1. Minor errors/typos

(1) Line 7 after (3.4), Tan^α should be Tan_α.
(2) Eq. (3.12), line before and two lines after: the integration variable should be y (and not x) and the domain of the integrals should be $B_{\rho r_i}(x)$.
(3) Last two lines of Proposition 3.12: the argument shows the inclusion $\supset$, rather than $\subset$.
(4) Displayed eq. before (3.13): the definition of $E_{i,j,k}$ should ask for $x \in E$.
(5) Inequality (3.14): a factor r^α is missing.
(6) Second displayed equation after (3.14), a factor $r_i^{-\alpha}$ is missing in the first $\limsup$.
(7) First line of page 26: $\mathrm{supp}\,(r)$ is actually $\mathrm{supp}\,(\nu)$.
(8) In page 97 the integrand in the first displayed formula is missing the factor $e^{-s|z|^2}$.
(9) Page 5, line 12: dy should be $d\mu(y)$.
(10) Page 5, line -8: "for every n" should be "for every j".
(11) Page 6, line -2: "uniformly" should be added after $f_{i(j)} \to f$ to ensure the argument in page 7.
(12) Page 7, line -9 : "$f_i \to f$ uniformly" should be "$f_i \to \varphi$ uniformly".
(13) Page 15, line 3: The statement of Lemma 3.6 should have also "for μ-a.e. $x \in E$, since the proof relies in Step 3 on Proposition 3.4, where μ-a.e. $x \in E$ is needed.
(14) Page 17, line -6: In the statement of Proposition 3.12, $f \in L^1(\mu)$ should be $f \in L^1_{loc}(\mu)$. Otherwise, the argument of Remark 3.13 may fail, e.g. when μ is just the Lebesgue measure.
(15) Page 20, line -6: μ_{x,r_j} should be $r_j^{-\alpha}\mu_{x,r_j}$.
(16) Page 25, line 1: add "similar argument in" before "Propostion 2.7".

Many thanks to Federico Glaudo, Simone Steinbrüchel, and Cheng Lingxiao.

2. Substantial errors

(i) The proof at page 92 of (8.21) for the case $m = 2$ is wrong: the implication suggested in the last line (namely that the existence of the vectors y and z implies $\alpha_2 \geq 1$ by simple linear algebra) is incorrect. A correct argument goes as follows. We consider the plane W spanned by y and z and from the identitities derived thus far we certainly conclude that $\mathrm{tr}\, b_2^{(1)} \llcorner W = 2$. In particular from Lemma 8.7 we conclude $\mathrm{tr}\, b_2^{(1)} \llcorner W^\perp = 0$. We can now use the identity (8.22) (which is valid for any m) to derive that λ is supported in the plane W, which in turn implies $\lambda = \mathcal{H}^2 \llcorner W$. This is already the desired conclusion to achieve Proposition 8.5, rather than (8.21). However (8.21) follows as well.

Many thanks to Federico Glaudo for spotting the error and suggesting the alternative argument!

(ii) Federico has also pointed out to me that I missed the following point: the product of two uniform measures is a uniform measure as well. For this reason Question 10.19 in the book has the simple answer that, if $C \times C \subset \mathbb{R}^8$ is the product of two light cones, then $\mathcal{H}^6 \llcorner C \times C$ is indeed a uniform measure. The question should have therefore been stated in the following way:

(Q) Are there k-uniform measures which are not the product of light cones and flat measures?

Recently Dali Nimer has classified all three-dimensional cones $Z \subset \mathbb{R}^n$ for which $\mathcal{H}^3 \llcorner Z$ is a uniform measure, showing that there are indeed infinitely many. Dali's work provides thus a positive answer even to the above adjusted version of Question 10.19. The interested reader may find Dali's work here:

https://arxiv.org/abs/1608.02604

I have not been able to find a reference in the literature to Federico's remark that the product of two uniform measures is a uniform measure, although it was known to Preiss. I attach Federico's proof for the reader's convenience.

