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Some fine properties of currents and applications to distributional Jacobians

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We study fine properties of currents in the framework of geometric measure theory on metric spaces developed by Ambrosio and Kirchheim, and we prove a rectifiability criterion for flat currents of finite mass. We apply these tools to study the structure of the distributional Jacobians of functions in the space BnV, defined by Jerrard and Soner. We define the subspace of *special functions of bounded higher variation* and we prove a closure theorem.

1. Introduction

In this paper we generalize some tools of geometric measure theory on metric spaces developed by Ambrosio and Kirchheim in [4] and we apply them to the space BnV. This space, which has been defined by Jerrard and Soner in [11], is composed, roughly speaking, by those functions such that their weak Jacobians are measures.

If $u \in C^1(\mathbb{R}^m, \mathbb{R}^n)$, with $m \geqslant n$, then the Jacobian of u can be seen as the differential form $\omega = \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n$. Of course, this notion can be easily extended to functions $u \in W^{1,n}$, but the main idea for a broader extension is based on the fact that $\omega = \mathrm{d}(u_1\, \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n)$. Indeed, we need less summability on the derivatives of u to handle the form $\nu = u_1\, \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n$ and we can define the weak Jacobian of u as the exterior derivative of ν in the distributional sense. A lot of attention has been devoted to this notion in the last years and we refer to [11] for an account of its applications and of the main papers on the argument.

In this work we propose to think of $u_1\, \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n$ as a current T via the natural action

$$T(\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_{m-n+1}) = \int_{\mathbb{R}^m} u_1 \det(\nabla u_2, \ldots, \nabla u_n, \nabla w_1, \ldots, \nabla w_{m-n+1})\, \mathrm{d}\mathcal{L}^m.$$

Thus we can define the weak Jacobian $[Ju]$ as the boundary of T and the space BnV can be identified with those u such that $[Ju]$ is a normal current. Instead of working in the framework of classical geometric measure theory, we prefer to use the 'metric currents theory' of [4], because we think that it is much easier to handle and provides more powerful tools for studying the structure of weak Jacobians. The main idea of this approach, suggested by De Giorgi in [7], is to replace the duality with differential forms with the duality with $(k+1)$-ples of Lipschitz functions. We hope to show that in this way we simplify the notation and proofs. In the last section we define a new class of functions, called SBnV, that is a generalization of the space of special functions of bounded variations (see [1, 3]). We prove for SBnV a closure theorem that is a generalization of the closure theorem for SBV (see theorems 5.5 and 5.7).

The definition of SBnV is induced, as a particular case, by a more general decomposition of flat currents of finite mass, which is proposed in § 3. Indeed, we show that it is possible to decompose every k-dimensional flat metric current T of finite mass into two currents of finite mass T_{l} and T_{u} such that

(a) T_{l} is concentrated on a $\mathcal{H}^k$-rectifiable set S;

(b) the mass of T_{l} is absolutely continuous with respect to $\mathcal{H}^k \llcorner S$;

(c) T_{u} neglects all $\mathcal{H}^k$ σ-finite sets.

One of the consequences of this decomposition is the following criterion of rectifiability for flat metric currents.

Criterion A. A flat k-dimensional current T of finite mass on E is rectifiable if and only if, for every Lipschitz function $\pi : E \to \mathbb{R}^k$, almost every slice of T with respect to π is composed of atoms (see theorem 3.3).

This criterion has been already proved by Ambrosio and Kirchheim in [4] for normal metric currents, and by White in [16], with a different approach, for flat currents on Euclidean spaces with coefficients in normed groups.

The paper is organized as follows.

The next section contains the basic definitions and theorems (available in the first part of [4]) of geometric measure theory on metric spaces. We develop the main tools for proving criterion A and we introduce the notion of BV functions that take values in metric spaces (first defined by Ambrosio in [2]).

In the third section we define the decomposition of currents and we prove that the lower-dimensional part of a flat current is rectifiable. In order to prove this fact, we need a basic BV-estimate on the slicing of currents (first due to Jerrard and Soner in the Euclidean case and then developed by Ambrosio and Kirchheim).

In the fourth section we apply to BnV the tools just developed. Taking a function $u \in$ BnV, we single out a 'lower-dimensional part' $[Ju]_{\mathrm{l}}$ of the Jacobian and we prove that it is a rectifiable current. The remaining part of the Jacobian (namely $[Ju]-[Ju]_{\mathrm{l}}$) can be split further into two currents: one that is absolutely continuous with respect to the Lebesgue measure and the other that is singular (which we call the Cantor part, in analogy with the case of functions of bounded variation). Thanks to its flatness, the lower-dimensional part of $[Ju]$ can be represented as

$$\langle [Ju]_{\mathrm{l}}, \omega \rangle = \int_{S_{\mathrm{l}}} m(x) \langle \tau(x), \omega(x) \rangle \, \mathrm{d}\mathcal{H}^{m-n},$$

where S_{l} is a $\mathcal{H}^k$-rectifiable set, $\tau(x)$ is its approximate tangent space in x and ω is any smooth $(m-n)$-form.

Then we analyse the structure of the absolutely continuous part of the Jacobian and, extending a result of Müller (see [13]), we prove that it can be represented as

$$[Ju]_{\mathrm{a}} = H(\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n)\mathcal{L}^m,$$

where H is the Hodge star operator. Thus $[Ju]_{\mathrm{l}} + [Ju]_{\mathrm{a}}$ can be represented as $\nu \mathrm{d}\mu$, where ν is a simple covector and μ is a measure. We conjecture that even the Cantor part has a similar structure, but we are not able to prove it.

In the last section we define the functions of special bounded higher variation as those BnV functions whose Jacobian has zero Cantor part. Finally, we prove that under suitable conditions (i.e. equi-integrability of the absolutely continuous part and equiboundedness of the Hausdorff measure of the singular supports), a closure property holds for SBnV.

2. Metric currents

Throughout the paper, (E, d) is a complete metric space and $\mathrm{Lip}_{\mathrm{b}}(E)$ is the space of Lipschitz and bounded real functions on E. We denote by $\mathcal{D}^k(E)$ the set of all $(k+1)$-ples $(f, g_1, \ldots, g_k)$ of functions such that $f, g_1, \ldots, g_k \in \mathrm{Lip}_{\mathrm{b}}(E)$, and we refer to it as the space of k-dimensional differential forms (or simply k-forms). For every k-form $\omega = (f, g_1, \ldots, g_k)$, we define its exterior derivative as the $(k+1)$-form

$$\mathrm{d}\omega = (1, f, g_1, \ldots, g_k). \tag{2.1}$$

If $\phi : F \to E$ is Lipschitz and bounded (and F is a complete metric space), we define the pull-back of ω as the k-form on F given by

$$\phi^{\#}\omega = (f \circ \phi, g_1 \circ \phi, \ldots, g_k \circ \phi). \tag{2.2}$$

If $\omega_1 = (f, g_1, \ldots, g_n)$ and $\omega_2 = (w, h_1, \ldots, h_k)$, then their exterior product is the $(n+k)$-form

$$\omega_1 \wedge \omega_2 := (fw, g_1, \ldots, g_n, h_1, \ldots, h_k).$$

Let us fix $\omega = (f, g_1, \ldots, g_n) \in \mathcal{D}^n(E)$. For every i, we define

$$\mathcal{C}_i := \{C \text{ open} \mid g_i \text{ is constant in every connected component of } C\}.$$

After setting $C_i := E \setminus (\bigcup\{U \in \mathcal{C}_i\})$, we define the closed set

$$\mathrm{supp}(\omega) = \mathrm{supp}(f) \cap \bigcap_{i=1}^{n} C_i, \tag{2.3}$$

and we refer to it as *support* of ω.

Definition 2.1. Let $k \in \mathbb{N}$. A k-dimensional current in E is a functional

$$T : \mathcal{D}^k(E) \to \mathbb{R}$$

such that

(a) $\lim_i T(f, g_1^i, \ldots, g_k^i) = T(f, g_1, \ldots, g_k)$ if $g_k^i \to g_k$ pointwise and $(\mathrm{Lip}(g_k^i))$ is bounded for every k;

(b) T is multilinear with respect to $(f, g_1, \ldots, g_k)$;

(c) $T(f, g_1, \ldots, g_k) = 0$ if $\mathrm{supp}((f, g_1, \ldots, g_k)) = \emptyset$.

We denote by $\mathcal{M}_k(E)$ the vector space of k-dimensional currents.

Remark 2.2. We could replace $\mathcal{D}^k(E)$ with $\mathcal{D}^k_{\mathrm{c}}(E)$, namely the set of differential forms with compact support, and we could also define a k-dimensional 'local current' as a linear functional that satisfies conditions (b) and (c) above and condition (a′) below,

(a′) $\lim_i T(f, g_1^i, \dots, g_k^i) = T(f, g_1, \dots, g_k)$ if $g_k^i \to g_k$ pointwise, $(\mathrm{Lip}(g_k^i))$ is bounded for every k and $\mathrm{supp}((f, g_1^i, \dots, g_k^i))$ is contained on a compact subset K for every i.

All the definitions and theorems of this paper also work with slight modifications. Moreover, in the applications to distributional Jacobians, we will use local currents.

Definition 2.3. Let T be a k-dimensional current. If there exists a finite positive measure μ such that

$$T(f, g_1, \dots, g_k) \leqslant \prod_{i=1}^{k} \mathrm{Lip}(g_i) \int_E |f| \, \mathrm{d}\mu, \tag{2.4}$$

then we say that T is of *finite mass*. We call the *mass* of the current T the minimal μ that satisfies (2.4), and we denote it by $\|T\|$. We say that T is concentrated on a Borel set B if $\|T\|(E \setminus B) = 0$.

We denote by $\boldsymbol{M}_k(E)$ the vector space of k-dimensional currents of finite mass.

From now on, given a current T of finite mass, we will denote by $\boldsymbol{M}(T)$ the total variation of $\|T\|$ in E. If T has not finite mass, we set $\boldsymbol{M}(T) = \infty$.

Remark 2.4. We will always assume that $\|T\|$ is concentrated on a σ-compact set. However, as observed in [4], this fact can be proved if E is separable or if the cardinality of E is a Ulam number. The assumption that the cardinality of any set E is a Ulam number is consistent with the standard ZFC theory.

Definition 2.5. Given a sequence $(T_n) \subset \mathcal{M}_k(E)$, we say that

$$T_n \rightharpoonup T \in \mathcal{M}_k(E) \tag{2.5}$$

if $T_n(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^k(E)$.

Sometimes we will write $\langle T, \omega \rangle$ for $T(\omega)$. As we can see in [4], from the assumptions of definition 2.1, it follows that a k-dimensional current is always alternating in $(g_1, \dots, g_k)$; hence we use, for differential forms, the usual notation,

$$f \, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n.$$

Sometimes, for the sake of simplicity, we will denote by g the n-tuple $(g_1, \dots, g_n)$ and we will write $f \, \mathrm{d}g$ for $f \, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n$. A trivial computation shows that if $\omega \in \mathcal{D}^n(E)$, $\nu \in \mathcal{D}^k(E)$ and $T \in \mathcal{M}_{n+k}(E)$, then

$$T(\mathrm{d}(\omega \wedge \nu)) = T(\mathrm{d}\omega \wedge \nu) + (-1)^n T(\omega \wedge \mathrm{d}\nu).$$

Moreover, every current satisfies the usual chain rule,

$$T(f \, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n) + T(g_1 \, \mathrm{d}f \wedge \cdots \wedge \mathrm{d}g_n) = T(1 \, \mathrm{d}(f g_1) \wedge \cdots \wedge \mathrm{d}g_n).$$

If $T \in \mathcal{M}_k(\mathbb{R}^k)$, then, for every $g \in C^1_{\mathrm{c}}(\mathbb{R}^k, \mathbb{R}^k)$ and $f \in \mathrm{Lip}(\mathbb{R}^k)$, we have

$$T(f \,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n) = T(f \det(\nabla g)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$$

(with x_i, we denote the projection on the ith coordinate of the canonical system of $\mathbb{R}^k$).

