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ODES WITH SOBOLEV COEFFICIENTS: THE EULERIAN AND THE LAGRANGIAN APPROACH

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Abstract. In this paper we describe two approaches to the well-posedness of Lagrangian flows of Sobolev vector fields. One is the theory of renormalized solutions which was introduced by DiPerna and Lions in the eighties. In this framework the well-posedness of the flow is a corollary of an analogous result for the corresponding transport equation. The second approach has been recently introduced by Gianluca Crippa and the author and it is instead based on suitable estimates performed directly on the lagrangian formulation.

1. **Introduction.**

These notes stem from a series of four lectures which the author was kindly invited to give in the nice atmosphere of the Paseky Spring School in Fluid Dynamics in 2007. The lectures focused on two different approaches to the well–posedness of ordinary differential equations with Sobolev coefficients.

Consider first a smooth vector field $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and the associated flow Φ solving

$$\begin{cases} \partial_t \Phi(x,t) \;=\; b(t, \Phi(x,t)) \\ \\ \Phi(0,x) \;=\; x\,. \end{cases} \tag{1}$$

Here we will always regard Φ as a one-parameter family of maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. In fact, when b is smooth, Φ is a family of diffeomorphisms and we will denote by $\Phi^{-1}(t,\cdot)$ the inverse of $\Phi(t,\cdot)$.

A classical observation is that a smooth function $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is constant along the paths $\Phi(\cdot, x)$ if and only if u solves the transport equation $\partial_t u + b\cdot\nabla_x u = 0$. Indeed, differentiating $g(t) = u(t,\phi(t)) = u(t,\Phi(t,x))$ we find

$$\frac{dg}{dt} \;=\; \partial_t u(t,\phi(t)) + \dot\phi(t)\cdot\nabla_x u(t,\phi(t)) \;=\; \partial_t u(t,\phi(t)) + b(t,\phi(t))\cdot\nabla_x u(t,\phi(t)) \;=\; 0\,.$$

Therefore, the unique solution of the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla_x u(t,x) \;=\; 0 \\ \\ u(0,x) \;=\; \overline{u}(x) \end{cases} \tag{2}$$

is given, when b is sufficiently smooth, by the formula $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$.

When b is Lipschitz, existence and uniqueness of solutions to (1) are guaranteed by the classical Cauchy–Lipschitz Theorem, but for less regular b this elegant and

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elementary picture breaks down. On the other hand, many physical phenomena lead naturally to consider transport and ordinary differential equations with discontinuous coefficients. The literature related to this kind of problems is huge and I will not try to give an account of it here.

It is therefore desirable to have a theory of solutions for ODEs and transport equations which allows for non–smooth coefficients. The Sobolev spaces $W^{1,p}$ (given by functions $u \in L^p$ with distributional derivatives in L^p) are probably the most popular spaces of irregular functions in partial differential equations. In their groundbreaking paper [11], motivated by their celebrated work on the Boltzmann equation, DiPerna and Lions introduced a theory of generalized solutions for ODEs with Sobolev coefficients. Loosely speaking, this was done at the loss of a "pointwise" point of view into an "almost everywhere" point of view.

The approach of DiPerna and Lions relies on an "eulerian approach": one proves indeed well-posedness for (2) and, as a byproduct, gets a corresponding result for (1). This approach is based on the principle that any reasonable concept of flow for (1) is linked to solutions of (2). In recent years the problem of ODEs with rough coefficients has gained again a lot of attention because of a groundbreaking result of Ambrosio (see [1] and [2]), who succeeded in extending the DiPerna–Lions theory to BV coefficients. Motivated by situations where, so far, no extension of the DiPerna–Lions theory has been shown to exist, different authors raised the following question: can we reach some of the conclusions of that theory via a direct lagrangian approach? More precisely, is it possible to prove sufficiently strong a-priori estimates in order to derive existence, uniqueness and compactness properties of flows for (1) from a direct "lagrangian" point of view?

This is indeed possible in many cases, and it has been shown for the first time in a joint work of myself with Gianluca Crippa (see [8]). Besides giving a different derivation of the results of DiPerna and Lions, our new approach has a number of interesting corollaries such as, for instance, a partial answer to a conjecture raised by Bressan in [7]. Moreover, in a recent paper by Bouchut and Crippa (see [6]), the ideas of [8] have been extended to a setting where the eulerian point of view has been, so far, unsuccessful.

In these notes I will present both the DiPerna–Lions theory and the approach of [8]. I will do it in a simplified setting, in order to avoid the technical difficulties of the most general result and with the hope of highlighting at the same time the most important ideas.

2. Regular Lagrangian flows and a modified conjecture of Bressan.

2.1. Regular Lagrangian flows.

We start by defining a generalized notion of flow for (1).

Definition 2.1 (Regular Lagrangian flows). Let $b \in L^\infty([0,\infty[\times\mathbb{R}^n, \mathbb{R}^n)$. A map $\Phi : [0,\infty[\times\mathbb{R}^n \to \mathbb{R}^n$ is a regular Lagrangian flow for b if

(a) There exists a constant C such that $|\{x : \Phi(t,x) \in A\}| \leq C|A|$ for $\mathscr{L}^1$–a.e. t.
(b) The identity (1) is valid in the sense of distributions.

The smallest constant C which fulfills (a) will be called the *compressibility constant of the flow* Φ.

Note that assumption (a) guarantees that $b(t,\Phi(t,x))$ is well defined. Indeed, if $\hat{b} = b$ $\mathscr{L}^{n+1}$–a.e., then $\hat{b}(t,\Phi(t,x)) = b(t,\Phi(t,x))$ for $\mathscr{L}^{n+1}$–a.e. (t,x).

These notes provide two different proofs of the following Theorem (in fact, the second proof is restricted to the case $p > 1$, cp. with the comments below).

Theorem 2.2 (Existence, uniqueness and stability of regular Lagrangian flows)**.** *Let $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n)) \cap L^\infty$ with bounded divergence and $p \geq 1$. Then there exists a unique regular Lagrangian flow Φ for b. Moreover, if b_k is a sequence of smooth vector fields converging strongly in $L^1_{\rm loc}$ to b such that $\|{\rm div}_x b\|_\infty$ is uniformly bounded, then the flows of b_k converge strongly in $L^1_{\rm loc}$ to Φ.*

This theorem is just a prototype of what can be proved with the DiPerna–Lions theory or with the direct lagrangian approach of [8]. Indeed, its hypotheses can be relaxed in several ways. We refer the interested reader to the various survey articles [3], [9]. In the rest of the notes we will: first prove Theorem 2.2 using the DiPerna-Lions theory; then prove the theorem under the assumption $p > 1$ using the estimates of [8]. It must be noted that so far we have not be able to extend the estimates of [8] when $p = 1$, which would encompass also the BV case. However, a direct lagrangian proof of Theorem 2.2 can be achieved even in the $p = 1$ case proving weaker estimate (see [5] and [15]; these estimates, however, do not include the BV case). Since this extension would require a certain amount of technicality, we do not present it here.

2.2. **A modified conjecture of Bressan.** The final section of this note will give an application of [8] which cannot be reached by the DiPerna–Lions theory: a proof of an L^p version of a conjecture of Bressan. Before stating it, we need to introduce some notation. Consider the two-dimensional torus $K = \mathbb{R}^2/\mathbb{Z}^2$. Fix coordinates $(x_1, x_2) \in [0,1[\times[0,1[$ on K and consider the set

$$A = \big\{(x_1, x_2)\ :\ 0 \leq x_1 \leq 1/2\big\} \subset K\,.$$

If $b : [0,1] \times K \to \mathbb{R}^2$ is a smooth time-dependent vector field, we denote by $\Phi(t,x)$ the flow of b and by $X : K \to K$ the value of the flow at time $t = 1$. We assume that the flow is nearly incompressibile, so that for some $\kappa' > 0$ we have

$$\kappa'|\Omega| \leq |\Phi(t,\Omega)| \leq \frac{1}{\kappa'}|\Omega| \tag{3}$$

for all $\Omega \subset K$ and all $t \in [0,1]$. For a fixed $0 < \kappa < 1/2$, we say that X *mixes the set A up to scale ε* if for every ball $B_\varepsilon(x)$ we have

$$\kappa|B_\varepsilon(x)| \leq |B_\varepsilon(x) \cap X(A)| \leq (1-\kappa)|B_\varepsilon(x)|\,.$$

Then in [7] the following conjecture is proposed:

Conjecture 2.3 (Bressan's mixing conjecture)**.** *Under these assumptions, there exists a constant C depending only on κ and κ' such that, if X mixes the set A up to scale ε, then*

$$\int_0^1 \int_K |D_x b|\, dxdt \geq C|\log\varepsilon| \qquad \textit{for every } 0 < \varepsilon < 1/4.$$

In the last section of these notes, we will prove the following result:

Theorem 2.4. *Let $p > 1$. Under the previous assumptions, there exists a constant C depending only on κ, κ' and p such that, if X mixes the set A up to scale ε, then*

$$\int_0^1 \|D_x b\|_{L^p(K)}\, dt \geq C|\log\varepsilon| \qquad \textit{for every } 0 < \varepsilon < 1/4. \tag{4}$$

3. **Renormalized solutions to transport equations.** In this section I discuss the first key idea of [11]: the notion of renormalized solutions and its link to the uniqueness and stability for (2).

3.1. **Distributional solutions.** Let us start by rewriting (2) in the following way:

$$\begin{cases} \partial_t u + \operatorname{div}_x(ub) - u\operatorname{div}_x b \;=\; 0 \\ \\ u(0,x) \;=\; \overline{u}(x)\,. \end{cases} \tag{5}$$

Here and in what follows I denote by $\operatorname{div}_x b$ the divergence (in space) of the vector b. Clearly any classical solution of (5) is a solution of (2) and viceversa. However, equation (5) can be understood in the distributional sense under very mild assumptions on u and b. This is stated more precisely in the following definition.

