## On Nearly Umbilical Hypersurfaces

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Diese Arbeit ist all jenen gewidmet, die nie in den Genuss einer höheren Ausbildung kamen, obwohl sie weitaus mehr Talent gehabt hätten als ich.

A university is very much like a coral reef. It provides calm waters and food particles for delicate yet marvellously constructed organisms that could not possibly survive in the pounding surf of reality, where people ask questions like 'Is what you do of any use?' and other nonsense.

Terry Pratchett, Ian Stewart and Jack Cohen, The Science of Discworld, revised ed., Ebury Press, London, 2002, 150-151

## Zusammenfassung

Im Jahr 2005 erzielten C. De Lellis und S. Müller eine quantitative Stabilitätsaussage über den klassischen Nabelpunktsatz der Differentialgeometrie. Für glatte, geschlossene und zusammenhängende Flächen $\Sigma$ im $\mathbb{R}^{3}$, bewiesen sie folgende Ab schätzung im kritischen Exponenten Zwei:
([DLM05, (1)])

$$
\inf _{\lambda \in \mathbb{R}}\|A-\lambda \mathrm{id}\|_{L^{2}(\Sigma)} \leq C\left\|A-\frac{\operatorname{tr} A}{2} \mathrm{id}\right\|_{L^{2}(\Sigma)}
$$

Dabei bezeichnet $A$ die zweite Fundamentalform von $\Sigma$, und $C>0$ ist eine von der Fläche unabhängige Konstante.

Ziel der vorliegenden Arbeit ist es, obige Abschätzung auf höhere Dimensionen $n$, sowie allgemeine, nicht-kritische Exponenten $p$ der Lebesgue-Norm zu erweitern. Wir betrachten glatte, geschlossene und zusammenhängende Hyperflächen des $\mathbb{R}^{n+1}$ und unterscheiden die beiden Fälle $1<p \leq n$ und $p>n$. Im ersten Fall gelingt uns die Verallgemeinerung für konvexe Hyperflächen, wohingegen wir im zweiten Fall die zusätzliche Annahme treffen müssen, dass die $L^{p}$-Norm der zweiten Fundamentalform einer (von uns bestimmbaren) Schranke genügt.

Des Weiteren ermitteln wir in obiger $L^{2}$-Ungleichung die optimale Konstante für konvexe Hyperflächen des $\mathbb{R}^{n+1}$.

Schliesslich beweisen wir noch die Notwendigkeit der Konvexitätsannahme, wann immer $1 \leq p<n$ ist.


#### Abstract

In 2005 , C. De Lellis and S. Müller obtained a quantitative rigidity result regarding the classical theorem of differential geometry about surfaces all whose points are umbilical. For smooth, closed and connected surfaces $\Sigma$ in $\mathbb{R}^{3}$, they proved the following estimate in the critical exponent two: ([DLM05, (1)]) $$
\inf _{\lambda \in \mathbb{R}}\|A-\lambda \mathrm{id}\|_{L^{2}(\Sigma)} \leq C\left\|A-\frac{\operatorname{tr} A}{2} \mathrm{id}\right\|_{L^{2}(\Sigma)}
$$

Here, $A$ denotes the second fundamental form of $\Sigma$, and $C>0$ is a constant which is independent of the surface.

The goal of the present work is to generalise the above estimate to higher dimensions $n$ and general non-critical exponents $p$ of the Lebesgue norm. We consider smooth, closed and connected hypersurfaces in $\mathbb{R}^{n+1}$ and distinguish the two cases $1<p \leq n$ and $p>n$. In the first case, we obtain the generalisation for convex hypersurfaces, whereas in the second we need to make the additional assumption that the $L^{p}$-norm of the second fundamental form satisfy some bound (which we are able to preset).

Furthermore, we establish the optimal constant in the above $L^{2}$-inequality for convex hypersurfaces in $\mathbb{R}^{n+1}$.

Finally, we also prove that the hypothesis of convexity is necessary, whenever $1 \leq p<n$.


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## Preface

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## 1. Introduction

1.1. The Nabelpunktsatz. A point on a smooth surface in Euclidean space is called umbilical if the two principal curvatures in this point coincide. According to the classical Nabelpunktsatz (German for "umbilical theorem"), if all the points of a smooth, connected surface are umbilical, then this surface is either a subset of a plane or of a sphere (see, e.g., [dC76, §3.2, Prop.4, p.147], [Str88, §3.5(2), p.122], [Küh08, Thm.3.14, p.51] or [Pre10, Prop.8.2.9, p.191]). This is one of the first rigidity results in differential geometry of type local to global, in that a local (or even pointwise) condition has a global consequence. The word rigidity can be interpreted in two ways here. Either by saying that the two obvious cases are the only ones occurring, or by pointing out that the curvature cannot vary across the surface.

There are many generalisations of the Nabelpunktsatz in various directions. For instance, there is an $n$-dimensional version (see [Spi99, Lem.1, p.8]), and the
smoothness hypothesis can be weakened (see, e.g., [BF36, §5], [Har47, Thm.1] and [Pau08]). Also, there are versions applying to submanifolds of higher co-dimension in $\mathbb{R}^{n+1}$ (see [Spi99, Thm.26, p.75]), or even in spaces of constant curvature (see [Spi99, Thms.27\&29, pp.75\&77]).

A natural question to ask about all rigidity results is whether the conclusion is stable. For the Nabelpunktsatz this shall mean: does the assumption that all points of a closed surface be nearly umbilical entail that this surface must be nearly a sphere? Here we exclude the planar conclusion by restricting attention to closed surfaces, i.e. compact ones without boundary. Of course, one needs to give a more precise meaning to the word nearly, both in the hypotheses and in the conclusion. To do this, we observe that the ratio of the principal curvatures of a sphere is identically one. Thus, for a strictly convex surface, we obtain a rough measure for a point $q$ not to be umbilical by considering $\eta(q)=\frac{\lambda_{\max }(q)}{\lambda_{\min }(q)}-1$, where $\lambda_{\min }(q)$ and $\lambda_{\max }(q)$ denote the minimal and maximal principal curvature in $q$, respectively. Note that we assume strict convexity in order to ensure that $\eta(q)$ is well-defined everywhere. On the other hand, we can say that the surface is close to a sphere, if it lies in a thin spherical shell, and we measure that by considering the difference $\rho=\frac{R}{r}-1$, where $0<r<R$ are the radii of the two bounding spheres. Now we can ask:

- Does uniform smallness of $\eta$ imply smallness of $\rho$ ? (qualitative question)
- Is there a universal constant $C>0$ such that $\rho \leq C \sup _{q \in \Sigma} \eta(q)$, for all compact, strictly convex surfaces $\Sigma \subset \mathbb{R}^{3}$ ? (quantitative question)
These questions are obviously well-posed in higher dimensions, as well.
1.2. The Russian school. A. V. Pogorelov in [Pog67], complemented by H. Guggenheimer [Gug69], answers both of these questions in the affirmative - the monograph [Pog73, §VII.9] revisits these results. Here, one should also mention the work by Yu. E. Borovskiŭ [Bor67,Bor68], who achieves a positive answer to the first question with different methods, as well as Yu. A. Volkov and N. S. Nevmeržickiǐ ([Vol63], [Nev69] and reference 134 in [Res94]), whose papers we were not able to see (like A. I. Fet's related stability result [Fet63] - see also work by V. I. Diskant, such as [Dis71], as well as further references given in [Sch89]).

After these first results, several mathematicians endeavoured in generalising A. V. Pogorelov's theorem, among which D. Koutroufiotis [Kou71], J. D. Moore [Moo73] and, much later, K. Leichtweiss [Lei99], who all consider curvature-quantities other than the ratio between the principal curvatures (the latter author gives an explicit constant estimating the global deviation given the local one - see also R. Schneider [Sch88, Thm.2] and B. Andrews [And94, Thm.5.1\&Lem.5.4], who obtain related results as corollaries).

Others took more interest in weakening the sense in which the quantity $\eta$ is small. One possibility of doing so is to assume this condition to hold only in some form of average, for instance by replacing the sup-norm with an $L^{p}-$ norm. Then one asks whether smallness of $\|\eta\|_{L^{p}}$ implies smallness of $\rho$, or even seeks a precise estimate of the form $\rho \leq C\|\eta\|_{L^{p}}$. For convex hypersurfaces of $\mathbb{R}^{n+1}$, Yu. G. Reshetnyak [Res68] and S. K. Vodop'yanov [Vod70] achieve this for a slightly different control-quantity

- a detailed exposition is given in Reshetnyak's book [Res94, Ch.6]. Not only do they conclude the qualitative stability of the Nabelpunktsatz in this weak setting, but they also obtain a quantitative estimate under a suitable smallness assumption on the right-hand side.
1.3. G. Huisken's question. In 2003, G. Huisken asked C. De Lellis and S . Müller whether one could establish a similar result when replacing $\|\eta\|_{L^{2}}$ by the $L^{2}-$ norm of the traceless part of the second fundamental form of a smooth, closed and connected, but otherwise arbitrary surface in $\mathbb{R}^{3}$. The question was motivated by applications to foliations of asymptotically flat three-manifolds by surfaces of prescribed mean curvature (see [Met07] and subsequent works [LMS09, LM10], where the result mentioned below is crucial for ensuring that the leaves be close to spheres).

We now expose why one might want to look at this quantity. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth hypersurface endowed with the Riemannian metric $g$ induced by the ambient Euclidean space. Recall that the second fundamental form $A$ is the quadratic form on the tangent bundle whose eigenvalues are the principal curvatures, and the mean curvature $H$ is its trace. Consequently, the traceless part $\AA=A-\frac{H}{n} g$ of the second fundamental form constitutes a measure for the deviation of each single principal curvature from the arithmetic mean of all of them. Clearly, an umbilical point $q$ is then characterised by the vanishing of $\AA(q)$, and spheres are characterised by being closed and having $A$ vanish globally, as a consequence of the Nabelpunktsatz.

For a sphere of radius $R$, all the principal curvatures are equal to $\frac{1}{R}$, and thus, putting $\lambda=\frac{\bar{H}}{n}$, where $\bar{H}$ is the mean of $H$ over $\Sigma$, the quadratic form $A-\lambda g$ vanishes precisely on spheres. From the viewpoint of stability, then, it is some norm of this quadratic form which one seeks to control with some norm of the traceless part of the second fundamental form. In [DLM05], C. De Lellis and S. Müller were able to provide such an estimate in an $L^{2}$-sense for surfaces, thereby answering G. Huisken's question in the affirmative. In addition, and under an appropriate smallness assumption on $\|\AA\|_{L^{2}}$, they obtained in a quantitative way the closeness to a sphere in the norm of the Sobolev space $W^{2,2}$. Only one year later, they sharpened that previous result by proving even quantitative $C^{0}$-closeness (see [DLM06]). Their remarkable theorems are quite involved and require a very subtle analysis. This is due to the fact that the quantities considered are measured in the so-called critical norm, which we now explain.
1.4. Critical, sub-critical and super-critical exponents. The way we understand the distinction between critical, sub-critical and super-critical in this context is as follows. If the $L^{p}$-norm of a quantity tends to zero when we enlarge $\Sigma$ homothetically (we say we are "blowing up"), then we call the exponent $p$ super-critical. If, however, that norm explodes under this rescaling, the exponent is called sub-critical. The critical case is characterised by the scaling invariance of the $L^{p}$-norm of that quantity. Intuitively, in the super-critical case, high-frequency oscillations in the quantity get over-compensated by the rescaled norm under blowing up. Put differently, our quantity might have slightly better regularity than we could originally expect. On the other hand, in the sub-critical case, the regularity might
be poor. Thus, it may require additional assumptions to be able to prove theorems in the sub-critical situation which are valid in the super-critical one. The borderline case (i.e. the critical one) is, usually, the hardest one to treat, since neither of the non-critical cases gives any indication on what the minimal assumptions could be. From this point of view alone, the work of C. De Lellis and S. Müller is, indeed, astounding.
1.5. The main estimate. The aim of this thesis is to provide a generalisation of the estimate outlined above ([DLM05, (1)]) to arbitrary dimensions and all noncritical exponents. The sought-after result would be of the form: For every $n \geq 2$ and $p \in(1,+\infty)$, there exists a constant $C>0$, such that, for every closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the inequality
(MAIN EStimate)

$$
\inf _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C\|\AA\|_{L^{p}(\Sigma)}
$$

holds. Note that this estimate is, in general, not true, but we can show it under certain additional hypotheses on $\Sigma$. A detailed description of our results is given in the next section.
1.6. Historical note. The Nabelpunktsatz is sometimes attributed to G. Darboux (e.g. in [Res94]). However, it appears that the first author to prove this result is J.-B.-C.-M. Meusnier de Laplace in his sole mathematical Mémoire [MdL85, Prob.III, §34, pp.500-502], presented in 1776 and printed in 1785. Moreover, Darboux himself gives credit to Meusnier regarding it ([Dar96, Vol.1, Ch.III.1, §175, p.270]). Meusnier, in turn, emphasises the influence of earlier work by L. Euler [Eul67]. G. Monge, one of Meusnier's teachers, also obtains the Nabelpunktsatz using related methods and at about the same time ([Mon00, no.26]; see also his later treatise [Mon50]) - this is why some authors like J. V. Boussinesq [Bou90, no.I.II.194, p.264] attribute the result to Monge. For further reading, we suggest [Hil20], [Eis20], [Str33] and [Tru96], as well as Darboux's Éloge on Meusnier [Dar12].

