Errata on "h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings"

The proof of Corollary 4 in the paper has a gap, pointed out to us by Reza Pakzad. The corrected argument yields the weaker conclusion of Corollary A below. Moreover, [1, Corollary 3] does not follow from Pogorelov's work and [1, Theorem 3], as claimed in the paper: in only to apply Theorem 9 of page 650 in [2] the map u must be assumed to be an *embedding* and not merely an *immersion* (as clarified at [2, Page 572], the term "smooth surface" in the book denotes what the modern literature would call embedded C^1 two-dimensional submanifolds).

Corollary A Let $\beta > 0$, $\overline{D} \subset \mathbb{R}^2$ be the closed unit disk and $g \in C^{2,\beta}$ a metric on $\overline{D}$ with positive Gauss curvature. Then there is a smooth short embedding $v \in C^\infty(\overline{D}, \mathbb{R}^3)$ and a positive δ with the following property. If $u \in C^{1,\alpha}(\overline{D}, \mathbb{R}^3)$ is an isometric embedding with $\alpha > \frac{2}{3}$, then $\|u - v\|_{C^0} \geq \delta$.

Moreover, in addition to the case of closed surfaces, the convexity can be concluded from the work of Pogorelov in another case:

Corollary B Let $\beta > 0$ and (Ω, g) be a complete 2-dimensional Riemannian $C^{2,\beta}$ manifold without boundary and with positive Gauss curvature. If $u \in C^{1,\alpha}(\Omega, \mathbb{R}^3)$ is a proper isometric embedding with $\alpha > \frac{2}{3}$, then $u(\Omega)$ is a complete unbounded convex surface of $\mathbb{R}^3$.

Corollary B follows from Theorem 3 in [1] and Theorem 2 at page 613 of Pogorelov's monograph [2].

Concerning the gap in the proof of [1, Corollary 4], note that, first of all, Sabitov's estimate in Step 3 of the proof is quoted incorrectly: the estimate is in the interior and depends on the distance to the boundary of the domain. This is immaterial: the argument could be changed by looking instead at the maximal open set U of the form $V \times]-a, a[$ such that $u(\Omega) \cap U$ is the graph of a (locally) C^2 function with $f|_{\partial V} = a$ and U has diameter smaller than c_0. The remaining argument is then correct: note however that it just implies that U has a sufficiently large diameter, but it does not imply that it contains a ball centered on y with radius controlled uniformly from below, as claimed in estimate (101): it might happen that such maximal U is very long and thin. We next show how this can however still be used to prove Corollary A above.

Proof of Corollary A. Let $\overline{D}$ and g be as in the statement and fix a $u \in C^{1,\alpha}$ isometric embedding of $(\overline{D}, g)$ into $\mathbb{R}^3$ with $\alpha > \frac{2}{3}$. Then the following alternative holds:

- (a) Either $u(D_{1/2})$ is locally uniformly convex;
- (b) Or there are points $x \in D_{1/2}$ and $y \in \partial D$ such that the straight segment $\sigma = [u(x), u(y)]$ is contained in $u(\Omega)$.

We first prove the alternative. Denote by N the Gauss map of the immersed surface. Using the terminology of Pogorelov, we say that $x \in D_{1/2}$ is a regular point if there is a neighborhood U of it with the property that $N(u(y)) \neq N(u(x))$ for every $y \notin U \setminus \{x\}$. As argued in the Appendix of [1] regular points are dense in D.

Moreover, following the explanation over there, u is uniformly convex in a neighborhood of $u(x)$ for any regular point x. Let π be the tangent plane to $u(\overline{D})$ at $u(x)$. To fix ideas assume that it is the plane $\{x_3 = 0\}$ and that locally around $u(x)$ the surface is the graph of a convex function $(x_1, x_2, f(x_1, x_2))$. It follows from [2] and [3] that f is a uniformly convex function. Consider the family of planes $\pi_t := \{x_3 = t\}$ for t positive. For small t's one of the connected components of $\pi_t \cap u(\overline{D})$ must be a curve γ_t which bounds a disk containing $u(x)$ in $u(\overline{D})$. We then argue as in the proof of Theorem 1 at page 613 in [2]: as we follow the evolution of γ_t, it either contracts to a point or it intersects with $u(\partial D)$, and as long as none of the two events happen, γ_t keeps being a convex curve which in $u(\overline{D})$ bounds a convex region containing $u(x)$. As described in the proof of Theorem 2 at page 615, in case γ_t becomes a point, we would find a portion of $u(\overline{D})$ homeomorphic to the 2-sphere, which is not possible. Since $u(\overline{D})$ is bounded, there must therefore be a $\bar{t}$ such that $\pi_{\bar{t}} \cap u(\overline{D})$ contains a curve $\gamma_{\bar{t}}$ with the following properties:

(i) $\gamma_{\bar{t}}$ is a convex curve;
(ii) $\gamma_{\bar{t}} \cap u(\partial D)$ is not empty;
(iii) $\gamma_{\bar{t}}$ bounds a convex region Ω in $u(\overline{D})$ which contains $u(x)$.

