Min-Max Constructions of Minimal Surfaces in Closed Riemannian Manifolds

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Abstract

We give a shorter proof of the existence of nontrivial closed minimal hypersurfaces in closed smooth $(n+1)$-dimensional Riemannian manifolds, a theorem proved first by Pitts for $2 \leq n \leq 5$ and extended later by Schoen and Simon to any n.

Our proof follows Pitts' original idea to implement a min-max construction. We introduce some new ideas that allow us to shorten parts of Pitts' proof – a monograph of about 300 pages – dramatically.

Pitts and Rubinstein announced an index bound for the minimal surface obtained by the min-max construction. To our knowledge a proof has never been published. We refine the analysis of our interpretation of the construction to draw some conclusions that could be helpful to prove the index bound.

Zusammenfassung

Wir geben einen kürzeren Beweis für die Existenz nicht-trivialer, geschlossener minimaler Hyperflächen in geschlossenen, glatten $(n+1)$-dimensionalen Riemannschen Mannigfaltigkeiten. Dieses Resultat wurde erstmals von Pitts für $2 \leq n \leq 5$ und später von Schoen und Simon für beliebige n bewiesen.

Unser Beweis folgt Pitts' ursprünglicher Idee, eine Min-Max Konstruktion durchzuführen. Wir führen einige neue Ideen ein, die es uns erlauben, Teile von Pitts' Beweis – einem Monographen von etwa 300 Seiten – dramatisch zu verkürzen.

Pitts und Rubinstein kündigten eine Indexschranke für die Minimalfläche, die man mit der Min-Max Konstruktion erhält, an. Unseres Wissens wurde ein Beweis nie veröffentlicht. Wir vertiefen die Analyse unserer Interpretation der Konstruktion, um einige Schlüsse zu ziehen, die bei einem Beweis der Indexschranke hilfreich sein könnten.

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Contents

1 Introduction

1.1 Historical introduction

Minimal surfaces have been the object of interest and investigation ever since the beginning of the calculus of variations. The first examples (apart from the trivial one, the plane), namely the catenoid, a rotated catenary, and the helicoid, a kind of double helix, have been discovered in the 18th century by Euler and Meusnier. A more systematic study was triggered by Lagrange's ideas that lead to the indirect method of the calculus of variations. Minimal surfaces from that point of view are critical points of the area functional. This approach – to Lagrange merely another instance to prove the power of his method – gave a way to study minimal surfaces via PDEs. A more geometric description of the Euler-Lagrange equations of the area functional was later given by Meusnier: a minimal surface is characterized by the vanishing of its mean curvature, i.e. $H = 0$. Following these two points of view, the analytic and the geometric, minimal surfaces have proven to be a prototypical object of study in the field of geometric analysis.

In the early 19th century the theory received a new boost of popularity that was due to the soap film experiments by the Belgian physicist Plateau. By dipping a wire frame into a soap solution and removing it he produced loads of soap films that can be viewed as models for minimal surfaces. The soap film spanned by the wire frame was assumed to take a shape that minimizes the surface energy (which is proportional to the surface area). Encouraged by the experiments Plateau claimed that every wire frame bounds a soap film. Formulated in a more mathematical way this statement was later known as the famous Plateau problem.

Problem 1. *Every simple closed curve γ in $\mathbb{R}^3$ bounds a surface of least area among all the surfaces with boundary γ.*

This formulation is still a bit vague since it does not specify the concept of surface. In the classical theory people studied regularly parametrized surfaces and graphs (sometimes referred to as non-parametric surfaces). Even though there was experimental evidence, a rigorous proof of the claim turned out to be very difficult and lead to many new developments in the calculus of variations. It was not until the early 1930s that Douglas and Radó independently provided a proof of the existence of a disk-type solution [29], [59]. For his achievement Douglas was awarded one of the inaugural Fields medals. The idea of his proof is an implementation of the so-called direct method of the calculus of variations. This abstract method to find minimizers of functionals can be sketched as follows: take a minimizing sequence and use some compactness arguments to extract a converging subsequence. If the functional is semicontinuous, the limit yields the desired minimizer. This method cannot be applied directly to the area functional for parametrized surfaces. There are basically two aspects of this problem. On the one hand there is the functional (semicontinuity), on the other hand there is the space of definition (compactness), usually a function space. Douglas – and Courant, who some years later gave a simpler proof along the same lines – studied functionals with better functional analytic properties – such as the Dirichlet energy in Courant's proof [23], [24]. Using the fact that for conformally parametrized disks the Dirichlet energy equals the area, the problem can be reduced to finding a minimizer of the Dirichlet energy and showing that it is conformally parametrized. These close connections between the area functional and the Dirichlet energy, or on a PDE level the minimal surface equation and the Laplace equation, have been very fruitful in the theory.

Even though the results of Douglas and Radó were fundamental they had certain flaws. First of all, the theory is purely two-dimensional. Attacking the generalized problem for higher dimensional surfaces requires different tools. Moreover, the approach sketched above fixes from the start the topology of the solution by the parametrization, disk-type in this case. Douglas later also studied surfaces of higher topology and with several boundary curves, a problem known as Douglas' problem [30]. But the topology has to be fixed there, too. In general, the topology of the absolute minimizer is not known in advance. Even more, Fleming gave an example of a boundary curve where the corresponding minimizer does not have finite topology [34]. A third issue concerns the regularity of the solution. Even with smooth boundary curves the Douglas solution is in general only immersed. Only in the 1970s, Osserman, Gulliver and Alt could exclude interior branch points of the Douglas solution in $\mathbb{R}^3$, whereas selfintersections are possible and often necessary due to the a priori choice of the topology [8], [9], [39], [52].

To overcome all these difficulties another approach had to be taken. Whereas in Douglas' solution the study of a different functional than the area functional lead to the goal, from the 1950s to the 1970s several concepts of generalized surfaces have been introduced by various people such as De Giorgi, Reifenberg, Federer, Fleming, Almgren and Allard to make the direct method of the calculus of variations work [1], [2], [3], [25], [33], [60]. In this thesis we will use some of those concepts, most notably the theory of Caccioppoli sets and the theory of varifolds, and therefore introduce them in detail later. These different concepts, such as Caccioppoli sets, currents or varifolds, have been developed to attack various variational problems related to Plateau's problem or minimal surfaces. In spite of their differences there is an important common feature. The spaces of currents or varifolds are compact with respect to certain weak topologies. Similarly to the study of weak solutions of PDEs in Sobolev spaces, weak solutions to Plateau's problem and related questions can be found by very abstract functional analytical arguments such as the direct method of the calculus of variations or, more relevant for this thesis, mountain-pass-type arguments. The main difficulty therefore lies in the regularity theory that investigates to what extent these weak objects are in fact classical surfaces or embedded manifolds. For the generalized Plateau problem this strategy has been very successful. In the 1960s, within the framework of the rectifiable currents of Federer and Fleming, the codimension 1 case could be settled quite completely in the works of De Giorgi, Fleming, Almgren, Simons, Federer and Simon [4], [25], [32], [35], [70], [71].

Up to dimension 7 the minimizers found by geometric measure theory methods are in fact embedded. In dimension 8 the first singular solution occurs, for instance the famous Simons' cone that was shown to be the absolute minimizer in a celebrated paper by Bombieri, De Giorgi and Giusti [15]. However, this regularity comes at the cost of a loss of control over the topology. In the special situation of a two-dimensional surface with a boundary curve lying on the boundary of a convex set, Almgren and Simon showed the existence of an embedded minimal disk [7], see also a result of Meeks and Yau [47]. The case of higher codimension is considerably harder and could be solved by Almgren in his Big Regularity Paper [5].

1.2 Unstable minimal surfaces

Another fundamental question that arose after the solution of Douglas and Radó and that lies in the core of this thesis is the question of the existence of unstable minimal surfaces. Plateau's problem asks to find the absolute minimizer among all suitable surfaces sharing a common given boundary. By definition, any critical point of the area functional, not only minimizers, is a minimal surface. Clearly, in Plateau's experiments unstable surfaces could barely be studied since the soap films sought stable configurations and did not persist in the unstable shape. A very classical example of an unstable minimal surface is the catenoid. Given two coaxial unit circles as boundary curves, there are different minimal surfaces spanned by them. First of all, there are of course the two plane disks which give a surface with two connected components. In addition, there are – depending on the distance of the two circles – zero, one or two catenoids, one of which is unstable in the latter case.

The general theory to study critical points of higher index, that is, unstable critical points, is Morse theory [49]. In the late 1930s Morse studied the applicability of his general theory to minimal surfaces [51], see also [68] for a similar result of Shiffman. The so-called Morse inequalities that relate the numbers of critical points of different indices provide a crucial tool. A particularly simple instance of the type of conclusions that can be drawn can also be achieved by mountain-pass-type arguments. We will give a precise and detailed description of this argument for the case of our primary interest later. Here, we only sketch the main ideas of the strategy. The statement of the mountain pass lemma, or min-max argument, is roughly the following: given two strict local minimizers (in the same connected component) of a functional F, there is a third critical point that is not a local minimizer, but unstable. To see this, one considers paths connecting the two local minimizers. Along each of these paths F takes a maximum point. Among all these paths one chooses a sequence such that the maxima of F converge to $\inf_\gamma \max_{t\in[0,1]} F(\gamma(t))$, a minimizing sequence. Exhibiting a converging subsequence one finds as limit the desired critical point. Again there is a compactness argument involved. The condition that is usually needed is the so-called condition C, also known as Palais-Smale condition, that requires that all critical sequences, i.e., sequences satisfying $\|DF(x_n)\| \to 0$, contain a converging subsequence. The length functional for curves satisfies the condition (on a suitable space of maps), whereas the area functional for higher dimensional surfaces or manifolds does not. This can be viewed as an indication that Morse theory for curves or geodesics is much more successful than for minimal surfaces.

The early results about unstable minimal surfaces by Morse-Tompkins and Shiffman still considered minimal surfaces bounded by curves. There is, however, another setting in which one might try to apply these techniques that is particularly interesting from a geometric point of view.

Problem 2. *Given a closed Riemannian manifold. Are there closed minimal submanifolds?*

As for the Plateau problem, one can distinguish various cases regarding dimension, codimension or topology. A very classical result in global differential geometry is the existence of closed geodesics in arbitrary closed Riemannian manifolds of dimension 2 by Lyusternik and Fet [43]. A famous result by Lyusternik and Shnirelman asserts that there are always at least three such closed geodesics on any manifold homeomorphic to the sphere [44]. The ellipsoid shows that this result is optimal. The min-max method has first been applied to produce closed geodesics by Birkhoff in 1917 [14]. He proved the existence of closed geodesics in manifolds that are homeomorphic to the sphere. In this setting the paths of the mountain pass argument correspond to 1-parameter families $\{\gamma_t\}$ of maps from S^1 into the manifold, where γ_0 and γ_1 are assumed to be constant maps (corresponding to the local minimizers). There are certain topological conditions and extra arguments involved to assure that the min-max critical point is not trivial (the minimizers are not strict), but the strategy that we have sketched above works very well since the condition C holds on a suitable choice of the space of maps.

Since the condition C does not hold for the area functional, a direct application of this method is not possible to produce closed minimal submanifolds. Nevertheless, there have been two successful attempts to implement the strategy in the early 1980s. Similarly to the discussion of Plateau's problem one attempt is studying conformal harmonic maps, whereas the other one uses geometric measure theory. In dimension 2 there has been a very influential work by Sacks and Uhlenbeck [61]. As in Courant's solution of Plateau's problem, minimal spheres can be characterized as conformal harmonic maps from a standard sphere into a manifold. The conformal invariance of the Dirichlet energy and the non-compactness of the conformal group lead to the failure of condition C. There is, however, a way to partially overcome this difficulty. The exponent 2 in the Dirichlet energy is critical in the sense that for any exponent $\alpha > 2$ the functional $E_\alpha(u) = \int |\nabla u|^\alpha$ will satisfy the condition C on a suitable Sobolev space. Therefore the strategy of Sacks and Uhlenbeck was to apply the mountain pass argument to the functionals E_α to find critical maps u_α and then study the behaviour of the sequences $\{u_\alpha\}$ as $\alpha \to 2$. Either a subsequence converges to a harmonic sphere or there is a concentration of energy. In the latter case, a blowup provides the harmonic sphere. This phenomenon, sometimes referred to as bubbling, occurs in many other geometric variational problems with conformal invariance, such as the Yamabe problem or the study of Yang-Mills connections (see for instance [74], [67]).

Similarly to the Douglas solution of the Plateau problem, there are certain drawbacks of the solution of Sacks and Uhlenbeck. Again, the theory is two-dimensional and does not apply to higher dimensions. Moreover, the resulting minimal sphere is merely immersed and cannot be shown to be embedded in general. In a later work Sacks and Uhlenbeck extended their results to surfaces of higher genus [62]. But again, the genus has to be specified in advance, which on the bright side also gives a control of the genus.

1.3 Geometric measure theory and min-max arguments

To meet these problems, starting with works of Almgren and then most notably one of his students, Pitts, there was a second attempt to prove the existence of closed minimal submanifolds that used tools from geometric measure theory [56]. In the case of the generalized Plateau problem the theory of currents (or in the codimension 1 case the subclass of Caccioppoli sets) turned out to provide the correct setting as there is a natural notion of boundary and the space has the functional analytic properties needed to apply the direct method of the calculus of variations. There is, however, a feature of the theory that makes currents unsuitable for a min-max argument, namely the fact that cancellation of mass can happen. Roughly speaking this is due to the fact that currents are oriented. In the min-max argument this could lead to trivial solutions if in the limit two copies of a surface with opposite orientations collapse. The theory of varifolds, developed by Almgren and Allard (first ideas in the direction date back to Young), can be seen as an unoriented version that avoids these problems.

In his monograph, Pitts implemented a version of the min-max argument to prove the existence of closed embedded minimal hypersurfaces. In a first step, he showed the existence of stationary varifolds (weak minimal surfaces) in arbitrary Riemannian manifolds. This result goes back to Almgren who showed it in arbitrary codimension [3]. The hard part of the proof is once again the regularity of these stationary varifolds. Unlike for area-minimizing currents, there is no general strong regularity result for stationary varifolds as simple counterexamples show. The best general result is the regularity in an open dense subset for rectifiable stationary varifolds due to Allard [1]. In general, examples show that the singular set might have codimension 1. In fact, even the rectifiability is not clear a priori. But in the case of the min-max argument there is more information about the stationary varifold since it has certain variational properties. Pitts used this information to find a clever local approximation by stable minimal hypersurfaces. Together with the curvature estimates for stable minimal hypersurfaces (and the subsequent compactness results) this was enough to establish the required regularity. A variant of this proof will be given in this thesis. Therefore we will describe the strategy in more detail later. Pitts' proof applies for hypersurfaces in manifolds of dimension less than or equal to 6, as he used a version of the curvature estimates due to Schoen, Simon and Yau [66] that holds in those dimensions. Later Schoen and Simon gave the final result by extending the curvature estimates to arbitrary dimensions (outside of a possible singular set, see [65]).

With these results the story in principle comes to an end. The monograph of Pitts, however, is very difficult and quite long. It is the main goal of this thesis to provide a proof that is considerably shorter and more accessible.

In the case of surfaces in 3-manifolds there has been an alternative proof by Simon and Smith, also reported in a survey by Colding and De Lellis [18], [72]. In the work of Simon and Smith the second goal was the proof of a genus bound that was recently completed by De Lellis and Pellandini [27]. The genus bounds have been claimed by Pitts and Rubinstein in [57] (these bounds are, however, stronger than the ones proved by De Lellis and Pellandini). This topological information was used to establish an analogous result to the Lyusternik-Shnirelman theorem about minimal surfaces: Jost proved that every metric on S^3 admits at least four embedded minimal 2-spheres. Examples of White show that this result is optimal [42], [76].

Since the existence parts in all these works essentially use a famous result due to Meeks, Simon and Yau [45] that is not available in dimensions (of the ambient manifold) higher than 3, the proofs cannot be transferred to higher dimensions. In this thesis we nevertheless follow in many aspects the arguments of the proofs in [18] to give a simplified version of Pitts' proof in the general case.

Seeing this result in the spirit of Morse theory, there is the natural question about the index of the embedded minimal surface found by the min-max argument. In the abstract setting of the mountain pass lemma the critical point has index 1 and the proof is quite simple, but relies heavily on the condition C [10]. Thus a straightforward conclusion cannot be drawn for the case at hand. For the approach of Sacks and Uhlenbeck there has been an important work by Micallef and Moore who proved index bounds [48]. As an important application they proved an improved version of the classical sphere theorem introducing a new curvature condition that was used by Brendle and Schoen in their recent proof of the differentiable sphere theorem [16]. Pitts and Rubinstein claimed that, for a generic metric of the ambient manifold, the surface obtained by Pitts' version of the min-max argument has index 1 (there is a more precise formulation of the claim in [57], see Claim 1 in Chapter 9). No proof of this claim has been published so far. The matter seems not so simple since the approximation by the critical sequence is merely in the varifold sense that is too weak to allow any direct conclusions about the index of the limit. It is a second goal of this thesis to make a further analysis of the (two-dimensional) embedded minimal hypersurface and its approximation with regard to index bounds.

1.4 Overview of the thesis

The main result of this thesis is a simplified proof of the following

Theorem 1. *Let M be an $(n+1)$-dimensional smooth closed Riemannian manifold. Then there is a nontrivial embedded minimal hypersurface $\Sigma \subset$*

M without boundary with a singular set $\operatorname{Sing}\Sigma$ *of Hausdorff dimension at most* $n-7$.

More precisely, Σ is a closed set of finite $\mathcal{H}^n$-measure and $\operatorname{Sing}\Sigma \subset \Sigma$ is the smallest closed set S such that $M \setminus S$ is a smooth embedded hypersurface ($\Sigma \setminus \operatorname{Sing}\Sigma$ is in fact analytic if M is analytic). In this thesis *smooth* will always mean C^∞. In fact, the result remains true for any C^4 Riemannian manifold M, Σ then will be of class C^2 (see [65]). Moreover $\int_{\Sigma\setminus\operatorname{Sing}\Sigma} \omega = 0$ for any exact n-form on M. The case $2 \leq n \leq 5$ was proved by Pitts in his groundbreaking monograph [56], an outstanding contribution which triggered all the subsequent research in the topic. The general case was proved by Schoen and Simon in [65], building heavily upon the work of Pitts.

The monograph [56] can be ideally split into two parts. The first half of the book implements a complicated existence theory for suitable "weak generalizations" of global minimal submanifolds, which is a version of the classical min-max argument introduced by Birkhoff for $n = 1$ (see [14]). The second part contains the regularity theory needed to prove Theorem 1. The curvature estimates of [66] for stable minimal surfaces are a key ingredient of this part: the core contribution of [65] is the extension of these fundamental estimates to any dimension, which enabled the authors to complete Pitts' program for $n > 5$.

[65] gives also a quite readable account of parts of Pitts' regularity theory. To our knowledge, there is instead no contribution to clarify other portions of the monograph, at least in general dimension. As discussed above, for $n = 2$ there has been a powerful variant of Pitts' approach, reported in [18].

This thesis gives a much simpler proof of Theorem 1. Our contribution draws heavily on the existing literature and follows Pitts in many aspects. However we introduce some new ideas which, in spite of their simplicity, allow us to shorten the proof dramatically.

As mentioned before, the second goal of the thesis is a refined study of the case $n = 2$ with regard to index bounds. In fact, we have no final result that would settle the issue. We prove the following theorem (this is Theorem 9.12, see Chapters 2 and 9 for the relevant definitions).

Theorem. *Consider* (S^3, g). *Let* Λ *be a family of regular sweepouts of type* S^2. *Then one of the following two cases holds:*

(i) *There is an embedded minimal* 2*-sphere with* Area $\leq \frac{m_0(\Lambda)}{2}$.

(ii) *There is an embedded minimal* 2*-sphere with* Area $\leq m_0(\Lambda)$ *and* Index + Nullity ≥ 1.

The ambiguity in the statement comes from the fact that the surface of Theorem 1 might have a multiplicity higher than one (coming from its construction). Hence the approximating critical sequence might consist of multiple coverings of the surface. Since the surface of Theorem 1 is obtained as the limit of a sequence that is defined in variational terms, we will derive most of its properties (regularity, index bound) as a consequence of properties of the approximating critical sequence.

The convergence of the critical sequence is only in the varifold sense (see [18], [27]). This convergence is too weak to prove an index bound with multiplicity or the genus bound with multiplicity (claimed by Pitts and Rubinstein, indeed the difference between the bounds of [27] and the bounds of [57] occurs only for multiplicity larger than 1). Therefore we study some improvements of the convergence of the critical sequence. For instance, we prove that suitable modifications allow us to deform the sequence into a new one that converges in the Hausdorff sense, but still carries the relevant variational information. In order to obtain a sequence that converges even smoothly, it seems that curvature estimates for stable surfaces up to the boundary would be needed (the regularity of area minimizing currents up to the boundary has been proved in [40]). To our knowledge no such estimates have been proved so far. General estimates at the boundary seem to be quite a delicate issue. In this thesis we study a simple situation, in which we can prove boundary estimates.

The thesis is organized as follows. Chapter 2 introduces the setting of our proof and necessary preliminaries. In Chapter 3 we recall some important curvature estimates for stable minimal surfaces. Usually, these estimates are interior estimates. We prove a curvature estimate up to the boundary for a particularly simple, but nevertheless interesting situation. Chapter 4 gives an overview of our proof of Theorem 1. We prefer to give a complete proof even though some parts are already clarified in the literature. The only major step we do not include is a proof of the curvature estimates. Chapters 5-8 contain the proof where our main contributions are contained in Chapters 6 and 7. Finally, in Chapter 9 we consider the case of surfaces in 3-manifolds. We discuss some improvements of the convergence of the critical sequence and a consequence concerning the index of the minimal min-max surface in a simple situation. Chapters 4, 6 (except Section 6.3), 7, 8 and parts of Chapter 2 are contained in our publication [28] (to appear in Journal of Differential Geometry). Chapter 5 is taken from [18] and we include it for the sake of completeness. Finally, Chapters 3 and 9 contain further unpublished results.

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2 Setting and Preliminaries

In this chapter we introduce the setting of our interpretation of Pitts' approach. Since the overall strategy of our proof will be quite similar, the correct setting is an important ingredient. In the first section we discuss both our setting and the corresponding setting of [18] for the case $n = 2$. We will need the latter in Chapter 9. At this stage the differences are technical and not well motivated. We will discuss their significance later. In the second section we collect the necessary preliminaries from geometric measure theory.

2.1 Min-max surfaces

In what follows M will denote an $(n+1)$-dimensional compact smooth Riemannian manifold without boundary. First of all we need to generalize slightly the standard notion of a 1-parameter family of hypersurfaces, allowing for some singularities.

Definition 2.1. *A family $\{\Gamma_t\}_{t\in[0,1]^k}$ of closed subsets of M with finite $\mathcal{H}^n$-measure is called a* generalized smooth family *if*

(s1) For each t there is a finite set $P_t \subset M$ such that Γ_t is a smooth hypersurface in $M \setminus P_t$;

(s2) $\mathcal{H}^n(\Gamma_t)$ depends smoothly on t and $t \mapsto \Gamma_t$ is continuous in the Hausdorff sense;

(s3) on any $U \subset\subset M \setminus P_{t_0}$, $\Gamma_t \stackrel{t\to t_0}{\longrightarrow} \Gamma_{t_0}$ smoothly in U.

$\{\Gamma_t\}_{t\in[0,1]}$ is a sweepout *of M if there exists a family $\{\Omega_t\}_{t\in[0,1]}$ of open sets such that*

(sw1) $(\Gamma_t \setminus \partial\Omega_t) \subset P_t$ for any t;

(sw2) $\Omega_0 = \emptyset$ and $\Omega_1 = M$;

(sw3) $\mathrm{Vol}(\Omega_t \setminus \Omega_s) + \mathrm{Vol}(\Omega_s \setminus \Omega_t) \to 0$ *as* $t \to s$.

Remark 2.2. *The convergence in (s3) means, as usual, that, if* $U \subset\subset M \setminus P_{t_0}$*, then there is* $\delta > 0$ *such that, for* $|t - t_0| < \delta$*,* $\Gamma_t \cap U$ *is the graph of a function* g_t *over* $\Gamma_{t_0} \cap U$*. Moreover, given* $k \in \mathbb{N}$ *and* $\varepsilon > 0$*,* $\|g_t\|_{C^k} < \varepsilon$ *provided* δ *is sufficiently small.*

We introduce the singularities P_t for two important reasons. They allow for the change of topology which, for $n > 2$, is a fundamental tool of the regularity theory. Moreover, it is easy to exhibit sweepouts as in Definition 2.1 as it is witnessed by the following proposition.

Proposition 2.3. *Let* $f : M \to [0,1]$ *be a smooth Morse function. Then* $\{\{f = t\}\}_{t\in[0,1]}$ *is a sweepout.*

The obvious proof is left to the reader. For any generalized family $\{\Gamma_t\}$ we set

$$\mathcal{F}(\{\Gamma_t\}) := \max_{t\in[0,1]} \mathcal{H}^n(\Gamma_t). \tag{2.1}$$

A key property of sweepouts is an obvious consequence of the isoperimetric inequality.

Proposition 2.4. *There exists* $C(M) > 0$ *such that* $\mathcal{F}(\{\Gamma_t\}) \geq C(M)$ *for every sweepout.*

Proof. Let $\{\Omega_t\}$ be as in Definition 2.1. Then, there is $t_0 \in [0,1]$ such that $\mathrm{Vol}(\Omega_{t_0}) = \mathrm{Vol}(M)/2$. We then conclude

$$\mathcal{H}^n(\Gamma_{t_0}) \geq c_0^{-1}(2^{-1}\mathrm{Vol}\,(M))^{\frac{n}{n+1}},$$

where c_0 is the isoperimetric constant of M. □

For any family Λ of sweepouts we define

$$m_0(\Lambda) := \inf_{\Lambda} \mathcal{F} = \inf_{\{\Gamma_t\}\in\Lambda} \left[\max_{t\in[0,1]} \mathcal{H}^n(\Gamma_t)\right]. \tag{2.2}$$

By Proposition 2.4, $m_0(\Lambda) \geq C(M) > 0$. A sequence $\{\{\Gamma_t\}^k\} \subset \Lambda$ is *minimizing* if

$$\lim_{k\to\infty} \mathcal{F}(\{\Gamma_t\}^k) = m_0(\Lambda).$$

A sequence of surfaces $\{\Gamma^k_{t_k}\}$ is a *min-max sequence* if $\{\{\Gamma_t\}^k\}$ is minimizing and $\mathcal{H}^n(\Gamma^k_{t_k}) \to m_0(\Lambda)$. The min-max construction is applied to families of sweepouts which are closed under a very natural notion of homotopy.

Definition 2.5. *Two sweepouts $\{\Gamma^0_s\}$ and $\{\Gamma^1_s\}$ are homotopic if there is a generalized family $\{\Gamma_t\}_{t\in[0,1]^2}$ such that $\Gamma_{(0,s)} = \Gamma^0_s$ and $\Gamma_{(1,s)} = \Gamma^1_s$. A family Λ of sweepouts is called* homotopically closed *if it contains the homotopy class of each of its elements.*

Ultimately, in this thesis we give a proof of the following theorem, which, together with Proposition 2.3, implies Theorem 1 for $n \geq 2$ (recall that Morse functions exist on every smooth compact Riemannian manifold without boundary; see Corollary 6.7 of [49]).

Theorem 2. *Let $n \geq 2$. For any homotopically closed family Λ of sweepouts there is a min-max sequence $\{\Gamma^k_{t_k}\}$ converging (in the sense of varifolds) to an embedded minimal hypersurface Σ as in Theorem 1. Multiplicity is allowed.*

The smoothness assumption on the metric g can be relaxed easily to C^4. The ingredients of the proof where this regularity is needed are: the regularity theory for Plateau's problem, the unique continuation for classical minimal surfaces and the Schoen-Simon compactness theorem. C^4 suffices for all of them.

2.1.1 The case $n = 2$

For $n = 2$ there is a variant of the above that is even more powerful. Following [27] we have

Definition 2.6. *A family $\{\Sigma_t\}_{t\in[0,1]}$ of surfaces of M is said to be* continuous *if*

(c1) $\mathcal{H}^2(\Sigma_t)$ is a continuous function of t;

(c2) $\Sigma_t \to \Sigma_{t_0}$ in the Hausdorff topology whenever $t \to t_0$.

A family $\{\Sigma_t\}_{t\in[0,1]}$ of subsets of M is said to be a generalized family *of surfaces if there are a finite subset T of $[0,1]$ and a finite set of points P in M such that* *(g1) (c1) and (c2) hold;*

(g2) Σ_t *is a surface for every* $t \notin T$*;*

(g3) for $t \in T$, Σ_t *is a surface in* $M \setminus P$.

In [18] this definition was used to prove the analogon of Theorem 2 (see also Theorem 9.3). For the genus bound that we will use in Chapter 9, in [27] there is still a narrower concept needed.

Definition 2.7. *A generalized family* $\{\Sigma_t\}$ *as in Definition 2.6 is said to be* smooth *if:*

(s1) Σ_t *varies smoothly in* t *on* $[0, 1] \setminus T$*;*

(s2) For $t \in T$, $\Sigma_\tau \to \Sigma_t$ *smoothly in* $M \setminus P$.

Here P *and* T *are the sets of requirements (g2) and (g3) of Definition 2.6. We assume further that* Σ_t *is orientable for any* $t \notin T$.

With a small abuse of notation, we shall use the word "surface" even for the sets Σ_t with $t \in T$.

Remark 2.8. *The term* generalized smooth family *has been used twice (in Definition 2.1 and in Definition 2.7) to denote different concepts. Since it will always be clear in the context which definition is used, we keep the ambiguous name.*

Given a generalized family $\{\Sigma_t\}$ we can generate new generalized families via the following procedure. Take an arbitrary map $\psi \in C^\infty([0, 1] \times M, M)$ such that $\psi(t, \cdot) \in \mathrm{Diff}_0$ (the identity component of the diffeomorphism group) for each t and define $\{\Sigma'_t\}$ by $\Sigma'_t = \psi(t, \Sigma_t)$. We will say that a set Λ of generalized families is *saturated* if it is closed under this operation. Note that, if a set Λ consists of smooth generalized families, then the elements of its saturation are still smooth generalized families.

Remark 2.9. *For technical reasons we require an additional property for any saturated set* Λ *considered in [18]: the existence of some* $N = N(\Lambda) < \infty$ *such that for any* $\{\Sigma_t\} \in \Lambda$*, the set* P *in Definition 2.6 consists of at most* N *points.*

The argument of Proposition 2.4 applies also in this situation. Moreover, we can argue analogously as above to find Λ with $m_0(\Lambda) > 0$.

Remark 2.10. *At this point there are two crucial differences between the case $n = 2$ and the general situation to be pointed out. First of all, sweepouts in the sense of Definition 2.1 allow for finitely many singularities at* each *time t, whereas smooth generalized families in the sense of Definition 2.7 only allow finitely many points on the manifold where singularities might occur at a finite number of times t. So far these singularities were only used to find sweepouts. We will see later that in the higher dimensional situation these singularities are crucial for the regularity theory (see Chapter 7).*

The second difference is in the definition of the set Λ. In both cases it is defined as the closure under some class of deformations. In the higher dimensional case deformations with the same regularity assumptions as the original family itself are allowed. In the two-dimensional case only deformations by isotopies are admissible. This is crucial for the two-dimensional regularity theory using [45]. In Chapter 9, we will consider a situation where in dimension 2 not only the deformations, but also the original family is induced by an isotopy. The existence of sets Λ with $m_0(\Lambda) > 0$ of such families is not covered by the discussion in this chapter. But in the special situations that we will study the same argument can be applied (the ambient manifold will be a 3-sphere and the sweepouts by 2-spheres or tori). In [57] Pitts and Rubinstein claimed that such Λ should always exist.

Remark 2.11. *In [56] Pitts studies families of much less regular objects, namely currents. To prove that a critical sequence converges in area (or mass) to a strictly positive value, he uses an isomorphism of Almgren [6] between homotopy groups of integral cycle groups (currents) and homology groups of the manifold. In view of the simplicity of Proposition 2.4, here a first advantage of our approach is evident.*

2.2 Preliminaries

2.2.1 Notation

Throughout this thesis our notation will be consistent with the one introduced in Section 2 of [18]. We summarize it in the following table.

$\mathrm{Inj}\,(M)$	the injectivity radius of M;
$B_\rho(x)$, $\overline{B}_\rho(x)$, $\partial B_\rho(x)$	the open and closed ball, the distance sphere in M;
$\mathrm{diam}(G)$	the diameter of $G \subset M$;
$\mathrm{d}(G_1, G_2)$	$\inf_{x\in G_1, y\in G_2} \mathrm{d}(x,y)$;
$\mathcal{B}_\rho$	the ball of radius ρ and centered in 0 in $\mathbb{R}^{n+1}$;
$\exp_x$	the exponential map in M at $x \in M$;
$An(x,\tau,t)$	the open annulus $B_t(x) \setminus \overline{B}_\tau(x)$;
$\mathcal{AN}_r(x)$	the set $\{An(x,\tau,t)$ with $0 < \tau < t < r\}$;
$\mathcal{X}(M)$, $\mathcal{X}_c(U)$	smooth vector fields, smooth vector fields compactly supported in U.

Remark 2.12. *In [18] the authors erroneously define* d *as the Hausdorff distance. However, for the purposes of both this thesis and that paper, the correct definition of* d *is the one given here, since in both cases the following fact plays a fundamental role:* $\mathrm{d}(A,B) > 0 \Longrightarrow A \cap B = \emptyset$ *(see also Lemma 6.3). Note that, unlike the Hausdorff distance,* d *is not a distance on the space of compact sets.*

2.2.2 Caccioppoli sets and Plateau's problem

We give here a brief account of the theory of Caccioppoli sets. A standard reference is [38]. Let $E \subset M$ be a measurable set and consider its indicator function $\mathbf{1}_E$ (taking the value 1 on E and 0 on $M \setminus E$). The perimeter of E is defined as

$$\mathrm{Per}\,(E) := \sup\left\{ \int_M \mathbf{1}_E \,\mathrm{div}\,\omega \;:\; \omega \in \mathcal{X}(M),\ \|\omega\|_{C^0} \le 1 \right\}.$$

A *Caccioppoli set* is a set E for which $\mathrm{Per}\,(E) < \infty$. In this case the distributional derivative $D\mathbf{1}_E$ is a Radon measure and $\mathrm{Per}\,E$ corresponds to its total variation. As usual, the perimeter of E in an open set U, denoted by $\mathrm{Per}\,(E,U)$, is the total variation of $D\mathbf{1}_E$ in the set U.