Definition 3.1. Let b and $\overline{u}$ be locally summable functions such that the distributional divergence of b is locally summable. We say that $u \in L^\infty_{\mathrm{loc}}$ is a distributional solution of (5) if the following identity holds for every test function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$

$$\int_0^\infty \int_{\mathbb{R}^n} u\,[\partial_t\varphi + b\cdot\nabla_x\varphi + \varphi\operatorname{div}_x b]\,dx\,dt \;=\; -\int_{\mathbb{R}^n} \overline{u}(x)\varphi(0,x)\,dx\,. \tag{6}$$

Of course for classical solutions the identity (6) follows from a simple integration by parts. The existence of weak solutions under quite general assumptions is an obvious corollary of the maximum principle for transport equations combined with a standard approximation argument.

Lemma 3.2 (Maximum Principle)**.** *Let b be smooth and let u be a smooth solution of (5). Then, for every t we have $\sup_{x\in\mathbb{R}^n} u(t,x) \le \sup_{x\in\mathbb{R}^n}\overline{u}(x)$ and $\inf_{x\in\mathbb{R}^n} u(t,x) \ge \inf_{x\in\mathbb{R}^n}\overline{u}(x)$. Hence $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|\overline{u}\|_\infty$.*

Proof. The lemma is a trivial consequence of the method of characteristics. Indeed, arguing as in the introduction, $u(t,x) = \overline{u}(\Phi^{-1}(t,x))$, where Φ is the solution of (1). From this representation formula the inequalities follow trivially. □

Theorem 3.3 (Existence of distibutional solutions)**.** *Let $b \in L^p$, $p \ge 1$, with $\operatorname{div}_x b \in L^1_{loc}$ and let $\overline{u} \in L^\infty$. Then there exists a distributional solution of (5).*

Proof. Consider a standard family of mollifiers ζ_ε and η_ε respectively on $\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{R}^n$. Let $b_\varepsilon = b * \eta_\varepsilon$ and $\overline{u}_\varepsilon = \overline{u} * \zeta_\varepsilon$ be the corresponding regularizations of b and $\overline{u}$. Then $\|\overline{u}_\varepsilon\|_\infty$ is uniformly bounded. Consider the classical solutions u_ε of

$$\begin{cases} \partial_t u_\varepsilon + b_\varepsilon\cdot\nabla_x u_\varepsilon \;=\; 0 \\ \\ u_\varepsilon(0,\cdot) \;=\; \overline{u}_\varepsilon\,. \end{cases} \tag{7}$$

Note that such solutions exist because we can solve the equation with the method of characteristics: indeed each b_ε is Lipschitz and we can apply the classical Cauchy–Lipschitz theorem to solve (1). By Lemma 3.2 we conclude that $\|u_\varepsilon\|_\infty$ is uniformly bounded. Hence there exists a subsequence converging weakly* to a function $u \in L^\infty(\mathbb{R}^+\times\mathbb{R}^n)$. Let us fix a test function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$. Since the u_ε are classical solutions of (7), the identity (6) is satisfied if we replace u, b and $\overline{u}$ with u_ε, b_ε and $\overline{u}_\varepsilon$. On the other hand, since $b_\varepsilon \to b$, $\operatorname{div}_x b_\varepsilon \to \operatorname{div}_x b$ and $\overline{u}_\varepsilon \to \overline{u}$ strongly in L^1_{loc}, we can pass into the limit in such identities to achieve (6) for u, $\overline{u}$ and b. □

3.2. **Renormalized solutions.** Of course the next relevant questions are whether such distributional solutions are unique and stable. Under the general assumptions above the answer is negative, as it is for instance witnessed by the elegant example of [10]. However, DiPerna and Lions in [11] proved stability and uniqueness when $b \in W^{1,p} \cap L^\infty$, $p \geq 1$, and $\mathrm{div}_x b \in L^\infty$.

Theorem 3.4 (Uniqueness and stability)**.** *Let $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n)) \cap L^\infty$, $p \geq 1$, with bounded divergence. Then for every $\overline{u} \in L^\infty$ there exists a unique distributional solution of (5). Moreover, let b_k and $\overline{u}_k$ be two smooth approximating sequences converging strongly in L^1_{loc} to b and $\overline{u}$ such that $\|\overline{u}_k\|_\infty$ is uniformly bounded. Then the solutions u_k of the corresponding transport equations converge strongly in L^1_{loc} to u.*

In order to understand their proof, we first go back to classical solutions u of (5), and we observe that, whenever $\beta : \mathbb{R} \to \mathbb{R}$ is a C^1 function, $\beta(u)$ solves

$$\begin{cases} \partial_t[\beta(u)] + \mathrm{div}_x[\beta(u)b] - \beta(u)\,\mathrm{div}_x b \;=\; 0 \\ \\ [\beta(u)] \;=\; \beta(\overline{u})\,. \end{cases} \tag{8}$$

This can be seen, for instance, using the chain rule for differentiable functions, i.e. $\partial_t \beta(u) + b \cdot \nabla_x \beta(u) \;=\; \beta'(u)[\partial_t u + b \cdot \nabla_x u]$. Otherwise, one can observe that, since u must be constant along the trajectories (1), so must be $\beta(u)$. Motivated by this observation, we introduce the following terminology.

Definition 3.5 (Renormalized solutions)**.** Let $b \in L^1_{\mathrm{loc}}$ with $\mathrm{div}_x b \in L^1_{\mathrm{loc}}$. A bounded distributional solution of (5) is said *renormalized* if $\beta(u)$ is a solution of (8) for any $\beta \in C^1$. The field b is said to have the *renormalization property* if every bounded distributional solution of (5) is renormalized.

When b and u are not regular we can use nor the chain rule, neither the theory of characteristics. Therefore, whether a distributional solution is renormalized might be a nontrivial question. Actually, for quite general b, there do exist distributional solutions which are not renormalized (see again [10]). The proof of Theorem 3.4 given by DiPerna and Lions consists of two parts. The first one, which is "soft" can be stated as follows.

Proposition 3.6 (Soft Part of Theorem 3.4)**.** *If $b \in L^\infty$ has the renormalization property and its divergence is bounded, then the uniqueness and stability properties of Theorem 3.4 hold.*

The second one, which is the "hard" part of the proof, states essentially that $W^{1,p}$ fields have the renormalization property.

Theorem 3.7 (Hard Part of Theorem 3.4)**.** *Any $b \in L^1([0,\infty[, W^{1,p}(\mathbb{R}^n))$ with $p \geq 1$ has the renormalization property.*

We postpone the "hard part" to the next section and come first to Proposition 3.6.

Proof. **Uniqueness.** Fix a u_0 and let u and v be two distributional solutions of (5). It then follows that $w = u - v$ is a distributional solution of the same transport equation with initial data 0. By the renormalization property so is w^2, i.e.

$$\begin{cases} \partial_t w^2 + \mathrm{div}_x(w^2 b) \;=\; w^2\,\mathrm{div}_x b \\ \\ w^2(0,\cdot) \;=\; 0\,. \end{cases} \tag{9}$$

Integrating (9) "formally" in space we obtain

$$\partial_t \int_{\mathbb{R}^n} w^2(t,x)\,dx = \int_{\mathbb{R}^n} w^2(t,x)\,\mathrm{div}_x b \;\le\; \|\mathrm{div}_x b\|_\infty \int_{\mathbb{R}^n} w^2(t,x)\,.$$

Since $\int_{\mathbb{R}^n} w^2(0,x)\,dx = 0$, by Gronwall's Lemma we would conclude that for every t, $\int_{\mathbb{R}^n} w^2(t,x)\,dx = 0$. We sketch how to make rigorous this formal argument. Assume for simplicity $\|b\|_\infty \le 1$. Let $T, R > 0$ be given and choose a smooth cut–off function $\varphi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$ such that:

- $\varphi = 1$ on $[0,T]\times B_R(0)$ and
- $\partial_t\varphi \le -|\nabla_x\varphi|$ on $[0,2T]\times\mathbb{R}^n$.

Now let $\psi \in C^\infty_c(]-2T,2T[)$ be nonnegative and test (9) with $\psi(t)\varphi(t,x)$. Define $f(t) = \int_{\mathbb{R}^n} w^2(t,x)\,\varphi(t,x)\,dx$ and use Fubini's Theorem to get

$$\begin{aligned}
-\int_0^\infty f(t)\partial_t\psi(t)\,dt =& \int_0^\infty\int \psi(t)\varphi(t,x)\,w^2(t,x)\,\mathrm{div}_x b(t,x)\,dx\,dt\\
&+\int_0^\infty\int \psi(t)w^2(t,x)\left[\partial_t\varphi(t,x) + b(t,x)\cdot\nabla_x\varphi(t,x)\right]dx\,dt\,.
\end{aligned}$$

Note that the second integral in the right hand side is nonpositive, whereas the first one can be estimated by $\|\mathrm{div}_x b\|_\infty \int f(t)\psi(t)\,dt$. We conclude that f satisfies a "distributional" form of Gronwall's inequality for $t\in[0,2T[$. It can be easily seen that this implies $f=0$. Thus $w=0$ a.e. on $[0,T]\times B_R(0)$, and by the arbitrariness of R and T we conclude $w=0$.