## 2. Presentation of our results

In this work, unless otherwise stated, $n \geq 2$ and $\Sigma \subset \mathbb{R}^{n+1}$ denotes a smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$. Thus, $\Sigma$ is orientable. In order to simplify the presentation of our arguments, we additionally require the hypersurface to have unit $n$-dimensional volume. This bears no restriction, however, since all our results are easily reformulated to apply to hypersurfaces of arbitrary area.
2.1. Chapter 1. The super-critical case $p>n$. We set out to prove the main estimate for general dimensions $n$ and exponents $p>n$, and show $C^{0}$-closeness to a sphere if $\|\AA\|_{L^{p}(\Sigma)}$ is small (N.B.: in contrast to the results of C. De Lellis and S. Müller, no quantitative estimate was obtained). Unfortunately, we could not establish the main estimate for a constant on the right-hand side that is fully independent of the hypersurface under consideration. More precisely, we prove that the estimate holds true with a constant $C$ depending only on $n, p$ and some $c_{0}>0$, whenever $\|A\|_{L^{p}(\Sigma)} \leq c_{0}$.

The idea of our proof is as follows. We first show that an inequality analogous to the main estimate holds locally, i.e. in "appropriate" charts. This is done by establishing a partial differential equation satisfied by the components of the second fundamental form $A$ in terms of the components of its traceless part $\AA$. Then we invoke the classical Calderón-Zygmund inequality ( $[\mathbf{C Z 5 6}]$ ), which basically tells us that the $L^{p}$-norm of the Hessian of a function can be bounded by the $L^{p}$-norm of its Laplacian. Incidentally, the use of this famous result is also a key step in S. K. Vodop' yanov's approach ( $[$ Res 94, ch.6]), which otherwise relies on methods completely different from ours. Using the bound on $\|A\|_{L^{p}(\Sigma)}$, we show that we can cover $\Sigma$ with a controlled number of such "appropriate" charts with large enough overlap. The global main estimate then follows from the local one.
2.2. Chapter 2. The sub-critical case $1 \leq p<n$. The idea here is to apply the same strategy as in the super-critical case. While in Chapter 1 we could conclude as described above, we need to make additional assumptions in the case at hand. It turns out that assuming convexity of $\Sigma$ is sufficient.

Observing that the main estimate in the present situation is trivially fulfilled whenever its right-hand side is not small, we can assume without loss of generality that some preset bound $c_{0}^{\prime}>0$ on $\|\AA\|_{L^{p}(\Sigma)}$ be given. But in the non-super-critical, convex case this implies a bound $c_{0}$ on $\|A\|_{L^{p}(\Sigma)}$, thus eliminating the disappointing restriction encountered before.

However, the passage from local to global, namely the generation of a suitable covering, proved to be fairly laborious. This is due to the fact that we could no longer use the additional regularity obtained for $p>n$ to show that the desired charts have a certain minimal size. To achieve the same here, we need to rule out the possibility that the hypersurface be close to degenerate. We do this by proving that $\Sigma$ must be contained in a spherical shell whose radii depend only on $n, p$ and our preset bound $c_{0}$. Notice, though, that we did not show a quantitative version of this circumstance, i.e., we do not know how the deviation of the ratio of these two radii from one is controlled by the $L^{p}$-norm of $\AA$. Moreover, even if we feel that the non-degeneracy should hold in the case $p=1$ as well, we could not establish it there. In turn, our considerations work equally well for proving the main estimate in the critical case $p=n$, even though requiring convexity in that situation seems exaggerated.
2.3. Chapter 3. $L^{2}$-theory. Here we treat a special non-super-critical case, namely the one when $p=2$. Our central result is the main estimate with an explicit constant for hypersurfaces $\Sigma$ of non-negative Ricci curvature (which, in the Euclidean case, is equivalent to convexity - see Proposition 3.2). The proof of this theorem is very short and based on an elegant strategy employed by C. De Lellis and P. M. Topping in [DLT10]. The key idea is to solve a suitable Poisson problem on $\Sigma$.

Afterwards, we exhibit a geometric flow approach due to G. Huisken (privately communicated to C. De Lellis) which is tailored to the critical two-dimensional case. This yields an alternative proof of our estimate and applies to strongly mean convex boundaries of star-shaped domains. These assumptions are, in fact, strictly weaker than ours, and they are enough to retrieve the same constant we obtained before.

We then endeavour to adapt this technique to the higher dimensional situation, where we face a sub-critical problem. Despite assuming convexity again, our reasoning only delivers a much larger constant than expected in the main estimate. Nevertheless, we find the argument instructive.
2.4. Chapter 4. Optimality. In this chapter, we subject our results to some optimality considerations. We start by demonstrating that the constant found in our $L^{2}$-estimate of Chapter 3 is optimal among Ricci-positive hypersurfaces. We do this by showing the existence of a suitable deformation of a round sphere. Afterwards, by constructing an explicit counter-example, we prove that the assumption of having non-negative Ricci curvature is optimal for all sub-critical exponents $p \in[1, n)$. Finally, restricting ourselves to the critical two-dimensional case, we show that our optimal constant cannot work for generic surfaces, thereby relating our result to the one of C. De Lellis and S. Müller. That last counter-example is due in part to P. M. Topping and C. De Lellis, whereas for the previous, the latter and S. Müller should be credited as well.
2.5. Complements. We conclude this work with two appendices. The first one contains three little lemmas which are all true in a slightly more general context than the one in which they are applied in the text and might be of independent interest. The second appendix, in contrast, contains work which has no effect on the topics just described. It is concerned with a preliminary step towards generalising the present results to hypersurfaces in Riemannian manifolds of non-negative Ricci curvature. More precisely, we give several $L^{2}$-integral quantities on a spherical cap or on its boundary, and calculate their second variation under a volume-preserving deformation. The chosen quantities appear both in the "Almost-Schur Lemma" of C. De Lellis and P. M. Topping ([DLT10]), as well as in this thesis. Admittedly, the obtained formulæ are rather unmanageable, even in a special case in which a lot of simplifications occur. In fact, not even that situation gave us a hint on how to proceed, and no result is obtained so far. We produce these calculations in spite of that, just in case someone might find them useful.

## 3. Discussion of our work, open problems

3.1. Weaknesses. In the previous section, we already mentioned two major shortcomings of our results, most prominently the necessity in the super-critical case to require a bound on the $L^{p}$-norm of the second fundamental form. At this point, we have no hope to overcome this restriction using our techniques. Also, we pointed out our inability to prove a $C^{0}$-estimate for the closeness to a sphere, something which we expect to be of importance in possible applications.

For this last objection, however, we already made a first step into the right direction, by concluding qualitative $C^{0}$-closeness from our main estimate. In contrast to the first drawback, then, this can be considered work in progress.
3.2. Implications. Perhaps the most satisfactory parts of the present work are the two theorems treating the sub-critical cases. Our optimality results of Chapter 4 indicate their strengths. Indeed, not only were we able to provide the best possible
constant in the $L^{2}$-estimate of Chapter 3 , but we also proved that assuming convexity is necessary for general sub-critical exponents, as treated in Chapter 2. Both our main theorems of these chapters are thus given under optimal hypotheses when $p<n$, and the $L^{2}$-theorem even gives an optimal statement.

In contrast, the main theorem of Chapter 1 about the general super-critical case lacks this feature. Nevertheless, it appears to step out from the usual (convex) context which we found in the literature. It would therefore seem a worthwhile goal to strengthen this result.
3.3. On the optimal constant. Also, one should be able to extract useful information by determining in general the optimal constant in the main estimate. Regarding this constant, another route one might want to pursue is a further analysis of the critical two-dimensional case. In view of the results we present, one might wonder about the most general hypotheses under which the constant we and G. Huisken found would apply. We know now that mean convex and star-shaped is sufficient, but this does not preclude the possibility of a more general condition under which the optimal constant is valid. On a related note, it would be interesting to investigate the size of the universal constant appearing in the original work of C. De Lellis and S. Müller.
3.4. Beyond Euclidean? Finally, a widely open problem is whether it is possible to combine our work with the one of C. De Lellis and P. M. Topping [DLT10], who prove an $L^{2}$-estimate analogous to ours (note that the inequality below appears to have been obtained already by B. Andrews in unpublished work - see [CLN06, §B.3, pp.517-519] for an exposition; however, [DLT10] also show the optimality of the constant appearing on the right-hand side). More precisely, for closed Riemannian manifolds of non-negative Ricci curvature and dimension larger than two, they prove

$$
\left\|\operatorname{Ric}-\frac{\overline{\text { Scal }}}{n} g\right\|_{L^{2}} \leq \frac{n}{n-2} \| \text { Ric }-\frac{\text { Scal }}{n} g \|_{L^{2}},
$$

and show that the constant on the right-hand side is optimal (we wish to repeat at this point, that the proof of our $L^{2}$-estimate is, in fact, a simple adaptation of theirs). A first, albeit very small, step in this direction is attached as the second appendix.

## Notations and conventions

The present work assumes a certain familiarity with Riemannian geometry and generally follows standard notation - for background references, the reader may consult the books recommended below.

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General remarks. In the whole text, the dimension $n$ is assumed to be at least two. The symbol $\Sigma$ denotes a smooth, closed (i.e. compact, without boundary) and connected hypersurface in $\mathbb{R}^{n+1}$. The Riemannian metric $g$ and the second fundamental form $A$ on $\Sigma$ are the ones inherited from Euclidean $\mathbb{R}^{n+1}$. We will, in general, abuse notation for objects of the tangent bundle, insofar as we will identify them with their respective push-forwards via the embedding of $\Sigma$ into $\mathbb{R}^{n+1}$. Also, whenever we take norms or traces of quantities in the tensor bundle, we imply the appropriate usage of the metric $g$ (however, we often write $\operatorname{tr}_{g}$ for the trace-operator). The same holds true for derivatives, unless stated otherwise. Moreover, whenever we put indices on quantities of the tensor bundle without specifying the chart we use, they will refer to some generic system of coordinates, and the metric $g$ will be used to raise and lower those indices (except for some cases we will point out).

Background reading. The author got his geometric education from several different sources, among which [BG92], [dC76], [Mi197], [Lee97], [GHL04], [dC92] and $[\mathbf{N i c} 07 \mathbf{b}]$. He also liked some expositions in $[\mathbf{B e s 8 7}]$ and $\left[\mathbf{O}^{\prime} \mathbf{N 8 3}\right]$, and enjoyed reading parts of [Ber03]. For analytical questions of a general nature, he learnt a lot from [Rud87], [Bar95], [Eva98], [GT01] and [EG92], whereas he recommends the outstanding [Aub98], as well as [Jos08], for more specific questions on analysis on manifolds (but see also the introductory text [Spi65]). Regarding convexity, he found [Sch93] extremely helpful, as well as [Roc70]. Finally, for topics in functional analysis, he usually uses [Bre83] and [Rud91].

