1. From (3.10) to (3.11)

The proof of (3.11) uses the claim that from (3.10) we can conclude

$$e(r) \le C r^{\varepsilon/2}\,.$$

It is indeed correct that (3.10) implies the latter estimate, but we could have given some more details about its derivation. First of all recall that, by the monotonicity formula

$$C_0 r^{\varepsilon} + e(r) \ge 0 \qquad\qquad (\dagger)\,.$$

for a suitable constant C_0. Hence consider

$$\bar{e}(r) := \max\{e(r), 0\}\,,$$

and we claim that, from (3.10), it follows that

$$\bar{e}(s) \le \left(\frac{s}{r}\right)^a \bar{e}(r) + \bar{C} r^{\varepsilon} \qquad \forall 0 < s < r \le r_0\,. \qquad (*)$$

Indeed we distinguish two cases:

- $e(s) = \bar{e}(s)$; in this case it is trivial because $e(r) \le \bar{e}(r)$;
- $e(s) \neq \bar{e}(s)$; then $\bar{e}(s) = 0$, but on the other hand the right hand side is certainly positive if $\bar{C}$ is chosen bigger than C_0 because of ($\dagger$).

Next set $\tilde{e}(r) := \bar{e}(r) + \tilde{C} r^{\varepsilon}$. We then claim that, if $\tilde{C}$ is chosen sufficiently large,

$$\tilde{e}\left(\frac{r}{2}\right) \le \frac{1}{2^{\varepsilon/2}} \tilde{e}(r) \qquad\qquad \forall 0 < r \le r_0\,.$$

This is indeed equivalent to

$$\bar{e}\left(\frac{r}{2}\right) \le \frac{1}{2^{\varepsilon/2}} \bar{e}(r) + \tilde{C}\left(\frac{1}{2^{\varepsilon/2}} - \frac{1}{2^{\varepsilon}}\right) r^{\varepsilon}\,,$$

which can be derived by $(*)$ recalling that $\varepsilon/2 \le a$ and $\bar{e}(r) \ge 0$ and ensuring that

$$\tilde{C}\left(\frac{1}{2^{\varepsilon/2}} - \frac{1}{2^{\varepsilon}}\right) \ge \bar{C}\,.$$

Iterating now the inequality for $\tilde{e}$ we conclude

$$\tilde{e}(s) \le \left(\frac{2s}{r_0}\right)^{\varepsilon_2} \max\left\{\tilde{e}(r) : \frac{r_0}{2} \le r \le r_0\right\} \qquad \forall s < r_0\,,$$

which clearly implies

$$\tilde{e}(s) \le C s^{\varepsilon/2} \qquad \forall s \le r_0$$

(for an appropriately chosen constant C). Since $e(s) \le \tilde{e}(s)$, the desired claim readily follows.

We in fact note that the argument implies also

$$|e(s)| \le C s^{\varepsilon/2}\,.$$