Stability. Arguing as in Theorem 3.3, we easily conclude that, up to subsequences, u_k converges weakly* in L^∞ to a distributional solution u of (5). However, by the uniqueness part of the Theorem, this solution is unique, and hence the whole sequence converges to u. Since the b_k and the u_k are both smooth, u_k^2 solves the corresponding transport equations with initial data $\overline{u}^2$. Arguing as above, u_k^2 must then converge, weakly* in L^∞, to the unique solution of (5) with initial data $\overline{u}^2$. But by the renormalization property this solution is u^2. Summarizing, $u_k \overset{*}{\rightharpoonup} u$ and $u_k^2 \overset{*}{\rightharpoonup} u^2$ in L^∞, which clearly implies the strong convergence in L^1_{loc}. □

4. The commutator estimate of DiPerna and Lions.

In this section we come to the "hard part", i.e. Theorem 3.7. We first prove a milder conclusion, neglecting the initial conditions, which will be adjusted later.

Proposition 4.1. *Assume $b\in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n))$ with $p\ge 1$ and let $u\in L^\infty$ solve*

$$\partial_t u + \mathrm{div}_x(ub) - u\,\mathrm{div}_x b = 0 \tag{10}$$

distributionally on $\mathbb{R}^+\times\mathbb{R}^n$. Then, for every $\beta\in C^1$,

$$\partial_t[\beta(u)] + \mathrm{div}_x(\beta(u)b) - \beta(u)\,\mathrm{div}_x b \;=\; 0\,. \tag{11}$$

4.1. Commutators.

Let us fix u and b as in Proposition 4.1 and consider a standard smooth and even kernel ρ in $\mathbb{R}^n$. By a slight abuse of notation we denote by $u*\rho_\varepsilon$ the convolution in the x variable, that is $[u*\rho_\varepsilon](t,x) = \int u(t,y)\rho_\varepsilon(x-y)dy$. Mollify (10) to obtain $0 = \partial_t u*\rho_\varepsilon + [\mathrm{div}_x(bu)]*\rho_\varepsilon - [u\,\mathrm{div}_x b]*\rho_\varepsilon$. We rewrite this identity as

$$\partial_t u*\rho_\varepsilon + b\cdot(\nabla_x u*\rho_\varepsilon) \;=\; [(u\,\mathrm{div}_x b)*\rho_\varepsilon - (u*\rho_\varepsilon)\,\mathrm{div}_x b] - R_\varepsilon \tag{12}$$

where R_ε are simply the commutators

$$R_\varepsilon \;=\; [\mathrm{div}_x(bu)] * \rho_\varepsilon - \mathrm{div}_x[b(u*\rho_\varepsilon)]\,. \tag{13}$$

Since R_ε is a locally summable function, the identity (12) implies that $\partial_t u * \rho_\varepsilon$ is also locally summable. Thus, $u*\rho_\varepsilon$ is a Sobolev function in space and time, and we can use the chain rule for Sobolev functions (see for instance Section 4.2.2 of [13]) to compute

$$\partial_t[\beta(u*\rho_\varepsilon)] + b\cdot\nabla_x\big[\beta(u*\rho_\varepsilon)\big] = \beta'(u*\rho_\varepsilon)\big[\partial_t u*\rho_\varepsilon + b\cdot\nabla_x(u*\rho_\varepsilon)\big]\,.$$

Inserting (12) in this identity we get

$$\partial_t[\beta(u*\rho_\varepsilon)]+b\cdot\nabla_x\big[\beta(u*\rho_\varepsilon)\big]=\beta'(u*\rho_\varepsilon)\big\{[(u\,\mathrm{div}_x b)*\rho_\varepsilon-(u*\rho_\varepsilon)\,\mathrm{div}_x b]-R_\varepsilon\big\}\,. \tag{14}$$

Now, the left hand side of (14) converges distributionally to the left hand side of (11). Recall that $\|\beta'(u_\varepsilon)\|_\infty$ and $\|u*\rho_\varepsilon\|_\infty$ are uniformly bounded, whereas

$$\big[(u\,\mathrm{div}_x b)*\rho_\varepsilon - (u*\rho_\varepsilon)\,\mathrm{div}_x b\big] \qquad\longrightarrow\qquad 0$$

strongly in L^1_{loc}. Therefore, in order to prove Proposition 4.1 we just need to show that $\beta'(u*\rho_\varepsilon)R_\varepsilon$ converges to 0. This is implied by the following lemma.

Lemma 4.2 (Commutator estimate)**.** *Let $b\in L^1(\mathbb{R}^+,W^{1,p}(\mathbb{R}^n))$ with $p\geq 1$ and $u\in L^\infty$. If R_ε is defined as in (13), then $R_\varepsilon\to 0$ in L^1_{loc}.*

Observe that Lemma 4.2 is valid for general b and u: the proof does not exploit the fact that u solves (10).

4.2. **The commutator estimate of DiPerna and Lions.**

Proof of Lemma 4.2. Without loss of generality we assume that the kernel ρ is supported in $B_1(0)$. First we use the elementary identity

$$R_\varepsilon = \sum_i (ub_i)*\partial_{x_i}\rho_\varepsilon - \sum_i b_i(u*\partial_{x_i}\rho_\varepsilon) - (u*\rho_\varepsilon)\,\mathrm{div}_x b$$

and we expand the convolutions to obtain

$$R_\varepsilon(t,x) \;=\; \int u(t,y)(b(t,y)-b(t,x))\cdot\nabla\rho_\varepsilon(x-y)\,dy - \big[(u*\rho_\varepsilon)\,\mathrm{div}_x b\big](t,x)\,. \tag{15}$$

Since $\nabla\rho_\varepsilon(\xi)=\varepsilon^{-n-1}\nabla\rho(\xi/\varepsilon)$, we perform the change of variables $z=(y-x)/\varepsilon$ to get

$$R_\varepsilon(t,x) \;=\; \int u(t,x+\varepsilon z)\frac{b(t,x+\varepsilon z)-b(t,x)}{\varepsilon}\cdot\nabla\rho(-z)\,dz - \big[(u*\rho_\varepsilon)\,\mathrm{div}_x b\big](t,x) \tag{16}$$

Next, fix a compact set K. By standard properties of Sobolev functions (see for instance Section 5.8.2 of [12]), the difference quotients

$$d_{\varepsilon,z}(t,x) \;=\; \frac{b(t,x+\varepsilon z)-b(t,x)}{\varepsilon} \tag{17}$$

are bounded in $L^p(K)$ independently of $z\in B_1(0)$ and $\varepsilon\in]0,1[$. We now let $\varepsilon\downarrow 0$. For each fixed z, $d_{\varepsilon,z}$ converges strongly in $L^p(K)$ to $\partial_z b$. The functions $u_{z,\varepsilon}(t,x)=u(t,x+\varepsilon z)$ are instead uniformly bounded in L^∞, and, by the L^1–continuity of the translation, they converge strongly in $L^1(K)$ to u.

Therefore we conclude that R_ε converges strongly in $L^1_{\rm loc}$ to

$$\begin{aligned} R_0(t,x) &= u(t,x)\int \partial_z b(t,x)\cdot\nabla\rho(-z)\,dz - \big[u\,{\rm div}_x b\big](t,x)\\ &= u(t,x)\sum_{i,j}\partial_i b^j(t,x)\int z_i\partial_{z_j}\rho(-z)\,dz - u(t,x)\,{\rm div}_x b(t,x)\,. \end{aligned}$$

Integrating by parts we have $\int z_i\partial_{z_j}\rho(-z)\,dz = \delta_{ij}$. So $R_0=0$, which completes the proof. □

4.3. **The initial condition.** In order to prove Theorem 3.7 we still need to show that $\beta(u)$ takes the initial condition $[\beta(u)](0,\cdot)=\beta(\overline{u})(\cdot)$. This is achieved with a small trick.

Proof of Theorem 3.7. Consider b and u as in Theorem 3.7 and extend both of them to negative times by setting $b(t,x)=0$ and $u(t,x)=\overline{u}(x)$ for $t<0$. It is then immediate to check that $\partial_t u+{\rm div}_x(bu)=u\,{\rm div}_x b$ distributionally on the whole space–time $\mathbb{R}\times\mathbb{R}^n$. On the other hand the proof of Proposition 4.1 remains valid if we replace $\mathbb{R}^+$ with $\mathbb{R}$ (actually the proof remains the same on any open set $\Omega\subset\mathbb{R}\times\mathbb{R}^n$). Therefore

$$\partial_t[\beta(u)]+{\rm div}_x[b\beta(u)]=\beta(u)\,{\rm div}_x b$$

distributionally on $\mathbb{R}\times\mathbb{R}^n$. We test this equation with a $\varphi\in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$, recalling that $[\beta(u)](t,x)=\beta(\overline{u}(x))$ and $b(t,x)=0$ for $t<0$. We then conclude

$$\int_0^\infty\int_{\mathbb{R}^n}\beta(u)\big[\partial_t\varphi+b\cdot\nabla_x\varphi+{\rm div}_x b\,\varphi\big]\,dx\,dt \;=\; -\int_{\mathbb{R}^n}\beta(\overline{u}(x))\int_{-\infty}^0\partial_t\varphi(t,x)\,dt\,dx\,. \tag{18}$$

On the other hand, since φ is smooth, we can integrate by parts in t in the right hand side of (18) in order to get $-\int\beta(\overline{u}(x))\varphi(0,x)dx$. This concludes the proof. □

5. **First proof of Theorem 2.2.** We start by defining the density of a regular Lagrangian flow Φ. First, we denote by μ_Φ the measure $({\rm id}\,,\Phi)_\#\mathscr{L}^{n+1}\llcorner([0,\infty[\times\mathbb{R}^n)$, i.e. the push–forward via the map $(t,x)\mapsto(t,\Phi(t,x))$ of the Lebesgue $(n+1)$–dimensional measure on $[0,\infty[\times\mathbb{R}^n$. Such push–forward is simply defined by the property

$$\int_{[0,\infty[\times\mathbb{R}^n}\psi(t,x)\,d\mu_\Phi(t,x) \;=\; \int_{[0,\infty[\times\mathbb{R}^n}\psi(t,\Phi(t,x))\,d\mathscr{L}^{n+1}(t,x)$$

valid for every $\psi\in C_c(\mathbb{R}\times\mathbb{R}^n)$. Observe that (a) implies the absolute continuity of μ_Φ with respect to the Lebesgue measure, and hence the existence of a $\rho\in L^\infty([0,\infty[\times\mathbb{R}^n)$ such that $\mu_\Phi=\rho\mathscr{L}^{n+1}$.

