Partial Differential Equations

An extension of the identity $\text{Det} = \text{det}$

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1. Introduction

We first define the notion of distributional Jacobian and of $BnV$ function:

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^m$ be an open set, assume $p$ and $q$ satisfy:

$$p \geq n - 1, \quad \frac{1}{q} + \frac{n - 1}{p} \leq 1. \quad (1)$$

For $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^n)$ with $m \geq n$, we let $j(u)$ be the $(m - n + 1)$-current given by the action $(j(u), \omega) := (-1)^n \int_\Omega u^1 du^2 \wedge \cdots \wedge du^n \wedge \omega$ on forms $\omega$ in $\mathcal{C}^\infty_c(\Omega)$. The distributional Jacobian of $u$ is the $(m - n)$-current $[Ju] := \partial j(u)$. We say that a map $u \in W^{1,p} \cap L^q$ belongs to $BnV$ if its distributional Jacobian $[Ju]$ has finite mass (and hence it can be represented by a Radon Measure).

If $m = n$, $[Ju]$ is a distribution and a simple calculation gives that $[Ju] = \frac{1}{m} \text{div}\text{Cof}(\nabla u)u$, where Cof$(\nabla u)$ is the matrix of cofactors of $\nabla u$. This case of Definition 1.1 was first introduced by Ball in [2]. Subsequent works by Šverák [17] and Müller and Spector [15] were devoted to analyze the regularity properties of such maps and their applications to problems in elasticity. A powerful theory for these variational problems has been developed by Giaquinta, Modica and Souček (see [9]).
Lemma 2.3. Let $\Omega \subset \mathbb{R}^m$ be an open set and let $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^m)$ be a BnV map. Let $v$ be the density of the absolutely continuous part of the distributional Jacobian $Ju$ with respect to the Lebesgue measure: $Ju = v\mathcal{L}^m + [Ju]^q = [Ju]^q + [Ju]^q$. Then $v(x) = \det Ju(x)$ for $\mathcal{L}^m$-almost every $x \in \Omega$.

Proof. Let $\phi_{\delta,t}$ be a standard Lipschitz cut-off, taking the value 1 for $|x| \leq r - \delta$ and 0 for $|x| \geq r$, with $\phi_{\delta,t}(x) = (r - |x|)/\delta$ for $r - \delta \leq |x| \leq r$. Let $f(r) = \int_{B_1} u^1 \, dx \ldots \wedge du^n$: then $f \in L^1([0,1])$ because of (1) and Fubini’s Theorem. This implies that $L^1$-a.e. $r$ is a Lebesgue point, that is: $\int_{r-\delta}^{r+\delta} f(s) - f(r) \, ds = o(\delta)$. Moreover $\langle Ju, \phi_{\delta,t} \rangle = \langle j(u), \phi_{\delta,t} \rangle = \int - u^1 \, du^1 \wedge \ldots \wedge du^n + \frac{1}{\delta} \int f_{r-\delta}^{r+\delta} \int_{B_1} u^i \, dx \ldots \wedge du^n \, dx1(t)$. Hence at every Lebesgue point $\langle Ju, \phi_{\delta,t} \rangle \to \int_{B_1} u^1 \, dx \ldots \wedge du^n$; on the other hand, by dominated convergence, $\langle [Ju], \phi_{\delta,t} \rangle \to [Ju](B_1)$, that proves the proposition. $\square$

Definition 2.2. Let $u \in BnV(\Omega, \mathbb{R}^m)$ and let $x_0 \in B_r \subset \Omega$. We define $u_{\varepsilon}(y) := (u(x_0 + \varepsilon y) - u(x_0))/\varepsilon$.

Lemma 2.3. Let $u$ be as above and set $\delta \varepsilon(x) := a(x - x_0)$. Then $\langle Ju_{\varepsilon} \rangle = \frac{1}{\varepsilon} \delta_{\varepsilon}\frac{\partial}{\partial x} [Ju]$.

Proof. Let $\phi \in C_c^\infty(B_1)$ be a test function. Since $\langle Ju_{\varepsilon} \rangle, \phi = \langle j(u_{\varepsilon}), \phi \rangle$ we have:

$$\langle [Ju_{\varepsilon}] \rangle, \phi = (-1)^n \int_{B_1} \frac{u^1(x_0 + \varepsilon y) - u^1(x_0)}{\varepsilon} \, dx \ldots \wedge dx \nabla u^n(x_0 + \varepsilon y), \nabla \phi(y) \rangle \, dy$$

$$= (-1)^n \int_\Omega \frac{u^1(x) - u^1(x_0)}{\varepsilon^{n+1}} \, dx \ldots \wedge dx \nabla u^n(x), \nabla \phi\left(\frac{x - x_0}{\varepsilon}\right) \rangle \, dx$$

$$= \frac{1}{\varepsilon^n} \int_{B_1} \hat{\phi}\left(\frac{x - x_0}{\varepsilon}\right) \right) \, dx \, [Ju_{\varepsilon}] \phi \left(\frac{x - x_0}{\varepsilon}\right) \rangle.$$