We follow De Giorgi and, given a Caccioppoli set $\Omega \subset M$ and an open set $U \subset M$, we consider the class

$$\mathcal{P}(U,\Omega) \;:=\; \{\Omega' \subset M \;:\; \Omega' \setminus U = \Omega \setminus U\}\,. \qquad (2.3)$$

The theorem below states the fundamental existence and interior regularity theory for De Giorgi's solution of Plateau's problem, which summarizes results of De Giorgi, Fleming, Almgren, Simons and Federer (see [38] for the case $M = \mathbb{R}^{n+1}$ and Section 37 of [69] for the general case).

Theorem 2.13. *Let $U, \Omega \subset M$ be, respectively, an open and a Caccioppoli set. Then there exists a Caccioppoli set $\Xi \in \mathcal{P}(U, \Omega)$ minimizing the perimeter. Moreover, any such minimizer is, in U, an open set whose boundary is smooth outside of a singular set of Hausdorff dimension at most $n-7$.*

2.2.3 Theory of varifolds

We recall here some basic facts from the theory of varifolds; see for instance Chapters 4 and 8 of [69] for further information. Varifolds are a convenient way of generalizing surfaces to a category that has good compactness properties. An advantage of varifolds, over other generalizations (like currents), is that they do not allow for cancellation of mass. This last property is fundamental for the min-max construction. If U is an open subset of M, any Radon measure on the Grassmannian $G(U)$ of unoriented n-planes on U is said to be an *n-varifold in U*. The space of n-varifolds is denoted by $\mathcal{V}(U)$ and we endow it with the topology of the weak* convergence in the sense of measures. Therefore, a sequence $\{V^k\} \subset \mathcal{V}(U)$ converges to V if

$$\lim_{k\to\infty} \int \varphi(x,\pi)\, dV^k(x,\pi) \;=\; \int \varphi(x,\pi)\, dV(x,\pi)$$

for every $\varphi \in C_c(G(U))$. Here π denotes an n-plane of T_xM. If $U' \subset U$ and $V \in \mathcal{V}(U)$, then $V \llcorner U'$ is the restriction of the measure V to $G(U')$. Moreover, $\|V\|$ is the nonnegative measure on U defined by

$$\int_U \varphi(x)\, d\|V\|(x) \;=\; \int_{G(U)} \varphi(x)\, dV(x,\pi) \qquad \forall \varphi \in C_c(U)\,.$$

The support of $\|V\|$, denoted by $\operatorname{supp}(\|V\|)$, is the smallest closed set outside which $\|V\|$ vanishes identically. The number $\|V\|(U)$ will be called the *mass of V in U*.

Recall also that an n-dimensional rectifiable set is the countable union of closed subsets of C^1 surfaces (modulo sets of $\mathcal{H}^n$-measure 0). If $R \subset U$ is an n-dimensional rectifiable set and $h : R \to \mathbb{R}^+$ is a Borel function, then the *varifold V induced by R* is defined by

$$\int_{G(U)} \varphi(x,\pi)\, dV(x,\pi) = \int_R h(x)\varphi(x, T_x R)\, d\mathcal{H}^n(x) \tag{2.4}$$

for all $\varphi \in C_c(G(U))$. Here $T_x R$ denotes the tangent plane to R in x. If h is integer-valued, then we say that V is an *integer rectifiable varifold.* If $\Sigma = \bigcup n_i \Sigma_i$, then by slight abuse of notation we use Σ for the varifold induced by Σ via (2.4).

If $\psi : U \to U'$ is a diffeomorphism and $V \in \mathcal{V}(U)$, $\psi_\sharp V \in \mathcal{V}(U')$ is the varifold defined by

$$\int \varphi(y,\sigma)\, d(\psi_\sharp V)(y,\sigma) \;=\; \int J\psi(x,\pi)\, \varphi(\psi(x), d\psi_x(\pi))\, dV(x,\pi)\,,$$

where $J\psi(x,\pi)$ denotes the Jacobian determinant (i.e. the area element) of the differential $d\psi_x$ restricted to the plane π; cf. equation (39.1) of [69]. Obviously, if V is induced by a C^1 surface Σ, V' is induced by $\psi(\Sigma)$.

Given $\chi \in \mathcal{X}_c(U)$, let ψ be the isotopy generated by χ, i.e. $\frac{\partial\psi}{\partial t} = \chi(\psi)$. The first and second variation of V with respect to χ are defined as

$$\begin{aligned} [\delta V](\chi) \;&=\; \frac{d}{dt}\,(\|\psi(t,\cdot)_\sharp V\|)(U)\Big|_{t=0} \\ \text{and} \qquad [\delta^2 V](\chi) \;&=\; \frac{d^2}{dt^2}\,(\|\psi(t,\cdot)_\sharp V\|)(U)\Big|_{t=0}\,, \end{aligned}$$

cf. Sections 16 and 39 of [69]. V is said to be *stationary* (resp. *stable*) in U if $[\delta V](\chi) = 0$ (resp. $[\delta^2 V](\chi) \geq 0$) for every $\chi \in \mathcal{X}_c(U)$. If V is induced by a surface Σ with $\partial\Sigma \subset \partial U$, V is stationary (resp. stable) if and only if Σ is minimal (resp. stable, see Section 3.1).

Stationary varifolds in a Riemannian manifold satisfy the monotonicity formula, i.e. there exists a constant Λ (depending on the ambient manifold M) such that the function

$$f(\rho) \;:=\; e^{\Lambda\rho}\,\frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \tag{2.5}$$

is nondecreasing for every x (see Theorem 17.6 of [69]; $\Lambda = 0$ if the metric of M is flat). This property allows us to define the *density* of a stationary varifold V at x, by

$$\theta(x, V) \; = \; \lim_{r \to 0} \frac{\|V\|(B_r(x))}{\omega_n r^n} \, .$$

3 Curvature estimates

In this chapter we discuss stable minimal surfaces and curvature estimates. In the first section we introduce the stability operator and some consequences for stable surfaces. In the second section we collect different versions of curvature estimates that we will need in later chapters. In the third section we study curvature estimates at the boundary. To our knowledge this situation has not been studied in the literature so far. We follow an idea of Brian White to prove curvature estimates up to the boundary in a special situation.

3.1 Stable surfaces

In this section we assume that $\Sigma \subset M$ is an orientable minimal hypersurface of dimension n. The second variation along a normal vectorfield $X = F_t$ (for a variation F; w.l.o.g. $F_t^\top \equiv 0$) is given by

$$\frac{d^2}{dt^2}\operatorname{Area}(F(\Sigma,t))\Big|_{t=0} = -\int_\Sigma g(F_t, LF_t)\,.$$

Here L is the *stability operator* that can be written as an operator on functions in the case of orientable hypersurfaces, namely, for $F_t = \eta N$, where N is the unit normal vectorfield defined by the orientation,

$$L\eta = \Delta_\Sigma \eta + |A|^2\eta + \operatorname{Ric}_M(N,N)\eta\,.$$

In particular, for $M = \mathbb{R}^{n+1}$ we have $L = \Delta + |A|^2$.

Definition 3.1. *A minimal hypersurface $\Sigma \subset M$ is called* stable *if for all variations F with fixed boundary*

$$\frac{d^2}{dt^2}\operatorname{Area}(F(\Sigma,t))\Big|_{t=0} = -\int_\Sigma g(F_t, LF_t) \geq 0\,.$$

Moreover, the Morse index *of a compact minimal surface is the number of negative eigenvalues of the stability operator. In particular, Σ is stable if and only if the Morse index is zero.*

For this definition the assumptions of codimension one and orientability are not needed. However, under these assumptions we have (see Section 1.8 in [19])

Lemma 3.2. *Suppose that $\Sigma \subset M$ is an orientable stable minimal hypersurface. Then for all Lipschitz functions η with compact support*

$$\int_\Sigma (\inf_M \mathrm{Ric}_M + |A|^2)\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2 \,. \tag{3.1}$$

This inequality indicates that for stable minimal hypersurfaces a certain a priori control of the (total) curvature is possible.

Examples of stable minimal surfaces are given by minimal graphs or the solutions of Plateau's problem.

3.2 Curvature estimates

In this section we collect various versions of curvature estimates. In the regularity theory in the proof of Theorem 1 the curvature estimate for stable minimal surfaces of Schoen-Simon will be crucial. On the other hand, for surfaces of dimension 2 there are also versions where stability is not required. These results will be needed in Chapter 9. A standard consequence of curvature estimates are compactness results for the corresponding spaces of (stable) surfaces.

3.2.1 Schoen-Simon curvature estimates

Consider an orientable $U \subset M$. We look here at closed sets $\Gamma \subset M$ of codimension 1 satisfying the following regularity assumption:

(SS) $\Gamma \cap U$ is a smooth embedded hypersurface outside a closed set S with $\mathcal{H}^{n-2}(S) = 0$.

Γ induces an integer rectifiable varifold V. Thus Γ is said to be minimal (resp. stable) in U with respect to the metric g of U if V is stationary (resp. stable). The following compactness theorem, a consequence of the Schoen-Simon curvature estimates (see Theorem 2 of Section 6 in [65]), is a fundamental tool in this thesis.

Theorem 3.3. *Let U be an orientable open subset of a manifold and $\{g^k\}$ and $\{\Gamma^k\}$, respectively, sequences of smooth metrics on U and of hypersurfaces $\{\Gamma^k\}$ satisfying (SS). Assume that the metrics g^k converge smoothly to a metric g, that each Γ^k is stable and minimal relative to the metric g^k and that $\sup \mathcal{H}^n(\Gamma^k) < \infty$. Then there are a subsequence of $\{\Gamma^k\}$ (not relabeled), a stable stationary varifold V in U (relative to the metric g) and a closed set S of Hausdorff dimension at most $n-7$ such that*

(a) V is a smooth embedded hypersurface in $U \setminus S$;

(b) $\Gamma^k \to V$ in the sense of varifolds in U;

(c) Γ^k converges smoothly to V on every $U' \subset\subset U \setminus S$.

Remark 3.4. *The precise meaning of (c) is as follows: fix an open $U'' \subset U'$ where the varifold V is an integer multiple N of a smooth oriented surface Σ. Choose a normal unit vector field on Σ (in the metric g) and corresponding normal coordinates in a tubular neighborhood. Then, for k sufficiently large, $\Gamma^k \cap U''$ consists of N disjoint smooth surfaces Γ^k_i which are graphs of functions $f^k_i \in C^\infty(\Sigma)$ in the chosen coordinates. Assuming, w.l.o.g., $f^k_1 \leq f^k_2 \leq \ldots \leq f^k_N$, each sequence $\{\Gamma^k_i\}_k$ converges to Σ in the sense of Remark 2.2.*

Note the following obvious corollary of Theorem 3.3: if Γ is a stationary and stable surface satisfying (SS), then the Hausdorff dimension of $\operatorname{Sing}\Gamma$ is, in fact, at most $n-7$. Since we will deal very often with this type of surfaces, we will use the following notational convention.

Definition 3.5. *Unless otherwise specified, a* hypersurface $\Gamma \subset U$ *is a closed set of codimension* 1 *such that* $\overline{\Gamma} \setminus \Gamma \subset \partial U$ *and* $\operatorname{Sing}\Gamma$ *has Hausdorff dimension at most* $n-7$. *The words "stable" and "minimal" are then used as explained at the beginning of this subsection. For instance, the surface* Σ *of Theorem 1 is a* minimal hypersurface.

3.2.2 Curvature estimates in 3-manifolds

In 3-manifolds there are curvature estimates for minimal surfaces with small total curvature, small area or small excess, where there is no requirement regarding stability (see Chapter 2 of [19]). On the other hand there is a curvature estimate for stable minimal surfaces without any further assumptions (the Schoen-Simon curvature estimate related to the discussion of the previous subsection needs some extra conditions, see Theorem 3 of [65]). Many of these curvature estimates were used extensively by Colding and Minicozzi in their study of the space of embedded minimal surfaces in 3-manifolds (see [22] for an overview and the references therein).

Notation 3.6. *In the remaining sections of this chapter we use the notation $B_r(p)$ for the intrinsic ball on the surfaces $\Sigma, \Gamma, \ldots$. In the rest of the thesis this notation is used for the geodesic ball in the ambient manifold M. In this section the ambient manifold is $\mathbb{R}^3$ and we can use our usual notation $\mathcal{B}_r(x)$ for Euclidean balls.*

We will need the following theorem by Schoen (see [63]) in the next section. There is a corresponding version for arbitrary 3-manifolds with bounded sectional curvature, but we only need and state the version for $M = \mathbb{R}^3$.

Theorem 3.7 (Schoen). *Let $\Sigma \subset \mathbb{R}^3$ be a stable, immersed, orientable minimal surface with $B_{r_0}(x) \subset \Sigma \setminus \partial\Sigma$. Then there is a constant $C > 0$ such that*

$$\sup_{y \in B_{r_0 - \sigma}(x)} |A|^2(y) \leq \frac{C}{\sigma^2}\,. \tag{3.2}$$

Note that one immediately gets $\mathrm{dist}_\Sigma(x, \partial\Sigma)|A|^2(x) \leq C$.

Remark 3.8. *Since intrinsic balls are contained in extrinsic balls, the same estimate holds for extrinsic balls and distance* d*, with a slightly different constant.*

In Chapter 9 we will need the following compactness theorem that is a consequence of the curvature estimate for minimal surfaces with small total curvature, a version of a theorem by Choi-Schoen [17], see also Section 5.5 in [19].

Theorem 3.9 (Choi-Schoen). *Let M be a closed Riemannian 3-manifold and let $\{\Sigma^j\}$ be a sequence of varifolds. Assume there is $J \in \mathbb{N}$ such that, for $j \geq J$, Σ^j satisfies the following*

(i) Σ^j *is a smooth embedded minimal surface;*

(ii) $\sup_{j \geq J} \mathbf{g}(\Sigma^j) < \infty$*;*

(iii) $\sup_{j \geq J} \mathcal{H}^2(\Sigma^j) < \infty$.

Then there is a subsequence of $\{\Sigma^j\}$ (not relabeled) and a varifold Σ such that

(a) Σ *is a smooth embedded minimal surface;*

(b) $\mathbf{g}(\Sigma) \leq \liminf_{j \to \infty} \mathbf{g}(\Sigma^j)$*;*

(c) $\Sigma^j \to \Sigma$ *in the sense of varifolds.*

In fact, the convergence in (c) is much better, namely smooth except in finitely many points.

3.3 Curvature estimates at the boundary: a toy problem

We will see in later chapters that the curvature estimates of Schoen-Simon, Theorem 3.3, will be a crucial ingredient of the regularity theory in the proof of Theorem 2. It will allow us to deform the critical sequence (min-max sequence) *locally* to obtain a stronger approximation and subsequently a better regularity of the limit.

If one wishes to deduce some more information about the min-max surface, such as genus bounds ($n = 2$) or index bounds, it would be useful to perform these deformations *globally*. One strategy could be to paste together local deformations in such a way that the critical sequence becomes a multiple cover of the limit. In order to do that a decomposition of stable surfaces into graphs (see Remark 3.4) *up to the boundary* could be helpful. This motivates the study of curvature estimates up to the boundary in this context. The problem, however, is of independent interest.

In this section we study a toy problem and prove curvature estimates up to the boundary in this situation. We start by looking at a very classical example, the catenoid. The standard catenoid is defined by

$$C = \left\{ x \in \mathbb{R}^3 \,|\, \sqrt{x_1^2 + x_2^2} = \cosh(x_3) \right\} .$$

Let $C_t = \{x \in \mathbb{R}^3 \,|\, tx \in C\}$ denote rescalings of C by $t > 0$. Then, for $t \to \infty$, $C_t \cap (D_1 \times \mathbb{R})$ converges in the sense of varifolds to the disk $D_1 \times \{0\}$ with multiplicity two. On the boundary of the solid cylinder, $S^1 \times \mathbb{R}$, the rescaled catenoids converge smoothly to twice the circle. But the curvature of the rescalings blows up near the origin, along the neck (see Figure 3.1). As a consequence, we can note that by Schoen's curvature estimate, Theorem 3.7, for t large enough, the rescaled catenoid C_t is not stable. In fact, more precise statements about the stability of catenoids are possible (see for instance [13] and the references therein). On the other hand, Theorem 3.7 is an interior estimate. Thus, in principle, it could be possible that a sequence of stable surfaces with the same boundary behaviour as the rescaled catenoids have curvature blowing up towards the boundary (e.g. a sequence of necks pinching off at the boundary, see Figure 3.1).

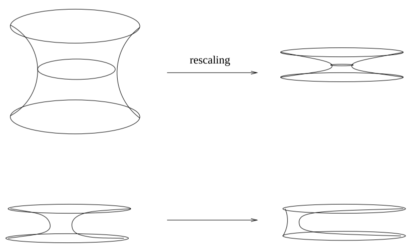

Figure 3.1: Above we see the curvature blowing up near the neck when we rescale the catenoid. Below a neck is pinching off at the boundary.

In this particular example, where the boundary curves on the cylinder are two coaxial circles that converge to twice $S^1 \times \{0\}$, this situation can be excluded. It is known that the only possible minimal surfaces spanned by these two circles are either catenoids or pairs of disks (see Corollary 3 in [64]). For more general boundary curves lying on the cylinder, however, no such classification is known. Therefore the blowing up of curvature at the boundary cannot be excluded in a similar way.

The main result of this section is the following

Theorem 3.10. *Let* $D_1 := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. *Assume that* $f_1^k, f_2^k : \partial D_1 \to \mathbb{R}$ *are two sequences of functions such that* $f_1^k(x) > f_2^k(x)$ *for all* x *and* $\|f_i^k\|_{C^{2,\alpha}} \to 0$ *as* $k \to \infty$ *for some* $\alpha > 0$. *Let* σ_i^k *be the graphs of* f_i^k *in* $\partial D_1 \times \mathbb{R} \subset \mathbb{R}^3$ *and let* Σ^k *be orientable embedded stable minimal surfaces in* $D_1 \times \mathbb{R}$ *such that*

- $\partial\Sigma^k = \sigma_1^k \cup \sigma_2^k$;
- $\Sigma^k \to 2[[D_1 \times \{0\}]]$ *in the sense of varifolds.*

Then, for k *large enough,* Σ^k *is the union of two disks. More precisely, there are two sequences of* $C^{2,\beta}$ *(for* $\beta < \alpha$*) functions* $g_i^k : D_1 \to \mathbb{R}$ *with* $g_i^k|_{\partial D_1} = f_i^k$ *such that* Σ^k *is the union of the graphs of* g_1^k *and* g_2^k, *and* $\|g_i^k\|_{C^{2,\beta}} \to 0$.

Clearly, a similar result for more than two boundary curves would be desirable. Our proof however, does not work for this situation anymore. Nevertheless we discuss it in Section 3.3.6.

The convergence result of Theorem 3.10 can be reduced to the following curvature estimate.

Theorem 3.11. *Assume that* Σ^k *are as above. Then*

$$\max_{x \in \Sigma^k} |A| \leq C, \qquad \text{for all } k, \tag{3.3}$$

where $C > 0$ *is a constant not depending on* k.

It is a by now fairly standard consequence of the uniform curvature estimates as in (3.3) that the sequence Σ^k is compact in the smooth ($C^{2,\beta}$, for $\beta < \alpha$) category yielding the statement of the first theorem. A proof of this fact can be found in [19] (see Lemma 2.4 and the second part of the proof of Proposition 5.17). Note that in that book the results are proved in the interior. Due to the regularity of the boundary in our case, the proofs can be adjusted to give the result up to the boundary.

3.3.1 Proof of Theorem 3.11: Setup

We argue by contradiction and consider a sequence $x_k \in \Sigma^k$ such that

$$a_k := \max_{x \in \Sigma^k} |A|(x) = |A|(x_k).$$

Then, by assumption, up to subsequences $a_k \to \infty$. By the interior curvature estimates for stable minimal surfaces, Theorem 3.7 and Remark 3.8, we have

$$a_k \, \mathrm{d}(x_k, \partial\Sigma^k) \le C \tag{3.4}$$

for some $C > 0$ not depending on k. Thus, for some $i \in \{1, 2\}$, there is a subsequence, not relabeled, such that

$$a_k \, \mathrm{d}(x_k, \sigma_i^k) \le C\,. \tag{3.5}$$

After possibly making a reflection, we can assume that $i = 1$ in (3.5). Let therefore $y_k \in \sigma_1^k$ such that $|y_k - x_k| = \mathrm{d}(x_k, \sigma_1^k)$. After a translation (along the vertical axis) and a rotation (around the vertical axis) we can assume that $y_k = (-1, 0, 0)$ for all k. Next consider the rescaled surfaces

$$\Gamma^k := a_k(\Sigma^k - y_k) = a_k(\Sigma^k - (-1, 0, 0))\,.$$

First of all, observe that

$$\max_{x \in \Gamma^k} |A|(x) = 1\,.$$

With the same kind of arguments as in the reduction of Theorem 3.10 to Theorem 3.11 we can assume (after the extraction of a subsequence) that Γ^k converges locally in $C^{2,\beta}$ for all $\beta < \alpha$ to a surface Γ. Note that the boundary regularity can only be improved in the blowup, so again this convergence is up to the boundary. The strategy of the proof will be to show that a blowup like Γ cannot exist. There is one minor case that has to be excluded in advance, namely when Γ is the empty set. This will be done in the next subsection.

3.3.2 Exclusion of the empty set

Assume that the two curves of $\partial\Gamma^k$ collapse to a single line in the limit (see Lemma 3.15 for a proof that the boundary of the blowup consists of parallel lines), but Γ^k converges to the empty set. We show that in this case the curvature of the original sequence $\{\Sigma^k\}$ does not blow up. To see this we rescale the sequence $\{\Sigma^k\}$ by the distance of the curves σ_i^k, i.e. $\rho_k = 1/\mathrm{d}(\sigma_1^k, \sigma_2^k)$ and

$$\Delta^k := \rho_k(\Sigma^k - \bar{y}_k) = \rho_k(\Sigma^k - (-1,0,0)) .$$

Here we choose the points $\bar{y}_k$ in such a way that (after the rotation and translation that brings them all to $(-1,0,0)$) in these rescalings the points $\bar{x}_k$ corresponding to the points x_k of maximal curvature lie in the half-plane $\{x_1 > 0, x_2 = 0\}$. Up to subsequence we have $\Delta^k \to \Delta$ in $C^{2,\beta}$.

We have the following information

- $|A_{\Delta^k}| = \mathrm{d}(\sigma_1^k, \sigma_2^k)|A_{\Sigma^k}| \to 0$;
- $\mathrm{d}(\partial\Delta_1^k, \partial\Delta_2^k) \geq 1$.

Moreover, Δ^k is squeezed between the rescalings of the two area minimizing disks (graphs) spanned by the σ_i^k (note that these curves lie on the cylinder above a *convex* set). These rescaled disks are very flat for k large and almost parallel to the plane $\{x_3 = 0\}$.

Now, by the assumptions on the sequence Γ^k, we can fix a point $z = (\zeta, 0, 0)$ and conclude that $\mathcal{B}_{\frac{\zeta}{2}}(z) \cap \Gamma^k = \emptyset$ for k large enough. In the rescalings this ball is travelling to ∞. Next, choose a catenoid centered at the center of the ball and such that the part of it lying between the two area minimizing disks is contained in the ball. Letting move the catenoid in x_1-direction, we can conclude with the maximum principle that $\partial\Delta$ consists of two lines $\{x_1 = 0, x_3 = h_i\}$ with $h_1 \neq h_2$ and Δ is the flat strip between these two lines. Therefore, in a slab $\{x_2 \in (-\gamma, \gamma)\}$, Δ^k is (in the right system of coordinates) a graph of a function u_k satisfying the minimal surface equation (see Figure 3.2). Note that the width 2γ of the slab does not depend on k.

By Schauder estimates and interpolation inequalities we have

$$|A_{\Delta^k}| \leq C\|\eta_1^k\|_{C^{2,\alpha}} \leq C(\|(\eta_1^k)''\|_{C^0} + [(\eta_1^k)'']_{0,\alpha}) . \tag{3.6}$$

Here η_i^k are the rescalings of the parametrizations f_i^k of σ_i^k. Moreover we have assumed that the point $\bar{x}_k$ is closer to η_1^k (therefore this part of the boundary appears in the estimate). Now we scale back inequality (3.6) and obtain

$$\frac{|A_{\Sigma^k}|}{\rho_k} \leq C \left(\frac{\|(f_1^k)''\|_{C^0}}{\rho_k} + \frac{[(f_1^k)'']_{0,\alpha}}{\rho_k^{1+\alpha}} \right) . \tag{3.7}$$

Now we multiply (3.7) by ρ_k and invoke the uniform bound on the $C^{2,\alpha}$ norms of f_1^k to get a uniform bound on $|A_{\Sigma^k}|$, which is what we have claimed (recall that $\rho_k \to \infty$). Therefore in the rest of this chapter we can assume that Γ is not empty.

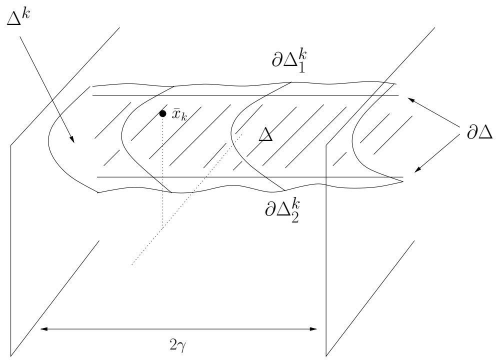

Figure 3.2: The surfaces Δ^k are graphical over the plane $\{x_1 = 0\}$. The boundary $\partial\Delta^k$ consists of two parts which are both graphs over the lines $\{x_1 = 0, x_3 = h_i\}$ with small $C^{2,\alpha}$ norms. The point $\bar{x}_k$ corresponds to the point of maximal curvature and lies always in the slab.

3.3.3 Properties of Γ

Lemma 3.12. *(a) Γ is a stable minimal surface;*

(b) Γ is contained in the region $\{x_3 \leq 0\} \cap \{x_1 \geq 0\}$.

Proof. (a) is a direct consequence of the smooth convergence and the stability and minimality of the surfaces Γ^k. So, we only need to prove (b). Note that Σ^k is contained in the cylinder $\{x_1^2 + x_2^2 \leq 1\}$. Therefore $\Sigma^k - (-1,0,0) \subset \{x_1 \geq 0\}$ and consequently also $\Gamma^k = a_k(\Sigma^k - (-1,0,0)) \subset \{x_1 \geq 0\}$. And so $\Gamma \subset \{x_1 \geq 0\}$.

Consider next the curve $\sigma_1^k - (-1,0,0)$. For $k \to \infty$, this curve converges in C^2 to the curve $\{(x_1-1)^2 + x_2^2 = 1\}$. Thus we can find a sequence of numbers $\alpha_k > 0$ converging to 0 such that $\sigma_1^k - (-1,0,0) \subset \{x_3 \leq \alpha_k x_1\}$. Obviously, since σ_2^k lies below σ_1^k on the cylinder $\partial D_1 \times \mathbb{R}$, we conclude that

$$\partial\Gamma^k = a_k\big((\sigma_1^k - (-1,0,0)) \cup (\sigma_2^k - (-1,0,0))\big) \subset \{x_3 \leq \alpha_k x_1\}\,.$$

By the maximum principle $\Gamma^k \subset \{x_3 \leq \alpha_k x_1\}$. Letting $k \to \infty$, we conclude $\Gamma \subset \{x_3 \leq 0\}$. □

Lemma 3.13. *There is a point $p \in \Gamma$ such that*

$$1 = |A|(p) = \sup_{x\in\Gamma} |A|(x)\,.$$

Proof. The inequality $\sup_{x\in\Gamma} |A|(x) \leq 1$ follows from $\max_{x\in\Gamma_k} |A|(x) = 1$ and the $C^{2,\beta}$ convergence of the surfaces Γ^k. Recall the scaling of the second fundamental form

$$|A_{\Gamma^k}|(a_k(x_k - (-1,0,0))) = \frac{|A_{\Sigma^k}|(x_k)}{a_k} = 1\,.$$

On the other hand, set $p_k = a_k(x_k - (-1,0,0))$. Then

$$|p_k| = a_k|x_k - y_k| = a_k\,\mathrm{d}(x_k, \partial\Sigma^k) \leq C\,,$$

where the last inequality follows from (3.4). So, up to subsequences, we can assume $p_k \to p$ for some $p \in \Gamma$ and by the $C^{2,\beta}$ convergence

$$|A_\Gamma|(p) = \lim_{k\to\infty} |A_{\Gamma^k}|(p_k) = 1\,.$$

□

Lemma 3.14. Γ *has quadratic area growth, i.e. there is a constant $C > 0$ such that*

$$\mathrm{Area}(\Gamma \cap \mathcal{B}_r(p)) \leq Cr^2, \qquad \forall\, p \in \Gamma,\ \forall\, r \geq 0\,.$$

Proof. By the varifold convergence of Σ^k, for every $\varepsilon > 0$ there is k large enough such that

$$\mathrm{Area}(\Sigma^k) \leq 4\pi + \varepsilon\,. \tag{3.8}$$

Using (3.8), the regularity of the boundary $\partial\Sigma^k$ and the monotonicity formula for minimal surfaces with boundary we conclude

$$\mathrm{Area}(\Sigma^k \cap \mathcal{B}_r(p)) \le Cr^2, \qquad \forall\, p \in \Sigma^k,\ \forall\, r \ge 0\,. \tag{3.9}$$

Here $C > 0$ is a constant that does not depend on k. Since Γ^k is obtained from Σ^k by rotations, translations and scaling, this area bound continues to hold and we have

$$\mathrm{Area}(\Gamma^k \cap \mathcal{B}_r(p)) \le Cr^2, \qquad \forall\, p \in \Gamma^k,\ \forall\, r \ge 0\,. \tag{3.10}$$

Now fix a point $p \in \Gamma$. Then there is a sequence $p_k \in \Gamma^k$ such that $p_k \to p$. By the $C^{2,\beta}$ convergence of Γ^k, for all $r \ge 0$, we conclude

$$\mathrm{Area}(\Gamma \cap \mathcal{B}_r(p)) = \lim_{k\to\infty} \mathrm{Area}(\Gamma^k \cap \mathcal{B}_r(p_k)) \le Cr^2\,.$$

□

Lemma 3.15. *$\partial\Gamma$ is*

(a) either $\{x_3 = x_1 = 0\}$;

(b) or $\{x_1 = x_3 = 0\} \cup \{x_1 = 0, x_3 = h\}$ for some negative h.

In the latter case we have

$$\Gamma \subset \{x_3 \ge h\}\,. \tag{3.11}$$

Proof. Consider the point $z_k := (-1, 0, h'_k) \in \sigma_2^k$. Note that there is a unique such point due to the graphicality assumption about the curve σ_2^k. Let $\zeta_k := a_k(z_k - y_k) =: (0, 0, h_k)$. Clearly $h_k < 0$. We distinguish three cases, covering (up to subsequences) all possibilities.

(i) $h_k \to -\infty$;

(ii) $h_k \to h < 0$;

(iii) $h_k \to 0$.

(*i*) and (*ii*) will lead to (*a*) and (*b*) of the claim, respectively, whereas (*iii*) will be excluded.

Case (i). Observe that

$$\partial\Gamma^k = a_k\big((\sigma_1^k - (-1,0,0)) \cup (\sigma_2^k - (-1,0,0))\big) =: \gamma_1^k \cup \gamma_2^k\,.$$

For each i, k we let $D^k := \{(x_1 - a_k)^2 + x_2^2 \leq a_k^2, x_3 = 0\}$ and $g_i^k : \partial D^k \to \mathbb{R}$ be the functions such that $\gamma_i^k = \mathrm{Graph}(g_i^k) \subset \partial D^k \times \mathbb{R}$. If we regard g_i^k as functions on $\mathbb{R}$ with period $2\pi a_k$ and f_i^k as functions on $\mathbb{R}$ with period 2π, we have

$$\|(g_i^k)'\|_{C^0} = \|(f_i^k)'\|_{C^0} \to 0 \qquad k \to \infty\,. \tag{3.12}$$

Taking into account that $g_1^k(0) = 0$ and $g_2^k(0) = h_k \to -\infty$ we can conclude that

$$\partial\Gamma^k \to \{x_1 = x_3 = 0\} \tag{3.13}$$

in the Hausdorff sense in any compact subset of $\mathbb{R}^3$.

We want to show that $\partial\Gamma = \{x_1 = x_3 = 0\}$. Therefore, let $q \in \Gamma \setminus \{x_1 = x_3 = 0\}$. Again, there is a sequence $q_k \in \Gamma^k$ with $q_k \to q$. By (3.13) there is a constant $c > 0$ such that $\mathcal{B}_c(q_k) \cap \partial\Gamma^k = \emptyset$ for k large enough. By the $C^{2,\beta}$ convergence of Γ^k to Γ we conclude that $\mathcal{B}_c(q) \cap \partial\Gamma = \emptyset$. This implies $\partial\Gamma \subset \{x_1 = x_3 = 0\}$. On the other hand, $\{x_1 = x_3 = 0\} \subset \Gamma$ and, by (b) in Lemma 3.12 and the maximum principle, a point $q \in \{x_1 = x_3 = 0\}$ cannot be an interior point of Γ.