Definition 5.1. The ρ defined above will be called the *density* of the regular Lagrangian flow Φ.

When b is smooth and Φ is the classical solution of (1), $t\mapsto\Phi(t,\cdot)$ is a one–parameter family of diffeomorphisms. For each t let us denote by $\Phi^{-1}(t,\cdot)$ the inverse of $\Phi(t,\cdot)$. Then ρ can be explicitly computed as $\rho(t,x)=\det\nabla_x\Phi(t,\Phi^{-1}(t,x))$ and the classical Liouville Theorem states that ρ solves the continuity equation $\partial_t\rho+{\rm div}_x(\rho b)=0$. Moreover, since $\Phi(0,x)=x$, the initial condition for ρ is $\rho(0,x)=1$. This property remains true for regular Lagrangian flows and it is simply the special case $\overline{\zeta}=1$ in the following Proposition.

Proposition 5.2 (Weak Liouville Theorem)**.** *Let Φ be a regular Lagrangian flow for a field b. Let $\overline{\zeta} \in L^\infty(\mathbb{R}^n)$ set $\mu = (\mathrm{id}\,,\Phi)_\#(\overline{\zeta}\mathscr{L}^{n+1})$. Then there exists $\zeta \in L^\infty([0,\infty[\times\mathbb{R}^n)$ such that $\mu = \zeta\mathscr{L}^{n+1}$. This ζ solves (distributionally)*

$$\begin{cases} \partial_t\zeta + \mathrm{div}_x(\zeta b) \;=\; 0 \\ \\ \zeta(0,\cdot) \;=\; \overline{\zeta}\,. \end{cases} \tag{19}$$

Proof. First of all, notice that $\mu \le \|\overline{\zeta}\|_\infty \mu_\Phi$ and the existence of ζ is an easy corollary of the reasoning above. Now, let $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^n)$ be any given test function. Our goal is to show that

$$-\int_{[0,\infty[\times\mathbb{R}^n} \zeta(t,x)\big(\partial_t\psi(t,x)+b(t,x)\cdot\nabla_x\psi(t,x)\big)\,dx\,dt \;=\; \int_{\mathbb{R}^n}\overline{\zeta}(x)\psi(0,x)\,dx\,. \tag{20}$$

By definition, the left hand side of (20) is equal to

$$-\int_{\mathbb{R}^n}\overline{\zeta}(x)\left[\int_0^\infty \big(\partial_t\psi(t,\Phi(t,x)) + \nabla_x\psi(t,\Phi(t,x))\cdot b(t,\Phi(t,x))\big)\,dt\right]dx\,. \tag{21}$$

The proof would follow if we could integrate by parts in t, since $\psi(0,\Phi_x(0)) = \psi(0,x)$ and $\psi(T,\Phi_x(T)) = 0$ for any T large enough (because ψ is compactly supported). On the other hand this integration by parts is easy to justify for a.e. x, since (1) implies that the curve $t \mapsto \Phi(t,x)$ is Lipschitz for a.e. x. □

5.1. **Uniqueness of solutions to the continuity equation.** Next, let us assume that $\mathrm{div}_x b$ is bounded in L^∞. Then we would expect, formally, that the density of Φ is bounded away from 0 and $+\infty$. Indeed, assume that b and Φ are both smooth and rewrite the continuity equation as $\partial_t\rho + b\cdot\nabla_x\rho + \rho\,\mathrm{div}_x b = 0$. Fix x and differentiate the function $\omega(t) = \rho(t,\Phi(t,x))$ to get

$$\begin{aligned} \frac{d\omega}{dt}(t) &= \partial_t\rho(t,\Phi(t,x)) + \partial_t\Phi(t,x)\cdot\nabla_x\rho(t,\Phi(t,x)) \\ &= \partial_t\rho(t,\Phi(t,x)) + b(t,\Phi(t,x))\cdot\nabla_x\rho(t,\Phi(t,x)) \\ &= -\mathrm{div}_x b(t,\Phi(t,x))\rho(t,\Phi(t,x)) \\ &= -\mathrm{div}_x b(t,\Phi(t,x))\omega(t)\,. \end{aligned}$$

Since $-\|\mathrm{div}_x b\|_\infty \le -\mathrm{div}_x b(t,\Phi(t,x)) \le \|\mathrm{div}_x b\|_\infty$ and $\omega(0)=1$, we can use Gronwall's Lemma to conclude $\exp(-T\|\mathrm{div}_x b\|_\infty) \le \omega(T) \le \exp(T\|\mathrm{div}_x b\|_\infty)$. But $\Phi(T,\cdot)$ is surjective, because it is a diffeomorphism. Therefore we conclude

$$\exp(-T\|\mathrm{div}_x b\|_\infty) \le \rho \le \exp(T\|\mathrm{div}_x b\|_\infty). \tag{22}$$

We cannot use this formal argument on the density of a general regular Lagrangian flow. On the other hand, by a standard approximation procedure, we can show the following Lemma.

Lemma 5.3 (Existence of a density)**.** *Let $b \in L^\infty$, $p \ge 1$, with bounded divergence. Then there exists a $\tilde\rho \in L^\infty_{\mathrm{loc}}$ satisfying the bounds (22) and solving*

$$\begin{cases} \partial_t\tilde\rho + \mathrm{div}_x(\tilde\rho b) \;=\; 0 \\ \\ \tilde\rho(0,\cdot) \;=\; 1\,. \end{cases} \tag{23}$$

Proof. Let φ be a standard convolution kernel, and consider $b_k = b*\varphi_{k^{-1}}$. Consider the densities ρ_k of the classical flows of b_k. Equation (19) holds with b and $\tilde\rho$ replaced by b_k and ρ_k. On the other hand, for ρ_k we can argue as above and get the bounds

$\exp(-\|\mathrm{div}_x b_k\|_\infty) \le \rho_k(t,x) \le \exp(-\|\mathrm{div}_x b_k\|_\infty)$. Since $\|\mathrm{div}_x b_k\|_\infty \le \|\mathrm{div}_x b\|_\infty$, there exists a subsequence of ρ_k which converges weakly* in L^∞ to a $\tilde\rho$ satisfying (22). Arguing as in Theorem 3.3 we obtain (23) by passing into the limit in the continuity equations for $\tilde\rho_k$. □

If we knew the uniqueness of solutions to the continuity equation, this existence result would become a proof of the formal bound (22) for the density of any regular Lagrangian flow. As usual, we consider the case of b smooth in order to get some insight. Let ρ and $\tilde\rho$ be two smooth solutions of (23), with $\tilde\rho > 0$, and define $u = \rho/\tilde\rho$. Then we could use the chain rule to compute

$$\partial_t u + b\cdot\nabla_x u \;=\; \tilde\rho^{-2}\big\{\tilde\rho\big[\partial_t\rho + b\cdot\nabla_x\rho\big] - \rho\big[\partial_t\tilde\rho + b\cdot\nabla_x\tilde\rho\big]\big\}\,.$$

Adding and subtracting $\tilde\rho^{-2}(\rho\tilde\rho\,\mathrm{div}_x b)$, we achieve

$$\partial_t u + b\cdot\nabla_x u \;=\; \tilde\rho^{-2}\big\{\tilde\rho\big[\partial_t\rho + \mathrm{div}_x(\rho b)\big] - \rho\big[\partial_t\tilde\rho + \mathrm{div}_x(\tilde\rho b)\big]\big\} \;=\; 0\,.$$

But since $u(0,x) = \rho(0,x)/\tilde\rho(0,x) = 1$, we conclude $u(t,x) = 1$ for every t and x.

The computations above are very similar, in spirit, to the renormalization property. It is therefore not a surprise that the theorem below follows from suitable modifications of the proof of Theorem 3.4.

Theorem 5.4. *Let $b \in L^1(\mathbb{R}^+, W^{1,p}(\mathbb{R}^n)) \cap L^\infty$ with bounded divergence and let $\tilde\rho$ and ζ be L^∞ functions solving respectively (23) and (19). If $\tilde\rho \ge C > 0$, then $u = \zeta/\tilde\rho$ is a distributional solution of*

$$\begin{cases} \partial_t u + \mathrm{div}_x(ub) - u\,\mathrm{div}_x b \;=\; 0 \\[2ex] u(0,\cdot) \;=\; \overline{\zeta}\,. \end{cases} \tag{24}$$

By minor modifications of the ideas of Section 1, Lemma 5.3 and Theorem 5.4 yield the desired uniqueness for solutions of the continuity equations.

Corollary 5.5 (Uniqueness of the density)**.** *Let b be as in Theorem 5.4. Then there exists a unique $\zeta \in L^1_{\mathrm{loc}}$ solving (19). Therefore, if Φ is a regular Lagrangian flow for b, the density of Φ coincides with the density $\tilde\rho$ of Lemma 5.3 and hence satisfies the bounds (22).*

5.2. **Uniqueness and stability of regular Lagrangian flows.** The uniqueness of solutions of the continuity equations yields easily the uniqueness and stability of regular Lagrangian flows.

