Surveys in Differential Geometry
IX, International Press

# Errata to “The min–max construction of minimal surfaces”

## 1. Varia

Many thanks to Alessandro Pigati for pointing out the following mistakes.

(1) The Definition 1.1 does not seem to be strong enough to ensure $\{\psi(t, \Sigma_t)\}_t$ is again a generalized family when $\psi \in C^\infty([0,1] \times M, M)$ is a smooth path of diffeomorphisms. As I do in some subsequent works one should strengthen the assumption of continuity in the parameter $t$ in some way. A possibility which covers all the applications of the papers is the following:
  - For everty given $\tau$ there is a discrete set $P(\tau)$ such that, as $t \to \tau$, $\Sigma_t$ converges in $C^1$ to $\Sigma_\tau$ locally on $M \setminus P(\tau)$.

  An even weaker sufficient condition would be convergence in the sense of varifolds.

(2) The argument of Lemma B.1 is wrong. In particular (70) is not at all guaranteed by the fact that $\psi_T(y) \cdot \nabla\varphi(y) \geq 0$. The test function $\chi$ should indeed be chosen rather differently. One should consider first a slightly larger strictly convex set $K'$ which contains $K$ and such that $\partial K \cap \partial K' = \{x\}$. Then we should take the test vector field

$$\chi(x) := \varphi(\mathrm{dist}(x, \partial K'))n(\pi(x))\,,$$

  where
  - $\varphi : [0,1] \to [1,0]$ is identically 1 in 0, $\varphi' < 0$ on $[0,\varepsilon]$ for some positive $\varepsilon$ and then vanishes identically on $[\varepsilon, \infty]$;
  - $n$ is the inward unit normal to $\partial K'$;
  - $\pi$ is the orthogonal projection onto $\partial K'$.

  For $\varepsilon$ sufficiently small this is an admissible test vector field because it can be multiplied by a cut-off function $\alpha(x)$ so to make it compactly supported without changing its values in a neighborhood of the support of the varifolds: note that $\partial K' \cap \partial K = \{x\}$ is used crucially here.

Then it can be checked that the strict convexity of $K'$ guarantees the following. For a sufficiently small $\varepsilon$ the trace of $D\chi(y)$ on any 2-plane is nonnegative for any $y$ and it is strictly positive for $y$ in a neighborhood of $x$.

## 2. Pull-tight lemma

There are (minor) issues with Proposition 4.1 (some passages are wrong as written and some would require more explanations). Moreover, some things are unnecessarily complicated. I have therefore decided to rewrite its proof from scratch. Thanks to Bill Allard for pointing out the mistakes in the original version.

PROPOSITION 2.1. *Assume $\Lambda$ is a saturated family. Then there exists a minimizing sequence $\{\{\Sigma_t\}^n\} \subset \Lambda$ such that, if $\{\Sigma^n_{t_n}\}$ is a min–max sequence, then $\mathfrak{d}\,(\Sigma^n_{t_n}, \mathcal{V}_\infty) \to 0$.*

PROOF. **Step 1: A map from $X$ to the space of vector fields.** For $k \in \mathbb{Z}$ define the annular neighborhood of $\mathcal{V}_\infty$

$$\mathcal{V}_k \;=\; \left\{V \in X | 2^{-k+1} \geq \mathfrak{d}\,(V, \mathcal{V}_\infty) \geq 2^{-k-2}\right\}\,.$$

For every $V \in \mathcal{V}_k$ choose a smooth vector field $\chi_V$ such that

$$\delta V(\chi_V) < 0\,.$$

By linearity we can assume that

$$\|\chi_V\|_{C^k} \leq \frac{1}{k} \qquad \text{for } k \in \mathbb{N} \setminus \{0\}.$$

Next, for each such $V$ choose a metric ball $B_{2\rho}(V)$ (with respect to the distance $\mathfrak{d}$ ) such that

$$\delta W(\chi_V) \leq \frac{1}{2}\delta V(\chi_V) \qquad \forall W \in B_{2\rho}(V)\,.$$

Cover $\mathcal{V}_k$ with a finite number of balls $B_{\rho_i}(V_i)$. Moreover, for each $i$ choose a continuous function $\psi_i \in C_c(B_{2\rho_i}(V_i))$ which is identically equal to 1 on $B_{\rho_i}(V_i)$ and satisfies $0 \leq \psi_i \leq 1$. We then define the map

$$\mathcal{V}_k \ni V \quad \mapsto \quad H^k_V := \frac{\sum_i \psi_i(V)\chi_{V_i}}{\sum_i \psi_i(V)} \in C^\infty(M, TM)\,.$$

Observe that:

(a) $\delta V(H^k_V) < 0$ for every $V \in \mathcal{V}_k$;
(b) $V \mapsto H^k_V$ is continuous;
(c) $\|H^k_V\|_{C^k} \leq \frac{1}{k}$ for every $k \in \mathbb{N}$ and for every $V \in \mathcal{V}_k$.