List of symbols. What follows is a list of generic symbols which we will use frequently and, on many occasions, without specifying their meaning again. We sometimes use subscripts to emphasise the context to which these symbols belong.

| $B_{r}(x)$ | The ball of radius $r$ around $x$ in the ambient space (usually $\mathbb{R}^{n+1}$ ) |
| :---: | :---: |
| $D_{r}(y)$ | The ball of radius $r$ around $y$ in coordinate space (normally $\mathbb{R}^{n}$ ) |
| $\operatorname{vol}_{m}$ | The $m$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ |
| $\langle\cdot, \cdot\rangle_{P}$ | The Euclidean scalar product in a linear subspace $P \subset \mathbb{R}^{n+1}$ |
| id | The identity (1,1)-tensor |
| $\delta^{i}{ }_{j}$ | The Kronecker delta |
| D | The coordinate derivative of Euclidean space |
| $g$ | The Riemannian metric of $\Sigma$ |
| $d \mathrm{vol}_{g}$ | The volume form associated to $g$ |
| $\|\cdot\|_{g}$ | The Hilbert-Schmidt norm with respect to $g$, acting on sections of the tensor bundle - we usually drop the subscript $g$ |
| $\operatorname{tr}_{g}$ | The trace with respect to $g$ |
| $\nabla$ | The Levi-Civita connection of $g$ |
| $\Delta$ | The Laplace-Beltrami operator with respect to $\nabla$, but sometimes also the usual Laplace operator |
| div | The covariant divergence operator acting on symmetric two-tensor fields or vector fields, but occasionally also the Euclidean divergence operator |
| Riem | The Riemann curvature tensor associated to $g$ |
| Ric | The Ricci tensor obtained from Riem |
| Scal | The scalar curvature obtained from Ric |
| $\nu$ | The outer unit normal vector field to $\Sigma$ in $\mathbb{R}^{n+1}$, also called Gauss map |
| A | The second fundamental form of $\Sigma$ in $\mathbb{R}^{n+1}$ |
| H | The mean curvature of $\Sigma, H=\operatorname{tr}_{g} A$ |
| $\stackrel{\circ}{\text { A }}$ | The traceless part of $A, A=A-\frac{H}{n} g$ |
| $B: C$ | The full contraction of the two smooth, symmetric two-tensor fields $B$ and $C$, i.e., in coordinates, $B: C=\sum_{i, j, k, l=1}^{n}\left(g^{-1}\right)^{i k}\left(g^{-1}\right)^{j l} B_{i j} C_{k l}$ |

Also, for smooth functions $\varphi: \Sigma \rightarrow \mathbb{R}$, we set

$$
\bar{\varphi}=f_{\Sigma} \varphi d \operatorname{vol}_{g}=\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} \varphi d \operatorname{vol}_{g}
$$

to denote the average of $\varphi$ over the hypersurface $\Sigma$, but usually we do not specify the volume form when we write an integral.

Sign conventions. If $X, Y$ and $Z$ denote any smooth vector fields on $\Sigma$, extended to a neighbourhood of $\Sigma$, then we put

$$
\operatorname{Riem}(X, Y) Z=\nabla_{Y} \nabla_{X} Z-\nabla_{X} \nabla_{Y} Z-\nabla_{\nabla_{Y} X-\nabla_{X} Y} Z
$$

and

$$
A(X, Y)=-\left\langle D_{X} Y, \nu\right\rangle_{\mathbb{R}^{n+1}}
$$

These signs are chosen in such a way that the usual $n$-sphere $S_{R}^{n}$ of radius $R>0$ has scalar curvature $\operatorname{Scal}_{S_{R}^{n}}=\frac{n(n+1)}{R}$, and that the second fundamental form $A$ of $\Sigma$ has non-negative eigenvalues, whenever $\Sigma$ bounds a convex domain.

## CHAPTER 1

## The super-critical case for generic hypersurfaces

In this chapter, we prove our main estimate for generic $n$-dimensional hypersurfaces of $\mathbb{R}^{n+1}$ in the case $p>n \geq 2$, and show that it implies qualitative $C^{0}$-closeness to a sphere. Unfortunately, the constant on the right-hand side of the estimate depends on the $L^{p}$-norm over the hypersurface of the second fundamental form of the hypersurface. At this point it is unclear to the author how to mend that.

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## 1. The main theorem of this chapter

Our goal is to prove
Theorem 1.1. Let $n \geq 2, p \in(n,+\infty)$ and $c_{0}>0$ be given. Then there is a constant $C>0$, depending only on $n, p$ and $c_{0}$, such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected $n$-dimensional hypersurface with induced Riemannian metric $g$ and such that
(a) $\operatorname{vol}_{n}(\Sigma)=1$
and
(b) $\|A\|_{L^{p}(\Sigma)}=\left(\int_{\Sigma}|A|^{p}\right)^{\frac{1}{p}} \leq c_{0}$,
then

$$
\begin{equation*}
\min _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C\|\AA\|_{L^{p}(\Sigma)} \tag{1.1}
\end{equation*}
$$

We prove the theorem in the next section, and show in Section 3 how it implies
Corollary 1.2 (to Theorem 1.1). Let $n \geq 2, p \in(n,+\infty), c_{0}>0$ and $\epsilon>0$ be given. Then there is a constant $\delta>0$, depending only on $n, p, c_{0}$ and $\epsilon$, such that: if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed and connected $n$-dimensional hypersurface such that
(a) $\operatorname{vol}_{n}(\Sigma)=1$,
(b) $\|A\|_{L^{p}(\Sigma)} \leq c_{0}$
and
(c) $\|\AA\|_{L^{p}(\Sigma)}<\delta$,
then

$$
d_{\mathrm{HD}}\left(\Sigma, \partial B_{\rho_{0}}(x)\right)<\epsilon, \quad \text { for some } x \in \mathbb{R}^{n+1}
$$

where $\rho_{0}=\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{-\frac{1}{n}}$ and $d_{\mathrm{HD}}$ denotes the Hausdorff distance in Euclidean $\mathbb{R}^{n+1}$.

The idea of the proof of Theorem 1.1 (performed in Section 2) is as follows. We will show that an estimate analogous to (1.1) holds in charts where a portion of $\Sigma$ is given as the graph of a smooth function. This will be done by examining the differential equation that each component of $A$ satisfies in such charts in terms of derivatives of $\AA$ and applying the Calderón-Zygmund inequality. In order to get the global estimate, we show that $\Sigma$ can be covered by a controlled number of geodesic balls with a certain size and overlap and such that each ball is contained in the graph over a tangential plane of a smooth Lipschitz function. As will become clear in the proof, we have to assume an upper bound for $\|A\|_{L^{p}(\Sigma)}$ as well as $p>n$, so that we can "patch together" the local estimates to obtain the global one.

In order to get the local estimate, we will make use of the following, rather surprising result, which states that the partial derivatives (with respect to the Cartesian coordinates of a chart in which a portion of $\Sigma$ is given as a graph) of the second fundamental form $A$ are entirely determined by the partial derivatives of its traceless part $\AA$. The proof of this will be given in Section 4.

Lemma 1.3. Let $U \subset \mathbb{R}^{n}, n \geq 2$, be an open set and assume $\Sigma$ is the graph of a smooth function $u: U \rightarrow \mathbb{R}\left(\Sigma\right.$ is thus a smooth hypersurface in $\left.\mathbb{R}^{n+1}\right)$. Let $\phi: U \rightarrow \mathbb{R}^{n+1}, x \mapsto(x, u(x))$ be the corresponding parametrisation. Denote by $D$ the derivation with respect to the Cartesian coordinates of $\mathbb{R}^{n}$ using the chart $\phi$. Then the partial derivatives in $U$ of the second fundamental form $A$ of $\Sigma$ satisfy

$$
\begin{equation*}
D_{k} A_{j}^{i}=D_{k} \AA_{j}^{i}+\frac{1}{n-1}\left(\sum_{l=1}^{n} D_{l} \AA^{l}{ }_{k}\right) \delta_{j}^{i}, \quad \forall i, j, k, \tag{1.2}
\end{equation*}
$$

where $\AA^{i}{ }_{j}=A^{i}{ }_{j}-\frac{1}{n} \sum_{l=1}^{n} A^{l}{ }_{l} \delta^{i}{ }_{j}$ are the components of the traceless part $\AA$ of $A$.

Remark 1.4. Notice that the statement of Lemma 1.3 would be a rather obvious consequence of the Codazzi equations, if (1.2) were given with respect to the LeviCivita connection $\nabla$. The point here is, of course, that the identity holds for the usual "Euclidean partial derivatives" in the corresponding chart.

We will think of expression (1.2) as a system of partial differential equations, where the unknowns are the components of $A$. The following proposition then yields the local estimate. Its proof, which is deferred to Section 5, is done by taking the trace of (1.2) to get an equation of the form $D \mathfrak{u}=\operatorname{div} \mathfrak{f}$ which, by the CalderónZygmund inequality, admits the desired estimate.

Proposition 1.5. Let $U \subset \mathbb{R}^{n}$, $n \geq 2$, be an open set with $0 \in U$, and assume $\Sigma$ is as in Lemma 1.3. Let $R>0$ be such that $D_{R}(0) \subset U$ and assume $p \in(1,+\infty)$ be given. Then there exists a constant $C>0$, which depends only on $n$ and $p$, and there exists a $\lambda \in \mathbb{R}$ such that

$$
\begin{equation*}
\|A-\lambda g\|_{L^{p}\left(D_{R / 4}(0)\right)} \leq C\|\stackrel{\circ}{A}\|_{L^{p}\left(D_{R}(0)\right)} . \tag{1.3}
\end{equation*}
$$

Remark 1.6. Notice that, in contrast to (1.1), the constant $C$ on the right-hand side of (1.3) is independent of the second fundamental form $A$ of the hypersurface $\Sigma$. We will make use of this fact in the next chapter. Here, it is the lower bound on $R$, required to be able to apply Lemma 1.7 (see below), that will introduce this dependence. In addition, it is also this lower bound that will restrict the global theorem to the cases $p>n$.

Now, to obtain the global estimate, we need the technical lemma below. It states that we can cover $\Sigma$ by a controlled number of geodesic balls in which a portion of $\Sigma$ is represented as a Lipschitz graph. As every smooth hypersurface can locally be parametrised as the graph of a Lipschitz map, the important assumption will be that there is a uniform upper bound on the Lipschitz constants of these maps, as well as a uniform lower bound on the size of the domain on which the maps are defined. As we will see in Section 2, the assumptions of Theorem 1.1 imply those of the lemma. The proof of the latter, performed in Section 6, is based on the observation that a geodesic sphere with small enough radius $\rho$ is contained in the graph of one of the Lischitz maps over a ring with radii related to $\rho$. Consequently, the volume of small geodesic balls on $\Sigma$ is controlled by the volume of Euclidean balls. Since $\Sigma$ has normalised area and is compact, the existence of the aforementioned cover is then assured.

Lemma 1.7. Let $\Sigma \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a smooth closed hypersurface with normalised area, $\operatorname{vol}_{n}(\Sigma)=1$, and let $r_{0}>0$ and $L>0$ be given. Assume that, for each point $q \in \Sigma$, there is an isometry $\Phi_{q}$ of $\mathbb{R}^{n+1}$ and a smooth Lipschitz function $u_{q}: D_{r_{0}}(0) \subset$ $\mathbb{R}^{n} \rightarrow \mathbb{R}$ with Lipschitz constant at most $L$, such that $\Phi_{q}\left(D_{r_{0}}(0), u_{q}\left(D_{r_{0}}(0)\right)\right) \subset \Sigma$ and $\Phi_{q}\left(0, u_{q}(0)\right)=q$.
Then, for every $q \in \Sigma$, the geodesic ball $\mathcal{B}_{r_{0}}^{g}(q) \subset \Sigma$ of radius $r_{0}$ around $q$ is contained in $\Phi_{q}\left(D_{r_{0}}(0), u_{q}\left(D_{r_{0}}(0)\right)\right)$, and there is a constant $C$, depending only on $n$, such
that $\Sigma$ can be covered with $N$ such geodesic balls, where

$$
\begin{equation*}
N \leq C \frac{(1+L)^{2 n}}{r_{0}^{n}} \tag{1.4}
\end{equation*}
$$

## 2. Proof of Theorem 1.1

In this section we deduce the global rigidity estimate (1.1) from the local rigidity estimate (Proposition 1.5) and the existence of a covering as described in Lemma 1.7.
2.1. Construction of Lipschitz charts. So let $\Sigma$ be as in Theorem 1.1 and pick any point $q \in \Sigma$. Without loss of generality, we may assume that $q=0 \in \mathbb{R}^{n+1}$ and $T_{q} \Sigma=\mathbb{R}^{n} \times\{0\} \subset \mathbb{R}^{n+1}$. By smoothness, a portion of $\Sigma$ is then given as the graph of a smooth function $u: U \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$ with $u(0)=0$ and $D u(0)=0$, where $D$ denotes derivation with respect to the Cartesian coordinates of $\mathbb{R}^{n}$. We can assume that $U$ is maximal, in the sense that if a portion of $\Sigma$ can be represented as a graph over $V \supset U$, then $V=U$ necessarily.

