NON-STANDARD SOLUTIONS TO THE EULER SYSTEM OF ISENTROPIC GAS DYNAMICS

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Abstract

This thesis aims at shining some new light on the *terra incognita* of multi-dimensional hyperbolic systems of conservation laws by means of techniques new for the field. Our concern focuses in particular on the isentropic compressible Euler equations of gas dynamics, the oldest but yet most prominent paradigm for this class of equations. The theory of the Cauchy problem for hyperbolic systems of conservation laws in more than one space dimension is still in its dawning and has been facing some basic issues so far: do there exist weak solutions for any initial data? how to prove well-posedness for weak solutions? which is a good space for a well-posedness theory? are entropy inequalities good selection criteria for uniqueness? Inspired by these interesting questions, we obtained some new results here collected. First, we present a counterexample to the well-posedness of entropy solutions to the multi-dimensional compressible Euler equations: in our construction the entropy condition is not sufficient as a selection criteria for unique solutions. Furthermore, we show that such a non-uniqueness theorem holds also for a classical Riemann datum in two space dimensions. Our results and constructions build upon the method of convex integration developed by De Lellis-Székelyhidi [**DLS09, DLS10**] for the incompressible Euler equations and based on a revisited "h-principle".

Finally, we prove existence of weak solutions to the Cauchy problem for the isentropic compressible Euler equations in the particular case of regular initial density. This result indicates the way towards a more general existence theorem for generic initial data. The proof ultimately relies once more on the methods developed by De Lellis and Székelyhidi in [**DLS09**]-[**DLS10**].

Zusammenfassung

In dieser Doktorarbeit studiere ich

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Introduction

The main topic of this thesis is the study of the compressible Euler equations of isentropic gas dynamics

$$\text{(0.1)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla[p(\rho)] \;=\; 0 \\ \rho(0,\cdot) \;=\; \rho^0 \\ v(0,\cdot) \;=\; v^0 \end{cases}$$

whose unknowns are the density ρ and the velocity v of the gas, while p is the pressure which depends on the density ρ. In particular, we are concerned with the Cauchy problem (0.1) and with its possible (or not) well-posedness theory.

The isentropic compressible Euler equations (0.1) are an archetype for systems of hyperbolic conservation laws. Conservation laws model situations in which the change of amount of a physical quantity in some domain is due only to an income or an outcome of that quantity across the boundary of the domain. Indeed, this is the case also for system (0.1), where the equations involved state the balance laws for mass and for linear momentum.

The apparent simplicity of conservation laws, and in particular of system (0.1), contrasts with the difficulties encountered when solving the Cauchy problem. To illustrate the mathematical difficulties, let us say that there has not been so far a satisfactory result concerning the existence of a solution of the Cauchy problem. The well-posedness theory for hyperbolic conservation laws is presently understood only in the scalar case (one equation) thanks to seminal work of Kruzkov [**Kru70**], and in the one–dimensional case (one space–dimension) via the Glimm scheme [**Gli65**] or the more recent vanishing–viscosity method of Bianchini and Bressan [**BB05**]. On the contrary, the general case is very far from being understood.
For this reason, a wise approach is to tackle some particular examples, in hope of getting some general insight.

This motivates our interest on the paradigmatic system of conservation laws (0.1). On the one hand, we can obtain a partial result on the existence of weak solutions of (0.1) for general initial momenta and regular initial density, on the other hand, building upon the same methods (see [**DLS09**]-[**DLS10**]), we can prove non–uniqueness for entropy solutions of (0.1) even for Riemann initial data. Our conclusions provide some answers in the understanding of multi-dimensional hyperbolic systems of conservation laws, yet raises and lives open other ones.

In this introductory chapter, we frame our dissertation presenting an overview of the theory of hyperbolic systems of conservation laws and highlighting open problems and challenges of the subject. Finally, we will present the main results contained in this thesis and we will outline its structure.

0.1. Hyperbolic systems of conservation laws

Hyperbolic systems of conservation laws are systems of partial differential equations of evolutionary type which arise in several problems of continuum mechanics. One of their characteristics is the appearance of singularities (known as *shock waves*) even starting from smooth initial data. In the last decades a very successful theory has been developed in one–space dimension but little is known about the general Cauchy problem in more than one–space dimension after the appearance of singularities. Recently, building on some new advances on the theory of transport equations, well-posedness for a particular class of systems has been proved. On the other hand, introducing techniques which are completely new in this context, it has been possible to establish an ill-posedness result for bounded entropy solutions of the Euler system of isentropic gas dynamics (0.1). Connected to these recent advances, there have been various open questions: how to conjecture well-posedness for general systems of conservation laws in several space dimensions? in which functional space? what structural properties of the Euler system of isentropic gas dynamics underlie the mentioned ill-posedness result? and in which class of initial data does this result hold? This thesis was inspired by such challenging questions and attempted to move some steps forward in the process of answering them.

0.1.1. Survey on the classical theory. The theory of nonlinear hyperbolic systems of conservation laws traces its origins to the mid 19th century and has developed over the years conjointly with continuum physics. The great number of books on the theoretical and numerical analysis published in recent years is an evidence of the vitality of the field. But, what does the denomination "hyperbolic systems of conservation laws" encode? They are systems of nonlinear, divergence structure first-order partial differential equations of evolutionary type, which are typically meant to model *balance laws*. In fact, the vast majority of noteworthy hyperbolic systems of conservation laws came up in physics, where differential equations were derived from corresponding statements of balance of an extensive physical quantity coupled with constitutive relations for a material body (see for instance [**Daf00**]). In the most general framework, the field equation resulting from this *coupling process* reads as

$$\partial_t U + \operatorname{div}[F(U)] = 0 \tag{0.1}$$

where the unknown is a vector valued function

$$U = U(t,x) = (U^1(t,x), ..., U^k(t,x)) \quad ((t,x) \in \Omega \subset \mathbb{R}_t \times \mathbb{R}^m_x), \tag{0.2}$$

the components of which are the densities of some conserved variables in the physical system under investigation, while the *flux* function F controls the rate of loss or increase of U through the spatial boundary and satisfies suitable "hyperbolicity conditions", namely that for every fixed U and $\nu \in S^{m-1}$, the $k \times k$ matrix

$$\sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}F_\alpha(U)$$

has real eigenvalues and k linearly independent eigenvectors.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profile of compression waves gets progressively steeper and eventually breaks, generating jump discontinuites which propagate on as *shocks*. This behavior is demonstrated by the simplest example of a nonlinear hyperbolic conservation law in one space variable, namely the *Burgers equation*

$$\partial_t U(t,x) + \partial_x \left(\frac{1}{2} U^2(t,x) \right) = 0. \tag{0.3}$$

The appearance of singularities, even when starting from regular initial data, drives the theory to deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. To see this, one can consider the Cauchy problem for the Burgers equation (0.3), with initial data

$$u(0,x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases} \tag{0.4}$$

The problem (0.3), (0.4) admits infinitely many solutions, including the family

$$u_\beta(t,x) = \begin{cases} -1, & -\infty < x \le -t \\ \frac{x}{t}, & -t < x \le -\beta t \\ -\beta, & -\beta t < x \le 0 \\ \beta, & 0 < x \le \beta t \\ \frac{x}{t}, & \beta t < x \le t \\ 1, & x > 0, \end{cases}$$

for any $\beta \in [0,1]$.

It thus becomes necessary to devise proper criteria for weeding out unstable, physically irrelevant, or otherwise undesirable solutions, in hope of singling out admissible weak solutions. The issue of admissibility of weak solutions to hyberbolic systems of conservation laws is a central question of the theory and stirred up a debate quite early in the development of the subject. Continuum physics naturally induces such admissibility criteria through the Second Law of Thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. In the framework of the general theory of hyperbolic systems of conservation laws, the use of entropy inequalities to characterize admissible solutions was first proposed by Kruzkov [**Kru70**] and then elaborated by Lax [**Lax71**]. The idea of regarding inviscid gases as viscous gases with vanishingly small viscosity is quite old; there are hints even in the seminal paper by Stokes [**Sto48**]. The important contributions of Rankine [**Ran70**], Hugoniot [**Hug89**] and Rayleigh [**Ray10**] helped to clarify the issue.

From the standpoint of analysis, a very elegant, definitive theory is available for the case of scalar conservation laws, in one or several space dimensions. The special feature that sets the scalar balance law apart from systems of more than one equation is the size of its family of entropies: in the scalar case the abundance of entropies induces an effective characterization of admissible weak solutions as well as very strong L^1-stability and L^∞-monotonicity properties. Armed with such powerful a priori estimates, one can construct admissible solutions in a number of ways. In the one-dimensional case the qualitative theory was first developed in the 1950's by the Russian school, headed by Oleinik [**Ole54, O.A57, O.A59**], while the first existence proof in several space dimensions was established a few years later by Conway and Smoller [**CS96**], who recognized the relevance of the space BV. The definitive treatment in the space BV was later given by Volpert [**Vol67**]; building on Volpert's work, Kruzkov [**Kru70**] proved the well-posedness for admissible weak solutions. As a consequence of Kruzkov's results when the initial data are functions of locally bounded variation then so are the solutions. Remarkably, even solutions that are merely in L^∞ exhibit the same geometric structure as BV functions, with jump discontinuities assembling on "manifolds" of codimension one (see [**DLOW03**], [**DLR03**] and [**DLG05**])

By contrast, when dealing with systems of conservation laws, it is still a challenging mathematical problem to develop a theory of well-posedness for the Cauchy problem of (0.1) which includes the formation and evolution of shock waves. In one space dimension, namely when $m = 1$ in (0.2), this problem has found recently a quite satisfactory and general answer, thanks to the efforts of generations of mathematicians: the general mathematical framework of the theory was set in the seminal paper of Lax [**Lax57**]; the first existence result is due to Glimm [**Gli65**] in the sixties; Bianchini and Bressan [**BB05**] finally proved a well-posedness result. Glimm's scheme gives the sole result of any generality concerning Cauchy problems and makes use of functions of bounded total variation on $\mathbb{R}$. The higher dimensional case is terra incognita: how to conjecture stability is still an open problem. Indeed, in several space dimensions, the situation is clearly less favourable: the success of the spaces L^∞ and BV in one–dimensional space is due to the fact they are algebras allowing for the treatment of the rather strong non–linearity of the equations; however, the works of Brenner [**Bre66**]

and Rauch [**Rau86**], which are concerned with linear systems, show that these spaces cannot be adapted to the multi–dimensional case. We are thus in presence of a paradox which has up to the present not been resolved: *to find a function space which is an algebra, probably constructed on L^2 and which contains enough discontinuous solutions.* Moreover, even a general existence result for weak solutions in more than one space dimension is missing so far. The theory is in its infancy.

0.1.2. Recent results. Recently, some attention was devoted to a first "toy example" falling in the class (0.1). This system, called Keyfitz–Kranzer system, is clearly very peculiar and, compared to the most relevant systems coming from the physical literature, has many more features. It reads as

$$\begin{cases} \partial_t u + \sum_{j=1}^m \frac{\partial}{\partial x_j}(f^j(|u|)u) \;=\; 0 \\ u(0,\cdot) \;=\; u^0 \end{cases} \tag{0.5}$$

where for any $j = 1, \dots, m$ the map $f^j : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be smooth. In this case the non–linearity depends only on the modulus of the solutions. Most notably the system (0.5) decouples into a nonlinear conservation law for the modulus of $\rho := |u|$

$$\partial_t \rho + \mathrm{div}(\mathrm{f}(\rho)\rho) \;=\; 0$$

and a system of linear transport equations for the angular part $\theta := u/|u|$

$$\partial_t \theta + f(\rho) - \nabla\theta \;=\; 0.$$

However, it does develop singularities in finite time and a theory of well–posedness of singular solutions was still lacking up to few years ago. Thanks to a groundbreaking paper of Ambrosio (see [**Amb04**]), it was possible to solve this problem in a very general and satisfactory way (see [**ADL03**]): well posedness of *renormalized entropy solutions* in the class of maps $u \in L^\infty([0,T] \times \mathbb{R}^m; \mathbb{R}^k)$ with $|u|\, in BV_{loc}$ has been proven by Ambrosio, Bouchut and De Lellis in [**ABDL04**]. Moreover, the problem has been used to show that, even in this very particular case, there is no hope of getting estimates in some of the classical function spaces which are used in the one-dimensional theory.

In a recent work De Lellis and Székelyhidi found striking counterexamples to the well-posedness of bounded entropy solutions to the isentropic system of gas-dynamics (0.1) (see [**DLS10**]). These examples

build on a previous work ([**DLS09**]) where they introduced techniques from the theory of differential inclusions to construct very irregular solutions to the incompressible Euler equations. The isentropic system of gas dynamics in Eulerian coordinates (0.1) is the oldest hyperbolic system of conservation laws. The hyperbolicity condition for system (0.1) reduces to the monotonicity of the pressure as a function of the density: $p'(\rho) > 0$. In this thesis the pressure p will always satisfy this assumption. Weak solutions of (0.1) are bounded functions which solve the system in the sense of distributions. Admissible (or entropy) solutions can be characterized as those weak solutions which satisfy an additional inequality, coming from the conservation law for the energy of the system. In the paper [**DLS10**], De Lellis and Székelyhidi show L^∞ initial data, with strictly positive piecewise constant density, which allow for infinitely many admissible solutions of (0.1) in more than one space dimension, all with strictly positive density:

THEOREM 0.1.1 (Theorem from [**DLS10**]). *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (0.1) with $\rho \geq c > 0$.*

This result proves that the space L^∞ is ill-suited for well-posedness of entropy solutions; moreover it makes us believe that admissibility inequalities are not the "right" selection criteria. Of course, from Theorem 0.1.1 arises a cascade of questions which point in many grey areas; some of these open questions are at the core of this dissertation.

0.1.3. Motivating problems. In this paragraph, we summarize the main issues which motivated the research presented in this thesis. They do not exhaust the immeasurable amount of open problems in the field of hyperbolic systems of conservation laws but they are indicative of the topicality of this branch of mathematics. Moreover, as illustrated by the following motivating questions, our concern is not only for the novelty of the results such questions could lead to, but also for the techniques involved.

- *Existence results*: the lack of satisfactory existence results for weak solution of multi–dimensional systems of conservation laws is a glaring symptom of the difficulties underlying the theory. In particular, for bounded initial data, but of arbitrary size, only 2×2 systems in one–space dimension have

been tackled by the method of *compensated compactness*. Is it possible to prove some existence result at least for the particular system of isentropic gas dynamics?

- *Well-posedness of admissible solutions*: how to separate the wheat from the chaff, the solutions observed in nature (the *physically admissible* ones) from those that are only mathematical artefacts is one of the central questions in the theory of multi–dimensional systems of conservation laws. In particular, are entropy (admissibility) inequalities efficient as selection criteria? Theorem 0.1.1 seems to give a negative answer to the aforemethioned question. Should this inefficiency of entropy inequalities believed to be a "universal law"?

- *Ill-posedness for the isentropic Euler system*: the surprising result 0.1.1 from [**DLS10**] left unsolved the question whether the system (0.1) directly allows for the construction of the paper [**DLS10**]. Such a issue is connected not only to the efficiency of admissible criteria to wheed out non–physical solutions, but also to a development of the techniques on which [**DLS10**] bases.

- *Ill-initial data*: relying only on [**DLS10**] one could argue that phenomena of ill-posedness could be restricted to very particular initial data and that for a large class of them, one could hope for a uniqueness theorem. We aimed at understanding better for which initial data such constructions are possible. In particular we questioned the case of Riemann initial data.

- *Further applications*: the new method introduced in [**DLS09**] has been extended in a quite direct way to the isentropic system of gas-dynamics. Another interesting question concerns possible extensions and further applications of the idea coming from [**DLS10**]; for instance could it be applied to other systems of conservation laws?

- *Suitable functional spaces*: The inadeguacy of the spaces L^∞ or BV for multi–dimensional systems of conservation laws raises

the issue of finding another functional space for the study of weak solutions. But no satisfactory space has been suggested until now.

0.2. Main results and outline of the thesis

This thesis consists of five chapters whose content we are going to disclose.

The results here presented represent new developments and applications of the innovative approach introduced by De Lellis and Székelyhidi in [**DLS09**]-[**DLS11**]. Their methods brought in the realm of fluid dynamics techniques coming from Gromov's convex integration [**Gro86**] and strategies from the theory for differential inclusions [**KMS03**] and remarkably combined these tools to construct non-standard solutions to the incompressible Euler equations and to the compressible ones as well. Since our achievements strongly rely on this new approach, we devote Chapter 1 to a brief compendium on the related background theory. In Chapter 1 we aim at introducing the reader to the interesting theory lying behind Theorem 0.1.1. In particular, we will explain how Gromov's work on partial differential relations and on convex integration together with Kirchheim, Müller and Sverák's approach to study the properties of nonlinear partial differential equations can concur to construct solutions to equations from fluid dynamics. We will also hint the analogies between problems in differential geometry, where the idea of convex integration originally arose, and the incompressible Euler equations. Finally, we will give an overview of the recent results in fluid dynamics obtained as further advancements of De Lellis and Székelyhidi's ideas.

The Introduction together with the first chapter provides a preface to the core of the thesis: indeed, in the subsequent chapters, the tools introduced in Chapter 1 will be applied to the compressible Euler equations (0.1) allowing for new results.

Chapter 2 contains the first important theorem of the thesis:

THEOREM 0.2.1 (Non-uniqueness of entropy solutions with arbitrary density). *Let $n \geq 2$. Then, for every periodic $\rho^0 \in C^1$ with $\rho^0 \geq c > 0$ and for any given function p, there exist an initial velocity $v^0 \in L^\infty$ and a time $T > 0$ such that there are infinitely many*

bounded admissible solutions (ρ, v) *of* (0.1) *on* $\mathbb{R}^n \times [0, T[$*, all with density bounded away from* 0*.*

This theorem is an improvement of Theorem 0.1.1: using the same techniques as in Theorem 0.1.1, we can show that the same non-uniqueness result holds for any choice of the initial density (see also [**Chi11**]). This highlights that the main role in the loss of uniqueness is due to the velocity field. While the proof of Theorem 0.1.1 relies on a non-uniqueness result for the incompressible Euler equations, and hence yields "piecewise incompressible" solutions, Theorem 0.2.1 is achieved by applying directly to (0.1) De Lellis and Székelyhidi's ideas. Yet, the solutions constructed in Theorem 0.2.1 allow for wild oscillations only in the velocity. The general case which would include wild oscillations in the density as well is presently under investigation (cf. [**CDLK**]).

The initial data v^0 in Theorem 0.2.1 are ad hoc constructed and in principle could be very irregular. A surprising corollary of De Lellis and Székelyhidi's result on the incompressible Euler equations is that a classical Riemann datum follows in the class of initial data allowing for very wild solutions:

Theorem 0.2.2 (Non-standard solutions of a Riemann problem). *For certain choices of the pressure law* p *with* $p' > 0$ *and for some specific Riemann initial data there exist infinitely many bounded entropy solutions of the compressible Euler equations* (0.1) *in two space dimensions, all with density bounded away from* 0*.*

The proof of Theorem 0.2.2 is the content of Chapter 3. Theorem 0.2.2 entails that the classical entropy inequality does not ensure uniqueness of the solutions even for this very natural initial condition. Though the solutions of Theorem 0.2.2 are very irregular, it is rather unclear which could be a good space for well-posedness. Let us note that among the pressure laws for which Theorem 0.2.2 holds, there is the physically relevant pressure $p(\rho) = \rho^2$. Indeed the quadratic pressure is predicted from classical kinetic theory in two space dimensions. The inspiration for such a result came from a recent paper by Székelyhidi [**Sz1**], where the vortex sheet initial datum for the incompressible Euler system is investigated. In Chapter 3 we also present an alternative proof of Székelyhidi's result. However, the different role

of the pressure term in the compressible Euler system with respect to the incompressible one requires new ideas and constructions involved in the proof of Theorem 0.2.2.

In Chapter 4 we restrict our attention to the 1-*dimensional Riemann problem* for the compressible Euler equations with the same choice of initial data allowing for Theorem 0.2.2 (which indeed depend only on one space variable): we show that such a problem admits unique self-similar solutions (even with the same choice of pressure law as in Theorem 0.2.2). The uniqueness of self–similar solutions is proven by direct construction of the admissible wave fun. Theorem 0.2.2 shows that as soon as the self–similarity assumption runs out, uniqueness is lost.

Chapter 5 is devoted to the last main result of the thesis:

THEOREM 0.2.3 (Existence of weak solution with arbitrary momentum). *Let $\rho^0 \in C^1$ and $v^0 \in L^2$. Then there exists a weak solution (ρ, v) (in fact, infinitely many) of the Cauchy problem for the compressible Euler equations* (0.1) *with intial data (ρ^0, m^0).*

Theorem 0.2.3 is just a first step towards the most desirable result of existence of weak solutions of (0.1) starting out from any given bounded initial density and momentum. Such an outcome would be of great impact, since so far no existence result for weak solutions of multi-dimensional hyperbolic systems of conservation laws with generic initial data is available. Of course Theorem 0.2.3 is able to deal only with regular initial densities, but the more general result is believed to hold building on the improvement of Theorem 0.2.1 (see [**CDLK**]).

The thesis is conceived in such a way that every chapter can be read both as the continuation of what precedes or independently.

CHAPTER 1

The h–principle and convex integration

The *h-principle* is an umbrella–concept forged by Gromov in 1969 ([**Gro86**]) to unify a series of counterintuitive results in topology and differential geometry. This principle is a strong property characterizing the set of solutions of *differential relations*: a differential relation is *soft* or *abides by the h-principle* if its solvability can be determined on the basis of purely homotopic calculus. By *differential relation* we mean a constraint on maps between two manifolds and on their derivatives as well. PDEs are examples of differential relations. It is striking that many differential relations, mostly rooted in differential geometry and topology, are soft. Two famous examples of the softness phenomenon are the Nash-Kuiper C^1 isometric embedding theory and the Smale's sphere eversion.

Why are we interested in the h–principle? It is surprising how the results on the isentropic Euler equations of gas dynamics presented in this thesis are based on a revisited h-principle. This new variant of h-principle has been first devised by De Lellis and Székelyhidi for the incompressible Euler equations (see [**DLS09**]) and lead to new developments for several equations in fluid dynamics as the ones of this thesis. Indeed, even if the original h-principle of Gromov pertains to various problems in differential geometry, De Lellis and Székelyhidi showed in their groundbreaking paper [**DLS09**] that the same principle and similar methods could be applied to problems in mathematical physics. The work by De Lellis and Székelyhidi found its breeding ground in the important paper by Müller and Švérak [**MS03**], where they extended the method of convex integration (introduced by Gromov to prove the h-principle) to Lipschitz mappings and noticed the strong connections between the existence theory for differential inclusions and the h-principle.

We do not pretend here to give an account of the extremely wide literature on this topic, but we rather prefer to illustrate some specific

instances of the h-principle as a jumping off point for a general understanding of the subject. In particular, we will spend some words on the method of convex integration which will be recalled in the next chapters for the arguments of our constructions.

1.1. Partial differential relations and Gromov's h-principle

In this section we will introduce the idea behind the h-principle by illustrating Gromov's original formalism.

A *partial differential relation* $\mathcal{R}$ is any condition imposed on the partial derivatives of an unknwon function. A solution of $\mathcal{R}$ is any function which satisfies this relation. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A simple example of differential relations are ordinary differential equations or inequalities. We can consider, for instance, the differential equation $y'(x) = y^2(x)$; then the underlying algebraic relation is obtained by introducing the new variable z in place of the derivative y': the resulting relation is simply $z = y^2$ seen as a constraint in $\mathbb{R}^3$ with coordinates (x, y, z). In this language, a solution of the corresponding algebraic relation is called a *formal solution* of the original differential relation $\mathcal{R}$. The difference between genuine and formal solutions in this specific example becomes clear as soon as we interpret genuine solutions as functions (in fact sections) $f : \mathbb{R} \to \mathbb{R}^3$, $f(x) = (x, y(x), y'(x))$ with $y'(x) = y^2(x)$ (this amounts to using the language of jets which we do not want to get into herein). Clearly, the existence of a formal solution is a necessary condition for the solvability of a differential relation $\mathcal{R}$. In the previous example, formal solutions are functions $g : \mathbb{R} \to \mathbb{R}^3$, $g(x) = (x, y(x), z(x))$ with $z(x) = y^2(x)$. The philosophy behind the h-principle consists in the following: before trying to solve $\mathcal{R}$ one should check whether $\mathcal{R}$ admits a formal solution. The problem of finding formal solutions is of purely homotopy-theoretical nature. It could seem, at first thought, that existence of a formal solution cannot be sufficient for the genuine solvability of $\mathcal{R}$. Indeed, finding a formal solution is an algebraic problem which is a dramatical simplification of the original differential problem. Thus it came as a big surprise when it was discovered in the second half of the twentieth cetury that there exist large and geometrically interesting classes of differential relations for which the solvability of the formal problem is sufficient for genuine solvability. Moreover, for many

of these relations the spaces of formal and genuine solutions turned out to be much more closely related than one could expect. This property was formalized by Gromov [**Gro86**] as the following:

- **Homotopy principle (*h*-principle)**. A differential relation $\mathcal{R}$ satisfies the h-principle, or the h-principle holds for solutions of $\mathcal{R}$, if every formal solution of $\mathcal{R}$ is homotopic to a genuine solution of $\mathcal{R}$ through a homotopy of formal solutions.

The term "h-principle" was introduced and popularized by M. Gromov in his book [**Gro86**]. It is now clear that the h-principle does not hold for the differential equation $y'(x) = y^2(x)$ (we consider global solutions), while we could prove the h-principle for the equation $y'(x) = y(x)$ since every formal solution $f(x) = (x, y(x), y(x))$ can be joined via a homotopy H_t of formal solutions to the genuine solution $h(x) = (x, \exp(x), \exp(x))$ simply choosing $H_t(x) = (1-t)f(x) + th(x)$. These examples are of course trivial and not typical of situations where the h-principle is useful. In fact, the h-principle is rather useless in the classical theory of (ordinary or partial) differential equations because there it fails or holds for some trivial, or at least well known reasons, as in the above examples.

By contrast, for many differential relations rooted in topology and geometry the notion of h-principle appeared to be fundamental. There are several amazing unexpected cases in which the h-priciples holds. A particular problem which abydes by the h-principle can also be called *soft*. As already mentioned, the softness phenomena was first discovered in the fifties by Nash [**Nas54**] for isometric C^1-immersions and by Smale [**Sma58**] for differential immersions. However, instances of the soft problems appeared earlier. In his dissertation and later in his book [**Gro86**], Gromov transformed Smale's and Nash's ideas into two powerful methods for solving partial differential relations: *continuous sheaves* method and the *convex integration* method. In the next section we will give an overview on Nash's contruction, where the "spirit" of convex integration originally arose.

In the language pertainig to Gromov, the idea lying behind convex integration can be illustrated through an easy example which is suggested in [**EM02**]. Let us call a path

$$r : I = [0, 1] \to \mathbb{R}^2,\ r(t) := (x(t), y(t)),$$

short if $x'(t)^2 + y'(t)^2 < 1$ a.e. $t \in I$. The inequality defining a short path is nothing else that a particular instance of partial differential relation. It easy to prove that any short path can be C^0-approximated by a solution of the equation $x'(t)^2 + y'(t)^2 = 1$ a.e. $t \in I$, which is another differential relation. This implies that the space of solutions $I \to \mathbb{R}^2$ of the differential equation $x'(t)^2 + y'(t)^2 = 1$ is C^0-dense in the space of solutions of the differential inequality $x'(t)^2 + y'(t)^2 < 1$. This elementary example illuminates the following idea at the core of convex integration: given a first order differential relation for maps $I \to \mathbb{R}^q$, it is useful to consider a "relaxed" differential relation which is the pointwise convex hull of the original relation

1.2. The h-principle for isometric embeddings

1.2.1. Isometric embedding problem. Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An *isometric immersion* of (M^n, g) into $\mathbb{R}^m$ is a map $u \in C^1(M^n; \mathbb{R}^m)$ such that the induced metric agrees with g. In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{1.1}$$

consisting of $n(n+1)/2$ equations in m unknowns. If in addition u is injective, it is an *isometric embedding*. Analougously, one defines a *short embedding* as a map $u : M^n \to \mathbb{R}^m$ such that the metric induced on M by u is shorter than g. In coordinates this translates into $(\partial_i u \cdot \partial_j u) \leq (g_{ij})$ in the sense of quadratic forms. Geometrically being short means that the embedding shrinks the length of curves. Equally, being isometric means that the length of curves is preserved.

The well-known result of Nash and Kuiper says that any short embedding in codimension one can be uniformly approximated by C^1 isometric embeddings.

THEOREM 1.2.1 (Nash-Kuiper theorem). *If $m \geq n+1$, then any short embedding can be uniformely approximated by isometric embeddings of class C^1.*

Note that Theorem 1.2.1 is not merely an existence theorem, but it shows that there exists a huge (essentially C^0-dense) set of solutions. Such a density of solutions is reminiscent of the example on short paths presented in the previous section. This type of abundance of solutions

is a central aspect of Gromov's h-principle, for which the isometric embedding problem is a primary example. Indeed, we could ask whether there exists a regular homotopy $f_t : S^2 \to \mathbb{R}^3$ which begins with the inclusion f_0 of the unit sphere and ends with an isometric immersion f_1 into the ball of radius $1/2$. One of the many counterintuitive implications of Nash and Kuiper's theorem is that we can answer positively to this question in case of C^1-immersions: S^2 can be C^1 isometrically embedded into an arbitrarily small ε-ball in Euclidean 3-space (for small ε there is no such C^2). The h-principle for isometric embeddings is rather striking, especially when compared to the classical rigidity result concerning the Weyl problem: if (S^2, g) is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion. Thus it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

The proof of Theorem 1.2.1 involves an iteration technique called *convex integration*.

1.2.2. Nash-Kuiper's general scheme. The general scheme of the construction upon which the main results of this thesis build are strongly inspired by the method of Nash and Kuiper. It is then interesting to sketch here the Nash-Kuiper scheme. For simplicity we assume g to be smooth. Let us set some notation: given an immersion $u : M^n \to \mathbb{R}^m$, we denote by $u^\sharp e$ the pullback of the standard Euclidean metric e through u, so that in local coordinates

$$(u^\sharp e)_{ij} = \partial_i u \cdot \partial_j u.$$

Moreover we define

$$n_* = \frac{n(n+1)}{2}.$$

A Riemannian metric g on $\mathbb{R}^n$ is said to be *primitive* if $g = \alpha(x)(dl)^2$, where $l = l(x)$ is a linear function on $\mathbb{R}^n$ and α is a non-negative function with compact support. A Riemannian metric g on a manifold M is called *primitive* if there exists a local parametrization $\phi : \mathbb{R}^n \to U \subset M$ such that supp $g \subset U$ and $\phi^\sharp g$ is a primitive metric on $\mathbb{R}^n$.

For the sake of clarity, we will give the ideas for the proof of the following simplified version of Theorem 1.2.1

THEOREM 1.2.2. *Let Ω be an open bounded subset of $\mathbb{R}^n$ equipped with the Riemannian metric g and let u be a smooth strictly short immersion of (Ω, g) into $(\mathbb{R}^m, e)$, with $m \geq n+2$. Then, for every $\varepsilon > 0$ there exists a C^1 isometric immersion $\widetilde{u} : \Omega \hookrightarrow \mathbb{R}^m$ such that $\|u - \widetilde{u}\|_{C^0(\Omega)} < \varepsilon$, i.e.*

- $\widetilde{u} \in C^1(\overline{\Omega})$;
- $\partial_i \widetilde{u} \cdot \partial_j \widetilde{u} = g_{ij}$ *in* Ω;
- $\|u - \widetilde{u}\|_{C^0(\overline{\Omega})} < \varepsilon$.

The proof of Theorem 1.2.2 is based on an iteration of *stages*, and each *stage* consists of several *steps* whose purpose we are going to unravel.
Starting from u one defines a first perturbation as follows

$$u_1(x) := u(x) + \frac{a(x)}{\lambda} \left(\sin(\lambda x \cdot \xi)\zeta(x) + \cos(\lambda x \cdot \xi)\eta(x)\right),$$

where $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and ζ, η are unit normal vectors to $u(\Omega)$, i.e.

- $\zeta \perp \eta$ and $|\zeta| = |\eta| = 1$;
- $\zeta \perp \partial_i u$ & $\eta \perp \partial_i u$ for $i = 1, \ldots, n$.

Let us note that the condition of isometry $\partial_i \widetilde{u} \cdot \partial_j \widetilde{u} = g_{ij}$ can be equivalently written in terms of the matrix differential $\nabla u = (\partial_j u^i)_{ij}$ as $\nabla \widetilde{u}^{\mathrm{T}} \nabla \widetilde{u} = g$. Now, by easy computations one obtains:

$$\begin{aligned} \nabla u_1(x) = {} & \nabla u(x) \\ & + a(x) \left(\cos(\lambda x \cdot \xi)\zeta(x) \otimes \xi - \sin(\lambda x \cdot \xi)\eta(x) \otimes \xi\right) \\ & + \mathrm{O}\left(\frac{1}{\lambda}\right), \end{aligned}$$

and hence

$$(\nabla u_1)^{\mathrm{T}} \nabla u_1 = (\nabla u)^{\mathrm{T}} \nabla u + a^2(x) \xi \otimes \xi + \mathrm{O}\left(\frac{1}{\lambda}\right).$$

Picture 1 gives a geometric intuition of the perturbation introduced in u_1 in the case $n = 1$.