Proof of the uniqueness and stability parts in Theorem 2.2. **Uniqueness.** Let Φ and Ψ be two regular Lagrangian flows for b. Fix a $\overline{\zeta} \in C_c(\mathbb{R}^n)$ and consider the unique solution ζ of (19). According to Proposition 5.2 we have $(\mathrm{id}\,,\Phi)_\#(\overline{\zeta}\mathscr{L}^{n+1}) = \zeta\mathscr{L}^{n+1} = (\mathrm{id}\,,\Psi)_\#(\overline{\zeta}\mathscr{L}^{n+1})$. This identity means that

$$\int \varphi(t,\Phi(t,x))\overline{\zeta}(x)\,dt\,dx \;=\; \int \varphi(t,\Psi(t,x))\overline{\zeta}(x)\,dt\,dx$$

for every test function $\varphi \in C_c(\mathbb{R}\times\mathbb{R}^n)$. But since $\overline{\zeta}$ has compact support, one can infer the equality even when $\varphi(t,y) = \chi(t)y_i$ for $\chi \in C_c(\mathbb{R})$. So

$$\int \Phi_i(t,x))\chi(t)\overline{\zeta}(x)\,dt\,dx \;=\; \int \Psi_i(t,x))\chi(t)\overline{\zeta}(x)\,dt\,dx$$

for any pair of functions $\chi \in C_c(\mathbb{R})$ and $\overline{\zeta} \in C_c(\mathbb{R}^n)$. This easily implies $\Phi_i = \Psi_i$ a.e..

Stability. Consider a sequence $\{b_k\}$ as in the statement of the Theorem and let Φ^k be the corresponding classical flows. Fix a $\overline{\zeta} \in C_c(\mathbb{R}^n)$ and consider the ζ_k and u_k solving, respectively, the continuity equations and the transport equations with coefficients b_k and initial data $\overline{\zeta}$. Recall that, if ρ_k are the densities of Φ^k, then $\zeta_k = u_k \rho_k$. The u_k are essentially bounded functions, and by the bounds in Subsection 5.1, the ρ_k are locally uniformly bounded. Therefore the ζ_k are locally uniformly bounded and, up to subsequences, they converge, weakly* in $L^\infty_{\rm loc}$, to some ζ. Arguing as in Theorem 3.3, this ζ must be the unique distributional solution of (19). So, fixing a test function $\varphi \in C_c(\mathbb{R} \times \mathbb{R}^n)$ and arguing as in the uniqueness part, we get

$$\lim_{k\uparrow\infty} \int \varphi(t, \Phi^k(t,x))\overline{\zeta}(x)\, dt\, dx \;=\; \int \varphi(t, \Phi(t,x))\overline{\zeta}(x)\, dt\, dx\,,$$

where we are allowed to test with $\varphi(t,y) = \chi(t) y_i$: this gives the weak* convergence of Φ^k to Φ in $L^\infty_{\rm loc}$. Testing with $\varphi(t,y) = \chi(t)|y|^2$, we conclude as well the weak* convergence of $|\Phi^k|^2$ to $|\Phi|^2$. This implies of course the strong $L^1_{\rm loc}$ convergence. □

5.3. **Existence of regular Lagrangian flows.** The proof of the existence of regular Lagrangian flows follows from an approximation argument. Indeed, let b_k be a standard regularization of b, with $\|b_k\|_\infty + \|{\rm div}_x b_k\|_\infty$ bounded by a constant C and $b_k \to b$ strongly in $L^1_{\rm loc}$. Consider the flows Φ^k of b_k. By the bounds of Subsection 5.1,

$$\exp(-Ct) \le \det \nabla_x \Phi^k(t,x) \le \exp(Ct)\,,$$

which translates into the bounds $\exp(-Ct)|A| \le |\Phi^k(t,A)| \le \exp(Ct)|A|$ for every Borel set A. Assume for the moment that we could prove the strong convergence of Φ^k to a map Φ. Then, clearly $\exp(-Ct)|A| \le |\Phi(t,A)| \le \exp(Ct)|A|$, and hence Φ satisfies condition (a) in Definition 2.1. It is then an exercise in elementary measure theory to show that $b_k(t, \Phi^k(t,x))$ converges to $b(t,\Phi(t,x))$ strongly in $L^1_{\rm loc}$. Since Φ^k solves

$$\begin{cases} \partial_t \Phi^k(t,x) \;=\; b_k(t, \Phi^k(t,x)) \\ \\ \Phi^k(0,x) \;=\; x \end{cases}$$

it is straightforward to conclude that Φ solves (1) distributionally.

The main point is therefore to show the strong convergence of Φ_k. This follows from the stability of the corresponding transport equations.

Proof of the strong convergence of Φ^k. Consider, backward in time, the ODE

$$\begin{cases} \partial_t \Lambda^k(t,x) \;=\; b_k(t, \Lambda^k(t,x)) \\ \\ \Lambda^k(T,x) \;=\; x\,. \end{cases} \tag{25}$$

Let $\Gamma^k(t,\cdot)$ be the inverse of the diffeomorphism $\Lambda^k(t,\cdot)$. If $\overline{u} \in L^\infty(\mathbb{R}^n)$, then $u_k(t,x) = \overline{u}(\Gamma^k(t,x))$ is the unique (backward) solution of the transport equation

$$\begin{cases} \partial_t u_k + {\rm div}_x(b_k u_k) \;=\; u_k\, {\rm div}_x b_k \\ \\ u_k(T,\cdot) \;=\; \overline{u}(\cdot)\,. \end{cases}$$

By Theorem 3.4 and Proposition 3.6, u_k converges strongly in $L^1_{\rm loc}$ to the unique (backward) solution u of

$$\begin{cases} \partial_t u + {\rm div}_x (bu) \;=\; u\,{\rm div}_x b \\ \\ u(T,\cdot) \;=\; \overline{u}(\cdot)\,. \end{cases}$$

Choose $\overline{u}(x) = \chi(x) x_i$, where χ is a smooth cutoff function. Since $u^k(t,x) = \chi(\Gamma^k(t,x))\Gamma^k_i(t,x)$ we infer easily the strong $L^1_{\rm loc}$ convergence of the components Γ^k_i. This implies that Γ^k converges to a map Γ strongly in $L^1_{\rm loc}([0,T]\times\mathbb{R}^n)$. On the other hand, for any given x, $\Gamma^k(\cdot,x)$ is a Lipschitz curve with Lipschitz constant bounded independently of k. It is then easy to see that $\Gamma^k(t,\cdot)$ is a Cauchy sequence in $L^1(A)$ for every bounded A and every $t\in[0,T]$. In particular, $\Gamma^k(0,\cdot)$ converges to some map strongly in $L^1_{\rm loc}$.

Now, $\Gamma^k(0,\cdot)$ is the inverse of $\Lambda^k(0,\cdot)$, which in view of (25) is the inverse of $\Phi^k(T,\cdot)$. Therefore we conclude that for each T there exists a map $\Phi(T,\cdot)$ such that $\Phi^k(T,\cdot)\to\Phi(T,\cdot)$ strongly in $L^1_{\rm loc}$. Again, using the fact that, for each x, $\Phi^k(\cdot,x)$ is a Lipschitz curve with Lipschitz constant bounded independently of k, it is not difficult to see that Φ^k is a Cauchy sequence in $L^1(A)$ for any bounded $A\subset\mathbb{R}^+\times\mathbb{R}^n$. This concludes the proof. □

6. **Estimates for regular lagrangian flows.** In this section we present two prototypical estimates for regular lagrangian flows, proved first in [8]. These estimates were inspired by previous computations of Ambrosio, Lecumberry and Maniglia in [4] (see Remark 1).

6.1. **Integral estimate.** In order to motivate the next estimates, we start by observing an interesting inequality for flows Φ of smooth fields as in (1). Indeed, differentiate (1) in x to get the following identity

$$\partial_t D_x\Phi \;=\; D_x b(t,\Phi)\cdot D_x\Phi$$

which, in turn, can be transformed in the inequality

$$\partial_t |D_x\Phi| \;\le\; |D_x b(t,\Phi)||D_x\Phi|$$

and hence into

$$\partial_t \log(|D_x\Phi|+1) \;\le\; |D_x b(t,\Phi)|\,. \tag{26}$$

Observe next that $D_x\Phi(0,x) = Id$. Integrating (26) in x and t on $[0,T]\times K$ one easily achieves the estimate

$$\int_K \log(|D_x\Phi(T,x)|+1)\,dx \;\le\; c|K| + C\|D_x b\|_{L^1([0,T]\times K')} \tag{27}$$

where C is the compressibility constant and K' is a set large enough so that $\Phi(t,K)\subset K'$ for every $t\in[0,T]$.

Unfortunately (27) is not very useful, because of instability of differentiation under the slow growth control. However, several other integral quantities have the structure of the RHS of (27) (and allow therefore for similar estimates). In particular, for every $p>1$ define the following integral quantity:

$$A_p(R,\Phi) = \left[\int_{B_R(0)}\left(\sup_{0\le t\le T}\;\sup_{0<r<2R}\;⨍_{B_r(x)} \log\left(\frac{|\Phi(t,x)-\Phi(t,y)|}{r}+1\right)dy\right)^p dx\right]^{\frac{1}{p}}.$$

Remark 1. A small variant of the quantity $A_1(R,X)$ was first introduced in [4] and studied in an Eulerian setting in order to prove the approximate differentiability of regular Lagrangian flows.

One basic observation of [4] is that a control of $A_1(R,X)$ implies the Lipschitz regularity of X outside of a set of small measure. This elementary Lipschitz estimate is shown in Proposition 6.4. The novelty of the point of view in [8] is that a direct Lagrangian approach allows to derive uniform estimates as in (28) below. These uniform estimates are then exploited in the next section to show existence, uniqueness, stability and regularity of the regular Lagrangian flow.