In the proof of Lemma 1.3 (Section 4), we will see that, in the coordinates at hand, the metric $g$ of $\Sigma$, its inverse and the second fundamental form $A$ of $\Sigma$ are given by (see equations (1.13), (1.14) and (1.15)):

$$
g_{i j}=\delta_{i j}+D_{i} u D_{j} u, \quad g^{i j}=\delta^{i j}-\frac{D^{i} u D^{j} u}{1+|D u|^{2}}, \quad A_{i j}=\frac{D_{i} D_{j} u}{\sqrt{1+|D u|^{2}}}
$$

respectively. Let

$$
v=\frac{D u}{\sqrt{1+|D u|^{2}}}
$$

Notice that, by definition, $|v|<1$. Moreover, if $0<c<1$ and $|v| \leq c$ then $|D u| \leq c / \sqrt{1-c^{2}}<+\infty$, implying that $u$ is Lipschitz with Lipschitz constant $L \leq c / \sqrt{1-c^{2}}$.

Define

$$
R=\sup \left\{r>0\left|\sup _{e \in \partial D_{1}(0)}\right| v(r e) \left\lvert\, \leq \frac{1}{2}\right.\right\}
$$

as the maximal radius of a ball $D_{R}(0) \subset \mathbb{R}^{n}$ such that the length of $v$ is uniformly bounded by $1 / 2$. Clearly, $R>0$ since $v(0)=0$ and $v$ is continuous. Also, $R<+\infty$ since otherwise $u$ would be defined on the whole of $\mathbb{R}^{n}$ and $\Sigma$ would not be compact. In fact, in view of equation (1.13) of Section 4 and our assumption that $\operatorname{vol}_{n}(\Sigma)=1$, we must have $R \leq\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{-\frac{1}{n}}$. Thus $R$ is well-defined and $u$ is at least defined on $D_{R}(0)$. Moreover $|v| \leq 1 / 2$ holds throughout $D_{R}(0)$, implying that $u$ is uniformly Lipschitz on $D_{R}(0)$ with constant $1 / \sqrt{3}$.
2.2. The local estimate. Applying Proposition 1.5 for $p>n$, we obtain a constant $C$, depending only on $n$ and $p$, and some $\lambda \in \mathbb{R}$ such that

$$
\|A-\lambda g\|_{L^{p}\left(D_{r / 4}(0)\right)} \leq C\|\AA\|_{L^{p}\left(D_{r}(0)\right)}, \quad \forall r \leq R
$$

Remark 1.8. As we will see in the proof of Lemma 1.7 (Section 6), any geodesic ball $\mathcal{B}_{r}^{g}(q)$ with centre $q$ and radius $r \leq R$ is contained in the graph of $u$ over $D_{r}(0)$. Moreover, since $u$ is Lipschitz, the area $\operatorname{vol}_{n}\left(\mathcal{B}_{r}^{g}(q)\right)$ of such a geodesic ball is controlled by $\operatorname{vol}_{n}\left(D_{r}(0)\right)$. This will be useful later.

The first part of the remark yields the following local estimate: For all $q \in \Sigma$ there is a $\lambda \in \mathbb{R}$ such that for $r \leq R / 4$

$$
\begin{equation*}
\|A-\lambda g\|_{L^{p}\left(\mathcal{B}_{r}^{g}(q)\right)} \leq C\|\AA\|_{L^{p}(\Sigma)} \tag{1.5}
\end{equation*}
$$

where $C$ depends only on $n$ and $p$.
2.3. A lower bound on the size of the Lipschitz charts. In order to apply Lemma 1.7 to get the global analogue of the above estimate, we now show that $R$ is bounded from below. For this we calculate:

$$
\begin{align*}
D_{j} v^{i} & =D_{j} \frac{D^{i} u}{\sqrt{1+|D u|^{2}}}=\frac{D_{j} D^{i} u}{\sqrt{1+|D u|^{2}}}-\frac{\sum_{l=1}^{n} D^{i} u D^{l} u D_{j} D_{l} u}{(1+|D u|)^{3 / 2}} \\
& =\sum_{l=1}^{n}\left(\delta^{i l}-\frac{D^{i} u D^{l} u}{1+|D u|^{2}}\right) \frac{D_{l} D_{j} u}{\sqrt{1+|D u|^{2}}}=\sum_{l=1}^{n} g^{i l} A_{l j}=A_{j}^{i} \tag{1.6}
\end{align*}
$$

Thus $|D v|=|A|$ in every point of $D_{R}(0)$. We apply the Morrey-type estimate found in Lemma A. 1 of the appendix to $v$ at $x=0$, and we get with identity (1.6)

$$
\sup _{y \in D_{R}(0)} \frac{|v(y)|}{|y|^{(p-n) / p}} \leq C\|D v\|_{L^{p}\left(D_{R}(0)\right)}=C\|A\|_{L^{p}\left(D_{R}(0)\right)}
$$

Now, by the maximality of $R$, there exists an $e \in \partial D_{1}(0)$ such that $\lim _{r \nearrow R}|v(r e)|=$ $1 / 2$. For $y=R e$, we therefore obtain

$$
\begin{equation*}
\frac{1 / 2}{R^{(p-n) / p}} \leq C\|A\|_{L^{p}\left(D_{R}(0)\right)} \tag{1.7}
\end{equation*}
$$

Since $\|A\|_{L^{p}(\Sigma)} \leq c_{0}$ by assumption, we infer that there is a constant $C$, depending only on $n$ and $p$, such that, if we define

$$
\begin{equation*}
R_{0}=C c_{0}^{-p /(p-n)} \tag{1.8}
\end{equation*}
$$

then $R \geq R_{0}$.
Remark 1.9. Conversely, in view of the upper bound $\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{-\frac{1}{n}}$ on $R$ coming from the assumption that $\operatorname{vol}_{n}(\Sigma)=1$ (see (1.13) in Section 4), we infer from inequality (1.7) that

$$
\|A\|_{L^{p}(\Sigma)} \geq \frac{\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{\frac{n-p}{n p}}}{2 C^{\prime}}
$$

where $C^{\prime}$ is the constant of Lemma A. 1 that depends only on $n$ and $p$. This observation will be useful in the proof of the qualitative $C^{0}$-closeness in Section 3.
2.4. Local to global. Taking $r_{0}=R_{0} / 8$ and $L=1 / \sqrt{3}$, we can now apply Lemma 1.7 (twice - once both statements, then only the first) to obtain a covering $\left\{\mathcal{B}_{r_{0}}^{g}\left(q_{j}\right)\right\}_{1 \leq j \leq N}$ of $\Sigma$ by geodesic balls of radius $r_{0}$ such that, in each $\mathcal{B}_{2 r_{0}}^{g}\left(q_{j}\right)$, the local estimate (1.5) holds for all $r \leq 2 r_{0}=R_{0} / 4$. By the triangle inequality, any two balls of the covering that intersect will have the property that the balls with same centres but twice the radius have an overlap that contains, at least, a geodesic ball of radius $r_{0}$. This will be useful to "patch together" the local estimates in order to obtain the global one, since, obviously, $\lambda$ depends on the geodesic ball (cf. the proof of Proposition 1.5 in Section 5). Indeed, given that the covering of $\Sigma$ by $\left\{\mathcal{B}_{2 r_{0}}^{g}\left(q_{j}\right)\right\}_{1 \leq j \leq N}$ has sufficiently large overlaps, the difference of the $\lambda$ s in two neighbouring balls is controlled.

In fact, let $\Omega_{1}, \Omega_{2} \subset \Sigma$ with $\operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)>0$ and assume that there are $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ such that $\left\|A-\lambda_{1} g\right\|_{L^{p}\left(\Omega_{1}\right)} \leq \beta$ and $\left\|A-\lambda_{2} g\right\|_{L^{p}\left(\Omega_{2}\right)} \leq \beta$ for some $\beta$ independent of $\Omega_{1}$ and $\Omega_{2}$. We have

$$
\begin{aligned}
\left|\lambda_{1}-\lambda_{2}\right| & =\left(\left|\lambda_{1}-\lambda_{2}\right|^{p}\right)^{\frac{1}{p}}=\left(\frac{1}{\operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)} \int_{\Omega_{1} \cap \Omega_{2}}\left|\lambda_{1}-\lambda_{2}\right|^{p}\right)^{\frac{1}{p}} \\
& =\frac{1}{\operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)^{\frac{1}{p}}}\left\|\lambda_{1}-\lambda_{2}\right\|_{L^{p}\left(\Omega_{1} \cap \Omega_{2}\right)} \\
& =\frac{1}{n \operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)^{\frac{1}{p}}}\left\|\lambda_{1} g-A+A-\lambda_{2} g\right\|_{L^{p}\left(\Omega_{1} \cap \Omega_{2}\right)} \\
& \leq \frac{1}{n \operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)^{\frac{1}{p}}}\left(\left\|A-\lambda_{1} g\right\|_{L^{p}\left(\Omega_{1} \cap \Omega_{2}\right)}+\left\|A-\lambda_{2} g\right\|_{L^{p}\left(\Omega_{1} \cap \Omega_{2}\right)}\right) \\
& \leq \frac{2 \beta}{n \operatorname{vol}_{n}\left(\Omega_{1} \cap \Omega_{2}\right)^{\frac{1}{p}}} .
\end{aligned}
$$

In particular, if $\Omega_{1}$ and $\Omega_{2}$ are two intersecting geodesic balls of radius $r_{0}$ from the cover, such that the intersection of the balls with doubled radius contains a geodesic ball of radius $r_{0}$, then the local estimate (1.5), together with Remark 1.8, yields a constant $C$, depending only on $n$ and $p$, such that

$$
\left|\lambda_{1}-\lambda_{2}\right| \leq C r_{0}^{-\frac{n}{p}}\|\AA\|_{L^{p}(\Sigma)}
$$

Consider a path joining the ball in the cover with the smallest $\lambda$, say $\lambda_{\min }$, to the one with the largest $\lambda$, say $\lambda_{\max }$. Since the path can cross at most $N$ distinct balls, we find that

$$
\begin{equation*}
\left|\lambda_{\max }-\lambda_{\min }\right| \leq C r_{0}^{-\frac{n}{p}} N\|\AA\|_{L^{p}(\Sigma)} \tag{1.9}
\end{equation*}
$$

where the constant $C$ depends only on $n$ and $p$. Let $\mathcal{B}_{j}^{g}, j=1, \ldots, N$, denote the geodesic balls of the cover and $\lambda_{j}$ their corresponding $\lambda \mathrm{s}$. By virtue of (1.9) above
and the local estimate (1.5) we then have for any $\lambda$ between $\lambda_{\min }$ and $\lambda_{\max }$,

$$
\begin{aligned}
\|A-\lambda g\|_{L^{p}(\Sigma)} & \leq \sum_{j=1}^{N}\|A-\lambda g\|_{L^{p}\left(\mathcal{B}_{j}^{g}\right)} \\
& =\sum_{j=1}^{N}\left\|A-\lambda_{j} g+\lambda_{j} g-\lambda g\right\|_{L^{p}\left(\mathcal{B}_{j}^{g}\right)} \\
& \leq \sum_{j=1}^{N}\left(\left\|A-\lambda_{j} g\right\|_{L^{p}\left(\mathcal{B}_{j}^{g}\right)}+\left\|\lambda_{j} g-\lambda g\right\|_{L^{p}\left(\mathcal{B}_{j}^{g}\right)}\right) \\
& \leq \sum_{j=1}^{N}\left(\left\|A-\lambda_{j} g\right\|_{L^{p}\left(\mathcal{B}_{j}^{g}\right)}+n\left|\lambda_{\max }-\lambda_{\min }\right| \operatorname{vol}_{n}\left(\mathcal{B}_{j}^{g}\right)^{\frac{1}{p}}\right) \\
& \leq \sum_{j=1}^{N} C_{1}\|\AA\|_{L^{p}(\Sigma)}\left(1+C_{2} N r_{0}^{-\frac{n}{p}}\left(r_{0}^{n}\right)^{\frac{1}{p}}\right) \\
& \leq \sum_{j=1}^{N} C_{3}(1+N)\|\AA\|_{L^{p}(\Sigma)} \\
& \leq C N^{2}\|\AA\|_{L^{p}(\Sigma)},
\end{aligned}
$$

where the constants $C_{1}, C_{2}, C_{3}$ and $C$ depend only on $n$ and $p$. Using the upper bound (1.4) on the number $N$ of balls in the cover and the expression (1.8) for $4 r_{0}$ we finally obtain

$$
\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C c_{0}^{\frac{2 n p}{p-n}}\|\AA\|_{L^{p}(\Sigma)}
$$

where, again, $C$ depends only on $n$ and $p$. This proves Theorem 1.1.