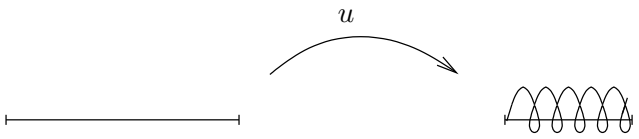

FIGURE 1. Geometric picture

Now, the purpose of a *stage* is to correct the error $g - u^\sharp e = u - (\nabla u)^{\mathrm{T}}\nabla u$. In order to achieve this correction, the error is locally decomposed into a sum of primitive metrics as follows

$$g - (\nabla u)^{\mathrm{T}}\nabla u \ = \ \sum_{k=1}^{n_*} a_k^2 \xi_k \otimes \xi_k \qquad \text{(locally)}.$$

Therefore, by iterating the procedure illustrated in the construction of u_1, i.e. by adding repeatedly "spirally perturbations" to u it should be possible to achieve u_N such that

- $(\nabla u_N)^{\mathrm{T}}\nabla u_N \ = \ g + \mathrm{O}\left(\frac{1}{\lambda}\right)$;
- $\|\nabla u_N - \nabla u\|_{C^0(\Omega)} = \sum_k \|a_k\|_{C^0(\Omega)} + \mathrm{O}\left(\frac{1}{\lambda}\right) \sim \ \left\|g - (\nabla u)^{\mathrm{T}}\nabla u\right\|_{C^0(\Omega)}^{1/2}$;
- $\|u_N - u\|_{C^0(\Omega)} \ = \ \mathrm{O}\left(\frac{1}{\lambda}\right)$.

Let us draw the attention to the fact that, when introducing a perturbation as in u_1, a, ζ and η may vary as x varies but that is not the case for ξ which is a fixed vector: this prevents us from correcting the error simply by taking the eigenvectors of $g(x) - \nabla u(x)^{\mathrm{T}}\nabla u(x)$. This involves the use of a "partition of unity" of the set of positive definite matrices. which we will not expound here. The previous considerations show which kind of estimates are involved when adding a primitive metric. Hence, the general Nash-Kuiper's scheme lies in the following iterations:

- *step*: a step involves adding one primitive metric; in other words the goal of a step is the metric change
$$u^\sharp e \to u^\sharp e + \sum a^2 \xi \otimes \xi;$$
- *stage*: a stage consists in decomposing the error into primitive metrics and adding them successively in steps.

The number of *steps* in a *stage* equals the number of primitive metrics in the above decomposition which interact. This equals n_* for the local construction and $(n+1)n_*$ for the global construction. Therefore iterating the estimates for one step over a single stage and then over the stages leads to the desired result.

1.2.3. Connection to the Euler equations. There is an interesting analogy between isometric immersions in low codimension and the incompressible and compressible Euler equations. In [**DLS09**] a method, which is very closely related to convex integration, was introduced to construct highly irregular energy-dissipating solutions of the

incompressible Euler equations. In general the regularity of solutions obtained using convex integration agrees with the highest derivatives appearing in the equations. Being in conservation form, the "expected" regularity space for convex integration for the incompressible Euler equations should be C^0. In [**DLS09**] a weaker version of convex integration was applied, to produce solutions in L^∞ (see also [**DLS10**] for a slightly better space) and to show that a weak version of the h-principle holds (even if there is no homotopy there). Recently, De Lellis and Székelyhidi have proved the existence of continuous and even Hölder continuous solutions which dissipate the kinetic energy. Moreover the same method devised in [**DLS10**] led to new developments in fluid dynamics in particular for the Euler system of isentropic compressible gas dynamics. When comparing the Euler equations (both compressible and incompressible) and the Nash-Kuiper result, the reader should take into account that, in this analogy, the velocity field of the Euler equations corresponds to the differential of the embedding in the isometric embedding problem. All these aspects are surveyed in the note [**DLS11**].

The understanding of Nash's construction is in a way a starting point for the approach developed by De Lellis and Székelyhidi in [**DLS09**]. As in the case of Nash, the solution of the incompressible Euler equations is generated by an iteration scheme: at each stage of this iteration a subsolution is produced from the previous one by adding some special perturbations, which oscillate quite fast. Hence, the final result of the iteration scheme is the superposition of infinitely many perturbations which converge suitably to an exact solution.

1.3. The h-principle and the equations of fluid dynamics

In a recent note [**DLS11**], De Lellis and Székelyhidi disclosed the analogy between recent outcomes in fluid dynamics (included the ones presented in this thesis) and some h-principle-type results in differential geometry, as the previously presented Nash-Kuiper Theorem. More precisely, the survey by De Lellis and Székelyhidi aims at showing how the theorems in fluid mechanics represent a suitable variant of Gromov's h-principle. In this section, we will retrace the main points of [**DLS11**] so to place the results of this thesis in a more general context.

1.3.1. The general framework. Kirchheim, Müller and Šverák in [**KMS03**] outlined an approach to study the properties of nonlinear partial differential equations through the geometric properties of a set in the space of $m \times n$ matrices which is naturally associated to the equation. This approach draws heavily on Tartar's work on oscillations in nonlinear PDEs and compensated compactness and on Gromov's work on partial differential relations and convex integration.

What does this method consist of? Following Tartar's framework [**Tar79**], many nonlinear systems of PDEs for a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ can be naturally espressed as a combination of a linear system of PDEs of the form

$$\sum_{i=1}^{n} A_i \partial_i z = 0 \text{ in } \Omega \tag{1.2}$$

and a pointwise nonlinear constraint

$$z(x) \in K \text{ a.e. } x \in \Omega, \tag{1.3}$$

where

- $z : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ is the unknown state variable;
- A_i are constant $m \times d$ matrices;
- $K \subset \mathbb{R}^d$ is a closed set.

Plane waves are solutions of (1.2) of the form

$$z(x) = ah(x \cdot \xi), \tag{1.4}$$

where $h : \mathbb{R} \to \mathbb{R}$. Then, one defines the *wave cone* Λ related to one–dimensional solutions and given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (1.4) solves (1.2):

$$\Lambda := \left\{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^n \setminus \{0\} \text{ with } \sum_{i=1}^{n} \xi_i A_i a = 0 \right\}. \tag{1.5}$$

Equivalently Λ characterizes the directions of one–dimensional oscillations compatible with (1.2). Given a cone Λ we say that a set S is Λ-convex if for any two points $A, B \in S$ with $B - A \in \Lambda$ the whole segment $[A, B]$ belongs to S. The Λ-convex hull of K, K^Λ, is the smallest Λ-convex set containing K. In some sense the Λ-convex hull K^Λ constitutes a relaxation of the initial set K. Then one defines *subsolutions* as solutions of the relaxed system, i.e. as solutions z of the linear relations (1.2) which satisfy the relaxed condition $z \in K^\Lambda$. Already

at this stage, the concept of subsolutions is reminiscent of the previously introduced concept of short maps for the isometric embedding problem. More precisely, equations (1.1) which define an isometric immersion u can be formulated for the deformation gradient $A := \nabla u$ as the coupling of the linear constraint

$$\operatorname{curl} A = 0$$

with the nonlinear relation

$$A^{\mathrm{T}} A = g.$$

With this interpretation, short maps are "subsolutions" to the isometric embedding problem.

The method of convex integration introduced by Gromov represents a generalization of Nash-Kuiper's result and is based on the upshot that (1.2)-(1.3) admit many interesting solutions if K^{Λ} is "big enough". Indeed, the key point of convex integration is to reintroduce oscillations by adding suitable localized versions of (1.4) to the subsolutions and to recover a solution of (1.2)-(1.3) iterating this process. The idea of adding oscillatory perturbations can be implemented either in an "implicite" way by the so called Baire category method or in a more constructive way. Both approaches provide the key to prove some h-principle-type results for systems of nonlinear evolutionary partial differential equations: they allow to show that, under suitable assumptions on the relaxed set K^{Λ}, the existence of subsolutions leads to the existence of solutions.

1.3.1.1. *Baire category method.* The Baire category method is a method of enforcing the idea of convex integration and relies on the surprising fact that, in a Baire generic sense, most solutions of the "relaxed system", i.e. most subsolutions, are actually solutions of the original system. Here, we recall the main steps underlying this approach following the "jargon" introduced by Kirchheim in [**Kir03**] (see also [**DLS11**]). In Kirchheim's formalisation, the space of subsolutions arises from a nontrivial open set $\mathcal{U} \subset \mathbb{R}^d$ ($\mathcal{U}$ plays the role of K^{Λ} in the previous section) satisfying the following perturbation property.

Perturbation Property (P): There is a continuous function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varepsilon(0) = 0$ with the following property: for every $z \in \mathcal{U}$ there is a sequence of solutions $z_j \in C_c^{\infty}(B_1)$ of (1.2) such that

- $z_j \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^n)$;
- $z + z_j(x) \in \mathcal{U}\ \forall x \in \mathbb{R}^n$;
- $\int |z_j(x)|^2\, dx \geq \varepsilon(dist(z, K))$.

The set $\mathcal{U}$ should be thought of as a relaxation of the initial set K, which, according to Kirchheim's jargon, "is stable only near K". Next, define X_0 as follows

$$X_0 := \{z \in C_c^\infty(\Omega) :\ z \text{ satisfies (1.2) and } z(x) \in \mathcal{U} \text{ for all } x \in \Omega\},$$

so that X_0 is the set of smooth compactly supported subsolutions of (1.2)-(1.3). Thanks to the *perturbation property*, X_0 consists of functions which are *perturbable* in an open subdomain $O \subset \Omega$. Then let X be the closure of X_0 with respect to the weak L^2-topology. Assuming that K is bounded, the set X_0 is bounded in L^2 and the topology of weak L^2 convergence is metrizable on X, making it into a complete space. Denote its metric by $d_X(\cdot, \cdot)$. An easy covering argument, together with property (P), results in the following lemma:

LEMMA 1.3.1. *There is a continuous function $\widetilde{\varepsilon} : \mathbb{R}^+ \to \mathbb{R}$ with $\widetilde{\varepsilon}(0) = 0$ such that for every $z \in X_0$ there is a sequence $z_j \in X_0$ with $z_j \overset{d_X}{\to} z$ in X and*

$$\int_\Omega |z_j - z|^2\, dx \geq \widetilde{\varepsilon}\left(\int_\Omega dist(z(x), K) dx\right).$$

Since the map $I : z \mapsto \int_\Omega |z|^2\, dx$ is a Baire 1-function on X, an easy application of the Baire category theorem gives that the set

$$Y := \{z \in X :\quad I \text{ is continuous at } z\}$$

is residual in X. By virtue of the previous lemma we can prove that $z \in Y$ implies $z(x) \in K$ for almost every $x \in \Omega$:

THEOREM 1.3.2. *Assuming the perturbation property to hold, the set*

$$Z := \{z \in X :\ z(x) \in K \ \text{a.e. } x \in \Omega\}$$

is residual in X.

Proof. In order to prove Theorem 1.3.2 it suffices to show that $Y \subset Z$. We will proceed by contradiction. So, let $z \in Y$ and let $z_j \in X_0$ such that $z_j \overset{d_X}{\to} z$ in X. Now, let us assume by absurd that $\int_\Omega \text{dist}(z(x), K) dx =: \delta > 0$. Thanks to lemma 1.3.1 we can pass (up

to a diagonal argument) to a new sequence $\widetilde{z}_j$ with $\widetilde{z}_j \overset{d_X}{\to} z$ and such that

$$\int_\Omega |z_j - \widetilde{z}_j|^2 \, dx \geq \widetilde{\varepsilon}\left(\int_\Omega dist(z_j(x), K)dx\right). \tag{1.6}$$

Since z is a point of continuity of I, it follows that $z_j \to z$ strongly in L^2 as well as $\widetilde{z}_j \to z$ strongly in L^2. This implies in particular that

$$\int_\Omega |z_j - \widetilde{z}_j|^2 \, dx \to 0 \quad \text{as } j \to +\infty. \tag{1.7}$$

Moreover the strong convergence in L^2 of the sequence z_j to z together with the hypothesis of absurd, allow us assume to that, from a certain $\overline{j}$ on, $\int_\Omega \operatorname{dist}(z_j(x), K)dx > \delta/2 > 0$ whence $\widetilde{\varepsilon}\big(\int_\Omega \operatorname{dist}(z_j(x), K)dx\big) > \alpha > 0$ for every $j > \overline{j}$ and for some α. This inequality together with (1.7) contradicts (1.6).

□

1.3.1.2. *Constructive convex integration.* In the previous section we presented the so called Baire category method, which is in some sense non contructive. However, the same idea of adding oscillatory perturbations can be implemented in a constructive way as well. In a nutshell the idea is to define a sequence of subsolutions $z_k \in K^\Lambda$ recursively as

$$z_{k+1}(x) \, = \, z_k(x) + Z_k(x, \lambda_k x), \tag{1.8}$$

where

$$Z_k(x, \xi)$$

is a periodic plane-wave (see (1.4)) solution of (1.2) in the variable ξ, parametrized by x and λ_k is a large frequency to be chosen. The aim is to choose the plane-wave Z_k and the frequency λ_k iteratively in such a way that

- z_k continues to satisfy (1.2) (strictly speaking this requires an additional corrector term in the scheme (1.8));
- z_k belongs to the relaxed constitutive set K^Λ;
- $z_k \to z$ in $L^2(\Omega)$ with $z \in K$ a.e..

The convergence of this constructive scheme is improved by choosing the frequencies λ_k higher and higher. On the other hand clearly any (fractional) derivative or Hölder norm of z_k gets worse by such a choice of λ_k. The best regularity corresponds to the slowest rate at which the frequencies λ_k tend to infinity while still leading to convergence.

1.3.2. Non–standard solutions in fluid dynamics.

1.3.2.1. *Incompressible Euler equations: non-uniqueness results.* The first and leading example of the h-principle in the realm of fluid dynamics is due to De Lellis and Székelyhidi and pertains to the incompressible Euler equations:

$$\begin{cases} \operatorname{div}_x v \;=\; 0 \\ \partial_t v + \operatorname{div}_x (v \otimes v) + \nabla_x p \;=\; 0 \\ v(\cdot, 0) \;=\; v^0 \end{cases} . \tag{1.9}$$

Here the unknowns v and p are, respectively, a vector field and a scalar field defined on $\mathbb{R}^n \times [0, T)$. These fundamental equations were derived 250 years ago by Euler and since then have played a major role in fluid dynamics. There are several outstanding problems connected to (1.9). In particular, weak solutions are known to be badly behaved in the sense of Hadamard's well-posedness: in the groundbreaking paper [**Sch93**] proved the existence of a nontrivial weak solution compactly supported in time. Thanks to the intuition of De Lellis and Székelyhidi such a nonuniqueness result has been explained as a suitable variant of the original h-principle by use of the method of convex integration. Moreover, such an approach allowed them to go way beyond the result of Scheffer and it has lead to new developments for several equations in fluid dynamics included the one presented in this work.

As already mentioned, the first nonuniqueness result for weak solutions of (1.9) is due to Scheffer in [**Sch93**]. The main theorem of [**Sch93**] states the existence of a nontrivial weak solution in $L^2(\mathbb{R}^2 \times \mathbb{R})$ with compact support in space and time. Later on Shnirelman in [**Shn97**] gave a different proof of the existence of a nontrivial weak solution in space-periodic setting and with compact support in time. In these constructions it is not clear wheter the solution belongs to the energy space. In the paper [**DLS09**], De Lellis and Székelyhidi provided a relatively simple proof of the following stronger statement.

Theorem 1.3.3 (Non-uniqueness of weak solutions to the incompressible Euler equations). *There exist infinitely many compactly supported weak solutions of the incompressible Euler equations* (1.9) *in any space dimension greater or equal to 2. In particular there are infinitely many solutions $v \in L^\infty \cap L^2$ to* (1.9) *for $v^0 = 0$ and arbitrary $n \geq 2$.*

The proof of Theorem 1.3.3 is based on the notion of subsolution. The spirit behind the notion of subsolutions in this context is the same

as the one outlined in the previous sections for general evolutionary partial differential equations. On the other hand, the definition of subsolution for the incompressible Euler system (1.9) can be made explicit and can be motivated in terms of the Reynold stress (see [**DLS11**] for more details on the connection between Reynold stress and subsolutions). In particular, if one writes (1.9) as the coupling of a linear system of PDEs and a pointwise non-linear constraint (as in (1.2)-(1.3)), then subsolutions are solutions of this linear system which belong pointwise to the convex hull of the non-linear constraint set. In other words:

DEFINITION 1.3.4 (Subsolution of incompressible Euler). *Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple*

$$(v, u, q) : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}^{n\times n}_0 \times \mathbb{R}$$

with the following properties:

- *$v \in L^2_{loc}$, $u \in L^1_{loc}$, q is a distribution;*
- $$\begin{cases} \operatorname{div}_x v &= 0 \\ \partial_t v + \operatorname{div}_x u + \nabla_x q &= 0 \end{cases} \tag{1.10}$$
  in the sense of distributions;
- $$v \otimes v - u \leq \frac{2}{n}\overline{e} Id \text{ a.e.}$$

Observe that subsolutions automatically satisfy $\frac{1}{2}|v|^2 \leq \overline{e}$ a.e. If in addition, the equality sign $\frac{1}{2}|v|^2 = \overline{e}$ a.e. holds true, then the v component of the subsolution is in fact a weak solution of the incompressible Euler equations. From the previous definition we can grasp even better the analogy between the velocity field of the Euler equations and the differential of the embedding in the isometric embedding problem and hence between subsolutions and short maps. Also the heuristic behind the two results shows striking similarities. The key point in De Lellis and Székelyhidi's approach to prove Theorem 1.3.3 is that, starting from a subsolution, an appropriate iteration process reintroduce the high frequencies oscillations. In the limit of this process one obtains weak solutions to (1.9). However, since the oscillations are reintroduced in a very non-unique way, in fact this generates *several* solutions from the same subsolution. The relevant iteration scheme has been already

outlined in the general setting in Section 1.3.1. The following theorem comes from Proposition 2 in [**DLS10**] and is a precise formulation of the previous discussion.

THEOREM 1.3.5 (Subsolution criterion). *Let $\overline{e} \in C(\mathbb{R}^n \times (0,T))$ and $(\overline{v}, \overline{u}, \overline{q})$ a smooth, strict subsolution, i.e.*

$$(\overline{v}, \overline{u}, \overline{q}) \in C^\infty(\mathbb{R}^n \times (0,T)) \text{ satisfies } (1.10)$$

and

$$v \otimes v - u < \frac{2}{n}\overline{e} Id \text{ a.e. on } \mathbb{R}^n \times (0,T). \tag{1.11}$$

Then there exist infinitely many weak solutions $v \in L^\infty_{loc}(\mathbb{R}^n \times (0,T))$ of the incompressible Euler equations (1.9) with pressure $p = \overline{q} - \frac{2}{n}\overline{e}$ and such that

$$\frac{1}{2}|v|^2 = \overline{e}$$

for a.e. (x,t). Infinitely many among these belong to $C((0,T), L^2)$. If in addition

$$\overline{v}(\cdot, t) \rightharpoonup v^0(\cdot) \text{ in } L^2_{loc}(\mathbb{R}^n) \text{ as } t \to 0,$$

then all the v's so constructed solve (1.9).

The previous results show that weak solutions of the Euler equations are in general highly non-unique. Moreover the kinetic energy density $\frac{1}{2}|v|^2$ can be prescribed as an independent quantity. Since classical C^1 solutions of the incompressible Euler equations are characterized by conservation of the total kinetic energy

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{|v|^2}{2}(x,t)dx = 0,$$

one can complement the notion of weak solution to (1.9) with several admissibility criteria defined as "relaxations" (in a proper sense) of the energy conservation. Let us denote by $L^2_w(\mathbb{R}^n)$ the space $L^2(\mathbb{R}^n)$ endowed with the weak topology. We recall that any weak solution of (1.9) can be modified on a set of measure zero so to get $v \in C([0,T), L^2_w(\mathbb{R}^n))$. Consequently v has a well-defined trace at every time. This allows to introduce the following admissibility criteria for weak solutions:

(a)

$$\int |v|^2(x,t)dx \leq \int |v_0|^2(x)dx \qquad \text{for every } t.$$

(b)

$$\int |v|^2 (x,t)dx \leq \int |v|^2 (x,s)dx \qquad \text{for every } t > s.$$

(c) If in addition $v \in L^3_{loc}$, then

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \leq 0$$

in the sense of distributions.

The first two criteria are of course suggested by the conservation of kinetic energy of classical solutions, while condition (c) has been proposed for the first time by Duchon and Robert in [**DR00**] and it resembles the admissibility criteria which are popular in the literature on hyperbolic conservation laws. However, none of these criteria restore the uniqueness of weak solutions.

THEOREM 1.3.6 (Non-uniqueness of admissible weak solutions to the incompressible Euler equations). *There exist initial data $v^0 \in L^\infty \cap L^2$ for which there are infinitely many bounded solutions of* (1.9) *which are strongly L^2-continuous and satisfy (a), (b) and (c).*

The conditions (a), (b) and (c) hold with the equality sign for infinitely many of these solutions, whereas for infinitely many other they hold as strict inequalities.

This theorem has been stated and proved in [**DLS10**]. The second part of the statement generalizes the intricate construction of Shnirelman in [**Shn97**], which produced the first example of a weak solution in 3d of (1.9) with strict inequalities in (a) and (b).

1.3.2.2. *Incompressible Euler equations: h-principle.* De Lellis and Székelyhidi in [**DLS12**] were able to extend the previous results: they devised a new iteration scheme which produces continuous and even Hölder continuous solutions on $\mathbb{T}^3$. Indeed, for the incompressible Euler equations, the natural space for convex integration is C^0. The method used in [**DLS09**] producing solutions in L^∞ was a weak form of convex integration. The new iteration scheme of [**DLS12**] is closer to the approach of Nash [**Nas54**] for the isometric embedding problem.

Recently, A. Choffrut [**Cho12**] established optimal h-principles in two and three space dimensions. More specifically he identifies all

subsolutions (defined in a suitable sense) which can be approximated in the H^1-norm by exact solutions of the incompressible Euler equations. For the precise statement we refer the reader directly to [**Cho12**].

1.3.2.3. *Incompressible Euler equations: Wild initial data.* The initial data v^0 constructed in Theorem 1.3.6 are obviously not regular, since for regular initial data the local existence theorems and the weak-strong uniqueness (see [**BDLS11**]) ensure local uniqueness under condition (a). One might therefore ask how large is the set of these "wild" initial data. A consequence of De Lellis and Székelyhidi's method is the following density theorem proved by Székelyhidi and Wiedemann in [**SW11**].

THEOREM 1.3.7 (Density of wild initial data). *The set of initial data v^0 for which the conclusions of Theorem 1.3.6 holds is dense in the space of L^2 solenoidal vectorfields.*

Another surprising corollary is that the usual shear flow is a "wild initial datum". More precisely, if we consider the following solenoidal vector field in $\mathbb{R}^2$

$$v^0(x) := \begin{cases} (1,0) & \text{if } x_2 > 0, \\ (-1,0) & \text{if } x_2 < 0, \end{cases} \tag{1.12}$$

then:

THEOREM 1.3.8 (Wild vortex sheet). *For v^0 as in* (1.12)*, there are infitely many weak solutions of* (1.9) *on $\mathbb{R}^2 \times [0,\infty)$ which satisfy (c).*

This theorem has been proven in [**Sz1**] using Proposition 1.3.5 and hence the proof amounts to showing the existence of a suitable subsolution. We will further discuss this result in Chapter 3.

1.3.2.4. *Incompressible Euler equations: global existence of weak solutions.* A further application of Theorem (1.3.5) is due to Wiedemann [**Wie11**]. E. Wiedemann considered an arbitrary initial datum $v^0 \in L^2_{loc}(\mathbb{R}^n)$ and constructed a smooth triple $(\overline{v}, \overline{u}, \overline{q}) \in C^\infty(\mathbb{R}^n \times (0,T))$ which solves (1.10) with initial datum v^0 and is a subsolution for a proper choice of the profile of $\overline{e}$. In particular, by constructing a subsolution with bounded energy, Wiedemann in [**Wie11**] recently obtained the following:

COROLLARY 1.3.9 (Global existence for weak solutions). *Let $v^0 \in L^2(\mathbb{T}^n)$ be a solenoidal vector field. Then there exist infinitely many global weak solutions of* (1.9) *with bounded energy.*

1.3.2.5. *Active scalar equations.* Active scalar equations are a class of systems of evolutionary partial differential equations in n space dimensions. The unknowns are the "active" scalar function θ and the velocity field v, which, for simplicity, is a divergence-free vector field. The equations are

$$\begin{cases} \operatorname{div}_x v \;=\; 0 \\ \partial_t \theta + v \cdot \nabla_x \theta \;=\; 0 \end{cases} \tag{1.13}$$

and v and θ are coupled by an integral operator, namely

$$v \;=\; T[\theta]. \tag{1.14}$$

Several systems of partial differential equations in fluid dynamics fall into this class. One can rewrite (1.13)-(1.14) in the spirit of Section 1.3.1, as a system of linear relations

$$\begin{cases} \operatorname{div}_x v \;=\; 0 \\ \partial_t \theta + \operatorname{div}_x q \;=\; 0 \\ v \;=\; T[\theta] \end{cases} \tag{1.15}$$

coupled with the nonlinear constraint

$$q \;=\; \theta v. \tag{1.16}$$

The initial value problem for the system (1.15)-(1.16) amounts to prescribing $\theta(x,0) \;=\; \theta_0(x)$. As described in Section 1.3.1 a key point is that the linear relations (1.15) admit a large set of plane wave solutions. Note that these linear relations are not strictly speaking of the form (1.2) and in order to define a suitable analogue of the plane waves in this setting, the linear operator T can be assumed to be translation invariant. Let $m(\xi)$ be its corresponding Fourier multiplier. Then one requires in addition that $m(\xi)$ is $0-$homogeneous so that (1.15) has the same scaling invariance as (1.2). Furthermore, the constraint $\operatorname{div}_x v \;=\; 0$ implies that $\xi \cdot m(\xi) \;=\; 0$. In spite of this restriction, several interesting equations fall into this category. Perhaps the best

known examples are the surface quasi geostrophic and the incompressible porous medium equations, corresponding respectively to

$$m(\xi) = i\,|\xi|^{-1}\,(-\xi_2, \xi_1) \quad \text{and} \tag{1.17}$$

$$m(\xi) = |\xi|^{-2}\,(\xi_1\xi_2, -\xi_1^2). \tag{1.18}$$

In [**CFG09**] Cordoba, Faraco and Gancedo proved the following theorem.

THEOREM 1.3.10. *Assume m is given by* (1.18). *Then there exist infinitely many weak solutions of* (1.15) *and* (1.16) *in* $L^\infty(\mathbb{T}^2\times[0,+\infty[)$ *with* $\theta_0 = 0$.

This was generalized by Shvydkoy to all even $m(\xi)$ satisfying a mild additional regularity assumption. We refer to the original paper [**Shv11**] for the details.

The proof of Theorem 1.3.10 in [**CFG09**], as well as the proof by Shvydkoy in [**Shv11**], relies on some refined tools which were developed in the theory of laminates and differential inclusions and they present some substantial differences with the methods of De Lellis and Székelyhidi in [**DLS09**] and [**DLS10**]. Indeed, the method used in [**CFG09**] is still based on understanding the equation as a differential inclusion in the spirit of Tartar [**Tar79**], but in the context of the porous media equation the situation differs from the setting of [**DLS09**]-[**DLS10**] and the authors had to take different routes in several steps. First, we would like to recall that a central point is to find an open set $\mathcal{U}$ satisfying the perturbation property (P). One possible candidate would be to take the largest open set $\mathcal{U}_{max}$ satisfying (P). Obviously this set is particularly meaningful since it gives the largest possible space X for which genericity conclusions holds. Moreover, this has the advantage that - at least in many relevant cases - the set $\mathcal{U}_{max}$ coincides with the interior of the Λ–convex hull K^Λ, which in turn can be characterized by separation arguments. For instance, in Theorem 1.3.5 condition (1.11) characterizes precisely the interior of K^Λ. Furthermore, in this case the interior of K^Λ is the interior of the convex hull K^{co}.

In [**CFG09**] and [**Shv11**] the authors avoid calculating the full Λ–convex hull and instead restrict themselves to exhibiting a nontrivial (but possibly much smaller) open set $\mathcal{U}$ satisfying (P). Opposite to

the incompressible Euler equations, in [**CFG09**] and [**Shv11**] the Λ–convex hull does not agree with the convex hull and more relevant $K \subset \partial K^{\Lambda}$. This is of course an obstruction for the available versions of convex integration as presented in Section 1.3.1 (the ones based on Baire category and the direct constructions). So the argument in [**CFG09**] and [**Shv11**] suggests a more systematic approach: instead of fixing a set and computing the hull, they pick a reasonable matrix A and compute $(A+\Lambda) \cap K$. Then by the results in [**Kir03**] it is enough to find a set $\widetilde{K} \subset (A+\Lambda) \cap K$ such that $A \in \widetilde{K}^{co}$ to find what are called degenerate T_4 configurations (see [**KMS03**]) supported in $\widetilde{K}^{co}$). However, in exchange they are forced to use much more complicated sequences z_j. Indeed, the z_j's used in [**DLS10**] are localizations of simple plane waves, whereas the ones used in [**CFG09**] and [**Shv11**] arise as an infinite nested sequence of repeated plane waves.

The obvious advantage of the method introduced in [**CFG09**] and used in [**Shv11**] is that it seems to be fairly robust and general. It is useful in cases where an explicit computation of the hull K^{Λ} is out of reach due to the high complexity and high dimensionality. Anyway, the constructions carried out in this dissertation are analogue to the ones by De Lellis and Székelyhidi and they do not require any understanding of the ideas by [**CFG09**] and [**Shv11**].

CHAPTER 2

Non–uniqueness with arbitrary density

In this chapter we present and prove the first result stated in the Introduction of the thesis, i.e. Theorem 0.2.1: given any continuosly differentiable initial density, we can construct bounded initial momenta for which admissible solutions to the isentropic compressible Euler equations are not unique in more than one space dimension.

The content of this chapter corresponds to the subject of the paper [**Chi11**] written by the author during the PhD studies. In particular the structure of the chapter is as follows. In the first section, we introduce the problem and the setting we will work with and we state the main result: even if the equations under investigation have already been presented in the Introduction of the thesis, we chose to recall them herein so that the chapter will be self-contained. Section 2 is an overview on the definitions of weak and admissible solutions and gives a first glimpse on how Theorem 0.2.1 is achieved. Section 3 is devoted to the reformulation of a simplified version of the isentropic compressible Euler equations as a differential inclusion and to the corresponding geometrical analysis. In Section 4 we state and prove a criterion (Proposition 2.4.1) to select initial momenta allowing for infinitely many solutions. The proof builds upon a refined version of the Baire category method for differential inclusions developed in [**DLS10**] and aimed at yielding weakly continuous in time solutions. Section 5 and 6 contain the proofs of the main tools used to prove Proposition 2.4.1. In Section 7, we show initial momenta satisfying the requirements of Proposition 2.4.1. Finally, in Section 8 we prove Theorem 0.2.1 (here stated in the first section as Theorem 2.1.1) by applying Proposition 2.4.1.

2.1. Introduction

We deal with the Cauchy Problem for the isentropic compressible Euler equations in the space-periodic setting.

We first introduce the isentropic compressible Euler equations of gas dynamics in n space dimensions, $n \geq 2$ (cf. Section 3.3 of [**Daf00**]). They are obtained as a simplification of the full compressible Euler equations, by assuming the entropy to be constant. The state of the gas will be described through the state vector

$$V = (\rho, m)$$

whose components are the density ρ and the linear momentum m. In contrast with the formulation of the problem given in the Introduction of the thesis, here the equations are written in terms of the linear momentum field which allows to write the equations in the *canonical* form (see [**Daf00**]). The balance laws in force are for mass and linear momentum. The resulting system, which consists of $n+1$ equations, takes the form (cf. (0.1)):

$$\text{(2.1)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}_x m \;=\; 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x [p(\rho)] \;=\; 0 \\ \rho(\cdot, 0) \;=\; \rho^0 \\ m(\cdot, 0) \;=\; m^0 \end{cases} .$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas in question. A common choice is the polytropic pressure law

$$p(\rho) = k\rho^\gamma$$

with constants $k > 0$ and $\gamma > 1$. The set of admissible values is $P = \{\rho > 0\}$ (cf. [**Daf00**] and [**Ser99**]). As already explained, the system is hyperbolic if

$$p'(\rho) > 0.$$

In addition, thermodynamically admissible processes must also satisfy an additional constraint coming from the energy inequality

$$\text{(2.2)} \quad \partial_t \left(\rho\varepsilon(\rho) + \frac{1}{2} \frac{|m|^2}{\rho} \right) + \operatorname{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2} \frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \;\leq\; 0$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2 \varepsilon'(r)$. The physical region for (2.1) is $\{(\rho, m) | \, |m| \leq R\rho\}$, for some constant $R > 0$. For $\rho > 0$, $v = m/\rho$ represents the velocity of the fluid.