Cases (ii)-(iii). As in the proof of Lemma 3.12 (b) we find a vanishing sequence of negative numbers α_k such that $\Gamma^k \subset \{x_3 \geq h_k + \alpha_k x_1\}$. Let $h := \lim_{k\to\infty} h_k$. Then we conclude $\Gamma \subset \{x_3 \geq h\}$. This proves the last claim in situation (b) and rules out (iii). Indeed, if (iii) holds, then Γ is contained in $\{x_3 = 0, x_1 \geq 0\}$. On the other hand, arguing as in case (i) we obtain $\partial\Gamma \subset \{x_1 = x_3 = 0\}$. Thus, Γ must be the half-plane $\{x_3 = 0, x_1 \geq 0\}$ (Γ is not empty by assumption). But then $|A_\Gamma| \equiv 0$, contradicting Lemma 3.13.

It remains the case (ii). If $h < 0$, we can argue as in case (i) to conclude that $\partial\Gamma^k$ converges to $\{x_1 = x_3 = 0\} \cup \{x_1 = 0, x_3 = h\}$. In the same way we can also deduce $\partial\Gamma = \{x_1 = x_3 = 0\} \cup \{x_1 = 0, x_3 = h\}$. □

3.3.4 Classification of blowups

With this information on the boundary of Γ we can get a complete picture of all possible blowups. In order to do so, let $\tilde{\Gamma}$ be a connected component of Γ. First of all we point out that, by the unique continuation for smooth minimal surfaces (see Theorem 8.3), the multiplicity of Γ is constant on $\tilde{\Gamma}$. Moreover, the multiplicity at the boundary components cannot be less than on the corresponding interiors. Therefore we can conclude that, if $\tilde{\Gamma}$ has multiplicity 2 and $\partial\tilde{\Gamma} \neq \emptyset$, then it can only be bounded by a single line with multiplicity 2 since the total multiplicity of the boundary curves cannot exceed 2. In this case we can disregard the multiplicity and assume without loss of generality that $\tilde{\Gamma}$ has multiplicity 1 if $\partial\tilde{\Gamma} \neq \emptyset$.

Next we can show that $\partial\tilde{\Gamma} \neq \emptyset$. We argue by contradiction and assume that this is not the case. Then $\tilde{\Gamma}$ is a complete stable minimal surface in $\mathbb{R}^3$ without boundary and with quadratic area growth. By Proposition 1.34 and Corollary 1.36 in [19], $\tilde{\Gamma}$ must be a plane. Since $\tilde{\Gamma} \subset \{x_1 \geq 0, x_3 \leq 0\}$ by Lemma 3.12 (b), this is a contradiction.

Based on Lemma 3.15 and the above remarks we can summarize the possible blowups:

(a) Γ is connected, has multiplicity 1 and is bounded by a single line with multiplicity 1;

(b1) Γ has two connected components with multiplicity 1 each of them bounded by a single line with multiplicity 1;

(b2) Γ is connected, has multiplicity 1 and is bounded by two lines with multiplicity 1.

Since $(b1)$ reduces to (a), we are left with two cases.

Remark 3.16. *The strategy of the proof will be to show that both these situations lead to contradictions. The argument for both cases is very similar and consists in making reflections along certain boundary lines to produce immersed complete minimal surfaces. The contradiction will be achieved by finding that the image of the Gauss map leaves out a big enough set to conclude that the surface needs to be a plane – which cannot be true due to the fact that $|A|$ is not zero everywhere. The key observation will be that the normal along the boundary lines can only take certain directions using Lemma 3.12 (b) and* (3.11)*. For (a) we will give two proofs – with and without using Lemma 3.12 (b).*

With the previous remark, we can note, summarizing Lemmas 3.12, 3.13, 3.14 and 3.15, that the assumption that the statement of Theorem 3.11 is false leads to one of the following two possibilities.

(A) This corresponds to the case (a).

There is an orientable analytic surface Γ in $\mathbb{R}^3$ such that

(A1) Γ is a connected stable minimal surface;

(A2) $\Gamma \subset \{x_1 \geq 0\} \cap \{x_3 \leq 0\}$ and $\partial\Gamma = \{x_1 = x_3 = 0\}$;

(A3) Γ has quadratic area growth,

$$\text{Area}(\Gamma \cap \mathcal{B}_r(p)) \leq Cr^2, \qquad \forall\, p \in \Gamma,\ \forall\, r \geq 0\,;$$

(A4) there is $q \in \Gamma$ such that $|A|(q) = \max_{x\in\Gamma} |A|$;

(A5) $|A|(q) > 0$.

(B) This corresponds to case $(b2)$ (after a translation and rescaling).

There is an orientable analytic surface Γ in $\mathbb{R}^3$ such that $(A1)$, $(A3)$, $(A4)$ and $(A5)$ hold and $(A2)$ is replaced by

(B2) $\Gamma \subset \{x_1 \geq 0,\ -1 \leq x_3 \leq 1\}$ and $\partial\Gamma = \{x_1 = 0,\ x_3 \in \{-1, 1\}\}$.

The proof of Theorem 3.11 is completed if we can exclude the existence of surfaces as in (A) and (B). We have the following two propositions.

Proposition 3.17. *A surface satisfying* $(A1)-(A4)$ *must be a half-plane. In particular, there is no surface as in* (A) *since* $(A5)$ *cannot hold.*

Proof. This is a direct consequence of a theorem of Perez (see Theorem 1.1 in [55]) that says that the only properly embedded orientable stable minimal surfaces bounded by a straight line and having quadratic area growth are the half-plane and half of Enneper's surface. Note that orientability in our case follows from the orientability of the Γ^k and the smooth convergence. Since in our case the surface also has to lie in a half-space, it must be the half-plane. □

As mentioned before, we will give a second proof of this result using Lemma 3.12 (b) (note that the proof above only used that Γ is contained in a half-space). This will not require much extra work and clarifies the proof of

Proposition 3.18. *There is no surface as in* (B).

3.3.5 Proofs of Propositions 3.17 and 3.18

Preliminary results

Proposition 3.19. *Let* Γ *be as in* (*A*) *or* (*B*) *and let* $v := (0,1,0)$. *If* N *is the Gauss map, then*

$$\begin{cases} \text{either} & N\cdot v \le 0 \quad \text{everywhere}\,, \\ \text{or} & N\cdot v \ge 0 \quad \text{everywhere}\,. \end{cases}$$

Proof. First of all note that it is well-known that for any constant vector w the function $\varphi = N\cdot w$ on a minimal surface Σ satisfies the Jacobi equation $\Delta_\Sigma\varphi + |A|^2\varphi = 0$. In order to see this, we introduce the following notation: $\Delta = \Delta_\Sigma$, D is the Euclidean connection on $\mathbb{R}^3$, ∇ is the Levi-Civita connection on Σ. Let E_1, E_2 be an orthonormal frame in normal coordinates. Then with the Einstein summation convention

$$\begin{aligned} \Delta\varphi &= E_i(E_i(N\cdot w)) - D_{\nabla_{E_i}E_i}(N\cdot w) \\ &= E_i(D_{E_i}N\cdot w) \\ &= -E_i(a(E_i, w^\top)) + E_i(D_{E_i}N\cdot w^N)\,, \end{aligned}$$

where we have used that $\nabla_{E_i}E_i = 0$ in normal coordinates. Moreover, since $D_{E_i}N$ is tangential, the second term vanishes. The first one equals

$$-(\nabla_{E_i}a)(E_i, w^\top) - a(\nabla_{E_i}E_i, w^\top) - a(E_i, \nabla_{E_i}w^\top)\,.$$

The first term vanishes by minimality and the Codazzi equations, whereas the second one again because of the normal coordinates. The last one we can write as

$$\begin{aligned} -a(E_i, \nabla_{E_i}(E_j\cdot w)E_j) &= -a(E_i, (D_{E_i}E_j\cdot w)E_j) \\ &\quad -a(E_i, E_j\cdot w\nabla_{E_i}E_j) \\ &= -a(E_i,E_j)\big((\nabla_{E_i}E_j)\cdot w^\top + (D_{E_i}E_j)\cdot N\varphi\big) \\ &= -|A|^2\varphi\,, \end{aligned}$$

where we used the normal coordinates several times. Note that, for $w = v$, we have moreover $\varphi|_{\partial\Gamma} = 0$ since v is tangential to $\partial\Gamma$.

If Γ were compact, the claim would follow from the stability and the fact that the eigenvectors for the smallest eigenvalue cannot change sign (see Section 1.8 in [19]). However, this argument cannot be applied to our situation. Using logarithmic cut-off functions and the quadratic area growth the result can still be achieved. We only sketch here the argument, a detailed computation can be found in the proof of Theorem 4.1 in [75] whose argument we follow (see also Proposition 5.1 in [55]).

Assume φ is not constantly equal to zero. Moreover, assume that φ changes sign in $\Gamma \cap \mathcal{B}$, where $\mathcal{B}$ is some ball (without loss of generality assume that the radius is 1). Denote by φ^+ the positive part of φ. For $R > 1$ define the logarithmic cut-off function

$$\phi(x) = \begin{cases} 1 & \text{if} \quad |x| \leq 1\,, \\ 1 - \frac{\log|x|}{\log R} & \text{if} \quad 1 \leq |x| \leq R\,, \\ 0 & \text{if} \quad |x| \geq R\,. \end{cases}$$

Then one computes with $Q(\eta) = \int_\Gamma |\nabla \eta|^2 - |A|^2\eta^2$

$$Q(\phi\varphi^+) = \int_\Gamma (\varphi^+)^2 |\nabla \phi|^2 \leq \int_{\Gamma \cap \{\varphi^+ \neq 0\}} |\nabla \phi|^2 \leq \frac{C}{\log^2 R} + \frac{C}{\log R}\,,$$

where the last inequality is justified in the reference given above. A key information used there is the quadratic area growth, Lemma 3.14. Note that $\phi\varphi^+$ is compactly supported on Γ since $\varphi = 0$ on $\partial\Gamma$.

Let now φ' minimize Q among all Lipschitz functions that coincide with $\phi\varphi^+$ outside $\mathcal{B}$. Since, by assumption, φ changes sign in $\mathcal{B}$, that is, φ^+ vanishes on an open ball contained in $\mathcal{B}$, $\phi\varphi^+ \neq \varphi'$. Thus

$$Q(\phi\varphi^+) - Q(\varphi') = \varepsilon > 0$$

and

$$Q(\varphi') = Q(\phi\varphi^+) - \varepsilon \leq \frac{C}{\log^2 R} + \frac{C}{\log R} - \varepsilon\,.$$

Since φ' and therefore also ε is independent of R, we can choose R so large that $Q(\varphi') < 0$, contradicting the stability of Γ (see Lemma 3.2). □

Proposition 3.20. *Let Γ be as in (A) or (B) and $v = (0, 1, 0)$. Let N be the Gauss map, then*

$$\begin{cases} \textit{either} & N \cdot v < 0 \quad \textit{on}\, \Gamma \setminus \partial\Gamma\,, \\ \quad \textit{or} & N \cdot v > 0 \quad \textit{on}\, \Gamma \setminus \partial\Gamma\,. \end{cases}$$

Proof. As above set $\varphi := N\cdot v$. Then from Proposition 3.19 (after possibly changing the orientation) we can assume $\varphi \geq 0$ and $\Delta\varphi + |A|^2\varphi = 0$. Thus, $\Delta\varphi \leq 0$. Assume now that $p \in \Gamma \setminus \partial\Gamma$ and $\varphi(p) = 0$. Then clearly p is a minimum and, by the strong maximum principle, φ vanishes in a neighborhood of p. On the other hand, iterating this argument we can conclude that

$$\begin{cases} \text{either} & \varphi > 0 \quad \text{on } \Gamma \setminus \partial\Gamma\,, \\ \text{or} & \varphi \equiv 0 \quad \text{on a connected component of } \Gamma \setminus \partial\Gamma\,. \end{cases}$$

We want to exclude the second alternative. Since $\Gamma \setminus \partial\Gamma$ is connected, the second alternative implies $\varphi \equiv 0$ on Γ. Consider the circle $\gamma := \{\nu \in S^2 : \nu \cdot v = 0\} \subset S^2$. Thus, $\varphi \equiv 0$ implies $N(\Gamma) \subset \gamma$. Let $p \in \Gamma \setminus \partial\Gamma$ and denote by k_1 and k_2 the principal curvatures of Γ at p. Then, by minimality, $k_1 + k_2 = H = 0$ and hence $|A|^2 = 2k_1^2 = -2K_G$, where K_G is the Gauss curvature of Γ at p. Assume now that $K_G(p) < 0$, then the Gauss map $N : U \to S^2$ is injective for some neighborhood U of p by the inverse function theorem (recall that K_G is the Jacobian of the Gauss map). Using the area formula this gives $|N(U)| = \int_U |K_G| > 0$ which is a contradiction to $N(\Gamma) \subset \gamma$. Summarizing, this means that $\varphi \equiv 0$ implies $|A| \equiv 0$. This however, contradicts ($A5$). Thus, $\varphi > 0$ on $\Gamma \setminus \partial\Gamma$ which proves the claim. □

In the proofs of Proposition 3.17 and Proposition 3.18 we will need the notion of logarithmic capacity. We follow [41] (see p. 280) and give the following

Definition 3.21. *Let $E \subset \mathbb{C}$. Let $\mathcal{P}(E) :=$ {probability measures on E} and define*

$$\begin{aligned} I[\mu] &:= \lim_{n\to\infty} \int_E \int_E \min\left\{\log \frac{1}{|z_1 - z_2|}, n\right\} d\mu(z_1)\, d\mu(z_2)\,; \\ V(E) &:= \inf_{\mu \in \mathcal{P}(E)} I[\mu]\,; \\ c(E) &:= e^{-V(E)}\,. \end{aligned}$$

$c(E)$ is called logarithmic capacity*. Moreover, we define the logarithmic capacity of a subset of S^2 to be the logarithmic capacity of its image under stereographic projection.*

Note that $c(E) > 0$ if and only if there is $\mu \in \mathcal{P}(E)$ with $I[\mu] < \infty$.

Lemma 3.22. *Let E be a bounded interval. Then $c(E) > 0$.*

Proof. We have to show that there is $\mu \in \mathcal{P}(E)$ with $I[\mu] < \infty$. We can assume without loss of generality that $E = (0,1) \subset \mathbb{R}$. Let μ be the uniform measure on E. We can compute

$$\begin{aligned} I[\mu] &= \int_0^1 \int_0^1 -\log|s-t|\,ds\,dt \\ &= -2\int_0^1 \int_t^1 \log(s-t)\,ds\,dt \\ &= -2\int_0^1 [(s-t)\log(s-t) - (s-t)]\,|_t^1\,dt \\ &= -2\int_0^1 t\log t - t\,dt \\ &= \left(-t^2\log t + \frac{3}{2}t^2\right)\Big|_0^1 = \frac{3}{2} < \infty\,. \end{aligned}$$

□

PROOF OF PROPOSITION 3.17

Consider the normal N on the line $\partial\Gamma = \{x_1 = x_3 = 0\}$. By Lemma 3.12 (b) the tangent plane $\pi(p)$ to Γ at any $p \in \partial\Gamma$ is of the form $\{\cos\theta x_1 + \sin\theta x_3 = 0\}$ for some $\theta \in [0, \pi/2]$. So, $N(p)$ is of the form $(\cos\theta, 0, \sin\theta)$ for some $\theta \in [0, \pi/2]$ or for some $\theta \in [\pi, 3\pi/2]$. Since $\partial\Gamma$ is connected and N is continuous, we are either for all $p \in \partial\Gamma$ in the first situation or always in the second situation. Thus, after possibly changing the orientation of Γ, we can assume that we are in the first case. Thus, by Proposition 3.20,

$$\begin{aligned} N(\Gamma) &= N(\partial\Gamma) \cup N(\Gamma \setminus \partial\Gamma) \\ &\subset \{(\cos\theta, 0\sin\theta) : \theta \in [0, \pi/2]\} \cup \{(\nu_1, \nu_2, \nu_3) \in S^2 : \nu_2 > 0\}\,. \end{aligned}$$

Next, we construct from Γ a complete surface in $\mathbb{R}^3$ by a Schwarz reflection. More precisely, we define the map

$$S : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3)$$

and set $\Sigma := \Gamma \cup S(\Gamma)$. Observe that,

$$N(p) = (\nu_1, \nu_2, \nu_3) \quad \Rightarrow \quad N(S(p)) = (\nu_1, -\nu_2, \nu_3)\,.$$

Therefore $N(\Sigma) \subset S^2 \setminus \alpha$, where α is the arc

$$\{(\cos\theta, 0, \sin\theta) : \theta \in (\pi/2, 2\pi)\}\,.$$

On the other hand, Σ is a complete minimal surface in $\mathbb{R}^3$. By the results of [53] (see (3) in the introduction and Section 2, see also Theorem 8.2 in [54]) we have

$$\begin{cases} \text{either} & S^2 \setminus N(\Sigma) \text{ has zero logarithmic capacity}\,, \\ \text{or} & \Sigma \text{ is a plane}\,. \end{cases}$$

The first alternative implies that the logarithmic capacity of α is 0. But note that the image of α under stereographic projection is a bounded interval. Hence, by Lemma 3.22 we get a contradiction. Therefore we see that the only surface with $(A1) - (A4)$ is the half-plane. Clearly, $(A5)$ in that case does not hold, which gives the second part of the claim.

Proof of Proposition 3.18

We argue in a similar way and start by considering $p \in \partial\Gamma$. Let $\theta(p) \in [0, 2\pi]$ be such that $N(p) = (\cos\theta(p), 0, \sin\theta(p))$. There are the following possibilities:

$$\begin{aligned} \theta(\partial\Gamma) &\subset [0, \pi/2] \cup [0, \pi]\,; \\ \theta(\partial\Gamma) &\subset [0, \pi/2] \cup [\pi, 2\pi]\,; \\ \theta(\partial\Gamma) &\subset [\pi, 3\pi/2] \cup [0, \pi]\,; \\ \theta(\partial\Gamma) &\subset [\pi, 3\pi/2] \cup [\pi, 2\pi]\,. \end{aligned}$$

As in the previous proof we construct from Γ a complete minimal surface Σ using inifitely many Schwarz reflections (see Figure 3.3).

Arguing as before, $S^2 \setminus N(\Sigma)$ contains at least an open interval of length $\pi/2$ which is a quarter of the equator $\{(\cos\theta, 0, \sin\theta) : \theta \in [0, 2\pi)\} \subset S^2$. Again by the results in [53] and Lemma 3.22 we conclude that Σ is a plane, contradicting $(A5)$.

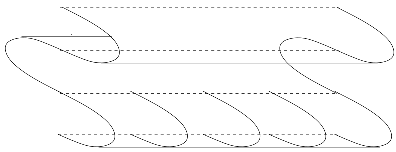

Figure 3.3: Γ is the part bounded by the two dashed lines on the bottom. Two Schwarz reflections have been made along the second and third dashed line.

3.3.6 About Theorem 3.10 with more than two boundary curves

The fact that originally we had only two boundary curves was used in the proof in the following way: we needed that, if we blow up by curvature, the blowup has a component that is either bounded by a single line (Case (a)) and lies in a half-space or the normals along all the boundary components have to leave out a non-trivial interval.

As soon as there are more than two boundary lines, in general, the above arguments do not apply anymore. We discuss here what happens for three boundary lines.

Blowing up by curvature in the case of three lines produces a wider variety of possible blowups. Two boundary lines could collapse or (if we rescale by the distance to the middle curve) there could be one single boundary line and the two outer curves diverge to $\pm\infty$ so that we only know that Γ is contained in a half-space, but no further information as in Lemma 3.12 (b) is known. In fact, the first situation does not cause any additional difficulty (at least in the case of three curves) and the latter is covered by our first proof of Proposition 3.17. In the situation of four (or more) curves, where the two middle ones stay at comparable distance and the two outer ones diverge, our argument does not work any more.

In the case of three curves there is, however, a new case that we could not exclude: if the blowup has three boundary lines (with multiplicity one). We collect here some of the properties of this blowup (without proofs). Denote by $l_i = \{(0, t, h_i)\}$ the three boundary lines.

(1) Generalizing the proof of Proposition 3.3 in [55], one can show that Γ is a graph over $\{x_1 > 0, x_3 \in (h_1, h_3)\}$.

(2) Defining Γ_R as the translation by R in x_2-direction, one finds that by the curvature estimates $\Gamma_R \to \Gamma_\infty$ in $C^{2,\beta}$ (similarly, one defines $\Gamma_{-\infty}$). Morever, one can show that the asymptotic behaviour essentially can be reduced to the situation where Γ_∞ is the half-plane $\{x_1 \geq 0, x_3 = l_1\}$ and the strip in $\{x_1 = 0\}$ between l_2 and l_3, and $\Gamma_{-\infty}$ is the half-plane at l_3 and the strip between l_1 and l_2 (see Figure 3.4).

(3) Using the fact that Γ is a disk and the quadratic area growth, one can show that it has finite total curvature (note that the boundary components are geodesics). With the Gauss-Bonnet theorem one can even show that the total curvature is 2π. As a consequence (degree theory) one can see that the normal covers a hemisphere injectively.

A possible model for such a surface – outside of a large ball – could be the helicoid. We do not expect such a surface to exist, but we were not able to exclude it. In fact, already the situation of Proposition 3.18 could be seen as the two-line analogon. One faces the same kind of difficulties if one wishes to prove the proposition without using the reflection argument.

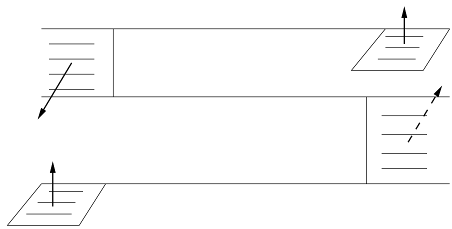

Figure 3.4: On the left there are the components of $\Gamma_{-\infty}$ and on the right the components of Γ_∞. The arrows indicate the normals coming from a choice of the orientation.

4 Proof of Theorem 2

In this chapter we discuss the various steps in the proof of Theorem 2. As already remarked, our proof follows in many aspects the corresponding proof of the case $n = 2$ in [18] (see also Theorem 9.3). Therefore we indicate where our proof differs from that one. In Section 2.1 we discussed the differences in the setup. In this chapter it will be apparent why in higher dimensions the narrower concept of Definition 2.6 is not sufficient.

4.1 Isotopies and stationarity

It is easy to see that not all min-max sequences converge to stationary varifolds (see Figure 4.1).

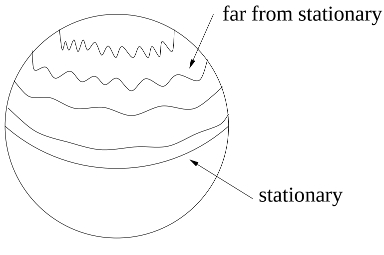

Figure 4.1: All the curves have the same length. Picking the wrong slice of the sweepout, one could construct a min-max sequence that does not come close to a stationary varifold.

In general, for any minimizing sequence $\{\{\Gamma_t\}^k\}$ there is at least one min-max sequence converging to a stationary varifold. For technical reasons, it is useful to consider minimizing sequences $\{\{\Gamma_t\}^k\}$ with the additional property that *any* corresponding min-max sequence converges to a stationary varifold. The existence of such a sequence, which roughly speaking follows from "pulling tight" the surfaces of a minimizing sequence, is an important conceptual step and goes back to Birkhoff in the case of geodesics and to the fundamental work of Pitts in the general case (see also [20] and [21] for other applications of these ideas). In order to state it, we need some terminology.

Definition 4.1. *Given a smooth map* $F : [0,1] \to \mathcal{X}(M)$, *for any* $t \in [0,1]$ *we let* $\Psi_t : [0,1] \times M \to M$ *be the 1-parameter family of diffeomorphisms generated by the vectorfield* $F(t)$. *If* $\{\Gamma_t\}_{t\in[0,1]}$ *is a sweepout, then* $\{\Psi_t(s,\Gamma_t)\}_{(t,s)\in[0,1]^2}$ *is a homotopy between* $\{\Gamma_t\}$ *and* $\{\Psi_t(1,\Gamma_t)\}$. *These will be called* homotopies induced by ambient isotopies.

We recall that the weak* topology on the space $\mathcal{V}$ (varifolds with mass bounded by $4m_0$) is metrizable and we choose a metric $\mathcal{D}$ which induces it. Moreover, let $\mathcal{V}_s \subset \mathcal{V}$ be the (closed) subset of stationary varifolds.

Proposition 4.2. *Let* Λ *be a family of sweepouts which is closed under homotopies induced by ambient isotopies. Then there exists a minimizing sequence* $\{\{\Gamma_t\}^k\} \subset \Lambda$ *such that, if* $\{\Gamma^k_{t_k}\}$ *is a min-max sequence, then* $\mathcal{D}(\Gamma^k_{t_k}, \mathcal{V}_s) \to 0$.

This proposition is Proposition 4.1 of [18]. Though stated for the case $n = 2$, this assumption, in fact, is never used in the proof given in that paper. For the sake of completeness we will include the proof in Chapter 5.

4.2 Almost minimizing varifolds

It is well-known that a stationary varifold can be far from regular. To overcome this issue, we introduce the notion of almost minimizing varifolds.

Definition 4.3. *Let* $\varepsilon > 0$ *and* $U \subset M$ *open. A boundary* $\partial\Omega$ *in* M *is called* ε*-almost minimizing (*ε*-a.m.) in* U *if there is NO* 1*-parameter family of boundaries* $\{\partial\Omega_t\}$, $t \in [0,1]$ *satisfying the following properties:*

$$\text{(s1), (s2), (s3), (sw1) and (sw3) of Definition 2.1 hold;} \tag{4.1}$$

$$\Omega_0 = \Omega \text{ and } \Omega_t \setminus U = \Omega \setminus U \text{ for every } t; \tag{4.2}$$

$$\mathcal{H}^n(\partial\Omega_t) \leq \mathcal{H}^n(\partial\Omega) + \tfrac{\varepsilon}{8} \text{ for all } t \in [0,1]; \tag{4.3}$$

$$\mathcal{H}^n(\partial\Omega_1) \leq \mathcal{H}^n(\partial\Omega) - \varepsilon. \tag{4.4}$$

A sequence $\{\partial\Omega^k\}$ *of hypersurfaces is called* almost minimizing in U *if each* $\partial\Omega^k$ *is* ε_k*-a.m. in* U *for some sequence* $\varepsilon_k \to 0$.

Roughly speaking, $\partial\Omega$ is a.m. if any deformation which eventually brings down its area is forced to pass through some surface which has substantially larger area (see Figure 4.2). A similar notion was introduced for the first time in the pioneering work of Pitts and a corresponding one is given in [72] using isotopies (see Section 3.2 of [18]). Following in part Section 5 of [18] (which uses a combinatorial argument inspired by a general one of [3] reported in [56]), we prove in Chapter 6 the following existence result.

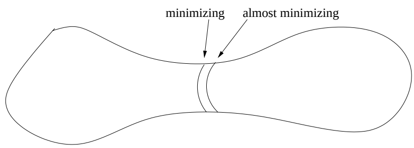

Figure 4.2: The length of the almost minimizing curve can only be decreased by a certain amount if the curve is slid over one of the “spheres”. In this process the length first has to be increased substantially.

Proposition 4.4. *Let* Λ *be a homotopically closed family of sweepouts. There are a function* $r : M \to \mathbb{R}^+$ *and a min-max sequence* $\Gamma^k = \Gamma^k_{t_k}$ *such that*

(a) $\{\Gamma^k\}$ *is a.m. in every* $An \in \mathcal{AN}_{r(x)}(x)$ *with* $x \in M$*;*

(b) Γ^k *converges to a stationary varifold* V *as* $k \to \infty$.

In this part we introduce, however, a new ingredient. The proof of Proposition 4.4 has a variational nature: assuming the nonexistence of such a min-max sequence we want to show that on an appropriate minimizing sequence $\{\{\Gamma_t\}^k\}$, the energy $\mathcal{F}(\{\Gamma_t\}^k)$ can be lowered by a fixed amount, contradicting its minimality. Note, however, that we have 1-parameter families of surfaces, whereas the variational notion of Definition 4.3 focuses on a single surface. Pitts (who in turn has a stronger notion of almost minimality) avoids this difficulty by considering discretized families and this, in our opinion, makes his proof quite hard. Instead, our notion of almost minimality allows us to stay in the smooth category: the key technical point is the "freezing" presented in Section 6.2 (see Lemma 6.1). In [18] this issue is much simpler since the almost minimality is defined in terms of isotopies. Therefore this "freezing" can be achieved simply by composing the isotopy with a cut-off function in the parameter t.

4.3 Replacements

We complete the program in Chapters 7 and 8 showing that our notion of almost minimality is still sufficient to prove regularity. As a starting point, as in the theory of Pitts, we consider *replacements*.

Definition 4.5. *Let $V \in \mathcal{V}(M)$ be a stationary varifold and $U \subset M$ be an open set. A stationary varifold $V' \in \mathcal{V}(M)$ is called a replacement for V in U if $V' = V$ on $M \setminus \bar{U}$, $\|V'\|(M) = \|V\|(M)$ and $V \llcorner U$ is a stable minimal hypersurface Γ.*

We show in Chapter 7 that almost minimizing varifolds do posses replacements.

Proposition 4.6. *Let $\{\Gamma^j\}$, V and r be as in Proposition 4.4. Fix $x \in M$ and consider an annulus $An \in \mathcal{AN}_{r(x)}(x)$. Then there are a varifold $\tilde{V}$, a sequence $\{\tilde{\Gamma}^j\}$ and a function $r' : M \to \mathbb{R}^+$ such that*

(a) $\tilde{V}$ is a replacement for V in An and $\tilde{\Gamma}^j$ converges to $\tilde{V}$ in the sense of varifolds;

(b) $\tilde{\Gamma}^j$ is a.m. in every $An' \in \mathcal{AN}_{r'(y)}(y)$ with $y \in M$;

(c) $r'(x) = r(x)$.

The strategy of the proof is the following. Fix an annulus An. We would like to substitute $\Gamma^j = \partial\Omega^j$ in An with the surface minimizing the area among all those which can be continuously deformed into Γ^j according to our homotopy class: we could appropriately call it a solution of the $(8j)^{-1}$ *homotopic Plateau problem.* As a matter of fact, we do not know any regularity for this problem. However, if we consider a corresponding minimizing sequence $\{\partial\Omega^{j,k}\}_k$, we will show that it converges, up to subsequences, to a varifold V^j which is regular in An. This regularity is triggered by the following observation: on any sufficiently small ball $B \subset An$, $V^j \llcorner B$ is the boundary of a Caccioppoli set Ω^j which solves the Plateau problem in the class $\mathcal{P}(\Omega^j, B)$ (in the sense of Theorem 2.13).

In fact, by standard blowup methods of geometric measure theory, V^j is close to a cone in any sufficiently small ball $B = B_r(y)$. For k large, the same property holds for $\partial\Omega^{j,k}$. Modifying suitably an idea of [72], this property can be used to show that any (sufficiently regular) competitor $\tilde{\Omega} \in \mathcal{P}(\Omega^{j,k}, B)$ can be homotopized to $\Omega^{j,k}$ without passing through a surface of large energy. In other words, minimizing sequences of the homotopic Plateau problem are in fact minimizing for the usual Plateau problem at sufficiently small scales.

This step is basically the main reason why the proof of [18] does not work in higher dimensions. In [18] it is shown that the competitor can be homotopized to $\Omega^{j,k}$ by *isotopies*. The important (and hard) result that is needed to achieve this is a theorem by Meeks, Simon and Yau [45] that roughly says that the minimizer in the isotopy class of a two-dimensional surface is smooth. No such result is known in higher dimensions.

Having shown the regularity of V^j in An, we use the Schoen-Simon compactness theorem to show that V^j converges to a varifold $\tilde{V}$ which in An is a stable minimal hypersurface. A suitable diagonal sequence $\Gamma^{j,k(j)}$ gives the surfaces $\tilde{\Gamma}^j$.

4.4 Regularity of V

One would like to conclude that, if V' is a replacement for V in an annulus contained in a convex ball, then $V = V'$ (and hence V is regular in An). However, two stationary varifolds might coincide outside of a convex set and be different inside: the standard unique continuation property of classical minimal surfaces fails in the general case of stationary varifolds (see the appendix of [18] for an example). We need more information to conclude the regularity of V. Clearly, applying Proposition 4.6 three times we conclude

Proposition 4.7. *Let V and r be as in Proposition 4.4. Fix $x \in M$ and $An \in \mathcal{AN}_{r(x)}(x)$. Then:*

(a) V has a replacement V' in An such that

(b) V' has a replacement V'' in any

$$An' \in \mathcal{AN}_{r(x)}(x) \cup \bigcup_{y \neq x} \mathcal{AN}_{r'(y)}(y)$$

such that

(c) V'' has a replacement V''' in any $An'' \in \mathcal{AN}_{r''(y)}(y)$ with $y \in M$.

r' and r'' are positive functions (which might depend on V' and V'').

In fact, the process could be iterated infinitely many times. However, it turns out that three iterations are sufficient to prove regularity, as stated in the following proposition. Its proof is given in Chapter 8, where we basically follow [65] (see also [18]).

Proposition 4.8. *Let V be as in Proposition 4.7. Then V is induced by a minimal hypersurface Σ (in the sense of Definition 3.5).*

5 The existence of stationary varifolds

In this chapter we prove Proposition 4.2. We give the proof of [18] (adding a few details) that is valid literally in our situation even though stated for $n = 2$ in that paper. In particular, we do not need the specific setting of Definition 2.1. The same argument works for the narrower concepts discussed in Section 2.1.1. Note that the homotopically closed family of sweepouts Λ of Theorem 2 is larger than the family of sweepouts that is closed under homotopies induced by ambient isotopies in Proposition 4.2.

5.1 Proof of Proposition 4.2

The key idea of the proof is building a continuous map $\Psi : \mathcal{V} \to \mathfrak{Is}$, where $\mathfrak{Is}$ is the set of smooth isotopies, such that :

- if V is stationary, then Ψ_V is the trivial isotopy;
- if V is not stationary, then Ψ_V decreases the mass of V.