We denote by $d(x)$ the value $\bar{t}$, by $\Omega(x)$ the region Ω and by $\gamma(x)$ the curve $\gamma_{\bar{t}}$.

Now, if every point $x \in D_{1/2}$ is regular, then alternative (a) holds. Assume therefore that there is a nonregular point x. Since regular points are dense, select a sequence x_k of regular points converging to x. The numbers $d(x_k)$ must converge to 0, otherwise, for a subsequence (not relabeled), the regions $\Omega(x_k)$ would converge to a nontrivial convex region $\Omega(x)$ containing x, namely there would be a neighborhood U of x such that $u(U)$ is convex and hence uniformly convex, contradicting the assumption that x is not a regular point.

We next claim that the curves $\gamma(x_k)$ must converge to a segment. Otherwise a subsequence would converge to a nontrivial convex curve γ in the tangent plane π to $u(\overline{D})$ at $u(x)$. The region bounded by such curve γ in π would have to be a subset of $u(\overline{D})$, but all its counterimages would consist of nonregular points, while we already know that regular points are dense in D. The limit of the curves $\gamma(x_k)$ is thus a straight segment, which must contain the point $u(x)$ and a point p in $u(\partial D)$. We conclude that alternative (a) holds.

We now come to the claim of the Corollary, which will be shown for the embedding $(x_1, x_2) \mapsto v(x_1, x_2) = (4\eta x_1, 4\eta x_2, 0)$ when $\eta > 0$ is sufficiently small. Clearly v is a short embedding if δ is small enough. Let now u be an isometric embedding into $\mathbb{R}^3$ of class $C^{1,\alpha}$ for $\alpha > \frac{2}{3}$ with $\|u - v\|_0 \leq \delta$. If alternative (b) were to hold, the segment σ would be long at most $4\eta + \delta$. However, the counterimage of this segment is a curve joining a point $x \in D_{1/2}$ and a point $y \in \partial D_1$ (which is in fact a geodesic for the metric g): the length of the curve with respect to the metric g is thus larger than a constant $c_0 > 0$ depending only on g. Thus, if $\eta < \frac{c_0}{5}$ and δ is sufficiently small, alternative (b) is excluded.

Fix therefore such an $\eta \leq \frac{c_0}{5}$ and assume that there is a sequence of isometric embeddings u_k of class $C^{1,\alpha}$ with $\alpha > \frac{2}{3}$ converging uniformly to it. We can assume withot loss of generality that each $u_k(D_{1/2})$ is locally convex. Let w_k be the planar maps consisting of the first two components of u_k. For sufficiently large k the degree of the map w_k in the square $[\eta, \eta]^2$ equals the degree of the map (v_1, v_2), which is 1. Assume without loss of generality that this holds for every k.

Consider now an infinite strip of the form

$$S_t := \{(x_1, x_2, x_3) : x_1 = t, -\eta \leq x_2 \leq \eta\}.$$

By Sard's theorem for a.e. t the intersection $S_t \cap u_k(\overline{D}_1)$ is transversal to $u(\overline{D}_1)$. Select a $t \in [-\eta, \eta]$ such that this holds true for every k. Then $S_t \cap u_k(\overline{D}_1)$ consists of finitely many closed curves and finitely many arcs, the endpoints of the latter being contained in the two lines $\ell_\pm(t) = \{(t, \pm\eta, x_3)\}$. Since the degree of the map w_k is 1 on $[-\eta, \eta]^2$, there must be at least one arc which is connecting the two lines (otherwise the degree of the map would be 0 on the whole segment $\{(t, x_2) : |x_2| \leq \eta\}$). Denote by α_k the arc, by q_k and p_k its two endpoints, by β_k the arc $u_k^{-1} \circ \alpha_k$, and by $\bar{q}_k$ and $\bar{p}_k$ the two endpoints of the latter. Observe that, by the uniform converge of u_k to v, the arc β_k must be contained in $D_{1/2}$ for k large enough. Since $u(D_{1/2})$ is locally uniformly convex, β_k is then given by $\{(t, s, g_k(s)), -\eta \leq s \leq \eta\}$, where the function g_k is C^2 and either convex or concave. Observe however that g_k converges uniformly to 0: in particular the lengths of the β_k' are converging to 2η. On the other hand the points $\bar{q}_k$ and $\bar{p}_k$ converge, respectively, to $\bar{q} = (t, -\frac{1}{4})$ and $\bar{p} = (t, \frac{1}{4})$. The geodesic distance, in the metric g, between these two points would have to be at most 2η. On the other hand such distance can be bounded from below with a positive constant depending only on g, and this would not be possible if η is chosen sufficiently small. $\square$

References

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