We will consider, from now on, the case of general pressure laws given by a function p on $[0,\infty[$, that we always assume to be continuously differentiable on $[0,\infty[$. The crucial requirement we impose upon p is that it has to be strictly increasing on $[0,\infty[$. Such a condition is meaningful from a physical viewpoint since it is a consequence of the principles of thermodynamics.

Using some techniques introduced by De Lellis-Székelyhidi (cf. [**DLS09**] and [**DLS10**]) we can consider any continuously differentiable periodic initial density ρ^0 and exhibit suitable periodic initial momenta m^0 for which space-periodic weak admissible solutions of (2.1) are not unique on some finite time-interval.

THEOREM 2.1.1. *Let $n \geq 2$. Then, for any given function p and any given continuously differentiable periodic initial density ρ^0, there exist a bounded periodic initial momentum m^0 and a positive time $\overline{T}$ for which there are infinitely many space-periodic admissible solutions (ρ, m) of (2.1) on $\mathbb{R}^n \times [0,\overline{T}[$ with $\rho \in C^1(\mathbb{R}^n \times [0,\overline{T}[)$.*

REMARK 2.1.2. *Indeed, in order to prove Theorem 2.1.1, it would be enough to assume that the initial density is a Hölder continuous periodic function: $\rho^0 \in C^{0,\alpha}(\mathbb{R}^n)$ (cf. Proof of Proposition 2.7.1).*

Some connected results are obtained in [**DLS10**] (cf. Theorem 2 therein) as a further consequence of their analysis on the incompressible Euler equations. Inspired by their approach, we adapt and apply directly to (2.1) the method of convex integration combined with Tartar's programme on oscillation phenomena in conservation laws (see [**Tar79**] and [**KMS03**]). In this way, we can show failure of uniqueness of admissible solutions to the compressible Euler equations starting from any given continuously differentiable initial density.

2.2. Weak and admissible solutions to the isentropic Euler system

The deceivingly simple-looking system of first-order partial differential equations (2.1) has a long history of important contributions over more than two centuries. We recall a few classical facts on this system (see for instance [**Daf00**] for more details).

- If ρ^0 and m^0 are "smooth" enough (see Theorem 5.3.1 in [**Daf00**]), there exists a maximal time interval $[0,T[$ on which

there exists a unique "smooth" solution (ρ, m) of (2.1) (for $0 \leq t < T$). In addition, if $T < \infty$, and this is the case in general, (ρ, m) becomes discontinuous as t goes to T.

- If we allow for discontinuous solutions, i.e., for instance, solutions $(\rho, m) \in L^\infty$ satisfying (2.1) in the sense of distributions, then solutions are neither unique nor stable. More precisely, one can exhibit sequences of such solutions which converge weakly in $L^\infty - *$ to functions which do not satisfy (2.1).
- In order to restore the stability of solutions and (possibly) the uniqueness, one may and should impose further restrictions on bounded solutions of (2.1), restrictions which are known as (Lax) entropy inequalities.

In this chapter we address the problem of better understanding the efficiency of entropy inequalities as selection criteria among weak solutions.

Here, we have chosen to emphasize the case of the flow with *space periodic* boundary conditions. For *space periodic* flows we assume that the fluid fills the entire space $\mathbb{R}^n$ but with the condition that m, ρ are periodic functions of the space variable. The space periodic case is not a physically achievable one, but it is relevant on the physical side as a model for some flows. On the mathematical side, it retains the complexities due to the nonlinear terms (introduced by the kinematics) and therefore it includes many of the difficulties encountered in the general case. However the former is simpler to treat because of the absence of boundaries. Furthermore, using Fourier transform as a tool simplifies the analysis.

Let $Q = [0,1]^n$, $n \geq 2$ be the unit cube in $\mathbb{R}^n$. We denote by $H^m_p(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H^m_{loc}(\mathbb{R}^n)$ and which are periodic with period Q:

$$m(x+l) = m(x) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and every } l \in \mathbb{Z}^n.$$

For $m = 0$, $H^0_p(Q)$ coincides simply with $L^2(Q)$. Analogously, for every functional space X we define $X_p(Q)$ to be the space of functions which are locally (over $\mathbb{R}^n$) in X and are periodic of period Q. The functions in $H^m_p(Q)$ are easily characterized by their Fourier series expansion

(2.3)

$$H^m_p(Q) = \left\{ m \in L^2_p(Q) : \sum_{k \in \mathbb{Z}^n} |k|^{2m} \left|\widehat{m}(k)\right|^2 < \infty \text{ and } \widehat{m}(0) = 0 \right\},$$

where $\widehat{m} : \mathbb{Z}^n \to \mathbb{C}^n$ denotes the Fourier transform of m. We will use the notation $H(Q)$ for $H^0_p(Q)$ and $H_w(Q)$ for the space $H(Q)$ endowed with the weak L^2 topology.

Let T be a fixed positive time. By a *weak solution* of (2.1) on $\mathbb{R}^n \times [0,T[$ we mean a pair $(\rho, m) \in L^\infty([0,T[; L^\infty_p(Q))$ satisfying

$$|m(x,t)| \leq R\rho(x,t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0, \tag{2.4}$$

and such that the following identities hold for every test functions $\psi \in C^\infty_c([0,T[; C^\infty_p(Q))$, $\phi \in C^\infty_c([0,T[; C^\infty_p(Q))$:

$$\int_0^T \int_Q \left[\rho \partial_t \psi + m \cdot \nabla_x \psi\right] dxdt + \int_Q \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{2.5}$$

$$\int_0^T \int_Q \left[m \cdot \partial_t \phi + \left\langle \frac{m \otimes m}{\rho}, \nabla_x \phi \right\rangle + p(\rho)\, \mathrm{div}_x\, \phi \right] dxdt$$

$$+ \int_Q m^0(x) \cdot \phi(x,0)dx \;=\; 0. \tag{2.6}$$

For $n \geq 2$ the only non-trivial entropy is the total energy $\eta = \rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}$ which corresponds to the flux $\Psi = \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right) m$.

Then a bounded weak solution (ρ, m) of (2.1) satisfying (3.21) in the sense of distributions, i.e. satisfying the following inequality

$$\int_0^T \int_Q \left[\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) \partial_t \varphi + \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \cdot \nabla_x \varphi \right]$$

$$+ \int_Q \left(\rho^0 \varepsilon(\rho^0) + \frac{1}{2}\frac{|m^0|^2}{\rho} \right) \varphi(\cdot, 0) \;\geq\; 0, \tag{2.7}$$

for every nonnegative $\varphi \in C^\infty_c([0,T[; C^\infty_p(Q))$, is said to be an *entropy* (or *admissible*) solution of (2.1).

The lack of entropies is one of the essential reasons for a very limited understanding of compressible Euler equations in dimensions greater than or equal to 2.

The recent paper [**DLS10**] by De Lellis-Székelyhidi gives an example in favour of the conjecture that entropy solutions to the multi-dimensional compressible Euler equations are in general not unique: see Theorem 0.1.1 in the Introduction of the thesis. Showing that this conjecture is true has far-reaching consequences. The entropy condition

is not sufficient as a selection principle for physical/unique solutions. The non-uniqueness result by De Lellis-Székelyhidi is a byproduct of their new analysis of the incompressible Euler equations based on its formulation as a differential inclusion. They first show that, for some bounded compactly supported initial data, none of the classical admissibility criteria singles out a unique solution to the Cauchy problem for the incompressible Euler equations. As a consequence, by constructing a piecewise constant in space and independent of time density ρ, they look at the compressible isentropic system as a "piecewise incompressible" system (i.e. still incompressible in the support of the velocity field) and thereby exploit the result for the incompressible Euler equations to exhibit bounded initial density and bounded compactly supported initial momenta for which admissible solutions of (2.1) are not unique (in more than one space dimension).

Inspired by their techniques, we give a further counterexample to the well-posedeness of entropy solutions to (2.1). Our result differs in two main aspects: here the initial density can be any given "regular" function and remains "regular" forward in time while in [**DLS10**] the density allowing for infinitely many admissible solutions must be chosen as piecewise constant in space; on the other hand we are not able to deal with compactly supported momenta (indeed we work in the periodic setting), hence our non-unique entropy solutions are only locally L^2 in contrast with the global-L^2-in-space property of solutions obtained in [**DLS10**]. Moreover, we have chosen to study the case of the flow in a cube of $\mathbb{R}^n$ with *space periodic* boundary conditions. This case leads to many technical simplifications while retaining the main structure of the problem.

More precisely, we are able to analyze the compressible Euler equations in the framework of convex integration introduced in Chapter 1. We recall that this method works well with systems of nonlinear PDEs such that the convex envelope (in an appropriate sense) of each small domain of the submanifold representing the PDE in the jet-space (see [**EM02**] for more details) is big enough. In our case, we consider a simplification of system (2.1), namely the semi-stationary associated problem, whose submanifold allows a convex integration approach leading us to recover the result of Theorem 2.1.1.

We are interested in the semi-stationary Cauchy problem associated with the isentropic Euler equations (simply set to 0 the time derivative

of the density in (2.1) and drop the initial condition for ρ):

$$\begin{cases} \operatorname{div}_x m \;=\; 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m\otimes m}{\rho}\right) + \nabla_x[p(\rho)] \;=\; 0 \\ m(\cdot,0) \;=\; m^0. \end{cases} \tag{2.8}$$

A pair $(\rho, m) \in L^\infty_p(Q) \times L^\infty([0,T[; L^\infty_p(Q))$ is a *weak solution* on $\mathbb{R}^n \times [0,T[$ of (2.8) if $m(\cdot,t)$ is weakly-divergence free for almost every $0 < t < T$ and satisfies the following bound

(2.9)

$$|m(x,t)| \le R\rho(x) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0,$$

and if the following identity holds for every $\phi \in C^\infty_c([0,T[; C^\infty_p(Q))$:

$$\begin{aligned} &\int_0^T \int_Q \left[m \cdot \partial_t \phi + \left\langle \frac{m\otimes m}{\rho}, \nabla_x \phi \right\rangle + p(\rho) \operatorname{div}_x \phi \right] dxdt \\ &+ \int_Q m^0(x) \cdot \phi(x,0) dx \;=\; 0. \end{aligned} \tag{2.10}$$

A general observation suggests us that a non-uniqueness result for weak solutions of (2.8) whose momentum's magnitude satisfies some suitable constraint could lead us to a non-uniqueness result for entropy solutions of the isentropic Euler equations (2.1). Indeed, the entropy solutions we construct in Theorem 2.1.1 come from *some* weak solutions of (2.8).

THEOREM 2.2.1. *Let $n \ge 2$. Then, for any given function p, any given density $\rho_0 \in C^1_p(Q)$ and any given finite positive time T, there exists a bounded initial momentum m^0 for which there are infinitely many weak solutions $(\rho, m) \in C^1_p(Q) \times C([0,T]; H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \rho_0(x)$.*
In particular, the obtained weak solutions m satisfy

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0,T[, \tag{2.11}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n, \tag{2.12}$$

for some smooth function χ.

An easy computation shows how, by properly choosing the function χ in (2.11)-(2.12), the solutions (ρ_0, m) of (2.8) obtained in Theorem 2.2.1 satisfy the admissibility condition (3.26).

THEOREM 2.2.2. *Under the same assumptions of Theorem 2.2.1, there exists a maximal time* $\overline{T} > 0$ *such that the weak solutions* (ρ, m) *of* (2.8) *(coming from Theorem 2.2.1) satisfy the admissibility condition* (3.26) *on* $[0, \overline{T}[$.

Our construction yields initial data m^0 for which the nonuniqueness result of Theorem 2.1.1 holds on any time interval $[0, T[$, with $T \leq \overline{T}$. However, as pointed out before, for sufficiently regular initial data, classical results give the local uniqueness of smooth solutions. Thus, *a fortiori*, the initial momenta considered in our examples have necessarily a certain degree of irregularity.

2.3. Geometrical analysis

This section is devoted to a qualitative analysis of the isentropic compressible Euler equations in a semi-stationary regime (i.e. (2.8)).

As in [**DLS09**] we will interpret the system (2.8) in terms of a differential inclusion, so that it can be studied in the framework combining the plane wave analysis of Tartar, the convex integration of Gromov and the Baire's arguments. For a complete description of this general framework we refer to Chapter 1, Section 1.3.1. In Section 2.3.2 we will recall only the tools needed for our construction.

2.3.1. Differential inclusion. The system (2.8) can indeed be naturally expressed as a linear system of partial differential equations coupled with a pointwise nonlinear constraint, usually called *differential inclusion*.

The following Lemma, based on Lemma 2 in [**DLS10**], gives such a reformulation. We will denote by $\mathcal{S}^n$ the space of symmetric $n \times n$ matrices, by $\mathcal{S}^n_0$ the subspace of $\mathcal{S}^n$ of matrices with null trace, and by I_n the $n \times n$ identity matrix.

LEMMA 2.3.1. *Let* $m \in L^\infty([0,T]; L^\infty_p(Q; \mathbb{R}^n))$, $U \in L^\infty([0,T]; L^\infty_p(Q; \mathcal{S}^n_0))$ *and* $q \in L^\infty([0,T]; L^\infty(Q; \mathbb{R}^+))$ *such that*

$$\begin{aligned} \operatorname{div}_x m &= 0 \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0. \end{aligned} \tag{2.13}$$

If (m,U,q) *solve* (2.13) *and in addition there exists* $\rho \in L^\infty_p(\mathbb{R}^n;\mathbb{R}^+)$ *such that* (5.7) *holds and*

$$U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n \quad \text{a.e. in } \mathbb{R}^n \times [0,T],$$

$$q = p(\rho) + \frac{|m|^2}{n\rho} \quad \text{a.e. in } \mathbb{R}^n \times [0,T], \tag{2.14}$$

then m *and* ρ *solve* (2.8) *distributionally. Conversely, if* m *and* ρ *are weak solutions of* (2.8)*, then* m*,* $U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$ *and* $q = p(\rho) + \frac{|m|^2}{n\rho}$ *solve* (2.13)-(2.14)*.*

In Lemma 2.3.1 we made clear the distinction between the augmented system (2.13), whose linearity allows a plane wave analysis, and the nonlinear pointwise constraint (2.14), which leads us to study the graph below.

For any given $\rho \in]0,\infty[$, we define the following graph

$$K_\rho := \left\{ (m,U,q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n, \right.$$

$$\left. q = p(\rho) + \frac{|m|^2}{n\rho} \right\}. \tag{2.15}$$

The key of the forthcoming analysis is the behaviour of the graph K_ρ with respect to the wave vectors associated with the linear system (2.13): are differential and algebraic constraints in some sense compatible?

For our purposes, it is convenient to consider "slices" of the graph K_ρ, by considering vectors m whose modulus is subject to some ρ-depending condition. Thus, for any given $\chi \in \mathbb{R}^+$, we define:

$$K_{\rho,\chi} := \left\{ (m,U,q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n, \right.$$

$$\left. q = p(\rho) + \frac{|m|^2}{n\rho}, |m|^2 = \rho\chi \right\}. \tag{2.16}$$

2.3.2. Wave cone. For the sake of completeness, we remind the notions of *planewave* solutions and *wave cone*, previously introduced in

Section 1.3.1. According to Tartar's framework [**Tar79**], we consider a system of first order linear PDEs (see (1.2))

$$\sum_i A_i \partial_i z = 0 \tag{2.17}$$

where z is a vector valued function and the A_i are matrices. Then, *planewave* solutions to (2.17) are solutions of the form

$$z(x) = a h(x \cdot \xi), \tag{2.18}$$

with $h : \mathbb{R} \to \mathbb{R}$. In order to find such solutions, we have to solve the relation $\sum_i \xi_i A_i a = 0$, where ξ_i is the oscillation frequency in the direction i. The set of directions a for which a solution $\xi \neq 0$ exists is called *wave cone* Λ of the system (2.17): equivalently Λ characterizes the directions of one dimensional oscillations compatible with (2.17).

The system (2.13) can be analyzed in this framework. Consider the $(n+1) \times (n+1)$ symmetric matrix in block form

$$M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{2.19}$$

Note that, with the new coordinates $y = (x, t) \in \mathbb{R}^{n+1}$, the system (2.13) can be easily rewritten as $\operatorname{div}_y M = 0$ (the divergence of M in space-time is zero). Thus, the wave cone associated with the system (2.13) is equal to

$$\Lambda = \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : \det \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix} = 0 \right\}. \tag{2.20}$$

Indeed, the relation $\sum_i \xi_i A_i a = 0$ for the system (2.13) reads simply as $M \cdot (\xi, c) = 0$, where $(\xi, c) \in \mathbb{R}^n \times \mathbb{R}$ (ξ is the space-frequency and c the time-frequency): this equation admits a non-trivial solution if M has null determinant, hence (2.20).

2.3.3. Convex hull and geometric setup. Since it will be of great importance in this chapter, we formulate once more the definition of Λ-convex hull already given in Section 1.3.1.

Given a cone Λ, we say that a set S is convex with respect to Λ (or Λ-convex) if, for any two points $A, B \in S$ with $B - A \in \Lambda$, the whole segment $[A, B]$ belongs to S. The Λ-convex hull of $K_{\rho,\chi}$ is the smallest Λ-convex set $K^{\Lambda}_{\rho,\chi}$ containing $K_{\rho,\chi}$, i.e. the set of states obtained by mixture of states of $K_{\rho,\chi}$ through oscillations in Λ-directions (Gromov [**Gro86**], who works in the more general setting of jet bundles, calls

this the P- convex hull). The key point in Gromov's method of convex integration (which is a far reaching generalization of the work of Nash [**Nas54**] and Kuiper [**Kui95**] on isometric immersions) is that (2.17) coupled with a pointwise nonlinear constraint of the form $z \in K$ a.e. admits many interesting solutions provided that the Λ-convex hull of K, K^{Λ}, is sufficiently large. In applications to elliptic and parabolic systems we always have $K^{\Lambda} = K$ so that Gromov's approach does not directly apply. For other applications to partial differential equations it turns out that one can work with the Λ-convex hull defined by duality. More precisely, a point does not belong to the Λ- convex hull defined by duality if and only if there exists a Λ-convex function which separates it from K. A crucial fact is that the second notion is much weaker. This surprising fact is illustrated in [**KMS03**].

In our case, the wave cone is quite large, therefore it is sufficient to consider the stronger notion of Λ-convex hull, indeed it coincides with the whole convex hull of $K_{\rho,\chi}$.

LEMMA 2.3.2. *For any $S \in \mathcal{S}^n$ let $\lambda_{max}(S)$ denote the largest eigenvalue of S. For $(\rho, m, U) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n_0$ let*

$$e(\rho, m, U) := \lambda_{max}\left(\frac{m \otimes m}{\rho} - U\right). \tag{2.21}$$

Then, for any given $\rho, \chi \in \mathbb{R}^+$, the following holds

(i) *$e(\rho, \cdot, \cdot) : \mathbb{R}^n \times \mathcal{S}^n_0 \to \mathbb{R}$ is convex;*

(ii) *$\frac{|m|^2}{n\rho} \leq e(\rho, m, U)$, with equality if and only if $U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$;*

(iii) *$|U|_\infty \leq (n-1)e(\rho, m, U)$, with $|U|_\infty$ being the operator norm of the matrix;*

(iv) *the $\frac{\chi}{n}$-sublevel set of e defines the convex hull of $K_{\rho,\chi}$, i.e.*

$$K^{co}_{\rho,\chi} = \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : e(\rho, m, U) \leq \frac{\chi}{n}, \right.$$

$$\left. q = p(\rho) + \frac{\chi}{n} \right\} \tag{2.22}$$

and $K_{\rho,\chi} = K^{co}_{\rho,\chi} \cap \{|m|^2 = \rho\chi\}$.

For the proof of (i)-(iv) we refer the reader to the proof of Lemma 3.2 in [**DLS10**]: the arguments there can be easily adapted to our case.

We observe that, for any $\rho, \chi \in \mathbb{R}^+$, the convex hull $K^{co}_{\rho,\chi}$ lives in the hyperplane H of $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ defined by $H := \left\{(m,U,q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : q = p(\rho) + \frac{\chi}{n}\right\}$. Therefore, the interior of $K^{co}_{\rho,\chi}$ as a subset of $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ is empty. This seems to prevent us from working in the classical framework of convex integration, but we can overcome this apparent obstacle.

For any $\rho, \chi \in \mathbb{R}^+$, we define the *hyperinterior* of $K^{co}_{\rho,\chi}$, and we denote it with "hint $K^{co}_{\rho,\chi}$", as the following set

$$\text{hint } K^{co}_{\rho,\chi} := \left\{ (m,U,q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : e(\rho,m,U) < \frac{\chi}{n}, \right.$$

$$\left. q = p(\rho) + \frac{\chi}{n} \right\}. \tag{2.23}$$

In the framework of convex integration, the larger the Λ-convex hull of $K_{\rho,\chi}$ is, the bigger the breathing space will be. How to "quantify" the meaning of a "large" Λ-convex hull in our context? The previous definition provides an answer: the Λ-convex hull of $K_{\rho,\chi}$ will be "large" if its hyperinterior is nonempty. The wave cone of the semi-stationary Euler isentropic system is wide enough to ensure that the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$ and has a nonempty hyperinterior. As a consequence, we can construct irregular solutions oscillating along any fixed direction. For our purposes, it will be convenient to restrict to some *special directions* in Λ, consisting of matrices of rank 2, which are not stationary in time, but are associated with a constant pressure.

LEMMA 2.3.3. *Let $c, d \in \mathbb{R}^n$ with $|c| = |d|$ and $c \neq d$, and let $\rho \in \mathbb{R}^+$.*
Then $\left(c-d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$.

Proof. Since the vector $\left(c+d, -\left(\frac{|c|^2 + c\cdot d}{\rho}\right)\right)$ is in the kernel of the matrix

$$C = \begin{pmatrix} \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho} & c-d \\ c-d & 0 \end{pmatrix},$$

C has indeed determinant zero, hence $\left(c-d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$. $\square$

Now, we introduce some important tools: they allow us to prove that $K^{\Lambda}_{\rho,\chi} = K^{co}_{\rho,\chi}$ is sufficiently large, thus providing us room to find many solutions for (2.13)-(2.14).

As first, we define the *admissible segments* as segments in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ whose directions belong to the wave cone Λ for the linear system of PDEs (2.13) and are indeed *special directions* in the sense specified by Lemma 2.3.3.

DEFINITION 2.3.4. *Given $\rho, \chi \in \mathbb{R}^+$ we call σ an admissible segment for (ρ, χ) if σ is a line segment in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ satisfying the following conditions:*

- *σ is contained in the hyperinterior of $K^{co}_{\rho,\chi}$;*
- *σ is parallel to $\left(c-d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right)$ for some $c, d \in \mathbb{R}^n$ with $|c|^2 = |d|^2 = \rho\chi$ and $c \neq \pm d$.*

The admissible segments defined above correspond to suitable plane-wave solutions of (2.13). The following Lemma ensures that, for any $\rho, \chi \in \mathbb{R}^+$, the hyperinterior of $K^{co}_{\rho,\chi}$ is " sufficiently round " with respect to the special directions: given any point in the hyperinterior of $K^{co}_{\rho,\chi}$, it can be seen as the midpoint of a sufficiently large admissible segment for (ρ, χ).

LEMMA 2.3.5. *There exists a constant $F = F(n) > 0$ such that for any $\rho, \chi \in \mathbb{R}^+$ and for any $z = (m, U, q) \in$ hint $K^{co}_{\rho,\chi}$ there exists an admissible line segment for (ρ, χ)*

$$\sigma = \left[(m, U, q) - (\overline{m}, \overline{U}, 0), (m, U, q) + (\overline{m}, \overline{U}, 0)\right] \tag{2.24}$$

such that

$$|\overline{m}| \geq \frac{F}{\sqrt{\rho\chi}} \left(\rho\chi - |m|^2\right).$$

The proof rests on a clever application of Carathéodory's theorem for convex sets and can be carried out, with minor modifications, as in [**DLS10**] (cf. Lemma 6 therein).

As an easy consequence of the previous Lemma, we can finally establish that the Λ-convex hull of $K_{\rho,\chi}$ coincides with $K^{co}_{\rho,\chi}$.

PROPOSITION 2.3.6. *For all given $\rho, \chi \in \mathbb{R}^+$, the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$.*

Proof. Recall that, given $\rho, \chi \in \mathbb{R}^+$, we denote the Λ-convex hull of $K_{\rho,\chi}$ with $K^{\Lambda}_{\rho,\chi}$. Of course $K^{\Lambda}_{\rho,\chi} \subset K^{co}_{\rho,\chi}$, hence we have to prove the opposite inclusion, i.e. $K^{co}_{\rho,\chi} \subset K^{\Lambda}_{\rho,\chi}$. For every $z \in K^{co}_{\rho,\chi}$ we can follow the procedure in the proof of Lemma 2.3.5 (cf. [**DLS10**]) and write it as $z = \sum_j \lambda_j z_j$, with $(z_j)_{1\le j\le N+1}$ in $K_{\rho,\chi}$, $(\lambda_j)_{1\le j\le N+1}$ in $[0,1]$ and $\sum_j \lambda_j = 1$. Again, we can assume that $\lambda_1 = \max_j \lambda_j$. In case $\lambda_1 = 1$ then $z = z_1 \in K_{\rho,\chi} \subset K^{\Lambda}_{\rho,\chi}$ and we can already conclude. Otherwise (i.e. when $\lambda_1 \in (0,1)$) we can argue as in Lemma 2.3.5 so to find an admissible segment σ for (ρ,χ) of the form (2.24). Since we aim at writing z as a Λ-barycenter of elements of $K_{\rho,\chi}$, we "play" with these admissible segments by prolongations and iterative constructions until we get segments with extremes lying in $K_{\rho,\chi}$. More precisely: we extend the segment σ until we meet $\partial\, \mathrm{hint} K^{co}_{\rho,\chi}$ thus obtaining z as the barycenter of two points (w_0, w_1) with $(w_0 - w_1) \in \Lambda$ and such that every $w_i = (m_i, U_i, q_i)$, $i = 0, 1$, satisfies either $|m_i|^2 = \rho\chi$ or $|m_i|^2 < \rho\chi$ and $e(\rho, m_i, U_i) = \chi/n$.

In the first case, $U_i - \left(\frac{m_i \otimes m_i}{\rho} - \frac{|m_i|^2}{n\rho} I_n\right) \ge 0$, and since it is a null-trace-matrix it is identically zero, whence $w_i \in K_{\rho,\chi}$ $\Big($note that in the construction of σ the q-direction remains unchanged, hence $q_i = p(\rho) + \frac{\chi}{n}\Big)$.

In the second case, i.e. when $|m_i|^2 < \rho\chi$ and $e(\rho, m_i, U_i) = \chi/n$, we apply again Lemma 2.3.5 and a limit procedure to express w_i as barycentre of $(w_{i,0}, w_{i,1})$ with $(w_{i,0} - w_{i,1}) \in \Lambda$ and such that every $w_{i,k} = (m_{i,k}, U_{i,k}, q_{i,k})$, $k = 0, 1$, will satisfy either $|m_{i,k}|^2 = \rho\chi$ or $\lambda_2(\rho, m_{i,k}, U_{i,k}) = e(\rho, m_{i,k}, U_{i,k}) = \chi/n$, where $\lambda_1(\rho, m, U) \ge \lambda_2(\rho, m, U) \ge$ ..... $\ge \lambda_n(\rho, m, U)$ denote the ordered eigenvalues of the matrix $\frac{m\otimes m}{\rho} - U$ (note that $\lambda_1(\rho, m, U) = e(\rho, m, U)$). Now, we iterate this procedure of constructing suitable admissible segments for (ρ,χ) until we have written z as Λ-barycenter of points (m, U, q) satisfying either $|m|^2 = \rho\chi$ or $\lambda_n(\rho, m, U) = \chi/n$ and therefore all belonging to $K_{\rho,\chi}$ as desired. □

2.4. A criterion for the existence of infinitely many solutions

The following Proposition provides a criterion to recognize initial data m^0 which allow for many weak admissible solutions to (2.1). Its proof relies deeply on the geometrical analysis carried out in Section 2.3. The underlying idea comes from convex integration. The general

principle of this method, developed for partial differential equations by Gromov [**Gro86**] and for ordinary differential equations by Filippov [**Fil67**], consists in the following steps (cf. with Section 1.3.1.2): given a nonlinear equation $\mathcal{E}(z)$,

- (i) we rewrite it as $(\mathcal{L}(z) \wedge z \in K)$ where $\mathcal{L}$ is a linear equation;
- (ii) we introduce a strict subsolution z_0 of the system, i.e. satisfying a relaxed system $(\mathcal{L}(z_0) \wedge z_0 \in \mathcal{U})$;
- (iii) we construct a sequence $(z_k)_{k\in\mathbb{N}}$ approaching K but staying in $\mathcal{U}$;
- (iv) we pass to the limit, possibly modifying the sequence (z_k) in order to ensure a suitable convergence.

Step (i) has already been done in Section 2.1. The choice of z_0 will be specified in Sections 2.7-2.8. Here, we define the notion of subsolution for an appropriate set $\mathcal{U}$, we construct an improving sequence and we pass to the limit. The way how we construct the approximating sequence will be described in Section 2.6 using some tools from Section 2.5.

One crucial step in convex integration is the passage from open sets K to general sets. This can be done in different ways, e.g. by the Baire category theorem (cf. [**Oxt90**]), a refinement of it using Baire-1 functions or the Banach-Mazur game [**Kir03**] or by direct construction [**Syc01**]. Whatever approach one uses the basic theme is the same: at each step of the construction one adds a highly oscillatory correction whose frequency is much larger and whose amplitude is much smaller than those of the previous corrections.

In this section, we achieve our goals following some Baire category arguments as in [**DLS09**]: they are morally close to the methods developed by Bressan and Flores in [**BF94**] and by Kirchheim in [**Kir03**] (see Section 1.3.1.1).

In our framework the initial data will be constructed starting from solutions to the convexified (or relaxed) problem associated to (2.8), i.e. solutions to the linearized system (2.13) satisfying a "relaxed" nonlinear constraint (2.14) (i.e. belonging to the hyperinterior of the convex hull of the "constraint set"), which we will call *subsolutions*.

As in [**DLS09**], our application shows that the Baire theory is comparable in terms of results to the method of convex integration and they

have many similarities: they are both based on an approximation approach to tackle problems while the difference lies only in the limit arguments, i.e. on the way the exact solution is obtained from better and better approximate ones. These similarities are clarified by Kirchheim in [**Kir03**], where the continuity points of a first category Baire function are considered; a comparison between the two methods is drawn by Sychev in [**Syc01**].

Here, the topological reasoning of Baire theory is preferred to the iteration technique of convex integration, since the first has the advantage to provide us directly with infinitely many different solutions.

PROPOSITION 2.4.1. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function and let T be any finite positive time. Assume there exist (m_0, U_0, q_0) continuous space-periodic solutions of (2.13) on $\mathbb{R}^n\times]0,T[$ with*

$$m_0 \in C([0,T];H_w(Q)), \tag{2.25}$$

and a function $\chi \in C^\infty([0,T];\mathbb{R}^+)$ such that

(2.26)
$$e(\rho_0(x), m_0(x,t), U_0(x,t)) < \frac{\chi(t)}{n} \qquad \text{for all } (x,t) \in \mathbb{R}^n\times]0,T[,$$

(2.27)
$$q_0(x,t) = p(\rho_0(x)) + \frac{\chi(t)}{n} \qquad \text{for all } (x,t) \in \mathbb{R}^n\times]0,T[.$$

Then there exist infinitely many weak solutions (ρ, m) of the system (2.8) in $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \rho_0(x)$ and such that

$$m \in C([0,T];H_w(Q)), \tag{2.28}$$

$$m(\cdot,t) = m_0(\cdot,t) \qquad \text{for } t = 0, T \text{ and for a.e. } x \in \mathbb{R}^n, \tag{2.29}$$

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n\times]0,T[. \tag{2.30}$$

2.4.1. The space of subsolutions. We define the space of *subsolutions* as follows. Let ρ_0 and χ be given as in the assumptions of Proposition 2.4.1. Let m_0 be a vector field as in Proposition 2.4.1 with associated modified pressure q_0 and consider space-periodic momentum fields $m : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ which satisfy

$$\operatorname{div} m = 0, \tag{2.31}$$

the initial and boundary conditions

$$m(x,0) = m_0(x,0), \tag{2.32}$$

$$m(x,T) = m_0(x,T), \tag{2.33}$$

(2.34)

and such that there exists a continuous space-periodic matrix field $U : \mathbb{R}^n \times]0,T[\to \mathcal{S}_0^n$ with

$$\begin{aligned} &e(\rho_0(x), m(x,t), U(x,t)) < \frac{\chi}{n} \quad \text{for all } (x,t) \in \mathbb{R}^n \times]0,T[, \\ &\partial_t m + \operatorname{div}_x U + \nabla_x q_0 = 0 \quad \text{in } \mathbb{R}^n \times [0,T]. \end{aligned} \tag{2.35}$$

Definition 2.4.2. *Let X_0 be the set of such linear momentum fields, i.e.*

$$X_0 = \Big\{ m \in C^0(]0,T[; C_p^0(Q)) \cap C([0,T]; H_w(Q)) : \tag{2.36}$$
$$(2.31) - (2.35) \text{ are satisfied} \Big\}$$

and let X be the closure of X_0 in $C([0,T]; H_w(Q)$. Then X_0 will be the space of strict subsolutions.