(iii) Gao-Feng Zheng pointed out an error in the argument for Lemma 3.8. The first displayed equation at page 25 is incorrect and should really be

$$c(\rho) = \lim_{r\downarrow 0} \frac{b(\rho r_i)}{r_i}\,.$$

As a result, the subsequent chain of equalities and inequalities is missing a factor r^{-1} and the conclusion $\langle c(\rho), z\rangle = 0$ for every z in the support of $\tilde{\nu}$ is not valid any more.

Here is how to fix the problem. First of all the claim of Lemma 3.8 should be modified. Rather than claiming that every tangent measure $\tilde{\nu}$ to ν at 0 is supported in the hyperplane $\{x_1 = 0\}$, the claim should be that every tangent measure is contained in *some hyperplane* (not necessarily $\{x_1 = 0\}$ and possibly depending on the tangent measure). This statement is still good enough to prove Theorem 3.1, Proposition 3.5, and Corollary 3.9.

In order to show this modified Lemma 3.8 we argue as follows. First of all, define $b(r)$ as in (3.17). Assume first that for some $b(r) \neq 0$ for some r. Then every tangent measure $\tilde{\nu}$ to ν at 0 is supported in the hyperplane $\{z : \langle b(r), z\rangle = 0\}$. In fact if $z \in \operatorname{supp}(\tilde{\nu})$ then there is $r_i \downarrow 0$ and $y_i \in \operatorname{supp}(\nu)$ such that $\frac{y_i}{r_i} \to z$. Then, using (3.19)

$$|\langle b(r), z\rangle| = \lim_{i\to\infty} \frac{1}{r_i} |\langle b(r), y_i\rangle| \leq C \lim_{i\to\infty} \frac{|y_i|^2}{r_i} = 0\,.$$

It remains to examine the case when $b(r) = 0$ for every r. But this implies that, for $c(\rho)$ as defined as in (3.18),

$$c(\rho) = \lim_{i\to\infty} \frac{b(\rho r_i)}{r_i} = 0\,.$$

As observed in the lines after (3.18), combined with the information that $\operatorname{supp}(\tilde{\nu}) \subset \{x_1 \geq 0\}$, the latter property easily gives $\operatorname{supp}(\tilde{\nu}) \subset \{z_1 = 0\}$.

3. Additional explanations

This was pointed to me by Max Goering: showing that the set D at page 64 is μ-measurable does not seem obvious. I suggest a proof in this section, even though I think it needs some checking, I believe the strategy is correct. I thought at the time I wrote the book (and gave the lectures) that I checked pretty much all the claims, even though I did not write all the details in the book. But honestly this seems more complicated than I remember, so either I overlooked the subtlety of the issue or the 28-years-old version of me was a smarter guy than I am now.

There is quite some deal of technical complication with supports, otherwise the main point is to use the following fact, which is a consequence of the measurable projection theorem.

Theorem 3.1. *Assume $E \subset \mathbb{R}^m \times \mathbb{R}^k$ is a Borel set and $\mathbf{p}$ denotes the projection on the first factor. Than $\mathbf{p}(E)$ is a μ-measurable set for every Radon measure μ on $\mathbb{R}^m$.*

The set D in the book is described as follows. First of all μ is a Radon measure for which $\theta^m(\mu, x)$ is positive, finite and exists almost everywhere. Then define the set of points B is the set of points x where

$$\{R^{-1} \leq \theta^m_*(\mu, x) \leq \theta^{m,*}(\mu, x) \leq R\},$$

and G is an open ball (in some appropriate metric d which metrizes the weak* convergence of measures on some bounded set) in the space of Radon measures. Then D is the set of points $a \in B$ such that

(a) there are $\nu^a \in \mathrm{Tan}_m(\mu, a) \cap G$
(b) and $x \in \mathrm{spt}\,(\nu_a)$
(c) such that

$$d(\nu^a_{x,1}, r^{-m}\mu_{a,r}) \geq \frac{1}{k} \qquad \forall r < \frac{1}{j}\,.$$

Allow me to:

- substitute G with a closed ball of radius 1, which I will denote by $\bar{B}(\zeta, 1)$;
- substitute B with the set

$$B = \{\tfrac{1}{2}r^m \leq \mu(B_r(x)) \leq 2r^m \quad \forall r \leq 1\}\,; \tag{1}$$

- Set $k = j = 1$;
- impose also $x \in \overline{B}_1(0)$;
- assume $\mathrm{spt}\,(\mu) \subset B_1(0)$.