Since each Ψ_V is an isotopy, and thus is itself a map from $[0,1] \times M \to M$, to avoid confusion we use the subscript V to denote the dependence on the varifold V. The map Ψ will be used to deform a minimizing sequence $\{\{\Sigma_t\}^k\} \subset \Lambda$ into another minimizing sequence $\{\{\Gamma_t\}^k\}$ such that :

For every $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\text{if} \quad \left\{ \begin{array}{rc} & k > N \\ \text{and} & \mathcal{H}^n(\Gamma^k_{t_k}) > m_0 - \delta \end{array} \right\}, \quad \text{then} \quad \mathcal{D}(\Gamma^k_{t_k}, \mathcal{V}_s) < \varepsilon\,. \tag{5.1}$$

Such a $\{\{\Gamma_t\}^k\}$ would satisfy the requirement of the proposition.

The map Ψ_V should be thought of as a natural "shortening process" of varifolds which are not stationary. If the mass (considered as a functional on the space of varifolds) were smoother, then a gradient flow would provide a natural shortening process like Ψ_V. However, this is not the case; even if we start with smooth initial datum, in very short time the motion by mean curvature, i.e. the gradient flow of the area functional on smooth submanifolds, gives surfaces which are not isotopic to the initial one.

Step 1: A map from $\mathcal{V}$ to the space of vector fields. The isotopies Ψ_V will be generated as 1-parameter families of diffeomorphisms satisfying certain ODEs. In this step we associate to any V a suitable vector field, which in Step 2 will be used to construct Ψ_V.

For $l \in \mathbb{Z}$ we define the annuli

$$\mathcal{V}_l = \{V \in \mathcal{V} : 2^{-l+1} \geq \mathcal{D}(V, \mathcal{V}_s) \geq 2^{-l-1}\}.$$

These $\mathcal{V}_l$ are compact. Therefore there are constants $c(l) > 0$ only depending on l such that for all $V \in \mathcal{V}_l$ there is a smooth vector field χ_V with

$$\|\chi_V\|_\infty \leq 1, \quad \delta V(\chi_V) \leq -c(l).$$

For, if not, then there is a sequence $\{V_j\}$ of varifolds in $\mathcal{V}_l$ such that for all vector fields χ with $\|\chi\|_\infty \leq 1$

$$\delta V_j(\chi) \leq \frac{1}{j}.$$

Therefore $\|\delta V_j\| \to 0$. Compactness and the lower semi-continuity of the first variation then yield that a subsequence converges to a stationary varifold. But this is a contradiction to the definition of $\mathcal{V}_l$. This proves the existence of the constants $c(l)$.

Next, we want to show that the associated vector fields χ_V can be chosen in a continuous dependence on V. For this first note that we have

$$\begin{aligned} \delta W(\chi_V) &\leq \delta V(\chi_V) + \delta(W - V)(\chi_V) \\ &\leq -c(l) + \|\delta(W - V)\|. \end{aligned}$$

Thus, by the lower semicontinuity of the first variation there is $r > 0$ such that

$$\delta W(\chi_V) \leq -\frac{c(l)}{2}, \quad W \in U_r(V),$$

where $U_r(V)$ denotes the ball in $\mathcal{V}$. Using again compactness we can find for any $l \in \mathbb{Z}$ balls $\{U_i^l\}_{i=1}^{N(l)}$ and corresponding vector fields χ_i^l such that

- the balls $\tilde{U}_i^l$, concentric to U_i^l with half the radii, cover $\mathcal{V}_l$;
- for all $W \in U_i^l$ we have $\delta W(\chi_i^l) \leq -\frac{c(l)}{2}$;
- the balls U_i^l are disjoint from $\mathcal{V}_j$ for $|j - l| \geq 2$.

The balls $\{U_i^l\}_{i,l}$ form a locally finite covering of $\mathcal{V} \setminus \mathcal{V}_s$. So we can pick a continuous partition of the unity $\{\varphi_i^l\}$ subordinate to this covering. Then we define the vector fields $H_V := \sum_{i,l} \varphi_i^l(V)\chi_i^l$. The map

$$H : \mathcal{V} \to C^\infty(M, TM), \quad V \mapsto H_V$$

is continuous and $\|H_V\|_\infty \leq 1$ for all $V \in \mathcal{V}$.

Step 2: A map from $\mathcal{V}$ to the space of isotopies. Let $V \in \mathcal{V}_l$. Then, by the above covering, V is contained in at least one ball $\tilde{U}_i^l$. We denote by $r(V)$ the radius of the smallest such ball. As there are only finitely many such balls, we can find $r(l)$ only depending on l such that $r(V) \geq r(l) > 0$. By the properties of the covering, moreover

$$\delta W(H_V) \leq -\frac{1}{2} \min\{c(l-1), c(l), c(l+1)\}$$

for all $W \in U_{r(V)}(V)$. Thus, we have two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\delta W(H_V) \leq -g(\mathcal{D}(V, \mathcal{V}_s)) \quad \text{if} \quad \mathcal{D}(W, V) \leq r(\mathcal{D}(V, \mathcal{V}_s)). \tag{5.2}$$

The function $-g$ for instance can be obtained by dominating the step function depending on the $c(l)$ by a continuous function. By the compactness of M and the smoothness of each H_V we can construct for all V a 1-parameter family of diffeomorphisms

$$\Phi_V : [0, \infty) \times M \to M \quad \text{with} \quad \frac{\partial \Phi_V}{\partial t}(t, x) = H_V(\Phi_V(t, x)).$$

The key is now to prove that these diffeomorphisms decrease the mass of a varifold by an amount depending on its distance to the stationary varifolds. More precisely we claim the following: There are continuous functions $T : \mathbb{R}^+ \to [0, 1]$ and $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- if $\gamma = \mathcal{D}(V, \mathcal{V}_s) > 0$ and V' is obtained from V by the diffeomorphism $\Phi_V(T(\gamma), \cdot)$, then $\|V'\|(M) \leq \|V\|(M) - G(\gamma)$;

- $G(s)$ and $T(s)$ both converge to 0 for $s \to 0$.

For this we fix $V \in \mathcal{V} \setminus \mathcal{V}_s$. For all $r > 0$ there is $T > 0$ such that the curve

$$\{V(t) = (\Phi_V(t, \cdot))_\# V,\ t \in [0, T]\}$$

stays in $U_r(V)$. This implies the inequality

$$\|V(T)\|(M) - \|V\|(M) = \|V(T)\|(M) - \|V(0)\|(M) \le \int_0^T \delta V(t)(H_V)\, dt\,.$$

If we choose $r = r(\mathcal{D}(V, \mathcal{V}_s))$ as in (5.2), this yields

$$\|V(T)\|(M) - \|V\|(M) \le -Tg(\mathcal{D}(V, \mathcal{V}_s))\,,$$

or we can rewrite this as

$$\|V(T)\|(M) - \|V\|(M) \le -G(\mathcal{D}(V, \mathcal{V}_s))\,.$$

Moreover T and G are continuous. Clearly $T(s) \to 0$ as $s \to 0$. The boundedness of g then gives $G(s) \to 0$ as $s \to 0$. Arguing as in the first step, using a continuous partition of the unity, we can find a choice of T that is continous in V and depends only on $\mathcal{D}(V, \mathcal{V}_s)$.

Step 3: Construction of the competitor. Let $V \in \mathcal{V}$ be such that $\mathcal{D}(V, \mathcal{V}_s) = \gamma$. We renormalize the diffeomorphisms Φ_V, namely we set

$$\Psi_V(t, \cdot) = \Phi_V(T(\gamma)t, \cdot), \quad t \in [0, 1]\,.$$

Then by the definition of T the varifolds $(\Psi_V(t, \cdot))_\# V$ stay in $U_\gamma(V)$ for all $t \in [0, 1]$. By the second step of this proof we get a strictly increasing function $L : \mathbb{R} \to \mathbb{R}$ with $L(0) = 0$ and

$$\|V'\|(M) \le \|V\|(M) - L(\gamma),$$

where V' is the varifold that is obtained from V by $\Psi_V(1, \cdot)$. The function G above is not necessarily strictly increasing, but all the choices can be made in such a way that this goal is achieved.

Now choose a sequence of families $\{\{\Sigma_t\}^k\} \subset \Lambda$ such that $\mathcal{F}(\{\Sigma_t\}^k) \le m_0 + \frac{1}{k}$. Then we define a new family by

$$\tilde{\Gamma}^k_t = \Psi_{\Sigma^k_t}(1, \Sigma^k_t), \qquad t \in [0, 1],\, k \in \mathbb{N}\,.$$

Clearly the surfaces $\tilde{\Gamma}^k_t$ are again smooth (apart from at most finitely many points). But, since the dependence of the vector field $\Psi_{\Sigma^k_t}$ in t is merely continuous, the new family is not necessarily (and in general not) smooth. Before addressing this problem we simply point out that we moreover have

$$\mathcal{H}^n(\tilde{\Gamma}^k_t) \leq \mathcal{H}^n(\Sigma^k_t) - L\left(\mathcal{D}(\Sigma^k_t, \mathcal{V}_s)\right) .$$

To get a generalized smooth family with the same property we first simplify our notation. We write Ψ_t for $\Psi_{\Sigma^k_t}$. Then the smooth vector field

$$h_t = T(\mathcal{D}(\Sigma^k_t, \mathcal{V}_s))H_{\Sigma^k_t}$$

generates Ψ_t. Thus, in other terms, if we endow the space of smooth vector fields with the topology of the C^m-seminorms, we have a continuous map

$$h : [0,1] \to \mathcal{X}(M) .$$

We can approximate this continuous map by a smooth map $\tilde{h}$. We denote the smooth 1-parameter family of diffeomorphisms generated by $\tilde{h}_t$ by $\tilde{\Psi}_t$. Then we consider the smooth family

$$\Gamma^n_t = \tilde{\Psi}^n_t(1, \Sigma^n_t) .$$

Whenever we have that $\sup_t \|h_t - \tilde{h}_t\|_{C^1}$ is small enough, the same calculations as before yield

$$\mathcal{H}^n(\Gamma^n_t) \leq \mathcal{H}^n(\Sigma^n_t) - \frac{L(\mathcal{D}(\Sigma^n_t, \mathcal{V}_s))}{2} . \tag{5.3}$$

Since $\{\{\Sigma_t\}^k\}$ is minimizing, so is $\{\{\Gamma_t\}^k\}$. By construction there is an increasing continuous map $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lambda(0) = 0$ and

$$\mathcal{D}(\Sigma^k_t, \mathcal{V}_s) \geq \lambda(\mathcal{D}(\Gamma^k_t, \mathcal{V}_s)) . \tag{5.4}$$

Note that, if $\mathcal{D}(\Sigma^k_t, \mathcal{V}_s) = 0$, then $\Sigma^k_t = \Gamma^k_t$, and so $\mathcal{D}(\Gamma^k_t, \mathcal{V}_s) = 0$.

To conclude the proof we fix $\varepsilon > 0$ and choose $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\frac{L(\lambda(\varepsilon))}{2} - \delta > \frac{1}{N} .$$

Assume now that there are $k > N$ and t such that

$$\mathcal{H}^n(\Gamma_t^k) > m_0 - \delta \quad \text{and} \quad \mathcal{D}(\Gamma_t^k, \mathcal{V}_s) > \varepsilon\,.$$

Then by (5.3) and (5.4) we get

$$\begin{aligned}\mathcal{H}^n(\Sigma_t^k) &\geq \mathcal{H}^n(\Gamma_t^k) + \delta + \frac{L(\lambda(\varepsilon))}{2} - \delta \\ &> m_0 + \frac{1}{N}\,.\end{aligned}$$

But this contradicts $\mathcal{F}(\{\{\Sigma_t\}^k\}) \leq m_0 + \frac{1}{k}$. Thus we have the following: for all $\varepsilon > 0$ there are $\delta > 0$ and $N \in \mathbb{N}$ such that, if $k > N$ and $\mathcal{H}^n(\Gamma_{t_k}^k) > m_0 - \delta$, then $\mathcal{D}(\Gamma_{t_k}^k, \mathcal{V}_s) < \varepsilon$. This of course gives the proposition.

6 The existence of almost minimizing varifolds

In this chapter we prove Proposition 4.4. At various steps in the regularity theory we will have to construct comparison surfaces which are deformations of a given surface. However, each initial surface will be just a member of a 1-parameter family and in order to exploit our variational properties we must in fact construct "comparison families". If we consider a family as a moving surface, it becomes clear that difficulties arise when we try to embed the deformation of a single "time-slice" into the dynamics of the family itself. The main new point of this chapter is therefore the following technical lemma, which allows to use the "static" variational principle of Definition 4.3 to construct a "dynamic" competitor.

Lemma 6.1. *Let $U \subset\subset U' \subset M$ be two open sets and $\{\partial\Xi_t\}_{t\in[0,1]}$ a sweepout. Given an $\varepsilon > 0$ and a $t_0 \in [0,1]$, assume $\{\partial\Omega_s\}_{s\in[0,1]}$ is a 1-parameter family of surfaces satisfying (4.1), (4.2), (4.3) and (4.4), with $\Omega = \Xi_{t_0}$. Then there is $\eta > 0$, such that the following holds for every a, b, a', b' with $t_0 - \eta \le a < a' < b' < b \le t_0 + \eta$. There is a* competitor sweepout *$\{\partial\Xi'_t\}_{t\in[0,1]}$ with the following properties:*

(a) $\Xi_t = \Xi'_t$ for $t \in [0,a] \cup [b,1]$ and $\Xi_t \setminus U' = \Xi'_t \setminus U'$ for $t \in (a,b)$;

(b) $\mathcal{H}^n(\partial\Xi'_t) \le \mathcal{H}^n(\partial\Xi_t) + \frac{\varepsilon}{4}$ for every t;

(c) $\mathcal{H}^n(\partial\Xi'_t) \le \mathcal{H}^n(\partial\Xi_t) - \frac{\varepsilon}{2}$ for $t \in (a',b')$.

Moreover, $\{\partial\Xi'_t\}$ is homotopic to $\{\partial\Xi_t\}$.

Bulding on Lemma 6.1, Proposition 4.4 can be proved using a clever combinatorial argument due to Pitts and Almgren. Indeed, for this part our proof follows literally the exposition of Section 5 of [18]. This chapter is therefore split into three parts. In the first one we use the Almgren-Pitts combinatorial argument to show Proposition 4.4 from Lemma 6.1, which will be proved in the second one. In the third one we discuss a generalization of these results to multi-parameter families that is proved in [36] building on our proof in this chapter.

6.1 Almost minimizing varifolds

Before coming to the proof, we introduce some further notation.

Definition 6.2. *Given a pair of open sets (U^1, U^2) we call a hypersurface $\partial\Omega$ ε-a.m. in (U^1, U^2) if it is ε-a.m. in at least one of the two open sets. We denote by $\mathcal{CO}$ the set of pairs (U^1, U^2) of open sets with*

$$\mathrm{d}(U^1, U^2) \geq 4 \min\{\mathrm{diam}(U^1), \mathrm{diam}(U^2)\}\,.$$

The following trivial lemma will be of great importance.

Lemma 6.3. *If (U^1, U^2) and (V^1, V^2) are such that*

$$\mathrm{d}(U^1, U^2) \geq 2 \min\{\mathrm{diam}(U^1), \mathrm{diam}(U^2)\}\,,$$
$$\mathrm{d}(V^1, V^2) \geq 2 \min\{\mathrm{diam}(V^1), \mathrm{diam}(V^2)\}\,,$$

then there are indices $i, j \in \{1, 2\}$ with $\mathrm{d}(U^i, V^j) > 0$.

Proof. Without loss of generality, assume that U_1 is, among U_1, U_2, V_1, V_2, the set with the smallest diameter. We claim that either $\mathrm{d}(U_1, V_1) > 0$ or $\mathrm{d}(U_1, V_2) > 0$. If this were false, then there would be a point $x \in \overline{U}_1 \cap \overline{V}_1$ and a point $y \in \overline{U}_1 \cap \overline{V}_2$. But then $\mathrm{d}(x, y) \leq \mathrm{diam}(U_1) \leq \min\{\mathrm{diam}(V_1), \mathrm{diam}(V_2)\}$, and hence

$$\mathrm{d}(V_1, V_2) \leq \mathrm{d}(x, y) \leq \min\{\mathrm{diam}(V_1), \mathrm{diam}(V_2)\}\,,$$

contradicting the assumption on (V_1, V_2). □

We are now ready to state the Almgren-Pitts combinatorial Lemma: Proposition 4.4 is indeed a corollary of it.

Proposition 6.4 (Almgren-Pitts combinatorial Lemma)**.** *Let Λ be a homotopically closed family of sweepouts. There is a min-max sequence $\{\Gamma^N\} = \{\partial\Omega^{k(N)}_{t_{k(N)}}\}$ such that*

- *Γ^N converges to a stationary varifold;*
- *for any $(U^1, U^2) \in \mathcal{CO}$, Γ^N is $1/N$-a.m. in (U^1, U^2), for N large enough.*

Proof of Proposition 4.4. We show that a subsequence of the $\{\Gamma^k\}$ in Proposition 6.4 satisfies the requirements of Proposition 4.4. For this fix $k \in \mathbb{N}$ and $r > 0$ such that $\mathrm{Inj}\,(M) > 9r > 0$. Then, $(B_r(x), M \setminus \overline{B}_{9r}(x)) \in \mathcal{CO}$ for all $x \in M$. Therefore we have that Γ^k is (for k large enough) $1/k$-almost minimizing in $B_r(x)$ or $M \setminus \overline{B}_{9r}(x)$. Therefore, having fixed $r > 0$,

(a) either $\{\Gamma^k\}$ is (for k large) $1/k$-a.m. in $B_r(y)$ for every $y \in M$;

(b) or there are a (not relabeled) subsequence $\{\Gamma^k\}$ and a sequence $\{x^k_r\} \subset M$ such that Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{9r}(x^k_r)$.

If for some $r > 0$ (a) holds, we clearly have a sequence as in Proposition 4.4. Otherwise there are a subsequence of $\{\Gamma^k\}$, not relabeled, and a collection of points $\{x^k_j\}_{k,j\in\mathbb{N}} \subset M$ such that

- for any fixed j, Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{1/j}(x^k_j)$ for k large enough;
- $x^k_j \to x_j$ for $k \to \infty$ and $x_j \to x$ for $j \to \infty$.

We conclude that, for any J, there is K_J such that Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{1/J}(x)$ for all $k \geq K_J$. Therefore, if $y \in M \setminus \{x\}$, we choose $r(y)$ such that $B_{r(y)} \subset\subset M \setminus \{x\}$, whereas $r(x)$ is chosen arbitrarily. It follows that $An \subset\subset M \setminus \{x\}$, for any $An \in \mathcal{AN}_{r(z)}(z)$ with $z \in M$. Hence, $\{\Gamma^k\}$ is $1/k$-a.m. in An, provided k is large enough, which completes the proof of the proposition. □

Proof of Proposition 6.4. First we pick a minimizing sequence $\{\{\Gamma_t\}^k\}$ satisfying the requirements of Proposition 4.2 and such that $\mathcal{F}(\{\Gamma_t\}^k) < m_0 + \frac{1}{8k}$. We then assert the following claim, which clearly implies the proposition.

Claim. *For N large enough, there exists $t_N \in [0,1]$ such that $\Gamma^N := \Gamma^N_{t_N}$ is $\frac{1}{N}$-a.m. in all $(U^1, U^2) \in \mathcal{CO}$ and $\mathcal{H}^n(\Gamma^N) \geq m_0 - \frac{1}{N}$.*

Define

$$K_N := \left\{ t \in [0,1] : \mathcal{H}^n(\Gamma^N_t) \geq m_0 - \frac{1}{N} \right\}.$$

Assume the claim is false. Then there is a sequence $\{N_k\}$ such that the assertion of the claim is violated for every $t \in K_{N_k}$. By a slight abuse of notation, we do not relabel the corresponding subsequence and from now on we drop the super- and subscripts N.

Thus, for every $t \in K$ we get a pair $(U_{1,t}, U_{2,t}) \in \mathcal{CO}$ and families $\{\partial\Omega_{i,t,\tau}\}_{\tau\in[0,1]}^{i\in\{1,2\}}$ such that

(i) $\partial\Omega_{i,t,\tau} \cap (U_{i,t})^c = \partial\Omega_t \cap (U_{i,t})^c$;

(ii) $\partial\Omega_{i,t,0} = \partial\Omega_t$;

(iii) $\mathcal{H}^n(\partial\Omega_{i,t,\tau}) \leq \mathcal{H}^n(\partial\Omega_t) + \frac{1}{8N}$;

(iv) $\mathcal{H}^n(\partial\Omega_{i,t,1}) \leq \mathcal{H}^n(\partial\Omega_t) - \frac{1}{N}$.

For every $t \in K$ and every $i \in \{1, 2\}$, we choose $U'_{i,t}$ such that $U_{i,t} \subset\subset U'_{i,t}$ and

$$\mathrm{d}(U'_{1,t}, U'_{2,t}) \geq 2\min\{\mathrm{diam}(U'_{1,t}), \mathrm{diam}(U'_{2,t})\}.$$

Then we apply Lemma 6.1 with $\Xi_t = \Omega_t$, $U = U_{i,t}$, $U' = U'_{i,t}$ and $\Omega_\tau = \Omega_{i,t,\tau}$. Let $\eta_{i,t}$ be the corresponding constant η given by Lemma 6.1 and let $\eta_t = \min\{\eta_{1,t}, \eta_{2,t}\}$.

Next, cover K with intervals $I_i = (t_i - \eta_i, t_i + \eta_i)$ in such a way that:

- $t_i + \eta_i < t_{i+2} - \eta_{i+2}$ for every i;
- $t_i \in K$ and $\eta_i < \eta_{t_i}$.

Step 1: Refinement of the covering. We are now going to refine the covering I_i to a covering J_l such that:

- $J_l \subset I_i$ for some $i(l)$;
- there is a choice of a U_l such that $U'_l \in \{U'_{1,t_{i(l)}}, U'_{2,t_{i(l)}}\}$ and
$$\mathrm{d}(U'_i, U'_j) > 0 \qquad \text{if } \overline{J}_i \cap \overline{J}_j \neq \emptyset\,; \tag{6.1}$$
- each point $t \in [0, 1]$ is contained in at most two of the intervals J_l.

The choice of our refinement is in fact quite obvious. We start by choosing $J_1 = I_1$. Using Lemma 6.3 we choose indices r, s such that $\mathrm{d}(U'_{r,t_1}, U'_{s,t_2}) > 0$. For simplicity we can assume $r = s = 1$. We then set $U'_1 = U'_{1,t_1}$. Next, we consider two indices ρ, σ such that $\mathrm{d}(U'_{\rho,t_2}, U'_{\sigma,t_3}) > 0$. If $\rho = 1$, we then set $J_2 = I_2$ and $U'_2 = U'_{1,t_2}$. Otherwise, we cover I_2 with two open intervals J_2 and J_3 with the property that $\overline{J}_2$ is disjoint from $\overline{I}_3$ and $\overline{J}_3$ is disjoint from $\overline{I}_1$. We then choose $U'_2 = U'_{1,t_2}$ and $U'_3 = U'_{2,t_2}$. From this we are ready to proceed inductively. Note therefore that, in our refinement of the covering, each interval I_j with $j \geq 2$ is either "split into two halves" or remains the same (see Figure 6.1, left).

Next, fixing the notation $(a_i, b_i) = J_i$, we choose $\delta > 0$ with the property:

(C) Each $t \in K$ is contained in at least one segment $(a_i + \delta, b_i - \delta)$ (see Figure 6.1, right).

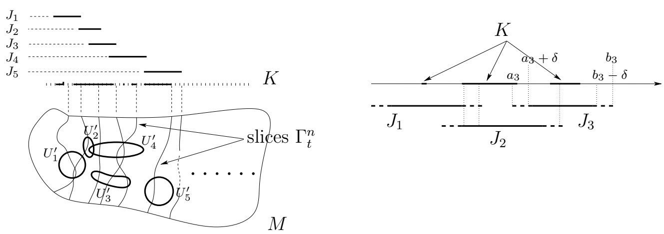

Figure 6.1: The left picture shows the refinement of the covering. We split I_2 into $J_2 \cup J_3$ because $U'_4 = U'_{1,t_3}$ intersects $U'_2 = U'_{1,t_2}$. The refined covering has the property that $U'_i \cap U'_{i+1} = \emptyset$. In the right picture the segments $(a_k, b_k) = J_k$ and $(a_k + \delta, b_k - \delta)$. Any point $\tau \in K$ belongs to at least one $(a_i + \delta, b_i - \delta)$ and to at most one $J_j \setminus (a_j + \delta, b_j - \delta)$.

Step 2: Conclusion. We now apply Lemma 6.1 to conclude the existence of a family $\{\partial\Omega_{i,t}\}$ with the following properties:

- $\Omega_{i,t} = \Omega_t$ if $t \notin (a_i, b_i)$ and $\Omega_{i,t} \setminus U'_i = \Omega_t \setminus U'_i$ if $t \in (a_i, b_i)$;
- $\mathcal{H}^n(\partial\Omega_{i,t}) \leq \mathcal{H}^n(\partial\Omega_t) + \frac{1}{4N}$ for every t;

- $\mathcal{H}^n(\partial\Omega_{i,t}) \leq \mathcal{H}^n(\partial\Omega_t) - \frac{1}{2N}$ if $t \in (a_i + \delta, b_i - \delta)$.

Note that, if $t \in (a_i, b_i) \cap (a_j, b_j)$, then $j = i + 1$ and in fact $t \notin (a_k, b_k)$ for $k \neq i, i+1$. Moreover, $\mathrm{d}(U'_i, U'_{i+1}) > 0$. Thus, we can define a new sweepout $\{\partial\Omega'_t\}_{t\in[0,1]}$

- $\Omega'_t = \Omega_t$ if $t \notin \cup J_i$;
- $\Omega'_t = \Omega_{i,t}$ if t is contained in a single J_i;
- $\Omega'_t = \left[\Omega_t \setminus (U'_i \cup U'_{i+1})\right] \cup [\Omega_{i,t} \cap U'_i] \cup \left[\Omega_{i+1,t} \cap U'_{i+1}\right]$ if $t \in J_i \cap J_{i+1}$.

In fact, it is as well easy to check that $\{\partial\Omega'_t\}_{t\in[0,1]}$ is homotopic to $\{\partial\Omega_t\}$ and hence belongs to Λ.

Next, we want to compute $\mathcal{F}(\{\partial\Omega'_t\})$. If $t \notin K$, then t is contained in at most two J_i's, and hence $\partial\Omega'_t$ can gain at most $2 \cdot \frac{1}{4N}$ in area:

$$t \notin K \quad \Rightarrow \quad \mathcal{H}^n(\partial\Omega'_t) \ \leq \ \mathcal{H}^n(\partial\Omega_t) + \frac{1}{2N} \ \leq \ m_0(\Lambda) - \frac{1}{2N}\,. \tag{6.2}$$

If $t \in K$, then t is contained in at least one segment $(a_i + \delta, b_i - \delta) \subset J_i$ and in at most a second segment J_l. Thus, the area of $\partial\Omega'_t$ looses at least $\frac{1}{2N}$ in U'_i and gains at most $\frac{1}{4N}$ in U'_l. Therefore we conclude

$$t \in K \quad \Rightarrow \quad \mathcal{H}^n(\partial\Omega'_t) \ \leq \ \mathcal{H}^n(\partial\Omega_t) - \frac{1}{4N} \ \leq \ m_0(\Lambda) - \frac{1}{8N}\,. \tag{6.3}$$

Hence $\mathcal{F}(\{\partial\Omega'_t\}) \leq m_0(\Lambda) - (8N)^{-1}$, which is a contradiction to $m_0(\Lambda) = \inf_\Lambda \mathcal{F}$. □

6.2 Proof of Lemma 6.1

The proof consists of two steps.

Step 1: Freezing. First of all we choose open sets A and B such that

- $U \subset\subset A \subset\subset B \subset\subset U'$;
- $\partial\Xi_{t_0} \cap C$ is a smooth surface, where $C = B \setminus \overline{A}$.

This choice is clearly possible since there are only finitely many singularities of $\partial\Xi_{t_0}$. Next, we fix two smooth functions φ_A and φ_B such that

- $\varphi_A + \varphi_B = 1$;
- $\varphi_A \in C_c^\infty(B)$, $\varphi_B \in C_c^\infty(M \setminus \overline{A})$.

Now, we fix normal coordinates $(z,\sigma) \in \partial\Xi_{t_0} \cap C \times (-\delta,\delta)$ in a regular δ-neighborhood of $C \cap \partial\Xi_{t_0}$. Because of the convergence of Ξ_t to Ξ_{t_0} we can fix $\eta > 0$ and an open $C' \subset C$, such that the following holds for every $t \in (t_0-\eta, t_0+\eta)$:

- $\partial\Xi_t \cap C$ is the graph of a function g_t over $\partial\Xi_{t_0} \cap C$;
- $\Xi_t \cap C \setminus C' = \Xi_{t_0} \cap C \setminus C'$;
- $\Xi_t \cap C' = \{(z,\sigma) : \sigma < g_t(z)\} \cap C'$,

(see Figure 6.2). Obviously, $g_{t_0} \equiv 0$. We next introduce the functions

$$g_{t,s,\tau} \; := \; \varphi_B g_t + \varphi_A((1-s)g_t + s g_\tau) \tag{6.4}$$

for $t,\tau \in (t_0-\eta, t_0+\eta)$, $s \in [0,1]$. Since g_t converges smoothly to g_{t_0} as $t \to t_0$, by choosing η arbitrarily small, we can make $\sup_{s,\tau} \|g_{t,s,\tau} - g_t\|_{C^1}$ arbitrarily small. Next, if we express the area of the graph of a function g over $\partial\Xi_{t_0} \cap C$ as an integral functional of g, this functional depends obviously only on g and its first derivatives. Thus, if $\Gamma_{t,s,\tau}$ is the graph of $g_{t,s,\tau}$, then we can choose η so small that

$$\max_{s,\tau} \mathcal{H}^n(\Gamma_{t,s,\tau}) \; \leq \; \mathcal{H}(\partial\Xi_t \cap C) + \frac{\varepsilon}{16}\,. \tag{6.5}$$

Now, given $t_0 - \eta < a < a' < b' < b < t_0 + \eta$, we choose $a'' \in (a,a')$ and $b'' \in (b',b)$ and fix:

- a smooth function $\psi : [a,b] \to [0,1]$ which is identically equal to 0 in a neighborhood of a and b and equal to 1 on $[a'',b'']$;
- a smooth function $\gamma : [a,b] \to [t_0-\eta, t_0+\eta]$ which is equal to the identity in a neighborhood of a and b and indentically t_0 in $[a'',b'']$.

Next, define the family of open sets $\{\Delta_t\}$ as follows:

- $\Delta_t = \Xi_t$ for $t \not\in [a,b]$;
- $\Delta_t \setminus \overline{B} = \Xi_t \setminus \overline{B}$ for all t;

- $\Delta_t \cap A = \Xi_{\gamma(t)} \cap A$ for $t \in [a, b]$;
- $\Delta_t \cap C \setminus C' = \Xi_{t_0} \cap C \setminus C'$ for $t \in [a, b]$;
- $\Delta_t \cap C' = \{(z, \sigma) : \sigma < g_{t,\psi(t),\gamma(t)}(z)\}$ for $t \in [a, b]$.

Note that $\{\partial\Delta_t\}$ is in fact a sweepout homotopic to $\partial\Xi_t$. In addition:

- $\Delta_t = \Xi_t$ if $t \notin [a, b]$, and Δ_t and Ξ_t coincide outside of B (and hence outside of U') for *every* t;
- $\Delta_t \cap A = \Xi_{\gamma(t)} \cap A$ for $t \in [a, b]$ (and hence $\Delta_t \cap U = \Xi_{\gamma(t)} \cap U$).

Therefore, $\Delta_t \cap U = \Xi_{t_0} \cap U$ for $t \in [a'', b'']$, i.e. $\Delta_t \cap U$ is *frozen* in the interval $[a'', b'']$. Moreover, because of (6.5),

$$\mathcal{H}^n(\partial\Delta_t \cap C) \le \mathcal{H}^n(\partial\Xi_t \cap C) + \frac{\varepsilon}{16} \qquad \text{for } t \in [a, b]. \tag{6.6}$$

Step 2: Dynamic competitor. Next, fix a smooth function $\chi : [a'', b''] \to [0, 1]$ which is identically 0 in a neighborhood of a'' and b'' and which is identically 1 on $[a', b']$. We set

- $\Xi'_t = \Delta_t$ for $t \notin [a'', b'']$;
- $\Xi'_t \setminus A = \Delta_t \setminus A$ for $t \in [a'', b'']$;
- $\Xi'_t \cap A = \Omega_{\chi(t)} \cap A$ for $t \in [a'', b'']$.

The new family $\{\partial\Xi'_t\}$ is also a sweepout, obviously homotopic to $\{\partial\Delta_t\}$ and hence homotopic to $\{\partial\Xi_t\}$ (see Figure 6.2). Next, we will estimate $\mathcal{H}^n(\partial\Xi'_t)$. For $t \notin [a, b]$, $\Xi'_t \equiv \Xi_t$ and hence

$$\mathcal{H}^n(\partial\Xi'_t) \;=\; \mathcal{H}^n(\partial\Xi_t) \qquad \text{for } t \notin [a, b]\,. \tag{6.7}$$

For $t \in [a, b]$, we anyhow have $\Xi'_t = \Xi_t$ on $M \setminus B$ and $\Xi'_t = \Delta_t$ on C. This shows the property (a) of the lemma. Moreover, for $t \in [a, b]$ we have

$$\begin{aligned} \mathcal{H}^n(\partial\Xi'_t) - \mathcal{H}^n(\partial\Xi_t) \quad &\le \quad [\mathcal{H}^n(\partial\Delta_t \cap C) - \mathcal{H}^n(\partial\Xi_t \cap C)] \\ &\quad\;\; +[\mathcal{H}^n(\partial\Xi'_t \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \\ &\overset{(6.6)}{\le} \quad \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial\Xi'_t \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \end{aligned} \tag{6.8}$$

To conclude, we have to estimate the part in A in the time interval $[a, b]$. We have to consider several cases separately.