Theorem 6.1. *Let b be a bounded vector field belonging to $L^1([0,T];W^{1,p}(\mathbb{R}^n))$ for some $p>1$. Let Φ be a regular Lagrangian flow associated to b and let L be its compressibility constant, as in Definition 2.1. Then we have*

$$A_p(R,\Phi) \le C\left(p,R,L,\|D_x b\|_{L^1(L^p)}\right) . \tag{28}$$

All the computations in the following proof can be justified using the definition of regular Lagrangian flow: the differentiation of the flow with respect to the time gives the vector field (computed along the flow itself), thanks to condition (b); condition (a) implies that all the changes of variable we are performing just give an L in front of the integral.

During the proof, we will use some tools borrowed from the theory of maximal functions. We recall that, for a function $f \in L^1_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$, the *local maximal function* is defined as

$$M_\lambda f(x) = \sup_{0<r<\lambda} \fint_{B_r(x)} |f(y)|\, dy\, .$$

For a proof of the first lemma we refer, for instance, to Chapter I of [16]. For the second Lemma 6.3 we refer to [14].

Lemma 6.2. *Let $\lambda>0$. The local maximal function of f is finite for a.e. $x\in\mathbb{R}^n$ and, for $p>1$ and $\rho>0$, we have*

$$\int_{B_\rho(0)} (M_\lambda f(y))^p\, dy \le c_{n,p} \int_{B_{\rho+\lambda}(0)} |f(y)|^p\, dy\, .$$

Lemma 6.3. *If $u\in W^{1,p}(\mathbb{R}^n)$ then there exists a negligible set $N\subset\mathbb{R}^n$ such that*

$$|u(x)-u(y)| \le c_n |x-y| \left(M_\lambda Du(x) + M_\lambda Du(y)\right)$$

for $x,y\in\mathbb{R}^n\setminus N$ with $|x-y|\le\lambda$.

We also recall the Chebyshev inequality:

$$|\{|f|>t\}| \le \frac{1}{t}\int_{\{|f|>t\}} |f(x)|\, dx \le \frac{|\{|f|>t\}|^{1/q}}{t}\|f\|_{L^p(\Omega)}\, ,$$

which implies

$$|\{|f|>t\}|^{1/p} \le \frac{\|f\|_{L^p(\Omega)}}{t}\, . \tag{29}$$

Proof of Theorem 6.1. For $0\le t\le T$, $0<r<2R$ and $x\in B_R(0)$ define

$$Q(t,x,r) := \fint_{B_r(x)} \log\left(\frac{|\Phi(t,x)-\Phi(t,y)|}{r}+1\right) dy\, .$$

From Definition 2.1(b) it follows that for a.e. x and for every $r > 0$ the map $t \mapsto Q(t,x,r)$ is Lipschitz and

$$\begin{aligned} \frac{dQ}{dt}(t,x,r) &\leq \fint_{B_r(x)} \left|\frac{d\Phi}{dt}(t,x) - \frac{d\Phi}{dt}(t,y)\right| \left(|\Phi(t,x)-\Phi(t,y)|+r\right)^{-1}\,dy \\ &= \fint_{B_r(x)} \frac{|b(t,\Phi(t,x))-b(t,\Phi(t,y))|}{|\Phi(t,x)-\Phi(t,y)|+r}\,dy\,. \end{aligned} \tag{30}$$

We now set $\tilde{R} = 4R + 2T\|b\|_\infty$. Since we clearly have $|\Phi(t,x)-\Phi(t,y)| \leq \tilde{R}$, applying Lemma 6.3 we can estimate

$$\begin{aligned} &\frac{dQ}{dt}(t,x,r) \\ \leq & c_n \fint_{B_r(x)} \left(M_{\tilde{R}}Db(t,\Phi(t,x)) + M_{\tilde{R}}Db(t,\Phi(t,y))\right) \frac{|\Phi(t,x)-\Phi(t,y)|}{|\Phi(t,x)-\Phi(t,y)|+r}\,dy \\ \leq & c_n M_{\tilde{R}}Db(t,\Phi(t,x)) + c_n \fint_{B_r(x)} M_{\tilde{R}}Db(t,\Phi(t,y))\,dy\,. \end{aligned}$$

Integrating with respect to the time, passing to the supremum for $0 < r < 2R$ and exchanging the supremums we obtain

$$\begin{aligned} &\sup_{0\leq t\leq T}\ \sup_{0<r<2R} Q(t,x,r) \\ \leq & c + c_n \int_0^T M_{\tilde{R}}Db(t,\Phi(t,x))\,dt + c_n \int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}}Db(t,\Phi(t,y))\,dy\,dt\,. \end{aligned}$$

Taking the L^p norm over $B_R(0)$ we get

$$A_p(R,\Phi) \leq c_{p,R} + c_n \left\|\int_0^T M_{\tilde{R}}Db(t,\Phi(t,x))\,dt\right\|_{L^p(B_R(0))} \tag{31}$$

$$+c_n \left\|\int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}}Db(t,\Phi(t,y))\,dydt\right\|_{L^p(B_R(0))}\,. \tag{32}$$

Recalling Definition 2.1(a) and Lemma 6.2, the integral in (31) can be estimated with

$$\begin{aligned} & c_n L^{1/p} \int_0^T \|M_{\tilde{R}}Db(t,x)\|_{L^p(B_{R+T\|b\|_\infty}(0))}\,dt \\ \leq & c_{n,p} L^{1/p} \int_0^T \|Db(t,x)\|_{L^p(B_{R+\tilde{R}+T\|b\|_\infty}(0))}\,dt\,. \end{aligned} \tag{33}$$

The integral in (32) can be estimated in a similar way with

$$
\begin{aligned}
& c_n \int_0^T \left\| \sup_{0<r<2R} \fint_{B_r(x)} \left[(M_{\tilde R} Db) \circ (t, \Phi(t,\cdot)) \right](y)\, dy \right\|_{L^p(B_R(0))} dt \\
= \;& c_n \int_0^T \left\| M_{2R} \left[(M_{\tilde R} Db) \circ (t, \Phi(t,\cdot)) \right](x) \right\|_{L^p(B_R(0))} dt \\
\leq \;& c_{n,p} \int_0^T \left\| \left[(M_{\tilde R} Db) \circ (t, \Phi(t,\cdot)) \right](x) \right\|_{L^p(B_{3R}(0))} dt \\
= \;& c_{n,p} \int_0^T \left\| (M_{\tilde R} Db) \circ (t, \Phi(t,x)) \right\|_{L^p(B_{3R}(0))} dt \\
\leq \;& c_{n,p} L^{1/p} \int_0^T \left\| M_{\tilde R} Db(t,x) \right\|_{L^p(B_{3R+T\|b\|_\infty}(0))} dt \\
\leq \;& c_{n,p} L^{1/p} \int_0^T \left\| Db(t,x) \right\|_{L^p(B_{3R+T\|b\|_\infty+\tilde R}(0))} dt \,. \qquad (34)
\end{aligned}
$$

Combining (31), (32), (33) and (34), we obtain the desired estimate for $A_p(R,X)$. □

6.2. **Lipschits estimate.** We now show how the estimate of the integral quantity can be used to show a quantitative Lusin-type theorem.

Proposition 6.4 (Lipschitz estimates)**.** *Let $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a measurable map. Then, for every $\varepsilon > 0$ and every $R > 0$, we can find a set $K \subset B_R(0)$ such that $|B_R(0) \setminus K| \leq \varepsilon$ and for any $0 \leq t \leq T$ we have*

$$
\mathrm{Lip}\,\left(X(t,\cdot)|_K \right) \leq \exp \frac{c_n A_p(R,X)}{\varepsilon^{1/p}} \,.
$$

Proof. Fix $\varepsilon > 0$ and $R > 0$. We can suppose that the quantity $A_p(R,X)$ is finite, otherwise the thesis is trivial; under this assumption, thanks to (29) we obtain a constant

$$
M = M(\varepsilon, p, A_p(R,X)) = \frac{A_p(R,X)}{\varepsilon^{1/p}}
$$

and a set $K \subset B_R(0)$ with $|B_R(0) \setminus K| \leq \varepsilon$ and

$$
\sup_{0\leq t\leq T} \sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1 \right) dy \leq M \qquad \forall x \in K \,.
$$

This clearly means that

$$
\fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1 \right) dy \leq M
$$

for every $x \in K$, $t \in [0,T]$ and $r \in]0, 2R[$. Now fix $x, y \in K$. Clearly $|x - y| < 2R$. Set $r = |x - y|$ and compute

$$
\begin{aligned}
&\log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)\\
= &\quad \fint_{B_r(x)\cap B_r(y)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)\,dz\\
\leq &\quad \fint_{B_r(x)\cap B_r(y)} \log\left(\frac{|X(t,x)-X(t,z)|}{r}+1\right)+\log\left(\frac{|X(t,y)-X(t,z)|}{r}+1\right)dz\\
\leq &\quad c_n \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,z)|}{r}+1\right)\,dz\\
&\qquad\qquad +c_n \fint_{B_r(y)} \log\left(\frac{|X(t,y)-X(t,z)|}{r}+1\right)\,dz\\
\leq &\quad c_n M = \frac{c_n A_p(R,X)}{\varepsilon^{1/p}}\,.
\end{aligned}
$$

This implies that

$$
|X(t,x)-X(t,y)| \leq \exp\left(\frac{c_n A_p(R,X)}{\varepsilon^{1/p}}\right)|x-y| \qquad \text{for every } x,y\in K.
$$

Therefore

$$
\mathrm{Lip}\,(X(t,\cdot)|_K) \;\leq\; \exp\frac{c_n A_p(R,X)}{\varepsilon^{1/p}}\,.
$$

□

7. **Second proof of Theorem 2.2.** In this section we use the estimates of Theorem 6.1 and Proposition 6.4 to prove Theorem 2.2 in the case $p>1$.

7.1. **Compactness and existence.** In this first subsection we apply our estimates to get a compactness result for regular lagrangian flows. Combining this compactness with a standard approximation procedure, it is easy to prove the existence part of Theorem 2.2.