With just some notational adjustments (i.e. substitute the 1's and 2's with some constants) this should be equivalent to the measurability of a family $\mathcal{F}$ of sets where the 1's and 2's are general independent real parameters. Then the original D should be the result of applying a finite number of operations, each of which consists of taking either countable intersections or countable unions, and the first operation starts with sets of this form. Hopefully I am not guessing wrong here. Note that B is compact.

Next let me consider a new D', which, given all the substitutions/assumptions above, is the set of points $a \in B$ for which there is a point $x \in \bar{B}_1$ and a sequence $\rho_\ell \downarrow 0$ with the following properties:

(a') $\rho_\ell \le \frac{1}{\ell}$;
(b') $\mathrm{dist}\,(x, a + \rho_\ell B) \le \frac{1}{\ell}$ and $\rho_\ell^{-m} \mu_{a,\rho_\ell} \in \bar{B}(\zeta, 1 + \ell^{-1})$;
(c') For every $r < 1$ we have

$$d(\rho_\ell^{-m} (\mu_{a,\rho_\ell})_{x,1}, r^{-m} \mu_{a,r}) \ge 1 - \frac{1}{\ell}\,. \tag{2}$$

First of all, given a point $a \in D'$ we see that it is indeed in D by taking ν^a to be the weak* limit of the sequence $\rho_\ell^{-m} \mu_{a,\rho_\ell}$. It is not true that $D \subset D'$, however. The reason is that, given the ν^a we can find a sequence $\rho_\ell^{-m} \mu_{a,\rho_\ell}$ converging to it, but (b') will not necessarily be satisfied by x. However, I claim that $D \setminus D'$ has μ-measure zero. The point is that, if I look at the points $a \in B$ of μ-density 1 for μ, then $\mathrm{Tan}_m(\mu, a) = \mathrm{Tan}_m(\mu \llcorner B, a)$, and then $\rho_\ell^{-m} (\mu \llcorner B)_{a,\rho_\ell}$ is also converging to ν^a. This in particular will guarantee (b') at those points. However what is left out are the points a of B which do not have μ-density 1, and that is a set of measure zero.

Next we introduce the set E of pairs $(x, a) \in \bar{B}_1 \times B$ for which there is a sequence $\rho_\ell \downarrow 0$ satisfying (a'), (b'), (c'). If I can show that E is Borel, then I use Theorem 3.1 to prove that D' is μ-measurable.

First of all for each ℓ let E_ℓ be the set of pairs $(x, a) \in \bar{B}_1 \times B$ such that there is $\rho \in]0, \frac{1}{\ell}]$ such that

(b") $\mathrm{dist}\,(x, a + \rho B) \le \frac{1}{\ell}$ and $\rho^{-m} \mu_{a,\rho} \in \bar{B}(\zeta, 1 + \ell^{-1})$;
(c") For every $r < 1$ we have

$$d(\rho^{-m} (\mu_{a,\rho})_{x,1}, r^{-m} \mu_{a,r}) \ge 1 - \frac{1}{\ell}\,. \tag{3}$$

Clearly $E = \bigcap_\ell E_\ell$. Then for every $s \in \mathbb{N}, s > \ell$ let $E_{\ell,s}$ be the set of pairs (x, a) such that

(a$^{(3)}$) $\exists \rho \in [s^{-1}, \ell^{-1}]$ such that (b") and (c')) hold.

Then $E_\ell = \bigcup_s E_{\ell,s}$. Now, if I am not mistaken, $E_{\ell,s}$ is a closed set using standard arguments.