We can define a boundary operator $\partial : \mathcal{M}_k \to \mathcal{M}_{k-1}$ with the duality relation $\partial T(\omega) := T(\mathrm{d}\omega)$; it is not difficult to see that ∂T satisfies conditions (a), (b) and (c) of definition 2.1, but it can fail to be of finite mass, even if T itself has finite mass.

Definition 2.6. If T and ∂T are currents of finite mass, then we call T *normal*. We denote by $\mathcal{N}^k(E)$ the vector space of normal currents.

Remark 2.7. Given $T \in \mathcal{N}^k(E)$, we can define

$$\|T\|_N := \|T\|(E) + \|\partial T\|(E).$$

It is easy to check that $\mathcal{N}^k(E)$, endowed with the norm $\|\cdot\|_N$, is a Banach space.

Definition 2.8. Let T be a k-dimensional current on E. We define the flat norm $\boldsymbol{F}(T)$ as

$$\inf\{\boldsymbol{M}(T - \partial S) + \boldsymbol{M}(\partial S) \mid S \text{ is a } (k+1)\text{-dimensional current}\}.$$

Definition 2.9. Let T be a k-dimensional current. We say that T is a flat current if there exists a sequence of normal currents (T_n) such that

$$\lim_{n\to\infty} \boldsymbol{F}(T_n - T) = 0.$$

It is easy to see that a current T of finite mass is flat if and only if there exists a sequence of normal currents (T_n) such that $\boldsymbol{M}(T_n - T) \to 0$. Indeed, one implication is trivial because, for every current S, we have $\boldsymbol{F}(S) \leqslant \boldsymbol{M}(S)$. Moreover, if T is a flat current of finite mass, then, for every n, there exist $T_n \in \mathcal{N}_k(E)$ and $S_n \in \mathcal{M}_k(E)$ such that

$$\boldsymbol{M}(T - T_n - \partial S_n) + \boldsymbol{M}(\partial S_n) \leqslant \frac{1}{n}.$$

So we have that $T_n' := T_n + \partial S_n$ is a normal current and $\boldsymbol{M}(T - T_n') \leqslant 1/n$. A useful consequence of the last statement is that, for every current T, we can find a sequence of normal current T_n such that

$$\lim_{n\to\infty} \boldsymbol{M}\left(T - \sum_{i=1}^{n} T_n\right) = 0 \tag{2.6}$$

and

$$\sum_{i=1}^{\infty} \boldsymbol{M}(T_n) < \infty. \tag{2.7}$$

If (2.6) and (2.7) hold, we simply write

$$T = \sum_{i=1}^{\infty} T_n.$$

Definition 2.10. We say that a k-dimensional current T of finite mass is rectifiable if it is concentrated on a k-dimensional rectifiable set and $\|T\| \ll \mathcal{H}^k$.

As for the notion of boundary, we can define by duality the push-forward of currents. Indeed, given a Lipschitz and bounded map $\phi : E \to F$ and a k-dimensional current T on E, it is not difficult to check that $\phi_\# T$, defined by

$$\langle \phi_\# T, \omega \rangle := T(\phi^\# \omega), \tag{2.8}$$

is a k-dimensional current. Moreover, if T is a current of finite mass, then $\phi_\# T$ has finite mass and

$$\|\phi_\#(T)\| \leqslant \phi_\# \|T\|.$$

(We recall that if μ is a measure, then its push-forward $\phi_\# \mu$ is defined by $\phi_\# \mu(U) = \mu(\phi^{-1}(U))$.)

From these definitions, one can develop a self-contained theory of normal currents in E that is equivalent to the classical theory in the Euclidean case. Hereafter, we study the aspects that are useful for our purposes. We begin with the definitions of restriction and slicing.

Definition 2.11. Let $T \in \mathcal{M}_k(E)$ and $\omega \in \mathcal{D}^h(E)$, with $h \leqslant k$. We define the *restriction of T to ω* as the $(k-h)$-dimensional current given by

$$T \llcorner \omega(\nu) := T(\omega \wedge \nu).$$

Remark 2.12. If T is a current of finite mass, then we can extend its action to the $(k+1)$-ples $(f, g_1, \ldots, g_k)$ such that $g_i \in \mathrm{Lip}_\mathrm{b}(E)$ and f is bounded and Borel measurable. Indeed, $T \llcorner \mathrm{d}g$ is a 0-dimensional current of finite mass, and so there exists a finite measure μ_g such that

$$\langle T, w\, \mathrm{d}g \rangle = \langle T \llcorner \mathrm{d}g, w \rangle = \int_E w\, \mathrm{d}\mu_g \quad \text{for every } w \in \mathrm{Lip}_\mathrm{b}(E) \tag{2.9}$$

and

$$\|\mu_g\|_{\mathrm{var}} \leqslant \|T \llcorner \mathrm{d}g\|(E) \leqslant \left(\prod_{i=1}^k \mathrm{Lip}(g_i)\right) \|T\|(E).$$

Using equation (2.9), the action of $T \llcorner \mathrm{d}g$ can be easily extended to every Borel measurable and bounded function. Of course, if $f^k \to f$ uniformly, $g_i^k \to g_i$ pointwise and $(\mathrm{Lip}(g_i^k))$ is bounded for every i, then

$$\langle T, f^i \mathrm{d}g_1^i \wedge \cdots \wedge \mathrm{d}g_k^i \rangle \to \langle T, f\, \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_k \rangle.$$

From this last remark it follows that if T is a current of finite mass, then, for every Borel set A, we can define the current $T \llcorner A$,

$$\langle T \llcorner A, f\, \mathrm{d}g \rangle := \langle T \llcorner \chi_A, f\, \mathrm{d}g \rangle = \langle T, f \chi_A\, \mathrm{d}g \rangle.$$

Moreover, $\|T \llcorner \chi_A\| \leqslant \|T\|$.

Theorem 2.13. *Let T be a k-dimensional normal current in E and π a Lipschitz function from E to $\mathbb{R}^h$, with $h \leqslant k$. Then there exist normal $(k-h)$-dimensional currents $\langle T, \pi, x \rangle$ such that*

(i) *$\langle T,\pi,x\rangle$ and $\partial\langle T,\pi,x\rangle$ are concentrated on $E\cap\pi^{-1}(x)$;*

(ii) *for every $\psi\in C_{\mathrm{c}}(\mathbb{R}^h)$,*

$$\int_{\mathbb{R}^h}\langle T,\pi,x\rangle\psi(x)\,\mathrm{d}\mathcal{L}^h = T\llcorner(\psi\circ\pi)\,\mathrm{d}\pi; \tag{2.10}$$

(iii)

$$\int_{\mathbb{R}^h}\|\langle T,\pi,x\rangle\|\,\mathrm{d}\mathcal{L}^h = \|T\llcorner\mathrm{d}\pi\|. \tag{2.11}$$

We refer to [4] for the proof. Such a map $\langle T,\pi,x\rangle$ is called a *slicing of T with respect to π*. The previous theorem can be easily extended to flat currents.

THEOREM 2.14. *Let T be a k-dimensional flat current of finite mass on E and $\pi:E\to\mathbb{R}^h$ a Lipschitz function (with $h\leqslant k$). Then there exist $(k-h)$-dimensional flat currents $\langle T,\pi,x\rangle$ of finite mass such that*

(i) *$\langle T,\pi,x\rangle$ is concentrated on $E\cap\pi^{-1}(x)$;*

(ii) *for every $\psi\in C_{\mathrm{c}}(\mathbb{R}^h)$,*

$$\int_{\mathbb{R}^h}\langle T,\pi,x\rangle\psi(x)\,\mathrm{d}\mathcal{L}^h = T\llcorner(\psi\circ\pi)\,\mathrm{d}\pi; \tag{2.12}$$

(iii)

$$\int_{\mathbb{R}^h}\|\langle T,\pi,x\rangle\|\,\mathrm{d}\mathcal{L}^h = \|T\llcorner\mathrm{d}\pi\|. \tag{2.13}$$

Proof. Let T_n be a sequence of normal currents such that

$$T=\sum_{i=1}^{\infty}T_n.$$

From theorem 2.13, we have that there exist normal $(k-h)$-dimensional currents $\langle T_n,\pi,x\rangle$ that verify conditions (a), (b) and (c) above. Let us think of $\langle T_n,\pi,x\rangle$ as an L^1 function of x that takes values on the Banach space $\boldsymbol{M}_{k-h}(E)$ (endowed with the norm $\boldsymbol{M}$). Condition (c) and inequality (2.7) imply that

$$\sum_{i=1}^{\infty}\langle T_n,\pi,\cdot\rangle \tag{2.14}$$

is a totally convergent series in $L^1(\mathbb{R}^h,\boldsymbol{M}_{k-h}(E))$. We define $\langle T,\pi,\cdot\rangle$ as the sum of (2.14). It is easy to check that T verifies conditions (i), (ii) and (iii). Moreover, we can extract a subsequence $T_{j(n)}$ such that, for $\mathcal{L}^h$ a.e. $x\in\mathbb{R}^h$,

$$\lim_{n\to\infty}\boldsymbol{M}\biggl(\langle T,\pi,x\rangle-\sum_{i=1}^{n}\langle T_{j(n)},\pi,x\rangle\biggr)=0.$$

We conclude that, for $\mathcal{L}^h$ a.e. x, $\langle T,\pi,x\rangle$ is a flat current of finite mass. $\square$

As we will see at the end of this section, the slicing map of a normal current has a remarkable property. In order to state it, we need the definition of a map of bounded variation from an open set of $\mathbb{R}^n$ to a weakly separable metric space (M, d) (see [2,4]).

Definition 2.15. We say the metric space (M, d) is weakly separable if there exists a countable family $\mathcal{F} \subset \mathrm{Lip_b}(M)$ such that

$$d(x, y) = \sup_{\varphi \in \mathcal{F}} |\varphi(x) - \varphi(y)| \quad \text{for every } x, y \in M. \tag{2.15}$$

Weakly separable metric spaces can be seen as a suitable generalization of separable Banach spaces. Indeed, if E is a separable Banach space, then we can choose as $\mathcal{F}$ in (2.15) a family of linear functionals; in particular, in the case $E = \mathbb{R}^n$, one can choose the projections on the coordinates of some coordinate system. Hence a natural definition for a map of bounded variation that take values in a weakly separable metric space would be the following:

$$\begin{gathered} u : \mathbb{R}^n \to E \text{ is MBV if, for every } \varphi \in \mathcal{F}, \\ \text{the function } \varphi \circ u \text{ is a function of bounded variation.} \end{gathered} \tag{A}$$

In particular, this is one of the possible definitions of BV functions when $E = \mathbb{R}^n$. However, another very natural requirement on a map of bounded variation would be that the measures $\|D(\varphi \circ u)\|$ enjoy some kind of 'global control', independent of the choice of $\varphi \in \mathcal{F}$. In the Euclidean space, this property is a byproduct of (A) because we can choose finite families $\mathcal{F}$ satisfying (2.15), but in more general cases this is not true. Hence we require this 'global control' in the definition. But first we need to introduce the concept of supremum of a family of measures.