## 3. Proof of Corollary 1.2

In this section we want to show that any hypersurface $\Sigma$ that fulfils the assumptions of Theorem 1.1 has to be $C^{0}$-close to a sphere, whenever $\|\AA\|_{L^{p}(\Sigma)}$ is small enough. We do this through a contradiction argument.
3.1. Preliminaries. Assume Corollary 1.2 were false. Then we would find a sequence $\left(\Sigma_{k}\right)_{k \in \mathbb{N}}$ of smooth, closed and connected hypersurfaces of $\mathbb{R}^{n+1}$, satisfying $\operatorname{vol}_{n}\left(\Sigma_{k}\right)=1$ and $\|A\|_{L^{p}\left(\Sigma_{k}\right)} \leq c_{0}$ independently of $k$, and such that

$$
\lim _{k \rightarrow \infty}\|\AA\|_{L^{p}\left(\Sigma_{k}\right)}=0
$$

and the hypersurfaces $\Sigma_{k}$ do not converge (in the Hausdorff topology) to a ball (notice that we do not even claim that there is a limit set). We shall show that this is impossible.

We begin with the following observations. Applying Theorem 1.1 to each $\Sigma_{k}$, we get, for every $k \in \mathbb{N}$, a $\lambda_{k} \in \mathbb{R}$ such that

$$
\left\|A-\lambda_{k} g\right\|_{L^{p}\left(\Sigma_{k}\right)} \leq C\|\AA\|_{L^{p}\left(\Sigma_{k}\right)},
$$

where $C>0$ depends only on $n, p$ and $c_{0}$. The sequence $\left(\lambda_{k}\right)_{k \in \mathbb{N}}$ is bounded in $\mathbb{R}$, since

$$
\begin{align*}
\left|\lambda_{k}\right| & =\frac{1}{\sqrt{n}}\left\|\lambda_{k} g\right\|_{L^{p}\left(\Sigma_{k}\right)} \leq \frac{1}{\sqrt{n}}\|A\|_{L^{p}\left(\Sigma_{k}\right)}+\frac{1}{\sqrt{n}}\left\|A-\lambda_{k} g\right\|_{L^{p}\left(\Sigma_{k}\right)} \\
& \leq \frac{1}{\sqrt{n}}\|A\|_{L^{p}\left(\Sigma_{k}\right)}+\frac{C}{\sqrt{n}}\|\AA\|_{L^{p}\left(\Sigma_{k}\right)} \leq \frac{c_{0}}{\sqrt{n}}+\frac{C}{\sqrt{n}}\|A\|_{L^{p}\left(\Sigma_{k}\right)} \tag{1.10}
\end{align*}
$$

and the second term on the right-hand side converges to zero as $k \rightarrow \infty$. Hence, modulo picking a subsequence, we might without loss of generality assume that $\lim _{k \rightarrow \infty} \lambda_{k}=\bar{\lambda} \in \mathbb{R}$. Notice also that, in view of Remark 1.9 in the proof of Theorem 1.1 (Section 2), we have for each $k$, that $\|A\|_{L^{p}\left(\Sigma_{k}\right)} \geq \delta$, where $\delta>0$ depends only on $n$ and $p$. It follows that

$$
\begin{align*}
\left|\lambda_{k}\right| & =\frac{1}{\sqrt{n}}\left\|\lambda_{k} g\right\|_{L^{p}\left(\Sigma_{k}\right)} \geq \frac{1}{\sqrt{n}}\|A\|_{L^{p}\left(\Sigma_{k}\right)}-\frac{1}{\sqrt{n}}\left\|A-\lambda_{k} g\right\|_{L^{p}\left(\Sigma_{k}\right)} \\
& \geq \frac{\delta}{\sqrt{n}}-\frac{C}{\sqrt{n}}\|\AA\|_{L^{p}\left(\Sigma_{k}\right)} \tag{1.11}
\end{align*}
$$

whence $|\bar{\lambda}| \geq \delta / \sqrt{n}>0$.
Returning to the main argument, we show how our assumptions imply that, locally, the $\Sigma_{k}$ s have to converge to portions of spheres.
3.2. Local convergence. We pick, for each $k \in \mathbb{N}$, an arbitrary point $q_{k} \in \Sigma_{k}$. Modulo translations and rotations, we can without loss of generality assume that $q_{k}=0 \in \mathbb{R}^{n+1}$, and that $T_{q_{k}} \Sigma_{k}=\mathbb{R}^{n} \times\{0\} \subset \mathbb{R}^{n+1}$ (cf. the proof of Theorem 1.1). Then, from Section 2, we know that each $\Sigma_{k}$ has a portion given as the graph of a smooth $1 / \sqrt{3}$-Lipschitz function $u_{k}: \overline{D_{R}(0)} \rightarrow \mathbb{R}$, where $R$ depends only on $n, p$ and $c_{0}$ (notice that, by our construction in the proof of Theorem 1.1, the functions $u_{k}$ are Lipschitz up to the boundary of $\left.D_{R}(0) \subset \mathbb{R}^{n}\right)$. The sequence $\left(u_{k}\right)_{k \in \mathbb{N}}$ is therefore a pointwise bounded, equicontinuous sequence in the space of continuous, real-valued functions on the compact domain $\overline{D_{R}(0)}$. Then the Ascoli-Arzelà-Theorem (see, e.g., [Rud91, Thm.A5, p.394]) implies the existence of a subsequence $\left(u_{k_{l}}\right)_{l \in \mathbb{N}} \subset\left(u_{k}\right)_{k \in \mathbb{N}}$ that converges uniformly on $\overline{D_{R}(0)}$ to a continuous function $\bar{u}$.

Now define for each $l$, as in the proof of Theorem 1.1,

$$
v_{l}=\frac{D u_{k_{l}}}{\sqrt{1+\left|D u_{k_{l}}\right|^{2}}}
$$

There, we had seen that $D_{j}\left(v_{l}\right)^{i}=\left(A_{u_{k_{l}}}\right)^{i}$ for all $i, j \in\{1, \ldots, n\}$ (see eq. (1.6)). Then, since $\left|D u_{k_{l}}\right| \leq \frac{1}{\sqrt{3}}$ (implying that $\left|v_{l}\right| \leq \frac{1}{2}$ ) and, using our bound on the $L^{p}$-norm of $A$,

$$
\begin{aligned}
\left\|D v_{l}\right\|_{L^{1}\left(\overline{D_{R}(0)}\right)} & \leq\left(\operatorname{vol}_{n}\left(\overline{D_{R}(0)}\right)\right)^{1-\frac{1}{p}}\left\|A_{u_{k_{l}}}\right\|_{L^{p}\left(\overline{D_{R}(0)}\right)} \\
& \leq c_{0}\left(\operatorname{vol}_{n}\left(\overline{D_{R}(0)}\right)\right)^{1-\frac{1}{p}}
\end{aligned}
$$

$\left(v_{l}\right)_{l \in \mathbb{N}}$ is bounded in $W^{1,1}\left(\overline{D_{R}(0)} ; \mathbb{R}^{n}\right)$. Consequently, by Rellich-Kondrachov (see, e.g, [Eva98, Thm.1, §5.7, p.272]), there is a subsequence $\left(v_{l_{m}}\right)_{m \in \mathbb{N}} \subset\left(v_{l}\right)_{l \in \mathbb{N}}$ and a (vector-valued) function $\left.\bar{v} \in \underline{L^{1}\left(\overline{D_{R}(0)}\right.} ; \mathbb{R}^{n}\right)$ to which the $v_{l_{m}}$ converge in $L^{1}$.

For each $m \in \mathbb{N}$ and $y \in \overline{D_{R}(0)}$, let $w_{m}(y)=v_{l_{m}}(y)-\bar{\lambda} y$. Then the $w_{m}$ converge in $L^{1}$ to $\bar{w}=(\bar{v}-\bar{\lambda} \cdot)$. But thanks to Theorem 1.1, we also have

$$
\begin{aligned}
\left\|D w_{m}\right\|_{L^{1}\left(\overline{D_{R}(0)}\right)} & =\left\|D v_{l_{m}}-\bar{\lambda} \mathrm{id}\right\|_{L^{1}\left(\overline{D_{R}(0)}\right)} \\
& \leq\left(\operatorname{vol}_{n}\left(\overline{D_{R}(0)}\right)\right)^{1-\frac{1}{p}}\left\|A_{u_{k_{l_{m}}}}-\bar{\lambda} g_{u_{k_{l_{m}}}}\right\|_{L^{p}\left(\overline{D_{R}(0)}\right)} \\
& \leq\left(\operatorname{vol}_{n}\left(\overline{D_{R}(0)}\right)\right)^{1-\frac{1}{p}}\left(\|\AA\|_{L^{p}\left(\Sigma_{k_{l_{m}}}\right)}+\left\|\lambda_{k_{l_{m}}} g-\bar{\lambda} g\right\|_{L^{p}\left(\Sigma_{k_{l_{m}}}\right)}\right) \\
& \xrightarrow{m \rightarrow \infty} 0 .
\end{aligned}
$$

It follows, that $D \bar{w}=0$ in the sense of distributions (see, e.g., [GT01, Thm.7.4, p.150]), implying that $\bar{w}=c$ almost everywhere, for some $c \in \mathbb{R}^{n}$ (see, e.g., [LL97, Thm.6.11, p.138]). By the $L^{1}$-convergence of the $v_{l_{m}}$ to $\bar{v}$, we may then, after picking a subsequence, assume without loss of generality that for almost every $y \in \overline{D_{R}(0)}$,

$$
\begin{equation*}
v_{l_{m}}(y)=\frac{D u_{k_{l_{m}}}(y)}{\sqrt{1+\left|D u_{k_{l_{m}}}(y)\right|^{2}}} \stackrel{m \rightarrow \infty}{ } \quad \bar{\lambda} y+c . \tag{1.12}
\end{equation*}
$$

Since all the $v_{l_{m}}$ are bounded in modulus by $\frac{1}{2}$, we first observe that $|\bar{\lambda} y+c| \leq \frac{1}{2}$ for all $y \in \overline{D_{R}(0)}$, necessarily. Moreover, since the map $z \mapsto \frac{z}{\sqrt{1-|z|^{2}}}$ is (even uniformly) continuous on $\overline{D_{\frac{1}{2}}(0)} \subset D_{1}(0)$, it follows that

$$
D u_{k_{l_{m}}}(y) \quad \stackrel{m \rightarrow \infty}{\longrightarrow} \frac{\bar{\lambda} y+c}{\sqrt{1-|\bar{\lambda} y+c|^{2}}}, \quad \text { for almost every } y \in \overline{D_{R}(0)} .
$$

Since the $D u_{k_{l_{m}}}$ are uniformly bounded, the dominated convergence theorem (see, e.g., [Rud87, Thm.1.34, p.26]) then yields that the above convergence also holds in $L^{1}\left(\overline{D_{R}(0)}\right)$. In addition, the $u_{k_{l_{m}}}$ themselves obviously converge in $L^{1}$ to $\bar{u}$, as well. Consequently, as before, we conclude that

$$
D \bar{u}=\frac{\bar{\lambda} \cdot+c}{\sqrt{1-|\bar{\lambda} \cdot+c|^{2}}}
$$

in the sense of distributions. But then we argue once more that there must be constant $b \in \mathbb{R}$ such that, for almost every $y \in \overline{D_{R}(0)}$,

$$
\bar{u}(y)=b-\sqrt{1-|\bar{\lambda} y+c|^{2}}
$$

Since we established already that $\bar{u}$ is continuous, the above identity holds in all of $\overline{D_{R}(0)}$. Thus, indeed, $\bar{u}$ parametrises a portion of a sphere of radius $|\bar{\lambda}|^{-1}$ and centre $\left(-\frac{c}{\bar{\lambda}}, b\right)$.