Taking the supremum over $\phi \in C_c^\infty(B_1)$: $\|\phi\|_\infty \leq 1$ we conclude $\|Ju_{\varepsilon}\| = \frac{1}{\varepsilon^n} \delta_{\varepsilon}\frac{\partial}{\partial x} [Ju]$. Since the Radon–Nikodym decomposition commutes with the push forward, $\|Ju_{\varepsilon}\| = \frac{1}{\varepsilon^n} \delta_{\varepsilon}\frac{\partial}{\partial x} [Ju]^q$ and $\|Ju_{\varepsilon}\|^q = \frac{1}{\varepsilon^n} \delta_{\varepsilon}\frac{\partial}{\partial x} [Ju]^q$, which allows to conclude

$$\|Ju_{\varepsilon}\|^q(B_r(x_0)) = \frac{\|Ju\|^q(B_r(x_0))}{\varepsilon^n} \quad \forall r > 0.
$$

(3)
Proof of Theorem 1.2. To simplify the notation we use \( u_h \) for the function \( u_{h-1} \) given by Definition 2.2. We use formula (2) to the blow-up sequence \( (u_h) \) around a “good” point \( x_0 \) to get \( [J u_h](B_\rho(x_0)) = \int_{\partial B_\rho(x_0)} u_h^1 \, \text{d} u_h^n \wedge \cdots \wedge \text{d} u_h^n \), and hence we let \( h \uparrow \infty \) to obtain

\[
\nu(x_0)|B_\rho| = \int_{\partial B_\rho(x_0)} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = \int_{\partial B_\rho(x_0)} (L \cdot x)^1 \text{cof}(L)_k^1 \cdot \eta^k = \text{det}(L)|B_\rho|,
\]

where \( L := \nabla u(x_0) \) and \( \eta \) is the exterior unit normal to \( \partial B_\rho \).

Step 1: By the standard theory of Sobolev functions (see [7]), a.e. \( x_0 \in \Omega \) satisfies the following properties:

(a) \( \lim_{r \downarrow 0} \frac{1}{r^n} \left[ \left\| [J u_h^2]^{1}([B_r(x_0))] + \int_{B_r(x_0)} [\nu(x) - \nu(x_0)] \, \text{d} x \right\| \right] = 0 \);

(b) \( \nabla u \) is approximately continuous at \( x_0 \) and in particular \( \int_{B_r(x_0)} |\nabla u(x) - \nabla u(x_0)|^p \, \text{d} x = o(r^n) \).

From now on we fix \( x_0 \) satisfying (a) and (b) and, without loss of generality, we assume \( x_0 = 0 \). Observe first of all that condition (a) and Eq. (3) imply:

\[
[J u_h_1]([B_r(0))] = h^n [J u_1]([B_r(0))] = o(1) + h^n \int_{B_r(0)} [\nu(y) \, \text{d} y \to [\nu(0)])[B_r(0)] \quad \forall r > 0.
\]

Step 2: We observe that, being \( (u_h) \) a sequence, there is a set of radii \( \rho \in (0, 1) \) of full measure such that (2) holds for every \( h \). Moreover by (b), using Fubini’s and Fatou’s Theorems, for a.e. \( \rho \) there exists a subsequence (not relabeled and possibly depending on \( \rho \)) such that \( \nabla u_h \to L := \nabla u(0) \) in \( L^p(\partial B_\rho) \). We fix now a radius \( \rho \) with all the properties above and we do not relabel the relevant subsequence. Hence \( du_h^1 \wedge \cdots \wedge du_h^n \to L^2 \wedge \cdots \wedge L^n \) in \( L^{2^*} (\partial B_\rho) \), since

\[
\text{det} \frac{\partial u_h}{\partial x} \to \text{det} \frac{\partial u}{\partial x} = \frac{L^2}{L^n}.
\]

In the borderline case \( p = n - 1 \), the convergence is improved to the first Hardy space \( \mathcal{H}^1(\partial B_\rho) \) because of the Coifman–Lions–Meyer–Semmes estimate (see [3]):

\[
\left\| (\text{d} v^2 \wedge \cdots \wedge \text{d} v^n, \tau) \right\|_{\mathcal{H}^1(\partial B_\rho)} \leq C \left\| \text{d} v^2 \right\|_{L^{2^n - 1}(\partial B_\rho)} \cdots \left\| \text{d} v^n \right\|_{L^{2^n - 1}(\partial B_\rho)}.
\]

Suppose first of all that \( p > n - 1 \). Then by the Poincaré’s inequality and the Sobolev embedding theorem, the sequence \( (u_h) \) is equicontinuous, with the estimate \( \|u_h - L \cdot x - C_h\|_{C^a(\partial B_\rho)} \leq C \|\nabla u_h - L\|_{L^p(\partial B_\rho)} \to 0 \). Here \( C_h \) is the average of \( u_h \) on \( \partial B_\rho \). Since \( \int_{\partial B_\rho} du_h^2 \wedge \cdots \wedge du_h^n = 0 \), we conclude,

\[
[J u_h_1]([B_r(0))] = \int_{\partial B_r(0)} (u_h^1 - C_{h_1}^1) \, \text{d} u_h^n \wedge \cdots \wedge \text{d} u_h^n \to \int_{\partial B_r(0)} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = \text{det}(L)|B_\rho|.
\]