Choose next a function $\psi^k \in C_c(\mathcal{V}_k)$ with the properties that $0 \leq \psi^k \leq 1$ and $\psi^k \equiv 1$ on $\{V \in \mathcal{V}_k : 2^{-k} \geq \mathfrak{d}\,(V, \mathcal{V}_\infty) \geq 2^{-k-1}\}$. On $X \setminus \mathcal{V}_\infty$ we define the continuous function

$$V \quad \mapsto \quad H_V := \frac{\sum_k \psi^k(V) H^k_V}{\sum_k \psi^k(V)}\,.$$

Observe that:

(a') $\delta V(H_V) < 0$ for every $V \in X \setminus \mathcal{V}_\infty$;
(b') $V \mapsto H_V$ is continuous;
(c') $\|H_V\|_{C^{k-1}} \leq \frac{1}{k-1}$ if $\mathfrak{d}\,(V, \mathcal{V}_\infty) \leq 2^{-k}$ and $k \in \mathbb{N} \setminus \{0,1\}$.

Extend the map $V \mapsto H_V$ to $X$ by setting it identically equal to 0 on $\mathcal{V}_\infty \cap X$. By property (c') it turns out that such extension is continuous in the $C^k$ norm. By the arbitrariness of $k$ we thus conclude that $V \mapsto H_V$ is actually continuous in the $C^\infty$ space.

**Step 2: A map from $X$ to the space of isotopies.** For each $V \in X$ let $\Psi_V$ be the one-parameter family of diffeomorphisms generated by $H_V$. For each $V \in X \setminus \mathcal{V}_\infty$ there is a positive time $\sigma_V$ such that

$$\delta(\Psi(s, \cdot)_\sharp V)(H_V) \leq \frac{1}{2}\delta V(H_V) < 0 \qquad \forall s \in [0, \sigma_V]\,.$$

By the continuity of the map $H_V$, the map $(s, V) \mapsto \delta(\Psi(s, \cdot)_\sharp V)(H_V)$ is also continuous. Thus we conclude the existence of a radius $\rho_V$ such that

$$\delta(\Psi(s, \cdot)_\sharp W)(H_W) \leq \frac{1}{4}\delta V(H_V) < 0 \qquad \forall s \in [0, \sigma_V]\,, \forall W \in B_{2\rho_V}(V)\,.$$

Arguing as in the first step we then can construct a continuous function $\sigma : X \to [0, \infty]$ such that:

(i) $\sigma = 0$ on $\mathcal{V}_\infty$;
(ii) $\sigma$ is positive on $X \setminus \mathcal{V}_\infty$;
(iii) $\max_{t\in[0,\sigma(V)]} \delta(\Psi(s, \cdot)_\sharp V)(H_V) < 0$ for every $V \in X \setminus \mathcal{V}_\infty$ and for every $s \in [0, \sigma(V)]$.

Let us now redefine a new $H_V$ by multiplying the old one by $\sigma(V)$. For this newly defined $H_V$ (which remains continuous and vanishes identically on $\mathcal{V}_\infty$) the property (iii) becomes

$$\max_{s\in[0,1]} \delta(\Psi(s, \cdot)_\sharp V)(H_V) < 0 \qquad \text{for every } V \in X \setminus \mathcal{V}_\infty. \tag{1}$$

**Step 3: Conclusion.** We next take a minimizing sequence $\{\{\Gamma^n_t\}_t\}^n \subset \Lambda$ and consider newly defined families $\{\Xi^n_t\}_t$ by setting

$$\Xi^n_t = \Psi_{\Gamma^n_t}(1, \Gamma^n_t)\,.$$

Never mind, for the moment, that $\{\Xi^n_t\}_t$ is not necessarily an element of $\Lambda$ (because the map $(t,x) \mapsto \Psi_{\Gamma^n_t}(1,x)$ is not known to be smooth in the parameter $t$). We wish first to show that the sequence $\{\{\Xi^n_t\}_t\}^n$ has the other property claimed by the Proposition. To this aim we choose a sequence $\{t_n\}$ such that $\lim_n \mathcal{H}^2(\Xi^n_{t_n}) = m_0$ and, by possibly passing to a subsequence, assume without loss of generality that the sequence $\{\Gamma^n_{t_n}\}_n$ (namely the counterpart of $\{\Xi^n_{t_n}\}_n$ in the original minimizing sequence $\{\{\Gamma^n_t\}_t\}^n$) converges to some varifold $V$. Note that, by continuity of the map $\Psi$, we would then have that

$$\Xi^n_{t_n} = \Psi_{\Gamma^n_{t_n}}(1,\cdot)_\sharp \Gamma^n_{t_n} \quad \text{converges to} \quad \Psi_V(1,\cdot)_\sharp V \tag{2}$$

(in the varifold sense and hence in the metric $\mathfrak{d}$). If $V$ itself is stationary, we are then finished because $\Psi_V$ is the 1-parameter family generated by the vector field $H_V = 0$ and thus $\Psi_V(1,\cdot)_\sharp V = V$.