We now show how we can easily obtain the global statement from Lemma 1.7.
3.3. Local to global. Applying the same technique as in the proof of Theorem 1.1 (Section 2), we cover each $\Sigma_{k_{l_{m}}}$ with geodesic balls of radius $2 r_{0}=R / 4$ such that neighbouring balls have large enough overlap. Then our local argument shows that the centres of the two spherical portions to which $\left(\Sigma_{k_{l_{m}}}\right)_{m \in \mathbb{N}}$ converges in neighbouring balls have to coincide. We conclude immediately that the whole sequence has to converge to a sphere of radius $|\bar{\lambda}|^{-1}$, which contradicts our assumptions. This finishes the proof of the corollary.

## 4. Proof of Lemma 1.3

Let $\Sigma \subset \mathbb{R}^{n+1}$ be as in Lemma 1.3. $\Sigma$ is embedded into $\mathbb{R}^{n+1}$ by the map

$$
f: U \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto f^{\alpha}(x)= \begin{cases}x^{\alpha}, & \alpha \in\{1, \ldots, n\} \\ u(x), & \alpha=n+1\end{cases}
$$

The metric of $\Sigma$ in the given coordinates $x^{i}$ is obtained from $g_{i j}=\sum_{\alpha=1}^{n+1} D_{i} f^{\alpha} D_{j} f_{\alpha}$. We adopt the convention that Greek indices run from 1 to $n+1$ (representing coordinates in the ambient space $\mathbb{R}^{n+1}$ ), whereas Latin ones run from 1 to $n$ (representing coordinates in the coordinate space $\mathbb{R}^{n}$ ). Since

$$
D_{i} f^{\alpha}= \begin{cases}\delta_{i}{ }^{\alpha}, & \alpha \in\{1, \ldots, n\} \\ D_{i} u, & \alpha=n+1\end{cases}
$$

we get

$$
\begin{equation*}
g_{i j}=\sum_{k=1}^{n} \delta_{i}^{k} \delta_{k j}+D_{i} u D_{j} u=\delta_{i j}+D_{i} u D_{j} u \tag{1.13}
\end{equation*}
$$

It is easy to verify that the inverse of $g$ is then given by

$$
\begin{equation*}
g^{i j}=\delta^{i j}-\frac{D^{i} u D^{j} u}{1+|D u|^{2}} \tag{1.14}
\end{equation*}
$$

where $|D u|^{2}=\sum_{i=1}^{n} D^{i} u D_{i} u$.
Remark 1.10. In the above expressions (1.13) and (1.14), the indices on the righthand side refer to the coordinates in $\mathbb{R}^{n+1}$. Therefore, they are raised and lowered with the ambient metric $\delta$. On the left-hand side, however, the indices are with respect to the coordinate space $\mathbb{R}^{n}$. Those indices are raised and lowered with $g$.

The Gauss map of $\Sigma$ is given by

$$
\nu^{\alpha}=\frac{1}{\sqrt{1+|D u|^{2}}} \begin{cases}D^{\alpha} u, & \alpha \in\{1, \ldots, n\} \\ -1, & \alpha=n+1\end{cases}
$$

where we have chosen the orientation such that $\operatorname{det}(D f, \nu)<0$. Consequently,

$$
D_{i} \nu^{\alpha}= \begin{cases}\frac{D_{i} D^{\alpha} u}{\sqrt{1+|D u|^{2}}}+\frac{-D^{\alpha} u\left(\sum_{k=1}^{n}\left(\left(D_{i} D^{k} u\right) D_{k} u+D^{k} u\left(D_{i} D_{k} u\right)\right)\right)}{2\left(1+|D u|^{2}\right)^{3 / 2}} \\ \frac{2 \sum_{k=1}^{n} D^{k} u\left(D_{i} D_{k} u\right)}{2\left(1+|D u|^{2}\right)^{3 / 2}}, & \alpha \in\{1, \ldots, n\}, \\ & \alpha=n+1,\end{cases}
$$

from which we calculate the second fundamental form $A$ of $\Sigma$ :

$$
\begin{equation*}
A_{i j}=\sum_{\alpha=1}^{n+1} D_{i} \nu^{\alpha} D_{j} f_{\alpha}=\frac{D_{i} D_{j} u}{\sqrt{1+|D u|^{2}}} \tag{1.15}
\end{equation*}
$$

To prove (1.2), we first calculate the Christoffel symbols of $g$ :

$$
\Gamma_{i j}^{k}=\frac{1}{2} \sum_{l=1}^{n} g^{k l}\left(D_{i} g_{j l}+D_{j} g_{i l}-D_{l} g_{i j}\right)=\frac{D^{k} u D_{i} D_{j} u}{1+|D u|^{2}}=v^{k} A_{i j}
$$

where $v=\frac{D u}{\sqrt{1+|D u|^{2}}}$. Note that $v$ is the projection of the Gauss map $\nu$ onto the tangent plane to $U$. Denoting by $\nabla$ the Levi-Civita connection of $g$, we have for any indices $i, j$ and $k$

$$
\nabla_{k} A_{j}^{i}=D_{k} A_{j}^{i}+\sum_{l=1}^{n} \Gamma_{k l}^{i} A_{j}^{l}-\sum_{l=1}^{n} \Gamma_{k j}^{l} A_{l}^{i} .
$$

Using the Codazzi equations,

$$
\nabla_{k} A_{j}^{i}=\nabla_{j} A_{k}^{i},
$$

we obtain

$$
D_{k} A_{j}^{i}=D_{j} A_{k}^{i}-\sum_{l=1}^{n} \Gamma_{k l}^{i} A_{j}^{l}+\sum_{l=1}^{n} \Gamma_{j l}^{i} A_{k}^{l},
$$

which, inserting the expression for the Christoffel symbols, reads

$$
\begin{aligned}
D_{k} A^{i}{ }_{j} & =D_{j} A^{i}{ }_{k}-v^{i} \sum_{l, s=1}^{n} A_{k l} g^{l s} A_{s j}+v^{i} \sum_{l, t=1}^{n} A_{j l} g^{l t} A_{t k} \\
& =D_{j} A^{i}{ }_{k} .
\end{aligned}
$$

Now denote by $\AA$ the traceless part of $A$, i.e. $\AA^{i}{ }_{j}=A^{i}{ }_{j}-\frac{1}{n} \sum_{k=1}^{n} A^{k}{ }_{k} \delta^{i}{ }_{j}$. Clearly, if $i \neq j$, we have for all $k$

$$
D_{k} A^{i}{ }_{j}=D_{k} \stackrel{\circ}{A}_{j}^{i} .
$$

Consequently, we have for all $k \neq i$

$$
D_{k} A^{i}{ }_{i}=D_{i} A^{i}{ }_{k}=D_{i} \AA^{i}{ }_{k}
$$

We finally calculate

$$
\begin{aligned}
D_{i} A_{i}^{i} & =D_{i} \stackrel{\circ}{A}_{i}+\frac{1}{n} \sum_{k=1}^{n} D_{i} A_{k}^{k} \\
& =D_{i} \stackrel{\circ}{A}_{i}{ }_{i}+\frac{1}{n} \sum_{\substack{k=1 \\
k \neq i}}^{n} D_{i} A_{k}^{k}+\frac{1}{n} D_{i} A_{i}^{i} \\
& =\frac{1}{1-\frac{1}{n}}\left(D_{i} \AA_{i}^{i}+\frac{1}{n} \sum_{\substack{k=1 \\
k \neq i}}^{n} D_{k} \AA^{k}{ }_{i}\right) \\
& =\frac{n}{n-1}\left(\left(1-\frac{1}{n}\right) D_{i} \AA_{i}^{i}{ }_{i}+\frac{1}{n} \sum_{k=1}^{n} D_{k} \AA^{k}{ }_{i}\right) \\
& =D_{i} \stackrel{\circ}{A}_{i}^{i}+\frac{1}{n-1} \sum_{k=1}^{n} D_{k} \AA^{k}{ }_{i} .
\end{aligned}
$$

This yields for arbitrary indices $i, j$ and $k$ :

$$
\begin{aligned}
D_{k} A^{i}{ }_{j} & =D_{k} \AA^{i}{ }_{j}+\frac{1}{n} \sum_{\substack{l=1 \\
l \neq k}}^{n} D_{k} A^{l}{ }_{l} \delta^{i}{ }_{j}+\frac{1}{n} D_{k} A^{k}{ }_{k} \delta^{i}{ }_{j} \\
& =D_{k} \AA^{i}{ }_{j}+\frac{1}{n} \sum_{\substack{l=1 \\
l \neq k}}^{n} D_{l} \AA^{l}{ }_{k} \delta^{i}{ }_{j}+\frac{1}{n} D_{k} \AA^{k}{ }_{k} \delta^{i}{ }_{j}+\frac{1}{n(n-1)} \sum_{l=1}^{n} D_{l}{ }_{l}{ }^{l}{ }_{k} \delta^{i}{ }_{j} \\
& =D_{k} \AA^{i}{ }_{j}+\frac{1}{n} \sum_{l=1}^{n} D_{l} \AA^{l}{ }_{k} \delta^{i}{ }_{j}+\frac{1}{n(n-1)} \sum_{l=1}^{n} D_{l} \AA^{l}{ }_{k} \delta^{i}{ }_{j} \\
& =D_{k} \AA^{i}{ }_{j}+\frac{1}{n-1} \sum_{l=1}^{n} D_{l} \AA^{l}{ }_{k} \delta^{i}{ }_{j} .
\end{aligned}
$$

This proves (1.2) and thus Lemma 1.3.

## 5. Proof of Proposition 1.5

Proposition 1.5 will follow directly from the following
Proposition 1.11. Let $n \geq 2, R>0$ and $1<p<+\infty$. Let $\mathfrak{u} \in C^{3}\left(B_{R}(0)\right) \cap$ $L^{p}\left(B_{R}(0)\right)$ and $\mathfrak{f} \in C^{3}\left(B_{R}(0) ; \mathbb{R}^{n \times n}\right) \cap L^{p}\left(B_{R}(0) ; \mathbb{R}^{n \times n}\right)$ be such that $\mathfrak{u}$ solves on $B_{R}(0) \subset \mathbb{R}^{n}$

$$
\begin{equation*}
D \mathfrak{u}=\operatorname{div} \mathfrak{f} \tag{1.16}
\end{equation*}
$$

i.e.

$$
D_{i} \mathfrak{u}=\sum_{k=1}^{n} D_{k} \mathfrak{f}_{i}^{k}
$$

Then there is a constant $C$, depending only on $n$ and $p$, such that

$$
\begin{equation*}
\left\|\mathfrak{u}-f_{B_{R / 4}(0)} \mathfrak{u}\right\|_{L^{p}\left(B_{R / 4}(0)\right)} \leq C\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)} \tag{1.17}
\end{equation*}
$$

where $f_{B_{R / 4}(0)} \mathfrak{u}=\operatorname{vol}_{n}\left(B_{R / 4}(0)\right)^{-1} \int_{B_{R / 4}(0)} \mathfrak{u}$ denotes the average of $\mathfrak{u}$ on $B_{R / 4}(0)$.
Indeed, Proposition 1.11 implies Proposition 1.5 if we take

$$
\mathfrak{u}=\sum_{k=1}^{n} A_{k}^{k} \quad \text { and } \quad \mathfrak{f}=\frac{n}{n-1} \AA .
$$

For then, by equation (1.2) of Lemma $1.3, \mathfrak{u}$ satisfies $D \mathfrak{u}=\operatorname{div} f$, and (1.17) yields

$$
\left\|\sum_{k=0}^{n} A_{k}^{k}-f_{D_{R / 4}(0)} \sum_{k=0}^{n} A_{k}^{k}\right\|_{L^{p}\left(D_{R / 4}(0)\right)} \leq C\|\AA\|_{L^{p}\left(D_{R}(0)\right)}
$$

for some constant $C$ that depends only on $n$ and $p$. Writing

$$
\lambda=\frac{1}{n} f_{D_{R / 4}(0)} \sum_{k=1}^{n} A_{k}^{k},
$$

we obtain

$$
\begin{aligned}
\|A-\lambda g\|_{L^{p}\left(D_{R / 4}(0)\right)} & =\left\|\AA+\frac{1}{n}\left(\sum_{k=1}^{n} A^{k}{ }_{k}\right) g-\lambda g\right\|_{L^{p}\left(D_{R / 4}(0)\right)} \\
& \leq\|\AA\|_{L^{p}\left(D_{R / 4}(0)\right)}+\frac{n}{n}\left\|\sum_{k=1}^{n} A_{k}^{k}-f_{D_{R / 4}(0)} \sum_{k=1}^{n} A_{k}^{k}\right\|_{L^{p}\left(D_{R / 4}(0)\right)} \\
& \leq C\|\AA\|_{L^{p}\left(D_{R}(0)\right)},
\end{aligned}
$$

which proves Proposition 1.5.
Proof of Proposition 1.11. Let $\varphi \in C^{\infty}(\mathbb{R} ;[0,1])$ be a smooth function with the following properties:
(i) $\forall x \in \mathbb{R}, \varphi(-x)=\varphi(x)$,
(ii) $\forall|x| \leq 1 / 2, \varphi(x)=1$,
(iii) $\forall|x| \geq 1, \varphi(x)=0$,
and
(iv) If $|x|=1$, then $\varphi(x)=\varphi^{\prime}(x)=\varphi^{\prime \prime}(x)=0$.