As ρ_0 is continuous and periodic on $\mathbb{R}^n$ and χ is smooth on $[0,T]$, there exists a constant G such that $\chi(t)\int_Q \rho_0(x)dx \le G$ for all $t \in [0,T]$. Since for any $m \in X_0$ with associated matrix field U we have that (see Lemma 2.3.2- (ii))

$$\begin{aligned} \int_Q |m(x,t)|^2 \, dx &\le \int_Q n\rho_0(x) e(\rho_0(x), m(x,t), U(x,t)) dx \\ &< \chi(t) \int_Q \rho_0(x) dx \text{ for all } t \in [0,T], \end{aligned}$$

we can observe that X_0 consists of functions $m : [0,T] \to H(Q)$ taking values in a bounded subset B of $H(Q)$. Without loss of generality, we can assume that B is weakly closed. Then, B in its weak topology is metrizable and, if we let d_B be a metric on B inducing the weak topology, we have that (B, d_B) is a compact metric space. Moreover, we can define on $Y := C([0,T], (B, d_B))$ a metric d naturally induced by d_B via

$$d(f_1, f_2) := \max_{t \in [0,T]} d_B(f_1(\cdot,t), f_2(\cdot,t)). \tag{2.37}$$

Note that the topology induced on Y by d is equivalent to the topology of Y as a subset of $C([0,T];H_w)$. In addition, the space (Y,d) is complete. Finally, X is the closure in (Y,d) of X_0 and hence (X,d) is as well a complete metric space.

LEMMA 2.4.3. *If $m \in X$ is such that $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$, then the pair (ρ_0, m) is a weak solution of (2.8) in $\mathbb{R}^n \times [0,T[$ satisfying (2.28)-(2.29)-(2.30).*

Proof. Let $m \in X$ be such that $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$. By density of X_0, there exists a sequence $\{m_k\} \subset X_0$ such that $m_k \overset{d}{\to} m$ in X. For any $m_k \in X_0$ let U_k be the associated smooth matrix field enjoying (2.35). Thanks to Lemma 2.3.2 (iii) and (2.35), the following pointwise estimate holds for the sequence $\{U_k\}$

$$|U_k|_\infty \leq (n-1)e(\rho_0, m_k, U_k) < \frac{(n-1)-\chi}{n}.$$

As a consequence, $\{U_k\}$ is uniformly bounded in $L^\infty([0,T];L^\infty_p(Q))$; by possibly extracting a subsequence, we have that

$$U_k \overset{*}{\rightharpoonup} U \text{ in } L^\infty([0,T];L^\infty_p(Q)).$$

Note that $\overline{\text{hint}K^{co}_{\rho_0,\chi}} = K^{co}_{\rho_0,\chi}$ is a convex and compact set by Lemma 2.3.2-(i)-(ii)-(iii). Hence, $m \in X$ with associated matrix field U solves (2.13) on $\mathbb{R}^n \times [0,T]$ for $q = q_0$ and (m, U, q_0) takes values in $K^{co}_{\rho_0,\chi}$ almost everywhere. If, in addition, $|m(x,t)|^2 = \rho_0(x)\chi(t)$, then $(m,U,q_0)(x,t) \in K_{\rho,\chi}$ a.e. in $\mathbb{R}^n \times [0,T]$ (cf. Lemma 2.3.2-(iv)). Lemma 2.3.1 allows us to conclude that (ρ_0, m) is a weak solution of (2.8) in $\mathbb{R}^n \times [0,T[$. Finally, since $m_k \to m$ in $C([0,T];H_w(Q))$ and $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$, we see that m satisfies also (2.28)-(2.29)-(2.30). □

Now, we will argue as in [**DLS09**] exploiting Baire category techniques to combine weak and strong convergence (see also [**Kir03**]).

LEMMA 2.4.4. *The identity map $I : (X,d) \to L^2([0,T];H(Q))$ defined by $m \to m$ is a Baire-1 map, and therefore the set of points of continuity is residual in (X,d).*

Proof. Let $\phi_r(x,t) = r^{-(n+1)}\phi(rx,rt)$ be any regular spacetime convolution kernel. For each fixed $m \in X$, we have

$$\phi_r * m \to m \quad \text{strongly in } L^2(H) \text{ as } r \to 0.$$

On the other hand, for each $r > 0$ and $m_k \in X$,

$$m_k \xrightarrow{d} m \text{ implies } \phi_r * m_k \to \phi_r * m \text{ in } L^2(H).$$

Therefore, each map $I_r : (X, d) \to L^2(H)$, $m \to \phi_r * m$ is continuous, and $I(m) = \lim_{r\to 0} I_r(m)$ for all $m \in X$. This shows that $I : (X, d) \to L^2(H)$ is a pointwise limit of continuous maps; hence it is a Baire-1 map. As a consequence, the set of points of continuity of I is residual in (X, d) (cf. [**Oxt90**]). □

2.4.2. Proof of Proposition 2.4.1. We aim to show that all points of continuity of the identity map correspond to solutions of (2.8) enjoying the requirements of Proposition 2.4.1: Lemma 2.4.4 will then allow us to prove Proposition 2.4.1 once we know that the cardinality of X is infinite. In light of Lemma 2.4.3, for our purposes it suffices to prove the following claim:

CLAIM. If $m \in X$ is a point of continuity of I, then

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \text{ for almost every } (x,t) \in \mathbb{R}^n \times]0,T[. \tag{2.38}$$

□

Note that proving (2.38) is equivalent to prove that $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx\right)^{1/2}$, since for any $m \in X$ we have $|m(x,t)|^2 \leq \rho_0(x)\chi(t)$ for almost all $(x,t) \in \mathbb{R}^n \times [0,T]$. Thanks to this remark, the claim is reduced to the following lemma (cf. Lemma 4.6 in [**DLS09**]), which provides a strategy to move towards the boundary of X_0: given $m \in X_0$, we will be able to approach it with a sequence inside X_0 but closer than m to the boundary of X_0.

LEMMA 2.4.5. *Let ρ_0, χ be given functions as in Proposition 2.4.1. Then, there exists a constant $\beta = \beta(n)$ such that, given $m \in X_0$, there exists a sequence $\{m_k\} \subset X_0$ with the following properties*

$$\|m_k\|^2_{L^2(Q\times[0,T])} \geq \|m\|^2_{L^2(Q\times[0,T])} + \beta\left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])}\right)^2 \tag{2.39}$$

and

$$m_k \to m \text{ in } C([0,T], H_w(Q)). \tag{2.40}$$

The proof is postponed to Section 2.6. Let us show how Lemma 2.4.5 implies the claim. As in the claim, assume that $m \in X$ is a point of continuity of the identity map I. Let $\{m_k\} \subset X_0$ be a fixed sequence that converges to m in $C([0,T], H_w(Q))$. Using Lemma 2.4.5 and a standard diagonal argument, we can find a second sequence $\{\widetilde{m}_k\}$ yet converging to m in X and satisfying

$$\liminf_{k\to\infty} \|\widetilde{m}_k\|^2_{L^2(Q\times[0,T])} \geq \liminf_{k\to\infty} \left(\|m_k\|^2_{L^2(Q\times[0,T])} + \beta \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m_k\|^2_{L^2(Q\times[0,T])} \right)^2 \right).$$

According to the hypothesis, I is continuous at m, therefore both m_k and $\widetilde{m}_k$ converge strongly to m and

$$\|m\|^2_{L^2(Q\times[0,T])} \geq \|m\|^2_{L^2(Q\times[0,T])} + \beta \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])} \right)^2.$$

Hence $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx \right)^{1/2}$ and the claim holds true. Finally, since the assumptions of Proposition 2.4.1 ensure that X_0 is nonempty, by Lemma 2.4.5 we can see that the cardinality of X is infinite whence the cardinality of any residual set in X is infinite. In particular, the set of continuity points of I is infinite: this and the claim conclude the proof of Proposition 2.4.1.

2.5. Localized oscillating solutions

The wild solutions are made by adding one dimensional oscillating functions in different directions $\lambda \in \Lambda$. For that it is needed to localize the waves. More precisely, the proof of Lemma 2.4.5 relies on the construction of solutions to the linear system (2.13), localized in space-time and oscillating between two states in $K^{co}_{\rho_0,\chi}$ along a given special direction $\lambda \in \Lambda$. Aiming at compactly supported solutions, one faces the problem of localizing vector valued functions: this is bypassed thanks to the construction of a "localizing" potential for the conservation laws (2.13). This approach is inherited from [**DLS10**]. As in [**DLS09**] it could be realized for every $\lambda \in \Lambda$, but in our framework it is

convenient to restrict only to *special* Λ-directions (cf. [**DLS10**]): this restriction will allow us to localize the oscillations at constant pressure.

Why oscillations at constant pressure are meaningful for us and needed in the proof of Lemma 2.4.5?

Owing to Section 2.3, in the variables $y = (x, t) \in \mathbb{R}^{n+1}$, the system (2.13) is equivalent to $\operatorname{div}_y M = 0$, where $M \in \mathcal{S}^{n+1}$ is defined via the linear map

$$\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R} \ni (m, U, q) \longmapsto M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{2.41}$$

More precisely, this map builds an identification between the set of solutions (m, U, q) to (2.13) and the set of symmetric $(n+1) \times (n+1)$ matrices M with $M_{(n+1)(n+1)} = 0$ and $\operatorname{tr}(M) = q$.

Therefore, solutions of (2.13) with $q \equiv 0$ correspond to matrix fields $M : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\operatorname{div}_y M = 0, \quad M^{\mathrm{T}} = M, \quad M_{(n+1)(n+1)} = 0, \quad \operatorname{tr}(M) = 0. \tag{2.42}$$

Moreover, given a density ρ and two states $(c, U_c, q_c), (d, U_d, q_d) \in K_\rho$ with non collinear momentum vector fields c and d having same magnitude ($|c| = |d|$), and hence same pressure ($q_c = q_d$), then the corresponding matrices M_c and M_d have the following form

$$M_c = \begin{pmatrix} \frac{c \otimes c}{\rho} + p(\rho) I_n & c \\ c & 0 \end{pmatrix} \text{ and } M_d = \begin{pmatrix} \frac{d \otimes d}{\rho} + p(\rho) I_n & d \\ d & 0 \end{pmatrix}$$

and satisfy

$$M_c - M_d = \begin{pmatrix} \frac{c \otimes c}{\rho} - \frac{d \otimes d}{\rho} & c - d \\ c - d & 0 \end{pmatrix}.$$

Finally note that $\operatorname{tr}(M_c - M_d) = 0$ and $M_c - M_d \in \Lambda$ corresponds to a *special direction.*

The following Proposition provides a potential for solutions of (2.13) oscillating between two states M_c and M_d at constant pressure. It is an easy adaptation to our framework of Proposition 4 in [**DLS10**].

Proposition 2.5.1. *Let $c, d \in \mathbb{R}^n$ such that $|c| = |d|$ and $c \neq d$. Let also $\rho \in \mathbb{R}$. Then there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order* 3

$$A(\partial) : C^\infty_c(\mathbb{R}^{n+1}) \to C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{(n+1)\times(n+1)})$$

such that $M = A(\partial)\phi$ satisfies (2.42) *for all $\phi \in C^\infty_c(\mathbb{R}^{n+1})$. Moreover there exists $\eta \in \mathbb{R}^{n+1}$ such that*

- η *is not parallel to* e_{n+1}*;*
- *if* $\phi(y) = \psi(y \cdot \eta)$*, then*

$$A(\partial)\phi(y) = (M_c - M_d)\psi'''(y \cdot \eta).$$

We also report Lemma 7 from [**DLS10**]: it ensures that the oscillations of the planewaves generated in proposition 2.5.1 have a certain size in terms of an appropriate norm-type-functional.

LEMMA 2.5.2. *Let* $\eta \in \mathbb{R}^{n+1}$ *be a vector which is not parallel to* e_{n+1}*. Then for any bounded open set* $B \subset \mathbb{R}^n$

$$\lim_{N\to\infty} \int_B \sin^2(N\eta \cdot (x,t))dx = \frac{1}{2}|B|$$

uniformly in $t \in \mathbb{R}$*.*

For the proof we refer the reader to [**DLS10**].

2.6. The improvement step

We are now about to prove one of the cornerstones of the costruction. Before moving forward, let us resume the plan. We have already identified a relaxed problem by introducing *subsolutions*. Then, we have proved a sort of "h-principle" (even if there is no homotopy here) according to which, the space of *subsolutions* can be "reduced" to the space of solutions or, equivalently, the typical (in Baire's sense) *subsolution* is a solution. Once assumed that a *subsolution* exists, the proof of our "h-principle" builds upon Lemma 2.4.5 combined with Baire category arguments. Indeed, we could also prove Proposition 2.4.1 by applying iteratively Lemma 2.4.5 and thus constructing a converging sequence of subsolutions approaching $K_{\rho,\chi}$: this would correspond to the constructive convex integration approach (see Section 1.3.1). So two steps are left in order to conclude our argument: showing the existence of a "starting" *subsolution* and prove Lemma 2.4.5.

This section is devoted to the second task, the proof of Lemma 2.4.5, while in next section we will exhibit a "concrete" *subsolution*.

What follows will be quite technical, therefore we first would like to recall the plan: we will add fast oscillations in allowed directions so to let $|m|^2$ increase in average. The proof is inspired by [**DLS09**]-[**DLS10**].

Proof. [Proof of Lemma 2.4.5] Let us fix the domain $\Omega := Q \times [0,T]$. We look for a sequence $\{m_k\} \subset X_0$, with associated matrix fields $\{U_k\}$, which improves m in the sense of (2.39) and has the form

$$(m_k, U_k) = (m, U) + \sum_j (\widetilde{m}_{k,j}, \widetilde{U}_{k,j}) \tag{2.43}$$

where every $z_{k,j} = (\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$ is compactly supported in some suitable ball $B_{k,j}(x_{k,j}, t_{k,j}) \subset \Omega$. We proceed as follows.

Step 1. Let $m \in X_0$ with associated matrix field U. By Lemma 2.3.5, for any $(x,t) \in \Omega$ we can find a line segment $\sigma_{(x,t)} := [(m(x,t), U(x,t), q_0(x)) - \lambda_{(x,t)}, (m(x,t), U(x,t), q_0(x)) + \lambda_{(x,t)}]$ admissible for $(\rho_0(x), \chi(t))$ and with direction

$$\lambda_{(x,t)} = (\overline{m}(x,t), \overline{U}(x,t), 0)$$

such that

$$|\overline{m}(x,t)| \geq \frac{F}{\sqrt{\rho_0(x)\chi(t)}} \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right). \tag{2.44}$$

Since $z := (m, U)$ and $K^{co}_{\rho_0,\chi}$ are uniformly continuous in (x,t), there exists an $\varepsilon > 0$ such that for any $(x,t), (x_0,t_0) \in \Omega$ with $|x - x_0| + |t - t_0| < \varepsilon$, we have

$$(z(x,t), q_0(x)) \pm (\overline{m}(x_0,t_0), \overline{U}(x_0,t_0), 0) \subset \mathrm{hint} K^{co}_{\rho_0,\chi}. \tag{2.45}$$

Step 2. Fix $(x_0,t_0) \in \Omega$ for the moment. Now, let $0 \leq \phi_{r_0} \leq 1$ be a smooth cutoff function on Ω with support contained in a ball $B_{r_0}(x_0,t_0) \subset \Omega$ for some $r_0 > 0$, identically 1 on $B_{r_0/2}(x_0,t_0)$ and strictly less than 1 outside. Thanks to Proposition 2.5.1 and the identification $(m,U,q) \to M$, for the admissible line segment $\sigma_{(x_0,t_0)}$, there exist an operator A_0 and a direction $\eta_0 \in \mathbb{R}^{n+1}$ not parallel to e_{n+1}, such that for any $k \in \mathbb{N}$

$$A_0 \left(\frac{\cos(k\eta_0 \cdot (x,t))}{k^3} \right) = \lambda_{(x_0,t_0)} \sin(k\eta_0 \cdot (x,t)),$$

and such that the pair $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ defined by

$$(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})(x,t) := A_0 \left[\phi_{r_0}(x,t) k^{-3} \cos(k\eta_0 \cdot (x,t)) \right]$$

satisfies (2.13) with $q \equiv 0$. Note that $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ is supported in the ball $B_{r_0}(x_0, t_0)$ and that

$$\begin{aligned} &\left\| (\widetilde{m}_{k,0}, \widetilde{U}_{k,0}) - \phi_{r_0}\left(\overline{m}(x_0,t_0), \overline{U}(x_0,t_0)\right) \sin(k\eta_0 \cdot (x,t)) \right\|_\infty \\ &\leq const\,(A_0, \eta_0, \|\phi_0\|_{C^3})\,\frac{1}{k} \end{aligned} \tag{2.46}$$

since A_0 is a linear differential operator of homogeneous degree 3. Furthermore, for all $(x,t) \in B_{r_0/2}(x_0,t_0)$, we have

$$|\widetilde{m}_{k,0}(x,t)|^2 = |\overline{m}(x_0,t_0)|^2 \sin^2(k\eta_0 \cdot (x,t)).$$

Since $\eta_0 \in \mathbb{R}^{n+1}$ is not parallel to e_{n+1}, from Lemma 2.5.2 we can see that

$$\lim_{k\to\infty} \int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2\,dx = \frac{1}{2} \int_{B_{r_0/2}(x_0,t_0)} |\overline{m}(x_0,t_0)|^2\,dx$$

uniformly in t. In particular, using (2.44), we obtain

$$\begin{aligned} &\lim_{k\to\infty} \int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2\,dxdt \geq \\ &\frac{F^2}{2\rho_0(x_0)\chi(t_0)} \left(\rho_0(x_0)\chi(t_0)) - |m(x_0,t_0)|^2\right)^2 \left|B_{r_0/2}(x_0,t_0)\right|. \end{aligned} \tag{2.47}$$

Step 3. Next, observe that since m is uniformly continuous, there exists an $\bar{r} > 0$ such that for any $r < \bar{r}$ there exists a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with $r_j < r$ such that

$$\begin{aligned} &\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt \leq \\ &2\sum_j \left(\rho_0(x_j)\chi(t_j) - |m(x_j,t_j)|^2\right)^2 \left|B_{r_j}(x_j,t_j)\right|. \end{aligned} \tag{2.48}$$

Fix $s > 0$ with $s < \min\{\bar{r}, \varepsilon\}$ and choose a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with radii $r_j < s$ such that (2.48) holds. In each ball $B_{2r_j}(x_j,t_j)$ we apply the construction of *Step 2* to obtain, for every $k \in \mathbb{N}$, a pair $(\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$.

Final step. Letting (m_k, U_k) be as in (2.43), we observe that the sum therein consists of finitely many terms. Therefore from (2.45) and (2.46) we deduce that there exists $k_0 \in \mathbb{N}$ such that

$$m_k \in X_0 \text{ for all } k \geq k_0. \tag{2.49}$$

Moreover, owing to (2.47) and (2.48) we can write

$$\lim_{k\to\infty}\int_\Omega |m_k(x,t)-m(x,t)|^2\,dxdt = \lim_{k\to\infty}\sum_j \int_\Omega |\widetilde{m}_{k,j}(x,t)|^2\,dxdt$$

$$\geq \sum_j \frac{F^2}{2\rho_0(x_j)\chi(t_j)}\left(\rho_0(x_j)\chi(t_j)) - |m(x_j,t_j)|^2\right)^2 \left|B_{r_j}(x_j,t_j)\right|$$

$$\geq C\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt. \tag{2.50}$$

Since $m_k \xrightarrow{d} m$, due to (2.50) we have

$$\liminf_{k\to\infty}\|m_k\|^2_{L^2(\Omega)} = \|m\|_2^2 + \liminf_{k\to\infty}\|m_k-m\|_2^2$$

$$\geq \|m\|_2^2 + C\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right)^2 dxdt, \tag{2.51}$$

which gives (2.39) with $\beta = \beta(n) = \beta(F(n))$. □

2.7. Construction of suitable initial data

In this section we show the existence of a *subsolution* in the sense of Definition 3.4.1. Since the subsolution we aim to construct has to be space-periodic, it will be enough to work on the building brick Q and then extend the costruction periodically to $\mathbb{R}^n$.

The idea to work in the space-periodic setting has been recently adopted by Wiedemann [**Wie11**] in order to construct global solutions to the incompressible Euler equations, i.e. to prove Theorem 1.3.9.

PROPOSITION 2.7.1. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 2.4.1 and let T be any given positive time. Then, there exist a smooth function $\widetilde{\chi}:\mathbb{R}\to\mathbb{R}^+$, a continuous periodic matrix field $\widetilde{U}:\mathbb{R}^n\to\mathcal{S}^n_0$ and a function $\widetilde{q}\in C^1(\mathbb{R};C^1_p(\mathbb{R}^n))$ such that*

$$\operatorname{div}_x \widetilde{U} + \nabla_x \widetilde{q} = 0 \qquad \text{on} \quad \mathbb{R}^n\times\mathbb{R} \tag{2.52}$$

and

$$e(\rho_0(x),0,\widetilde{U}(x)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t)\in\mathbb{R}^n\times[0,T[\tag{2.53}$$

$$\widetilde{q}(x,t) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } x\in\mathbb{R}^n\times\mathbb{R}. \tag{2.54}$$

Proof. [Proposition 2.7.1] Let us define $\widetilde{U}$ componentwise by its Fourier transform as follows:

$$\widehat{\widetilde{U}}_{ij}(k) := \left(\frac{nk_ik_j}{(n-1)\left|k\right|^2}\right)\widehat{p(\rho_0(k))} \text{ if } i \neq j,$$

$$\widehat{\widetilde{U}}_{ii}(k) := \left(\frac{nk_i^2 - \left|k\right|^2}{(n-1)\left|k\right|^2}\right)\widehat{p(\rho_0(k))}. \tag{2.55}$$

for every $k \neq 0$, and $\widehat{\widetilde{U}}(0) = 0$. Clearly $\widehat{\widetilde{U}}_{ij}$ thus defined is symmetric and trace-free. Moreover, since $p(\rho_0) \in C^1_p(\mathbb{R}^n)$, standard elliptic regularity arguments allow us to conclude that $\widetilde{U}$ is a continuous periodic matrix field. Next, notice that

$$\left\|e(\rho_0(x), 0, \widetilde{U}(x))\right\|_\infty = \left\|\lambda_{max}(-\widetilde{U})\right\|_\infty = \widetilde{\lambda} \tag{2.56}$$

for some positive constant $\widetilde{\lambda}$. Therefore, we can choose any smooth function $\widetilde{\chi}$ on $\mathbb{R}$ such that $\widetilde{\chi} > n\widetilde{\lambda}$ on $[0,T]$ in order to ensure (2.53). Now, let $\widetilde{q}$ be defined exactly as in (2.54) for the choice of $\widetilde{\chi}$ just done. It remains to show that (2.52) holds. In light of (2.54), we can write equation (2.52) in Fourier space as

$$\sum_{j=1}^{n} k_j \widehat{\widetilde{U}}_{ij} = k_i \widehat{p(\rho_0)} \tag{2.57}$$

for $k \in \mathbb{Z}^n$. It is easy to check that $\widehat{\widetilde{U}}$ as defined by (2.55) solves (2.57) and hence $\widetilde{U}$ and $\widetilde{q}$ satisfy (2.52)

□

REMARK 2.7.2. *We note that the Hölder continuity of ρ_0 would be enough to argue as in the previous proof in order to infer the continuity of $\widetilde{U}$.*

PROPOSITION 2.7.3. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 2.4.1 and let T be any given positive time. There exist triples $(\overline{m}, \overline{U}, \overline{q})$ solving* (2.13) *distributionally on $\mathbb{R}^n \times \mathbb{R}$ enjoying the*

following properties:

(2.58)

$$(\overline{m}, \overline{U}, \overline{q}) \text{ is continuous in } \mathbb{R}^n \times (\mathbb{R}\backslash\{0\}) \text{ and } \overline{m} \in C(\mathbb{R}; H_w(\mathbb{R}^n)),$$

(2.59)

$$\overline{U}(\cdot, t) = \widetilde{U}(\cdot) \text{ for } t = -T, T$$

and

(2.60)

$$\overline{q}(x) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

(2.61)

$$e(\rho_0(x), \overline{m}(x,t), \overline{U}(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times ([-T, 0[\cap]0, T]).$$

Moreover

$$|\overline{m}(x,0)|^2 = \rho_0(x)\chi(0) \text{ a.e. in } \mathbb{R}^n. \tag{2.62}$$

Proof. [Proposition 2.7.3] We first choose $\overline{q} := \tilde{q}$ given by Proposition 2.7.1. This choice already yields (2.60).

Now, in analogy with Definition 3.4.1 we consider the space X_0 defined as the set of continuous vector fields $m : \mathbb{R}^n \times]-T, T[\to \mathbb{R}^n$ in $C^0(]-T, T[; C^0_p(Q))$ to which there exists a continuous space-periodic matrix field $U : \mathbb{R}^n \times]-T, T[\to \mathcal{S}^n_0$ such that

$$\begin{aligned} \operatorname{div}_x m &= 0, \\ \partial_t m + \operatorname{div}_x U + \nabla_x \overline{q} &= 0, \end{aligned} \tag{2.63}$$

$$\operatorname{supp}(m) \subset Q \times [-T/2, T/2[\tag{2.64}$$

$$U(\cdot, t) = \widetilde{U}(\cdot) \text{ for all } t \in [-T, T[\backslash[-T/2, T/2] \tag{2.65}$$

and

$$e(\rho_0(x), m(x,t), U(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times]-T, T[. \tag{2.66}$$

As in Section 2.4.1, X_0 consists of functions $m :]-T, T[\to H$ taking values in a bounded set $B \subset H$. On B the weak topology of L^2 is metrizable, and correspondingly we find a metric d on $C(]-T, T[; B)$ inducing the topology of $C(]-T, T[; H_w(Q))$.

Next we note that with minor modifications the proof of Lemma 2.4.5 leads to the following claim:

Claim: Let $Q_0 \subset Q$ be given. Let $m \in X_0$ with associated matrix field U and let $\alpha > 0$ such that

$$\int_{Q_0} \left[|m(x,0)|^2 - (\rho_0(x)\widetilde{\chi}(0)) \right] dx < -\alpha$$

Then, for any $\delta > 0$ there exists a sequence $m_k \in X_0$ with associated smooth matrix field U_k such that

$$\operatorname{supp}(m_k - m, U_k - U) \subset Q_0 \times [-\delta, \delta],$$

$$m_k \overset{d}{\to} m,$$

and

$$\liminf_{k\to\infty} \int_{Q_0} |m_k(x,0)|^2 \geq \int_{Q_0} |m(x,0)|^2 \, dx + \beta\alpha^2.$$

Fix an exhausting sequence of bounded open subsets $Q_k \subset Q_{k+1} \subset Q$, each compactly contained in Ω, and such that $|Q_{k+1} \backslash Q_k| \leq 2^{-k}$. Let also γ_ε be a standard mollifying kernel in $\mathbb{R}^n$ (the unusual notation γ_ε for the standard mollifying kernel is aimed at avoiding confusion between it and the density function). Using the claim above we construct inductively a sequence of momentum vector fields $m_k \in X_0$, associated matrix fields U_k and a sequence of numbers $\eta_k < 2^{-k}$ as follows.

First of all let $m_1 \equiv 0$, $U_1(x,t) = \widetilde{U}(x)$ for all $(x,t) \in \mathbb{R}^{n+1}$ and having obtained (m_1, U_1), ..., (m_k, U_k), $\eta_1, ..., \eta_{k-1}$ we choose $\eta_k < 2^{-k}$ in such a way that

$$\|m_k - m_k * \gamma_{\eta_k}\|_{L^2} < 2^{-k}. \tag{2.67}$$

Then, we set

$$\alpha_k = -\int_{Q_k} [|m_k(x,0)|^2 - \rho_0(x)\widetilde{\chi}(0))] dx.$$

Note that (2.66) ensures $\alpha_k > 0$. Then, we apply the claim with Q_k, $\alpha = \alpha_k$ and $\delta = 2^{-k}T$ to obtain $m_{k+1} \in X_0$ and associated smooth matrix field U_{k+1} such that

$$\operatorname{supp}(m_{k+1} - m_k, U_{k+1} - U_k) \subset Q_k \times [-2^{-k}T, 2^{-k}T], \tag{2.68}$$

$$d(m_{k+1}, m_k) < 2^{-k}, \tag{2.69}$$

$$\int_{Q_k} |m_{k+1}(x,0)|^2\,dx \geq \int_{Q_k} |m_k(x,0)|^2\,dx + \beta\alpha_k^2. \tag{2.70}$$

Since d induces the topology of $C(]-T,T[;H_w(\Omega))$ we can also require that

$$\left\|(m_k - m_{k+1}) * \gamma_{\eta_j}\right\|_{L^2(\Omega)} < 2^{-k} \text{ for all } j \leq k \text{ for } t = 0. \tag{2.71}$$

From (2.7) we infer the existence of a function $\overline{m} \in C(]-T,T[,H_w(\Omega))$ such that

$$m_k \overset{d}{\to} \overline{m}.$$

Besides, (2.68) implies that for any compact subset S of $Q\times]-T,0[\cup]0,T[$ there exists k_0 such that $(m_k,U_k)|_S = (m_{k_0},U_{k_0})|_S$ for all $k > k_0$. Hence (m_k,U_k) converges in $C^0_{\text{loc}}(Q\times]-T,0[\cup]0,T[)$ to a continuous pair $(\overline{m},\overline{U})$ solving equations (2.63) in $\mathbb{R}^n\times]-T,0[\cup]0,T[$ and such that (2.58)-(2.61) hold. In order to conclude, we show that also (2.62) holds for $\overline{m}$.

As first, we observe that (2.70) yields

$$\alpha_{k+1} \leq \alpha_k - \beta\alpha_k^2 + |Q_{k+1}\backslash Q_k| \leq \alpha_k - \beta\alpha_k^2 + 2^{-k},$$

from which we deduce that

$$\alpha_k \to 0 \text{ as } k \to \infty.$$

This, together with the following inequality

$$0 \geq \int_Q \left[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\right] dx \geq -(\alpha_k + C\,|Q\backslash Q_k|) \geq -(\alpha_k + C2^{-k}),$$

implies that

$$\lim_{k\uparrow\infty} \int_\Omega \left[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\right] dx = 0. \tag{2.72}$$

On the other hand, owing to (2.67) and (2.71), we can write for $t = 0$ and for every k

$$\begin{aligned}
&\|m_k - \overline{m}\|_{L^2} \\
&\leq \|m_k - m_k * \gamma_{\eta_k}\|_{L^2} + \|m_k * \gamma_{\eta_k} - \overline{m} * \gamma_{\eta_k}\|_{L^2} + \|\overline{m} * \gamma_{\eta_k} - \overline{m}\|_{L^2} \\
&\leq 2^{-k} + \sum_{j=0}^{\infty} \|m_{k+j} * \gamma_{\eta_k} - m_{k+j+1} * \gamma_{\eta_k}\|_{L^2} + 2^{-k} \\
&\leq 2^{-(k-2)}.
\end{aligned} \tag{2.73}$$

Finally, (2.73) implies that $m_k(\cdot, 0) \to \overline{m}(\cdot, 0)$ strongly in $H(Q)$ as $k \to \infty$, which together with (2.72) gives

$$|\overline{m}(x, 0)|^2 = \rho_0(x)\chi(0) \text{ for almost every } x \in \mathbb{R}^n.$$

□

2.8. Proof of the main Theorems

Proof. [Proof of Theorem 2.2.1] Let T be any finite positive time and $\rho_0 \in C_p^1(Q)$ be a given density function. Let also $(\overline{m}, \overline{U}, \overline{q})$ be as in Proposition 2.7.3. Then, define $\chi(t) := \widetilde{\chi}(t)$, $q_0(x) := \overline{q}(x)$,

$$m_0(x, t) = \begin{cases} \overline{m}(x, t) \text{ for } t \in [0, T] \\ \overline{m}(x, t - 2T) \text{ for } t \in [T, 2T], \end{cases} \tag{2.74}$$

$$U_0(x, t) = \begin{cases} \overline{U}(x, t) \text{ for } t \in [0, T] \\ \overline{U}(x, t - 2T) \text{ for } t \in [T, 2T]. \end{cases} \tag{2.75}$$

For this choices, the quadruple (m_0, U_0, q_0, χ) satisfies the assumptions of Proposition 2.4.1. Therefore, there exist infinitely many solutions $m \in C([0, 2T], H_w(Q))$ of (2.8) in $\mathbb{R}^n \times [0, 2T[$ with density ρ_0, such that

$$m(x, 0) = \overline{m}(x, 0) = m(x, 2T) \text{ for a.e. } x \in \Omega$$

and

$$|m(\cdot, t)|^2 = \rho_0(\cdot)\chi(0) \text{ for almost every } (x, t) \in \mathbb{R}^n \times]0, 2T[. \tag{2.76}$$

Since $|m_0(\cdot, 0)|^2 = \rho_0(\cdot)\chi(0)$ a.e. in $\mathbb{R}^n$ as well, it is enough to define $m^0(x) = m_0(x, 0)$ to satisfy also (2.12) and hence conclude the proof. □

Proof. [Proof of Theorem 2.2.2] Under the assumptions of Theorem 2.2.1, we have proven the existence of a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0,T];H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0,T[$ with density ρ_0. Moreover, the proof (see Proof of Proposition 2.7.1) showed that for any smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$ with $\chi > n\widetilde{\lambda} > 0$ the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0,T[, \tag{2.77}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n. \tag{2.78}$$

Now, we claim that there exist constants $C_1, C_2 > 0$ such that choosing the function $\chi(t) > n\widetilde{\lambda}$ on $[0,T[$ among solutions of the following differential inequality

$$\chi'(t) \leq -C_1\chi^{1/2}(t) - C_2\chi^{3/2}(t), \tag{2.79}$$

then the weak solutions (ρ_0, m) of (2.8) obtained in Theorem 2.2.1 will also satisfy the admissibility condition (3.26) on $\mathbb{R}^n \times [0,T[$. Of course, there is an issue of compatibility between the differential inequality (2.79) and the condition $\chi > n\widetilde{\lambda}$: this motivates the existence of a time $\overline{T} > 0$ defining the maximal time-interval in which the admissibility condition indeed holds.