Definition 2.16. Let $\{\mu_i\}_{i \in I}$ be a family of positive measures μ on E. Then, for every Borel subset of E, we define

$$\bigvee_{i \in I} \mu_i(B) := \sup \left\{ \sum_{i \in J} \mu_i(B_i) \mid B_i \text{ are pairwise disjoint and } \bigcup_{i \in J} B_i = B \right\},$$

where J runs through all countable subsets of I.

Definition 2.17. Let $U \subset \mathbb{R}^k$ be an open set, (M, d) a weakly separable metric space and $u : U \to M$. We say that u is of metric bounded variation if

(a) $\varphi \circ u$ is of locally bounded variation for every $\varphi \in \mathcal{F}$;

(b) $\|Du\|_{\mathrm{MBV}} := \bigvee_{\varphi \in \mathcal{F}} |D(\varphi \circ u)|(\Omega) < \infty$.

We remark that this definition does not depend on the choice of $\mathcal{F}$ and that

$$\|Du\|_{\mathrm{MBV}} = \bigvee_{\varphi \in \mathrm{Lip_b}(M)} |D(\varphi \circ u)|(\Omega)$$

(see [4] for the proofs). From now on, we will denote the measure $\bigvee |D(\varphi \circ u)|$ by $\|Du\|$.

The key of the proof of theorem 3.2 in the next section is the fact that the slicing map of a k-dimensional normal current T with respect to $\pi \in \mathrm{Lip}_{\mathrm{b}}(E, \mathbb{R}^k)$ is a map of metric bounded variation if we endow $\boldsymbol{M}_0(E)$ with the flat norm

$$\boldsymbol{F}(T) = \sup\{\langle T, \phi\rangle \mid \phi \in \mathrm{Lip}_{\mathrm{b}}(E),\ \mathrm{Lip}(\phi) \leqslant 1\}.$$

This observation, due to Jerrard and Soner in the case of weak Jacobians [11], has been developed by Ambrosio and Kirchheim [4] in the framework of normal currents. (With a little effort, one can see that the last definition of flat norms coincide with that given in definition 2.9 when $E = \mathbb{R}^n$.)

Theorem 2.18. *Let E be a weak separable metric space, T a normal n-dimensional current in E and $\pi : E \to \mathbb{R}^n$ a Lipschitz map. Then the slicing map*

$$\mathcal{S} : \mathbb{R}^n \ni x \to \langle T, \pi, x\rangle \in \boldsymbol{M}_0(E)$$

is metric bounded variation if we endow $\boldsymbol{M}_0(E)$ with the flat norm. Moreover, the MBV seminorm of $\langle T, \pi, x\rangle$ is bounded by the norm of T in $\mathcal{N}_k(E)$.

Proof. With a little effort, one can see that there is a countable family $\mathcal{F} \subset \mathrm{Lip}_{\mathrm{b}}(E)$ such that

$$\boldsymbol{F}(T) = \sup\{\langle T, \phi\rangle \mid \phi \in \mathcal{F}\}$$

and $\mathrm{Lip}(\phi) \leqslant 1$ for every $\phi \in \mathcal{F}$. We can think of $\phi \in \mathcal{F}$ as a Lipschitz real function defined on $\boldsymbol{M}_0$. Then (recall definition 2.17) we will show that

(a) for every such ϕ, $\phi \circ \mathcal{S}(x) = \langle \mathcal{S}(x), \phi\rangle$ is a function of a locally bounded variation (as a real-valued function of x);

(b) $\displaystyle\bigvee_{\phi\in\mathcal{F}} |D(\mathcal{S} \circ \phi)| \leqslant n\pi_{\#}\|T\| + n\pi_{\#}\|\partial T\|$.

Indeed, let us fix a bounded ϕ such that $\mathrm{Lip}(\phi) \leqslant 1$. If we consider a test function $\psi \in C^1_{\mathrm{c}}(\mathbb{R}^n)$, then

$$\begin{aligned}(-1)^{i-1} \int_{\mathbb{R}^n} \mathcal{S}(\phi(x))\partial_i\psi(x)\,\mathrm{d}x &= (-1)^{i-1} T \llcorner \mathrm{d}\pi(\phi\partial_i\psi \circ \pi)\\ &= T(\phi\,\mathrm{d}(\psi \circ \pi) \wedge \mathrm{d}\tilde{\pi}_i)\\ &= \partial T(\phi(\psi \circ \pi)\,\mathrm{d}\tilde{\pi}_i) - T(\psi \circ \pi \mathrm{d}\phi \wedge \mathrm{d}\tilde{\pi}_i)\\ &\leqslant \|\partial T\|(\psi \circ \pi) + \|T\|(\psi \circ \pi),\end{aligned}$$

where $\mathrm{d}\tilde{\pi} = \mathrm{d}\pi_1 \wedge \cdots \wedge \mathrm{d}\pi_{i-1} \wedge \mathrm{d}\pi_{i+1} \wedge \cdots \wedge \mathrm{d}\pi_{m-n}$. Then $\phi \circ S$ is a function of locally bounded variation and

$$|D(\phi \circ \mathcal{S})| \leqslant n\pi_{\#}\|T\| + n\pi_{\#}\|\partial T\|.$$

□

3. Decomposition of currents and the rectifiability theorem

Given a k-dimensional current T of finite mass, we can find a $\mathcal{H}^k$ σ-finite set L_T such that $\|T\|(F) = 0$ whenever $\mathcal{H}^k(F) < \infty$ and $L_T \cap F = \emptyset$.

We construct this set a follows. Let us consider

$$K = \sup\{\|T\|(L) \mid L \text{ is } \mathcal{H}^k\ \sigma\text{-finite}\}.$$

We choose a sequence (L_n) of $\mathcal{H}^k$ σ-finite sets such that $\|T\|(L_n) \uparrow K$ and we put $L_T = \cup L_n$. Then we have that L_T is $\mathcal{H}^k$ σ-finite and $\|T\|(L_T) = K$. Hence L_T has the desired properties.

Definition 3.1. Let T be a k-dimensional current of finite mass and L_T be defined as above. Then we define

$$T_{\mathrm{u}} := T \llcorner (E \setminus L_T), \qquad T_{\mathrm{l}} := T \llcorner (L_T) \tag{3.1}$$

and we refer to T_{l} as the lower-dimensional part of T.

Of course, $\|T_{\mathrm{u}}\|$ and $\|T_{\mathrm{l}}\|$ are mutually singular and $T_{\mathrm{u}} + T_{\mathrm{l}} = T$. Moreover, $\|T_{\mathrm{u}}\| \leqslant \|T\|$ and $\|T_{\mathrm{l}}\| \leqslant \|T\|$. If E is $\mathcal{H}^p$ σ-finite for some $p > k$, then we define $\|T\|_{\mathrm{a}}$ as the absolutely continuous part of $\|T\|$ with respect to $\mathcal{H}^p$. Of course, $\|T\|_{\mathrm{a}}$ and $\|T_{\mathrm{l}}\|$ are mutually singular, and so there exists a Borel set A_T, disjoint from L_T, such that $\|T\| \llcorner A_T = \|T\|_{\mathrm{a}}$. Therefore, we can define

$$T_{\mathrm{a}} = T_{\mathrm{u}} \llcorner A_T, \qquad T_{\mathrm{c}} = T_{\mathrm{u}} \llcorner (E \setminus A_T), \tag{3.2}$$

and we refer to T_{a} and T_{c} as, respectively, the *absolutely continuous part* and the *cantor part* of T. Notice that $T_{\mathrm{a}} + T_{\mathrm{c}} + T_{\mathrm{l}} = T$.

When T is a flat current of finite mass, it is easy to see that there is a Borel set R_T such that $\|T_{\mathrm{l}}\|$ is absolutely continuous with respect to the measure $\mathcal{H}^k \llcorner R_T$. The main result of this section is that, in this case, T_{l} is a rectifiable current. To prove this, we need only check that R_T is rectifiable.

Theorem 3.2. *If E is separable and T is a k-dimensional flat current of finite mass on E, then T_{l} is a rectifiable current.*

A consequence of this fact is the following criterion of rectifiability, obtained in another framework by White [16].

Theorem 3.3. *Let T be a k-dimensional flat current of finite mass on a separable metric space E. Then T is rectifiable if and only if, for every Lipschitz function $\pi : E \to \mathbb{R}^k$ and for $\mathcal{L}^k$ a.e. $x \in \mathbb{R}^k$, the sliced current $\langle T, \pi, x\rangle$ is supported on a finite number of points.*

We remark that one implication is trivial: if T is rectifiable and $\pi : E \to \mathbb{R}^k$ is Lipschitz, then $\langle T, \pi, x\rangle$ is concentrated on $\pi^{-1}(\{x\})$, which, for almost every x, consists of a finite number of points.

Before proving theorem 3.2 and the other implications of theorem 3.3, we need some tools.

Theorem 3.4. *Let E be a separable metric space and let us endow $\boldsymbol{M}_0(E)$ with the norm*

$$\boldsymbol{F}(T) = \sup\{\langle T, \phi\rangle \mid \phi \in \mathrm{Lip_b}(E),\ \mathrm{Lip}(\phi) \leqslant 1\}.$$

If $\mathcal{S} \in MBV(\mathbb{R}^k, E)$ and $K \subset E$ is a compact set, then there exists an $\mathcal{L}^k$-negligible set $A \in \mathbb{R}^k$ such that

$$S := \{y \in K \mid \|\mathcal{S}(x)\|(\{y\}) > 0 \text{ for some } x \in \mathbb{R}^k \setminus A\}$$

is countably $\mathcal{H}^k$-rectifiable.

Before proving this theorem, we introduce the notion of maximal functions for MBV mappings. Given a function $u \in \mathrm{MBV}(\mathbb{R}^k, M)$, where M is a weakly separable metric space, we set

$$MDu(x) := \sup_{\rho > 0} \frac{\|Du\|(B_\rho(x))}{\omega_k \rho^k}$$

(MDu is known in the literature as the *maximal function* of the measure $\|Du\|$; see, for example, [15]). It is not difficult to see that this function is finite for almost every x. In fact, we can estimate $\mathcal{L}^k(\{MDu > \lambda\})$ from above with a constant times $\|Du\|(\mathbb{R}^k)/\lambda$. As happens for classical real-valued functions of bounded variation, MDu provides a Lipschitz property for u.

Lemma 3.5. *Let (M, d) be weakly separable and let $\mathcal{S} : \mathbb{R}^k \to M$ be a map of metric bounded variation. Then there exists $N \subset \mathbb{R}^k$ of measure zero such that*

$$d(\mathcal{S}(x), \mathcal{S}(y)) \leqslant c(MD\mathcal{S}(x) + MD\mathcal{S}(y))|x - y| \quad \forall x, y \in \mathbb{R}^k \setminus N, \tag{3.3}$$

where c depends only on k

Proof. Let us choose a family $\mathcal{F}$ of weakly dense Lipschitz functions. Then, for every $\varphi \in \mathcal{F}$, we define L_φ as the set of Lebesgue points of $\varphi \circ \mathcal{S}$ (which is a real function on $\mathbb{R}^k$). For every $x, y \in L_\varphi$, we claim that inequality (3.3) holds, with $w = \varphi \circ \mathcal{S}$ in place of $\mathcal{S}$.