Define

$$
\widetilde{\mathfrak{f}}(x)= \begin{cases}\mathfrak{f}(x) \varphi\left(\frac{|x|}{R}\right), & |x|<R \\ 0, & |x| \geq R\end{cases}
$$

Then $\|\widetilde{\mathfrak{f}}\|_{L^{p}\left(\mathbb{R}^{n}\right)} \leq\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)}$ and $\widetilde{\mathfrak{f}}=D_{i} \widetilde{\mathfrak{f}}=0, \forall i$, on $\partial B_{R}(0)$. Moreover, $\mathfrak{u}$ solves $D \mathfrak{u}=\operatorname{div} \widetilde{\mathfrak{f}}$ in $B_{R / 2}(0)$. Let $\mathfrak{w}$ be the fundamental solution of

$$
\left\{\begin{aligned}
\Delta \mathfrak{w} & =\operatorname{div} \operatorname{div} \widetilde{\mathfrak{f}}, & & \text { in } B_{R}(0) \\
\mathfrak{w} & =0, & & \text { on } \partial B_{R}(0)
\end{aligned}\right.
$$

and denote by $K$ the standard Dirichlet kernel in $\mathbb{R}^{n}$. Then $\mathfrak{w}$ is given by

$$
\begin{aligned}
\mathfrak{w} & =\int_{B_{R}(0)} K(y-x) \operatorname{div}_{y} \operatorname{div}_{y} \widetilde{\mathfrak{f}}(y) d y \\
& =\int_{B_{R}(0)} \sum_{k, l=1}^{n} K(y-x) D_{y}^{k} D_{y}^{l} \widetilde{f}^{k}{ }_{l}(y) d y \\
& =\int_{B_{R}(0)} \sum_{k, l=1}^{n} D_{y}^{k} D_{y}^{l} K(y-x) \widetilde{\mathfrak{f}}^{k}{ }_{l}(y) d y \\
& =\int_{\mathbb{R}^{n}} \sum_{k, l=1}^{n} D_{y}^{k} D_{y}^{l} K(y-x) \widetilde{\mathfrak{f}}^{k}{ }_{l}(y) d y \\
& =\sum_{k, l=1}^{n} \frac{1}{n \widetilde{\omega}_{n}} \int_{\mathbb{R}^{n}} \frac{n(y-x)^{k}(y-x)^{l}-|y-x|^{2} \delta^{k l}}{|y-x|^{n+2}} \widetilde{\mathfrak{f}}^{k}{ }_{l}(y) d y
\end{aligned}
$$

where $\widetilde{\omega}_{n}=\operatorname{vol}_{n}\left(B_{1}(0)\right)$ denotes the volume of the unit ball in $\mathbb{R}^{n}$. It is straightforward to check that, for any $k$ and $l$, the map

$$
\Omega^{k l}: \mathbb{R}^{n} \rightarrow \mathbb{R}, \quad x \mapsto \frac{n x^{k} x^{l}-|x|^{2} \delta^{k l}}{|x|^{2}}
$$

which is homogeneous of degree 0 , satisfies the cancellation property,

$$
\int_{\partial B_{1}(0)} \Omega^{k l}=0
$$

and the smoothness condition,

$$
\int_{0}^{1} \frac{\sup _{\substack{\left|x-x^{\prime}\right| \leq \delta \\ x, x^{\prime} \in \partial B_{1}(0)}}\left|\Omega^{k l}(x)-\Omega^{k l}\left(x^{\prime}\right)\right|}{\delta} d \delta<+\infty
$$

required to apply the Calderón-Zygmund inequality as in Theorem 3 of Chapter 2, p.39, in [Ste70]. We get

$$
\begin{equation*}
\|\mathfrak{w}\|_{L^{p}\left(B_{R}(0)\right)} \leq C\|\widetilde{\mathfrak{f}}\|_{L^{p}\left(\mathbb{R}^{n}\right)} \leq C\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)} \tag{1.18}
\end{equation*}
$$

for some constant $C$ depending solely on $n$ and $p$.
Now, for $x \in B_{R / 2}(0)$, let

$$
\mathfrak{h}(x)=\mathfrak{u}(x)-\mathfrak{w}(x)
$$

Taking the Laplacian, we see that

$$
\Delta \mathfrak{h}=\Delta \mathfrak{u}-\Delta \mathfrak{w}=\operatorname{div}(\operatorname{div} \widetilde{\mathfrak{f}})-\operatorname{div} \operatorname{div} \widetilde{\mathfrak{f}}=0
$$

i.e. $\mathfrak{h}$ is harmonic in $B_{R / 2}(0)$. Moreover, $\mathfrak{h}$ solves in $B_{R / 2}(0)$

$$
D \mathfrak{h}=D \mathfrak{u}-D \mathfrak{w}=\operatorname{div} \widetilde{\mathfrak{f}}-D \mathfrak{w}=\operatorname{div}(\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id}),
$$

where id denotes the $n \times n$ identity matrix. Notice that, by our considerations above,

$$
\|\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id}\|_{L^{p}\left(B_{R / 2}(0)\right)} \leq\|\widetilde{\mathfrak{f}}\|_{L^{p}\left(B_{R / 2}(0)\right)}+n\|\mathfrak{w}\|_{L^{p}\left(B_{R / 2}(0)\right)} \leq C\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)},
$$

where $C$ depends only on $n$ and $p$. Since $\mathfrak{h}$ is harmonic, so is $D \mathfrak{h}$, and we have by the mean value property for all $x \in B_{R / 4}(0)$, all $\rho \in\left(\frac{R}{8}, \frac{R}{4}\right)$ and all $i$
$D_{i} \mathfrak{h}(x)=f_{B_{\rho}(x)} D_{i} \mathfrak{h}=f_{B_{\rho}(x)} \operatorname{div}(\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id})_{i}=\frac{n}{\rho} f_{\partial B_{\rho}(x)} \sum_{k=1}^{n}\left(\widetilde{\mathfrak{f}}^{k}{ }_{i}-\mathfrak{w} \delta^{k}{ }_{i}\right)\left(\nu^{\text {ext }}\right)^{k}$,
where the last equality follows from the Gauss Theorem and $\nu^{\text {ext }}$ denotes the outward unit normal to $\partial B_{\rho}(x)$. Thus, since $\rho>R / 8$, we have for all $x \in B_{R / 4}(0)$, all $\rho \in\left(\frac{R}{8}, \frac{R}{4}\right)$ and all $i$

$$
\left.\left.\left.\left|D_{i} \mathfrak{h}\right| \leq \frac{1}{\widetilde{\omega}_{n} \rho^{n}} \int_{\partial B_{\rho}(x)} \right\rvert\, \widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id}\right)_{i}\left|\leq \frac{8^{n}}{\widetilde{\omega}_{n} R^{n}} \int_{\partial B_{\rho}(x)}\right| \widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id}\right)_{i} \mid .
$$

Integrating with respect to $\rho$ and using the Hölder inequality we get

$$
\begin{aligned}
\frac{R}{8}\left|D_{i} \mathfrak{h}\right| & \left.\left.\leq \frac{8^{n}}{\widetilde{\omega}_{n} R^{n}} \int_{\frac{R}{8}}^{\frac{R}{4}} \int_{\partial B_{\rho}(x)} \right\rvert\, \widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id}\right)_{i} \mid d \rho \\
& =\frac{8^{n}}{\widetilde{\omega}_{n} R^{n}}\left\|(\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id})_{i}\right\|_{L^{1}\left(B_{R / 4}(x) \backslash B_{R / 8}(x)\right)} \\
& \leq \frac{8^{n}}{\widetilde{\omega}_{n} R^{n}}\left\|(\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id})_{i}\right\|_{L^{1}\left(B_{R / 2}(0)\right)} \\
& \leq \frac{8^{n}}{\widetilde{\omega}_{n} R^{n}}\left(\frac{\widetilde{\omega_{n}} R^{n}}{2^{n}}\right)^{1-1 / p}\left\|(\widetilde{\mathfrak{f}}-\mathfrak{w} \cdot \mathrm{id})_{i}\right\|_{L^{p}\left(B_{R / 2}(0)\right)} .
\end{aligned}
$$

It follows that there is a constant $C$, depending only on $n$ and $p$, such that for all $x \in B_{R / 4}(0)$

$$
|D \mathfrak{h}| \leq C R^{-1-n / p}\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)} .
$$

Using the Poincaré inequality for balls (see, e.f., [Eva98, Thm.2, §5.8.1, p.276]), we obtain

$$
\begin{aligned}
\| \mathfrak{h}-f_{B_{R / 4}(0)} & \mathfrak{h} \|_{L^{p}\left(B_{R / 4}(0)\right)}
\end{aligned} \quad \leq C_{1} R\|D \mathfrak{h}\|_{L^{p}\left(B_{R / 4}(0)\right)} \quad \leq C_{2} R^{-\frac{n}{p}} R^{\frac{n}{p}}\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)}=C_{2}\|\mathfrak{f}\|_{L^{p}\left(B_{R}(0)\right)}, ~ l
$$

for some constants $C_{1}$ and $C_{2}$ depending on $n$ and $p$. Since $\mathfrak{u}=\mathfrak{w}+\mathfrak{h}$ and with the help of the estimate (1.18) for the potential $\mathfrak{w}$, we arrive at

$$
\left\|\mathfrak{u}-f_{B_{R / 4}(0)} \mathfrak{u}\right\|_{L^{p}\left(B_{R / 4}(0)\right)} \leq C\|f\|_{L^{p}\left(B_{R}(0)\right)},
$$

for some constant $C$ depending on $n$ and $p$ alone. This proves Proposition 1.11.

## 6. Proof of Lemma 1.7

Let $\Sigma \subset \mathbb{R}^{n+1}$ be as in Lemma 1.7 and choose $q \in \Sigma$. We first show some estimates on the size of a geodesic ball $\mathcal{B}_{\rho}^{g}(q)$ with radius $\rho \leq r_{0}$ centred at $q$. Without loss of generality, we may assume that $q=0 \in \mathbb{R}^{n+1}$ and that $\Sigma$ is rotated in such a way that a portion of it is parametrised as the graph of a smooth Lipschitz function $u: D_{r_{0}}(0) \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$, with $u(0)=0$ and Lipschitz constant Lip $(u) \leq L$.

Claim 1. The geodesic sphere $\partial \mathcal{B}_{\rho}^{g}(q)$ with radius $\rho \leq r_{0}$ centred at $q$ is contained in the graph of $u$ over the closed ring $\overline{D_{\rho}(0)} \backslash D_{\rho /(1+L)}(0)$.