Proposition 7.1 (Compactness of the flow)**.** *Let $\{b_h\}$ be a sequence of vector fields equi-bounded in $L^\infty([0,T]\times\mathbb{R}^n)$ and in $L^1([0,T];W^{1,p}(\mathbb{R}^n))$ for some $p>1$. For each h, let Φ_h be a regular Lagrangian flow associated to b_h with compressibility constant L_h. Suppose that the sequence $\{L_h\}$ is equi-bounded. Then the sequence $\{\Phi_h\}$ is strongly precompact in $L^1_{\rm loc}([0,T]\times\mathbb{R}^n)$.*

The proof of the Proposition uses the following elementary lemma, whose proof we postpone at the end of this subsection.

Lemma 7.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Borel set and let $\{f_h\}$ be a sequence of maps into $\mathbb{R}^m$. Suppose that for every $\delta>0$ we can find a positive constant $M_\delta<\infty$ and, for every fixed h, a Borel set $B_{h,\delta}\subset\Omega$ with $|\Omega\setminus B_{h,\delta}|\leq\delta$ in such a way that*

$$
\|f_h\|_{L^\infty(B_{h,\delta})}\leq M_\delta
$$

and

$$
\mathrm{Lip}\,\left(f_h|_{B_{h,\delta}}\right)\leq M_\delta\,.
$$

Then the sequence $\{f_h\}$ is precompact in measure in Ω.

Proof of Proposition 7.1. Fix $\delta > 0$ and $R > 0$. Since $\{b_h\}$ is equi-bounded in $L^\infty([0,T]\times\mathbb{R}^n)$, we deduce that $\{\Phi_h\}$ is equi-bounded in $L^\infty([0,T]\times B_R(0))$: let $C_1(R)$ be an upper bound for these norms. Applying Proposition 6.4, for every h we find a Borel set $K_{h,\delta}$ such that $|B_R(0)\setminus K_{h,\delta}|\le\delta$ and

$$\mathrm{Lip}\left(\Phi_h(t,\cdot)|_{K_{h,\delta}}\right)\le\exp\frac{c_nA_p(R,\Phi_h)}{\delta^{1/p}}\qquad\text{for every } t\in[0,T].$$

Recall first Theorem 6.1 implies that $A_p(R,\Phi_h)$ is equi-bounded with respect to h, because of the assumptions of the corollary. Moreover, using Definition 2.1(b) and thanks again to the equi-boundedness of $\{b_h\}$ in $L^\infty([0,T]\times\mathbb{R}^n)$, we deduce that there exists a constant $C_2^\delta(R)$ such that

$$\mathrm{Lip}\left(\Phi_h|_{[0,T]\times K_{h,\delta}}\right)\le C_2^\delta(R)\,.$$

If we now set $B_{h,\delta}=[0,T]\times K_{h,\delta}$ and $M_\delta=\max\left\{C_1(R),C_2^\delta(R)\right\}$, we are in position to apply Lemma 7.2 with $\Omega=[0,T]\times B_R(0)$. Then the sequence $\{\Phi_h\}$ is precompact in measure in $[0,T]\times B_R(0)$, and by equi-boundedness in L^∞ we deduce that it is also precompact in $L^1([0,T]\times B_R(0))$. Using a standard diagonal argument it is possible to conclude that $\{\Phi_h\}$ is locally precompact in $L^1([0,T]\times\mathbb{R}^n)$. □

Proof of Theorem 2.2 for $p>1$: Existence part. Choose a convolution kernel in $\mathbb{R}^n$ and regularize b by convolution. It is simple to check that the sequence of smooth vector fields $\{b_h\}$ we have constructed satisfies the equi-bounds of the previous corollary. Moreover, since every b_h is smooth, for every h there is a unique regular Lagrangian flow associated to b_h. We can bound the compressibility constant as in Subsection 5.1: we therefore conclude that L_h is as well equibounded. Hence, we can apply Proposition 7.1. It is then easy to check that every limit point of $\{\Phi_h\}$ in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$ is a regular Lagrangian flow associated to b. □

Proof of Lemma 7.2. For every $j\in\mathbb{N}$ we find the value $M_{1/j}$ and the sets $B_{h,1/j}$ as in the assumption of the lemma, with $\delta=1/j$. Now, arguing component by component, we can extend every map $f_h|_{B_{h,1/j}}$ to a map f_h^j defined on Ω in such a way that the equi-bounds are preserved, up to a dimensional constant: we have

$$\|f_h^j\|_{L^\infty(\Omega)}\le M_{1/j}\qquad\text{for every } h$$

and

$$\mathrm{Lip}\left(f_h^j\right)\le c_nM_{1/j}\qquad\text{for every } h.$$

Then we apply Ascoli-Arzelà theorem (notice that by uniform continuity all the maps f_h^j can be extended to the compact set $\bar\Omega$) and using a diagonal procedure we find a subsequence (in h) such that for every j the sequence $\{f_h^j\}_h$ converges uniformly in Ω to a map f_∞^j.

Now we fix $\varepsilon>0$. We choose $j\ge 3/\varepsilon$ and we find $N=N(j)$ such that

$$\int_\Omega|f_i^j-f_k^j|\,dx\le\varepsilon/3\qquad\text{for every } i,k>N.$$

Keeping j and $N(j)$ fixed we estimate, for $i,k > N$

$$
\begin{aligned}
\int_\Omega 1\wedge |f_i - f_k|\,dx &\le \int_\Omega 1\wedge |f_i - f_i^j|\,dx + \int_\Omega 1\wedge |f_i^j - f_k^j|\,dx + \int_\Omega 1\wedge |f_k^j - f_k|\,dx\\
&\le |\Omega\setminus B_{i,1/j}| + \int_\Omega |f_i^j - f_k^j|\,dx + |\Omega\setminus B_{k,1/j}|\\
&\le \frac{1}{j} + \frac{\varepsilon}{3} + \frac{1}{j} \le \varepsilon\,.
\end{aligned}
$$

It follows that the given sequence has a subsequence which is Cauchy with respect to the convergence in measure in Ω. This implies the thesis. □

7.2. **Stability estimates and uniqueness.** In this subsection we show an estimate similar in spirit to that of Theorem 6.1, but comparing flows for different vector fields. The uniqueness and stability claimed in Therem 2.2 are, in the case $p > 1$, a direct corollary of it.

Theorem 7.3 (Stability of the flow)**.** *Let b and $\tilde b$ be bounded vector fields belonging to $L^1([0,T];W^{1,p}(\mathbb{R}^n))$ for some $p > 1$. Let Φ and $\tilde\Phi$ be regular Lagrangian flows associated to b and $\tilde b$ respectively and denote by L and $\tilde L$ the compressibility constants of the flows. Then, for every time $\tau\in[0,T]$, we have*

$$
\|\Phi(\tau,\cdot) - \tilde\Phi(\tau,\cdot)\|_{L^1(B_r(0))} \le C\left|\log\left(\|b-\tilde b\|_{L^1([0,\tau]\times B_R(0))}\right)\right|^{-1},
$$

where $R = r + T\|b\|_\infty$ and the constant C only depends on τ, r, $\|b\|_\infty$, $\|\tilde b\|_\infty$, L, $\tilde L$, and $\|D_x b\|_{L^1(L^p)}$.

Proof. Set $\delta := \|b-\tilde b\|_{L^1([0,\tau]\times B_R(0))}$ and consider the function

$$
g(t) := \int_{B_r(0)} \log\left(\frac{|\Phi(t,x)-\tilde\Phi(t,x)|}{\delta} + 1\right) dx\,.
$$

Clearly $g(0) = 0$ and after some standard computations we get

$$
\begin{aligned}
g'(t) &\le \int_{B_r(0)} \left|\frac{d\Phi(t,x)}{dt} - \frac{d\tilde\Phi(t,x)}{dt}\right| \Big(|\Phi(t,x)-\tilde\Phi(t,x)| + \delta\Big)^{-1} dx\\
&= \int_{B_r(0)} \frac{|b(t,\Phi(t,x)) - \tilde b(t,\tilde\Phi(t,x))|}{|\Phi(t,x)-\tilde\Phi(t,x)|+\delta}\,dx\\
&\le \frac{1}{\delta}\int_{B_r(0)} |b(t,\tilde\Phi(t,x)) - \tilde b(t,\tilde\Phi(t,x))|\,dx\\
&\quad + \int_{B_r(0)} \frac{|b(t,\Phi(t,x)) - b(t,\tilde\Phi(t,x))|}{|\Phi(t,x)-\tilde\Phi(t,x)|+\delta}\,dx\,.
\end{aligned}
\tag{35}
$$

We set $\tilde R = 2r + T(\|b\|_\infty + \|\tilde b\|_\infty)$ and we apply Lemma 6.3 to estimate the last integral as follows:

$$
\begin{aligned}
&\int_{B_r(0)} \frac{|b(t,\Phi(t,x)) - b(t,\tilde\Phi(t,x))|}{|\Phi(t,x)-\tilde\Phi(t,x)|+\delta}\,dx\\
\le\quad & c_n \int_{B_r(0)} M_{\tilde R} Db(t,\Phi(t,x)) + M_{\tilde R} Db(t,\tilde\Phi(t,x))\,dx\,.
\end{aligned}
$$

Inserting this estimate in (35), setting $\tilde r = r + T\max\{\|b\|_\infty, \|\tilde b\|_\infty\}$, changing variables in the integrals and using Lemma 6.2 we get

$$\begin{aligned} g'(t) &\le \frac{\tilde L}{\delta}\int_{B_{r+T\|\tilde b\|_\infty}(0)} |b(t,y)-\tilde b(t,y)|\,dy + \big(\tilde L + L\big)\int_{B_{\tilde r}(0)} M_{\tilde R} Db(t,y)\,dy \\ &\le \frac{\tilde L}{\delta}\int_{B_{r+T\|\tilde b\|_\infty}(0)} |b(t,y)-\tilde b(t,y)|\,dy + c_n \tilde r^{n-n/p}\big(\tilde L + L\big)\|M_{\tilde R} Db(t,\cdot)\|_{L^p} \\ &\le \frac{\tilde L}{\delta}\int_{B_{r+T\|\tilde b\|_\infty}(0)} |b(t,y)-\tilde b(t,y)|\,dy + c_{n,p} \tilde r^{n-n/p}\big(\tilde L + L\big)\|Db(t,\cdot)\|_{L^p}\,. \end{aligned}$$

For any $\tau \in [0,T]$, integrating the last inequality between 0 and τ we get

$$g(\tau) \;=\; \int_{B_r(0)} \log\left(\frac{|\Phi(\tau,x)-\tilde\Phi(\tau,x)|}{\delta} + 1\right) dx \;\le\; C_1\,, \tag{36}$$

where the constant C_1 depends on τ, r, $\|b\|_\infty$, $\|\tilde b\|_\infty$, L, $\tilde L$, and $\|D_x b\|_{L^1(L^p)}$.