Finally if \( p = n - 1 \) we use the John–Nirenberg embedding and Poincaré’s inequality to get \( \|u_h - C_h - L \cdot x\|_{BMO} + \|u_h - C_h - L \cdot x\|_{L^1} \leq C \|\nabla u_h - L\|_{L^{2^n - 1}(\partial B_\rho)} \to 0 \). Recall that, by Fefferman’s Theorem, \( BMO \) is the dual space of \( \mathcal{H}^1 \) and thus \( \int \text{f} \, \text{g} \| \leq C(\|f\|_{BMO} + \|f\|_{L^1}) \|g\|_{\mathcal{H}^1} \) whenever \( f \) is integrable (see [16], Chapter IV; take into account that the original Theorem of Fefferman, proved in \( \mathbb{R}^n \), must be suitably modified to our situation where the domain is a compact manifold, see [10]). We thus infer that \( \int_{\partial B_\rho} (u_h^1 - C_{h_1}^1) \, \text{d} u_h^n \wedge \cdots \wedge \text{d} u_h^n \to \int_{\partial B_\rho} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = \text{det}(L)|B_\rho| \). \( \square \)

3. Proof of Theorem 1.3

Given a normal current \( T \in N_k(\mathbb{R}^m) \) and a Lipschitz map \( \pi : \mathbb{R}^m \to \mathbb{R}^l \) with \( k \geq l \), we can define a weakly*-measurable map \( x \mapsto (T, \pi, x) \in N_{k-l}(\mathbb{R}^m) \), uniquely characterized by the validity of the identity \( \int_{\pi(T)} T(T, \pi, x) \psi(x) \, \text{d} x = T \pi(T, \pi, x) \, \text{d} \pi \) for every \( \psi \in C^1_c(\mathbb{R}^l) \) (this is the so-called “slicing of the current”, see for instance [8]). In [5], the first author proved a slicing theorem for Jacobians, namely:

Theorem 3.1. Let \( \iota^* : \mathbb{R}^k \to \{x\} \times \mathbb{R}^k \) be the natural injection of \( \mathbb{R}^k \) into \( \mathbb{R}^m \), and let \( \pi : \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^{m-k} \) a projection, with \( k \geq m \). Denote by \( u^* \) the trace \( u|_{\mathbb{R}^k} = u o \iota^* \). Then \( [J u^*], \pi, x) = (-1)^{(m-k)n} \iota^*[J u] \). Moreover this property holds separately for the absolutely continuous part and the singular part of \( [J u] \).

This theorem allows us to pass from Theorem 1.2 to Theorem 1.3.
Proof of Theorem 1.3. Set \( \pi(x) = (x^1, \ldots, x^{m-n}) \), and \( y = (x^{m-n+1}, \ldots, x^n) \). By Theorem 3.1, \( \langle [Ju]^a, f \rangle = \langle [Ju]^a \circ \pi, f \rangle \rangle \in L^{m-n}(\pi). \) Thus, using Theorem 1.2, we conclude
\[
\langle [Ju]^a, f \rangle = \int \left( \int (-1)^{(m-n)\alpha} \det(\nabla_y u(x, y)) f(x, y) \, d\mathcal{L}^n(y) \right) \, d\mathcal{L}^{m-n}(x)
\]
\[
= \int \det(\nabla_y u(x, y)) f(x, y) \, dy \wedge d\pi = \int f(\epsilon_1 \wedge \cdots \wedge \epsilon_m \, \mathbb{L} \, du_1 \wedge \cdots \wedge du_n, d\pi) \, d\mathcal{L}^m.
\]

It is easy to show that, for every \( A \in GL(n, \mathbb{R}) \), the identity \( \langle Ju \circ A \rangle = \det(A) \cdot (A^{-1})^* [Ju] \) holds, where \( \det(A) \) is the sign of the determinant of \( A \). If then \( I \) is a multiindex of length \( m-n \), and \( \pi^I(x) = (x^1, \ldots, x^{m-n}) \), we let \( A \) be a permutation matrix satisfying \( \pi = \pi^I \circ A \). Then
\[
\langle [Ju]^a, f_I \rangle = \deg(A) \int \mathbb{R}^m f_I \circ A(e_1 \wedge \cdots \wedge e_m \mathbb{L} \, du_1 \circ A) \wedge \cdots \wedge du^n \circ A) \, d\mathcal{L}^m
\]
\[
= \deg(A) \int \mathbb{R}^m A^* (f_I \, du_1 \wedge \cdots \wedge du^n \, \wedge d\pi^I) = \int \mathbb{R}^m f_I \, du_1 \wedge \cdots \wedge du^n \, \wedge d\pi^I.
\]

It is then sufficient to write a generic form as \( \omega = \sum_i f_I \, dx^I \) to conclude the proof. \( \square \)

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References