Indeed, let us choose a ball B of radius $R = \frac{1}{2}|x - y|$ centred at $\frac{1}{2}(x - y)$. We obtain

$$\begin{aligned}
|w(x) - w(y)| &= \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(y)|}{|x - y|} \, \mathrm{d}z \\
&\leqslant \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(z)|}{|x - z|} \, \mathrm{d}z + \frac{1}{\omega_k R^k} \int_B \frac{|w(z) - w(y)|}{|z - y|} \, \mathrm{d}z \\
&\leqslant \frac{1}{\omega_k R^k} \left(\int_{B_{2R}(x)} \frac{|w(x) - w(z)|}{|x - z|} \, \mathrm{d}z + \int_{B_{2R}(y)} \frac{|w(y) - w(z)|}{|y - z|} \, \mathrm{d}z \right).
\end{aligned}$$

Moreover, we have

$$\frac{1}{\omega_k (2R)^k} \int_{B_{2R}(x)} \frac{|w(x) - w(z)|}{|x - z|} \, \mathrm{d}z \leqslant \int_0^1 \frac{|Dw|(B_{tR}(x))}{\omega_k (tR)^k} \, \mathrm{d}t \leqslant MDw(x),$$

and the claim easily follows.

Now, if we consider $\bigcap_{\varphi \in \mathcal{F}} L_{\varphi \circ \mathcal{S}}$, recalling that

$$d(\mathcal{S}(x), \mathcal{S}(y)) = \sup_{\varphi \in \mathcal{F}} |\varphi \circ \mathcal{S}(x) - \varphi \circ \mathcal{S}(y)|$$

and

$$\begin{aligned} MD\mathcal{S}(x) &= \sup_{\rho>0} \frac{\|D\mathcal{S}\|(B_\rho(x))}{\omega_k \rho^k} \\ &\geqslant \sup_{\rho>0} \frac{\|D(\varphi\circ\mathcal{S})\|(B_\rho(x))}{\omega_k \rho^k} \\ &= MD(\varphi\circ\mathcal{S})(x), \end{aligned}$$

we obtain (3.3). □

Proof of theorem 3.4. First of all, we set

$$A := N_1 \cup \{x \in \mathbb{R}^k \mid MD\mathcal{S}(x) = \infty\},$$

where N_1 is the set of measure zero that plays the role of N in lemma 3.5. Of course, $\mathcal{H}^k(A) = 0$.

Following [4], we define $Z_{\varepsilon,\delta}$ as the set of points $z \in \mathbb{R}^k \setminus A$ such that

(a) $MD\mathcal{S}(z) \leqslant 1/(2\varepsilon)$;

(b) for every $x \in K$ such that $\|\mathcal{S}(z)\|(\{x\}) \geqslant \varepsilon$, there holds

$$\|\mathcal{S}(z)\|(B_{3\delta}(x)\setminus\{x\}) \leqslant \tfrac{1}{3}\varepsilon.$$

Next we define

$$R_{\varepsilon,\delta} := \{x \in E \mid \|\mathcal{S}(z)\|(\{x\}) \geqslant \varepsilon \text{ for any } z \in Z_{\varepsilon,\delta}\}.$$

Observing that

$$S = \bigcup_{\varepsilon,\delta} R_{\varepsilon,\delta},$$

we will prove that, for each ε, δ, the set $R_{\varepsilon,\delta}$ is $\mathcal{H}^k$-rectifiable. Indeed, for every $x, x' \in R_{\varepsilon.\delta}$ and every $z, z' \in Z_{\varepsilon,\delta}$ such that

(i) $\|\mathcal{S}(z)\|(\{x\}) \geqslant \varepsilon$, $\|\mathcal{S}(z)\|(\{x'\}) \geqslant \varepsilon$;

(ii) $d(x,x') \leqslant \delta$,

we have

$$d(x,x') \leqslant \frac{3c(\delta+1)}{\varepsilon^2}|z-z'|. \tag{3.4}$$

Before proving this estimate, we that remark it implies that $R_{\varepsilon,\delta}\cap B$ is the image of a Lipschitz function whenever $\operatorname{diam}(B) \leqslant \delta$. Indeed, for any $z \in Z_{\varepsilon,\delta}\cap B$, there is only one $x = f(z) \in R_{\varepsilon,\delta}$ such that $\|\mathcal{S}\|(x) \geqslant \varepsilon$. Moreover, f is Lipschitz and, if D is the domain of f, $f(D) = B \cap R_{\varepsilon,\delta}$.

Now we complete the proof by showing that (3.4) holds. Let us set $d = d(x,x')$ and consider a Lipschitz function $\phi : E \to \mathbb{R}$ such that

(a′) $\phi(y) = d(y,x)$ for every $y \in B_d(x)$;

(b′) $\phi \equiv 0$ in $\mathbb{R}^m \setminus B_{2\delta}(x)$;

(c′) $\sup|\phi| = d$ and $\operatorname{Lip}(\phi) \leqslant 1$.

We have $|(\mathcal{S}(z))(\phi)| \leqslant \frac{1}{3}\varepsilon\delta$ and $|(\mathcal{S}(z'))(\phi)| \geqslant \varepsilon\delta - \frac{1}{3}\varepsilon\delta$, so we get

$$\begin{aligned}\tfrac{1}{3}\varepsilon d(x,x')| &\leqslant |(\mathcal{S}(z'))(\phi) - (\mathcal{S}(Z))(\phi)| \\ &\leqslant (\delta+1)\boldsymbol{F}(\mathcal{S}(z) - \mathcal{S}(z')) \\ &\leqslant c(\delta+1)(MD\mathcal{S}(z) + MD\mathcal{S}(z'))|z-z'|.\end{aligned}$$

Recalling that $MD\mathcal{S}(z) \leqslant 1/(2\varepsilon)$, we obtain the desired estimate. ☐

Proof of theorem 3.2. We need only prove that T_{l} is concentrated on a $\mathcal{H}^k$ σ-rectifiable set. First let us fix a Lipschitz function $\pi : E \to \mathbb{R}^k$. We want to prove that $T_{\mathrm{l}} \llcorner \mathrm{d}\pi$ is concentrated on a rectifiable set. We set $\mathcal{S}_\pi(x) = \langle T, \mathrm{d}\pi, x\rangle$ and we make the following claim.

Claim S. There exists a set $N \subset \mathbb{R}^k$ such that $\mathcal{L}^k(N) = 0$ and

$$S_\pi := \{y \in E \mid \|\mathcal{S}_\pi(x)\|(\{y\}) > 0 \text{ for some } x \in \mathbb{R}^k \setminus N\}$$

is countably rectifiable.

To prove it, let us choose a sequence T_n of normal currents such that

(i) $\boldsymbol{M}(T - T_n) \to 0$;

(ii) there exists a set $N^\infty \subset \mathbb{R}^k$ such that $\mathcal{L}^k(N^\infty) = 0$ and

$$\lim_{n\to\infty} \boldsymbol{M}(\mathcal{S}(x) - \langle T_n, \pi, x\rangle) = 0$$

for every $x \in \mathbb{R}^k \setminus N^\infty$.

To simplify the notation, we write $\mathcal{S}_n(x) = \langle T_n, \pi, x\rangle$. We remark that if (μ_n) is a sequence of finite measures and there exists a measure μ such that $\|\mu_n - \mu\|_{\mathrm{var}} \to 0$, then the set of atoms of μ is contained in the union of the sets of atoms of μ_n. Recalling that $\boldsymbol{M}_0(E)$ can be represented as the space of finite measures on E, we conclude that, for almost every $x \in \mathbb{R}^k \setminus N^\infty$,

$$\{z \mid \|\mathcal{S}_\pi(x)\|(z) > 0\} \subset \bigcup_n \{z \mid \|\mathcal{S}_n(x)\|(z) > 0\}.$$

Using theorem 3.4, we infer that for every i, there exists a set $N^i \subset \mathbb{R}^k$ of measure zero such that

$$S^i := \{z \in E \mid \|\mathcal{S}_i(x)\|(z) > 0 \text{ for some } x \in \mathbb{R}^k \setminus N_i\}$$

is countably rectifiable. If, in criterion A, we set

$$N = N^\infty \cup \bigcup_i N_i,$$

then we have

$$S_\pi \subset \bigcup_{i=1}^\infty S^i.$$

We conclude that S_π is countably rectifiable.

Now let us prove that

$$\|T_{\mathrm{l}}\llcorner \mathrm{d}\pi\|(E\setminus S_\pi)=0. \tag{3.5}$$

Recalling definition 3.1 we must only check that $\|T\llcorner \mathrm{d}\pi\|(A)=0$ for every $\mathcal{H}^k$ σ-finite set A such that $A\cap S_\pi=\emptyset$.

Since A is $\mathcal{H}^k$ σ-finite for a.e. $x\in\mathbb{R}^k$, $(\pi^{-1}\{x\})\cap A$ contains at most a countable number of points. This fact, combined with $A\cap S_\pi=\emptyset$, implies that, for a.e. $x\in\mathbb{R}^k$, $\|\mathcal{S}_\pi(x)\|(A)=0$. Then we have

$$\|T\llcorner \mathrm{d}\pi\|(A)=\int_{\mathbb{R}^n}\|\mathcal{S}_\pi(x)\|(A)\,\mathrm{d}\mathcal{L}^k(x)=0,$$

and this proves (3.5).

Now we recall that $\|T\|$ is concentrated on a σ-compact set. Because of this fact, it can be proved (see [4, lemma 5.4]) that there exists a countable set $D\subset \mathrm{Lip}_1(E)\cap\mathrm{Lip}_{\mathrm{b}}(E)$ such that

$$\|T\|=\bigvee\{\|T\llcorner \mathrm{d}\pi\|\mid \pi_1,\dots,\pi_k\in D\}. \tag{3.6}$$

If we take the countably rectifiable set

$$S:=\bigcup\{S_\pi\mid \pi_1,\dots,\pi_k\in D\},$$

then, from (3.6), it follows that $\|T_{\mathrm{l}}\|(E/S)=0$. □

Notice that if we are in the hypotheses of theorem 3.3, then we can reason as in the previous case. In fact, we have that, for every π, $T\llcorner \mathrm{d}\pi$ is concentrated on a $\mathcal{H}^k$ rectifiable set. Then it follows that T is concentrated on a $\mathcal{H}^k$ rectifiable set and coincides with T_{l}.

Remark 3.6. Assuming that every set has a cardinality that is a Ulam number, we can drop the assumption that E is separable (see remark 2.4).

4. Distributional Jacobians and BnV functions

In this section we are going to transpose some definitions and concepts from [11] in the language introduced above. We will work with differential forms with compact support and local currents, but this does not create any problems, as observed in remark 2.2. Finally, we recall that what we call differential forms in this paper are not the usual Lipschitz differential forms; in terms of classical theory, $\mathcal{D}^k_{\mathrm{c}}(\mathbb{R}^n)$ is the set of Lipschitz *simple* differential forms with compact support.

Definition 4.1. We define the k-dimensional local current H_k in $\mathbb{R}^k$ as

$$H_k(f\,\mathrm{d}g)=\int_{\mathbb{R}^k} f\det(\nabla g_1,\dots,\nabla g_k)\,\mathrm{d}\mathcal{L}^k.$$

The continuity axiom (definition 2.1, condition (c)) is satisfied because the Jacobian determinant is weakly* continuous in $W^{1,\infty}$. We remark that the classical Hodge star-operator assigns to every $\omega\in\mathcal{D}^k_{\mathrm{c}}(\mathbb{R}^n)$ the local $(n-k)$-dimensional current given by $H_n\llcorner\omega$.