Let T be any finite positive time. As first, we aim to prove the claim. Since $m \in C([0,T];H_w(Q))$ is divergence-free and fulfills (2.77)-(2.78) and ρ_0 is time-independent, (3.26) reduces to the following inequality

$$\frac{1}{2}\chi'(t) + m\cdot\nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right) + \frac{\chi(t)}{2} m\cdot\nabla\left(\frac{1}{\rho_0(x)}\right) \leq 0, \tag{2.80}$$

intended in the sense of (space-periodic) distributions on $\mathbb{R}^n \times [0,T]$. As $\rho_0 \in C^1_p(Q)$, there exists a constant c_0^2 with $\rho_0 \leq c_0^2$ on $\mathbb{R}^n$, whence (see (2.77)-(2.78))

$$|m(x,t)| \leq c_0\sqrt{\chi(t)} \text{ a.e. on } \mathbb{R}^n \times [0,T[. \tag{2.81}$$

Similarly we can find constants $c_1, c_2 > 0$ with

$$\left|\nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right)\right| \leq c_1 \text{ a.e. in } \mathbb{R}^n \tag{2.82}$$

$$\left|\nabla\left(\frac{1}{\rho_0(x)}\right)\right| \leq c_2 \text{ a.e. in } \mathbb{R}^n. \tag{2.83}$$

As a conseguence of (2.81)-(2.83), (2.80) holds as soon as χ satisfies

$$\chi'(t) \leq -2c_1c_0\chi^{1/2}(t) - c_2c_0\chi^{3/2}(t) \text{ on } [0,T[.$$

Therefore, by choosing $C_1 := 2c_1c_0$ and $C_2 := c_2c_0$ we can conclude the proof of the claim.

Now, it remains to show the existence of a function χ as in the claim, i.e. that both the differential inequality (2.79) and the condition $\chi > n\widetilde{\lambda}$ can hold true on some suitable time-interval. To this aim, we can consider the equality in (2.79), couple it with the initial condition $\chi(0) = \chi_0$ for some constant $\chi_0 > n\widetilde{\lambda}$ and then solve the resulting Cauchy problem. For the obtained solution χ, there exists a positive time $\overline{T}$ such that $\chi(t) > n\widetilde{\lambda}$ on $[0,\overline{T}[$.

Finally, applying the claim on the time-interval $[0,\overline{T}[$ we conclude that the admissibility condition holds on $\mathbb{R}^n \times [0,\overline{T}[$ as desired. □

Proof. [Proof of Theorem 2.1.1] The proof of Theorem 2.1.1 strongly relies on Theorems 2.2.1-2.2.2. Given a continuously differentiable initial density ρ^0 we apply Theorems 2.2.1-2.2.2 for $\rho_0(x) := \rho^0(x)$ thus obtaining a positive time $\overline{T}$ and a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0,T];H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0,\overline{T}[$ with density ρ^0 and such that the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0,\overline{T}[, \tag{2.84}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n, \tag{2.85}$$

for a suitable smooth function $\chi : [0,\overline{T}] \to \mathbb{R}^+$. Now, define $\rho(x,t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$. This shows that (3.25) holds. To prove (3.24) observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < \overline{T}$. Therefore, the pair (ρ, m) is a weak solution of (2.1) with initial data (ρ^0, m^0). Finally, we can also prove (3.26): each solution obtained is also admissible. Indeed, for $\rho(x,t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$, (3.26) is ensured by Theorem 2.2.2. □

CHAPTER 3

Non–standard solutions to the compressible Euler equations with Riemann data

3.1. Introduction

In this chapter, we will focus on the Cauchy problem for the isentropic compressible Euler equations of gas dynamics in two space space dimensions: we will consider Riemann data having a specific form and we will prove the following surprising result which corresponds to Theorem 0.2.2 in the Introduction of the thesis.

THEOREM 3.1.1. *For some choices of the pressure law p with $p' > 0$ (among which $p(\rho) = \rho^2$) and for some specific Riemann initial data, there exist infinitely many bounded entropy solutions of the isentropic compressible Euler equations in two space dimensions, all with density bounded away from zero.*

The proof builds upon the methods of [**DLS09**]-[**DLS10**] where De Lellis and Székelyhidi had already shown that the admissibility (entropy) condition does not imply uniqueness of L^∞ solutions of the Cauchy problem and upon the developments achieved in [**Chi11**] and illustrated in Chapter 2. However, the examples in the papers [**DLS10**] and [**Chi11**] had very rough initial data and it was not at all clear whether more regular data could be achieved.

The previous result was inspired by the recent work of Székelyhidi, who in [**Sz1**] recasts the vortex-sheet problem of incompressible fluid dynamics in the framework of [**DLS11**].

We therefore chose to review the construction of Székelyhidi in [**Sz1**] in next section (Section 3.2) before illustrating the new ideas and strategies devised to prove Theorem 3.1.1.

The rest of the chapter is organized as follows. In Section 3.3 we give the precise statement of Theorem 3.1.1 by describing the form of the initial data allowing for such a result and by characterizing the set

of allowed pressure laws. In Section 3.4 we overview the basic definitions and the main ingredients of our setting. Section 3.5 explains how the convex integration approach introduced by De Lellis-Székelyhidi applies to our framework and allows us to prove Theorem 3.1.1. Section 3.6 and 3.7 are devoted to the final argument for the proof of Theorem 3.1.1: the construction of a subsolution.

3.2. Weak solutions to the incompressible Euler equations with vortex sheet initial data

Recently Székelyhidi constructed infinitely many admissible weak solutions to the incompressible Euler equations with initial data given by the classical vortex sheet. He considered the Cauchy problem for the incompressible Euler equations (see Section 1.3.2),

$$\begin{cases} \operatorname{div}_x v \;=\; 0 \\ \partial_t v + \operatorname{div}_x (v \otimes v) + \nabla_x p \;=\; 0 \\ v(\cdot, 0) \;=\; v^0 \end{cases} . \tag{3.1}$$

where the unknowns v and p are the velocity vector and the pressure. His construction is based on the "convex integration" method introduced recently in [**DLS10**]. His result inspired us and in particular suggested that a similar approach could be of interest also for the compressible Euler system thus leading to Theorem 3.1.1.

The starting point of Székelyhidi's construction lies in the approach of [**DLS09**]-[**DLS10**] towards the construction of weak solutions to the incompressible Euler equations (3.1) in order to recover the celebrated non–uniqueness results of Scheffer [**Sch93**] and Shnirelman [**Shn97**] (see Section 1.3.2.1). The method in [**DLS09**]-[**DLS10**] is a revisitation of convex integration and Baire category arguments. In particular, in [**DLS10**] the strategy behind the construction of "admissible" weak solutions to the initial value problem was based on the notion of subsolution. Thanks to this strategy, in [**DLS10**] it was shown that admissibility by itself does not imply uniqueness for the incompressible Euler system (3.1). In other words there exist initial data v^0, for which there exist infinitely many distinct admissible weak solutions of the incompressible Euler equations (3.1). Such initial data are called *wild* initial data in [**DLS10**]. In particular, one can show the existence of infinitely many weak solutions satisfying the Duchon-Robert

admissibility condition (see section 1.3.2)

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \leq 0. \tag{3.2}$$

Obviously, wild initial data have to possess a certain amount of irregularity. This follows from the weak- strong uniqueness and classical local existence results. From the construction of wild initial data in [**DLS10**] it was not clear how bad this irregularity needs to be. In [**Sz1**] Székelyhidi showed that the classical vortex-sheet with a flat interface is a wild initial data. More precisely, consider the following solenoidal vector field in $\mathbb{R}^2$

$$v^0(x) := \begin{cases} v^+ := (1,0) & \text{if } x_2 > 0, \\ v^- := (-1,0) & \text{if } x_2 < 0, \end{cases} \tag{3.3}$$

then the following theorem holds:

THEOREM 3.2.1 (The vortex sheet is wild). *For v^0 as in* (3.3) *there are infinitely many weak solutions of* (3.1) *on $\mathbb{R}^2 \times [0,\infty)$ which satisfy the admissibility condition* (3.2).

As already explained in section 1.3.2, Theorem 3.2.1 is proved in [**Sz1**] using an adapted version of Proposition 1.3.5: hence the proof essentially consisted in finding a suitable subsolution for the incompressible Euler system which takes the right initial values. For the sake of completeness, we recall the relevant definitions and results in the case of two space-dimensions:

DEFINITION 3.2.2 (*Incompressible* subsolutions). *A subsolution to the incompressible Euler equations with respect to the kinetic energy $\overline{e}$ is a triple $(\widetilde{v}, \widetilde{u}, \widetilde{q}) : \mathbb{R}^2 \times]0,\infty[\to (\mathbb{R}^2, \mathcal{S}_0^{2\times 2}, \mathbb{R}^+)$ with $\widetilde{v} \in L^2_{loc}$, $\widetilde{u} \in L^1_{loc}$, $\widetilde{q} \in \mathcal{D}'$ and such that*

$$\begin{cases} \partial_t \widetilde{v} + \operatorname{div}_x \widetilde{u} + \nabla_x \widetilde{q} \ = \ 0 \\ \operatorname{div}_x \widetilde{v} \ = \ 0 \end{cases} \tag{3.4}$$

in the sense of distributions and

$$\widetilde{v} \otimes \widetilde{v} - \widetilde{u} \leq \overline{e} Id \ a.e.. \tag{3.5}$$

Proposition 1.3.5 is here recasted as follows:

THEOREM 3.2.3 (Proposition 2 in [**DLS10**]). *Let $(\widetilde{v}, \widetilde{u}, \widetilde{q})$ be a subsolution to the incompressible Euler equations on $\mathbb{R}^2 \times]0,T[$. Assume*

that $\mathcal{W} \subset \mathbb{R}^2 \times]0,T[$ *is an open set such that* $(\widetilde{v}, \widetilde{u}, \widetilde{q})$ *are continuous on* $\mathcal{W}$ *and*

$$\widetilde{v} \otimes \widetilde{v} - \widetilde{u} < \overline{e} Id \text{ on } \mathcal{W}. \tag{3.6}$$

Then, there exist infinitely many $(\underline{v}, \underline{u}) \in L^\infty_{loc}(\mathbb{R}^2 \times \mathbb{R})$ *with* $\underline{v} \in C(\mathbb{R}; L^2_w(\mathbb{R}^2))$ *such that* $(\underline{v}, \underline{u}, 0)$ *satisfies* (3.4)*,* $(\underline{v}, \underline{u}) = 0$ *a.e. on* $\mathcal{W}^c$*, and*

$$(\widetilde{v} + \underline{v}) \otimes (\widetilde{v} + \underline{v}) - (\widetilde{u} + \underline{u}) = \overline{e} Id \text{ a.e. in } \mathcal{W}. \tag{3.7}$$

In particular, if $v := \widetilde{v} + \underline{v}$ in the above theorem, then v satisfies the incompressible Euler equations in the space-time region $\mathcal{W}$ with pressure q, where

$$|v|^2 = 2\overline{e}, \qquad q = \widetilde{q} - \overline{e}. \tag{3.8}$$

In other words, if the subsolution is a continuous, strict subsolution in some subregion, then it is possible (in a highly non-unique way) to add a perturbation so that the sum is a solution of the incompressible Euler equations (3.1) with prescribed kinetic energy density. This theorem is essentially the content of Proposition 2 in [**DLS10**] with an almost identical proof, except that in the proof one needs to perform the covering inside the region $\mathcal{W}$ rather than in all of space-time. Theorem 3.2.3 leads to the following criterion for the initial datum (3.3) to be a wild datum. In what follows we will work in the simple case of $\overline{e} = \frac{1}{2}$.

THEOREM 3.2.4. *Let* v^0 *be as in* (3.3)*. Assume that there exists a subsolution* $(\overline{v}, \overline{u}, \overline{q})$ *for the incompressible Euler equations with*

$$\overline{v} \in C([0,T]; L^2_w(\mathbb{R}^2)), \qquad \overline{v}(0) = v^0. \tag{3.9}$$

Furthermore, assume that there exists an open set $\mathcal{W} \subset \mathbb{R}^2 \times]0,T[$ *such that* $(\overline{v}, \overline{u}, \overline{q})$ *is continuous on* $\mathcal{W}$ *and*

$$\overline{v} \otimes \overline{v} - \overline{u} < \frac{1}{2} Id \text{ in } \mathcal{W}. \tag{3.10}$$

$$\overline{v} \otimes \overline{v} - \overline{u} = \frac{1}{2} Id \text{ a.e. in } \mathcal{W}^c. \tag{3.11}$$

Then, there exist infinitely many weak solutions to the incompressible Euler equations on $\mathbb{R}^2 \times]0,T[$ *satisfying* (3.2) *and with initial data* v^0*.*

In [**Sz1**] the construction of a subsolution as required by Theorem 3.2.4 follows an idea introduced in [**Szé11**] for the incompressible porous media equation. Here we will show that the existence of such a

subsolution can be achieved also in a more direct way which will be of use in the compressible case as well.

3.2.1. Direct proof of Theorem 3.2.1. The aim of this section is to apply Theorem 3.2.4 in order to prove Theorem 3.2.1. Clearly we need to find a triple $(\overline{v}, \overline{u}, \overline{q})$ satisfying (3.4), (3.9) and (3.10) for some open set $\mathcal{W} \subset \mathbb{R}^2 \times]0, T[$. We will denote the space variable as $x = (x_1, x_2) \in \mathbb{R}^2$. To this aim, we consider potential subsolutions of the following form:

$$\begin{aligned}(\overline{v}, \overline{u}, \overline{q}) &= (v^-, u^-, q^-)\mathbf{1}_{R^-} \\ &\quad + (\overline{v}, \overline{u}, \overline{q})\mathbf{1}_R \\ &\quad + (v^+, u^+, q^+)\mathbf{1}_{R^+},\end{aligned} \tag{3.12}$$

with

$$R^- := \left\{ 0 < t < \frac{x_2}{\nu_1} \right\},$$

$$R := \left\{ t > \frac{x_2}{\nu_1} \text{ and } t > \frac{x_2}{\nu_2} \right\},$$

$$R^+ := \left\{ 0 < t < \frac{x_2}{\nu_2} \right\}$$

and

$$\overline{v} = (\alpha, 0), \tag{3.13}$$

$$u^- = u^+ = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.14}$$

$$q^- = q^+ = \frac{1}{2}, \tag{3.15}$$

$$\overline{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \tag{3.16}$$

for some constants $\nu_1 < 0 < \nu_2$, α, β, γ and $\overline{q}$. Inside each of the three regions R^-, R and R^+ the equations defining a subsolution are trivially satisfied; hence they need to be imposed only along *fronts* which do not depend on x_1. Moreover condition (3.9) trivially holds.

Since the divergence free condition is trivially satisfied for our choice of $\overline{v}$, the system (3.4) simply reads as

$$\nu_2(\alpha - 1) = \gamma, \tag{3.17a}$$

$$\beta = \overline{q}, \tag{3.17b}$$

$$\nu_1(\alpha + 1) = \gamma. \tag{3.17c}$$

Finally, if we choose $\mathcal{W}$ to coincide with R, then the requirement (3.10) amounts to the condition

$$\begin{pmatrix} \alpha^2 - \beta & -\gamma \\ -\gamma & \beta \end{pmatrix} < \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tag{3.18}$$

which is equivalent to the following couple of inequalities

$$\alpha^2 \leq 1, \tag{3.19a}$$

$$\frac{1}{4} - \frac{1}{2}\alpha^2 + \beta\alpha^2 - \beta^2 - \gamma^2 \geq 0. \tag{3.19b}$$

With the choice $\alpha = 0$, $\nu_1 = -\nu_2 = \gamma < 0$ and $-\gamma = \beta = \overline{q} = \frac{1}{4}$. all the conditions (3.17a)-(3.19b) are satisfied. Hence a subsolution as in the hypothesis of Theorem 3.5.1 exists. Note that for the initial datum v^0 as given by (3.3) we already have: $|v^0|^2 = 2\overline{e} = 1$ for our choice of $\overline{e}$. This together with the fact the pressure does not depend on x_1 (since $\overline{q}$ and $\overline{e} = \frac{1}{2}$ do not depend on x_1) justifies why the admissibility condition (3.2) will be satisfied by the so constructed infinitely many weak solutions of the incompressible Euler equations.

3.3. The compressible system: main results

The motivation to the results herein presented has already been disclosed in the Introduction to the chapter. We emphasize that [**DLS10**] inspired the construction carried out in Chapter 2 (see also [**Chi11**]), where the compressible Euler equations are analyzed in the framework of convex integration: by choosing a suitable concept of subsolution, a similar strategy as in [**DLS09, DLS10**] can be performed and failure of uniqueness of admissible weak solutions to the Cauchy problem for the compressible Euler equations starting from any regular initial density is shown. This further result confirms the conjecture that admissibility by itself does not imply uniqueness for the compressible Euler equations and highlights the main role of the velocity in this failure of uniqueness. As for the incompressible Euler equations, there exist

initial data with infinitely many distinct admissible weak solutions of the compressible Euler equations. These initial data can be denominated as *wild initial data*, coherently with the incompressible case. As explained in Chapter 2, wild initial data need to be *irregular*: if the initial data are smooth enough, then a classical weak solution exists on a maximal time interval and is unique within the broader class of weak solutions (cf. [**Daf00**]).

A natural question then arises: how bad does this irregularity needs to be? Here we give a partial answer to this question: we show that a classical Riemann initial datum is a *wild initial datum* in the two dimensional case. More precisely, we consider the Cauchy problem for the compressible Euler equations in two space dimensions and we exhibit a Riemann initial datum allowing for infinitely many admissible weak solutions.

We recall once more in the thesis, for the first time in this chapter, the isentropic compressible Euler equations of gas dynamics in two space dimensions. The unknowns of the equations are the density ρ and the velocity v (see (0.1)). The Cauchy problem for the compressible Euler system, which consists of 3 scalar equations in the two-space-dimensional case, takes the form:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \ = \ 0 \\ \partial_t(\rho v) + \operatorname{div}_x\left(\rho v \otimes v\right) + \nabla_x[p(\rho)] \ = \ 0 \\ \rho(\cdot, 0) \ = \ \rho^0 \\ v(\cdot, 0) \ = \ v^0 \end{cases} . \tag{3.20}$$

The pressure p is a function of ρ whose analytical form depends on the gas under investigation. A common choice is the polytropic pressure law

$$p(\rho) = k\rho^\gamma$$

with constants $k > 0$ and $\gamma > 1$. The first main result will be concerned with the case $p(\rho) = k\rho^2$. This choice is consistent with classical kinetic theory which predicts $\gamma = 1 + 2/d$ where d is the number of degrees of freedom: indeed in the two space-dimensional case we have exactly $d = 2$ hence the chosen value of γ.

Entropy or admissible solutions are those satisfying an additional constraint (also called *entropy inequality*) coming from the energy inequality

$$\partial_t \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} \right) + \operatorname{div}_x \left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho) \right) v \right] \leq 0 \tag{3.21}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$.
If we consider Riemann data ρ^0 and v^0 having the following very specific form

$$\rho^0(x) := \begin{cases} \rho^+ & \text{if } x_2 > 0, \\ \rho^- & \text{if } x_2 < 0, \end{cases} \tag{3.22}$$

$$v^0(x) := \begin{cases} v^+ := (1,0) & \text{if } x_2 > 0, \\ v^- := (-1,0) & \text{if } x_2 < 0, \end{cases} \tag{3.23}$$

then the following theorems hold (precise statements of Theorem 3.1.1).

THEOREM 3.3.1. *With the choice of pressure law $p(\rho) = \rho^2$, there exist constants $\rho^\pm$ for which there exist infinitely many admissible bounded solutions (ρ, v) of* (3.20), (3.22)-(3.23) *with* $\inf \rho > 0$.

THEOREM 3.3.2. *For some specific smooth pressure laws p with $p' > 0$, there exist constants $\rho^\pm$ for which there exist infinitely many admissible bounded solutions (ρ, v) of* (3.20), (3.22)-(3.23) *with* $\inf \rho > 0$.

If we restrict our attention to pairs (ρ, v) which are admissible solutions of (3.20), (3.22)-(3.23) and depend only on (x_2, t), then we will be dealing with a classical Riemann problem for (3.20) in one space-variable (only x_2) which admits self-similar solutions. Since the pioneering work of Riemann it is known that, under the hypothesis of "self-similarity" of (ρ, v) (dependence only on $\frac{x_2}{t}$) and with some other assumptions, there is a unique solution of (3.20), (3.22)-(3.23) (see for instance [**Ser99**]). We will show that, such a uniqueness result holds also for the choice of pressure law dictated by Theorem 3.3.1: this will be the content of next chapter. But Theorems 3.3.1 and 3.3.2 show that uniqueness is completely lost if we drop the requirement that (ρ, v) depends only on $\frac{x_2}{t}$.

Moreover, Theorems 3.3.1 and 3.3.2 deserve some further comments. The pressure laws p which allow our constructions are not a *condition sine qua non* for the non-uniqueness results to hold. An interesting question correlated to our result is whether Theorem 3.3.2 might hold for regular initial data: so far the problem is open since the data of Theorem 3.3.2 cannot be generated by Lipschitz compression waves.

Finally, though the solutions of Theorems 3.3.1 and 3.3.2 are very irregular, it is rather unclear where one wishes to set a boundary. On the one hand the space of BV functions does not seem suitable for an existence theory in more than one space dimension (see [**Rau86**]). On the other hand the recent paper [**DLS12**] shows the existence of continuous solutions to the incompressible Euler equations which dissipate the kinetic energy. This may suggest that the framework of [**DLS09**]-[**DLS10**] is likely to produce "strange" piecewise continuous solutions to hyperbolic systems of conservation laws.

3.4. Basic definitions

Let T be a fixed positive time. By a *weak solution* of (3.20) on $\mathbb{R}^2 \times [0,T[$ we mean a pair $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0,T[)$ such that the following identities hold for every test functions $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,T[)$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,T[)$:

$$\int_0^T \int_{\mathbb{R}^2} \left[\rho \partial_t \psi + \rho v \cdot \nabla_x \psi\right] dxdt + \int_{\mathbb{R}^2} \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{3.24}$$

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2} \left[\rho v \cdot \partial_t \phi + \langle \rho v \otimes v, \nabla_x \phi \rangle + p(\rho)\,\mathrm{div}_x\,\phi\right] dxdt \\ &+ \int_{\mathbb{R}^2} \rho^0(x) v^0(x) \cdot \phi(x,0)dx \;=\; 0. \end{aligned} \tag{3.25}$$

For $n = 2$ the only non-trivial entropy is the total energy $\eta = \rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}$ which corresponds to the flux $\Psi = \left(\varepsilon(\rho) + \rho\frac{|v|^2}{2} + \frac{p(\rho)}{\rho}\right) v$.

Then a bounded weak solution (ρ, v) of (3.20) satisfying (3.21) in the sense of distributions, i.e. satisfying the following inequality

$$\int_0^T \int_{\mathbb{R}^2} \left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} \right) \partial_t\varphi + \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x\varphi \right]$$

(3.26)

$$+ \int_{\mathbb{R}^2} \left(\rho^0\varepsilon(\rho^0) + \rho^0\frac{|v^0|^2}{2} \right) \varphi(\cdot, 0) \ \geq \ 0,$$

for every nonnegative $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0, T[)$, is said to be an *entropy* (or *admissible*) solution of (3.20).

3.4.1. Subsolutions. In order to prove Theorem 3.3.2 we first recall the underlying strategy. To start with, we make precise the definition of subsolution in our context. Here $\mathcal{S}_0^{2\times 2}$ denotes the set of symmetric traceless 2×2 matrices and Id is the identity matrix.

Definition 3.4.1 (*Compressible* subsolutions). *A subsolution to the compressible Euler equations* (3.20) *is a triple* $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times]0, \infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$*, with* $\overline{\rho} \in L^\infty$*,* $\overline{v} \in L^2_{loc}$ *and* $\overline{u} \in L^1_{loc}$*, satisfying the following properties:*

(1) $(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^N (\overline{\rho}_i, \overline{v}_i, \overline{u}_i)\mathbf{1}_{P_i}$ *for some partition* $\{P_1, ..., P_N\}$ *of* $\mathbb{R}^2 \times]0, \infty[$ *in finitely many open sets and for some constants* $(\overline{\rho}_i, \overline{v}_i, \overline{u}_i)$*;*

(2) $\forall\ 1 \leq i \leq N$*, one of the following two conditions holds on* P_i*: either*

$$|\overline{v}_i| = 1 \text{ and } \overline{u}_i = \overline{v}_i \otimes \overline{v}_i - \frac{|\overline{v}_i|^2}{2} Id \text{ a.e.}, \tag{3.27}$$

or

$$|\overline{v}_i|^2 < C_i \text{ and } \overline{v}_i \otimes \overline{v}_i - \overline{u}_i < \frac{C_i}{2} Id \text{ a.e.}. \tag{3.28}$$

for some positive constant C_i*,*

and solving

(3.29)

$$\begin{cases} \partial_t\overline{\rho} + \operatorname{div}_x(\overline{\rho}\,\overline{v}) \ = \ 0 \\ \partial_t(\overline{\rho}\,\overline{v}) + \operatorname{div}_x(\overline{\rho}\,\overline{u}) + \nabla_x \left(p(\overline{\rho}) + \sum_{i\in I_C} C_i \frac{\overline{\rho}}{2} \mathbf{1}_{P_i} + \sum_{i\in I_1} \frac{\overline{\rho}}{2} \mathbf{1}_{P_i} \right) \ = \ 0 \\ \overline{\rho}(\cdot, 0) \ = \ \rho^0 \\ \overline{v}(\cdot, 0) \ = \ v^0 \end{cases}.$$

in the sense of distributions, where the sets of indexes I_1 *and* I_C *are defined as follows:*

$$I_1 := \{i \in [1,N] \cap \mathbb{N} \text{ such that (3.27) holds on } P_i\},$$
$$I_C := \{i \in [1,N] \cap \mathbb{N} \text{ such that (3.28) holds on } P_i\}.$$

Now, we will introduce a completely new definition which will be the starting point for the convex integration argument.

DEFINITION 3.4.2 (*Admissible* subsolutions). *A subsolution* $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times]0, \infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ *to the compressible Euler equations* (3.20) *is said to be admissible if it satisfies the following inequality in the sense of distributions:*

$$\begin{aligned} &\partial_t \left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \operatorname{div}_x \left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho})\right)\overline{v}\right] \\ &+ \sum_{i\in I_C} \left[\partial_t \left(\overline{\rho}\,\frac{C_i - 1}{2}\,\mathbf{1}_{P_i}\right)\right] + \sum_{i\in I_C} \left[\operatorname{div}_x \left(\overline{\rho}\,\overline{v}\,\frac{C_i - 1}{2}\,\mathbf{1}_{P_i}\right)\right] \;\leq\; 0. \end{aligned} \tag{3.30}$$

For our purposes we will also make use of "incompressible subsolutions" as defined in Definition 3.2.2 with the particular choice $\overline{e} = C/2$ for some positive constant C, i.e.:

DEFINITION 3.4.3 (*Incompressible* subsolutions 2). *A subsolution to the incompressible Euler equations is a triple* $(\widetilde{v}, \widetilde{u}, \widetilde{q}) : \mathbb{R}^2 \times]0, \infty[\to (\mathbb{R}^2, \mathcal{S}_0^{2\times 2}, \mathbb{R}^+)$ *with* $\widetilde{v} \in L^2_{loc}$, $\widetilde{u} \in L^1_{loc}$, $\widetilde{q} \in \mathcal{D}'$ *and such that*

$$\begin{cases} \partial_t \widetilde{v} + \operatorname{div}_x \widetilde{u} + \nabla_x \widetilde{q} \;=\; 0 \\ \operatorname{div}_x \widetilde{v} \;=\; 0 \end{cases} \tag{3.31}$$

in the sense of distributions and

$$\widetilde{v} \otimes \widetilde{v} - \widetilde{u} < \frac{C}{2} Id \text{ a.e.}. \tag{3.32}$$

Note that subsolutions to the incompressible Euler equations automatically satisfy $|\widetilde{v}|^2 < C$ a.e. If, in addition, (3.32) is an equality a.e. then $\widetilde{v}$ is a weak solution of the incompressible Euler equations.

3.5. Convex integration

In this section, we will clarify how convex integration allows to construct admissible weak solutions to the compressible Euler system (3.20) with Riemann initial data (ρ^0, v^0) starting from *admissible* subsolutions. Given an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ for the compressible

Euler system as in definition 3.4.1-3.4.2, the density $\overline{\rho}$ is constant in every open set P_i of the partition; therefore, in those open sets P_i where (3.28) holds, we are indeed dealing with a subsolution for the incompressible Euler system in the sense of definition 3.4.3. More precisely, in the open sets P_i where (3.28) holds, the triple $(\overline{v}_i, \overline{u}_i, 0)$ is, by definition, a subsolution on P_i to the incompressible Euler system. This observation suggests that, in every open set P_i where (3.28) holds, we can exploit the results obtained in [**DLS10**] for the incompressible Euler equations: in every such open set it is indeed possible to run a suitable version of convex integration so to construct infinitely many weak solutions. To this aim we will make use of Proposition 3.2.3 from [**DLS10**] with the choice $\overline{e} = C/2$. Theorem 3.2.3 leads to the following criterion for wild initial data for the compressible Euler equations.

THEOREM 3.5.1. *Let (ρ^0, v^0) be as in (3.22)-(3.23). Assume that there exists an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ for the compressible Euler equations with*

$$\overline{v}(0) = v^0, \quad \overline{\rho}(0) = \rho^0. \tag{3.33}$$

Then, there exist infinitely many admissible weak solutions to the compressible Euler equations on $\mathbb{R}^2 \times]0, T[$ with initial data (ρ^0, v^0).

Proof. Let $(\overline{\rho}, \overline{v}, \overline{u})$ be an admissible subsolution for the compressible Euler equations as given by the assumptions of Theorem 3.5.1. In particular, we can assume that $(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^{N} (\overline{\rho}_i, \overline{v}_i, \overline{u}_i) \mathbf{1}_{P_i}$. For every open set P_i where (3.28) holds, we apply Theorem 3.2.3 to the triple $(\overline{v}_i, \overline{u}_i, 0)$ on the open set $\mathcal{W} = P_i$. Then, we find infinitely many functions $(\underline{v}_i, \underline{u}_i)$ satisfying (3.31) on $\mathbb{R}^2 \times]0, \infty[$ with $\widetilde{q} = 0$ and such that the following holds:

$$(\overline{v}_i + \underline{v}_i) \otimes (\overline{v}_i + \underline{v}_i) - (\overline{u}_i + \underline{u}_i) = \frac{C_i}{2} Id \text{ a.e. in } P_i, \tag{3.34}$$

$$(\underline{v}_i, \underline{u}_i) = 0 \text{ a.e. on } P_i^c. \tag{3.35}$$

Once Theorem 3.2.3 has been applied to every open set P_i where (3.28) holds (i.e. for every $i \in I_C$), we define v and u in a highly non-unique way as follows:

$$v := \overline{v} + \sum_{i=1}^{N} \underline{v}_i,$$

$$u := \overline{u} + \sum_{i=1}^{N} \underline{u}_i$$

Since $\overline{\rho}$ is constant on every P_i and thanks to (3.29), (3.30) and (3.34)-(3.35), the infinitely many couples $(\overline{\rho}, v)$ are admissible weak solutions of (3.20) on $\mathbb{R}^2 \times]0, T[$ with initial data (ρ^0, v^0). □

Thanks to Theorem 3.5.1, in order to prove Theorems 3.3.1 and 3.3.2 it will be enough to exhibit an admissible subsolution to the compressible Euler equations in the sense of Definition 3.4.2. The construction of such a subsolution will be the content of the next sections. At this point the chapter takes two different routes according to the theorem we are going to prove.

3.6. Non–standard solutions with quadratic pressure

3.6.1. Construction of the subsolution. In this section, we will prove the existence of an admissible subsolution for the quadratic pressure law satisfying (3.33).

THEOREM 3.6.1. *For $p(\rho) = \rho^2$ there are constants $\rho^{\pm}$ such that there exists an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times]0, \infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ to the compressible Euler equations with*

$$\overline{v}(0) = v^0, \quad \text{and} \quad \overline{\rho}(0) = \rho^0, \tag{3.36}$$

for ρ^0 and v^0 as in (3.22)-(3.23).

