The second part of the statement of Theorem 1.1, although correct, would need a much sharper argument to hold under the assumption $\|\mathring{A}\|^2_{L^2} \leq 8\pi$ than the one given in the paper. In fact the one given in the paper suffices to allow any constant strictly smaller than 8π. Moreover Lemma 2.2 is incorrect: the proof given in the paper is enough to show the lemma when $\delta^2 < 8\pi$, whereas using the solution of the Willmore Conjecture by Marques and Neves, the sharp condition is actually $\delta^2 < 4\pi^2$. Any torus in $\mathbb{R}^3$ which is the conformal image of the Clifford torus in $\mathbb{S}^3$ is a counterexample to Lemma 2.2 and shows the sharpness of the latter condition.

First of all, let me point out a mistake in equation (7). Assuming that we have diagonalized A and denoting by κ_1 and κ_2 the diagonal entries (i.e. the principal curvatures) we have

$$|\mathring{A}|^2 = 2\left(\frac{\kappa_1-\kappa_2}{2}\right)^2 = \frac{\kappa_1^2+\kappa_2^2}{2} - \kappa_1\kappa_2 = \frac{|A|^2}{2} - K_G\,.$$

In other words the left hand side of (7) is missing a factor 2. Thus (8) is missing a factor 2 in front of $\int_\Sigma |\mathring{A}|^2$ after the first equality sign and in front of δ^2 after the second equality sign.

0.1. **Proof of Lemma 2.2 with $\delta^2 < 8\pi$.** Taking the factor 2 above into account, the argument given in the paper for Lemma 2.2 can be modified so to prove the assertion when $\delta^2 < 8\pi$. Indeed, (9) becomes now

$$\int_\Sigma |\det A| \leq \frac{1}{2}\int_\Sigma |A|^2 = \delta^2 + 4\pi(1-\mathbf{g}(\Sigma))\,. \qquad (\dagger)$$

We then remark, as in the proof of the lemma given in the paper, that N is surjective and that, when the genus of Σ is not 0, N takes at least two values. Hence, if Σ is not a sphere, $\int |\det A|$ is at least 8π (by the area formula). Plugging this information in ($\dagger$) we get

$$8\pi \leq 4\pi(1-\mathbf{g}(\Sigma)) \leq \delta^2 \qquad \text{if } \mathbf{g}(\Sigma)\geq 1.$$

So $\delta^2 < 8\pi$ implies that the genus of Σ is necessarily 0.

0.2. **Proof of Lemma 2.2 with $\delta^2 < 4\pi^2$.** First of all notice that

$$\mathcal{W}(\Sigma) = \int_\Sigma \frac{(\kappa_1+\kappa_2)^2}{4}$$

is the Willmore energy, which, because of the proof of Marques and Neves of the Willmore conjecture, on surfaces of genus strictly larger than 0 is at least $2\pi^2$. Since

$$\frac{1}{2}\int_\Sigma |A|^2 = 2\mathcal{W}(\Sigma) - \int K_G = 2\mathcal{W}(\Sigma) + 4\pi(\mathbf{g}(\Sigma)-1)$$

we conclude the inequality

$$\frac{1}{2}\int_\Sigma |A|^2 \geq 4\pi^2 + 4\pi(\mathbf{g}(\Sigma)-1)$$

whenever $\mathbf{g}(\Sigma)\geq 1$. Plugging this in ($\dagger$) we achieve

$$4\pi^2 + 8\pi(\mathbf{g}(\Sigma)-1) \leq \delta^2 \qquad \mathbf{g}(\Sigma)\geq 1.$$

Note that, if Σ is the image of the Clifford torus in $\mathbb{S}^3$ through a conformal diffeomorphism $\Phi : \mathbb{S}^3\setminus\{P\}\to\mathbb{R}^3$ with pole $P\not\in\Sigma$, all inequalities are actually equalities showing that $4\pi^2$ is the optimal threshold.

0.3. **Further remarks.** The error in the equality (7) of the paper affects Section 3: in particular in the statements of Proposition 3.2 and Lemma 3.4 δ^2 appears again. The arguments can be taken literally if δ^2 were to denote the quantity $2\int_\Sigma |\mathring{A}|^2$ rather than $\int_\Sigma |\mathring{A}|^2$. A similar error appears on top of page 90 in the displayed line, where the factor $\frac{1}{2}$ in the right hand side should actually be 2 (note that in the left hand side of that equation Id is a typo and should be removed). In the remaining sections I do not believe any change is needed since all the estimates have a non explicit (and possibly very large) constant in front of δ.

Many thanks to Ruben Jakob for pointing out the mistakes in equations (7) and (8) of the paper and the fact that Lemma 2.2 is obvioulsy wrong, as stated, because a counterexample is immediately given by Willmore's examples of revolution tori which minimize the Willmore energy.