In the definition of $H_n \llcorner \omega$, the regularity assumptions on ω can be weakened. In particular, let us suppose that $\omega = f \, \mathrm{d}g$ satisfies

(1) $f \in L^p$;

(2) $g \in W^{1,q}(\mathbb{R}^n, \mathbb{R}^k)$;

(3) $\dfrac{1}{p} + \dfrac{k}{q} = \dfrac{1}{r} < 1$.

Then it is well known (see, for example, [9, 12]) that the map

$$F : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^{n-k}) \to L^r(\mathbb{R}^n)$$

given by

$$F(u) = f \det(\nabla g_1, \dots, \nabla g_k, \nabla u_1, \dots, \nabla u_{n-k})$$

is continuous if we endow $L^r(\mathbb{R}^n)$ with the weak topology and $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^{n-k})$ with the weak* one. Even in this case, with a slight abuse of notation, we define $H_n \llcorner \omega$ as the k-dimensional local current T given by

$$T(f \, \mathrm{d}g) = \int_{\mathbb{R}^n} f \det(\nabla g_1, \dots, \nabla g_k, \nabla u_1, \dots, \nabla u_{n-k}) \, \mathrm{d}\mathcal{L}^n.$$

We will see below that this is a crucial point in the definition of weak Jacobians. In the rest of this section, U will denote an open set.

Definition 4.2. Let $u \in W^{1,p}_{\mathrm{loc}}(U, \mathbb{R}^n) \cap L^\infty$, with $U \subset \mathbb{R}^m$, $p \geqslant n-1$ and $m \geqslant n$ (or $u \in W^{1,mn/(m+1)}$). We define $j(u)$ as the $(m-n+1)$-dimensional local current $(-1)^n H_m \llcorner (u^1 \wedge \mathrm{d}u_2 \wedge \dots \wedge \mathrm{d}u_n)$.

Definition 4.3. Let u be as in the previous definition. Then we define

$$[Ju] := \partial j(u).$$

We say that $u \in \mathrm{BnV}(U, \mathbb{R}^n)$ if $j(u)$ is a normal local current.

The last definition is motivated as follows. Let us put $\nu = u_1 \, \mathrm{d}u_2 \wedge \dots \wedge \mathrm{d}u_n$ and suppose that u is sufficiently regular (i.e. Lipschitz). Then we have

$$\begin{aligned}
\langle [Ju], \omega \rangle &= \langle j(u), \mathrm{d}\omega \rangle \\
&= (-1)^n H_m(\nu \wedge \mathrm{d}\omega) \\
&= (-1)^n H_m((-1)^{n-1}(\mathrm{d}(\nu \wedge \omega) - \mathrm{d}\nu \wedge \omega)) \\
&= -\partial H_m(\nu \wedge \omega) + H_m(\mathrm{d}\nu \wedge \omega) \\
&= H_m \llcorner \mathrm{d}\nu(\omega).
\end{aligned}$$

We remark that, in view of this fact, we could have defined $j(u)$ as

$$\mathrm{sgn}(\pi) H_m \llcorner (u_{\pi(1)} \mathrm{d}u_{\pi(2)} \wedge \dots \wedge \mathrm{d}u_{\pi(n)}),$$

where π is any permutation of the set $\{1,\dots,n\}$. Indeed, if u is a smooth function, then

$$
\begin{aligned}
(-1)^n\partial(H_m \llcorner u_1\,\mathrm{d}u_2\wedge\cdots\wedge\mathrm{d}u_n)& \\
&= H_m\llcorner \mathrm{d}(u_1\,\mathrm{d}u_2\wedge\cdots\wedge\mathrm{d}u_n)\\
&= H_m\llcorner(\mathrm{d}u_1\wedge\mathrm{d}u_2\wedge\cdots\wedge\mathrm{d}u_n)\\
&= H_m\llcorner \operatorname{sgn}(\pi)(\mathrm{d}u_{\pi(1)}\wedge\mathrm{d}u_{\pi(2)}\wedge\cdots\wedge\mathrm{d}u_{\pi(n)})\\
&= (-1)^n\operatorname{sgn}(\pi)H_m\llcorner \mathrm{d}(u_{\pi(1)}\mathrm{d}u_{\pi(2)}\wedge\cdots\wedge\mathrm{d}u_{\pi(n)})\\
&= (-1)^n)\operatorname{sgn}(\pi)\partial(H_m\llcorner u_{\pi(1)}\mathrm{d}u_{\pi(2)}\wedge\cdots\wedge\mathrm{d}u_{\pi(n)}). \qquad (4.1)
\end{aligned}
$$

These equalities follows from the fact that if T is a k-dimensional local current and ω is an h-dimensional form with $k\leqslant h-1$, then

$$\partial T\llcorner\omega=(\partial T)\llcorner\omega-T\llcorner\mathrm{d}\omega.$$

Now, approximating every $u\in \mathrm{BnV}$ by convolutions with standard mollifiers, we obtain the identity (4.1) in its full generality. Indeed, it is easy to see that if $u_n\to u$ in the strong Sobolev topology and $\|u_n\|_\infty\leqslant c$, then $j(u_n)$ converges to $j(u)$ as local current.

Actually, $j(u)$ satisfies a stronger continuity result: appropriate weak convergence of the functions induces weak convergence on the Jacobians. More precisely, we have the following.

THEOREM 4.4. *Suppose that $(u_n),u$ satisfy the conditions of definition 4.2 and*

(a) $u_k\rightharpoonup u$ *weakly in* W^{1,p_1}_{loc};

(b) $u_k\to u$ *strongly in* $L^{p_2}_{\mathrm{loc}}$;

(c) $(n-1)/p_1+1/p_2<1$

(*or* $u_k\rightharpoonup u$ *in* $W^{1,n-1}_{\mathrm{loc}}$, $u_k\in C(U)$ *and* $u_k\to u$ *uniformly on compact sets*).

Then $j(u_n)\rightharpoonup j(u)$ *as local current.*

(For the proof of this theorem, we refer to the weak continuity of Jacobian determinant maps [9, 12].) If the hypotheses of the previous theorem hold, then we have

$$\langle[Ju_n],\omega\rangle=\langle\partial j(u_n),\omega\rangle=\langle j(u_n),\mathrm{d}\omega\rangle\to\langle j(u),\mathrm{d}\omega\rangle=\langle[Ju],\omega\rangle.$$

Now let us see how the local current $[Ju]$ behaves with respect to slicing when $u\in\mathrm{BnV}$. Let us consider a projection π of $\mathbb{R}^m$ onto a subspace of dimension $m-k\leqslant m-n$. For the sake of simplicity, we choose a system of coordinates and we suppose that π is the projection on the first $m-k$ coordinates. We will adopt the notation $\mathbb{R}^m\ni z=(x,y)\in\mathbb{R}^{m-k}\times\mathbb{R}^k$.

We notice that, for a.e. $x\in\mathbb{R}^{m-k}$, $j(u(x,\cdot))$ is a $(k-n+1)$-dimensional local current in $\mathbb{R}^k$. Indeed, because of the Fubini–Tonelli theorem, for a.e. x, the map $u(x,\cdot)$ belongs to the appropriate Sobolev space that allows us to define

$$j(u(x,\cdot)):=(-1)^nH_k\llcorner u_1(x,\cdot)\,\mathrm{d}_yu(x,\cdot).$$

Definition 4.5. We denote by i^x the natural identification between $\mathbb{R}^k$ and the affine subspace $\{x\} \times \mathbb{R}^k$ of $\mathbb{R}^m$.

Theorem 4.6. *Let u be as in definition 4.2 and $\pi : \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^{m-k}$ a projection, with $k \geqslant n$. Then we have*

$$\langle j(u), \mathrm{d}\pi, x\rangle = (-1)^l (i^x)_\# j(u(x,\cdot)), \tag{4.2}$$

$$\langle [Ju], \mathrm{d}\pi, x\rangle = (-1)^r (i^x)_\# [Ju(x,\cdot)], \tag{4.3}$$

with $l = (m-k)(n-1)$ and $r = (m-k)n$.

Proof. We use the notation of the previous paragraph to simplify the calculations. We observe that

$$j(u) \llcorner \mathrm{d}\pi = H_m \llcorner (u_1 \,\mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{m-k}).$$

So, for every $f\,\mathrm{d}g \in \mathcal{D}_{\mathrm{c}}^{k-n+1}$, we have

$$\begin{aligned} &H_m \llcorner (u_1 \,\mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}\pi)(f\,\mathrm{d}g) \\ &\qquad = \int_{\mathbb{R}^m} f u_1 \det(\nabla u_2, \ldots, \nabla u_n, e_1, \ldots, e_{m-k} \nabla g_1, \ldots, \nabla g_{k-n+1})\,\mathrm{d}z \\ &\qquad = (-1)^{(m-k)(n-1)} \int_{\mathbb{R}^m} f u_1 \det(e_1, \ldots, e_{m-k}, \nabla\tilde{u}, \nabla g)\,\mathrm{d}z, \end{aligned} \tag{4.4}$$

where $e_1, \ldots, e_k$ are the first $m-k$ vectors of the canonical basis and $\tilde{u}$ denotes the vector $(u_2, \ldots, u_n)$.

We remark that the matrix $(e_1, \ldots, e_{m-k}, \nabla\tilde{u}, \nabla g)$ can be written as

$$\begin{pmatrix} \mathrm{Id} & \nabla_x \tilde{u} & \nabla_x g \\ 0 & \nabla_y \tilde{u} & \nabla_y g \end{pmatrix}$$

(where Id is the identical $k \times k$ matrix and 0 is the $(m-k) \times k$ null matrix). Therefore, $(\nabla_y \tilde{u}, \nabla_y g)$ is a $(k \times k)$ matrix and

$$\det(e_1, \ldots, e_{m-k}, \nabla\tilde{u}, \nabla g) = \det(\nabla_y \tilde{u}, \nabla_y g).$$

This means that (4.4) is equal to

$$(-1)^l \int_{\mathbb{R}^{m-k}} \int_{\mathbb{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\,\mathrm{d}y\mathrm{d}x. \tag{4.5}$$

Then the expression

$$\int_{\mathbb{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\,\mathrm{d}y$$

can be read as

$$\langle (i^x)_\# j(u(x,\cdot)), f\,\mathrm{d}g\rangle.$$

We conclude from (4.5) that

$$\mathbb{R}^k \ni x \to \mathcal{S}(x) = (-1)^l (i^x)_\# j(u(x,\cdot))$$

is the slicing for $j(u)$ with respect to $\mathrm{d}\pi$.

Notice that

$$[Ju]\llcorner \mathrm{d}\pi = (-1)^{m-k}(\partial(j(u)\llcorner \mathrm{d}\pi) - j(u)\llcorner \mathrm{d}(\mathrm{d}\pi)) = (-1)^{m-k}\partial(j(u)\llcorner \mathrm{d}\pi).$$

Hence $(-1)^r(i^x)_\#[Ju(x,\cdot)]$ is a slicing map for $[Ju]$. □

For the sake of simplicity, from now on we will identify the local current $[Ju(x,\cdot)]$ and its push-forward via i^x.

