$$

Therefore

$$\int_K |b_{2,\delta}| \;\le\; \int_{\mathbb{R}^n}\int_{\{x_n:(t,y,x_n)\in K\}} \left|(\widehat{D^s b_{t,y}})_\delta(x_n)\right| \, dx_n\, dy\, dt$$

$$\le \int_{\mathbb{R}^n} |D^s b_{t,y}|\,(\{x_n : (t,y,x_n)\in K_\delta\})\; dy\,dt \;=\; |D^s_x b\cdot e_n|(K_\delta) \;\le\; |D^s_x b|(K_\delta)\,. \tag{48}$$

Letting $\delta \downarrow 0$, this gives (18).

Note moreover that

$$\int_K |b_{1,\delta}| \;\le\; \int_{\mathbb{R}^n}\int_{\{x_n:(t,y,x_n)\in K\}} \left|(\widehat{b'_{t,y}\mathscr{L}^1})_\delta(x_n)\right| \, dx_n\, dy\, dt$$

$$\le \int_{K_\delta} |\nabla_x b\cdot e_n|(t,y,x_n)\, dy\,dt\,dx_n \;\le\; \int_{K_\delta} |\nabla_x b|(t,y,x_n)\,dy\,dt\,dx_n\,. \tag{49}$$

Adding the bounds (48) and (49) we get (19). □

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