REMARK 3.6.2. *The admissible subsolution of Theorem 3.6.1 will have the form*

$$(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^{3} (\overline{\rho}_i, \overline{v}_i, \overline{u}_i) \mathbf{1}_{P_i},$$

for some suitably chosen P_i, $1 \leq i \leq 3$.

What follows is devoted to the proof of Theorem 3.6.1.

3.6.1.1. *Subsolutions in three regions.* We aim at constructing an admissible subsolution to the compressible Euler equations of the following form (see Fig. 1):

$$\begin{aligned} (\overline{\rho}, \overline{v}, \overline{u}) = {} & (\rho^-, v^-, u^-) \mathbf{1}_{P^-} \\ & + (\overline{\rho}, \overline{v}, \overline{u}) \mathbf{1}_{P} \\ & + (\rho^+, v^+, u^+) \mathbf{1}_{P^+}, \end{aligned} \tag{3.37}$$

with

$$P^- := \left\{ 0 < t < \frac{x_2}{\nu_1} \right\},$$

$$P := \left\{ t > \frac{x_2}{\nu_1} \text{ and } t > \frac{x_2}{\nu_2} \right\},$$

$$P^+ := \left\{ 0 < t < \frac{x_2}{\nu_2} \right\}$$

and

$$\overline{v} = (a, b), \tag{3.38}$$

$$u^- = u^+ = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.39}$$

$$\overline{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \tag{3.40}$$

for some constants a, b, β, γ.

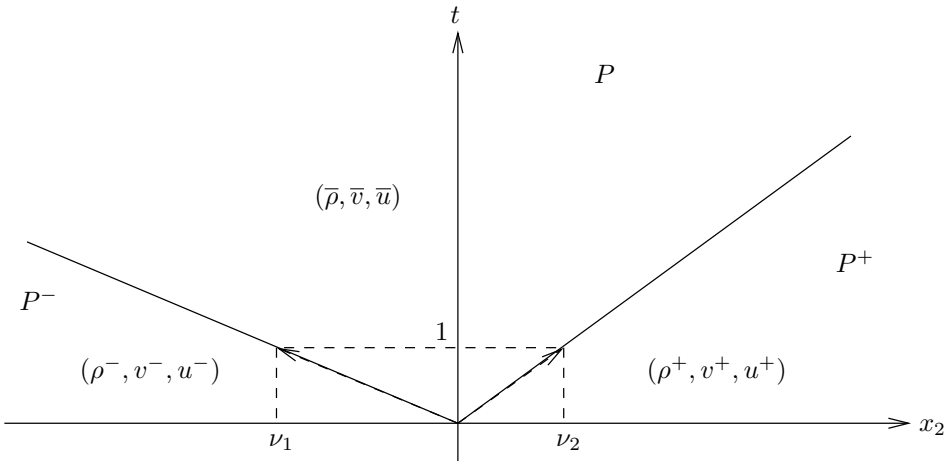

FIGURE 1. Subsolution in three regions

Hence, the condition (3.30) of admissibility for a subsolution takes the following easier form:

$$\begin{aligned} &\partial_t \left(\overline{\rho} \varepsilon(\overline{\rho}) \right) + \operatorname{div}_x \left[\left(\overline{\rho} \varepsilon(\overline{\rho}) + p(\overline{\rho}) \right) \overline{v} \right] \\ &+ \partial_t \left(\overline{\rho} \, \frac{C-1}{2} \, \mathbf{1}_P \right) + \operatorname{div}_x \left(\overline{\rho} \, \overline{v} \, \frac{C-1}{2} \, \mathbf{1}_P \right) \leq 0. \end{aligned} \tag{3.41}$$

Inside each of the three regions P^-, P and P^+ the equations defining a subsolution are trivially satisfied; hence they need to be imposed only along *fronts*. If we plug in the choice of pressure law $p(\rho) = \rho^2$, the system (3.29) then reads as

$$\nu_2(\rho^+ - \overline{\rho}) = -b\overline{\rho}, \tag{3.42a}$$

$$\nu_1(\overline{\rho} - \rho^-) = b\overline{\rho}, \tag{3.42b}$$

$$\nu_2(\rho^+ - \overline{\rho}a) = -\gamma\overline{\rho}, \tag{3.43a}$$

$$\nu_1(\overline{\rho}a + \rho^-) = \gamma\overline{\rho}, \tag{3.43b}$$

$$-\nu_2\overline{\rho}b = {\rho^+}^2 - \overline{\rho}^2 - \frac{C\overline{\rho}}{2} + \overline{\rho}\beta, \tag{3.43c}$$

$$\nu_1\overline{\rho}b = \overline{\rho}^2 - {\rho^-}^2 + \frac{C\overline{\rho}}{2} - \overline{\rho}\beta. \tag{3.43d}$$

Moreover, the relation between pressure and internal energy dictates the analytical form of $\varepsilon(\rho)$ in case $p(\rho) = \rho^2$: we will simply have $\varepsilon(\rho) = \rho$. Thus, the admissibility inequality (3.41) originates the following two inequalities

$$\nu_2\left(\overline{\rho}^2 - {\rho^+}^2 + \overline{\rho}\frac{C-1}{2}\right) - b\left(2\overline{\rho}^2 + \overline{\rho}\frac{C-1}{2}\right) \leq 0 \tag{3.44a}$$

$$\nu_1\left({\rho^-}^2 - \overline{\rho}^2 - \overline{\rho}\frac{C-1}{2}\right) + b\left(2\overline{\rho}^2 + \overline{\rho}\frac{C-1}{2}\right) \leq 0. \tag{3.44b}$$

Finally, the subsolution condition (3.28) in P translates into

$$\begin{pmatrix} a^2 - \beta & ab - \gamma \\ ab - \gamma & b^2 + \beta \end{pmatrix} < \begin{pmatrix} \frac{C}{2} & 0 \\ 0 & \frac{C}{2} \end{pmatrix}. \tag{3.45}$$

3.6.1.2. *Reduction of the admissibility condition.*

THEOREM 3.6.3. *Let us suppose that*

$$\nu_1 < 0 < \nu_2. \tag{3.46}$$

Then, for the pressure function $p(\rho) = \rho^2$, the admissibility conditions (3.44a)-(3.44b) *for a subsolution are implied by the following system of*

inequalities:

$$\overline{\rho} \leq \rho^+ - \sqrt{\rho^+ \frac{C-1}{2}}, \tag{3.47a}$$

$$\overline{\rho} \geq \rho^- + \sqrt{\rho^- \frac{C-1}{2}}, \tag{3.47b}$$

$$C > 1. \tag{3.47c}$$

Indeed, the condition of admissibility (3.44a)-(3.44b) for the subsolution will be reduced to (3.47a)-(3.47c).

Proof. First, owing respectively to (3.42a) and (3.42b), we can rewrite (3.44a) and (3.44b) as follows:

$$\nu_2 \left(\overline{\rho}^2 - {\rho^+}^2 + \overline{\rho} \frac{C-1}{2} \right) + \nu_2 (\rho^+ - \overline{\rho}) \left(2\overline{\rho} + \frac{C-1}{2} \right) \leq 0 \tag{3.48a}$$

$$\nu_1 \left({\rho^-}^2 - \overline{\rho}^2 - \overline{\rho} \frac{C-1}{2} \right) + \nu_1 (\overline{\rho} - \rho^-) \left(2\overline{\rho} + \frac{C-1}{2} \right) \leq 0. \tag{3.48b}$$

From the hypothesis (3.46) we can further reduce (3.48a)-(3.48b) to

$$\left(\overline{\rho}^2 - {\rho^+}^2 + \overline{\rho} \frac{C-1}{2} \right) + (\rho^+ - \overline{\rho}) \left(2\overline{\rho} + \frac{C-1}{2} \right) \leq 0 \tag{3.49a}$$

$$\left({\rho^-}^2 - \overline{\rho}^2 - \overline{\rho} \frac{C-1}{2} \right) + (\overline{\rho} - \rho^-) \left(2\overline{\rho} + \frac{C-1}{2} \right) \geq 0. \tag{3.49b}$$

Now, from (3.49a)- (3.49b), by simple algebra, we get

$$\overline{\rho}^2 - 2\rho^+ \overline{\rho} + {\rho^+}^2 - \rho^+ \frac{C-1}{2} \geq 0 \tag{3.50a}$$

$$\overline{\rho}^2 - 2\overline{\rho}\rho^- + {\rho^-}^2 - \rho^- \frac{C-1}{2} \geq 0. \tag{3.50b}$$

Clearly (3.50a) is satisfied if and only if

$$\overline{\rho} \leq \rho^+ - \sqrt{\rho^+ \frac{C-1}{2}} \qquad \vee \qquad \overline{\rho} \geq \rho^+ + \sqrt{\rho^+ \frac{C-1}{2}}, \tag{3.51}$$

while (3.50b) holds if and only if

$$\overline{\rho} \leq \rho^- - \sqrt{\rho^- \frac{C-1}{2}} \qquad \vee \qquad \overline{\rho} \geq \rho^- + \sqrt{\rho^- \frac{C-1}{2}}, \tag{3.52}$$

provided that $C > 1$, i.e. that (3.47c) holds. Observe from (3.42a)-(3.42b) that

$$\nu_2(\rho^+ - \overline{\rho}) = -\nu_1(\overline{\rho} - \rho^-).$$

Hence, in view of (3.46), either

$$\rho^- < \overline{\rho} < \rho^+ \tag{3.53}$$

or

$$\rho^+ < \overline{\rho} < \rho^- \tag{3.54}$$

have to hold.

We assume to be in the first case, i.e. when (3.53) holds. Then (3.51)-(3.52) respectively reduce to (3.47a)-(3.47b) as desired. □

3.6.2. Proof of Theorem 3.6.1. This section is devoted to the proof of Theorem 3.6.1: we will show that the set of all the conditions required for an admissible subsolution to the compressible Euler system has a solution. Since a lot of calculations are involved, we prefer to split the proof in different parts, each of which corresponds to one of the following subsections.

3.6.2.1. *Change of variables.* In this section we will rewrite the conditions required to be an admissible subsolution for the compressible Euler system in a new and more convenient set of variables.

First of all, by Theorem 3.6.3, the conditions of admissibility (3.44a)-(3.44b) for the subsolution can be replaced with (3.47a)-(3.47b).

Resuming, we are looking for constants $(\overline{\rho}, \rho^+, \rho^-, \nu_1, \nu_2, a, b, \beta, \gamma, C)$ satisfying (3.42a)-(3.42b), (3.43a)-(3.43d), (3.45), (3.47a)-(3.47c) and such that

$$\nu_1 < 0 < \nu_2 \tag{3.55}$$

$$0 < \rho^- < \overline{\rho} < \rho^+. \tag{3.56}$$

Without loss of generality we can assume $\overline{\rho} = 1$. Hence, (3.56) simplifies to

$$0 < \rho^- < 1 < \rho^+, \tag{3.57}$$

while, (3.42a)-(3.43d) become

$$\nu_2(\rho^+ - 1) = -b, \tag{3.58a}$$

$$\nu_1(1 - \rho^-) = b, \tag{3.58b}$$

$$\nu_2(\rho^+ - a) = -\gamma, \tag{3.59a}$$

$$\nu_1(a + \rho^-) = \gamma, \tag{3.59b}$$

$$-\nu_2 b = {\rho^+}^2 - 1 - \frac{C}{2} + \beta, \tag{3.59c}$$

$$\nu_1 b = 1 - {\rho^-}^2 + \frac{C}{2} - \beta. \tag{3.59d}$$

Finally, (3.47a)-(3.47b) take the form

$$\rho^- + \sqrt{\rho^- \frac{C-1}{2}} \leq 1 \tag{3.60a}$$

$$\rho^+ - \sqrt{\rho^+ \frac{C-1}{2}} \geq 1. \tag{3.60b}$$

Owing to (3.57), the admissibility conditions (3.60a)-(3.60b) change respectively into:

$$0 < \rho^- \leq \frac{1}{4}\left\{\sqrt{\left(\frac{C-1}{2} + 4\right)} - \sqrt{\frac{C-1}{2}}\right\}^2 \tag{3.61a}$$

$$\rho^+ \geq \frac{1}{4}\left\{\sqrt{\left(\frac{C-1}{2} + 4\right)} + \sqrt{\frac{C-1}{2}}\right\}^2. \tag{3.61b}$$

For the sake of simplicity, we introduce a new set of variables. We define:

$$\overline{C} := C/2, \quad \overline{b} := -b, \quad \overline{\gamma} := -\gamma, \quad \nu_+ := \nu_2, \quad \nu_- := -\nu_1$$

and

$$r^+ := \rho^+ \nu_+, \quad r^- := \rho^- \nu_-.$$

This allows us to rewrite the ensemble of conditions (3.55), (3.57), (3.58a)-(3.59d), (3.61a)-(3.61b) for an admissible subsolution as

$$\nu_-,\ \nu_+,\ \overline{b},\ r^-,\ r^+ > 0 \quad \text{and} \quad \overline{C} > 1/2, \tag{3.62}$$

$$r^+ - \nu_+ = \overline{b}, \tag{3.63}$$

$$r^- - \nu_- = -\overline{b}, \tag{3.64}$$

$$r^+ - a\nu_+ = \overline{\gamma}, \tag{3.65}$$

$$-r^- - a\nu_- = -\overline{\gamma}, \tag{3.66}$$

$$\nu_+\overline{b} = {\rho^+}^2 - 1 - \overline{C} + \beta, \tag{3.67}$$

$$\nu_-\overline{b} = 1 - {\rho^-}^2 + \overline{C} - \beta, \tag{3.68}$$

$$0 < \rho^- \leq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} - \sqrt{\overline{C} - \frac{1}{2}}\right\}^2, \tag{3.69}$$

$$\rho^+ \geq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} + \sqrt{\overline{C} - \frac{1}{2}}\right\}^2, \tag{3.70}$$

while the subsolution condition (3.45) in P translates into

$$\begin{pmatrix} \overline{C} - a^2 + \beta & a\overline{b} - \overline{\gamma} \\ a\overline{b} - \overline{\gamma} & \overline{C} - \overline{b}^2 - \beta \end{pmatrix} > 0. \tag{3.71}$$

To summarize, we are looking for constants $(r^+, r^-, \nu_+, \nu_-, a, \overline{b}, \beta, \overline{\gamma}, \overline{C})$ satisfying (3.62)-(3.71).

3.6.2.2. *Reduction of the equations.* In this section, we work on the system (3.62)-(3.71) so to simplify the conditions for an admissible subsolution.

From (3.63) and (3.65) we obtain

$$\nu_+ = \frac{\overline{\gamma} - \overline{b}}{1 - a}, \tag{3.72}$$

$$r^+ = \frac{\overline{\gamma} - a\overline{b}}{1 - a}. \tag{3.73}$$

Similarly (3.64) and (3.66) give

$$\nu_- = \frac{\overline{\gamma} + \overline{b}}{1 + a}, \tag{3.74}$$

$$r^- = \frac{\overline{\gamma} - a\overline{b}}{1 + a}. \tag{3.75}$$

Note that $r^+r^- = (\overline{\gamma} - a\overline{b})^2/(1-a^2)$ which together with (3.62) implies $a^2 < 1$. In view of this, the condition of positivity of ν_+ (see (3.62)) entails $\overline{\gamma} > \overline{b}$. On the other hand, if we assume $a^2 < 1$ and $\overline{\gamma} > \overline{b}$, then (3.72)-(3.75) respectively ensure the positivity of ν_+, r^+, ν_- and r^- required by (3.62). Thereby, (3.62)-(3.70) can be substituted by the following new systems of conditions:

$$\overline{\gamma} > \overline{b} > 0, \quad a^2 < 1, \quad \overline{C} > 1/2 \tag{3.76}$$

$$\left(\frac{\overline{\gamma} - \overline{b}}{1-a}\right)\overline{b} = \rho^{+2} - 1 - \overline{C} + \beta, \tag{3.77}$$

$$\left(\frac{\overline{\gamma} + \overline{b}}{1+a}\right)\overline{b} = 1 - \rho^{-2} + \overline{C} - \beta, \tag{3.78}$$

$$0 < \rho^- \leq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} - \sqrt{\overline{C} - \frac{1}{2}}\right\}^2, \tag{3.79}$$

$$\rho^+ \geq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} + \sqrt{\overline{C} - \frac{1}{2}}\right\}^2. \tag{3.80}$$

Finally, we note that for a symmetric 2×2-matrix M the condition $M > 0$ is equivalent to $\det M > 0$ and $\mathrm{trM} > 0$. Thus, we have reduced

the conditions for an admissible subsolution to:

$$\overline{\gamma} > \overline{b} > 0, \tag{3.81}$$

$$a^2 < 1, \tag{3.82}$$

$$\overline{C} > \frac{1}{2}, \tag{3.83}$$

$$\rho^{+2} = \left(\frac{\overline{\gamma} - \overline{b}}{1-a}\right)\overline{b} + 1 + \overline{C} - \beta, \tag{3.84}$$

$$\rho^{-2} = -\left(\frac{\overline{\gamma} + \overline{b}}{1+a}\right)\overline{b} + 1 + \overline{C} - \beta, \tag{3.85}$$

$$0 < \rho^{-} \leq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} - \sqrt{\overline{C} - \frac{1}{2}}\right\}^2, \tag{3.86}$$

$$\rho^{+} \geq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} + \sqrt{\overline{C} - \frac{1}{2}}\right\}^2, \tag{3.87}$$

$$a^2 + \overline{b}^2 < 2\overline{C}, \tag{3.88}$$

$$(\overline{C} - a^2 + \beta)(\overline{C} - \overline{b}^2 - \beta) > (\overline{\gamma} - a\overline{b})^2. \tag{3.89}$$

3.6.2.3. *Concluding argument.* In view of the previous sections, in order to prove Theorem 3.6.1 is enough to find constants $(a, \overline{b}, \beta, \overline{\gamma}, \overline{C})$

satisfying:

(3.90)

$$\overline{\gamma} > \overline{b} > 0,$$

(3.91)

$$a^2 < 1,$$

(3.92)

$$\overline{C} > \frac{1}{2},$$

(3.93)

$$0 < \sqrt{-\left(\frac{\overline{\gamma}+\overline{b}}{1+a}\right)\overline{b}+1+\overline{C}-\beta} \leq \frac{1}{4}\left\{\sqrt{\left(\overline{C}+\frac{7}{2}\right)}-\sqrt{\overline{C}-\frac{1}{2}}\right\}^2,$$

(3.94)

$$\sqrt{\left(\frac{\overline{\gamma}-\overline{b}}{1-a}\right)\overline{b}+1+\overline{C}-\beta} \geq \frac{1}{4}\left\{\sqrt{\left(\overline{C}+\frac{7}{2}\right)}+\sqrt{\overline{C}-\frac{1}{2}}\right\}^2,$$

(3.95)

$$a^2+\overline{b}^2 < 2\overline{C},$$

(3.96)

$$(\overline{C}-a^2+\beta)(\overline{C}-\overline{b}^2-\beta) > (\overline{\gamma}-a\overline{b})^2.$$

To this aim, we further introduce a new variable $\lambda := \overline{\gamma} - a\overline{b}$, so that (3.90)-(3.96) transform into the equivalent system:

(3.97)

$$\lambda > \overline{b}(1-a) > 0,$$

(3.98)

$$a^2 < 1,$$

(3.99)

$$\overline{C} > \frac{1}{2},$$

(3.100)

$$0 < \sqrt{-\frac{\lambda\overline{b}}{1+a} - \overline{b}^2 + 1 + \overline{C} - \beta} \leq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} - \sqrt{\overline{C} - \frac{1}{2}}\right\}^2,$$

(3.101)

$$\sqrt{\frac{\lambda\overline{b}}{1-a} - \overline{b}^2 + 1 + \overline{C} - \beta} \geq \frac{1}{4}\left\{\sqrt{\left(\overline{C} + \frac{7}{2}\right)} + \sqrt{\overline{C} - \frac{1}{2}}\right\}^2,$$

(3.102)

$$a^2 + \overline{b}^2 < 2\overline{C},$$

(3.103)

$$(\overline{C} - a^2 + \beta)(\overline{C} - \overline{b}^2 - \beta) > \lambda^2.$$

Let us notice that, in view of (3.102) and (3.103), the following two inequalities must hold

$$\overline{C} - a^2 + \beta > 0 \quad \text{and} \quad \overline{C} - \overline{b}^2 - \beta > 0.$$

Armed with the previous considerations, we now set $\overline{C} = 4/5\overline{b}^2$ and $\beta = -2/5\overline{b}^2$. Then, (3.97)-(3.103) transform into

(3.104)
$$\lambda > \overline{b}(1-a) > 0,$$

(3.105)
$$a^2 < 1,$$

(3.106)
$$\overline{b}^2 > \frac{5}{8},$$

(3.107)
$$0 < \sqrt{-\frac{\lambda\overline{b}}{1+a} + \frac{\overline{b}^2}{5} + 1} \leq \frac{1}{4}\left\{\sqrt{\left(\frac{4}{5}\overline{b}^2 + \frac{7}{2}\right)} - \sqrt{\frac{4}{5}\overline{b}^2 - \frac{1}{2}}\right\}^2,$$

(3.108)
$$\sqrt{\frac{\lambda\overline{b}}{1-a} + \frac{\overline{b}^2}{5} + 1} \geq \frac{1}{4}\left\{\sqrt{\left(\frac{4}{5}\overline{b}^2 + \frac{7}{2}\right)} + \sqrt{\frac{4}{5}\overline{b}^2 - \frac{1}{2}}\right\}^2,$$

(3.109)
$$a^2 < \frac{3}{5}\overline{b}^2,$$

(3.110)
$$\left(\frac{2}{5}\overline{b}^2 - a^2\right)\frac{\overline{b}^2}{5} > \lambda^2.$$

In conclusion, we are looking for $(a, \overline{b}, \lambda)$ such that (3.104)-(3.110) hold.

Let us make the choice $\overline{b} = 5$, so that if there exist a and λ satisfying the following set of conditions

(3.111)
$$\lambda > 5(1-a) > 0,$$

(3.112)
$$a^2 < 1,$$

(3.113)
$$0 < \sqrt{6 - \frac{5\lambda}{1+a}} \leq \frac{1}{4}\left(43 - \sqrt{1833}\right),$$

(3.114)
$$\sqrt{6 + \frac{5\lambda}{1-a}} \geq \frac{1}{4}\left(43 + \sqrt{1833}\right),$$

(3.115)
$$5\left(10 - a^2\right) > \lambda^2,$$

then we will obtain an admissible subsolution for the compressible Euler equations. But it easy to check that conditions (3.111)-(3.115) admit a solution (a, λ). Indeed, if we set $a = 1 - \varepsilon$ for some $\varepsilon > 0$ to be estimated, then (3.112) is immediately satisfied; on the other end the remaining inequalities become:

$$\lambda > 5\varepsilon > 0, \tag{3.116}$$

$$0 < \sqrt{6 - \frac{5\lambda}{2-\varepsilon}} \leq \frac{1}{4}\left(43 - \sqrt{1833}\right), \tag{3.117}$$

$$\sqrt{6 + \frac{5\lambda}{\varepsilon}} \geq \frac{1}{4}\left(43 + \sqrt{1833}\right), \tag{3.118}$$

$$5\left(9 + 2\varepsilon - \varepsilon^2\right) > \lambda^2, \tag{3.119}$$

Now, it is enough to plug in a small value for ε, for example $\varepsilon = 1/100$, to check by hands that the region of λ's satisfying (3.116)-(3.119) is non empty, hence the proof of Theorem 3.6.1 is concluded.

3.7. Non–standard solutions with specific pressure

3.7.1. Construction of the subsolution. In this section, we will prove the existence of an admissible subsolution for some specific smooth pressure laws with $p' > 0$, subsolution satisfying (3.33) too.

THEOREM 3.7.1. *There are a smooth pressure law p with $p' > 0$ and constants $\rho^{\pm}$ such that there exists an admissible subsolution $(\overline{\rho}, \overline{v}, \overline{u})$: $\mathbb{R}^2 \times]0, \infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ to the compressible Euler equations with*

$$\overline{v}(0) = v^0, \quad \text{and} \quad \overline{\rho}(0) = \rho^0, \tag{3.120}$$

for ρ^0 and v^0 as in (3.22)-(3.23).

REMARK 3.7.2. *The admissible subsolution of Theorem 3.6.1 will have the form*

$$(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^{3} (\overline{\rho}_i, \overline{v}_i, \overline{u}_i) \mathbf{1}_{P_i},$$

for some suitably chosen P_i, $1 \leq i \leq 3$ as in the case of quadratic pressure law.

What follows is devoted to the proof of Theorem 3.7.1.

3.7.1.1. *Subsolutions in three regions.* We aim at constructing an admissible subsolution to the compressible Euler equations of the following form (see Fig. 1):

$$\begin{aligned}(\overline{\rho},\overline{v},\overline{u}) &= (\rho^-,v^-,u^-)\mathbf{1}_{P^-}\\ &\quad+(\overline{\rho},\overline{v},\overline{u})\mathbf{1}_{P}\\ &\quad+(\rho^+,v^+,u^+)\mathbf{1}_{P^+},\end{aligned}\tag{3.121}$$

with

$$P^-:=\left\{0<t<\frac{x_2}{\nu_1}\right\},$$

$$P:=\left\{t>\frac{x_2}{\nu_1}\text{ and }t>\frac{x_2}{\nu_2}\right\},$$

$$P^+:=\left\{0<t<\frac{x_2}{\nu_2}\right\}$$

and

$$\overline{v}=(a,b),\tag{3.122}$$

$$u^-=u^+=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix},\tag{3.123}$$

$$\overline{u}=\begin{pmatrix}\beta & \gamma\\ \gamma & -\beta\end{pmatrix},\tag{3.124}$$

for some constants a,b,β,γ.

Hence, the condition (3.30) of admissibility for a subsolution takes the following easier form:

$$\begin{aligned}&\partial_t\left(\overline{\rho}\varepsilon(\overline{\rho})\right)+\operatorname{div}_x\left[\left(\overline{\rho}\varepsilon(\overline{\rho})+p(\overline{\rho})\right)\overline{v}\right]\\ &+\partial_t\left(\overline{\rho}\,\frac{C-1}{2}\,\mathbf{1}_P\right)+\operatorname{div}_x\left(\overline{\rho v}\,\frac{C-1}{2}\,\mathbf{1}_P\right)\;\leq\;0.\end{aligned}\tag{3.125}$$

Inside each of the three regions P^-, P and P^+ the equations defining a subsolution are trivially satisfied; hence they need to be imposed only along *fronts*. The system (3.29) then reads as

$$\nu_2(\rho^+-\overline{\rho})=-b\overline{\rho},\tag{3.126a}$$

$$\nu_1(\overline{\rho}-\rho^-)=b\overline{\rho},\tag{3.126b}$$

$$\nu_2(\rho^+ - \overline{\rho}a) = -\gamma\overline{\rho}, \tag{3.127a}$$

$$\nu_1(\overline{\rho}a + \rho^-) = \gamma\overline{\rho}, \tag{3.127b}$$

$$-\nu_2\overline{\rho}b = p(\rho^+) - p(\overline{\rho}) - \frac{C\overline{\rho}}{2} + \overline{\rho}\beta, \tag{3.127c}$$

$$\nu_1\overline{\rho}b = p(\overline{\rho}) - p(\rho^-) + \frac{C\overline{\rho}}{2} - \overline{\rho}\beta. \tag{3.127d}$$

The admissibility inequality (3.125) originates the following two inequalities

(3.128a)
$$\nu_2\left(\overline{\rho}\varepsilon(\overline{\rho}) - \rho^+\varepsilon(\rho^+) + \overline{\rho}\frac{C-1}{2}\right) - b\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}) + \overline{\rho}\frac{C-1}{2}\right) \leq 0$$

(3.128b)
$$\nu_1\left(\rho^-\varepsilon(\rho^-) - \overline{\rho}\varepsilon(\overline{\rho}) - \overline{\rho}\frac{C-1}{2}\right) + b\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}) + \overline{\rho}\frac{C-1}{2}\right) \leq 0.$$

Finally, the subsolution condition (3.28) in P translates into

$$\begin{pmatrix} a^2-\beta & ab-\gamma \\ ab-\gamma & b^2+\beta \end{pmatrix} < \begin{pmatrix} \frac{C}{2} & 0 \\ 0 & \frac{C}{2} \end{pmatrix}. \tag{3.129}$$

3.7.1.2. *Reduction of the admissibility condition.*

THEOREM 3.7.3. *Let us suppose that*

$$\nu_1 < 0 < \nu_2. \tag{3.130}$$

Then, there exist pressure functions $p \in C^\infty([0,+\infty[)$ *with* $p(0) = p'(0) = 0$ *and* $p' > 0$ *on* $]0,+\infty[$ *such that the admissibility conditions* (3.128a)-(3.128b) *for a subsolution are implied by the following system of inequalities:*

$$\big(p(\rho^+) - p(\overline{\rho})\big)\big(\rho^+ - \overline{\rho}\big) \geq \frac{C-1}{2}\rho^+\overline{\rho} \tag{3.131a}$$

$$\big(p(\overline{\rho}) - p(\rho^-)\big)\big(\overline{\rho} - \rho^-\big) \geq \frac{C-1}{2}\rho^-\overline{\rho}. \tag{3.131b}$$

The pressures p given by Theorem 3.7.3 will be chosen in Theorem 3.7.1 as the pressure law allowing room for the existence of an admissible subsolution. Indeed, thanks to the choice of such pressure laws the condition of admissibility (3.128a)-(3.128b) for the subsolution will be reduced to (3.131a)-(3.131b).

Proof. First, let us define $g(\rho) := \rho\varepsilon(\rho)$. In view of the relation $p(r) = r^2\varepsilon'(r)$, we obtain

$$g'(\rho) = \varepsilon(\rho) + \frac{p(\rho)}{\rho}.$$

Thus, owing respectively to (3.126a) and (3.126b), we can rewrite (3.128a) and (3.128b) as follows:

$$\nu_2(g(\overline{\rho}) - g(\rho^+)) + \nu_2(\rho^+ - \overline{\rho})g'(\overline{\rho}) + \nu_2\rho^+\frac{C-1}{2} \leq 0 \tag{3.132a}$$

$$\nu_1(g(\rho^-) - g(\overline{\rho})) + \nu_1(\overline{\rho} - \rho^-)g'(\overline{\rho}) - \nu_1\rho^-\frac{C-1}{2} \leq 0. \tag{3.132b}$$

From the hypothesis (3.130) we can further reduce (3.132a)-(3.132b) to

$$(g(\rho^+) - g(\overline{\rho})) - (\rho^+ - \overline{\rho})g'(\overline{\rho}) \geq \frac{C-1}{2}\rho^+ \tag{3.133a}$$

$$-(g(\overline{\rho}) - g(\rho^-)) + (\overline{\rho} - \rho^-)g'(\overline{\rho}) \geq \frac{C-1}{2}\rho^-. \tag{3.133b}$$

Moreover, we observe from (3.126a)-(3.126b) that

$$\nu_2(\rho^+ - \overline{\rho}) = -\nu_1(\overline{\rho} - \rho^-).$$

Hence, in view of (3.130), either

$$\rho^- < \overline{\rho} < \rho^+ \tag{3.134}$$

or

$$\rho^+ < \overline{\rho} < \rho^- \tag{3.135}$$

have to hold.

We assume to be in the first case, i.e. when (3.134) holds. Let us note that

$$(g(\sigma) - g(s)) - (\sigma - s)g'(s) = \int_s^\sigma \int_s^\tau g''(r)drd\tau$$

for every $s < \sigma$. On the other hand, by simple algebra, we can compute $g''(r) = p'(r)/r$. Hence, the following equalities hold for every $s < \sigma$:

$$(g(\sigma) - g(s)) - (\sigma - s)g'(s) = \int_s^\sigma \int_s^\tau \frac{p'(r)}{r}drd\tau$$

and

$$(g(s) - g(\sigma)) + (\sigma - s)g'(\sigma) = \int_s^\sigma \int_\tau^\sigma \frac{p'(r)}{r}drd\tau.$$

As a consequence, and in view of (3.134), we can rewrite (3.133a) and (3.133b) equivalently as

$$\int_{\overline{\rho}}^{\rho^+}\int_{\overline{\rho}}^{\tau}\frac{p'(r)}{r}drd\tau \geq \frac{C-1}{2}\rho^+, \tag{3.136a}$$

$$\int_{\rho^-}^{\overline{\rho}}\int_{\tau}^{\overline{\rho}}\frac{p'(r)}{r}drd\tau \geq \frac{C-1}{2}\rho^-. \tag{3.136b}$$

Now, we introduce two new variables q^+ and q^- defined by

$$q^+ := p(\rho^+) - p(\overline{\rho}),$$

$$q^- := p(\overline{\rho}) - p(\rho^-).$$

Proving the Theorem is then equivalent to show the existence of a pressure law p satisfying $p(\rho^+) - p(\overline{\rho}) = q^+$, $p(\overline{\rho}) - p(\rho^-) = q^-$ and for which the two inequalities (see (3.131a)-(3.131b))

$$q^+\left(\rho^+ - \overline{\rho}\right) \geq \frac{C-1}{2}\rho^+\overline{\rho} \tag{3.137a}$$

$$q^-\left(\overline{\rho} - \rho^-\right) \geq \frac{C-1}{2}\rho^-\overline{\rho}. \tag{3.137b}$$

imply (3.136a)-(3.136b) (which are equivalent to (3.128a)-(3.128b) as just shown). In the following we will prove that such a choice of pressure is possible. To this aim, we will introduce a measure theoretic setting.