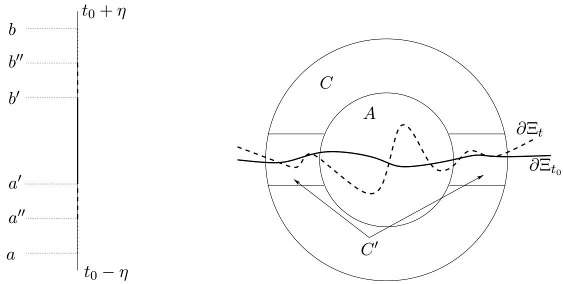

Figure 6.2: The left picture shows the intervals involved in the construction. If we focus on the smaller set A, then: the sets Ξ'_t coincide with Δ_t and evolve from Ξ_a to Ξ_{t_0} (resp. Ξ_{t_0} to Ξ_b) in $[a, a'']$ (resp. $[b'', b]$); they then evolve from Ξ_{t_0} to Ω_1 (resp. Ω_1 to Ξ_{t_0}) in $[a'', a']$ (resp. $[b', b'']$). On the right picture, the sets in the region C. Indeed, the evolution takes place in the region C' where we patch smoothly Ξ_{t_0} with $\Xi_{\gamma(t)}$ into the sets Δ_t.

(i) Let $t \in [a, a''] \cup [b'', b]$. Then $\Xi'_t \cap A = \Delta_t \cap A = \Xi_{\gamma(t)} \cap A$. However, $\gamma(t), t \in (t_0 - \eta, t_0 + \eta)$ and, having chosen η sufficiently small, we can assume

$$|\mathcal{H}^n(\partial\Xi_s \cap A) - \mathcal{H}^n(\partial\Xi_\sigma \cap A)| \ \leq \ \frac{\varepsilon}{16} \tag{6.9}$$

for every $\sigma, s \in (t_0-\eta, t_0+\eta)$. (Note: this choice of η is independent of a and b!). Thus, using (6.8), we get

$$\mathcal{H}^n(\partial\Xi'_t) \ \leq \ \mathcal{H}^n(\partial\Xi_t) + \frac{\varepsilon}{8}\,. \tag{6.10}$$

(ii) Let $t \in [a'', a'] \cup [b'', b']$. Then $\partial\Xi'_t \cap A = \partial\Omega_{\chi(t)} \cap A$. Therefore we can write, using (6.8),

$$\begin{aligned}\mathcal{H}^n(\partial\Xi'_t) - \mathcal{H}^n(\partial\Xi_t) \ &\leq \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial\Xi_{t_0} \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \\ &\quad + [\mathcal{H}^n(\partial\Omega_{\chi(t)} \cap A) - \mathcal{H}^n(\partial\Xi_{t_0} \cap A)] \\ &\overset{(4.3),(6.9)}{\leq} \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{\varepsilon}{8} \ = \ \frac{\varepsilon}{4}\,. \end{aligned} \tag{6.11}$$

(iii) Let $t \in [a', b']$. Then we have $\Xi'_t \cap A = \Omega_1 \cap A$. Thus, again using (6.8),

$$\begin{aligned}\mathcal{H}^n(\partial \Xi'_t) - \mathcal{H}^n(\partial \Xi_t) \;&\le \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial \Omega_1 \cap A) - \mathcal{H}^n(\partial \Xi_{t_0} \cap A)] \\ &\quad + \;[\mathcal{H}^n(\partial \Xi_{t_0} \cap A) - \mathcal{H}^n(\partial \Xi_t \cap A)] \\ &\overset{(4.4),(6.9)}{\le} \;\frac{\varepsilon}{16} - \varepsilon + \frac{\varepsilon}{16} \;<\; -\frac{\varepsilon}{2}\,. \end{aligned} \tag{6.12}$$

Gathering the estimates (6.7), (6.10), (6.11) and (6.12), we finally obtain the properties (b) and (c) of the lemma. This finishes the proof.

6.3 The multi-parameter situation

The min-max construction of Chapter 2 is based on the notion of 1-parameter families. The same construction could also be made with k-parameter families, where k is an arbitrary natural number. One motivation is the fact that the 1-parameter family construction should produce a minimal surface of index 1 (see Chapter 9 for more on this). Correspondingly, the construction with k-parameter families should produce a surface of index k. This might be helpful to prove that there are many non-trivial minimal surfaces in Riemannian manifolds. However, it is clearly not enough. One would have to exclude that the higher index is simply coming from the fact that the k-parameter family approach produces the same minimal surface as the 1-parameter family approach with a higher multiplicity. Therefore, some more refined arguments would be necessary (see [42] for some constructions in this direction).

Large parts of the existence proof do not rely on the choice of k, for example Chapters 7 and 8. The key step is to show that there is an almost minimizing min-max sequence. The proofs of this chapter can be mimicked also in the multi-parameter case. But some arguments get much more complicated. For instance, Step 1 in the proof of Proposition 6.4 becomes combinatorically far more difficult since the covering is no longer with intervals, but with k-dimensional cubes. In the master thesis of Fuchs all these complications are taken care of [36]. Moreover, both the settings of this thesis and [18] are treated.

The main result of [36] is the following

Proposition 6.5. *Let Λ be a homotopically closed set of k-parameter sweepouts. Then there are a natural number $\omega = \omega(k)$ and a min-max sequence $\{\Gamma^N\}$ such that*

- Γ^N *converges to a stationary varifold;*
- *for any $(U_1, \ldots, U_\omega) \in \mathcal{CO}_\omega$, Γ^N is $\frac{1}{N}$-a.m. in $(U_1, \ldots, U_\omega)$ for N large enough.*

Here $\mathcal{CO}_\omega$ is the obvious generalization of $\mathcal{CO} = \mathcal{CO}_2$. There is an analogous statement for the setting of [18] in the case $n = 2$. Moreover, there is a corresponding generalization of Proposition 4.4 to the multi-parameter situation.

7 The existence of replacements

In this chapter we fix $An \in \mathcal{AN}_{r(x)}(x)$ and prove the conclusion of Proposition 4.6.

7.1 Setting

For every j, consider the class $\mathcal{H}(\Omega^j, An)$ of sets Ξ such that there is a family $\{\Omega_t\}$ satisfying $\Omega_0 = \Omega^j$, $\Omega_1 = \Xi$, (4.1), (4.2) and (4.3) for $\varepsilon = 1/j$ and $U = An$. Consider next a sequence $\Gamma^{j,k} = \partial\Omega^{j,k}$ which is minimizing for the perimeter in the class $\mathcal{H}(\Omega^j, An)$: this is the minimizing sequence for the $(8j)^{-1}$-*homotopic Plateau problem* mentioned in Section 4.3. Up to subsequences, we can assume that

- $\Omega^{j,k}$ converges to a Caccioppoli set $\tilde{\Omega}^j$;
- $\Gamma^{j,k}$ converges to a varifold V^j;
- V^j (and a suitable diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$) converges to a varifold $\tilde{V}$.

The proof of Proposition 4.6 will then be broken into three steps. In the first one we show

Lemma 7.1. *For every j and every $y \in An$ there are a ball $B = B_\rho(y) \subset An$ and a $k_0 \in \mathbb{N}$ with the following property. Every open set Ξ such that*

- *$\partial\Xi$ is smooth except for a finite set;*
- *$\Xi \setminus B = \Omega^{j,k} \setminus B$;*
- *$\mathcal{H}^n(\partial\Xi) < \mathcal{H}^n(\partial\Omega^{j,k})$*

belongs to $\mathcal{H}(\Omega^j, An)$ *if* $k \geq k_0$.

In the second step we use Lemma 7.1 and Theorem 2.13 to show:

Lemma 7.2. *$\partial\tilde{\Omega}^j \cap An$ is a stable minimal hypersurface in An and $V^j \llcorner An = \partial\tilde{\Omega}^j \llcorner An$.*

Recall that in this chapter we use the convention of Definition 3.5. In the third step we use Lemma 7.2 to conclude that the sequence $\tilde{\Gamma}^j$ and the varifold $\tilde{V}$ meet the requirements of Proposition 4.6.

7.2 Proof of Lemma 7.1

The proof of the lemma is achieved by exhibiting a suitable homotopy between $\Omega^{j,k}$ and Ξ. The key idea is:

- First deform $\Omega^{j,k}$ to the set $\tilde{\Omega}$ which is the union of $\Omega^{j,k} \setminus B$ and the cone with vertex y and base $\Omega^{j,k} \cap \partial B$;
- then deform $\tilde{\Omega}$ to Ξ.

The surfaces of the homotopizing family do not gain too much in area, provided $B = B_\rho(y)$ is sufficiently small and k sufficiently large: in this case the area of the surface $\Gamma^{j,k} \cap B$ will, in fact, be close to the area of the cone. This "blow down-blow up" procedure is an idea which we borrow from [72] (see Section 7 of [18]).

Proof of Lemma 7.1. We fix $y \in An$ and $j \in \mathbb{N}$. Let $B = B_\rho(y)$ with $B_{2\rho}(y) \subset An$ and consider an open set Ξ as in the statement of the lemma. The choice of the radius of the ball $B_\rho(y)$ and of the constant k_0 (which are both independent of the set Ξ) will be determined at the very end of the proof.

Step 1: Stretching $\Gamma^{j,k} \cap \partial B_r(y)$. First of all, we choose $r \in (\rho, 2\rho)$ such that, for every k,

$$\Gamma^{j,k} \text{ is regular in a neighborhood of } \partial B_r(y) \text{ and intersects it transversally.} \tag{7.1}$$

In fact, since each $\Gamma^{j,k}$ has finitely many singularities, Sard's lemma implies that (7.1) is satisfied by a.e. r. We assume moreover that 2ρ is smaller than the injectivity radius. For each $z \in \overline{B}_r(y)$ we consider the closed geodesic arc $[y,z] \subset \overline{B}_r(y)$ joining y and z. As usual, (y,z) denotes $[y,z] \setminus \{y,z\}$. We let K be the open cone

$$K = \bigcup_{z \in \partial B \cap \Omega^{j,k}} (y,z) . \tag{7.2}$$

We now show that $\Omega^{j,k}$ can be homotopized through a family $\tilde{\Omega}_t$ to a $\tilde{\Omega}_1$ in such a way that

- $\max_t \mathcal{H}^n(\partial\tilde{\Omega}_t) - \mathcal{H}^n(\partial\Omega^{j,k})$ can be made arbitrarily small;
- $\tilde{\Omega}_1$ coincides with K in a neighborhood of $\partial B_r(y)$.

First of all consider a smooth function $\varphi : [0,2\rho] \to [0,2\rho]$ with

- $|\varphi(s) - s| \leq \varepsilon$ and $0 \leq \varphi' \leq 2$;
- $\varphi(s) = s$ if $|s - r| > \varepsilon$ and $\varphi \equiv r$ in a neighborhood of r.

Set $\Phi(t,s) := (1-t)s + t\varphi(s)$. Moreover, for every $\lambda \in [0,1]$ and every $z \in \overline{B}_r(y)$ let $\tau_\lambda(z)$ be the point $w \in [y,z]$ with $\mathrm{d}(y,w) = \lambda\,\mathrm{d}(y,z)$. For $1 < \lambda < 2$, we can still define $\tau_\lambda(z)$ to be the corresponding point on the geodesic that is the extension of $[y,z]$. (Note that by the choice of ρ this is well defined.) We are now ready to define $\tilde{\Omega}_t$ (see Figure 7.1, left).

- $\tilde{\Omega}_t \setminus An(y, r-\varepsilon, r+\varepsilon) = \Omega^{j,k} \setminus An(y, r-\varepsilon, r+\varepsilon)$;
- $\tilde{\Omega}_t \cap \partial B_s(y) = \tau_{s/\Phi(t,s)}(\Omega^{j,k} \cap \partial B_{\Phi(t,s)})$ for every $s \in (r-\varepsilon, r+\varepsilon)$.

Thanks to (7.1), for ε sufficiently small, $\tilde{\Omega}_t$ has the desired properties. Moreover, since Ξ coincides with $\Omega^{j,k}$ on $M \setminus B_\rho(y)$, the same argument can be applied to Ξ. This shows that

$$\text{w.l.o.g. we can assume } K = \Xi = \Omega^{k,j} \text{ in a neighborhood of } \partial B_r(y). \tag{7.3}$$

Step 2: The homotopy. We then consider the following family of open sets $\{\Omega_t\}_{t\in[0,1]}$ (see Figure 7.1, right):

- $\Omega_t \setminus \overline{B}_r(y) = \Omega^{j,k} \setminus \overline{B}_r(y)$ for every t;

- $\Omega_t \cap An(y, |1-2t|r, r) = K \cap An(y, |1-2t|r, r)$ for every t;
- $\Omega_t \cap \overline{B}_{(1-2t)r}(y) = \tau_{1-2t}(\Omega^{k,j} \cap \overline{B}_r(y))$ for $t \in [0, \frac{1}{2}]$;
- $\Omega_t \cap \overline{B}_{(2t-1)r}(y) = \tau_{2t-1}(\Xi \cap \overline{B}_r(y))$ for $t \in [\frac{1}{2}, 1]$.

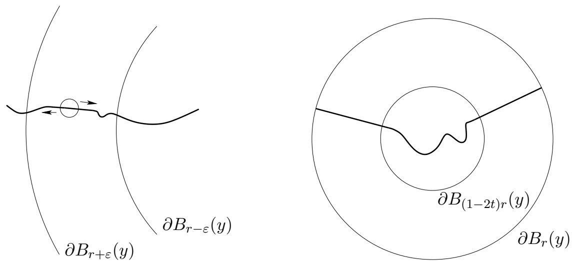

Figure 7.1: The left picture illustrates the stretching of $\Gamma^{j,k}$ into a cone-like surface in a neighborhood of $\partial B_r(y)$. The right picture shows a slice $\Omega_t \cap B_r(y)$ for $t \in (0, 1/2)$.

Because of (7.3), this family satisfies (s1)-(s3), (sw1) and (sw3). It remains to check,

$$\max_t \mathcal{H}^n(\partial\Omega_t) \leq \mathcal{H}^n(\partial\Omega^{j,k}) + \frac{1}{8j} \qquad \forall k \geq k_0 \tag{7.4}$$

for a suitable choice of ρ, r and k_0.

First of all we observe that, by the smoothness of M, there are constants μ and ρ_0, depending only on the metric, such that the following holds for every $r < 2\rho < 2\rho_0$ and $\lambda \in [0, 1]$:

$$\mathcal{H}^n(K) \leq \mu r \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y)); \tag{7.5}$$
$$\mathcal{H}^n([\partial(\tau_\lambda(\Omega^{j,k} \cap \overline{B}_r(y)))] \cap B_{\lambda r}(y)) \leq \mu \mathcal{H}^n(\partial\Omega^{j,k} \cap B_r(y)); \tag{7.6}$$
$$\mathcal{H}^n([\partial(\tau_\lambda(\Xi \cap \overline{B}_r(y)))] \cap B_{\lambda r}(y)) \leq \mu \mathcal{H}^n(\partial\Xi \cap B_r(y)); \tag{7.7}$$
$$\int_0^{2\rho} \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_\tau(y))\, d\tau \leq \mu \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)). \tag{7.8}$$

In fact, for ρ small, μ will be close to 1. (7.5), (7.6) and (7.7) give the obvious estimate

$$\max_t \mathcal{H}^n(\partial\Omega_t) - \mathcal{H}^n(\partial\Omega^{j,k}) \le \mu\mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)) + \mu r \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y)) . \tag{7.9}$$

Moreover, by (7.8) we can find $r \in (\rho, 2\rho)$ which, in addition to (7.9), satisfies

$$\mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y)) \le \frac{2\mu}{\rho}\mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)) . \tag{7.10}$$

Hence, we conclude

$$\max_t \mathcal{H}^n(\partial\Omega_t) \le \mathcal{H}^n(\partial\Omega^{j,k}) + (\mu + 4\mu^2)\mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)) . \tag{7.11}$$

Next, by the convergence of $\Gamma^{j,k} = \partial\Omega^{j,k}$ to the stationary varifold V^j, we can choose k_0 such that

$$\mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)) \le 2\|V^j\|(B_{4\rho}(y)) \qquad \text{for } k \ge k_0. \tag{7.12}$$

Finally, by the monotonicity formula,

$$\|V^j\|(B_{4\rho}(y)) \le C_M \|V^j\|(M)\rho^n . \tag{7.13}$$

We are hence ready to specify the choice of the various parameters.

- We first determine the constants μ and $\rho_0 < \mathrm{Inj}\,(M)$ (which depend only on M) which guarantee (7.5), (7.6), (7.7) and (7.8);
- we subsequently choose $\rho < \rho_0$ so small that

  $$2(\mu + 4\mu^2)C_M\|V^j\|(M)\rho^n < (8j)^{-1},$$

  and k_0 so that (7.12) holds.

At this point ρ and k are fixed and, choosing $r \in (\rho, 2\rho)$ satisfying (7.1) and (7.10), we construct $\{\partial\Omega_t\}$ as above, concluding the proof of the lemma.

7.3 Proof of Lemma 7.2

Fix $j \in \mathbb{N}$ and $y \in An$ and let $B = B_\rho(y) \subset An$ be the ball given by Lemma 7.1. We claim that $\tilde{\Omega}^j$ minimizes the perimeter in the class $\mathcal{P}(\tilde{\Omega}^j, B_{\rho/2}(y))$. Assume, by contradiction, that Ξ is a Caccioppoli set with $\Xi \setminus B_{\rho/2}(y) = \tilde{\Omega}^j \setminus B_{\rho/2}(y)$ and

$$\mathrm{Per}\,(\Xi) \;<\; \mathrm{Per}\,(\tilde{\Omega}^j) - \eta\,. \tag{7.14}$$

Note that, since $\mathbf{1}_{\Omega^{j,k}} \to \mathbf{1}_{\tilde{\Omega}^j}$ strongly in L^1, up to extraction of a subsequence we can assume the existence of $\tau \in (\rho/2, \rho)$ such that

$$\lim_{k\to\infty} \|\mathbf{1}_{\tilde{\Omega}^j} - \mathbf{1}_{\Omega^{j,k}}\|_{L^1(\partial B_\tau(y))} \;=\; 0\,. \tag{7.15}$$

We also recall that, by the semicontinuity of the perimeter,

$$\mathrm{Per}\,(\tilde{\Omega}^j) \;\leq\; \liminf_{k\to\infty} \mathcal{H}^n(\partial\Omega^{j,k})\,. \tag{7.16}$$

Define therefore the set $\Xi^{j,k}$ by setting

$$\Xi^{j,k} \;=\; (\Xi \cap B_\tau(y)) \cup (\Omega^{j,k} \setminus B_\tau(y))\,.$$

(7.14), (7.15) and (7.16) imply

$$\limsup_{k\to\infty} [\mathrm{Per}\,(\Xi^{j,k}) - \mathcal{H}^n(\partial\Omega^{j,k})] \;\leq\; -\eta\,. \tag{7.17}$$

Fix next k and recall the following standard way of approximating $\Xi^{j,k}$ with a smooth set. We first fix a compactly supported convolution kernel φ, then we consider the function $g_\varepsilon := \mathbf{1}_{\Xi^{j,k}} * \varphi_\varepsilon$ and finally look at a smooth level set $\Delta_\varepsilon := \{g_\varepsilon > t\}$ for some $t \in (\frac{1}{4}, \frac{3}{4})$. Then $\mathcal{H}^n(\partial\Delta_\varepsilon)$ converges to $\mathrm{Per}\,(\Xi^{j,k})$ as $\varepsilon \to 0$ (see [38] in the Euclidean case and [50] for the general one).

Clearly, Δ_ε does not coincide anymore with $\Omega^{j,k}$ outside $B_\rho(y)$. Thus, fix $(a, b) \subset (\tau, \rho)$ with the property that $\Sigma := \Omega^{j,k} \cap \overline{B}_b(y) \setminus B_a(y)$ is smooth. Fix a regular tubular neighborhood T of Σ and corresponding normal coordinates (ξ, σ) on it. Since $\Xi^{j,k} \setminus B_\tau(y) = \Omega^{j,k} \setminus B_\tau(y)$, for ε sufficiently small $\partial\Delta_\varepsilon \cap \overline{B}_b(y) \setminus B_a(y) \subset T$ and $T \cap \Delta_\varepsilon$ is the set $\{\sigma < f_\varepsilon(\xi)\}$ for some smooth function f_ε. Moreover, as $\varepsilon \to 0$, $f_\varepsilon \to 0$ smoothly.

Therefore, a patching argument entirely analogous to the one of the freezing construction (see Section 6.2) allows us to modify $\Xi^{j,k}$ to a set $\Delta^{j,k}$ with the following properties:

- $\partial\Delta^{j,k}$ is smooth outside of a finite set;
- $\Delta^{j,k} \setminus B = \Omega^{j,k} \setminus B$;
- $\limsup_k(\mathcal{H}^n(\partial\Delta^{j,k}) - \mathcal{H}^n(\partial\Omega^{j,k})) \leq -\eta < 0$.

For k large enough, Lemma 7.1 implies that $\Delta^{j,k} \in \mathcal{H}(\Omega^j, An)$, which would contradict the minimality of the sequence $\Omega^{j,k}$.

Next, in order to show that the varifold V^j is induced by $\partial\tilde{\Omega}^j$, it suffices to show that in fact $\mathcal{H}^n(\partial\Omega^{j,k})$ converges to $\mathcal{H}^n(\partial\tilde{\Omega}^j)$ (since we have not been able to find a precise reference for this well-known fact, we give a proof in Section 7.3.1; see Proposition 7.3). On the other hand, if this is not the case, then we have

$$\mathcal{H}^n(\partial\tilde{\Omega}^j \cap B_{\rho/2}(y)) \;<\; \limsup_{k\to\infty} \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{\rho/2}(y))$$

for some $y \in An$ and some ρ to which we can apply the conclusion of Lemma 7.1. We can then use $\tilde{\Omega}^j$ in place of Ξ in the argument of the previous step to contradict, once again, the minimality of the sequence $\{\Omega^{j,k}\}_k$. The stationarity and stability of the surface $\partial\tilde{\Omega}^j$ is, finally, an obvious consequence of the variational principle.

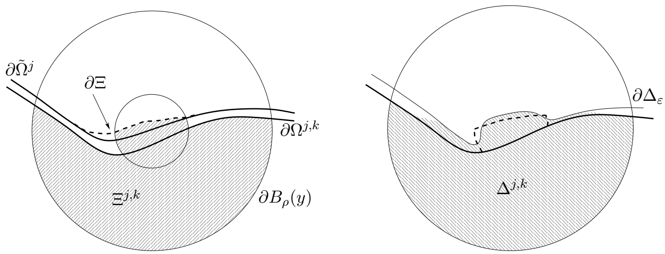

Figure 7.2: On the left, the set $\tilde{\Omega}^j$, the competitor Ξ, one set of the sequence $\{\Omega^{j,k}\}_k$ and the corresponding $\Xi^{j,k}$. On the right, the smoothing Δ_ε of $\Xi^{j,k}$ and the final set $\Delta^{j,k}$ (a competitor for $\Omega^{j,k}$).

7.3.1 Varifolds and Caccioppoli set limits

Proposition 7.3. *Let $\{\Omega^k\}$ be a sequence of Caccioppoli sets and U an open subset of M. Assume that*

(i) $D\mathbf{1}_{\Omega^k} \to D\mathbf{1}_\Omega$ *in the sense of measures in* U*;*

(ii) $\mathrm{Per}\,(\Omega^k, U) \to \mathrm{Per}\,(\Omega, U)$

for some Caccioppoli set Ω and denote by V^k and V the varifolds induced by $\partial^*\Omega^k$ and $\partial^*\Omega$. Then $V^k \to V$ in the sense of varifolds.

Proof. First, we note that by the rectifiability of the boundaries we can write

$$V^k \;=\; \mathcal{H}^n \llcorner \partial^*\Omega^k \otimes \delta_{T_x \partial^*\Omega^k} \;\; , \;\; V \;=\; \mathcal{H}^n \llcorner \partial^*\Omega \otimes \delta_{T_x \partial^*\Omega} \,, \tag{7.18}$$

where $\partial^*\Omega, \partial^*\Omega^k$ are the reduced boundaries and $T_x\partial^*\Omega$ is the approximate tangent plane to Ω in x (see Chapter 3 of [38] for the relevant definitions). With the notation $\mu \otimes \alpha_x$ we understand, as usual, the measure ν on a product space $X \times Y$ given by

$$\nu(E) \;=\; \int\!\!\int \mathbf{1}_E(x, y)\, d\alpha_x(y)\, d\mu(x)\,,$$

where μ is a Radon measure on X and $x \mapsto \alpha_x$ is a weak* μ-measurable map from X into $\mathcal{M}(Y)$ (the space of Radon measures on Y).

By (*ii*) we have $\|V^k\| \to \|V\|$ and hence there is $W \in \mathcal{V}(U)$ such that (up to subsequences) $V^k \to W$. In addition, $\|V\| = \|W\|$. By the disintegration theorem (see Theorem 2.28 in [11]) we can write $W = \mathcal{H}^n \llcorner \partial^*\Omega \otimes \alpha_x$. The proposition is proved, once we have proved

(Cl) $\alpha_{x_0} = \delta_{T_{x_0}\partial^*\Omega}$ for $\mathcal{H}^n$-a.e. $x_0 \in \partial^*\Omega$.

To prove this, we reduce the situation to the case where Ω is a half-space by a classical blowup analysis. Having fixed a point x_0, a radius r and the rescaled exponential maps $T_r^{x_0} : \mathcal{B}_1 \to B_r(x_0)$ (see Section 8.2), we define

- $V_r^k := (T_r^{x_0})_\sharp^{-1} V^k$ and $W_r := (T_r^{x_0})_\sharp^{-1} W$;
- $\Omega_r^k := (T_r^{x_0})^{-1}(\Omega^k)$ and $\Omega_r := (T_r^{x_0})^{-1}(\Omega)\,.$

Clearly, V_r^k and Ω_r^k are related by the same formulas as in (7.18). Next, let G be the set of radii r such that

$$\mathcal{H}^n(\partial^*\Omega^k \cap \partial B_r(x_0)) = \mathcal{H}^n(\partial^*\Omega \cap \partial B_r(x_0)) = 0$$

for every k and observe that the complement of G is a countable set. Denote by H the set $\{x_1 < 0\}$. Then, after a suitable choice of orthonormal coordinates in $\mathcal{B}_1$, we have

(a) $D\mathbf{1}_{\Omega_r^k} \to D\mathbf{1}_{\Omega_r}$ and $\mathrm{Per}\,(\Omega_r^k, \mathcal{B}_1) \to \mathrm{Per}\,(\Omega_r, \mathcal{B}_1)$ for $k \to \infty$ and $r \in G$;

(b) $D\mathbf{1}_{\Omega_r} \to D\mathbf{1}_H$ and $\mathrm{Per}\,(\Omega_r, \mathcal{B}_1) \to \mathrm{Per}\,(H, \mathcal{B}_1)$ for $r \to 0, r \in G$;

(c) $T_0\partial^* H = T_{x_0}\partial^*\Omega$;

(d) $V_r^k \to W_r$ for $k \to \infty$ and $r \in G$.

(The assumption $r \in G$ is essential: see Proposition 1.62 of [11] or Proposition 2.7 of [26]).

Next, for $\mathcal{H}^n$-a.e. $x_0 \in \partial^*\Omega$ we have in addition

(e) $W_r \to \mathcal{H}^n \llcorner \partial^* H \otimes \alpha_{x_0}$

(in fact, if $\mathcal{D} \subset C(\mathbb{P}^n\mathbb{R})$ is a dense set, the claim holds for every x_0 which is a point of approximate continuity for all the functions $x \mapsto \int \varphi(y) d\alpha_x(y)$ with $\varphi \in \mathcal{D}$).

By a diagonal argument we get sets $\tilde{\Omega}^k = \Omega^k_{r(k)}$ such that

(f) $D\mathbf{1}_{\tilde{\Omega}^k} \to D\mathbf{1}_H$ and $\mathrm{Per}\,(\tilde{\Omega}^k, \mathcal{B}_1) \to \mathrm{Per}\,(H, \mathcal{B}_1)$;

(g) $\mathcal{H}^n \llcorner \partial^*\tilde{\Omega}^k \otimes \delta_{T_x\partial^*\tilde{\Omega}^k} \to \mathcal{H}^n \llcorner \partial^* H \otimes \alpha_{x_0}$.

Let $e_1 = (1, 0, \ldots 0)$ and ν be the exterior unit normal to $\partial^*\Omega^k$. Then (f) implies

$$\lim_{k\to\infty} \int_{\partial^*\tilde{\Omega}^k} \|\nu - e_1\|^2 = \lim_{k\to\infty} \left(2\mathcal{H}^n(\partial^*\tilde{\Omega}^k) - 2\int_{\partial^*\tilde{\Omega}^k} \langle \nu, e_1 \rangle \right) \;=\; 0\,.$$

This obviously gives $\mathcal{H}^n \llcorner \partial^*\tilde{\Omega}^k \otimes \delta_{T_x\partial^*\tilde{\Omega}^k} \to \mathcal{H}^n \llcorner \partial^* H \otimes \delta_{T_0\partial^* H}$, which together with (c) and (g) gives $\alpha_{x_0} = \delta_{T_0\partial^* H} = \delta_{T_{x_0}\partial^*\Omega}$, which is indeed the Claim (Cl). □

7.4 Proof of Proposition 4.6

Consider the varifolds V^j and the diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$ of Section 7.1. Observe that $\tilde{\Gamma}^j$ is obtained from Γ^j through a suitable homotopy which leaves everything fixed outside An. Consider $An(x, \varepsilon, r(x) - \varepsilon)$ containing An. It follows from the a.m. property of $\{\Gamma^j\}$ that $\{\tilde{\Gamma}^j\}$ is also a.m. in $An(x, \varepsilon, r(x) - \varepsilon)$.

Note next that if a sequence is a.m. in an open set U and U' is a second open set contained in U, then the sequence is a.m. in U' as well. This trivial observation and the discussion above implies that $\tilde{\Gamma}^j$ is a.m. in any $An \in \mathcal{AN}_{r(x)}(x)$.

Fix now an annulus $An' = An(x, \varepsilon, r(x) - \varepsilon) \supset\supset An$. Then $M = An' \cup (M \setminus An)$. For any $y \in M \setminus An$ (and $y \neq x$) consider $r'(y) := \min\{r(y), \mathrm{d}(y, An)\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $\Gamma^j \cap An'' = \tilde{\Gamma}^j \cap An''$, and hence $\{\tilde{\Gamma}^j\}$ is a.m. in An''. If $y \in An'$, then we can set $r'(y) = \min\{r(y), \mathrm{d}(y, \partial An')\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $An'' \subset An'$ and, since $\{\tilde{\Gamma}^j\}$ is a.m. in An' by the argument above, $\{\tilde{\Gamma}^j\}$ is a.m. in An''.

We next show that $\tilde{V}$ is a replacement for V in An. By Theorem 3.3, $\tilde{V}$ is a stable minimal hypersurface in An. It remains to show that $\tilde{V}$ is stationary. $\tilde{V}$ is obviously stationary in $M \setminus An$, because it coincides with V there. Let next $An' \supset\supset An$. Since $\{An', M \setminus An\}$ is a covering of M, we can subordinate a partition of unity $\{\varphi_1, \varphi_2\}$ to it. By the linearity of the first variation, we get $[\delta\tilde{V}](\chi) = [\delta\tilde{V}](\varphi_1\chi) + [\delta\tilde{V}](\varphi_2\chi) = [\delta\tilde{V}](\varphi_1\chi)$. Therefore it suffices to show that $\tilde{V}$ is stationary in An'. Assume, by contradiction, that there is $\chi \in \mathcal{X}_c(An')$ such that $[\delta\tilde{V}](\chi) \leq -C < 0$ and denote by ψ the isotopy defined by $\frac{\partial\psi(x,t)}{\partial t} = \chi(\psi(x,t))$. We set

$$\tilde{V}(t) \; := \; \psi(t)_\sharp \tilde{V}\,, \qquad \Sigma^j(t) \; = \; \psi(t, \tilde{\Gamma}^j)\,. \tag{7.19}$$

By continuity of the first variation there is $\varepsilon > 0$ such that $\delta\tilde{V}(t)(\chi) \leq -C/2$ for all $t \leq \varepsilon$. Moreover, since $\Sigma^j(t) \to \tilde{V}(t)$ in the sense of varifolds, there is J such that

$$[\delta\Sigma^j(t)](\chi) \leq -\frac{C}{4} \quad \text{for } j > J \text{ and } t \leq \varepsilon\,. \tag{7.20}$$

Integrating (7.20) we conclude $\mathcal{H}^n(\Sigma^j(t)) \leq \mathcal{H}^n(\tilde{\Gamma}^j) - Ct/8$ for every $t \in [0, \varepsilon]$ and $j \geq J$. This contradicts the a.m. property of $\tilde{\Gamma}^j$ in An', for j large enough.

Finally, observe that $\mathcal{H}^n(\tilde{\Gamma}^j) \leq \mathcal{H}^n(\Gamma^j)$ by construction and

$$\liminf_n (\mathcal{H}^n(\tilde{\Gamma}^j) - \mathcal{H}^n(\Gamma^j)) \geq 0\,,$$

because otherwise we would contradict the a.m. property of $\{\Gamma^j\}$ in An. We thus conclude that $\|V\|(M) = \|\tilde{V}\|(M)$. □

8 The regularity of varifolds with replacements

In this chapter we prove Proposition 4.8. We recall that we adopt the convention of Definition 3.5. We first list several technical facts from geometric measure theory.

8.1 Maximum principle

The first one is just a version of the classical maximum principle.

Theorem 8.1. *(i) Let V be a stationary varifold in a ball $\mathcal{B}_r(0) \subset \mathbb{R}^{n+1}$. If $\mathrm{supp}(V) \subset \{z_{n+1} \geq 0\}$ and $\mathrm{supp}(V) \cap \{z_{n+1} = 0\} \neq \emptyset$, then $\mathcal{B}_r(0) \cap \{z_{n+1} = 0\} \subset \mathrm{supp}(V)$.*

(ii) Let W be a stationary varifold in an open set $U \subset M$ and K be a smooth strictly convex closed set. If $x \in \mathrm{supp}(V) \cap \partial K$, then $\mathrm{supp}(V) \cap B_r(x) \setminus K \neq \emptyset$ for every positive r.