Using the decomposition defined in the previous section, when $u \in \mathrm{BnV}$, we can consider $[Ju]_{\mathrm{a}}$, $[Ju]_{\mathrm{c}}$ and $[Ju]_{\mathrm{l}}$. Moreover, from theorem 3.2, it follows that $[Ju]_{\mathrm{l}}$ is a rectifiable local current. From now on, we define S_{u} as the set on which $[Ju]_{\mathrm{l}}$ is concentrated.

From classical theory, we know that there exists a Borel function ν from $\mathbb{R}^m$ to the linear space of $m-n$ covectors $\Lambda_{m-n}(\mathbb{R}^m)$ such that

$$\langle [Ju], \omega\rangle = \int_{\mathbb{R}^m} \langle \nu(x), \omega(x)\rangle \,\mathrm{d}\|Ju\| \tag{4.6}$$

(where $\|Ju\|$ is the mass of $[Ju]$). Of course, similar representations hold for the three parts of $[Ju]$,

$$\begin{aligned}
\langle [Ju]_{\mathrm{a}}, \omega\rangle &= \int_{\mathbb{R}^m} \langle \nu_{\mathrm{a}}(x), \omega(x)\rangle \,\mathrm{d}\|Ju\|_{\mathrm{a}},\\
\langle [Ju]_{\mathrm{c}}, \omega\rangle &= \int_{\mathbb{R}^m} \langle \nu_{\mathrm{c}}(x), \omega(x)\rangle \,\mathrm{d}\|Ju\|_{\mathrm{c}},\\
\langle [Ju]_{\mathrm{l}}, \omega\rangle &= \int_{S_{\mathrm{u}}} \langle \nu_{\mathrm{l}}(x), \omega(x)\rangle \,\mathrm{d}\mathcal{H}^{m-n}.
\end{aligned}$$

In fact, we can say a little more. From the fact that $[Ju]$ is a flat current, it follows that

$$\nu_{\mathrm{l}}(x) = m(x)\tau(x) \quad \text{for } \mathcal{H}^{m-n} \text{ a.e. } x,$$

where $\tau(x)$ is the approximate tangent plane to S_{u} in x and m is a Borel measurable real-valued function.

Even for the absolutely continuous part, we have a similar property. Indeed, let us suppose that $u \in \mathrm{BnV}(U, \mathbb{R}^n)$, with $U \subset \mathbb{R}^n$. Then we can define a notion of a 'pointwise determinant' of ∇u as follows. We choose a Borel function M, which is a pointwise representative of ∇u, and we define the pointwise Jacobian $\det(\nabla u)$ as the class of measurable functions f such that $\det M(x) = f(x)$ for $\mathcal{L}^n$ a.e. x. Then, from a result of Müller [13], we know that $\det(\nabla u) \in L^1_{\mathrm{loc}}$ and $[Ju]_{\mathrm{a}} = \det(\nabla u)\mathcal{L}^n$. In the following theorem we will prove a slight generalization of this result.

THEOREM 4.7. *Let u be a* BnV *function and let us choose a pointwise representative $\tilde{u}$ of u. Then there exists a Borel function $\nu_{\mathrm{a}} : \mathbb{R}^m \to \Lambda_{m-n}(\mathbb{R}^m)$ such that*

$$\langle [Ju]_{\mathrm{a}}, \omega\rangle = \int_{\mathbb{R}^m} \langle \nu_{\mathrm{a}}(x), \omega(x)\rangle \,\mathrm{d}\mathcal{L}^m \tag{4.7}$$

and

$$\nu_{\mathrm{a}}(x) = \mathrm{d}\tilde{u}_1(x) \wedge \cdots \wedge \mathrm{d}\tilde{u}_n(x) \quad \textit{for } \mathcal{L}^k \textit{ a.e. } x \in \mathbb{R}^m, \tag{4.8}$$

where $\mathrm{d}\tilde{u}_i(x)$ is the approximate differential of u_i at x.

Before proving the theorem, we address the special case when $m = n$.

LEMMA 4.8. *Let* $u \in \mathrm{BnV}(U, \mathbb{R}^n)$, *with* $U \subset \mathbb{R}^n$, *and let us choose a pointwise representative* $\tilde{u}$ *of* u. *Then* $\det(\nabla u)$ *is summable and*

$$\langle [Ju]_{\mathrm{a}}, \omega \rangle = \int_{\mathbb{R}^m} \omega(x) \det(\nabla u(x)) \, \mathrm{d}\mathcal{L}^m. \tag{4.9}$$

Proof. We follow the proof of Müller in [13] and, to simplify the notation, we identify u and $\tilde{u}$.

We know that $[Ju]_{\mathrm{a}}$ acts on 0-dimensional forms, i.e. on bounded measurable functions (recall remark 2.12 and the fact that $[Ju]_{\mathrm{a}}$ has finite mass). So we can write

$$\langle [Ju]_{\mathrm{a}}, f \rangle = \int_{\mathbb{R}^n} \nu(x) f(x) \, \mathrm{d}\mathcal{L}^n,$$

where $\nu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We only have to check

$$\nu(x) = \det(\nabla u(x)) \quad \text{for a.e. } x.$$

Therefore, let us fix $x_0 \in \mathbb{R}^n$ such that

(a) x_0 is a Lebesgue point for ν, $|\nabla u|^p$ and u (where p depends on the Sobolev space chosen in definition 4.2);

(b) $\displaystyle\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \|Ju\|_s(B_\varepsilon(x_0)) = 0$

(we recall that $\|Ju\|_s$ is the singular part of $\|Ju\|$).

Without loss of generality, we can suppose that $x_0 = 0$ and $u(x_0) = 0$, and we define the rescaled functions

$$u^\varepsilon := \frac{1}{\varepsilon} u(\varepsilon x).$$

We observe that they are BnV and they converge strongly (in the appropriate Sobolev space) to the linear function given by $\nabla u(0)$, which we denote by u^∞. So we have that $[Ju^\varepsilon]$ converges to $[Ju^\infty]$ as local current, and this implies that

$$\lim_{\varepsilon \to 0} \langle [Ju^\varepsilon], \omega \rangle = \langle [Ju^\infty], \omega \rangle = \int_{\mathbb{R}^m} \omega \det(\nabla u(0)) \, \mathrm{d}\mathcal{L}^m \tag{4.10}$$

for every Lipschitz function ω with compact support.

On the other hand, for every $f \, \mathrm{d}g = \omega \in \mathcal{D}^1_{\mathrm{c}}$, we also have

$$\langle j(u^\varepsilon), f \, \mathrm{d}g \rangle = \int_{\mathbb{R}^m} \frac{1}{\varepsilon} f(x) u_1(\varepsilon x) \det(\nabla u_2(\varepsilon x), \dots, \nabla u_n(\varepsilon x), \nabla g(x)) \, \mathrm{d}\mathcal{L}^m$$

and, by a change of variables,

$$= \frac{1}{\varepsilon^{n+1}} \int_{\mathbb{R}^m} u_1(y) f\left(\frac{y}{\varepsilon}\right) \det\left(\nabla u_2(y), \dots, \nabla u_n(y), \nabla g\left(\frac{y}{\varepsilon}\right)\right) \mathrm{d}y.$$

Thus we have obtained

$$\langle j(u^\varepsilon), \omega \rangle = \frac{1}{\varepsilon^n} \left\langle j(u), \omega\left(\frac{y}{\varepsilon}\right) \right\rangle,$$

from which it follows that

$$\begin{aligned}\langle [Ju^\varepsilon], \omega\rangle &= \langle j(u^\varepsilon), \mathrm{d}\omega\rangle\\ &= \frac{1}{\varepsilon^{n+1}}\left\langle j(u), (\mathrm{d}\omega)\left(\frac{y}{\varepsilon}\right)\right\rangle\\ &= \frac{1}{\varepsilon^{n}}\left\langle j(u), \mathrm{d}\left(\omega\left(\frac{y}{\varepsilon}\right)\right)\right\rangle\\ &= \frac{1}{\varepsilon^{n}}\left\langle [Ju], \omega\left(\frac{y}{\varepsilon}\right)\right\rangle.\end{aligned}$$

From condition (b), we have

$$\lim_{\varepsilon\to 0}\left|\frac{1}{\varepsilon^n}\left\langle [Ju]_s, \omega\left(\frac{y}{\varepsilon}\right)\right\rangle\right| \leqslant \lim_{\varepsilon\to 0}\frac{\mathrm{Lip}(\omega)}{\varepsilon^n}\|Ju\|_s(\varepsilon\,\mathrm{supp}(\omega)) = 0.$$

So we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle [Ju^\varepsilon], \omega\rangle &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\left\langle [Ju]_{\mathrm{a}}, \omega\left(\frac{y}{\varepsilon}\right)\right\rangle\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\nu(y)\omega\left(\frac{y}{\varepsilon}\right)\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\nu(\varepsilon x)\omega(x)\,\mathrm{d}x.\end{aligned}$$

Since 0 is a Lebesgue point for ν, $\nu(\varepsilon y)$ converges in L^1_{loc} to the function $\nu(0)$. Recalling (4.10), we have

$$\int_{\mathbb{R}^n}\nu(0)\omega\,\mathrm{d}x = \int_{\mathbb{R}^n}\det(\nabla u(0))\omega\,\mathrm{d}x$$

for every Lipschitz function ω with compact support. Then we conclude that

$$\nu(0) = \det(\nabla u(0)).$$

Now we remark that $\mathcal{L}^n$ a.e. x satisfies (a) and (b), and this completes the proof. □

Before proving theorem 4.7 in its full generality, we put on the space of covectors $\Lambda_{m-n}(\mathbb{R}^m)$ the norm

$$|\nu| = \sup\{\langle \nu, f_1\wedge\cdots\wedge f_n\rangle \mid f_i\in\mathbb{R}^m \text{ and } |f_i|\leqslant 1\}.$$

From the classical theory of currents, we know that $|\nu_{\mathrm{a}}(x)|\in L^1(\mathbb{R}^m)$. This fact and equation (4.8) imply that

$$\det(\nabla u_1,\dots,\nabla u_n,\nabla g_1,\dots,\nabla g_{m-n})\in L^1(\mathbb{R}^m)$$

for every $(m-n)$-tuple of Lipschitz functions $(g_1,\dots,g_{m-n})$.

Proof. First we choose a Borel function ν'_{a} such that

$$\langle [Ju]_{\mathrm{a}}, \omega\rangle = \int_{\mathbb{R}^m}\langle \nu'_{\mathrm{a}}(x), \omega(x)\rangle\,\mathrm{d}\|Ju\|_{\mathrm{a}}.$$

Recalling that $\|Ju\|_{\mathrm{a}}$ is absolutely continuous with respect to $\mathcal{L}^m$, we set

$$\nu_{\mathrm{a}} := \frac{\mathrm{d}\|Ju\|_{\mathrm{a}}}{\mathrm{d}\mathcal{L}^m}\nu'_{\mathrm{a}}.$$

Using lemma 4.8 and the slicing techniques introduced above, we will prove that ν_{a} satisfies equation (4.8). To simplify the notation, we identify u and $\tilde{u}$ and we put $m = n + k$.