First, we define the set of functions

$$\mathcal{L} := \{f \in \mathbb{C}^\infty([0,+\infty[) \text{ such that } f(0) = 0,\ f(r) > 0\,\forall r \geq 0 \text{ and}$$

$$\left.\int_{\rho^-}^{\overline{\rho}} f = q^-,\ \int_{\overline{\rho}}^{\rho^+} f = q^+\right\}$$

and the two functional defined on $\mathcal{L}$

$$L^+(f) := \int_{\overline{\rho}}^{\rho^+}\int_{\overline{\rho}}^{\tau}\frac{f(r)}{r}drd\tau,$$

$$L^-(f) := \int_{\rho^-}^{\overline{\rho}}\int_{\tau}^{\overline{\rho}}\frac{f(r)}{r}drd\tau.$$

Note that $\mathcal{L}$ is the set of derivatives of the possible pressure functions. A necessary condition to find a pressure function p with the properties above is that

$$l^+ := \sup_{f\in\mathcal{L}} L^+(f) > \frac{C-1}{2}\rho^+$$

and

$$l^- := \sup_{f\in\mathcal{L}} L^-(f) > \frac{C-1}{2}\rho^-.$$

Let us generalize the space $\mathcal{L}$ as follows. We introduce

$$\mathcal{M}^+ := \left\{\text{positive Radon measures } \mu \text{ on } [\overline{\rho},\rho^+] : \mu([\overline{\rho},\rho^+]) = q^+\right\},$$

$$\mathcal{M}^- := \left\{\text{positive Radon measures } \mu \text{ on } [\rho^-,\overline{\rho}] : \mu([\rho^-,\overline{\rho}]) = q^-\right\}.$$

Consistently, we extend the functionals L^+ and L^- defined on $\mathcal{L}$ to new functionals L_+ and L_- respectively defined on $\mathcal{M}^+$ and on $\mathcal{M}^-$. We define

$$L_+(\mu) := \int_{\overline{\rho}}^{\rho^+}\int_{\overline{\rho}}^{\tau} \frac{1}{r} d\mu(r)d\tau \quad \text{for } \mu\in\mathcal{M}^+,$$

$$L_-(\mu) := \int_{\rho^-}^{\overline{\rho}}\int_{\tau}^{\overline{\rho}} \frac{1}{r} d\mu(r)d\tau \quad \text{for } \mu\in\mathcal{M}^-.$$

Once introduced

$$m^+ := \max_{\mu\in\mathcal{M}^+} L_+(\mu)$$

and

$$m^- := \max_{\mu\in\mathcal{M}^-} L_-(\mu),$$

it is clear that

$$l^+ \le m^+ \quad \text{and} \quad l^- \le m^-.$$

Moreover, let us remark the existence of $m^\pm$ (i.e. that the maxima are achieved) due to the compactness of $\mathcal{M}^\pm$ with respect to the weak-$*$ topology. By a simple Fubini's type argument, we write

$$L_+(\mu) = \int_{\overline{\rho}}^{\rho^+} \frac{\rho^+ - r}{r}\, d\mu(r).$$

Hence, defining the function $h\in C([\overline{\rho},\rho^+])$ as $h(r) := (\rho^+ - r)/r$ allows us to express the action of the linear functional L_+ as a duality pairing; more precisely we have:

$$L_+(\mu) =< h,\mu > \qquad \text{for } \mu\in\mathcal{M}^+.$$

Analougously, if we define $g\in C([\rho^-,\overline{\rho}])$ as $g(r) := (r-\rho^-)/r$, we can express L_- as a duality pairing as well:

$$L_-(\mu) =< g,\mu > \qquad \text{for } \mu\in\mathcal{M}^-.$$

By standard functional analysis, we know that $m^\pm$ must be achieved at the extrem points of $\mathcal{M}^\pm$. The extreme points of $\mathcal{M}^\pm$ are the single-point measures, i.e. weighted Dirac masses. For $\mathcal{M}^+$ the set of extrem

points is then given by $E_+ := \{q^+\delta_\sigma \text{ for } \sigma \in [\overline{\rho}, \rho^+]\}$ while for $\mathcal{M}^-$ the set of extrem points is then given by $E_- := \{q^-\delta_\sigma \text{ for } \sigma \in [\rho^-, \overline{\rho}]\}$ In order to find $m^\pm$, it is enough to find the maximum value of $L^\pm$ on $E^\pm$. Clearly, we obtain

$$m^+ = \max_{\sigma\in[\overline{\rho},\rho^+]} \left\{ q^+ \frac{\rho^+ - \sigma}{\sigma} \right\} = q^+ \frac{\rho^+ - \overline{\rho}}{\overline{\rho}}$$

and

$$m^- = \max_{\sigma\in[\rho^-,\overline{\rho}]} \left\{ q^- \frac{\sigma - \rho^-}{\sigma} \right\} = q^- \frac{\overline{\rho} - \rho^-}{\overline{\rho}}.$$

Furthermore, for every $\varepsilon > 0$ there exists a function $f \in \mathcal{L}$ such that

$$L^+(f) > q^+ \frac{\rho^+ - \overline{\rho}}{\overline{\rho}} - \varepsilon$$

and

$$L_-(f) > q^- \frac{\overline{\rho} - \rho^-}{\overline{\rho}} - \varepsilon.$$

Such a function f is the derivative of the desired pressure function p. Indeed, since $f \in \mathcal{L}$, for $p'(\rho) := f(\rho)$ we have: $p(\rho^+) - p(\overline{\rho}) = q^+$ and $p(\overline{\rho}) - p(\rho^-) = q^-$. Finally, if (3.137a)-(3.137b) hold, then for every $\varepsilon > 0$

$$L^+(p') \geq \frac{C-1}{2}\rho^+ - \varepsilon$$

and

$$L_-(p') \geq \frac{C-1}{2}\rho^- - \varepsilon,$$

whence (3.128a)-(3.128b). □

3.7.2. Proof of Theorem 3.6.1. This section is devoted to the proof of Theorem 3.7.1: we will show that the pressure law suggested by Theorem 3.7.3 enable us to solve all the conditions required for an admissible subsolution to the compressible Euler system. Since a lot of calculations are involved, we will divide the proof in different parts corresponding to different subsections.

3.7.2.1. *Change of variables.* In this section we will rewrite the conditions required to be an admissible subsolution for the compressible Euler system in a new and more convenient set of variables.

First of all, by Theorem 3.7.3, there is a choice of pressure law such that the conditions of admissibility (3.128a)-(3.128b) for the subsolution can be replaced with (3.131a)-(3.131b).

Resuming, we are looking for constants $(\overline{\rho}, \rho^+, \rho^-, \nu^1, \nu^2, a, b, \beta, \gamma, q^+, q^-, C)$ satisfying (3.126a)-(3.126b), (3.127a)-(3.127d), (3.129), (3.131a)-(3.131b) and such that

$$\nu_1 < 0 < \nu_2 \tag{3.138}$$

$$0 < \rho^- < \overline{\rho} < \rho^+ \tag{3.139}$$

$$q^+, q^- > 0. \tag{3.140}$$

Without loss of generality we can assume $\overline{\rho} = 1$. Hence, (3.139) simplifies to

$$0 < \rho^- < 1 < \rho^+, \tag{3.141}$$

while, (3.126a)-(3.127d) become

$$\nu_2(\rho^+ - 1) = -b, \tag{3.142a}$$

$$\nu_1(1 - \rho^-) = b, \tag{3.142b}$$

$$\nu_2(\rho^+ - a) = -\gamma, \tag{3.143a}$$

$$\nu_1(a + \rho^-) = \gamma, \tag{3.143b}$$

$$-\nu_2 b = q^+ - \frac{C}{2} + \beta, \tag{3.143c}$$

$$\nu_1 b = q^- + \frac{C}{2} - \beta. \tag{3.143d}$$

Finally, (3.131a)-(3.131b) take the form

$$q^+\left(\rho^+ - 1\right) \geq \frac{C-1}{2}\rho^+ \tag{3.144a}$$

$$q^-\left(1 - \rho^-\right) \geq \frac{C-1}{2}\rho^-. \tag{3.144b}$$

From equations (3.142a)-(3.142b) we infer that

$$\rho^+ - 1 = -\frac{b}{\nu_2},$$

$$(1 - \rho^-) = \frac{b}{\nu_1}.$$

Owing to previous equalities, the admissibility conditions (3.144a)-(3.144b) change respectively into:

$$-\frac{b}{\nu_2}q^+ \geq \frac{C-1}{2}\rho^+ \tag{3.145a}$$

$$\frac{b}{\nu_1}q^- \geq \frac{C-1}{2}\rho^-. \tag{3.145b}$$

For the sake of simplicity, we introduce a new set of variables. We define:

$$\overline{C} := C/2, \quad \overline{b} := -b, \quad \overline{\gamma} := -\gamma, \quad \nu_+ := \nu_2, \quad \nu_- := -\nu_1$$

and

$$r^+ := \rho^+\nu_+, \quad r^- := \rho^-\nu_-.$$

This allows us to rewrite the ensemble of conditions (3.138), (5.22), (3.141), (3.142a)-(3.143d), (3.145a)-(3.145b) for an admissible subsolution as

$$\nu_-,\ \nu_+,\ \overline{b},\ q^-,\ q^+,\ r^-,\ r^+ > 0, \tag{3.146}$$

$$r^+ - \nu_+ = \overline{b}, \tag{3.147}$$

$$r^- - \nu_- = -\overline{b}, \tag{3.148}$$

$$r^+ - a\nu_+ = \overline{\gamma}, \tag{3.149}$$

$$-r^- - a\nu_- = -\overline{\gamma}, \tag{3.150}$$

$$\nu_+\overline{b} = q^+ - \overline{C} + \beta, \tag{3.151}$$

$$\nu_-\overline{b} = q^- + \overline{C} - \beta, \tag{3.152}$$

$$\overline{b}q^+ \geq \left(\overline{C} - \frac{1}{2}\right) r^+, \tag{3.153}$$

$$\overline{b}q^- \geq \left(\overline{C} - \frac{1}{2}\right) r^-, \tag{3.154}$$

while the subsolution condition (3.45) in P translates into

$$\begin{pmatrix} \overline{C} - a^2 + \beta & a\overline{b} - \overline{\gamma} \\ a\overline{b} - \overline{\gamma} & \overline{C} - \overline{b}^2 - \beta \end{pmatrix} > 0. \tag{3.155}$$

To summarize, we are looking for constants $(r^+, r^-, \nu_+, \nu_-, a, \overline{b}, \beta, \overline{\gamma}, q^+, q^-, \overline{C})$ satisfying (3.146)-(3.155).

3.7.2.2. *Reduction of the equations.* In this section, we work on the system (3.146)-(3.155) so to simplify the conditions for an admissible subsolution.
From (3.147) and (3.149) we obtain

$$\nu_+ = \frac{\overline{\gamma} - \overline{b}}{1-a}, \tag{3.156}$$

$$r^+ = \frac{\overline{\gamma} - a\overline{b}}{1-a}. \tag{3.157}$$

Similarly (3.148) and (3.150) give

$$\nu_- = \frac{\overline{\gamma} + \overline{b}}{1+a}, \tag{3.158}$$

$$r^- = \frac{\overline{\gamma} - a\overline{b}}{1+a}. \tag{3.159}$$

Note that $r^+r^- = (\overline{\gamma}-a\overline{b})^2/(1-a^2)$ which together with (3.146) implies $a^2 < 1$. In view of this, the condition of positivity of ν_+ (see (3.146)) entails $\overline{\gamma} > \overline{b}$. On the other hand, if we assume $a^2 < 1$ and $\overline{\gamma} > \overline{b}$, then (3.156)-(3.159) respectively ensure the positivity of ν_+, r^+, ν_- and r^- required by (3.146). Thereby, (3.146)-(3.154) can be substituted by the following new systems of conditions:

$$\overline{\gamma} > \overline{b} > 0, \quad a^2 < 1, \tag{3.160}$$

$$q^-,\, q^+ > 0, \tag{3.161}$$

$$\left(\frac{\overline{\gamma} - \overline{b}}{1-a}\right)\overline{b} = q^+ - \overline{C} + \beta, \tag{3.162}$$

$$\left(\frac{\overline{\gamma} + \overline{b}}{1+a}\right)\overline{b} = q^- + \overline{C} - \beta, \tag{3.163}$$

$$\overline{b}q^+ \geq \left(\overline{C} - \frac{1}{2}\right)\left(\frac{\overline{\gamma} - a\overline{b}}{1-a}\right), \tag{3.164}$$

$$\overline{b}q^- \geq \left(\overline{C} - \frac{1}{2}\right)\left(\frac{\overline{\gamma} - a\overline{b}}{1+a}\right). \tag{3.165}$$

Note that, if $\overline{C} > 1/2$, then (3.160) and (3.164)-(3.165) guarantee the positivity of $q^\pm$ (i.e. (3.161)). Hence, we can derive $q^\pm$ from (3.162)-(3.163) and substitute them in (3.164)-(3.165). Finally, we note that for a symmetric 2×2-matrix M the condition $M > 0$ is equivalent to $\det M > 0$ and $\mathrm{trM} > 0$. Thus, we have finally reduced the conditions

for an admissible subsolution to:

$$\overline{\gamma} > \overline{b} > 0, \tag{3.166}$$

$$a^2 < 1, \tag{3.167}$$

$$\overline{C} > \frac{1}{2}, \tag{3.168}$$

$$\overline{b}\left[\frac{\overline{b}(\overline{\gamma}-\overline{b})}{1-a} + \overline{C} - \beta\right] \geq \left(\overline{C} - \frac{1}{2}\right)\left(\frac{\overline{\gamma}-a\overline{b}}{1-a}\right), \tag{3.169}$$

$$\overline{b}\left[\frac{\overline{b}(\overline{\gamma}+\overline{b})}{1+a} - \overline{C} + \beta\right] \geq \left(\overline{C} - \frac{1}{2}\right)\left(\frac{\overline{\gamma}-a\overline{b}}{1+a}\right), \tag{3.170}$$

$$a^2 + \overline{b}^2 < 2\overline{C}, \tag{3.171}$$

$$(\overline{C} - a^2 + \beta)(\overline{C} - \overline{b}^2 - \beta) - (a\overline{b} - \overline{\gamma})^2 > 0. \tag{3.172}$$

3.7.2.3. *Concluding argument.* In view of the previous sections, in order to prove Theorem 3.7.1 is enough to find constants $(a, \overline{b}, \beta, \overline{\gamma}, q^+, q^-, \overline{C})$ satisfying (3.166)-(3.172). To this aim, we further introduce a new variable $\lambda := \overline{\gamma} - a\overline{b}$, so that (3.166)-(3.172) transform into the equivalent system:

$$\lambda > \overline{b}(1-a) > 0, \tag{3.173}$$

$$a^2 < 1, \tag{3.174}$$

$$\overline{C} > \frac{1}{2}, \tag{3.175}$$

$$\overline{b}(1-a)\left(\overline{C} - \overline{b}^2 - \beta\right) \geq \left(\overline{C} - \frac{1}{2} - \overline{b}^2\right)\lambda, \tag{3.176}$$

$$\overline{b}(1+a)\left(-\overline{C} + \overline{b}^2 + \beta\right) \geq \left(\overline{C} - \frac{1}{2} - \overline{b}^2\right)\lambda, \tag{3.177}$$

$$a^2 + \overline{b}^2 < 2\overline{C}, \tag{3.178}$$

$$(\overline{C} - a^2 + \beta)(\overline{C} - \overline{b}^2 - \beta) > \lambda^2. \tag{3.179}$$

Now, from (3.178)-(3.179) follows that $(\overline{C} - \overline{b}^2 - \beta) > 0$; hence (3.177) and (3.173) imply that $(\overline{C} - 1/2 - \overline{b}^2) < 0$, which automatically guarantees (3.176).

Let us notice that if there exists a quadruple $(a, \overline{b}, \overline{C}, \beta)$ satisfying

$$\overline{b} > 0 \tag{3.180}$$

$$\overline{C} > \frac{1}{2} \tag{3.181}$$

$$a^2 < 1, \tag{3.182}$$

$$\overline{C} - a^2 + \beta > 0 \tag{3.183}$$

$$\overline{C} - \overline{b}^2 - \beta > 0 \tag{3.184}$$

$$\left(\overline{b}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - a^2 + \beta} > \overline{b}(1+a)\sqrt{\overline{C} - \overline{b}^2 - \beta} \tag{3.185}$$

$$\sqrt{\left(\overline{C} - a^2 + \beta\right)\left(\overline{C} - \overline{b}^2 - \beta\right)} > (1-a)\overline{b}, \tag{3.186}$$

then choosing $\lambda := \sqrt{\left(\overline{C} - a^2 + \beta\right)\left(\overline{C} - \overline{b}^2 - \beta\right)} - \delta$ for $\delta > 0$ small enough, the quintuple $(a, \overline{b}, \overline{C}, \beta, \lambda)$ satisfies (3.173)-(3.179).

Furthermore if $(\overline{b}, \overline{C}, \beta)$ satisfy (3.180), (3.181), (3.184) and

$$\overline{C} - 1 + \beta > 0 \tag{3.187}$$

$$\left(\overline{b}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - 1 + \beta} > 2\overline{b}\sqrt{\overline{C} - \overline{b}^2 - \beta}, \tag{3.188}$$

then choosing $a := 1 - \varepsilon$ for $\varepsilon > 0$, they will satisfy (3.180)-(3.186).

Armed with the previous considerations, we now set $\overline{C} = 4/5\overline{b}^2$ and $\beta = -2/5\overline{b}^2$. Then (3.184) is automatically satisfied. The remaining conditions (3.180), (3.181) and (3.187) are satisfied as soon as

$$\overline{b} > \sqrt{\frac{5}{2}}. \tag{3.189}$$

Finally, with this choice of $\overline{C}$ and β, the inequality (3.188) is equivalent to:

$$\left(\frac{\overline{b}^2}{5} + \frac{1}{2}\right)\sqrt{\frac{2\overline{b}^2}{5} - 1} > \frac{2\overline{b}^2}{\sqrt{5}}, \tag{3.190}$$

which holds surely in the limit for $\overline{b} \to +\infty$. In conclusion, by choosing $\overline{b} > \sqrt{5/3}$ big enough, $\overline{C} = 4/5\overline{b}^2$, $\beta = -2/5\overline{b}^2$, $a = 1 - \varepsilon$ and $\lambda = \sqrt{\left(\overline{C} - a^2 + \beta\right)\left(\overline{C} - \overline{b}^2 - \beta\right)} - \delta$, for $\delta > 0$ small enough, all the conditions (3.173)-(3.179) are satisfied: we have proven the existence

of an admissible subsolution for a particular choice of the pressure law given by Theorem 3.7.3.

CHAPTER 4

Study of a classical Riemann problem for the compressible Euler equations

This chapter is a complement to Chapter 3. Here we restrict our attention to the 1-*dimensional Riemann problem* for the compressible Euler equations with the same choice of initial data allowing for the non–uniqueness theorems proven in Chapter 3 (see Theorem 0.2.2 or Theorem 3.1.1) (such data indeed depend only on one space variable): we show that such a problem admits unique self-similar solutions. In particular we will investigate more deeply the case of quadratic pressure law (as in Theorem 3.3.1) which is of easier treatise. The uniqueness of self–similar solutions is proven by direct construction of the admissible wave fun. Theorem 0.2.2 shows that as soon as the self–similarity assumption runs out, uniqueness is lost.

4.1. Solution of the Riemann problem via wave curves

The property of first order conservation laws in one space-dimension to be invariant under uniform stretching of the space-time coordinates induces the existence of self-similar solutions, which stay constant along straight-line rays emanating from the origin in space-time. This observation is at the core of the study of Riemann problems via wave fans.

Let us rewrite the isentropic Euler system (3.20) in canonical form, i.e. in terms of the state variables (ρ, m) where m denotes the linear momentum as was done in Chapter 2:

$$\left\{\begin{array}{l} \partial_t \rho + \mathrm{div}_x(m) \;=\; 0 \\ \partial_t m + \mathrm{div}_x\left(\frac{m\otimes m}{\rho}\right) + \nabla_x p(\rho) \;=\; 0 \\ \rho(\cdot,0) \;=\; \rho^0 \\ m(\cdot,0) \;=\; m^0. \end{array}\right. \tag{4.1}$$

With the new variables, the energy inequality (3.21) takes the following form:

$$\partial_t \left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) + \operatorname{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \leq 0. \tag{4.2}$$

Similarly, we can rewrite the initial data (3.22)-(3.23) used in Chapter 3 for (ρ, m) and obtain

$$\rho^0(x) := \begin{cases} \rho^+ & \text{if } x_2 > 0, \\ \rho^- & \text{if } x_2 < 0, \end{cases} \tag{4.3}$$

$$m^0(x) := \begin{cases} m^+ := (\rho^+, 0) & \text{if } x_2 > 0, \\ m^- := (-\rho^-, 0) & \text{if } x_2 < 0. \end{cases} \tag{4.4}$$

If we restrict our attention to pairs (ρ, v) which are admissible solutions of (4.1), (4.3)-(4.4) and depend only on (x_2, t), then we will be dealing with a classical Riemann problem for (4.1) substantially in one space-variable (only x_2) which admits self-similar solutions. Since the pioneering work of Riemann it is known that, under the hypothesis of "self-similarity" of (ρ, v) (dependence only on $\frac{x_2}{t}$), there is a unique solution of (4.1), (4.3)-(4.4) (see for instance [**Ser99**]). Surprisingly, Theorem 3.3.1 of Chapter 3 shows that uniqueness is completely lost if we drop the requirement that (ρ, v) depends only on $\frac{x_2}{t}$.

This section aims at investigating the Riemann problem (4.1), (4.3)-(4.4), whose object is the resolution of the initial jump discontinuity into wave fans: the solution will be constructed by the classical method of piecing together elementary centered solutions, i.e., constant states, shocks joining constant states, and centered rarefaction waves bordered by constant states or contact discontinuities. The wave fan which will be constructed will also satisfy the admissibility condition (3.26).

Now, the plan is to construct weak admissible solutions (ρ, m) to the Riemann problem (4.1)-(4.3)-(4.4) which depend only on the space variable x_2:

$$(\rho(x,t), m(x,t)) = (\rho(x_2,t), m(x_2,t)).$$

As already anticipated, we observe that (4.1) is invariant under coordinates stretching $(x,t) \mapsto (\alpha x, \alpha t)$. Hence (4.1) admits self-similar solutions defined on the $x_2 - t$ plane. Under our assumptions, it is convenient to make explicit the divergence operators in (4.1) and (4.2).

Indeed, for $\rho(x,t) = \rho(x_2,t)$ and $m(x,t) = (m_1(x_2,t), m_2(x_2,t))$, we can write the system (4.1) as

$$\begin{cases} \partial_t \rho + \partial_{x_2}(m_2) = 0 \\ \partial_t m_1 + \partial_{x_2}\left(\frac{m_1 m_2}{\rho}\right) = 0 \\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + p(\rho)\right) = 0 \\ \rho(\cdot,0) = \rho^0 \\ m(\cdot,0) = m^0, \end{cases} \tag{4.5}$$

while the energy inequality (4.2) becomes

$$\partial_t\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}\right) + \partial_{x_2}\left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right) m_2\right] \leq 0. \tag{4.6}$$

If (ρ, m) is a self-similar solution of (4.5), focused at the origin, its restriction to $t > 0$ admits the representation

$$(\rho, m)(x,t) = (R, M)\left(\frac{x_2}{t}\right), \quad -\infty < x < \infty, \quad 0 < t < \infty, \tag{4.7}$$

where (R, M) is a bounded measurable function on $(-\infty, \infty)$, which satisfies the ordinary differential equations

$$\begin{aligned} &[M_2(\xi) - \xi R(\xi)]^{\cdot} + R(\xi) = 0 \\ &\left[\frac{M_1(\xi)M_2(\xi)}{R(\xi)} - \xi M_1(\xi)\right]^{\cdot} + M_1(\xi) = 0 \\ &\left[\frac{M_2(\xi)^2}{R(\xi)} + p(R(\xi)) - \xi M_2(\xi)\right]^{\cdot} + M_2(\xi) = 0, \end{aligned}$$

in the sense of distributions.

Before describing the structure of the solution to the Riemann problem, let us illustrate some general concepts concerning system (4.5). If we define the state vector $U := (\rho, m_1, m_2)$, we can recast the system (4.5) in the general form

$$\partial_t U + \partial_{x_2} F(U) = 0,$$

where

$$F(U) := \begin{pmatrix} m_2 \\ \frac{m_1 m_2}{\rho} \\ \frac{{m_2}^2}{\rho} + p(\rho) \end{pmatrix}.$$

By definition (cf. [**Daf00**]) the system (4.5) is hyperbolic since the Jacobian matrix $DF(U)$

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-m_1 m_2}{\rho^2} & \frac{m_2}{\rho} & \frac{m_1}{\rho} \\ \frac{-{m_2}^2}{\rho^2} + p'(\rho) & 0 & \frac{2m_2}{\rho} \end{pmatrix}$$

has real eigenvalues

$$\lambda_1 = \frac{m_2}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m_2}{\rho}, \qquad \lambda_3 = \frac{m_2}{\rho} + \sqrt{p'(\rho)} \tag{4.8}$$

and 3 linearly independent eigenvectors

$$R_1 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} + \sqrt{p'(\rho)} \end{pmatrix}. \tag{4.9}$$

The eigenvalue λ_i of DF, $i = 1, 2, 3$, is called the *i-characteristic speed* of the system (4.5). On the part of the state space of our interest, with $\rho > 0$, the system (4.5) is indeed strictly hyperbolic. Finally, one can easily verify that the functions

$$w_1 = \frac{m_2}{\rho} + \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \qquad w_2 = \frac{m_1}{\rho}, \qquad w_3 = \frac{m_2}{\rho} - \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau \tag{4.10}$$

are, respectively, 1− and 2−, 1− and 3−, 2− and 3− Riemann invariants of the system (4.5) (for the relevant definitions see [**Daf00**]). In other words there exist two 1−Riemann invariants w_1 and w_2, two 2−Riemann invariants w_1 and w_3 and two 3−Riemann invariants w_2 and w_3.

We close this section with a key observation: note that the state variable m_1 appears only in the second equation of the system (4.5). We can thus "decouple" the study of the first and third equations in (4.5) from the study of the second one: this is possible by performing a sort of "projection" operation on the $\rho - m_2$-plane. More precisely, we will first construct solutions (ρ, m_2) and then analyze the behaviour of m_1 determined by the second equation in (4.5).

4.2. The Hugoniot locus

We focus our attention on the reduced system

$$\begin{cases} \partial_t \rho + \partial_{x_2}(m_2) \;=\; 0 \\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + p(\rho)\right) \;=\; 0 \\ \rho(\cdot,0) \;=\; \rho^0 \\ m_2(\cdot,0) \;=\; (m^0)_2, \end{cases} \tag{4.11}$$

obtained by discarding the second equation in (4.5). Note that, if we define the state variable $\tilde{U} = (\rho, m_2)$ and we formally recast the system (4.11) in the form $\partial_t \tilde{U} + \partial_{x_2} G(\tilde{U}) = 0$ for $\tilde{U} = P_{1,3}U$, then we have $G(\tilde{U}) = P_{1,3}F(U)$ and $\widetilde{D}G(\tilde{U}) = P_{1,3}DF(U)P_{1,3}^T$ ($\widetilde{D} = D_{\tilde{U}}$), where $P_{1,3}$ is the following matrix:

$$P_{1,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, the characteristic speeds of the system (4.11) are λ_1 and λ_3 with associated eigenvectors $P_{1,3}\cdot R_1$ and $P_{1,3}\cdot R_3$ (see (4.8)-(4.9)). The *Hugoniot locus* of the reduced system (4.11) is the set of points $\tilde{U}_2 = (\rho, m_2)$ that may be joined to a fixed point $\tilde{U}_1 = (\overline{\rho}, \overline{m_2})$ by a shock. In view of the previous remark, we can observe that Hugoniot loci of the reduced system (4.11) correspond to projections on the $\rho - m_2$- plane of Hugoniot loci of the full system (4.5). In our case, we can describe the Hugoniot locus for (4.11) explicitly by computing the *Rankine-Hugoniot* jump conditions:

$$G(\tilde{U}_2) - G(\tilde{U}_1) = \sigma(\tilde{U}_2 - \tilde{U}_1),$$

which can be written, for U_1, U_2 such that $\tilde{U}_1 = P_{1,3}U_1$ and $\tilde{U}_2 = P_{1,3}U_2$, as

$$P_{1,3}[F(U_2) - F(U_1)] = P_{1,3}\sigma(U_2 - U_1).$$

Note, that the set of equations $[F(U_2) - F(U_1)] = \sigma(U_2 - U_1)$ describes indeed the Hugoniot locus of the full system (4.5).

Now, we would like to investigate the Hugoniot locus of the state (ρ^-, m^-). We start by studying its projection on the $\rho - m_2$-plane, i.e. by computing the Hugoniot locus of system (4.11). The state $(\rho^-, (m^-)_2) = (\rho^-, 0)$ on the left is joined to the state (ρ, m_2) on the

right, by a shock of speed σ if the following equations hold:

$$\begin{cases} -\sigma(\rho - \rho^-) + m_2 &= 0 \\ -\sigma m_2 + \frac{{m_2}^2}{\rho} + p(\rho) - p(\rho^-) &= 0. \end{cases} \tag{4.12}$$

We can now restrict ourselves to the case of a polytropic pressure law, with adiabatic exponent $\gamma = 2$ for which the non-uniqueness result 3.3.1 stated in Chapter 3 holds. Thus, from now on we will assume that

$$p(r) = kr^2$$

for some positive constant k. As a consequence, the Rankine-Hugoniot conditions (4.12) take the form

$$\begin{cases} -\sigma(\rho - \rho^-) + m_2 &= 0 \\ -\sigma m_2 + \frac{{m_2}^2}{\rho} + k\rho^2 - k(\rho^-)^2 &= 0. \end{cases} \tag{4.13}$$

From (4.13) we infer that

$$\sigma = \pm\sqrt{\frac{k\rho(\rho + \rho-)}{\rho-}}. \tag{4.14}$$

Recalling the characteristic speeds λ_1 and λ_3 for the system (4.11), it is natural to call shocks propagating to the left ($\sigma_1 = -\sqrt{k\rho(\rho + \rho-)/\rho-} < 0$) 1-shocks and shocks propagating to the right ($\sigma_3 = \sqrt{k\rho(\rho + \rho-)/\rho-} > 0$) 3-shocks. Combining (4.13) with (4.14) we deduce that the Hugoniot locus of the point $(\rho^-, 0)$ in state space consists of two curves:

$$m_2 = \pm\sqrt{\frac{k\rho\,(\rho + \rho^-)}{\rho^-}}\,\left(\rho - \rho^-\right), \tag{4.15}$$

defined on the whole range of $\rho > 0$. Moreover a 1-shock joining $(\rho^-, 0)$ on the left to (ρ, m_2) on the right is admissible, i.e. it satisfies the entropy condition (4.2) if and only if $\rho^- < \rho$. While a 3-shock joining the state (ρ, m_2) on the left with the state $(\rho^+, 0)$ on the right is admissible if and only if $\rho > \rho^+$.

4.3. Rarefaction waves

In order to characterize rarefaction waves of the reduced system (4.11), we can refer to Theorem 7.6.6 from [**Daf00**]: every i-Riemann invariant is constant along any i-rarefaction wave curve of the system (4.11) and conversely the i- rarefaction wave curve, through a state

$(\overline{\rho}, \overline{m}_2)$ of genuine nonlinearity of the i-characteristic family, is determined implicitly by the system of equations $w_i(\rho, m_2) = w_i(\overline{\rho}, \overline{m}_2)$ for every $i-$Riemann invariant w_i. As an application of this Theorem, we obtain that the 1- and 3-rarefaction wave curves of the system (4.11) through the point $(\rho^-, 0)$ are determined respectively in terms of the Riemann invariants w_1 and w_3 by the equations

$$m_2 = 2\rho\sqrt{2k}(\sqrt{\rho^-} - \sqrt{\rho}), \qquad m_2 = 2\rho\sqrt{2k}(\sqrt{\rho} - \sqrt{\rho^-}). \tag{4.16}$$

Similarly, we can obtain the rarefaction waves through the point $(\rho^+, 0)$. As already discussed for the Hugoniot locus, the rarefaction waves here obtained for the reduced system (4.11) correspond to projections on the $\rho - m_2$-plane of rarefaction waves of (4.5). In order to determine the first component of the linear momentum along the rarefaction waves we will make use of the second Riemann invariant w_2 for the complete system (4.5).

4.4. Contact discontinuity

It is easy to verify that the 2-characteristic family of the system (4.5) is linearly degenerate, i.e. $D\lambda_2 \cdot R_2 = 0$. As a consequence, no centered 2-rarefaction waves for the system (4.5) exist and hence any 2-wave is necessarily a 2-contact discontinuity. A 2-contact discontinuity is a shock traveling at characteristic speed λ_2 (see (4.8)) and such that the only variable which experiences a jump is the first component of the linear momentum. The entropy condition holds trivially across the contact discontinuity, for the rate of production of entropy is indeed zero.