(i) is a very special case of the general result of [73]. (ii) is proved for $n = 2$ in Appendix B of [18]. The proof can be translated with the obvious modifications to our situation. For the reader's convenience we include the proof in Section 8.5.

8.2 Tangent cones

The second device is a fundamental tool of geometric measure theory. Consider a stationary varifold $V \in \mathcal{V}(U)$ with $U \subset M$ and fix a point $x \in \mathrm{supp}(V) \cap U$. For any $r < \mathrm{Inj}\,(M)$ consider the rescaled exponential map $T_r^x : \mathcal{B}_1 \ni z \mapsto \exp_x(rz) \in B_r(x)$, where $\exp_x$ denotes the exponential map with base point x. We then denote by $V_{x,r}$ the varifold $(T^x_r)^{-1}_\sharp V \in \mathcal{V}(\mathcal{B}_1)$. Then, as a consequence of the monotonicity formula, one concludes that for any sequence $\{V_{x,r_n}\}$ there exists a subsequence converging to a stationary varifold V^* (stationary for the Euclidean metric!), which in addition is a cone (see Corollary 42.6 of [69]). Any such cone is called *tangent cone to V in x*. For varifolds with the replacement property, the following is a fundamental step towards the regularity (first proved by Pitts for $n \leq 5$ in [56]).

Lemma 8.2. *Let V be a stationary varifold in an open set $U \subset M$ having a replacement in any annulus $An \in \mathcal{AN}_{r(x)}(x)$ for some positive function r. Then:*

- *V is integer rectifiable;*
- *$\theta(x, V) \geq 1$ for any $x \in U$;*
- *any tangent cone C to V at x is a minimal hypersurface for general n and (a multiple of) a hyperplane for $n \leq 6$.*

Proof. First of all, by the monotonicity formula there is a constant C_M such that

$$\frac{\|V\|(B_\sigma(x))}{\sigma^n} \leq C_M \frac{\|V\|(B_\rho(x))}{\rho^n} \tag{8.1}$$

for all $x \in M$ and all $0 < \sigma \leq \rho < \mathrm{Inj}\,(M)$. Fix $x \in \mathrm{supp}\,(\|V\|)$ and $0 < r < \min\{r(x)/2, \mathrm{Inj}\,(M)/4\}$. Next, we replace V with V' in the annulus $An(x, r, 2r)$. We observe that $\|V'\| \not\equiv 0$ on $An(x, r, 2r)$, otherwise there would be $\rho \leq r$ and ε such that $\mathrm{supp}\,(\|V'\|) \cap \partial B_\rho(x) \neq \emptyset$ and $\mathrm{supp}\,(\|V'\|) \cap An(x, \rho, \rho+\varepsilon) = \emptyset$. By the choice of ρ, this would contradict Theorem 8.1(ii).

Thus we have found that $V' \llcorner An(x, r, 2r)$ is a non-empty stable minimal hypersurface and hence there is $y \in An(x, r, 2r)$ with $\theta(y, V') \geq 1$. By (8.1),

$$\begin{aligned} \frac{\|V\|(B_{4r}(x))}{(4r)^n} &= \frac{\|V'\|(B_{4r}(x))}{(4r)^n} \geq \frac{\|V'\|(B_{2r}(y))}{(4r)^n} \\ &\geq \frac{\omega_n}{2^n C_M}\, \theta(y, V') \geq \frac{\omega_n}{2^n C_M}\,. \end{aligned}$$

Hence, $\theta(x, V)$ is uniformly bounded away from 0 on supp$(\|V\|)$ and Allard's rectifiability theorem (see Theorem 42.4 of [69]) gives that V is rectifiable.

Let C denote a tangent cone to V at x and $\rho_k \to 0$ a sequence with $V_{x,\rho_k} \to C$. Note that C is stationary. Let V_k' be a replacement of V in $An(x, \lambda\rho_k, (1-\lambda)\rho_k)$, where $\lambda \in (0, 1/4)$, and set $W_k' = (T^x_{\rho_k})^{-1}_\sharp V_k'$. Up to subsequences we have $W_k' \to C'$ for some stationary varifold C'. By the definition of a replacement we obtain

$$\begin{aligned} C' &= C \quad \text{in } \mathcal{B}_\lambda \cup An(0, 1-\lambda, 1)\,; \\ \|C'\|(\mathcal{B}_\rho) &= \|C\|(\mathcal{B}_\rho) \quad \text{for } \rho \in (0,\lambda) \cup (1-\lambda, 1)\,. \end{aligned} \tag{8.2}$$

Moreover, since C is cone,

$$\frac{\|C'\|(\mathcal{B}_\sigma)}{\sigma^n} = \frac{\|C'\|(\mathcal{B}_\rho)}{\rho^n} \quad \text{for all } \rho, \sigma \in (0,\lambda) \cup (1-\lambda, 1)\,. \tag{8.3}$$

By the monotonicity formula for stationary varifolds in Euclidean spaces, (8.3) implies that C' as well is a cone (see for instance 17.5 of [69]). Moreover, by Theorem 3.3, $C' \llcorner An(0, \lambda, 1-\lambda)$ is a stable embedded minimal hypersurface. Since C and C' are integer rectifiable, the conical structure of C implies that supp(C) and supp(C') are closed cones (in the usual meaning for sets) and the densities $\theta(\cdot, C)$ and $\theta(\cdot, C')$ are 0-homogeneous functions (see Theorem 19.3 of [69]). Thus (8.2) implies $C = C'$ and hence that C is a stable minimal hypersurface in $An(0, \lambda, 1-\lambda)$. Since λ is arbitrary, C is a stable minimal hypersurface in the punctured ball. Thus, if $n \leq 6$, by Simons' theorem (see Theorem B.2 in [69]) C is in fact a multiple of a hyperplane. If instead $n \geq 7$, since $\{0\}$ has dimension $0 \leq n-7$, C is a minimal hypersurface in the whole ball $\mathcal{B}_1$ (recall Definition 3.5). □

8.3 Unique continuation and two technical lemmas on varifolds

To conclude the proof we need yet three auxiliary results. All of them are justified in Section 8.5. The first one is a consequence of the classical unique continuation for minimal surfaces.

Theorem 8.3. *Let U be a smooth open subset of M and $\Sigma_1, \Sigma_2 \subset U$ two connected* smooth *embedded minimal hypersurfaces with $\partial\Sigma_i \subset \partial U$. If Σ_1 coincides with Σ_2 in some open subset of U, then $\Sigma_1 = \Sigma_2$.*

The other two are elementary lemmas for stationary varifolds.

Lemma 8.4. *Let $r < Inj(M)$ and V a stationary varifold. Then*

$$\mathrm{supp}(V) \cap \overline{B}_r(x) \;=\; \overline{\bigcup_{0<s<r} \mathrm{supp}(V \llcorner B_s(x)) \cap \partial B_s(x)}\,. \qquad (8.4)$$

Lemma 8.5. *Let $\Gamma \subset U$ be a relatively closed set of dimension n and S a closed set of dimension at most $n-2$ such that $\Gamma \setminus S$ is a smooth embedded hypersurface. Assume Γ induces a varifold V which is stationary in U. If Δ is a connected component of $\Gamma \setminus S$, then Δ induces a stationary varifold.*

8.4 Proof of Proposition 4.8

The proof consists of five steps.

Step 1: Setup. Let $x \in M$ and $\rho \le \min\{r(x)/2, \mathrm{Inj}\,(M)/2\}$. Then we choose a replacement V' for V in $An(x, \rho, 2\rho)$ coinciding with a stable minimal embedded hypersurface Γ'. Next, choose $s \in (0, \rho)$ and $t \in (\rho, 2\rho)$ such that $\partial B_t(x)$ intersects Γ' transversally. Then we pick a second replacement V'' of V' in $An(x, s, t)$, coinciding with a stable minimal embedded hypersurface Γ'' in the annulus $An(x, s, t)$. Now we fix a point $y \in \partial B_t(x) \cap \Gamma'$ that is a regular point of Γ' and a radius $r > 0$ sufficiently small such that $\Gamma' \cap B_r(y)$ is topologically an n-dimensional ball in M and $\gamma = \Gamma' \cap \partial B_t(x) \cap B_r(y)$ is a smooth $(n-1)$-dimensional surface. This can be done due to our regularity assumption on y. Then we choose a diffeomorphism $\zeta : B_r(y) \to \mathcal{B}_1$ such that

$$\zeta(\partial B_t(x)) \subset \{z_1 = 0\} \quad \text{and} \quad \zeta(\Gamma'') \subset \{z_1 > 0\}\,,$$

where $z_1, \ldots, z_{n+1}$ are orthonormal coordinates in $\mathcal{B}_1$. Finally suppose

$$\begin{aligned} \zeta(\gamma) &= \{(0, z_2, \ldots, z_n, g'((0, z_2, \ldots, z_n))\}\,, \\ \zeta(\Gamma') \cap \{z_1 \le 0\} &= \{(z_1, \ldots, z_n, g'((z_1, \ldots, z_n))\} \end{aligned}$$

for some smooth function g'. Note that

- any kind of estimates (like curvature estimates or area bound or monotonicity) for a minimal surface $\Gamma \subset B_r(y)$ translates into similar estimates for the surface $\zeta(\Gamma)$;
- varifolds in $B_r(y)$ are pushed forward to varifolds in $\mathcal{B}_1$ and there is a natural correspondence between tangent cones to V in ξ and tangent cones to $\zeta_\sharp V$ in $\zeta(\xi)$.

We will use the same notation for the objects in $B_r(y)$ and their images under ζ.

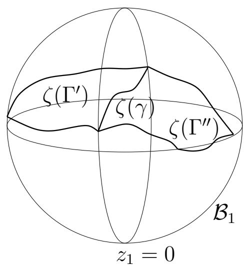

Figure 8.1: The surfaces Γ', Γ'' and γ in the coordinates z.

Step 2: Tangent cones. We next claim that any tangent cone to V'' at any point $w \in \gamma$ is a unique flat space. Note that all these w are regular points of Γ'. Therefore by our transversality assumption every tangent cone C at w coincides in $\{z_1 < 0\}$ with the half-space $T_w\Gamma' \cap \{z_1 < 0\}$. We wish to show that C coincides with $T_w\Gamma'$. By the constancy theorem (see Theorem 41.1 in [69]), it suffices to show $\text{supp}(C) \subset T_w\Gamma'$.

Note first that if $z \in T_w\Gamma' \cap \{z_1 = 0\}$ is a regular point for C, then by Theorem 8.3, C coincides with $T_w\Gamma'$ in a neighborhood of z. Therefore, if $z \in \text{supp}(C) \cap \{z_1 = 0\}$, either z is a singular point, or $C = T_w\Gamma'$ in a neighborhood of z. Assume now by contradiction that $p \in \text{supp}(C) \setminus T_w\Gamma'$. Since, by Lemma 8.2 and the fact that Γ'' has replacements due to Proposition 4.7, $\text{Sing}\, C$ has dimension at most $n-7$, we can assume that p is a regular point of C. Consider next a sequence N^j of smooth open neighborhoods of $\text{Sing}\, C$ such that $T_w\Gamma' \setminus \overline{N}^j$ is connected and $N^j \to \text{Sing}\, C$. Let Δ^j be the connected component of $C \setminus \overline{N}^j$ containing p.

Then Δ^j is a smooth minimal surface with $\partial\Delta^j \subset \partial N^j$. We conclude that Δ^j cannot touch $\{z_1 = 0\}$: it would touch it in a regular point of $\mathrm{supp}(C) \cap \{z_1 = 0\}$ and hence it would coincide with $T_w\Gamma' \setminus \overline{N}^j$, which is impossible because it contains p. If we let $\Delta = \cup\Delta^j$, then Δ is a connected component of the regular part of C, which does not intersect $\{z_1 = 0\}$. Let W be the varifold induced by Δ: by Lemma 8.5 W is stationary. Since C is a cone, W is also a cone. Thus $\mathrm{supp}(W) \ni 0$. On the other hand $\mathrm{supp}(W) \subset \{z_1 \geq 0\}$. Thus, by Theorem 8.1(i), $\{z_1 = 0\} \subset \mathrm{supp}(W)$. But this would imply that $\{z_1 = 0\} \cap T_w\Gamma'$ is in the singular set of C: this is a contradiction because the dimension of $\{z_1 = 0\} \cap T_w\Gamma'$ is $n-1$.

Step 3: Graphicality. In this step we show that the surfaces Γ' and Γ'' can be "glued" together at $\partial B_t(x)$, that is

$$\Gamma'' \subset \Gamma' \text{ in } B_t(x) \setminus B_{t-\varepsilon}(x) \text{ for some } \varepsilon > 0\,. \tag{8.5}$$

For this we fix $z \in \gamma$ and, using the notation of Step 2, consider the (exterior) unit normal $\tau(z)$ to the graph of g'. Let $T_r^z \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the dilation of the $(n+1)$-space given by

$$T_r^z(\bar{z}) = \frac{\bar{z} - z}{r}\,.$$

By Step 2 we know that any tangent cone to V'' at z is given by the tangent space $T_z\Gamma'$ and therefore the rescaled surfaces $\Gamma_r = T_r^z(\Gamma'')$ converge to the half-space $H = \{v \colon \tau(z) \cdot v = 0, v_1 > 0\}$. We claim that this implies that we have

$$\lim_{\bar{z} \to z, \bar{z} \in \Gamma''} \frac{|(\bar{z} - z) \cdot \tau(z)|}{|\bar{z} - z|} = 0 \tag{8.6}$$

uniformly on compact subsets of γ. We argue by contradiction and assume the claim is wrong. Then there is a sequence $\{z_j\} \subset \Gamma''$ with $z_j \to z$ and $|(z_j - z) \cdot \tau(z)| \geq k|z_j - z|$ for some $k > 0$. We can assume that z_j is a regular point of Γ'' for all $j \in \mathbb{N}$. We set $r_j = |z_j - z|$, then there is a positive constant $\bar{k}$ such that $\mathcal{B}_{2\bar{k}r_j}(z_j) \cap H = \emptyset$. This implies that $\mathrm{d}(H, \mathcal{B}_{\bar{k}r_j}(z_j)) \geq \bar{k}r_j$. By the minimality of Γ'' we can apply the monotonicity formula and find

$$\|V''\|(\mathcal{B}_{\bar{k}r_j}(z_j)) \geq C\bar{k}^n r_j^n$$

for some positive constant C depending on the diffeomorphism ζ. In other words, there is a considerable amount of the varifold that is far from the half-space H. But this contradicts the fact that the corresponding full space is the only tangent cone. We also point out that this convergence is uniform on compact subsets of γ.

Now we denote by ν the smooth normal field to Γ'' with $\nu\cdot(0,\ldots,0,1)\geq 0$. Let Σ be the space $\{(0,\alpha_1,\ldots,\alpha_n) : \alpha_i \in \mathbb{R}\}$. Then we assume that $z_j \to z$, set $r_j = \mathrm{d}(z_j,\Sigma)$ and define the rescaled hypersurfaces $\Gamma_j = T^{z_j}_{r_j}(\Gamma'' \cap \mathcal{B}_{r_j}(z_j))$. Then all the Γ_j are smooth stable minimal surfaces in $\mathcal{B}_1$, thus we can apply Theorem 3.3 to extract a subsequence that converges to a stable minimal hypersurface in the ball $\mathcal{B}_{1/2}$. But by (8.6) we know that this limit surface is simply $T_z\Gamma' \cap \mathcal{B}_{1/2}$. Since the convergence is in the C^1 topology we have

$$\lim_{\bar{z}\to z, \bar{z}\in\Gamma''} \nu(\bar{z}) = \tau(z)\,.$$

Again this convergence is uniform in compact subsets of γ.

For any $z \in \gamma$ Theorem 3.3 gives us a radius $\sigma > 0$ and a function $g'' \in C^2(\{z_1 \geq 0\})$ with

$$\begin{aligned} \Gamma'' \cap B_\sigma(z) &= \{(z_1,\ldots,z_n,g''(z_1,\ldots,z_n)) : z_1 > 0\}\,;\\ g''(0,z_2,\ldots,z_n) &= g'(0,z_2,\ldots,z_n)\,;\\ Dg''(0,z_2,\ldots,z_n) &= Dg'(0,z_2,\ldots,z_n)\,. \end{aligned}$$

Using elliptic regularity theory (see [37]), we conclude that g' and g'' are the restriction of a smooth function g giving a minimal surface Δ. Using now Theorem 8.3, we conclude that $\Delta \subset \Gamma'$, and hence that Γ'' is a subset of Γ' in a neighborhood of z. Since this is vaild for every $z \in \gamma$, we conclude (8.5).

Step 4: Regularity in the annuli. In this step we show that V is a minimal hypersurface in the punctured ball $B_\rho(x)\setminus\{x\}$. First of all we prove

$$\Gamma' \cap An(x,\rho,t) = \Gamma'' \cap An(x,\rho,t)\,.$$

Assume for instance that $p \in \Gamma''\setminus\Gamma'$. Without loss of generality we can assume that p is a regular point. Let then Δ be the connected component of $\Gamma''\setminus(\mathrm{Sing}\,\Gamma''\cup\mathrm{Sing}\,\Gamma')$ containing p. Δ is necessarily contained in $\overline{B}_{t-\varepsilon}(x)$, otherwise by (8.5) and Theorem 8.3, Δ would coincide with a connected component of $\Gamma'\setminus(\mathrm{Sing}\,\Gamma''\cup\mathrm{Sing}\,\Gamma')$ contradicting $p \in \Gamma''\setminus\Gamma'$. But then Δ induces, by Lemma 8.5, a stationary varifold V, with

$\text{supp}(V) \subset \overline{B}_{t-\varepsilon}(x)$. So, for some $s \le t-\varepsilon$, we have $\partial B_s(x) \cap \text{supp}(V) \neq \emptyset$ and $\text{supp}(V) \subset \overline{B}_s(x)$, contradicting Theorem 8.1(ii). This proves $\Gamma'' \subset \Gamma'$. Precisely the same argument can be used to prove $\Gamma' \subset \Gamma''$.

Thus we conclude that $\Gamma' \cup \Gamma''$ is in fact a minimal hypersurface in $An(x, s, 2\rho)$. Since s is arbitrary, this means that Γ' is in fact contained in a larger minimal hypersurface $\Gamma \subset B_{2\rho}(x) \setminus \{x\}$ and that, moreover, $\Gamma'' \subset \Gamma$ for any second replacement V'', whatever the choice of s (t being instead fixed) is.

Fix now such a V'' and note that $V'' \llcorner B_s(x) = V \llcorner B_s(x)$. Note, moreover, that by Theorem 8.1(ii) we necessarily conclude

$$\text{supp}(V \llcorner B_s(x)) \cap \partial B_s(x) \ \subset \ \overline{\Gamma''} \ \subset \ \Gamma \,.$$

Thus, using Lemma 8.4, we conclude $\text{supp}(V) \subset \Gamma$, which hence proves the desired regularity of V.

Step 5: Conclusion. The only thing left to analyize are the centers of the balls $B_\rho(x)$ of the previous steps. Clearly, if $n \ge 7$, we are done because by the compactness of M we only have to add possibly a finite set of points, that is a 0-dimensional set, to the singular set. In other words, the centers of the balls can be absorbed in the singular set.

If, on the other hand, $n \le 6$, we need to show that x is a regular point. If $x \notin \text{supp}\,(\|V\|)$, we are done, so we assume $x \in \text{supp}\,(\|V\|)$. By Lemma 8.2 we know that every tangent cone is a multiple $\theta(x, V)$ of a plane (note that $n \le 6$). Consider the rescaled exponential maps of Section 8.2 and note that the rescaled varifolds V_r coincide with $(T^x_r)^{-1}(\Gamma) = \Gamma_r$. Using Theorem 3.3 we get the C^1-convergence of subsequences in $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$ and hence the integrality of $\theta(x, V) = N$.

Fix geodesic coordinates in a ball $B_\rho(x)$. Thus, given any small positive constant c_0, if $K \in \mathbb{N}$ is sufficiently large, there is a hyperplane π_K such that, on $An(x, 2^{-K-2}, 2^{-K})$, the varifold V is the union of $m(K)$ disjoint graphs of Lipschitz functions over the plane π_K, all with Lipschitz constants smaller than c_0, counted with multiplicity $j_1(K), \ldots, j_m(K)$, with $j_1 + \ldots + j_m = N$. We do not know a priori that there is a *unique* tangent cone to V at x. However, if K is sufficiently large, it follows that the tilt between two consecutive planes π_K and π_{K+1} is small. Hence $j_i(K) = j_i(K+1)$ and the corresponding Lipschitz graphs do join, forming m disjoint smooth minimal surfaces in the annulus $An(x, 2^{-K-3}, 2^{-K})$, topologically equivalent to n-dimensional annuli. Repeating the process inductively, we find that $V \llcorner B_\rho(x) \setminus \{x\}$ is in fact the union of m smooth disjoint minimal hypersurfaces $\Gamma^1, \ldots, \Gamma^m$ (counted with multiplicities $j_1 + \ldots + j_m = N$), which are all, topologically, punctured n-dimensional balls.

Since $n \geq 2$, by Lemma 8.5, each Γ^i induces a stationary varifold. Every tangent cone to Γ^i at x is a hyperplane and, moreover, the density of Γ^i (as a varifold) is everywhere equal to 1. We can therefore apply Allard's regularity theorem (see [1]) to conclude that each Γ^i is regular. On the other hand, the Γ^i are disjoint in $B_r(x) \setminus \{x\}$ and they contain x. Therefore, if $m > 1$, we contradict the classical maximum principle. We conclude that $m = 1$ and hence that x is a regular point for V.

8.5 Proofs of the technical lemmas

8.5.1 Proof of (ii) in Theorem 8.1

For simplicity assume that $M = \mathbb{R}^{n+1}$. The proof can be easily adapted to the general case. Let us argue by contradiction; so assume that there are $x \in \text{supp}(\|W\|)$ and $B_r(x)$ such that $(B_r(x) \setminus \overline{K}) \cap \text{supp}(\|W\|) = \emptyset$. Given a vector field $\chi \in C^\infty_c(U, \mathbb{R}^{n+1})$ and an n-plane π we set

$$\text{Tr}\,(D\chi(x), \pi) \;=\; D_{v_1}\chi(x) \cdot v_1 + \cdots + D_{v_n}\chi(x) \cdot v_n$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal base for π. Recall that the first variation of W is given by

$$\delta W(\chi) \;=\; \int_{G(U)} \text{Tr}\,(D\chi(x), \pi)\, dW(x, \pi)\,.$$

Take a decreasing function $\eta \in C^\infty([0,1])$ which vanishes on $[3/4, 1]$ and is identically 1 on $[0, 1/4]$. Denote by φ the function given by $\varphi(x) = \eta(|y - x|/r)$ for $y \in B_r(x)$. Take the interior unit normal ν to ∂K in x, and let z_t be the point $x + t\nu$. If we define vector fields ψ_t and χ_t by

$$\psi_t(y) \;=\; \frac{y - z_t}{|y - z_t|} \qquad \text{and} \qquad \chi_t = \varphi\psi_t\,,$$

then χ_t is supported in $B_r(x)$ and $D\chi_t = \varphi D\psi_t + \nabla\varphi \otimes \psi_t$. Moreover, by the strict convexity of the subset K,

$$\nabla\varphi(y) \cdot \nu > 0 \qquad \text{if } y \in \overline{K} \cap B_r(x) \text{ and } \nabla\varphi(y) \neq 0\,.$$

Note that ψ_t converges to ν uniformly in $B_r(x)$, as $t \uparrow \infty$. Thus, $\psi_T(y) \cdot \nabla\varphi(y) \geq 0$ for every $y \in \overline{K} \cap B_r(x)$, provided T is sufficiently large. This yields that

$$\mathrm{Tr}\,(\nabla\varphi(y) \otimes \psi_T(y), \pi) \geq 0 \qquad \text{for all } (y,\pi) \in G(B_r(x) \cap \overline{K})\,. \tag{8.7}$$

Note that $\mathrm{Tr}\,(D\psi_t(y), \pi) > 0$ for all $(y,\pi) \in G(B_r(x))$ and all $t > 0$. Thus

$$\begin{aligned}
\delta W(\chi_T) &= \int_{G(B_r(x)\cap\overline{K})} \mathrm{Tr}\,(D\chi_T(y),\pi)\, dW(y,\pi) \\
&\overset{(8.7)}{\geq} \int_{G(B_r(x)\cap\overline{K})} \mathrm{Tr}\,(\varphi(y) D\psi_T(y),\pi)\, dW(y,\pi) \\
&\geq \int_{G(B_{r/4}(x)\cap\overline{K})} \mathrm{Tr}\,(D\psi_T(y),\pi)\, dW(y,\pi) \;>\; 0\,.
\end{aligned}$$

This contradicts that W is stationary and completes the proof.

8.5.2 Proof of Theorem 8.3

Let $W \subset U$ be the maximal open set on which Σ_1 and Σ_2 coincide. If $W \neq U$, then there is a point $p \in \overline{W} \cap U$. In a ball $B_\rho(p)$, Σ_2 is the graph of a smooth function w over Σ_1 (as usual, we use normal coordinates in a regular neighborhood of Σ_1). By a straightfoward computation, w satisfies a differential inequality of the form $|A^{ij} D^2_{ij} w| \leq C(|Dw| + |w|)$ where A is a smooth function with values in symmetric matrices, satisfying the usual ellipticity condition $A^{ij}\xi_i\xi_j \geq \lambda|\xi^2|$, where $\lambda > 0$. Let $x \in W$ be such that $\mathrm{d}(x,p) < \varepsilon$. Then w vanishes at infinite order in x and hence, according to the classical result of Aronszajn (see [12]), $w \equiv 0$ on a ball $B_r(x)$ where r depends on λ, A, C and $\mathrm{d}(x, \partial B_\rho(p))$, but not on ε. Hence, by choosing $\varepsilon < r$ we contradict the maximality of W.

8.5.3 Proof of Lemma 8.4

Let T be the set of points $y \in \mathrm{supp}(V)$ such that the approximate tangent plane to V in y is transversal to the sphere $\partial B_{|y-x|}(x)$. The claim follows from the density of T in $\mathrm{supp}(V)$. Again, this is proved for $n = 2$ in Appendix B of [18] (see Lemma B.2 therein). We include the proof here with the small (and obvious adjustments). Since V is integer rectifiable, V is supported on a rectifiable n-dimensional set R and there exists a Borel function $h : R \to \mathbb{N}$ such that $V = hR$. Assume that the lemma is false; then there exists $y \in B_\rho(x) \cap \mathrm{supp}(\|V\|)$ and $t > 0$ such that

- the tangent plane to R in z is tangent to $\partial B_{\mathrm{d}(z,x)}(x)$, for any $z \in B_t(y)$.

We choose t so that $B_t(y) \subset B_\rho(x)$. Denote the polar coordinates in $B_\rho(x)$ by $(r, \theta, \varphi_1, \dots, \varphi_{n-1})$ and let f be a smooth nonnegative function in $C^\infty_c(B_t(y))$ with $f = 1$ on $B_{t/2}(y)$. Denote by χ the vector field

$$\chi(r, \theta, \varphi_1, \dots, \varphi_{n-1}) = f(r, \theta, \varphi_1, \dots, \varphi_{n-1}) \tfrac{\partial}{\partial r} \,.$$

For every $z \in R \cap B_t(y)$, the plane π tangent to R in z is also tangent to the sphere $\partial B_{\mathrm{d}(z,x)}(x)$. Hence, an easy computation yields that

$$\mathrm{Tr}\,(D\chi, \pi)(z) = \frac{nf(z)}{\mathrm{d}(z,x)} \,.$$

This gives

$$[\delta V](\chi) = \int_{R \cap B_t(y)} \frac{nh(z) f(z)}{\mathrm{d}(z,x)} \, d\mathcal{H}^n(z) \; > \; C \|V\|(B_{t/2}(y)) \,,$$

for some positive constant C. Since $y \in \mathrm{supp}(\|V\|)$, we have

$$\|V\|(B_{t/2}(y)) \; > \; 0 \,.$$

This contradicts that V is stationary.

8.5.4 Proof of Lemma 8.5

Set $\Gamma_r := \Gamma \setminus S$ and denote by H the mean curvature of Γ_r and by ν the unit normal to Γ_r. Obviously $H = 0$. Let V' be the varifold induced by Δ. We claim that

$$[\delta V'](\chi) \; = \; \int_\Delta \mathrm{div}_\Delta \, \chi \; = \; - \int_\Delta H \chi \cdot \nu \tag{8.8}$$

for any vector field $\chi \in \mathcal{X}_c(U)$.

The first identity is the classical computation of the first variation (see Lemma 9.6 of [69]). To prove the second identity, fix a vector field χ and a constant $\varepsilon > 0$. Without loss of generality we assume $S \subset \Gamma$. By the definition of the Hausdorff measure, there exists a covering of S with balls $B_{r_i}(x_i)$ centered on $x_i \in S$ such that $r_i < \varepsilon$ and $\sum_i r_i^{n-1} \leq \varepsilon$. By the compactness of $S \cap \mathrm{supp}(\chi)$ we can find a finite covering $\{B_{r_i}(x_i)\}_{i\in\{1,\ldots,N\}}$. Fix smooth cut-off functions φ_i with

- $\varphi_i = 1$ on $M \setminus B_{2r_i}(x_i)$ and $\varphi_i = 0$ on $B_{r_i}(x_i)$;
- $0 \leq \varphi_i \leq 1$, $|\nabla\varphi_i| \leq C r_i^{-1}$.

(Note that C is in fact only a geometric constant.) Then $\chi_\varepsilon := \chi\Pi\varphi_i$ is compactly supported in $U \setminus S$. Thus,

$$\int_\Delta \mathrm{div}_\Delta\, \chi_\varepsilon \;=\; -\int_\Delta H\chi_\varepsilon \cdot \nu \tag{8.9}$$

The RHS of (8.9) obviously converges to the RHS of (8.8) as $\varepsilon \to 0$. As for the left hand side, we estimate

$$\begin{aligned} \int_\Delta |\mathrm{div}_\Delta(\chi - \chi_\varepsilon)| &\leq \sum_i \int_{B_{r_i}(x_i)\cap\Delta} (\|\nabla\chi\|_{C^0} + \|\chi\|_{C^0}\|\nabla\varphi_i\|_{C^0}) \\ &\leq \sum_i \|V\|(B_{r_i}(x_i))\|\chi\|_{C^1}(1 + Cr_i^{-1}) \qquad (8.10) \\ &\leq C\|\chi\|_{C^1}\sum_i (r_i^n + Cr_i^{n-1}) \;<\; C\varepsilon \end{aligned}$$

where the first inequality in the last line follows from the monotonicity formula. We thus conclude that the LHS of (8.9) converges to the LHS of (8.8).

9 Improved convergence and an index bound in 3-manifolds

In this chapter we restrict our discussion to the case $n = 2$, that is, surfaces in 3-manifolds. There is a special interest in the methods presented in the previous chapters for the case of 3-manifolds due to the rich interplay between minimal surface theory and the topology of 3-manifolds (see for instance [46], [58]). In [57] Pitts and Rubinstein claimed a bound on the genus of the minimal surface obtained by the min-max method in terms of the genera of an approximating critical sequence (see the precise statement below). Building on [72] this claim was finally proved in [27]. The second claim of [57] concerns the index of instability of the min-max surface (see the precise statement below). In the classical situation of the mountain pass lemma the critical point has index 1 (unless there is some nullity). It is therefore reasonable to expect that this will hold in the present situation. The proof, however, should be more involved since no Palais-Smale condition can be applied and the convergence of the critical sequence is in a very weak sense. In this chapter we refine the analysis of the min-max surface (under somewhat more restrictive conditions on the sweepouts). In the first section we collect the results of [27] and the claims of [57]. In the second section we show that it is possible to choose a minimizing sequence such that *every* min-max sequence (with a certain rate of convergence) converges to a smooth minimal surface. In the third section we use this result to prove in some very simple situations that the min-max surface has index at least one. Even though this result is still very far from a proof of the claims in [57], we introduce some ideas that might be helpful. Finally, in the fourth section we show that we can deform the min-max sequence in such a way that it converges in the Hausdorff sense and still has the almost minimality property.

9.1 Genus and index bounds: the claims

In Section 2.1.1 we introduced the relevant version of the min-max construction. Following the discussion of Remark 2.10, in this chapter we will consider smooth families that satisfy an additional regularity assumption.

Definition 9.1. *We call a generalized smooth family* $\{\Sigma_t\}$ *a* regular family of (type Σ) *if*

(r1) $T = \{0,1\}$*;*

(r2) there is an ambient isotopy $\Phi : (0,1)\times M \to M$ *and a smooth surface* Σ *such that* $\Sigma_t = \Phi(t,\Sigma)$*.*

The reason for this restriction is more of technical nature, to make the regularity of the families and their deformations coincide. We use it in some proofs. The results, we believe, should not depend on it.

Remark 9.2. *As pointed out in Remark 2.10, in the two-dimensional theory this notion is still sufficient. One only has to make sure that there is a saturated set* Λ *with* $m_0(\Lambda) > 0$*. In many situations our manifold* M *will be a sphere and the sweepouts consist of spheres or tori. In these cases, the same argument as in Section 2.1.1 works.*

In [18] a proof of the following theorem, the analogon of Theorem 2, is given.

Theorem 9.3. *Let* M *be a closed Riemannian* 3*-manifold. For any saturated set* Λ*, there is a min-max sequence* $\{\Sigma^j_{t_j}\}$ *converging in the sense of varifolds to a smooth embedded minimal surface* Σ *with area* $m_0(\Lambda)$ *(multiplicity is allowed).*

Now we write $\Sigma = \sum_{i=1}^N n_i\Gamma^i$, where the Γ^i are the connected components of Σ, counted without multiplicity, and $n_i \in \mathbb{N}\setminus\{0\}$. Moreover, we denote by $\mathcal{O}$ the set of orientable components and by $\mathcal{N}$ the set of the unorientable ones. With these notions we can state the main result of [27].