First we choose an orthogonal system $x_1, \dots, x_{n+k}$ and a particular partition of $\{1, \dots, n+k\}$ into two disjoint sets $I = \{i_1, \dots, i_k\}$ and $J = \{i_{k+1}, \dots, i_{k+n}\}$. We call $y_1, \dots, y_k$ the k coordinates $x_{i_1}, \dots, x_{i_k}$ and $z_1, \dots, z_n$ the remaining n. Moreover, we denote by π_I be the projection on the coordinates $y_1, \dots, y_k$. From lemma 4.8 we know that equation (4.8) holds when $m = n$. Hence, for a.e. $y \in \mathbb{R}^k$, we have

$$\langle [Ju(y,\cdot)]_{\mathrm{a}}, f(y,\cdot)\rangle = \int_{\mathbb{R}^n} \det(\nabla_z u(y,z)) f(y,z)\,\mathrm{d}\mathcal{L}^n(z). \tag{4.11}$$

Then, from the slicing property of the Jacobians applied to π_I, it follows that

$$\begin{aligned}\langle [Ju]_{\mathrm{a}} \llcorner \mathrm{d}\pi_I, f\rangle &= (-1)^{nk} \int_{\mathbb{R}^k} \langle [Ju(y,\cdot)]_{\mathrm{a}}, f(y,\cdot)\rangle \,\mathrm{d}\mathcal{L}^k(y) \\ &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} (-1)^{nk} \det(\nabla_z u(y,z)) f(y,z)\,\mathrm{d}\mathcal{L}^n(z)\,\mathrm{d}\mathcal{L}^k(y) \\ &= \int_{\mathbb{R}^{n+k}} f \det(\nabla u_1, \dots, \nabla u_n, e_{i_1}, \dots e_{i_k})\,\mathrm{d}\mathcal{L}^{n+k}.\end{aligned}$$

Of course, this means that $\det(\nabla u_1, \dots, \nabla u_n, e_{i_1}, \dots e_{i_k})$ is an L^1 function. Moreover, this fact is true for every choice of I and, from the multilinearity of the determinant, we argue that

$$\det(\nabla u_1, \dots, \nabla u_n, \nabla g_1, \dots, \nabla g_k)$$

is summable for every Lipschitz and bounded k-tuple $(g_1, \dots, g_k)$. The continuity of $[Ju]_{\mathrm{a}} \llcorner \mathrm{d}\pi_I$ for every choice of I and the multilinearity of the determinant give the continuity (as a k-dimensional local current) of $H_{n+k} \llcorner \mathrm{d}u$, which is defined by

$$\langle H_{n+k} \llcorner \mathrm{d}u, f\,\mathrm{d}g\rangle = \int_{\mathbb{R}^{n+k}} f \det(\nabla u_1, \dots, \nabla u_n, \nabla g_1, \dots, \nabla g_k)\,\mathrm{d}\mathcal{L}^{n+k}.$$

Of course, $H_{n+k} \llcorner du$ is the same local current as $[Ju]_{\mathrm{a}}$. □

Unfortunately, we are not able to prove that something similar holds for the Cantor part, i.e. that $\nu_{\mathrm{c}}(x)$ is a simple covector for $\|Ju\|_{\mathrm{c}}$ a.e. x.

5. SBnV

In analogy with the case of SBV functions (see [1,3]), we can define the space SBnV of special functions of bounded higher variation.

Definition 5.1. We say that a map $u \in \mathrm{BnV}(U, \mathbb{R}^n)$ is a 'special function of bounded higher variation' if $[Ju]_{\mathrm{c}} = 0$.

The next proposition provides an equivalent definition of SBnV functions.

PROPOSITION 5.2. *Let* Ind *be the collection of all subsets I of $\{1,\dots,m\}$ such that the cardinality of I is $m-n$. For every $I \in \mathrm{Ind}$, we denote by π_I the projection on the coordinates $\{x_i \mid i \in I\}$. A function $u \in \mathrm{BnV}(U,\mathbb{R}^n)$ is in* SBnV *if and only if*

$$\begin{aligned} &\text{for every set of indices } I := \{i_1,\dots,i_{m-n}\} \in \mathrm{Ind} \\ &\text{and for } \mathcal{L}^{m-n} \text{ a.e. } x,\ u(x_{i_1},\dots,x_{i_{m-n}},y) \text{ is an SBnV function of } y. \end{aligned} \tag{A}$$

Proof. The 'only if' part is an easy consequence of the slicing of currents. So, let us suppose that (A) holds. Then, for every $I \in \mathrm{Ind}$, we have $[Ju]_{\mathrm{c}} \llcorner \mathrm{d}\pi = 0$. If ω is an n-form, we can write

$$\omega = \sum_{I \in \mathrm{Ind}} g_I \, \mathrm{d}\pi_I,$$

and so we obtain $[Ju]_{\mathrm{c}} \llcorner \omega = 0$. We conclude that $[Ju]_{\mathrm{c}} = 0$. □

An interesting fact is that the special functions of bounded higher variation satisfy a closure theorem similar to that proved in [1] for SBV.

REMARK 5.3. From now on, when ν is a k-covector, we denote by $|\nu|$ the standard norm induced by its action on k-vectors (see the proof of theorem 4.6).

First of all, we prove the closure theorem in a particular case.

THEOREM 5.4. *Let us consider $(u_k) \subset \mathrm{BnV}(U,\mathbb{R}^n)$ and $u \in \mathrm{BnV}(U,\mathbb{R}^n)$, with $U \subset \mathbb{R}^n$. Let us suppose that*

(a) *$u_k \rightharpoonup u$ weakly in W^{1,p_1}_{loc}, $u_k \to u$ strongly in $L^{p_2}_{\mathrm{loc}}$ and*

$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$

(or $u_k \rightharpoonup u$ in $W^{1,n-1}_{\mathrm{loc}}$, $u_k \in C(U)$ and $u_k \to u$ uniformly on compact sets);

(b) *if we write*

$$[Ju_k] = m_k(x)\mathcal{H}^0 \llcorner E_k + H_n \llcorner \nu_k(x),$$

then $|\nu_k|$ are equi-integrable and $\mathcal{H}^0(E_k) \leqslant C < \infty$.

Then $u \in \mathrm{SBnV}(U,\mathbb{R}^n)$ and

$$[Ju_k]_{\mathrm{a}} \rightharpoonup [Ju]_{\mathrm{a}}, \qquad [Ju_k]_{\mathrm{l}} \rightharpoonup [Ju]_{\mathrm{l}}.$$

Proof. We follow the ideas of the proof of SBV closure in [1]. First we notice that, in this particular case, the weak Jacobians $[Ju_n]$ are distributions. From the fact that the functions ν_k are equi-integrable, we can find a subsequence that converges weakly in L^1 to a function ν. To simplify the notation, we will suppose that the whole sequence (ν_k) converges to ν.

We recall that, from the continuity of the Jacobians, $[Ju_k] \rightharpoonup [Ju]$ (which means

$$\lim_{k\to\infty} \langle [Ju_k], \omega \rangle = \langle [Ju], \omega \rangle$$

for every Lipschitz function ω with compact support). We notice that

$$[Ju_k]_{\mathrm{l}} = [Ju_k] - [Ju_k]_{\mathrm{a}} \rightharpoonup [Ju] - \nu\mathcal{L}^n$$

in the sense of distributions. Moreover, we can write, for some integer N,

$$[Ju_k]_{\mathrm{l}} = \sum_{i=1}^{N} a_N^k \delta_{x_N^k}.$$

Then we can find a subsequence $u_{k(r)}$ such that (possibly reordering each set $\{x_1^{k(r)}, \ldots, x_N^{k(r)}\}$ in a proper way)

$$\text{for every } j \in \{1, \ldots, N\}, \text{ either } x_j^{k(r)} \text{ converges to } x_j \in \bar{U} \text{ or } |x_j^{k(r)}| \text{ tends to infinity.} \tag{B}$$

Recalling that $[Ju] - \nu\mathcal{L}^n$ is the limit of $[Ju_{k(r)}]_{\mathrm{l}}$, we obtain that its support is a finite number of points. But we know that $[Ju] - \nu\mathcal{L}^n$ is a measure, so it is the sum of a finite number of Dirac masses. We can conclude that $[Ju]$ is the sum of an absolutely continuous measure and a finite number of Dirac masses.

Moreover, we have that

$$[Ju_k]_{\mathrm{a}} \rightharpoonup [Ju]_{\mathrm{a}} \tag{5.1}$$

and

$$[Ju_k]_{\mathrm{l}} = [Ju_k]_s \rightharpoonup [Ju]_s = [Ju]_{\mathrm{l}}. \tag{5.2}$$

(Actually, we have proved these last statements only for a subsequence. However, we notice that from every subsequence of u_k we can choose another subsequence such that (5.1) and (5.2) hold. Then (5.1) and (5.2) hold for the whole sequence (u_k).) □

From the slicing property of $[Ju]$, we are now able to prove the next theorem.

THEOREM 5.5. *Let us consider* $(u_k) \subset \mathrm{BnV}(U, \mathbb{R}^n)$ *and* $u \in BnV(U, \mathbb{R}^n)$, *with* $U \subset \mathbb{R}^m$. *Moreover, suppose that*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc}, $u_k \to u$ *strongly in* $L^{p_2}_{\mathrm{loc}}$ *and*

$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$

(*or* $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{\mathrm{loc}}$, $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets*);

(b) *if we write*

$$[Ju_k] = m_k(x)\tau_k(x)\mathcal{H}^{m-n} \llcorner E_k + H_m \llcorner \nu_k(x),$$

then $|\nu_k|$ *are equi-integrable and* $\mathcal{H}^{m-n}(E_k) \leqslant C < \infty$.

Then u *is of special higher bounded variation.*

During the proof, we will use the representations of the previous section. So the restriction of $[Ju]_{\mathrm{a}}$ to $\mathrm{d}g$ becomes

$$\langle [Ju]_{\mathrm{a}} \llcorner \mathrm{d}g, \omega \rangle = \int_{\mathbb{R}^m} \langle \nu_{\mathrm{a}}(z), \omega(w) \wedge \mathrm{d}g(w) \rangle \, \mathrm{d}\mathcal{L}^m(w)$$

and the slicing map with respect to the projection π on the first $m-n$ coordinates is given by

$$\langle \mathcal{S}(x), \omega \rangle = \int_{\{x\}\times\mathbb{R}^n} \langle \nu_{\mathrm{a}}(x,y), \omega(x,y) \wedge \mathrm{d}\pi(x,y) \rangle \, \mathrm{d}\mathcal{L}^n(y).$$

From the slicing property of Jacobians, we argue that for a.e. x we can find a 0-covector-valued function $\xi(x,\cdot)$ (i.e. a *real function*) such that

$$\langle [J(u(x,\cdot))]_{\mathrm{a}}, \omega \rangle = \int_{\mathbb{R}^n} \langle \xi(x,y), \omega(y) \rangle \, \mathrm{d}\mathcal{L}^n(y).$$

We denote $\xi(x,y)$ by $\nu_{\mathrm{a}}(x,y) \llcorner \mathrm{d}\pi$ and we remark that $|\xi(x,y)| \leqslant |\nu(x,y)|$ for a.e. (x,y).

Proof. We will prove that statement (A) in proposition 5.2 holds.

Let us fix $I \in \mathrm{Ind}$. Without loss of generality, we can suppose that

$$I = \{1, \ldots, m-n\}.$$

We denote by z the projection on the first $m-n$ coordinates. Then we can write, for a.e. $x \in \mathbb{R}^{m-n}$,

$$[J(u_k(x,\cdot))]_{\mathrm{a}} = (\nu_k(x,\cdot) \llcorner \mathrm{d}z)\mathcal{H}^n \llcorner x \times \mathbb{R}^n, \tag{5.3}$$

$$[J(u_k(x,\cdot))]_{\mathrm{l}} = m_k(x,\cdot)\mathcal{H}^0 \llcorner (S_{\mathrm{u}} \cap z^{-1}\{x\}), \tag{5.4}$$

and, of course, $[Ju_k(x,\cdot)]_{\mathrm{c}} = 0$.