Next we fix a second parameter $\eta > 0$ to be chosen later. Using Chebyshev inequality we find a measurable set $K \subset B_r(0)$ such that $|B_r(0)\setminus K| \le \eta$ and

$$\log\left(\frac{|\Phi(\tau,x)-\tilde\Phi(\tau,x)|}{\delta} + 1\right) \;\le\; \frac{C_1}{\eta} \qquad \text{for } x \in K.$$

Therefore we can estimate

$$\begin{aligned} &\int_{B_r(0)} |\Phi(\tau,x)-\tilde\Phi(\tau,x)|\,dx \\ \le\;& \eta\left(\|\Phi(\tau,\cdot)\|_{L^\infty(B_r(0))} + \|\tilde\Phi(\tau,\cdot)\|_{L^\infty(B_r(0))}\right) + \int_K |\Phi(\tau,x)-\tilde\Phi(\tau,x)|\,dx \\ \le\;& \eta C_2 + c_n r^n \delta\left(\exp(C_1/\eta)\right) \;\le\; C_3\left(\eta + \delta\exp(C_1/\eta)\right)\,, \end{aligned}$$

with C_1, C_2 and C_3 which depend only on T, r, $\|b\|_\infty$, $\|\tilde b\|_\infty$, L, $\tilde L$, and $\|D_x b\|_{L^1(L^p)}$. Without loss of generality we can assume $\delta < 1$. Setting $\eta = 2C_1|\log\delta|^{-1} = 2C_1(-\log\delta)^{-1}$, we have $\exp(C_1/\eta) = \delta^{-1/2}$. Thus we conclude

$$\int_{B_r(0)} |\Phi(\tau,x)-\tilde\Phi(\tau,x)|\,dx \;\le\; C_3\left(2C_1|\log\delta|^{-1} + \delta^{1/2}\right) \;\le\; C|\log\delta|^{-1}\,, \tag{37}$$

where C depends only on τ, r, $\|b\|_\infty$, $\|\tilde b\|_\infty$, L, $\tilde L$, and $\|D_x b\|_{L^1(L^p)}$. This completes the proof. □

8. **Proof of Theorem 2.4.** In this final section we prove Theorem 2.4. This was done in [8] using the Lipschitz estimate of Proposition 6.4. However, we follow here a more direct argument suggested by the referee. Besides being simpler, this argument gives an explicit constant C in (4).

Proof. We first assume that X mixes at a scale $\varepsilon \le 8^{-2}$. Set

$$B := \{(x_1,x_2) : 5/8 \le x_1 \le 7/8\}$$

and consider the set

$$U := \{(x,y) \in B\times A : |X(x)-X(y)| \le \varepsilon\}\,.$$

For every x and y define $U_x := \{y : (x,y) \in U\}$ and $U_y := \{x : (x,y) \in U\}$. Clearly, by the nearly incompressibility of the flow we have

$$|U_x| \;\leq\; |X^{-1}(B_\varepsilon(X(x))| \;=\; \kappa'^{-1}|B_\varepsilon(X(x))| \;=\; \frac{\pi\varepsilon^2}{\kappa'} \tag{38}$$

$$|U_y| \;\leq\; |X^{-1}(B_\varepsilon(X(y))| \;=\; \kappa'^{-1}|B_\varepsilon(X(y))| \;=\; \frac{\pi\varepsilon^2}{\kappa'}\,. \tag{39}$$

Moreover, the mixing property and the nearly incompressibility imply

$$|U_x| \;=\; |X^{-1}(X(A)\cap B_\varepsilon(X(x)))| \;\geq\; \kappa'|X(A)\cap B_\varepsilon(X(x))| \;\geq\; \pi\kappa\kappa'\varepsilon^2\,. \tag{40}$$

Observe next that, if $(x,y)\in U$, then $|x-y|\geq 8^{-1}$ and therefore, since $\varepsilon \leq 8^{-2}$,

$$\frac{1}{2}|\log\varepsilon| \;\leq\; -\log\varepsilon + \log 8 \;\leq\; -\log\varepsilon + \log|y-x| \;=\; \left|\log\frac{\varepsilon}{|y-x|}\right|\,. \tag{41}$$

Therefore, we conclude

$$\frac{\pi\kappa'\kappa\varepsilon^2}{8}|\log\varepsilon| \stackrel{(40)}{\leq} \frac{1}{2}\int_U |\log\varepsilon| \stackrel{(41)}{\leq} \int_U \log\left(\frac{|X(x)-X(y)|}{|x-y|}\right)dx\,dy\,. \tag{42}$$

On the other hand, by computations similar to the ones of Theorem 6.1:

$$\begin{aligned}
\log\left(\frac{|X(x)-X(y)|}{|x-y|}\right) &= \int_0^1 \frac{d}{dt}\log|\Phi(t,x)-\Phi(t,y)|\,dt\\
&\leq \int_0^1 \frac{|b(t,\Phi(t,y))-b(t,\Phi(t,x))|}{|\Phi(t,y)-\Phi(t,x)|}\,dt\\
&\leq C\int_0^1 \big[M|Db|(t,\Phi(t,x))+M|Db|(t,\Phi(t,y))\big]\,dt\,.
\end{aligned}\tag{43}$$

Inserting (43) in (42), we achieve

$$\begin{aligned}
&\frac{\pi\kappa'\kappa\varepsilon^2}{8}|\log\varepsilon|\\
\leq\;& C\int_0^1\int_U M|Db|(t,\Phi(t,x))\,dy\,dx\,dt + C\int_0^1\int_U M|Db|(t,\Phi(t,y))\,dx\,dy\,dt\\
\stackrel{(38),(39)}{\leq}\;& \frac{C\pi\varepsilon^2}{\kappa'}\int_0^1\int_A M|Db|(t,\Phi(t,x))\,dx\,dt + \frac{C\pi\varepsilon^2}{\kappa'}\int_B M|Db|(t,\Phi(t,y))\,dy\,dt\\
\leq\;& frac2C\pi\varepsilon^2\kappa'^2\|M|Db|\|_{L^1} \;\leq\; \frac{2C_p\pi\varepsilon^2}{\kappa'^2}\|Db\|_{L^p}\,.
\end{aligned}\tag{44}$$

We conclude therefore that, if X κ-mixes at a scale $\varepsilon\leq 8^{-2}$, then

$$\|Db\|_{L^p} \;\geq\; C\kappa\kappa'^3|\log\varepsilon|\,.$$

The constant C is independent of κ, κ' and ε, but of course depends on $p>1$.

We are now going to show the thesis for every $0<\varepsilon<1/4$. Indeed, suppose that the thesis is false. Then, we could find a sequence $\{b_h\}$ of vector fields and a sequence $\{\varepsilon_h\}$ with $8^{-2}<\varepsilon_h<1/4$ in such a way that

$$\|D_x b_h\|_{L^1([0,1];L^p(K))} \leq \frac{1}{h}|\log\varepsilon_h|$$

and the corresponding map X_h mixes the set A up to scale ε_h. Moreover, without loss of generality we can assume that $\int b_h(t,x)dx = 0$: if we subtract from b_h a function depending only on t, all the properties above remain true.

Our assumptions imply that

$$\|D_x b_h\|_{L^1([0,1];L^p(K))} \leq \frac{1}{h}|\log \varepsilon_h| \leq \frac{1}{h}|\log 8^{-2}| \longrightarrow 0 \quad \text{as } h \to \infty.$$

Up to an extraction of a subsequence, we can suppose that $\varepsilon_h \to \bar{\varepsilon} \geq 8^{-2} > 0$, that the flows Φ_h converge to a map Φ strongly in $L^1([0,1] \times K)$ and that the $X_h = \Phi_h(1, \cdot)$ converge, strongly in $L^1(K)$ to $X = \Phi(1, \cdot)$ For this, we can suitably modify the proof of the compactness result in Proposition 7.1, noticing that (3) gives a uniform control on the compressibility constants of the flows and that we do not need the assumption on the boundedness of the vector fields, since we are on the torus and then the flow is automatically uniformly bounded.

Next notice two things:

(a) Since $\|D_x b_h\|_{L^1([0,1];L^p(K))} \to 0$ and $\int_K b_h(t,x)dx = 0$, $b_h \to 0$ strongly in $L^1([0,1] \times K)$. But then $\partial_t \Phi = 0$ and we deduce that X is indeed the identity map.

(b) The mixing property is stable with respect to strong convergence: this means that X has to mix up to scale $8^{-2} \leq \bar{\varepsilon} \leq 1/4$.

However the identity map does not mix at a scale $\bar{\varepsilon} \leq 1/4$. This contradiction completes the proof. □

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