Theorem 9.4. *Let* Λ*,* $\Sigma^j_{t_j}$*,* Σ *as in Theorem 9.3. Then*

$$\sum_{\Gamma^i\in\mathcal{O}} \mathbf{g}(\Gamma^i) + \frac{1}{2}\sum_{\Gamma^i\in\mathcal{N}} (\mathbf{g}(\Gamma^i) - 1) \le \liminf_{j\to\infty} \mathbf{g}(\Sigma^j_{t_j})\,. \tag{9.1}$$

This result is not the one annouced in [57], where on the left hand side also the multiplicities appear, but see Section 10 of [27] for a discussion. The claim of [57] concerning the index is the following.

Claim 1. *Let Σ be as before. Then*

$$\sum_{\Gamma^i \in \mathcal{O}} n_i \mathrm{Index}(\Gamma^i) + \sum_{\Gamma^i \in \mathcal{N}} \frac{n_i}{2} \mathrm{Index}(\Gamma^i) \leq 1$$
$$\leq \sum_{\Gamma^i \in \mathcal{O}} n_i (\mathrm{Index}(\Gamma^i) + \mathrm{Nullity}(\Gamma^i)) + \sum_{\Gamma^i \in \mathcal{N}} \frac{n_i}{2} (\mathrm{Index}(\Gamma^i) + \mathrm{Nullity}(\Gamma^i)) \, .$$

In particular, together with the bumpy metric theorem of Brian White (see [76]) this would yield that on a Riemannian manifold the min-max method generically gives an index 1 embedded minimal surface.

9.2 Good minimizing sequences

One main step in the proof of Theorems 9.3 and 9.4 is the fact that there is a min-max sequence that is almost minimizing in sufficiently small annuli. This was the key ingredient to prove the smoothness of the limit. Recall that in the theory of [18] almost minimality is defined analogously to Definition 4.3 but using isotopies as the deformation families. In this section we show that we can modifiy the corresponding minimizing sequence in such a way that *all* min-max sequences that converge with a certain rate have smooth limits.

Basically, the idea is to use the techiques of Proposition 6.4 to deform all the surfaces that are not almost minimal to considerably less area such that in the end only those surfaces stay close in area to m_0 (depending on j: the rate) that are almost minimizing.

Unfortunately, there are some technical difficulties since it might happen that almost minimizing surfaces loose this property as collateral damage of the deformations.

We start by proving a version of the above sketched strategy. Due to the mentioned difficulties the formulation of the lemma is slightly more complicated.

Lemma 9.5. *Let $\{\{\Sigma_t\}^j\}$ a minimizing sequence such that*

(i) for all $j \in \mathbb{N}$ $\{\Sigma_t\}^j$ is a regular family of type Σ^j via an isotopy Φ_j;

(ii) $\mathcal{F}(\{\Sigma_t\}^j) \leq m_0 + \frac{1}{8j}$;

(iii) all min-max limits are stationary varifolds.

Then, for all j*, there is* $\varepsilon_j > 0$ *such that for all small* $\delta > 0$ *there is a family* $\{\Gamma_t\}^j$ *such that for* $t \in (0,1)$ *with* $\mathcal{H}^2(\Gamma_t^j) \geq m_0 - \varepsilon_j$ *there are two open sets* $U_{t,i} = U_{t,i}^{j,\delta}$, $i = 1, 2$, *with diameter less than* 4δ *such that*

(a) $\Gamma_t^j \cap U_{t,i}^c = \Sigma_t^j \cap U_{t,i}^c$, *for* $i = 1, 2$;

(b) Σ_t^j *is* $\frac{1}{j}$*-a.m. in all pairs of open sets in* $\mathcal{CO}$ *with diameter less than* 2δ.

Proof. The idea of the proof is to run a variant of the argument of the Almgren-Pitts combinatorial lemma (see Proposition 6.4, see also Proposition 5.3 in [18]). There, under the assumption that all the large time slices are not almost minimizing in pairs, the argument is used to bring down the area of all these large time slices to construct a competitor family that contradicts the definition of m_0. The variant we want to use here only brings down the area of the large time slices that are not almost minimizing in pairs. There is, however, a catch in this strategy. One can not guarantee that after the deformation a time slice that was almost minimizing in a specific set before still has this property (see Figure 9.1). Since we only change in the small sets $U_{t,i}$ at time t, the original surface needs to be almost minimizing even there because otherwise the area would have been brought down too much. That is the reason why the statement is about the original sequence. Taking sets of small diameter assures that the two sequences do not differ too much.

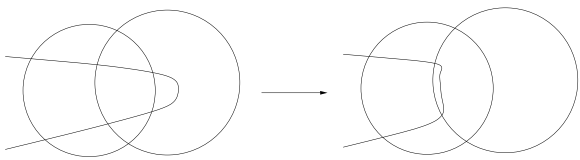

Figure 9.1: On the left, the curve is almost minimizing in the left ball, but not in the right one. After the deformation it is no longer almost minimizing in the left ball.

Step 1: The set K. To run the combinatorial argument we need a good finite covering of the set of bad time slices. In the original proof the set of large time slices (with a certain lower bound) is compact, so everything is straight forward. In our situation the set of times of non-almost-minimality is not compact. However, considering the (compact) closure still gives us a sort of non-almost-minimality with slightly different constants. More precisely, we define for a fixed $j \in \mathbb{N}$ and $\delta > 0$

$$K_0 := \left\{ t \in [0,1] : \Sigma_t^j \text{ is not } \frac{1}{j}\text{-a.m. in a pair of balls in } \mathcal{CO} \text{ of radius } \delta \right\}.$$

We denote by K the closure of K_0 and note that K is compact.

Claim. *Let $t \in K$ and $(B_{t,1}, B_{t,2})$ the pair of balls where Σ_t^j is not almost minimal. Then for every small $\varepsilon >$ and $\gamma > 0$ there are isotopies $\phi^i_{t,\varepsilon,\gamma}$, $i = 1, 2$, with the following properties:*

- $\phi^i_{t,\varepsilon,\gamma}(0, \Sigma_t^j) = \Sigma_t^j$;
- $\operatorname{supp}\left(\phi^i_{t,\varepsilon,\gamma}\right) \subset (B_{i,t})_\gamma = \{x \in M : \mathrm{d}(x, B_{i,t}) < \gamma\}, \quad s \in [0,1]$;
- $\mathcal{H}^2(\phi^i_{t,\varepsilon,\gamma}(s, \Sigma_t^j)) \leq \mathcal{H}^2(\Sigma_t^j) + \varepsilon + \frac{1}{8j}, \quad s \in [0,1]$;
- $\mathcal{H}^2(\phi^i_{t,\varepsilon,\gamma}(1, \Sigma_t^j)) \leq \mathcal{H}^2(\Sigma_t^j) + \varepsilon - \frac{1}{j}$.

To prove the claim let $t \in K$. Then there is a sequence $t_l \to t$ such that $t_l \in K_0$. To construct the isotopies $\phi^i_{t,\varepsilon,\gamma}$ the idea is to start from Σ_t^j, then follow the family to $\Sigma_{t_l}^j$ for some t_l close enough and then use the isotopies given by the non-almost-minimality of $\Sigma_{t_l}^j$.

Let $x_{l,1}$ and $x_{l,2}$ be the centers of balls of non-almost-minimality of $\Sigma_{t_l}^j$. By the compactness of M there is a subsequence of $\{t_l\}_l$ (we keep the notation) and two points x_1 and x_2 such that $x_{l,i} \to x_i$ for $i = 1, 2$. Thus for all $\gamma > 0$ there is $L \in \mathbb{N}$ with $B(x_{l,i}) \subset\subset B(x_i)_{\frac{\gamma}{2}}$ for $l \geq L$ where all the balls $B(y)$ have radius δ. By the isotopy version of the freezing lemma (Step 1 in Lemma 6.1) there are isotopies Ψ^i such that for τ close enough to t the following properties hold

- $\Psi^i(\tau, \Sigma^j) \cap B(x_i)^c_\gamma = \Sigma^j_\tau \cap B(x_i)^c_\gamma$;
- $\Psi^i(\tau, \Sigma^j) \cap B(x_i)_{\frac{\gamma}{2}} = \Sigma^j_t \cap B(x_i)_{\frac{\gamma}{2}}$;

- $|\mathcal{H}^2(\Psi^i(\tau, \Sigma^j)) - \mathcal{H}^2(\Sigma^j_t)| \leq \frac{\varepsilon}{2}$.

Next we choose $l \geq L$ such that t_l is close enough to t for the above to hold. Denote by ψ^i_l the isotopies given by the non-almost-minimality of $\Sigma^j_{t_l}$ in $B(x_{l,i}) \subset\subset B(x_i)_{\frac{\gamma}{2}}$. Then we glue the isotopies Ψ^i running from t to t_l and ψ^i_l. Finally, we smooth these isotopies in the time parameter (if necessary) with an error of area of at most $\frac{\varepsilon}{2}$. By construction these isotopies satisfy all the requirements of the claim.

Step 2: The combinatorial argument. Fix $\gamma > 0$ so small that for all pairs of balls $(B_1, B_2) \in \mathcal{CO}$ of radius δ the pair $((B_1)_\gamma, (B_2)_\gamma)$ is still in $\mathcal{CO}$. Then for all $t \in K$ there are a pair of balls $(B_{1,t}, B_{2,t}) \in \mathcal{CO}$ and isotopies φ^i_t such that the properties in the claim hold for our choice of γ and a fixed $\varepsilon > 0$ that is much smaller than $\frac{1}{8j}$. From this point we can copy the proof of Proposition 5.3 in [18] except for the fact that we replace $(B_{1,t}, B_{2,t})$ by $((B_{1,t})_\gamma, (B_{2,t})_\gamma)$. Hence, we find a regular family $\{\Gamma_t\}^j$ of type Σ^j such that

$$\begin{aligned} t \notin K \quad &\Rightarrow \quad \mathcal{H}^2(\Gamma^j_t) \leq \mathcal{H}^2(\Sigma^j_t) + \frac{1}{2j} \\ t \in K \quad &\Rightarrow \quad \mathcal{H}^2(\Gamma^j_t) \leq \mathcal{H}^2(\Sigma^j_t) - \frac{1}{4j}\,. \end{aligned}$$

In particular, the family $\{\Gamma_t\}^j$ has the following property: if $\mathcal{H}^2(\Gamma^j_t) \geq m_0 - \frac{1}{8j}$, then there are at most two balls $B(y_1)$ and $B(y_2)$ of radius $\delta + \gamma$ where Γ^j_t differs from Σ^j_t. Since all the non-almost-minimizing time slices are deformed to surfaces of area below the chosen threshold, Σ^j_t is indeed almost minimizing in all pairs in $\mathcal{CO}$ for t with $\mathcal{H}^2(\Gamma^j_t) \geq m_0 - \frac{1}{8j}$. Defining $U_{t,i} = B_{2\delta}(y_i)$ gives the result of the proposition for balls. Note that almost minimality is preserved under restriction to subsets. Thus, the conclusion holds not only for pairs of balls, but all pairs of open sets of diameter less than 2δ. □

Remark 9.6. *In general, Γ^j_t with $\mathcal{H}^2(\Gamma^j_t) \geq m_0 - \frac{1}{8j}$ is not necessarily almost minimizing in all pairs in $\mathcal{CO}$. Clearly it is where the surface coincides with Σ^j_t. However, even if Σ^j_t was almost minimizing in a ball, this property cannot be assumed to hold after the deformation if the ball in consideration is affected by the deformation (see Figure 9.1). Therefore* *the result of the previous lemma is the best we can hope for using this strategy.*

The following proposition is a version of Proposition 4.4 that shows that the almost minimality in pairs of sets of small diameter is still sufficient for the regularity theory.

Proposition 9.7. *Let* $0 < \delta < Inj(M)$ *and assume that* Σ^j *is* $\frac{1}{j}$*-almost minimizing in pairs* $(U, V) \in \mathcal{CO}$ *of open sets with*

$$\max\{\text{diam}(U), \text{diam}(V)\} < 2\delta\,.$$

Then there is a subsequence $\{\Sigma^{k(j)}\}$ *and a smooth embedded minimal surface* Σ *such that*

(i) $\Sigma^{k(j)} \to \Sigma$ *in the sense of varifolds;*

(ii) the genus bound (9.1) *holds.*

Proof. The proof of this lemma is basically contained in [18], [27]. The only difference is in the restriction to sets with diameter less than 2δ. It is sufficient to show that under this slightly weaker assumption we can still conclude almost minimality in sufficiently small annuli as the rest of the proof is then the same.

Fix $k \in \mathbb{N}$ and $r > 0$ with $0 < 9r < \delta$. Then for all $x \in M$ the pair $(B_r(x), B_\delta(x) \setminus \bar{B}_{9r}(x))$ is in $\mathcal{CO}$ and the sets have small enough diameter. Therefore, by assumption, Σ^j is $\frac{1}{j}$-a.m in $B_r(x)$ or $B_\delta(x) \setminus \bar{B}_{9r}(x)$. So, we have

(a) either Σ^j is $\frac{1}{j}$-a.m. in $B_r(y)$ for all $y \in M$;

(b) or there is $x^j_r \in M$ such that Σ^j is $\frac{1}{j}$-a.m. in $B_\delta(x^j_r) \setminus \bar{B}_{9r}(x^j_r)$.

If (a) holds for some $r > 0$ and some subsequence $\{\Sigma^{k(j)}\}$, we are done. Otherwise there are $\{x^l_k\}^\infty_{l,k=1}$ such that

(c) for k, l large enough, Σ^l is $\frac{1}{l}$-a.m. in $B_\delta(x^l_k) \setminus \bar{B}_{\frac{1}{k}}(x^l_k)$;

(d) $x^l_k \to x_k, \quad x_k \to x$.

We conclude that, for any K there is J_K such that Σ^j is $\frac{1}{j}$-a.m. in $B_\delta(x) \setminus \bar{B}_{\frac{1}{K}}(x)$ for all $j \geq J_K$. Therefore, if $y \in B_\delta(x) \setminus \{x\}$, we choose $r(y)$ such that $B_{r(y)}(y) \subset\subset B_\delta(x) \setminus \{x\}$, whereas $r(x)$ is chosen arbitrarily (but smaller than δ). It follows that $An \subset\subset B_\delta(x) \setminus \{x\}$, for any $An \in \mathcal{AN}_{r(z)}(z)$ with $z \in B_\delta(x)$. Hence, $\{\Sigma^j\}$ is $\frac{1}{j}$-a.m. in An, provided j is large enough. Repeating this argument finitely many times, starting with $M \setminus B_\delta(x)$, we get the almost minimality in sufficiently small annuli in all points $z \in M$. So far the proof was except for the restriction of the diameter a line by line copy of the proof of Proposition 4.4. In the two-dimensional case of [18], for the regularity theory (in particular to apply [45]), in addition one needs that in every annulus An Σ^j is a smooth surface for j large enough. We recall Remark 2.9. Each Σ^j is smooth except at finitely many points. We denote by P_j the set of singular points of Σ^j. After extracting another subsequence we can assume that P_j is converging, in the Hausdorff topology, to a finite set P. If $x \in P$ and An is any annulus centered at x, then $P_j \cap An = \emptyset$ for j large enough. If $x \notin P$ and An is any (small) annulus centered at x with outer radius less than $\mathrm{d}(x, P)$, then $P_j \cap An = \emptyset$ for j large enough. Thus, after possibly modifying the function r above, the sequence $\{\Sigma^j\}$ satisfies all the necessary conditions for the regularity theory of [18]. □

Notation 9.8. *To simplify the upcoming discussion we fix our notation for this section.*

- *We call a minimizing sequence* $\{\{\Sigma_t\}^j\}$ good *if it satisfies (i),(ii) and (iii) of Lemma 9.5.*
- *We denote by* $\{\{\Gamma_t^{j,\delta}\}\}$ *the minimizing sequence constructed from* $\{\{\Sigma_t\}^j\}$ *as in Lemma 9.5 with parameter* δ.
- *Moreover we set* $\varepsilon_j = \frac{1}{8j}$. *Finally we set*

$$G_{j,\delta} = \{t \in [0,1] : \mathcal{H}^2(\Gamma_t^{j,\delta}) \geq m_0 - \varepsilon_j\}.$$

Remark 9.9. *We can summarize the results obtained so far in this section as follows: Let* $\{\{\Sigma_t\}^j\}$ *be good,* $\delta > 0$. *If* $t_j \in G_{j,\delta}$, *then* $\Sigma^j_{t_j}$ *converges to a smooth embedded minimal surface and the genus bound holds.*

Since the condition encoded in $G_{j,\delta}$ is one for the deformed minimizing sequence and the conclusion is about the original one, this is still not satisfactory. The two sequences differ only on small sets of diameter 4δ. Therefore, we would like to let δ go to 0. To be able to do this, we will need the compactness theorem of Choi-Schoen (see Theorem 3.9).

Theorem 9.10. *Let M be a closed Riemannian 3-manifold and $\{\{\Sigma_t\}^j\}$ a good minimizing sequence. Then there is a minimizing sequence $\{\{\Gamma_t\}^j\}$ such that Γ_t^j is isotopic to Σ_t^j with the following property: If $\{\Gamma_{t_{k(j)}}^{k(j)}\}$ is a min-max sequence with*

(i) $\mathcal{H}^2(\Gamma_{t_{k(j)}}^{k(j)}) \geq m_0 - \varepsilon_{k(j)}$*;*

(ii) $\Gamma_{t_{k(j)}}^{k(j)} \to V$ *in the sense of varifolds;*

(iii) $g := \liminf_{j\to\infty} \mathbf{g}(\Gamma_{t_{k(j)}}^{k(j)}) < \infty$,

then V is a smooth embedded minimal surface and the genus bound (9.1) *holds.*

Proof. **Step 1: Approximation.** For the moment we fix $\delta > 0$. Then, if $\{t_j\}$ is a sequence with $t_j \in G_{j,\delta}$, $\{\Sigma_{t_j}^j\}$ is a min-max sequence with the property that $\Sigma_{t_j}^j$ is $\frac{1}{j}$-a.m. in *all* pairs of sets in $\mathcal{CO}$ with diameter less than 2δ. Thus, by Proposition 9.7, there is an embedded minimal surface Γ^δ satisfying the genus bound (9.1) such that $\Sigma_{t_j}^j \to \Gamma^\delta$ in the sense of varifolds. Note that the genus of Γ^δ (counted without multiplicity) is bounded independently from $\delta > 0$ by g once the subsequence $\{k(j)\}$ is fixed. We choose a (not relabeled) subsequence such that this holds and therefore can be assumed for any further subsequence we take. In view of (*iii*) we can assume that $g < \infty$. We introduce the following notation

$$\mathcal{M}_g = \{\text{smooth embedded minimal surfaces in } M \text{ with genus (when counted without multiplicity) bounded by } g\}\,.$$

Then we claim the following: For all $\delta > 0$ there is $k^\delta(j) \geq j$ such that for all $t \in G_{k^\delta(j),\delta}$ we have

$$\mathcal{D}(\Sigma_t^l, \mathcal{M}_g) < \frac{1}{j}, \quad l \geq k^\delta(j)\,.$$

For, if not, we can find sequences $t_j \in G_{j,\delta}$ such that $\Sigma^{k(j)}_{t_j}$ is bounded away from $\mathcal{M}_g$. A contradiction to the above argument.

Next, we take a sequence $\delta_j \to 0$ and consider $\{\Gamma^{j,\delta_j}_{t_j}\}$ with $t_j \in G_{j,\delta_j}$. We choose a subsequence according to the following iterative scheme:

1. Step: We set $k(1) = k^{\delta_1}(1)$.
2. Step: Assume that we have chosen $k(1), \ldots, k(j)$, then we choose

$$k(j+1) = \max\left\{k^{\delta_{j+1}}(j+1), k(j)+1\right\}\,.$$

In this way we obtain a sequence of smooth embedded minimal surfaces Γ^j and a subsequence $\Sigma^{k(j)}_{t_j}$ such that

$$\mathcal{D}\left(\Sigma^{k(j)}_{t_{k(j)}}, \Gamma^j\right) < \frac{1}{j}\,.$$

By construction, $\Gamma^{k(j),\delta_{k(j)}}_{t_{k(j)}}$ differs from $\Sigma^{k(j)}_{t_{k(j)}}$ only in two disjoint open sets $U^{j,i}$, $i = 1, 2$, with diameter less than $4\delta_{k(j)}$. Therefore we also have

$$\mathcal{D}\left(\Gamma^{k(j),\delta_{k(j)}}_{t_{k(j)}} \llcorner (U^{j,1} \cup U^{j,2})^c, \Gamma^j \llcorner (U^{j,1} \cup U^{j,2})^c\right) < \frac{1}{j}\,. \tag{9.2}$$

From now on we assume $k(j) = j$ to simplify the notation. Moreover we omit the reference to δ_j in the notation for $\Gamma^{j,\delta_j}_{t_j}$ as we now always consider this situation.

Step 2: Convergence. Assume we have a varifold V such that $\Gamma^j_{t_j} \to V$ in the sense of varifolds. This can always be achieved by taking a further subsequence. We would like to apply Theorem 3.9 to the sequence $\{\Gamma^j\}$ and conclude that the limit is smooth and coincides with V. We have

$$\Gamma^j = \sum_{k=1}^{N_j} m_{j,k} \Gamma^{j,k}\,,$$

where the $\Gamma^{j,k}$ are the connected components and $m_{j,k}$ are the multiplicities. We have $\text{Area}(\Gamma^j) \le C$ for all j large enough. We order the connected components by their area in decreasing order. There might be two problems. It might happen that $N_j \to \infty$ or $m_{j,k} \to \infty$ as $j \to \infty$

for a fixed k. In any of these cases $\mathrm{Area}(\Gamma^{j,k}) \to 0$. Now we parametrize $\Gamma^{j,k}$ conformally. Then the parametrization $s_{j,k}$ is harmonic and the area equals the energy. Then by a result in [61] there is $\varepsilon_0 > 0$ such that $E(s_{j,k}) < \varepsilon_0$ implies that $E(s_{j,k}) = 0$. Therefore $\Gamma^{j,k}$ is a point for j large. Since the varifold does not see points, we can assume that none of the two mentioned problems occurs and we have a uniform bound on the number of connected components and the multiplicities. On the other hand, we know that $\mathbf{g}(\Gamma^{j,k})$ (counted without multiplicity) is uniformly bounded by g. Thus we can apply Theorem 3.9 and find a smooth embedded minimal surface Γ such that $\Gamma^j \to \Gamma$. Moreover the genus bound holds, i.e.

$$\mathbf{g}(\Gamma) \leq \liminf_j \mathbf{g}(\Gamma^j) \leq \liminf_j \mathbf{g}(\Gamma^j_{t_j}), \tag{9.3}$$

where we know the second inequality to hold only if we disregard the multiplicities of Γ^j.

We still have to prove that $V = \Gamma$ (where we identify the surface with the associated varifold). In order to do so, we note that we can assume that $U^{j,i} = B_{2\delta_j}(x_{j,i})$. Taking a suitable subsequence (not relabeled), we have $x_{j,i} \to x_i$. Note that x_1 and x_2 might coincide. Now, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $U^{j,i} \subset\subset B_\varepsilon(x_i)$ for all $j \geq N$, $i = 1, 2$. Let $x \in \mathrm{supp}(V) \setminus \{x_1, x_2\}$. Then, for all j large enough and $r > 0$ small enough, $B_r(x) \cap (U^{j,1} \cup U^{j,2}) = \emptyset$. Therefore,

$$\begin{aligned}\mathcal{D}(\Gamma \llcorner B_r(x), V \llcorner B_r(x)) \quad \leq \quad & \mathcal{D}(\Gamma \llcorner B_r(x), \Gamma^j \llcorner B_r(x)) \\ & + \mathcal{D}(\Gamma^j \llcorner B_r(x), \Gamma^j_{t_j} \llcorner B_r(x)) \\ & + \mathcal{D}(\Gamma^j_{t_j} \llcorner B_r(x), V \llcorner B_r(x)) .\end{aligned}$$

The first term clearly converges to 0, so does the third one. Finally the second one converges to 0 due to (9.2). This shows that $\mathrm{supp}(V) \setminus \{x_1, x_2\}$ coincides with the smooth embedded minimal surface Γ. To conclude we need to show that x_1 and x_2 are removable singularities. If $x_i \in \overline{\Gamma \setminus \{x_1, x_2\}} = \Gamma$, then the singularity is clearly removable. But the monotonicity formula implies that this is the only possibility. This together with (9.3) gives the claimed result. $\square$

9.3 The index of min-max surfaces

In this section we show how to use Theorem 9.10 to deduce information about the index of instability of min-max surfaces. One key ingredient is the following result by Brian White [77].

Theorem 9.11. *Let M be a closed Riemannian manifold, $\Sigma \subset M$ a smooth compact embedded minimal hypersurface that is strictly stable. Then there are an open subset $U \subset M$ containing Σ such that* $\text{Area}(\Sigma) < \text{Area}(\Sigma')$ *for all currents Σ' homologuos to Σ in U.*

The result we want to prove is the following.

Theorem 9.12. *Consider (S^3, g). Let Λ be a family of regular sweepouts of type S^2. Then one of the following two cases holds:*

(i) There is an embedded minimal 2-sphere with $\text{Area} \leq \frac{m_0(\Lambda)}{2}$.

(ii) There is an embedded minimal 2-sphere with $\text{Area} \leq m_0(\Lambda)$ *and* $\text{Index} + \text{Nullity} \geq 1$.

Remark 9.13. *In particular, if the multiplicity of the min-max surface of the theorem is one, then situation (ii) occurs.*

Proof. Let $\{\{\Sigma_t\}^j\}$ a good minimizing sequence and $\{\{\Gamma_t\}^j\}$ the corresponding minimizing sequence of Theorem 9.10. Then we pick a min-max sequence $\{\Gamma^j_{t_j}\}$ such that $\text{Area}(\Gamma^j_{t_j}) \in (m_0 - \varepsilon_j, m_0)$. By Theorem 9.10 there are a subsequence (not relabeled) and a smooth embedded minimal 2-sphere or an embedded projective plane Γ such that $\Gamma^j_{t_j} \to \Gamma$ in the sense of varifolds. The latter is impossible for topological reasons. Thus we only need to consider the case of the sphere. Now there are two possibilities. Either Γ has multiplicity one or it has higher multiplicity. In the latter case, Γ counted without multiplicity one has $\text{Area} \leq \frac{m_0}{2}$. This gives case (i). Therefore from now on we can assume that the multiplicity is one. To deduce that in this situation we arrive at case (ii), we argue by contradiction. We assume strict stability of Γ and find a contradiction to Theorem 9.11.

Step 1: Surgery. The result of Theorem 9.11 only gives us information about competitors lying in an L^∞-neighborhood of Γ. The varifold convergence $\Gamma^j_{t_j} \to \Gamma$, however, is too weak to guarantee that $\Gamma^j_{t_j}$ lies in this neighborhood for j large. To handle this defect, we show that an appropriate surgery produces a valid competitor with good properties. A precise description of the type of surgery we use is given in Section 2 of [27]. More precisely, Proposition 2.3 therein asserts that, for each $\varepsilon > 0$ small enough and j large enough, we can find a surface $\tilde{\Gamma}^j_{t_j}$ obtained from $\Gamma^j_{t_j}$ through surgery (cutting away necks and discarding connected components) and satisfying the following properties:

(a) $\tilde{\Gamma}^j_{t_j}$ is contained in $T_{2\varepsilon}\Gamma$;

(b) $\tilde{\Gamma}^j_{t_j} \cap T_\varepsilon \Gamma = \Gamma^j_{t_j} \cap T_\varepsilon \Gamma$.

Here $T_\delta \Gamma$ denotes the tubular δ-neighborhood. If we do this procedure carefully, we can even get more

(c) $\mathcal{H}^2(\tilde{\Gamma}^j_{t_j}) \leq \mathcal{H}^2(\Gamma^j_{t_j})$.

To see this, we consider $\gamma^j_t = \Gamma^j_{t_j} \cap \partial T^+_t \Gamma$. $T^+_t \Gamma$ means that we only consider one of the two boundary components (the other one is treated in the same way). By Sard's lemma γ^j_t is the union of curves for a.e. $t \in (\varepsilon, 2\varepsilon)$. The coarea formula gives

$$\int_\varepsilon^{2\varepsilon} \mathrm{Length}(\gamma^j_t) \leq \eta \,.$$

Note that $\eta > 0$ can be chosen arbitrarily small due to the varifold convergence $\Gamma^j_{t_j} \to \Gamma$. If we now perform surgery at the level t (one of the a.e. "good" t), then we get

$$\mathcal{H}^2(\tilde{\Gamma}^j_{t_j}) \leq \mathcal{H}^2(\Gamma^j_{t_j}) + C\,\mathrm{Length}(\gamma^j_t)^2 - \int_t^{2\varepsilon} \mathrm{Length}(\gamma^j_\tau)\,d\tau + \alpha \,.$$

Here $C > 0$ is the isoperimetric constant and $\alpha > 0$ is an arbitrarily small constant coming from the fact that we might have to smooth the new surface. To obtain (c) we need to find a "good" time slice such that

$$C\,\mathrm{Length}(\gamma^j_t)^2 - \int_t^{2\varepsilon} \mathrm{Length}(\gamma^j_\tau)\,d\tau < 0 \,.$$

We argue by contradiction and assume that

$$C\,\mathrm{Length}(\gamma^j_t)^2 - \int_t^{2\varepsilon} \mathrm{Length}(\gamma^j_\tau)\,d\tau \geq 0 \quad \text{for a.e. } t \,. \tag{9.4}$$

We define $f(t) := \int_t^{2\varepsilon} \mathrm{Length}(\gamma_\tau^j)\, d\tau$. This gives $\mathrm{Length}(\gamma_t^j) = -f'(t) \geq 0$. Thus, by (9.4), we obtain the differential inequality $f(t) \leq C(-f'(t))^2$. This can be resolved to give

$$-\frac{d}{dt}\left(f(t)^{\frac{1}{2}}\right) = -\frac{f'(t)}{2f(t)^{\frac{1}{2}}} \geq \frac{1}{C'}\,.$$

Integrating between ε and 2ε we get

$$\eta \geq \int_\varepsilon^{2\varepsilon} \mathrm{Length}(\gamma_t^j)\, dt = f(\varepsilon) - f(2\varepsilon) \geq \left(\frac{\varepsilon}{C'}\right)^2\,.$$

This gives a contradiction because $\eta > 0$ can be chosen arbitrarily small (and C is independent of that choice). Therefore (c) is established. (A similar argument is contained in the filgree lemma in [7].)

Step 2: Currents. We want to show that the new sequence $\{\tilde{\Gamma}^j_{t_j}\}$ leads to a contradiction to Theorem 9.11. First we point out that due to (b) we still have $\tilde{\Gamma}^j_{t_j} \to \Gamma$ in the sense of varifolds. Moreover, we have assumed that the mulitiplicity of Γ is 1. Therefore in the limit no cancellation of mass can happen if we regard the $\tilde{\Gamma}^j_{t_j}$ as currents. To be more precise, consider V_j (resp. V) the varifold induced by $\tilde{\Gamma}^j_{t_j}$ (resp. Γ) and T_j (resp. T) the associated currents. Note that by assumption all the varifolds have multiplicity one. By the compactness of integer currents there is a subsequence (not relabeled) of $\{T_j\}$ and an integer current S such that $T_j \to S$ as currents.

Claim. $S = T$.

To prove this claim, denote the support of S by N. Clearly, $N \subset \Gamma$ due to the inequality $\|V(S)\| \leq \|V\|$ as measures that follows from the lower semicontinuity of the mass of currents. Here $V(S)$ denotes the varifold induced by S. Therefore, by the constancy theorem, $S = m[\Gamma]$, where $m \in \mathbb{Z}$. Again, by the semicontinuity of the mass of currents, $m \in \{-1, 0, 1\}$. Denote by π the projection onto Γ, orthogonal with respect to the normal coordinates given by the tubular neighborhood. After the choice of an orientation, above $\|T\|$-a.e. every point $x \in \Gamma$ there are an odd number of points of $\tilde{\Gamma}^j_{t_j}$ sitting in $\pi^{-1}(\{x\})$ that add up with signs to 1. This is essentially Sard's lemma (and degree theory).

Thus $\pi_\# T_j = T$. Therefore $\lim_{j\to\infty} \pi_\# T_j = T$. On the other hand, the projection commutes with the limit, that is,

$$S = \pi_\# S = \pi_\#(\lim_{j\to\infty} T_j) = \lim_{j\to\infty} \pi_\# T_j = T\,.$$

This establishes the claim.

Step 3: Conclusion. To apply Theorem 9.11, we need to know that the $\tilde{\Gamma}^j_{t_j}$ are homologous to Γ in $T_\varepsilon\Gamma$ (or at least one of them). We compute the homology groups of $T_\varepsilon\Gamma$. We make two observations:

(1) $T_{2\varepsilon}\Gamma$ is homeomorphic to $\Gamma \times (-1,1)$ (in fact even diffeomorphic);

(2) $\Gamma \times (-1,1)$ is homotopic to $\Gamma \times \{0\}$.

Combining these facts implies that the homology groups $H_i(\Gamma,\mathbb{Z})$ and $H_i(T_{2\varepsilon}\Gamma,\mathbb{Z})$ are isomorphic. Since Γ is a 2-sphere, this gives

$$H_i(T_{2\varepsilon}\Gamma,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,2 \\ 0 & \text{else}\,. \end{cases}$$

Moreover, Γ is a generator of $H_2(T_{2\varepsilon}\Gamma,\mathbb{Z})$. This is due to the fact that the multiplicity of Γ is one. Now assume that none of the $\tilde{\Gamma}^j_{t_j}$ is homologous to Γ. Because they have all multiplicity one, none of them can be homologous to a multiple cover of Γ and they are all in the homology class of the current $T' = 0$. But then also the limit is in the same homology class. But this is a contradiction to the fact that Γ is a generator of the homology group. So at least one $\tilde{\Gamma}^j_{t_j}$ is homologous to Γ in $T_{2\varepsilon}\Gamma$.