On the other hand by construction

$$\mathcal{H}^2(\Xi^n_t) \le \mathcal{H}^2(\Gamma^n_t)\,.$$

Hence, we conclude that $\|V\|(M) = \lim_n \mathcal{H}^2(\Gamma^n_{t_n}) = m_0$ (because $\{\{\Gamma^n_t\}_t\}^n$ was a minimizing sequence in $\Lambda$). Note however that, if $V$ were not stationary, then (2) and (1) would imply

$$\begin{aligned} m_0 = \lim_{n\to\infty} \mathcal{H}^2(\Xi^n_{t_n}) &= \|\Psi_V(1,\cdot)_\sharp V\|(M) \\ &= \|V\|(M) + \int_0^1 \delta(\Psi(s,\cdot)_\sharp V)(H_V)\,ds < m_0\,, \end{aligned}$$

reaching a contradiction.

We have thus proved that $\{\{\Gamma^n_t\}_t\}^n$ enjoys the second property of the proposition. We now wish to regularize each $\{\Gamma^n_t\}_t$ in the parameter $t$. To this aim, for each $n$ let $h^n_t$ denote the one parameter family of vector fields $H_{\Gamma^n_t}$. The map $(t,x) \mapsto h^n(t,x) := h^n_t(x)$ is continuous. However, in addition

$$\lim_{t\to\tau} \|h^n(t,\cdot) - h^n(\tau,\cdot)\|_{C^k} = \lim_{t\to\tau} \|H_{\Gamma^n_t} - H_{\Gamma^n_\tau}\|_{C^k} = 0$$

for every fixed $k$. By a standard smooth procedure (for instance by convolution with a standard kernel in the parameter $t$), we can construct a smooth map $(t,x) \mapsto \bar{h}^n(t,x)$ with the property

$$\max_t \|h^n(t,\cdot) - \bar{h}^n(t,\cdot)\|_{C^1} \le \frac{1}{n+1}\,. \tag{3}$$

Consider now for each fixed $n$ and $t$ the one-parameter family of diffeomorphisms $\Phi^n_t(s,\cdot)$ generated by $\bar{h}^n_t$ and the one-parameter family of diffeomorphisms $\Psi^n_t(s,\cdot)$ generated by $h^n_t$. Note that $\Xi^n_t = \Psi^n_t(1,\Gamma^n_t)$. Define thus correspondingly $\Sigma^n_t := \Phi^n_t(1,\Gamma^n_t)$. By the smoothness of the map $\bar{h}$ in the paremeter $t$, we now know that $\{\Sigma_t^n\}_t$ is indeed an element of $\Lambda$. On the other hand (3) implies the property that

$$\lim_{n\to\infty} \max_t \mathfrak{d}\,(\Sigma_t^n, \Xi_t^n) = 0\,.$$

We therefore can conclude that:

- First of all

  $$\begin{aligned}\limsup_n \max_t \mathcal{H}^2(\Sigma_t^n) &= \limsup_n \max_t \mathcal{H}^2(\Xi_t^n)\\ &\leq \limsup_n \max_t \mathcal{H}^2(\Gamma_t^n) \leq m_0\,.\end{aligned}$$

  Thus $\{\{\Sigma_t^n\}_t\}^n \subset \Lambda$ is a minimizing sequence.
- Secondly, if $\lim_n \mathcal{H}^2(\Sigma_{t_n}^n) = m_0$, then $\lim_n \mathcal{H}^2(\Xi_{t_n}^n) = m_0$. Therefore

  $$\limsup_n \mathfrak{d}\,(\Sigma_{t_n}^n, \mathcal{V}_\infty) = \limsup_n \mathfrak{d}\,(\Xi_{t_n}^n, \mathcal{V}_\infty) = 0\,.$$

  Namely $\{\{\Sigma_t^n\}_t\}^n$ inherits from $\{\{\Xi_t^n\}_t\}^n$ the second property claimed in the statement of the proposition.

$\{\{\Sigma_t^n\}_t\}^n$ is then the desired sequence. $\square$