We split the proof into several steps.

Step 1. First we suppose that $u_k \to u$ strongly in $L^{p_2}_{\mathrm{loc}}$ and weakly in W^{1,p_1}_{loc}. Let us fix an open set $V \subset\subset U$ and set $V_x = V \cap z^{-1}\{x\}$. Let us extract a subsequence u_k of u_n such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^{m-n}} \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)} \, \mathrm{d}x < \infty.$$

From the monotone convergence theorem, we infer that, for a.e. x,

$$\sum_k \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)}$$

is a convergent series. This implies that $u_k(x,\cdot) \to u(x,\cdot)$ in $L^{p_2}(V_x)$ for a.e. x. Let us choose a family of open sets $V_n \uparrow V$ such that $V_n \subset\subset V$. We reason as above for every V_n and we apply a diagonalization argument to conclude that there is a subsequence (u_k) such that $u_k(x,\cdot) \to u(x,\cdot)$ strongly in $L^{p_2}_{\mathrm{loc}}$ for a.e. x.

Step 2. From Fatou's lemma, we have that

$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} \, \mathrm{d}x \leqslant \liminf_{j\to\infty} \int_{\mathbb{R}^{m-n}} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty.$$

We conclude that for a.e. x we can extract a further subsequence (u_{l}) (possibly depending on x) such that $\|u_{\mathrm{l}}(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty$ for every open set $V \subset\subset U$.

Then, recalling that, for a.e. x, $(u_k(x,\cdot))$ converges strongly in $L^{p_2}_{\mathrm{loc}}$ to $u(x,\cdot)$, we have that $u_l(x,\cdot)$ converges weakly in W^{1,p_1}_{loc} to $u(x,\cdot)$.

Summarizing, we have proved that, for a.e. $x \in \mathbb{R}^n$, we can extract a subsequence (u_k) (possibly depending on x) such that

$$u_k(x,\cdot) \rightharpoonup u(x,\cdot) \text{ in } W^{1,p_1}_{\mathrm{loc}} \text{ and } u_k(x,\cdot) \to u(x,\cdot) \text{ strongly in } L^{p_2}_{\mathrm{loc}},\\ \text{with } (n-1)/p_1 + 1/p_2 < 1.$$

In a similar way, we can treat the case in which $u_k(x,\cdot) \rightharpoonup u(x,\cdot)$ in $W^{1,n-1}_{\mathrm{loc}}$ $u_k(x,\cdot) \to u(x,\cdot)$ uniformly on compact sets as continuous functions.

Step 3. From the Dunford–Pettis theorem on L^1 weakly compact sequences (see, for example, [6]), we know that $|\nu_k|$ belongs to some Orlicz space. So there exists a real convex function ϕ, with superlinear growth, such that

$$\int_{\mathbb{R}^m} \phi(|\nu_k|) \leqslant K < \infty.$$

Then we have

$$\begin{aligned} K &\geqslant \limsup_{k\to\infty} \int_{\mathbb{R}^m} \phi(|\nu_k \llcorner \mathrm{d}z|) \\ &\geqslant \limsup_{k\to\infty} \int_{\mathbb{R}^{m-n}} \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner \mathrm{d}z|)\,\mathrm{d}x\mathrm{d}y \\ &\geqslant \int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner \mathrm{d}z|)\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

This implies that, for a.e. x, we can find a subsequence $k(r)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \phi(|\nu_{k(r)}(x,y) \llcorner \mathrm{d}z|)\,\mathrm{d}x < \infty,$$

which means that $\nu_{k(r)}(x,\cdot) \llcorner \mathrm{d}z$ are equi-integrable (we remark that the chosen subsequence depends on x).

Step 4. Reasoning as in the previous cases, we have

$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}))\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^n} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\})).$$

Then, for a.e. x, we can extract a subsequence (u_l) (possibly depending on x) such that

$$(\mathcal{H}^0(S_{u_l} \cap z^{-1}\{x\}))$$

is bounded.

Step 5. Now we want to put together all the information of the previous steps. We notice that the subsequence extracted on the first step does not depend on x, whereas the choices of the other steps depend on x. However, for a.e. x, we can extract a subsequence that fulfils all the conditions. Indeed, let us define

$$f_k(x) := \mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}) + \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner \mathrm{d}z|)\,\mathrm{d}y + \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)}.$$

Then we have

$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} f_k(x)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^{m-n}} f_k(x)\,\mathrm{d}x$$
$$\leqslant \liminf_{k\to\infty} \left(\mathcal{H}^{m-n}(S_{u_k}) + \|u_k\|_{W^{1,p_1}} + \int_{\mathbb{R}^m} \phi(\nu_k) \right) < \infty. \tag{5.5}$$

We conclude that, for a.e. x, we can choose a subsequence u_r such that $(u_r(x,\cdot))$ and $u(x,\cdot)$ satisfy all the hypotheses of theorem 5.4. Then, for a.e. x, the Cantor part of $[J(u(x,\cdot))]$ is zero and from statement (A) it follows that u has no Cantor part. □

We end this section by proving that, in the same hypotheses of theorem 5.5, we have

$$[Ju_k]_{\mathrm{l}} \rightharpoonup [Ju]_{\mathrm{l}}, \qquad [Ju_k]_{\mathrm{a}} \rightharpoonup [Ju]_{\mathrm{a}}.$$

To do this, we need the next lemma.

Lemma 5.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$ and (ν_k) a weakly compact sequence in $L^1(\Omega, \mu)$. Then $\nu_k \rightharpoonup \nu$ if and only if*

$$\int_\Omega |w + \nu|\,\mathrm{d}\mu \leqslant \liminf_{k\to\infty} \int_\Omega |w + \nu_k|\,\mathrm{d}\mu \quad \forall w \in L^1(\Omega). \tag{5.6}$$

We refer to [1] for the proof.

Theorem 5.7. *Let us consider $(u_k) \subset \mathrm{BnV}(U, \mathbb{R}^n)$ and $ux \in \mathrm{BnV}(U, \mathbb{R}^n)$, with $U \subset \mathbb{R}^m$. If conditions (a) and (b) of theorem 5.5 hold, then*

$$[Ju_k]_{\mathrm{l}} \rightharpoonup [Ju]_{\mathrm{l}}, \qquad [Ju_k]_{\mathrm{a}} \rightharpoonup [Ju]_{\mathrm{a}} \tag{5.7}$$

Proof. We use the notation of theorem 5.5 and we reduce to prove

$$[Ju_k]_{\mathrm{a}} \llcorner \mathrm{d}\pi_I \rightharpoonup [Ju]_{\mathrm{a}} \llcorner \mathrm{d}\pi_I \tag{5.8}$$

for every $I \in \mathrm{Ind}$ (we recall that Ind is the collection of all subsets of $\{1, \dots, n\}$ that have cardinality $(m-n)$). Indeed, from this fact, we could conclude that $[Ju_k]_{\mathrm{a}} \rightharpoonup [Ju]_{\mathrm{a}}$ and

$$[Ju_k]_{\mathrm{l}} = [Ju_k] - [Ju_k]_{\mathrm{a}} \rightharpoonup [Ju] - [Ju]_{\mathrm{a}} = [Ju]_{\mathrm{l}}.$$

Therefore, we suppose that $I = \{1, \dots, m-n\}$ and we split the proof into several steps. To simplify the notation, we suppose that $U = \mathbb{R}^n$ and that global convergence hold on (u_k). The proof can be easily adapted to the local case.

Step 1. Let us fix a convex real function ϕ with superlinear growth and a real number z such that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner \mathrm{d}\pi|)\,\mathrm{d}\mathcal{L}^m < \infty.$$

For every $x \in \mathbb{R}^{m-n}$, let us put

$$\begin{aligned} J_1^x(u) &= \int_{\mathbb{R}^n} \phi(|z + \nu \llcorner \mathrm{d}\pi(x,y)|)\,\mathrm{d}y, \\ J_2^x(u) &= \|u(x,\cdot)\|_{W^{1,p_1}}, \\ J_3^x(u) &= \mathcal{H}^0(\pi^{-1}\{x\} \cap S_{u_k}) \end{aligned}$$

and fix a positive real number t. From Fatou's lemma, we have that

$$\liminf_{k\to\infty} J_1^x(u_k) + tJ_2^x(u_k) + tJ_3^x(u_k) = K(x) < \infty$$

for almost every x and, reasoning as in the last step of theorem 5.5, we infer that

$$J_1^x(u) \leqslant J_1^x(u) + tJ_2^x(u) + tJ_3^x(u) \leqslant K(x).$$

Integrating this inequality with respect to x, we obtain

$$\int_{\mathbb{R}^m} \phi(|z + \nu \llcorner \mathrm{d}\pi|)\,\mathrm{d}\mathcal{L}^m \leqslant \liminf_{k\to\infty}\biggl(\int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner \mathrm{d}\pi|)\,\mathrm{d}\mathcal{L}^m + t\|u_k\|_{W^{1,p_1}} + t\mathcal{H}^k(S_{u_k})\biggr).$$

Letting $t \downarrow 0$, we obtain

$$\int_{\mathbb{R}^m} \phi(|z + \nu \llcorner \mathrm{d}\pi|)\,\mathrm{d}\mathcal{L}^m \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner \mathrm{d}\pi|)\,\mathrm{d}\mathcal{L}^m. \tag{5.9}$$

We notice that the same arguments work if we replace $\mathbb{R}^m$ with an open set. Let us denote by $\mathcal{C}$ the class of functions $w \in L^1(\mathbb{R}^m)$ that can be written as

$$w = \sum_{i=1}^h \alpha_i \chi_{A_i}$$

for some open sets $A_1, \dots A_h$. Hence (5.9) holds for every function $z \in \mathcal{C}$.

Step 2. We know that there exists a convex real function ψ, with superlinear growth, such that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^m} \psi(|\nu_k \llcorner \mathrm{d}\pi|) < \infty.$$

Let us take a convex real function ϕ with superlinear growth such that $\phi(0) = 0$,

$$\lim_{t\to\infty} \frac{\psi(t)}{\phi(t)} = +\infty.$$

We can easily conclude that the sequence $\phi(|\nu_k \llcorner \mathrm{d}\pi|)$ is equi-integrable. Let us put

$$\phi_n(t) = \left(\frac{1}{n}\phi(t)\right) \vee t.$$

The equi-integrability of $\psi(|\nu_k \llcorner \mathrm{d}\pi|)$ and the fact that $\phi_n(t) \downarrow t$ imply that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^m} |\nu_k \llcorner \mathrm{d}\pi| = \lim_{n\to\infty} \liminf_{k\to\infty} \int_{\mathbb{R}^m} \phi_n(|\nu_k \llcorner \mathrm{d}\pi|).$$

From the previous step, we easily conclude that

$$\int_{\mathbb{R}^m} |z + \nu \llcorner \mathrm{d}\pi| \leqslant \int_{\mathbb{R}^m} \liminf_{k\to\infty} |z + \nu_k \llcorner \mathrm{d}\pi| \tag{5.10}$$

for every $z \in \mathcal{C}$.

STEP 3. By a standard approximation argument, we have that (5.10) holds for every $z \in L^1(\mathbb{R}^n)$. Then, applying lemma 5.6, we conclude that

$$\nu_k \llcorner \mathrm{d}\pi \rightharpoonup \nu \llcorner \mathrm{d}\pi.$$

□

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