4.5. Solution of the Riemann problem

According to Theorem 9.3.1 in [**Daf00**] any self-similar solution of the Riemann problem (4.5), (4.3)-(4.4), with shocks satisfying the entropy inequality, comprises 4 constant states $U_0 = (\rho_0, m_0) = (\rho^-, m^-)$, U_1, U_2, $U_3 = (\rho^+, m^+)$. For $i = 1, 2, 3$, U_{i-1} is joined to U_i by an i-wave.

In the following we will construct such a self-similar solution by piecing together shocks, rarefaction waves and contact discontinuities obtained in the previous sections. As in the literature, we will call *forward* (or *backward*) *i-wave fan curve through* $(\overline{\rho}, \overline{m})$ the Lipschitz curve $\Phi_i(\cdot, (\overline{\rho}, \overline{m}))$ (or $\Psi_i(\cdot, (\overline{\rho}, \overline{m}))$) describing the locus of states that may

be joined on the right (or left) of the fixed state $(\overline{\rho}, \overline{m})$ by an admissible i-wave fan. Clearly, we could obtain a solution of our Riemann problem starting from (ρ^-, m^-) and computing successively the states $(\rho_i, m_i) = \Phi_i(\cdot, (\rho_i, m_i))$ until we reach (ρ^+, m^+). In our circumstances, a "mixed" strategy proves to be advantageous. We will first consider the system (4.11) of two conservation laws in the two variables (ρ, m_2), then we will draw for it the forward 1-wave curve through the left state $(\rho^-, 0)$ and the backward 3-wave curve through the right state $(\rho^+, 0)$ and finally we will determine the intermediate state as the intersection of these two curves. In order to complete the picture, we will introduce a 2-contact discontinuity accounting for the jump in the first component of the linear momentum.

Recalling the form of the Hugoniot locus (4.15) and rarefaction wave curves (4.16) for system (4.11), we deduce that we can parametrize the wave curves employing ρ as the parameter. Thus, the forward 1-wave curve $m_2 = \Phi_1(\rho; (\rho^-, 0))$ through the point $(\rho^-, 0)$ consists of a 1-rarefaction wave for $\rho^- \geq \rho$ and an admissible 1-shock for $\rho^- < \rho$:

$$m_2 = \Phi_1(\rho; (\rho^-, 0)) = \begin{cases} 2\rho\sqrt{2k}(\sqrt{\rho^-} - \sqrt{\rho}) & \text{if } \rho^- \geq \rho \\ -\sqrt{\frac{k\rho(\rho+\rho^-)}{\rho^-}}\,(\rho - \rho^-) & \text{if } \rho^- < \rho. \end{cases} \tag{4.17}$$

On the other hand, the backward 3-wave curve through the point $(\rho^+, 0)$ is composed of a 3-rarefaction wave for $\rho^+ \geq \rho$ and an admissible 3-shock for $\rho^+ < \rho$:

$$m_2 = \Psi_3(\rho; (\rho^+, 0)) = \begin{cases} -2\rho\sqrt{2k}(\sqrt{\rho^+} - \sqrt{\rho}) & \text{if } \rho^+ \geq \rho \\ \sqrt{\frac{k\rho(\rho+\rho^+)}{\rho^+}}\,(\rho - \rho^+) & \text{if } \rho^+ < \rho. \end{cases} \tag{4.18}$$

The intermediate constant state (ρ_M, m_M) is determined on the $\rho - m_2$ plane as the intersection of the forward 1-wave curve $\Phi_1(\rho; (\rho^-, 0))$ with the backward 3-wave curve $\Psi_3(\rho; (\rho^+, 0))$ (see Fig. 1), namely by solving the equation

$$m_M = \Phi_1(\rho_M; (\rho^-, 0)) = \Psi_3(\rho_M; (\rho^+, 0)).$$

Note that such intersection is unique (if $\rho > 0$), since Φ_1 is a strictly decreasing function in ρ for $\rho > 4/9\rho^-$ with $\Phi_1 > 0$ for $0 < \rho < \rho^-$, while Ψ_3 is a strictly increasing function of ρ for $\rho > 4/9\rho^+$, always strictly convex and negative for every $0 < \rho < \rho^+$.

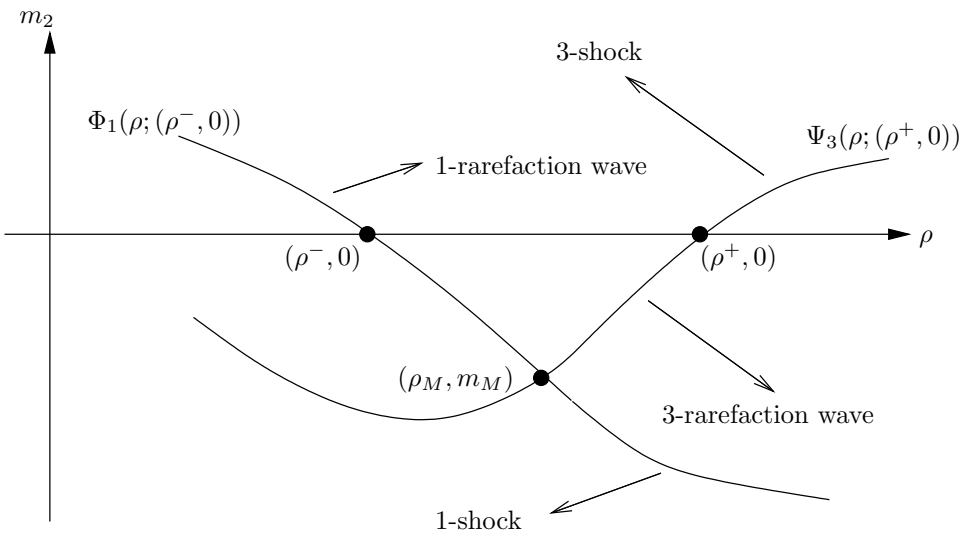

FIGURE 1. $\rho - m_2$ plane

So far, we have discussed the resolution of the reduced system (4.11), thus obtaining a complete description of the behavior of the density and of the second component of the linear momentum. Now, also the second equation of (4.5) governing the first component of the linear momentum has to come into play. By imposing the Rankine-Hugoniot condition corresponding to the second equation of (4.5) across the 1-shock traveling at speed σ_1, we can compute the uniquely determined value of the first component of the linear momentum of the second constant state U_1 and we obtain

$$U_1 = (\rho_M, (-\rho_M, m_M)).$$

The second state U_1 is then joined to the third state U_2 by a 2-contact discontinuity traveling at speed $\sigma_2 = m_M/\rho_M$. Note that $\sigma_2 \geq \sigma_1$. The entropy condition for the 2-contact discontinuity is always satisfied since across it the rate of production of entropy is zero. Moreover, the only variable which jumps across the 2-contact discontinuity is the first component of the linear momentum. Hence, U_2 has the form

$$U_2 = (\rho_M, (n, m_M)),$$

with n to be determined later. Finally, the third state U_2 will be joined to $U_3 = (\rho^+, m^+)$ by a 3- rarefaction wave. As previously explained, along a 3-rarefaction wave all the 3−Riemann invariants have to stay

constant. So far we have only considered the 3−Riemann invariant w_3. Now, in order to determine the behavior of the first component of the linear momentum along the 3−rarefaction wave, we can impose the condition $w_2(U_2) = w_2(U_3)$ which yields

$$\frac{n}{\rho_M} = 1,$$

whence $n = \rho_M$. Finally, the intermediate state U_2 will have the form

$$U_2 = (\rho_M, (\rho_M, m_M)).$$

According to the analysis carried out, the unique solution to the Riemann problem (4.5)-(4.3)-(4.4), with end-states (ρ^-, m^-) and (ρ^+, m^+), comprises a compressive 1-shock joining (ρ^-, m^-) with the state $(\rho_M, (-\rho_M, m_M))$, followed by a 2-contact discontinuity that joins the state $(\rho_M, (-\rho_M, m_M))$ with the state $(\rho_M, (\rho_M, m_M))$, and a 3-rarefaction wave , joining $(\rho_M, (\rho_M, m_M))$ with (ρ^+, m^+) (see Fig. 2).

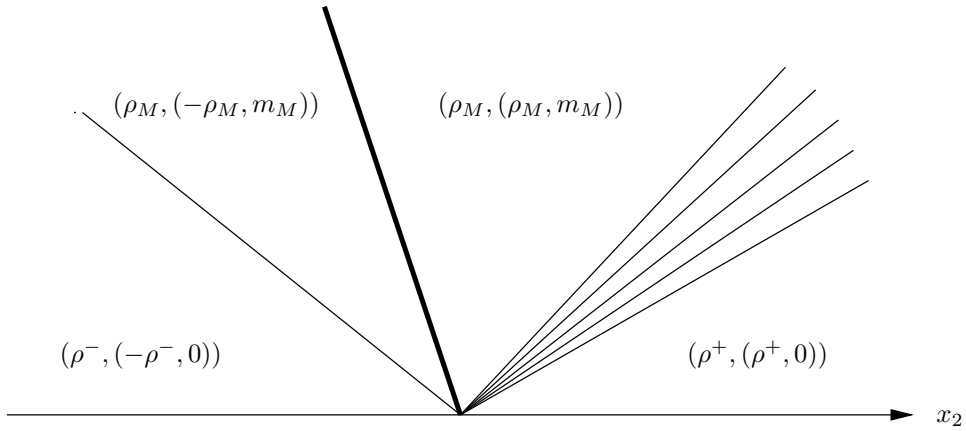

FIGURE 2. Solution

4.6. The case of general pressure

The same analysis carried out in the previous sections for the quadratic pressure law, can be done for general pressure laws. In this section, we present the relative computations.

4.6.1. The Hugoniot locus. As before, we deal with the reduced system (4.11). From (4.13) we infer that for general pressure p:

$$\sigma = \pm\sqrt{\frac{\rho(p(\rho) - p(\rho^-))}{\rho^-(\rho - \rho^-)}}. \tag{4.19}$$

Recalling the characteristic speeds λ_1 and λ_3 for the system (4.11), it is natural to call shocks propagating to the left $\left(\sigma_1 = -\sqrt{\frac{\rho(p(\rho)-p(\rho^-))}{\rho^-(\rho-\rho^-)}} < 0\right)$ 1-shocks and shocks propagating to the right $\left(\sigma_3 = \sqrt{\frac{\rho(p(\rho)-p(\rho^-))}{\rho^-(\rho-\rho^-)}} > 0\right)$ 3-shocks. Combining (4.13) with (4.19) we deduce that the Hugoniot locus of the point $(\rho^-, 0)$ in state space consists of two curves:

$$m_2 = \pm\sqrt{\frac{\rho(p(\rho) - p(\rho^-))}{\rho^-(\rho - \rho^-)}}\left(\rho - \rho^-\right), \tag{4.20}$$

defined on the whole range of $\rho > 0$. Moreover a 1-shock joining $(\rho^-, 0)$ on the left to (ρ, m_2) on the right is admissible, i.e. it satisfies the entropy condition (4.2) if and only if $\rho^- < \rho$. While a 3-shock joining the state (ρ, m_2) on the left with the state $(\rho^+, 0)$ on the right is admissible if and only if $\rho > \rho^+$.

4.6.2. Rarefaction waves. By use of Theorem 7.6.6 from [**Daf00**] the i- rarefaction wave curve, through a state $(\overline{\rho}, \overline{m}_2)$ of genuine nonlinearity of the i-characteristic family, is determined implicitly by the system of equations $w_i(\rho, m_2) = w_i(\overline{\rho}, \overline{m}_2)$. As an application of this Theorem, we obtain that the 1- and 3-rarefaction wave curves of the system (4.11) through the point $(\rho^-, 0)$ are respectively determined in terms of the Riemann invariants w_1 and w_3 by the equations

$$m_2 = \rho\int_{\rho}^{\rho^-} \frac{\sqrt{p'(\tau)}}{\tau}d\tau, \qquad m_2 = \rho\int_{\rho^-}^{\rho} \frac{\sqrt{p'(\tau)}}{\tau}d\tau. \tag{4.21}$$

The rarefaction waves through the point $(\rho^+, 0)$ can be obtained in a similar way.

4.6.3. Contact discontinuity. As already pointed out, the 2-characteristic family of the system (4.5) is linearly degenerate, i.e. $D\lambda_2 \cdot R_2 = 0$. Any 2-wave is a 2-contact discontinuity, i.e. a shock traveling at characteristic speed λ_2. Along the 2-contact discontinuity a jump in the first component of the linear momentum occurs. The entropy

condition holds trivially across the contact discontinuity, for the rate of production of entropy is indeed zero.

4.6.4. Solution of the Riemann problem. As done in Section 4.5, we will apply Theorem 9.3.1 in [**Daf00**]. The definitions as well as the strategies introduced in Section 4.5 are valid also herein.

Recalling the form of the Hugoniot locus (4.20) and rarefaction wave curves (4.21) for system (4.11) with general pressure laws, we deduce that we can parametrize the wave curves employing ρ as the parameter. Thus, the forward 1-wave curve $m_2 = \Phi_1(\rho; (\rho^-, 0))$ through the point $(\rho^-, 0)$ consists of a 1-rarefaction wave for $\rho^- \geq \rho$ and an admissible 1-shock for $\rho^- < \rho$:

$$m_2 = \Phi_1(\rho; (\rho^-, 0)) = \begin{cases} \rho \int_\rho^{\rho^-} \frac{\sqrt{p'(\tau)}}{\tau} d\tau & \text{if } \rho^- \geq \rho \\ -\sqrt{\frac{\rho(p(\rho)-p(\rho^-))}{\rho^-(\rho-\rho^-)}} \, (\rho - \rho^-) & \text{if } \rho^- < \rho. \end{cases} \tag{4.22}$$

On the other hand, the backward 3-wave curve through the point $(\rho^+, 0)$ is composed of a 3-rarefaction wave for $\rho^+ \geq \rho$ and an admissible 3-shock for $\rho^+ < \rho$:

$$m_2 = \Psi_3(\rho; (\rho^+, 0)) = \begin{cases} -\rho \int_\rho^{\rho^+} \frac{\sqrt{p'(\tau)}}{\tau} d\tau & \text{if } \rho^+ \geq \rho \\ \sqrt{\frac{\rho(p(\rho)-p(\rho^+))}{\rho^+(\rho-\rho^+)}} \, (\rho - \rho^+) & \text{if } \rho^+ < \rho. \end{cases} \tag{4.23}$$

The intermediate constant state (ρ_M, m_M) is determined on the $\rho - m_2$ plane as the intersection of the forward 1-wave curve $\Phi_1(\rho; (\rho^-, 0))$ with the backward 3-wave curve $\Psi_3(\rho; (\rho^+, 0))$, namely by the equation

$$m_M = \Phi_1(\rho_M; (\rho^-, 0)) = \Psi_3(\rho_M; (\rho^+, 0)).$$

We argue that such intersection is unique for the choice of pressure functions as in Theorem 3.7.3 of Chapter 3. Indeed m_M is uniquely determined, since Φ_1 is a strictly decreasing function in ρ with $\Phi_1 > 0$ for $0 < \rho < \rho^-$, while Ψ_3 is negative for every $0 < \rho < \rho^+$ with Ψ_3 convex for $\rho < \widetilde{\rho}$ and concave for $\rho > \widetilde{\rho}$, where $\widetilde{\rho}$ is a density-value in a small neighborhood of 1 (recall from Section 3.7.2 of Chapter 3 that in the construction of the subsolution we chose $\overline{\rho} = 1$).

By imposing the Rankine-Hugoniot condition corresponding to the second equation of (4.5) across the 1-shock traveling at speed σ_1, we can compute the uniquely determined value of the first component of

the linear momentum of the second constant state U_1 and we obtain

$$U_1 = (\rho_M, (-\rho_M, m_M)).$$

The second state U_1 is then joined to the third state U_2 by a 2-contact discontinuity traveling at speed $\sigma_2 = m_M/\rho_M$. Note that $\sigma_2 \geq \sigma_1$. The entropy condition for the 2-contact discontinuity is always satisfied since across it the rate of production of entropy is zero. Moreover, the only variable which jumps across the 2-contact discontinuity is the first component of the linear momentum. Hence, U_2 has the form

$$U_2 = (\rho_M, (n, m_M)),$$

with n to be determined later. Finally, the third state U_2 will be joined to $U_3 = (\rho^-, m^-)$ by a 3- rarefaction wave along which the Riemann invariant w_2 has to stay constant: $w_2(U_2) = w_2(U_3)$, i.e.

$$\frac{n}{\rho_M} = 1.$$

This implies that $n = \rho_M$, whence

$$U_2 = (\rho_M, (\rho^+, m_M)).$$

The unique solution to the Riemann problem (4.5)-(4.3)-(4.4) for pressure laws as in Theorem **??**, with end-states (ρ^-, m^-) and (ρ^+, m^+), comprises a compressive 1-shock joining (ρ^-, m^-) with the state $(\rho_M, (-\rho_M, m_M))$, followed by a 2-contact discontinuity that joins the state $(\rho_M, (-\rho_M, m_M))$ with the state $(\rho_M, (\rho^+, m_M))$, and a 3-rarefaction wave , joining $(\rho_M, (\rho^+, m_M))$ with (ρ^+, m^+) (see Fig. 2), as in the case of quadratic pressure.

CHAPTER 5

Existence of weak solutions

5.1. Introduction

The result presented in this Chapter stems from an idea recently explored by Emil Wiedemann for the incompressible Euler equations. In [**Wie11**] Wiedemann shows existence of weak solutions to the Cauchy problem for the incompressible Euler equations with general initial data (see Chapter 1). His proof combines some Fourier analysis with a clever application of the methods developed by De Lellis and Székelyhidi in [**DLS09**]-[**DLS10**] for the construction of non-standard solutions to the incompressible Euler equations. The conclusions achieved in [**Chi11**] and presented in Chapter 2 for the compressible Euler system gave hope that such an existence result could hold also for isentropic compressible gas dynamics in several space dimensions.

The existence of entropy solutions for the Cauchy problem associated with the isentropic compressible Euler equations in one space dimension was established, in the case of polytropic perfect gases first by DiPerna [**DP85**], Ding, Chen & Luo [**DCL85**], and Chen [**Che86**] based on compensated compactness arguments, and then, motivated by a kinetic formulation of hyperbolic conservation laws, by Lions, Perthame & Tadmor [**LPT94**], and Lions, Perthame & Souganidis [**LPS96**]. General pressure laws were covered first by Chen & LeFloch [**CL00**]. Unlike in the one-dimensional case, the existence problem for weak solutions of multi-dimensional isentropic gas dynamics has remained open so far.

The outcome of [**Wie11**] hints that the powerful approach by De Lellis and Székelyhidi is not only a "generator" of nonuniqueness, but can actually be exploited to construct weak solutions starting out from any initial data (see also [**DLS11**])! Here, we will follow such a hint and building upon results from [**Chi11**]-[**DLS10**]-[**Wie11**] we will show existence of weak solutions to the compressible Euler equations for any

Lipschitz continuous initial density and any bounded initial momentum.

THEOREM 5.1.1. *Let $\rho^0 \in C^1_p(Q;\mathbb{R}^+)$ and $m^0 \in H(Q)$. Then there exists a weak solution (ρ, m) (in fact, infinitely many) of the Cauchy problem for the compressible Euler equations with intial data (ρ^0, m^0).*

Of course, the optimal result would be existence of weak solutions starting out from any bounded initial data: Theorem 5.1.1 is a just a first step towards this.

5.2. The problem

In this section, we formulate the isentropic compressible Euler equations of gas dynamics in n space dimensions, $n \geq 2$ (cf. Section 3.3 of [**Daf00**]) and in canonical form (as in Chapter 2). The system, which consists of $n+1$ equations, takes the form:

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}_x m \ = \ 0 \\ \partial_t m + \mathrm{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x [p(\rho)] \ = \ 0 \\ \rho(\cdot, 0) \ = \ \rho^0 \\ m(\cdot, 0) \ = \ m^0 \end{array} \right. , \tag{5.1}$$

where ρ is the density and m the linear momentum field. The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas in question. The system is hyperbolic if

$$p'(\rho) > 0.$$

We will consider, from now on, the case of general pressure laws given by a function p on $[0, \infty[$, that we always assume to be continuously differentiable on $[0, \infty[$ and strictly increasing on $[0, \infty[$.

Here, as in Chapter 2, we work with *space periodic* boundary conditions. For *space periodic* flows we assume that the fluid fills the entire space $\mathbb{R}^n$ but with the condition that m, ρ are periodic functions of the space variable.

Let $Q = [0,1]^n$, $n \geq 2$ be the unit cube in $\mathbb{R}^n$. We denote by $H^m_p(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H^m_{loc}(\mathbb{R}^n)$ and which are periodic with period Q:

$$m(x+l) = m(x) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and every } l \in \mathbb{Z}^n.$$

For $m = 0$, $H^0_p(Q)$ coincides simply with $L^2(Q)$. Analogously, for every functional space X we define $X_p(Q)$ to be the space of functions which

are locally (over $\mathbb{R}^n$) in X and are periodic of period Q. The functions in $H_p^m(Q)$ are easily characterized by their Fourier series expansion

(5.2)
$$H_p^m(Q) = \left\{ m \in L_p^2(Q) : \sum_{k\in\mathbb{Z}^n} |k|^{2m} \, |\widehat{m}(k)|^2 < \infty \text{ and } \widehat{m}(0) = 0 \right\},$$

where $\widehat{m} : \mathbb{Z}^n \to \mathbb{C}^n$ denotes the Fourier transform of m. We will use the notation $H(Q)$ for $H_p^0(Q)$ and $H_w(Q)$ for the space $H(Q)$ endowed with the weak L^2 topology.

Let T be a fixed positive time. By a *weak solution* of (5.1) on $\mathbb{R}^n \times [0,T[$ we mean a pair $(\rho, m) \in L^\infty([0,T[; L_p^\infty(Q))$ satisfying

(5.3)
$$|m(x,t)| \le R\rho(x,t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0,$$

and such that the following identities hold for every test functions $\psi \in C_c^\infty([0,T[; C_p^\infty(Q))$, $\phi \in C_c^\infty([0,T[; C_p^\infty(Q))$:

$$\int_0^T \int_Q [\rho \partial_t \psi + m \cdot \nabla_x \psi] \, dxdt + \int_Q \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{5.4}$$

$$\int_0^T \int_Q \left[m \cdot \partial_t \phi + \left\langle \frac{m \otimes m}{\rho}, \nabla_x \phi \right\rangle + p(\rho) \operatorname{div}_x \phi \right] dxdt$$
$$+ \int_Q m^0(x) \cdot \phi(x,0)dx \;=\; 0. \tag{5.5}$$

In the following, we will be dealing also with the semi-stationary Cauchy problem associated with the isentropic Euler equations (simply set to 0 the time derivative of the density in (5.1) and drop the initial condition for ρ):

$$\begin{cases} \operatorname{div}_x m \;=\; 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m\otimes m}{\rho} \right) + \nabla_x [p(\rho)] \;=\; 0 \\ m(\cdot, 0) \;=\; m^0. \end{cases} \tag{5.6}$$

A pair $(\rho, m) \in L_p^\infty(Q) \times L^\infty([0,T[; L_p^\infty(Q))$ is a *weak solution* on $\mathbb{R}^n \times [0,T[$ of (2.8) if $m(\cdot,t)$ is weakly-divergence free for almost every $0 < t < T$ and satisfies the following bound

(5.7)
$$|m(x,t)| \le R\rho(x) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0,$$

and if the identity (3.25) holds for every $\phi \in C_c^\infty([0,T[; C_p^\infty(Q))$.

5.3. Existence of weak solutions

5.3.1. Background results. Before stating and proving the main theorem, we recall a Proposition from [**Chi11**] which represents the building block of our argument. For the sake of completeness of the chapter, we report Proposition 4.1 from [**Chi11**] which is Proposition 2.4.1 in Chapter 2. A similar criterion was proposed by De Lellis and Székelyhidi for the incompressible Euler equations (see [**DLS10**]) and used by Wiedemann in his proof of existence of weak solutions for incompressible Euler in [**Wie11**].

PROPOSITION 5.3.1. *Let $\overline{\rho} \in C^1_p(Q;\mathbb{R}^+)$ be any given density function and let T be any positive time.*
Assume there exist $(\overline{m}, \overline{U}, \overline{q})$ continuous space-periodic solutions of

$$\begin{aligned} \operatorname{div}_x \overline{m} &= 0 \\ \partial_t \overline{m} + \operatorname{div}_x \overline{U} + \nabla_x \overline{q} &= 0. \end{aligned} \tag{5.8}$$

on $\mathbb{R}^n \times]0,T[$ with

$$\overline{m} \in C([0,T]; H_w(Q)), \tag{5.9}$$

and a function $\chi \in C^\infty([0,T];\mathbb{R}^+)$ such that

$$\lambda_{max}\left(\frac{\overline{m}(x,t)\otimes\overline{m}(x,t)}{\overline{\rho}(x)} - \overline{U}(x,t)\right) < \frac{\chi(t)}{n} \quad \text{f.e. } (x,t) \in \mathbb{R}^n \times]0,T[, \tag{5.10}$$

$$\overline{q}(x,t) = p(\overline{\rho}(x)) + \frac{\chi(t)}{n} \qquad \text{for all } (x,t) \in \mathbb{R}^n \times]0,T[. \tag{5.11}$$

Then there exist infinitely many weak solutions (ρ, m) of the system (5.6) in $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \overline{\rho}(x)$ and such that

$$m \in C([0,T]; H_w(Q)), \tag{5.12}$$

$$m(\cdot,t) = \overline{m}(\cdot,t) \qquad \text{for } t = 0, T \text{ and for a.e. } x \in \mathbb{R}^n, \tag{5.13}$$

$$|m(x,t)|^2 = \overline{\rho}(x)\chi(t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times]0,T[. \tag{5.14}$$

Let us remark that, according to the terminology used so far, a triple $(\overline{m}, \overline{U}, \overline{q})$ satisfying the hypotesis of Proposition 5.3.1 is called a *subsolution* of (5.6).

In the arguments of [**Chi11**], and hence of Chapter 2, the previous Proposition represents a criterion to recognize initial data m^0 allowing

for many weak admissible solutions to (5.1). In our context it will play a different role: first, starting from any initial data (ρ^0, m^0) we will be able to construct a subsolution $(\overline{m}, \overline{U}, \overline{q})$ with the properties stated in the assumptions of Proposition 5.3.1 (with $\overline{\rho} := \rho^0$) and such that $\overline{m}(\cdot, 0) = m^0(\cdot)$, then it will be enough to apply Proposition 5.3.1 in order to prove existence of weak solutions (in fact, infinitely many) to the compressible Euler equations (5.1). Indeed, the solutions of (5.6) provided by Proposition 5.3.1 are also solutions of the full system (5.1).

5.3.2. Proof of Theorem 5.1.1. This section is devoted to the proof of Theorem 5.1.1, the main result of this note. For the sake of completeness we report here the statement.

THEOREM 5.3.2. *Let $\rho^0 \in C^1_p(Q; \mathbb{R}^+)$ and $m^0 \in H(Q)$. Then there exists a weak solution (ρ, m) (in fact, infinitely many) of the Cauchy problem for the compressible Euler equations* (5.1).

Proof. The idea behind the proof is to choose suitably a subsolution $(\overline{m}, \overline{U}, \overline{q})$ satisfying the assumptions of Proposition 5.3.1 (with $\overline{\rho} := \rho^0$) and such that $\overline{m}(\cdot, 0) = m^0(\cdot)$, so that it will be enough to apply Proposition 5.3.1 in order to prove Theorem 5.1.1: indeed the conclusions of Proposition 5.3.1 and in particular (5.13) will yield our claim.

We first define via Fourier transform the following functions:

$$\underline{\widehat{m}}(k,t) = e^{-|k|t}\widehat{m}^0(k), \tag{5.15}$$

$$\underline{\widehat{U}}_{i,j}(k,t) = -i\left(\frac{k_j}{|k|}\underline{\widehat{m}}_i(k,t) + \frac{k_i}{|k|}\underline{\widehat{m}}_j(k,t)\right) \tag{5.16}$$

for every $k \neq 0$, and $\underline{\widehat{U}}(0,t) = 0$. Clearly, for $t > 0$, $\underline{m}$ and $\underline{U}$ are smooth. Moreover, $\underline{U}$ is symmetric and trace-free. The definition of $\underline{\widehat{m}}$ and $\underline{\widehat{U}}$ is taken from the construction of Wiedemann in [**Wie11**]. Let us note that the couple $(\underline{\widehat{m}}, \underline{\widehat{U}})$ defined by (5.15)-(5.16) satisfies the following system of equations in Fourier space:

$$\begin{cases} \partial_t \underline{\widehat{m}}_i + i\sum_{j=1}^n k_j \underline{\widehat{U}}_{i,j} = 0 \\ k \cdot \underline{\widehat{m}} = 0, \end{cases} \tag{5.17}$$

for $k \in \mathbb{Z}^n$, $i = 1, \dots, n$. Hence $(\underline{m}, \underline{U})$ satisfies the system:

$$\begin{cases} \operatorname{div}_x \underline{m} \;=\; 0 \\ \partial_t \underline{m} + \operatorname{div}_x \underline{U} \;=\; 0. \end{cases} \tag{5.18}$$

Next, inspired by the proof of Proposition 7.1 in [**Chi11**], we define $\widetilde{U}$ componentwise by its Fourier transform as follows:

$$\widehat{\widetilde{U}}_{ij}(k) := \left(\frac{nk_ik_j}{(n-1)\,|k|^2}\right)\widehat{p(\rho^0(k))} \text{ if } i \neq j,$$

$$\widehat{\widetilde{U}}_{ii}(k) := \left(\frac{nk_i^2 - |k|^2}{(n-1)\,|k|^2}\right)\widehat{p(\rho^0(k))}. \tag{5.19}$$

for every $k \neq 0$, and $\widehat{\widetilde{U}}(0) = 0$. Also $\widetilde{U}$ thus defined is symmetric and trace-free. Moreover, since $p(\rho^0) \in C^1_p(\mathbb{R}^n)$, standard elliptic regularity arguments allow us to conclude that $\widetilde{U}$ is a continuous periodic matrix field. Next, notice that, by continuity of $\underline{m}$, ρ^0, $\underline{U}$ and $\widetilde{U}$, we have

$$\left\|\lambda_{max}\left(\frac{\underline{m}\otimes\underline{m}}{\rho^0} - \underline{U} - \widetilde{U}\right)\right\|_\infty = \widetilde{\lambda} \tag{5.20}$$

for some positive constant $\widetilde{\lambda}$. Therefore, we can choose any smooth function $\widetilde{\chi}$ on $\mathbb{R}$ such that $\widetilde{\chi} > n\widetilde{\lambda}$ on $[0,T]$ in order to ensure

$$\lambda_{max}\left(\frac{\underline{m}\otimes\underline{m}}{\rho^0} - \underline{U} - \widetilde{U}\right) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times [0,T[. \tag{5.21}$$

Now, let $\widetilde{q}$ be defined exactly as

$$\widetilde{q}(x,t) = p(\rho^0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } x \in \mathbb{R}^n \times \mathbb{R} \tag{5.22}$$

for the choice of $\widetilde{\chi}$ just done. In light of (5.22), we can write the equation

$$\operatorname{div}_x \widetilde{U} + \nabla_x \widetilde{q} = 0 \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \tag{5.23}$$

in Fourier space as

$$\sum_{j=1}^{n} k_j \widehat{\widetilde{U}}_{ij} = k_i \widehat{p(\rho^0)} \tag{5.24}$$

for $k \in \mathbb{Z}^n$, $i = 1, \ldots, n$. It is easy to check that $\widehat{\widetilde{U}}$ as defined by (5.19) solves (5.24) and hence $\widetilde{U}$ and $\widetilde{q}$ satisfy (5.23).

Now, given $\overline{\rho} := \rho^0$, we are ready to choose $(\overline{m}, \overline{U}, \overline{q})$. We set:

$$\overline{m}(x,t) := \underline{m}(x,t), \tag{5.25}$$

$$\overline{U}(x,t) := \underline{U}(x,t) + \widetilde{U}(x), \tag{5.26}$$

$$\overline{q}(x,t) := \widetilde{q}(x,t). \tag{5.27}$$

It remains to show that the subsolution defined by (**??**)-(**??**)-(**??**) satisfies the assumptions of Proposition 5.3.1.

First, we notice that system (5.8) is trivially satisfied by $(\overline{m}, \overline{U}, \overline{q})$ as a consequence of (5.18) and (5.23). Finally, with the choice $\chi := \widetilde{\chi}$, the subsolution $(\overline{m}, \overline{U}, \overline{q})$ will satisfy also (**??**)-(**??**) thanks to the definition of $\widetilde{q}$ in (5.22) and to the property (5.21) of $\widetilde{\chi}$. By Proposition 2.4.1 we find infinitely many solutions $m \in C([0,T]; H_w(Q))$ of (5.6) on $\mathbb{R}^n \times [0,T[$ with density ρ^0. Now, define $\rho(x,t) = \rho_0(x)\mathbf{1}_{[0,T[}(t)$. This shows that (**??**) holds. To prove (5.4) observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < T$. Therefore, the pair (ρ, m) is a weak solution of (5.1) with initial data (ρ^0, m^0) as desired. □

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