Finally, if we choose $\varepsilon > 0$ small enough, then $T_{2\varepsilon}\Gamma$ is contained in the L^∞-neighborhood of Theorem 9.11. But then the theorem gives

$$\mathcal{H}^2(\tilde{\Gamma}^j_{t_j}) > \mathcal{H}^2(\Gamma) = m_0 > \mathcal{H}^2(\Gamma^j_{t_j}) \geq \mathcal{H}^2(\tilde{\Gamma}^j_{t_j})\,.$$

This is clearly a contradiction. Therefore Γ cannot be strictly stable, which gives (*ii*) of the theorem. □

Corollary 9.14. *Consider* (S^3, g). *Let* Λ *be a family of regular sweepouts of type* T^2, *where* T^2 *is the torus. Then one of the following two cases holds:*

(i) There is an embedded minimal 2*-sphere or torus with* Area $\leq \frac{m_0(\Lambda)}{2}$.

(ii) *There is an embedded minimal* 2*-sphere or torus with* Area $\leq m_0(\Lambda)$ *and* Index + Nullity ≥ 1.

Proof. Again for topological reasons we know that the min-max surface contains no non-orientable connected component. We only have to see where in the proof of Theorem 9.12 we used the assumption that we considered sweepouts by spheres. Then it is clear that by the genus bound each connected component has to be a sphere or a torus, hence this ambiguity in the statements. For this argument we can assume that Γ is connected.

The surgery-step might decrease the genus, i.e. transform a torus into a sphere. But this does not infect the rest of the argument. The second step still works and we get the convergence as currents. Finally, the third step involves some soft topological argument. Assume that $\{\Gamma^j\}$ is the min-max sequence after surgery converging to Γ in the sense of currents. Since Γ is a sphere or a torus, in particular oriented, by the same argument as in the proof of Theorem 9.12 we have $H_i(\Gamma, \mathbb{Z}) \cong H_i(T_{2\varepsilon}\Gamma, \mathbb{Z})$. Note that in both cases $H_2(\Gamma, \mathbb{Z}) = \mathbb{Z}$ and Γ is a generator of $H_2(T_{2\varepsilon}\Gamma, \mathbb{Z})$ due to multiplicity one. Now the rest of the argument is completely analogous to the one in Theorem 9.12. □

The upper bound of Claim 1 is more difficult, even in this particular case, and we only discuss a formal, rough idea and point out the main difficulties.

In [77] the following characterization of minimal surfaces Σ with index $k > 0$ (and no nullity) is given: There is an open subset $U \subset M$ containing Σ and a k-parameter family of surfaces Σ_v, $v \in \bar{\mathcal{B}}^k_v(0)$ such that $\Sigma = \Sigma_0$ and Σ strongly maximizes the area in this k-paramter family. Moreover, these Σ_v are graphs over Σ of functions u in the k-dimensional subspace V of $W^{1,2}$ spanned by eigenvectors of the Jacobi operator with negative eigenvalues.

To prove the index bound, the following argument seems natural. Assume by contradiction that the index is ≥ 2. Choose a good minimizing sequence $\{\{\Sigma_t\}^j\}$. For j large enough, some $\Sigma^j_{t_1}$ will come close to Σ in the varifold sense. Use surgery, to come close in a stronger sense such that $\Sigma^j_{t_1}$ will be a $W^{1,2}$-graph over Σ. Project orthogonally on V and connect in V to some $\Sigma^j_{t_2}$ that connects to the family $\{\Sigma^j_t\}$ again via surgery. If we were able to perform all these steps in such a way that we never gain too much area, this would produce a competitor family with a maximal time slice with area less than m_0, a contradiction.

There are some difficulties the be faced. First of all, in each of the steps one has to keep track of the area. For instance, such a careful surgery has been used in the proof of Theorem 9.12. Second, that surgery does not provide surfaces that are graphs over Σ. In fact, to deform the surface to a graph with the necessary control of the area seems to be quite difficult. Finally, it is not immediately clear that the deformed (and surgically treated) family can be constructed such that it is still an admissible competitor.

We investigate a bit further the second issue, that is, how to do surgery and admissible deformations to obtain a graph over Σ while controlling the gain of area. This issue seems crucial for any further steps towards index or also genus bounds.

To deal with the case of higher multiplicity in Theorem 9.12 the above argument does not apply anymore. It can very well happen that, say, a sequence of spheres that are homotopically trivial in the tubular neighborhood converge as varifolds to the minimal sphere Σ with multiplicity 2 (see Figure 9.2).

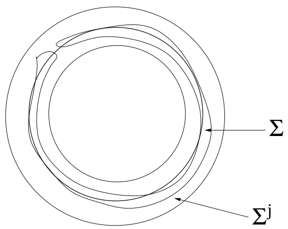

Figure 9.2: The sphere Σ with multiplicity 2 is approximated by a sphere Σ^j that is homotopically trivial in the tubular neighborhood.

Viewed as currents, however, a cancellation of mass yields that the limit would be the zero current. It would be desirable to modify this sequence by surgery and isotopies such that it becomes a double cover of Σ. More precisely, we would like to obtain two graphs over Σ, without having much more area. In this situation it is even more difficult since a priori the surfaces are not even in the same homology class in the tubular neighborhood.

Another situation where this procedure would be helpful is a proof of the genus bound (9.1) with the multiplicities on the left hand side of the inequality.

For all these reasons, we formulate the following

Goal. *There is a min-max sequence* $\{\Sigma^j_{t_j}\}$ *that can be deformed by surgery and isotopies into a sequence* $\{\tilde{\Sigma}^j_{t_j}\}$ *that converges smoothly to* Σ*. Morever, there is an appropriate control over the area in the process and the area of* $\tilde{\Sigma}^j_{t_j}$*.*

On a local level, that is, in annuli, such deformations have been used in the regularity theory of Theorem 9.3. The tools were the almost minimiality, the result of Meeks-Simon-Yau [45] about the regularity of minimizers in isotopy classes and the smooth convergence of stable surfaces as consequence of curvature estimates. For a similar global result, one possibility could be a careful pasting of local deformations. A prerequisite for such a strategy are the regularity result of Meeks-Simon-Yau and the curvature estimate of Schoen *up to the boundary.* The first has been proved in [27] (at least in balls), whereas the second is still open (but see Chapter 3.3).

9.4 Hausdorff convergence

As discussed in the previous section, in any further investigation the analysis of the convergence of the critical sequence is crucial. In general, we cannot improve this convergence. If, however, we allow modifications by suitable isotopies and surgeries on the critical sequence, then we can obtain a sequence of surfaces converging in the Hausdorff sense. In addition, and this is the improvement compared to the type of surgery applied in the previous section, the almost minimality in small annuli is conserved by these modifications. Even though this is still not sufficient to attack the questions discussed at the end of the previous section, it might be helpful for a proof of the goal that was formulated there.

First of all we prove a helpful lemma. For this we need to introduce the following notion. In the definition of almost minimality there was a ratio of the possible gain and the final loss of mass induced by an isotopy, namely $\frac{1}{8}$. This exact ratio is of course not necessary for the regularity theory as presented in [18], see also Chapters 6 and 7.

Definition 9.15. *We say that a hypersurface Σ has the (ε, δ)-almost minimizing property in $U \subset M$ if there DOES NOT exist any isotopy ψ supported in U such that $\psi(0, \cdot) = \mathrm{id}$ and*

$$\begin{aligned} \mathcal{H}^2(\psi(t, \Sigma)) &\leq \mathcal{H}^2(\Sigma) + \frac{\varepsilon}{8} - \frac{\delta}{8}\,; \\ \mathcal{H}^2(\psi(1, \Sigma)) &\leq \mathcal{H}^2(\Sigma) - \varepsilon - \delta\,. \end{aligned} \tag{9.5}$$

Now we can state and prove the following lemma that ascertains that the almost minimizing property is conserved (with slightly different constants) under singular isotopies.

Lemma 9.16. *Let $U \subset M$ open, Σ a hypersurface in M and $\Phi : [0, 1] \times M \to M$ an isotopy supported in U with*

$$\mathcal{H}^2(\Phi(t, \Sigma)) \leq \mathcal{H}^2(\Sigma) + \frac{\varepsilon}{8} - \frac{\delta_1}{8}\,, \quad t \in [0, 1]\,.$$

Let, moreover, $\Sigma' = \Sigma \llcorner U^c$, $\Gamma = \Sigma \llcorner U$ with

$$\Phi(t, \cdot)_\# \Sigma \to \Sigma' \cup (\Gamma' \cup \bar{\Gamma})\,, \quad \text{as } t \to 1\,,$$

in the sense of varifolds, where $\mathcal{H}^2(\bar{\Gamma}) = 0$ and $\Sigma' \cup \Gamma'$ is embedded, and $\mathcal{H}^2(\Sigma' \cup \Gamma') < \mathcal{H}^2(\Sigma)$.

Then the following holds: If Σ is (ε, δ_1)-almost minimizing in U, then there is $\delta_2 > \delta_1$ (arbitrarily close) such that $\Sigma' \cup \Gamma'$ is (ε, δ_2)-almost minimizing.

Proof. We argue by contradiction and assume that, for $\delta_2 > \delta_1$, there is an isotopy Ψ such that the properties (9.5) are satisfied. By the assumption that

$$\Phi(t, \cdot)_\# \Sigma \to \Sigma' \cup (\Gamma' \cup \bar{\Gamma})\,,$$

for all $\gamma > 0$, there is $\varepsilon_0 > 0$ such that

$$|\mathcal{H}^2(\Phi(1 - \varepsilon_0, \cdot)_\# \Sigma) - \mathcal{H}^2(\Sigma' \cup \Gamma')| < \gamma\,.$$

Moreover, by the continuity of the mass of varifolds we also obtain (choosing ε_0 possibly smaller)

$$|\mathcal{H}^2(\Psi(t, \cdot)_\# \Phi(1 - \varepsilon_0, \cdot)_\# \Sigma) - \mathcal{H}^2(\Psi(t, \cdot)_\# (\Sigma' \cup \Gamma'))| < \gamma$$

for all $t \in [0,1]$. Using these inequalities we get

$$\begin{aligned}\mathcal{H}^2(\Psi(t,\cdot)_\#\Phi(1-\varepsilon_0,\cdot)_\#\Sigma) &\leq \mathcal{H}^2(\Psi(t,\cdot)_\#(\Sigma'\cup\Gamma'))+\gamma \\ &\leq \mathcal{H}^2(\Sigma'\cup\Gamma')+\gamma+\frac{\varepsilon}{8}-\frac{\delta_2}{8} \\ &\leq \mathcal{H}^2(\Phi(1-\varepsilon_0,\cdot)_\#\Sigma)+2\gamma+\frac{\varepsilon}{8}-\frac{\delta_2}{8}\,.\end{aligned}$$

Analogously, we get

$$\mathcal{H}^2(\Psi(1,\cdot)_\#\Phi(1-\varepsilon_0,\cdot)_\#\Sigma) \leq \mathcal{H}^2(\Phi(1-\varepsilon_0,\cdot)_\#\Sigma)+2\gamma-\varepsilon-\delta_2\,.$$

Choosing $\gamma = \frac{\delta_2-\delta_1}{16}$, this gives that $\Phi(1-\varepsilon_0,\cdot)_\#\Sigma$ is not (ε,δ_1)-almost minimizing. By the assumption that $\mathcal{H}^2(\Sigma'\cup\Gamma') < \mathcal{H}^2(\Sigma)$ this also implies that Σ is not (ε,δ_1)-almost minimizing. This is a contradiction. Since $\delta_2 > \delta_1$ can be chosen arbitrarily close, this concludes the proof. □

The goal of this section is the proof of the following

Theorem 9.17. *Let Σ be the minimal hypersurface constructed by the min-max procedure. Assume that it is orientable. Then there is a sequence of smooth hypersurfaces $\{\Gamma^k\}$ such that*

- *(i)* Γ^k *is (ε_k,δ_k)-almost minimizing in sufficiently small annuli for some sequence $\varepsilon_k \to 0$ and a sequence $\delta_k \to 0$ of arbitrarily small $\delta_k < \varepsilon_k$;*
- *(ii)* $\Gamma^k \to \Sigma$ *in the sense of varifolds;*
- *(iii)* $\mathrm{supp}(\Gamma^k) \to \mathrm{supp}(\Sigma)$ *in the Hausdorf sense.*

Moreover, $\{\Gamma^k\}$ is obtained from a min-max sequence by isotopies and surgery.

We note that the critical sequence in the existence results of [18] and [28] satisfies the conditions (*i*) and (*ii*), whereas the sequence of Theorem 9.12 satisfies conditions (*ii*) and (*iii*). The key point therefore in the proof is to make sure that an appropriate surgery can be performed in such a way that the almost minimality is not lost.

Before we come to the proof, we recall some notation that will be used. Let $0 < 2r < \mathrm{Inj}\,(M)$, $x \in M$. For $y \in \bar{B}_r(x)$ we denote by $[x,y]$ the geodesic segment connecting x and y. This is well-defined due to the choice of r. Then, for $\lambda \in [0,1]$, we denote by $\tau_\lambda^x(z)$ the point $w \in [x,z]$ with $\mathrm{d}(x,w) = \lambda \mathrm{d}(x,z)$. For $1 < \lambda \leq 2$, we can still define $\tau_\lambda^x(z)$ by the corresponding point on the geodesic extension of $[x,z]$. Note that this is still well-defined. The map τ_λ^x is simply the homothetic shrinking or expansion with respect to the center x in the local Riemannian setting.

Proof of Theorem 9.17. We begin with some preliminary remarks.

Step 1: Setup. Let $\{\Sigma^k\}$ be the critical sequence of the existence theory. Then we know that there is a map $r : M \to \mathbb{R}^+$ such that Σ^k is $\frac{1}{k}$-a.m. in $\mathcal{AN}_{r(x)}(x)$. By assumption there is only a finite set P_k of points such that Σ^k is not $\frac{1}{k}$-a.m. in $B_{r(x)}(x)$.

Consider $\varepsilon_0 > 0$ such that the tubular neighborhood $T_{\varepsilon_0}\Sigma$ is diffeomorphic to $\Sigma \times (-1,1)$. In the following discussion we will use this diffeomorphism as an identification and by abuse of notation also write, for instance, $\Sigma \times \{t\}$ for its image under the diffeomorphism. Then the above remark implies that there is a sequence of levels $\{\Sigma \times \{t_l\}\}$ not intersecting $\bigcup_\mu P_\mu$ such that $t_l \to 0$. For the argument we fix such a level t and denote $\Sigma \times \{t\} = \tilde{\Sigma}$. By compactness there is a finite subcover of the cover $\bigcup_{x\in\tilde{\Sigma}} B_{r(x)}(x)$. We denote the balls by $B_1, \ldots, B_N$. Then there is $\alpha > 0$ such that $\Sigma \times (t-\alpha, t+\alpha)$ is still covered by $B_1, \ldots, B_N$.

Since Σ is a compact 2-manifold, there is a finite triangulation consisting of triangles $T_1, \ldots, T_M$. We denote, moreover, $\Delta_i = T_i \times (t-\alpha, t+\alpha)$. Then we have $T_{t+\alpha}\Sigma \setminus \overline{T_{t-\alpha}\Sigma} = \bigcup_{i=1}^M \Delta_i$. In fact, by barycentric subdivision we can also assume that for all $i \in \{1, \ldots, M\}$ there is $j \in \{1, \ldots, N\}$ with $\Delta_i \subset B_j$. Finally, we find $\beta > 0$ such that

$$\tilde{\Delta}_i = \Delta_{i,2\beta} = (T_i \times (t-\alpha, t+\alpha))_{2\beta} \subset B_j\,,$$

a 2β-neighborhood of Δ_i.

Step 2: Surgery. For every fixed $k \in \mathbb{N}$ we consider $\{\Sigma^{k,j}\}$, a minimizing sequence in $\mathcal{H}(\Sigma^k, \tilde{\Delta}_1)$, the set of isotopies that are supported in $\tilde{\Delta}_1$, are starting from Σ^k and have the property that the area of *no* time slice of the isotopy exeeds the area of Σ^k by more (or equal) than $\frac{1}{8k}$ (see Section 7.1). Using the Schoen-Simon curvature estimates we obtain that (up to subsequences) $\Sigma^{k,j} \to \Gamma^k$ $(j \to \infty)$ as varifolds, where

(i) $\Gamma^k \llcorner (\tilde{\Delta}_1)^c = \Sigma^k \llcorner (\tilde{\Delta}_1)^c$;

(ii) $\Gamma^k \llcorner \tilde{\Delta}_1$ is a smooth stable minimal hypersurface;

(iii) $\Gamma^k \to \Sigma$ as varifolds;

(iv) $\Gamma^k \llcorner \tilde{\Delta}_1 \overset{C^\infty}{\to} \Sigma \llcorner \tilde{\Delta}_1$.

Clearly, $\Sigma \llcorner \tilde{\Delta}_1 = 0$, thus by (iv) there is $K_1 \in \mathbb{N}$ such that for $k \geq K_1$ also $\Gamma^k \llcorner \tilde{\Delta}_1 = 0$. Hence, for $k \geq K_1$, $\lim_{j\to\infty} \mathcal{H}^2(\Sigma^{k,j} \llcorner \tilde{\Delta}_1) = 0$. Now the coarea formula gives that there is a geometric constant $C > 0$ such that, for any $\eta > 0$,

$$\int_\beta^{2\beta} \text{Length}(\Sigma^{k,j} \llcorner \partial\Delta_{1,\sigma})\, d\sigma \leq C\mathcal{H}^2(\Sigma^{k,j} \llcorner \tilde{\Delta}_1) < C\eta \tag{9.6}$$

for $j \in \mathbb{N}$ large enough (depending only on η). Thus

$$\text{Length}(\Sigma^{k,j} \llcorner \partial\Delta_{1,\sigma}) < \frac{2C\eta}{\beta} \tag{9.7}$$

for a set of σ of measure at least $\frac{\beta}{2}$. And by Sard's lemma we can find σ such that this inequality holds and $\Sigma^{k,j}$ intersects $\partial\Delta_{1,\sigma}$ transversally. Next, we note that there are constants $C_1 > 0$ and $\lambda > 0$ such that

(E) For any $s \in (0, 2\beta)$ the following holds: any simple curve γ lying on $\partial\Delta_{1,s}$ with $\text{Length}(\gamma) \leq \lambda$ bounds an embedded disk $D \subset \partial\Delta_{1,s}$ with $\text{diam}(D) \leq C_1 \text{Length}(\gamma)$.

Now we fix our choice of η. Let $C_M > 0$ be the isoperimetric constant in M. We choose $\eta > 0$ such that

$$\eta \leq \frac{1}{2C_M\left(\frac{2C}{\beta}\right)}\,; \tag{9.8}$$

$$8\eta < \mathcal{H}^2(\Sigma^k \llcorner \tilde{\Delta}_1)\,; \tag{9.9}$$

$$2\eta < \frac{1}{16k}\,; \tag{9.10}$$

$$\eta < \frac{\lambda\beta}{2C}\,. \tag{9.11}$$

Then we fix our choice of j such that (9.6) holds.

By construction, $\Sigma^{k,j} \llcorner \partial\Delta_{1,\sigma}$ is a finite collection of simple curves. Let $\Omega = \Delta_{1,\sigma+2\delta} \setminus \overline{\Delta_{1,\sigma-2\delta}}$. For $2\delta > 0$ sufficiently small, $\Sigma^{k,j} \llcorner \Omega$ is a finite collection cylinders with boundary curves lying on $\partial\Delta_{1,\sigma\pm 2\delta}$. For the sake of the argument we assume that there is only one cylinder $\mathcal{C}$ forming $\Sigma^{k,j} \llcorner \Omega$. Repeating Step 3 for each cylinder gives then the general result.

We replace $\Sigma^{k,j} \llcorner \Omega$ by the corresponding embedded disks D_1, D_2 lying on $\partial\Delta_{1,\sigma\pm 2\delta}$ whose existence is ensured by (E), (9.7) and (9.11). Note, however, that the choice of δ depends on j, but we will not indicate this dependence in the notation. We denote the new surfaces (after a little smoothing) by $\tilde{\Sigma}^{k,j}$. We know that we have, due to the isoperimetric inequality with constant $C_M > 0$,

$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}^{k,j}) &\leq \mathcal{H}^2(\Sigma^{k,j}) + \mathcal{H}^2(D_1) + \mathcal{H}^2(D_2) \\ &\leq \mathcal{H}^2(\Sigma^{k,j}) + 2C_M \left(\frac{2C\eta}{\beta}\right)^2 . \end{aligned}$$

Since $\tilde{\Sigma}^{k,j} \llcorner (\tilde{\Delta}_1)^c = \Sigma^{k,j} \llcorner (\tilde{\Delta}_1)^c$, this inequality reduces to

$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}^{k,j} \llcorner \tilde{\Delta}_1) &\leq \mathcal{H}^2(\Sigma^{k,j} \llcorner \tilde{\Delta}_1) + 2C_M \left(\frac{2C}{\beta}\right)^2 \eta^2 \\ &\leq \eta + 2C_M \left(\frac{2C}{\beta}\right)^2 \eta^2 \leq 2\eta , \end{aligned} \tag{9.12}$$

where we used (9.6) and (9.8).

Step 3: Isotopic approximation. To ensure that after surgery the surfaces $\tilde{\Sigma}^{k,j}$ still satisfy the almost minimizing property, we approximate the surgery by isotopies and apply Lemma 9.16.

We start by considering the Euclidean case (the Riemannian case follows up to a constant by taking normal coordinates in a small tubular neighborhood of $\partial\Delta_{1,\sigma}$ and choosing δ small enough). We look at the intersection of $\mathcal{C}$ with a horizontal plane. We obtain an embedded closed curve γ. We denote by u the Douglas solution of the two-dimensional Plateau problem with boundary curve γ. We keep the notation γ for the parametrization given by $u|_{S^1}$. Then u is a solution of the boundary value problem

$$\begin{cases} \Delta u = 0 \quad \text{on } D \\ u|_{S^1} = \gamma . \end{cases}$$

Moreover, u is weakly conformal in the interior. *Weakly* means that a priori there is a set B of isolated zeroes of ∂u (possible branch points). $u|_{S^1}$ is bijective and u in the interior is holomorphic outside B. Therefore the cardinality of the preimage is constant outside B, and we can conclude that u is a diffeomorphism on D and continuous up to the boundary (due to the fact that B is discrete). Hence all the curves $u(\partial D_s)$ are isotopic. The fact that u is not smooth up to the boundary does not cause any problem since it is only the parametrization that might be singular (but still bijective), not, however, the curve itself. (See [31] for all these facts about classical minimal surfaces.)

Then u is given by the Poisson Formula

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{\gamma(\varphi)}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi\,.$$

We compute

$$\begin{aligned}
\partial_\theta u(r,\theta) &= \frac{1-r^2}{2\pi}\int_0^{2\pi} \gamma(\varphi)\cdot\frac{d}{d\theta}(1-2r\cos(\varphi-\theta)+r^2)^{-1}\,d\varphi \\
&= \frac{1-r^2}{2\pi}\int_0^{2\pi} \gamma(\varphi)\cdot(-1)\frac{d}{d\varphi}(1-2r\cos(\varphi-\theta)+r^2)^{-1}\,d\varphi \\
&= \frac{1-r^2}{2\pi}\int_0^{2\pi} \gamma'(\varphi)\cdot(1-2r\cos(\varphi-\theta)+r^2)^{-1}\,d\varphi\,,
\end{aligned}$$

where we used integration by parts in the last step. Now we have, for $0<r<1$,

$$\begin{aligned}
\mathrm{Length}(u|_{\partial D_r}) &= \int_0^{2\pi} |\partial_\theta u(r,\theta)|\,d\theta \\
&\le \frac{1-r^2}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|\gamma'(\varphi)|}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi\,d\theta \\
&= \frac{1-r^2}{2\pi}\int_0^{2\pi} |\gamma'(\varphi)|\int_0^{2\pi} \frac{d\theta}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi \\
&= \int_0^{2\pi} |\gamma'(\varphi)|\,d\varphi = \mathrm{Length}(\gamma)\,.
\end{aligned}$$

In the step from the second last to the last line we used again the Poisson formula, this time for the harmonic function constant 1. Therefore the isotopy given by $\gamma_t = u(\partial D_{1-t})$ shrinks γ to a point in way that does not increase the length above the length of γ. Moreover, since all this argument depends smoothly on γ, we can derive, by choosing the height 4δ of the cylinder small enough, that we can "glue" these isotopies along the vertical axis to get an isotopy $\mathcal{C}_t$ of the cylinder shrinking it to a segment connecting the top and bottom disk of $\mathcal{C} = \mathcal{C}_0$ such that

$$\mathrm{Area}(\mathcal{C}_t) \leq 2\mathrm{Area}(\mathcal{C}), \quad t \in [0,1]. \tag{9.13}$$

The fact that we are not in a Euclidean situation (as assumed so far in this step) but in a Riemannian (at very small scale) is also absorbed in the constant 2. Now we can define

$$\hat{\Sigma}^{k,j}_s \llcorner A = \begin{cases} \Sigma^{k,j} \llcorner \bar{\Omega}^c & \text{if} \quad A = \bar{\Omega}^c, \\ \quad \mathcal{C}_s & \text{if} \quad A = \Omega, \\ D_{1,0} - D_{1,s} + D_{2,0} - D_{2,s} & \text{if} \quad A = \partial\Omega, \end{cases}$$

where $D_{i,s}$ denotes the top ($i = 1$) and the bottom ($i = 2$) of the cylinder $\mathcal{C}_s$ (note that $D_{i,0} = D_i$ in the previous notation). Thus $\hat{\Sigma}^{k,j}_s$ is a continuous deformation of $\Sigma^{k,j}$ with

(i) $\hat{\Sigma}^{k,j}_1 = \Sigma^{k,j} \llcorner \Omega^c \cup D_{1,0} \cup D_{2,0}$;

(ii) for all $s \in [0,1]$, by (9.6), (9.9), (9.12) and (9.13)

$$\begin{aligned} \mathcal{H}^2(\hat{\Sigma}^{k,j}_s) &\leq \mathcal{H}^2(\Sigma^{k,j}) + \mathcal{H}^2(D_{1,0} \cup D_{2,0}) + 2\mathcal{H}^2(\Sigma^{k,j} \llcorner \Omega) \\ &\leq \mathcal{H}^2(\Sigma^{k,j}) + 4\eta < \mathcal{H}^2(\Sigma^k). \end{aligned}$$

Smoothing as we did for the surgery and gluing this isotopy with the isotopy connecting Σ^k with $\Sigma^{k,j}$ gives for j large enough that all the hypotheses of Lemma 9.16 are satisfied for $\varepsilon = \frac{1}{k}$, $\delta_1 = 0$. Therefore $\tilde{\Sigma}^{k,j}$ is $\left(\frac{1}{k}, \varepsilon'\right)$-a.m. for arbitrarily small $\varepsilon' > 0$.

Step 4: Contraction. Let $\tilde{\Sigma}^{k,j}_c$ be the connected component of $\tilde{\Sigma}^{k,j}$ lying in $\Delta_{1,\sigma-\delta}$ and p the barycenter. Then there is a constant C_R (due to Riemannian effects) with

$$\mathcal{H}^2((\tau^p_{1-s})_\#(\tilde{\Sigma}^{k,j}_c)) \leq C_R \mathcal{H}^2(\tilde{\Sigma}^{k,j}_c)(1-s)^2 \leq C_R \mathcal{H}^2(\tilde{\Sigma}^{k,j}_c).$$

We can assume that $C_R \leq 2$ (since we consider small scales). We denote the isotopy of $\tilde{\Sigma}^{k,j}$ that results from the above homothetic shrinking of the component $\tilde{\Sigma}^{k,j}_c$ by $\tilde{\Sigma}^{k,j}_s$. Then the above inequality implies, for all $s \in [0,1]$,

$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}^{k,j}_s) &= \mathcal{H}^2(\tilde{\Sigma}^{k,j}_s \llcorner (\Delta_{1,\sigma+\delta})^c) + \mathcal{H}^2(\tilde{\Sigma}^{k,j}_s \llcorner \Delta_{1,\sigma-\delta}) \\ &\leq \mathcal{H}^2(\tilde{\Sigma}^{k,j}_s \llcorner (\Delta_{1,\sigma+\delta})^c) + 2\mathcal{H}^2(\tilde{\Sigma}^{k,j}_c) \\ &\leq \mathcal{H}^2(\tilde{\Sigma}^{k,j}) + \mathcal{H}^2(\tilde{\Sigma}^{k,j}_c) \\ &\overset{(9.12)}{\leq} \mathcal{H}^2(\tilde{\Sigma}^{k,j}) + 2\eta \leq \mathcal{H}^2(\tilde{\Sigma}^{k,j}) + \frac{1}{8k} - \frac{\varepsilon'}{8}\,. \end{aligned}$$

The last inequality follows from (9.10) and the fact that we can choose ε' arbitrarily small. Moreover, $(\tau^p_{1-s})_\#(\tilde{\Sigma}^{k,j}_c) \to 0$ as $s \to 1$. Therefore, discarding $\tilde{\Sigma}^{k,j}_c$ and invoking again Lemma 9.16 applied to $\tilde{\Sigma}^{k,j}$, we can summarize what we have achieved so far. By surgery, we can change $\Sigma^{k,j}$ such that

(a) $\tilde{\Sigma}^{k,j}$ does not intersect $\Delta_{1,\sigma}$;

(b) $\tilde{\Sigma}^{k,j} \llcorner (\tilde{\Delta}_1)^c = \Sigma^{k,j} \llcorner (\tilde{\Delta}_1)^c$;

(c) $\mathcal{H}^2(\tilde{\Sigma}^{k,j}) \leq \mathcal{H}^2(\Sigma^k)$;

(d) $\tilde{\Sigma}^{k,j}$ is $\left(\frac{1}{k}, \varepsilon\right)$-a.m. in $\tilde{\Delta}_1$.

Here $\varepsilon > \varepsilon'$ is arbitrarily close. In order to avoid unnecessary notational complications we kept the notation $\tilde{\Sigma}^{k,j}$ for $\tilde{\Sigma}^{k,j}_1$.

Step 5: Iteration. We want to iterate this procedure to get a sequence $\tilde{\Sigma}^k$ with the following properties:

(a') $\tilde{\Sigma}^k$ does not intersect $T_{t+\alpha}\Sigma \setminus \overline{T_{t-\alpha}\Sigma} = \bigcup_{i=1}^M \Delta_i$;

(b') $\tilde{\Sigma}^k \llcorner T_{t-\alpha}\Sigma = \Sigma^k \llcorner T_{t-\alpha}\Sigma$;

(c') $\mathcal{H}^2(\tilde{\Sigma}^k) \leq \mathcal{H}^2(\Sigma^k)$;

(d') $\tilde{\Sigma}^k$ is $\left(\frac{1}{k}, \varepsilon\right)$-a.m. in small annuli.

We denote by $\tilde{\Sigma}^k$ the surface $\tilde{\Sigma}^{k,j}$ for a j large enough such that $(a)-(d)$ of the previous step hold. Assume that $\tilde{\Delta}_1 \cap \tilde{\Delta}_2 \neq \emptyset$. If a neck passing through $\tilde{\Delta}_2$ has been cut away in the first surgery, it is very well possible that $\tilde{\Sigma}^k$ is no longer almost minimizing in $\tilde{\Delta}_2$. On the other hand, if we do the same procedure as before, that is, taking a minimizing sequence $\{\tilde{\tilde{\Sigma}}^{k,j}\}$ in $\mathcal{H}(\tilde{\Sigma}^k, \tilde{\Delta}_2)$, then, for j large enough, this will be $\frac{1}{k}$-a.m. in $\tilde{\Delta}_2$. This allows us to run the previous program iteratively. The only point we have to take care of is that we do not create any new pieces in Δ_1. To do so we consider $\tilde{\Delta}_2 \setminus \overline{\Delta_{1,\sigma}}$ instead of $\tilde{\Delta}_2$ and a minimizing sequence in $\mathcal{H}(\tilde{\Sigma}^k, \tilde{\Delta}_2 \setminus \overline{\Delta_{1,\sigma}})$. Applying Steps 2 to 4, we obtain that there is K_2 such that, for $k \geq K_2$, $\tilde{\Sigma}^k$ can be modified by suitable isotopies and surgery to give $\tilde{\tilde{\Sigma}}^k$ with the properties

(a") $\tilde{\tilde{\Sigma}}^k$ does not intersect $\Delta_{1,\sigma_1} \cup \Delta_{2,\sigma_2}$;

(b") $\tilde{\tilde{\Sigma}}^k \llcorner (\tilde{\Delta}_1 \cup \tilde{\Delta}_2)^c = \Sigma^k \llcorner (\tilde{\Delta}_1 \cup \tilde{\Delta}_2)^c$;

(c") $\mathcal{H}^2(\tilde{\tilde{\Sigma}}^k) \leq \mathcal{H}^2(\Sigma^k)$;

(d") $\tilde{\tilde{\Sigma}}^k$ is $\left(\frac{1}{k}, \varepsilon\right)$-a.m. in small annuli.

This shows that after finitely many iteration steps (corresponding to the M Δ_i) we indeed obtain $(a') - (d')$. In fact, if we discard the part of the modified surfaces lying in $M \setminus T_t\Sigma$, we can even obtain $\{\tilde{\Sigma}^{k,j}\}$ contained in $T_t\Sigma$ with $(b') - (d')$.

Step 6: Conclusion. So far, we have modified the initial sequence $\{\Sigma^k\}$, for t fixed, to get a sequence $\{\tilde{\Sigma}^k\}$ with the desired properties. In particular, there is $K = K(t) \in \mathbb{N}$ such that, for $k \geq K$, $\tilde{\Sigma}^k$ is contained in $T_t\Sigma$. Taking the sequence $\{t_l\}$ of Step 1 and repeating the argument of Steps 2 to 5 we find a sequence $\Gamma^l = \tilde{\Sigma}^{K(t_l)}$, where $\tilde{\Sigma}^{K(t_l)}$ is the corresponding surface lying in $T_{t_l}\Sigma$. Therefore, $\{\Gamma^l\}$ satisfies the requirements of the claim with $\varepsilon_l = \frac{1}{K(t_l)}$ and $\delta_l = \varepsilon$ of Step 5 corresponding to t_l. $\square$

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