Proof. Let $\bar{d}_{g}(x, y)$ denote the geodesic distance from $(x, u(x))$ to $(y, u(y))$. For each $y$ such that $(y, u(y)) \in \partial \mathcal{B}_{\rho}^{g}(q)$ we have $\bar{d}_{g}(0, y)=\rho$. Moreover,

$$
\bar{d}_{g}(0, y) \geq|(y, u(y))-(0,0)|=\sqrt{y^{2}+u(y)^{2}} \geq|y|
$$

and

$$
\bar{d}_{g}(0, y) \leq \int_{0}^{1} \sqrt{g\left(\partial_{t} \gamma, \partial_{t} \gamma\right)} d t
$$

where $\gamma$ is any curve $\gamma:[0,1] \rightarrow\left(D_{r_{0}}(0), u\left(D_{r_{0}}(0)\right)\right), t \mapsto \gamma(t)$, joining $(0,0)$ and $(y, u(y))$. The second estimate follows from the definition of $\bar{d}_{g}(0, y)$ as the infimum of the right-hand side taken over all such curves. Choosing $\gamma(t)=(t y, u(t y))$, we obtain with the help of equation (1.13) in the proof of Lemma 1.3 (Section 4) and using that $u$ is Lipschitz ( $D$ denoting derivation with respect to the Cartesian coordinates of $\mathbb{R}^{n}$ )

$$
\begin{aligned}
\int_{0}^{1} \sqrt{g\left(\partial_{t} \gamma, \partial_{t} \gamma\right)} d t & =\int_{0}^{1}\left(|y|^{2}+\left(\sum_{k=1}^{n} D_{k} u(t y) y^{k}\right)^{2}\right)^{1 / 2} d t \\
& \leq \int_{0}^{1}\left(|y|^{2}+|D u(t y)|^{2}|y|^{2}\right)^{1 / 2} d t \leq \int_{0}^{1}|y| \sqrt{1+L^{2}} d t \\
& \leq|y|(1+L)
\end{aligned}
$$

Thus we have

$$
|y| \leq \rho \quad \text { and } \quad|y| \geq \frac{\rho}{1+L}
$$

proving the claim.
It follows immediately
Claim 2. The volume $\operatorname{vol}_{n}\left(\mathcal{B}_{\rho}^{g}(q)\right)$ of the geodesic ball $\mathcal{B}_{\rho}^{g}(q)$ with radius $\rho \leq r_{0}$ and centre $q$ is bounded by

$$
\frac{\widetilde{\omega}_{n} \rho^{n}}{(1+L)^{n}} \leq \operatorname{vol}_{n}\left(\mathcal{B}_{\rho}^{g}(q)\right) \leq(1+L) \widetilde{\omega}_{n} \rho^{n}
$$

where $\widetilde{\omega}_{n}=\operatorname{vol}_{n}\left(D_{1}(0)\right)$ denotes the volume of the unit ball in $\mathbb{R}^{n}$.

Proof. From equation (1.13) in the proof of Lemma 1.3 (Section 4) follows that the volume form $\sigma_{g}$ on $\mathcal{B}_{\rho}^{g}(q)$ is given by

$$
\sigma_{g}=\sqrt{\operatorname{det} g}=\sqrt{1+|D u|^{2}}
$$

With Claim 1, we immediately get by the Lipschitz property of $u$

$$
\operatorname{vol}_{n}\left(\mathcal{B}_{\rho}^{g}(q)\right) \geq \int_{D_{\rho /(1+L)}(0)} \sqrt{1+|D u|^{2}} \geq \widetilde{\omega}_{n}\left(\frac{\rho}{1+L}\right)^{n}
$$

and

$$
\operatorname{vol}_{n}\left(\mathcal{B}_{\rho}^{g}(q)\right) \leq \int_{D_{\rho}(0)} \sqrt{1+|D u|^{2}} \leq \widetilde{\omega}_{n} \rho^{n} \sqrt{1+L^{2}} \leq(1+L) \widetilde{\omega}_{n} \rho^{n}
$$

which proves the Claim.
An argument similar to the one in the proof of Claim 1 shows
Claim 3. Let $x \in D_{r_{0}}(0), q^{\prime}=(x, u(x))$ and $0<\rho<r_{0}-|x|$. Then the geodesic ball $\mathcal{B}_{\rho}^{g}\left(q^{\prime}\right)$ is contained in the graph of $u$ over the open ring $D_{|x|+\rho}(0) \backslash \overline{D_{|x|-\rho}(0)}$.

Proof. For each $y$ such that $(y, u(y)) \in \mathcal{B}_{\rho}^{g}\left(q^{\prime}\right)$ we have $\bar{d}_{g}(x, y)<\rho$. Moreover, $\bar{d}_{g}(x, y) \geq|(y, u(y))-(x, u(x))|=\sqrt{|y-x|^{2}+|u(y)-u(x)|^{2}} \geq|y-x| \geq||y|-|x||$. Consequently,

$$
|x|-\rho<|y|<|x|+\rho,
$$

proving the Claim.
We now proceed with the proof of Lemma 1.7. Let $r_{1}=\frac{r_{0}}{4(1+L)}$ and let

$$
\mathfrak{B}=\left\{\mathcal{B}_{r_{1}}^{g}\left(q_{j}\right) \mid j \in J\right\}
$$

be a maximal collection of pairwise disjoint geodesic balls in $\Sigma$ of radius $r_{1}$. We will show that the number of balls in this collection is controlled. Moreover, we will show that the collection

$$
\widehat{\mathfrak{B}}=\left\{\mathcal{B}_{r_{0}}^{g}\left(q_{j}\right) \mid j \in J\right\}
$$

of geodesic balls of radius $r_{0}$ but with same centres $q_{j}$ has the properties claimed in the lemma to be proved. Clearly, the index set $J$ is finite since $\Sigma$ is compact. Moreover,

Claim 4. The cardinality $|J|$ of $J$ is controlled by

$$
|J| \leq \frac{4^{n}}{\widetilde{\omega}_{n}} \frac{(1+L)^{2 n}}{r_{0}^{n}}
$$

Proof. We know from Claim 2 that

$$
\operatorname{vol}_{n}\left(\mathcal{B}_{r_{1}}^{g}\left(q_{j}\right)\right) \geq \frac{\widetilde{\omega}_{n} r_{1}^{n}}{(1+L)^{n}}=\frac{\widetilde{\omega}_{n} r_{0}^{n}}{4^{n}(1+L)^{2 n}}
$$

Since $\operatorname{vol}_{n}(\Sigma)=1$ and the geodesic balls of radius $r_{1}$ are pairwise disjoint, the claim follows.

Claim 5. For every $q \in \Sigma$, there exist two distinct indices $j_{1}, j_{2} \in J, j_{1} \neq j_{2}$, such that

$$
q \in \mathcal{B}_{r_{0}}^{g}\left(q_{j_{1}}\right) \cap \mathcal{B}_{r_{0}}^{g}\left(q_{j_{2}}\right)
$$

(In particular $\Sigma \subset \cup_{j \in J} \mathcal{B}_{r_{0}}^{g}\left(q_{j}\right)$.)
Proof. Let $q \in \Sigma$ be arbitrary. Again, without loss of generality, we may assume that $q=0 \in \mathbb{R}^{n+1}$ and that a portion of $\Sigma$ is given as the graph of a smooth $L$-Lipschitz function $u: D_{r_{0}}(0) \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$ with $u(0)=0$.

By Claim 3, there must be a $j_{1} \in J$ such that the graph of $u$ over $D_{r_{1}}(0)$ intersects $\mathcal{B}_{r_{1}}^{g}\left(q_{j_{1}}\right)$ since, otherwise, by Claim 1 , we could add the ball $\mathcal{B}_{r_{1}}^{g}(q)$ to $\mathfrak{B}$ to get a bigger collection of pairwise disjoint balls, contradicting the maximality of $\mathfrak{B}$. Notice that, in view of Claim $3, \mathcal{B}_{r_{1}}^{g}\left(q_{j_{1}}\right)$ will be contained in the graph of $u$ over $D_{3 r_{1}}(0)$.

Similarly, there must be a $j_{2} \in J, j_{2} \neq j_{1}$, such that the graph of $u$ over $D_{5 r_{1}}(0)$ intersects $\mathcal{B}_{r_{1}}^{g}\left(q_{j_{2}}\right)$ since, otherwise, we could fit a ball $\mathcal{B}_{r_{1}}^{g}(\widetilde{q})$, for some $\widetilde{q} \in \Sigma$, in the graph of $u$ over the ring $D_{5 r_{1}}(0) \backslash \overline{D_{3 r_{1}}(0)}$. By virtue of Claim 3, that ball would then be disjoint from $\mathcal{B}_{r_{1}}^{g}\left(q_{j_{1}}\right)$, contradicting again the maximality of $\mathfrak{B}$.

Let $y_{1}$ and $y_{2}$ be such that $q_{j_{1}}=\left(y_{1}, u\left(y_{1}\right)\right)$ and $q_{j_{2}}=\left(y_{2}, u\left(y_{2}\right)\right)$. Then Claim 3 implies that $\left|y_{1}\right|<2 r_{1}$ and $\left|y_{2}\right|<4 r_{1}$.

By Claim 1, then, we have

$$
\bar{d}_{g}\left(0, y_{1}\right) \leq\left|y_{1}\right|(1+L)<\frac{r_{0}}{2}<r_{0}
$$

and

$$
\bar{d}_{g}\left(0, y_{2}\right) \leq\left|y_{2}\right|(1+L)<r_{0}
$$

Therefore $q$ is contained in both $\mathcal{B}_{r_{0}}^{g}\left(q_{j_{1}}\right)$ and $\mathcal{B}_{r_{0}}^{g}\left(q_{j_{2}}\right)$, finishing the proof of the Claim, and thus the proof of Lemma 1.7.

## CHAPTER 2

## The sub-critical and critical cases for convex hypersurfaces

In this chapter we prove our main estimate in the case $p \in(1, n](n \geq 2)$ for $n-$ dimensional hypersurfaces that are the boundary of some convex domain in $\mathbb{R}^{n+1}$. We also establish the qualitative $C^{0}$-closeness to a sphere. The ideas of the proofs are the same as in the super-critical case in Chapter 1. However, we need a different method to "patch together" the local estimates to obtain the global one. It is here then that convexity plays the fundamental role. In fact, we are going to prove a similar result to the one given by Pogorelov in $[\mathbf{P o g} 73]$, obtaining an $(n+1)$-dimensional ring of controlled inner and outer radius that contains the studied hypersurface. This will then enable us to apply Lemma 1.7 to conclude.

Due to the nature of the problem at hand, the restriction we faced in Chapter 1 regarding the necessity to preset a bound on $\|A\|_{L^{p}(\Sigma)}$ can be avoided.

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## 1. The main results of this chapter

Our goal is to prove

Theorem 2.1. Let $n \geq 2$ and $p \in(1, n]$ be given. Then there is a constant $C>0$, depending only on $n$ and $p$, such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed $n$-dimensional hypersurface with induced Riemannian metric $g$ and such that $\Sigma$ is the boundary of a convex domain in $\mathbb{R}^{n+1}$, then

$$
\begin{equation*}
\min _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C\|\AA\|_{L^{p}(\Sigma)} \tag{2.1}
\end{equation*}
$$

Since both sides in (2.1) scale identically and none of the assumptions in the theorem are scaling-dependent, we can without loss of generality assume that the $n$-dimensional volume of $\Sigma$ be normalised. It is thus sufficient to prove

Theorem 2.1'. Let $n \geq 2$ and $p \in(1, n]$ be given. Then there is a constant $C>0$, depending only on $n$ and $p$, such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed $n$-dimensional hypersurface with induced Riemannian metric $g$ and such that
(a) $\operatorname{vol}_{n}(\Sigma)=1$
and
(b) $\Sigma$ is the boundary of a convex domain in $\mathbb{R}^{n+1}$,
then

$$
\begin{equation*}
\min _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C\|\AA\|_{L^{p}(\Sigma)} \tag{2.1}
\end{equation*}
$$

Notice that, in view of the following Lemma which we prove at the end of this section, we can choose any constant $c_{0}>0$ and assume without loss of generality that $\|\AA\|_{L^{p}(\Sigma)} \leq c_{0}$ (the author is grateful to C. De Lellis for having brought this to his attention).
Lemma 2.2. Let $n \geq 2$ and $p \in[1, n]$ be given. Then there exists a constant $C>0$ depending only on $n$ and $p$ such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth hypersurface bounding a convex domain and such that $\operatorname{vol}_{n}(\Sigma)=1$, then

$$
\int_{\Sigma}|A|^{p} \leq C\left(1+\int_{\Sigma}|\AA|^{p}\right)
$$

Indeed, if $\|\AA\|_{L^{p}(\Sigma)}>c_{0}$, we would conclude

$$
\begin{aligned}
\min _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} & \leq\|A\|_{L^{p}(\Sigma)} \leq C^{\frac{1}{p}}\left(1+\|\AA\|_{L^{p}(\Sigma)}^{p}\right)^{\frac{1}{p}} \\
& \leq C^{\frac{1}{p}}\left(c_{0}^{-p}+1\right)^{\frac{1}{p}}\|\AA\|_{L^{p}(\Sigma)}
\end{aligned}
$$

and Theorem $2.1^{\prime}$ would be proved. It is therefore enough to prove the weaker
Theorem 2.3. Let $n \geq 2, p \in(1, n]$ and $c_{0}>0$ be given. Then there is a constant $C>0$, depending only on $n, p$ and $c_{0}$, such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed $n$-dimensional hypersurface with induced Riemannian metric $g$ and such that
(a) $\operatorname{vol}_{n}(\Sigma)=1$,
(b) $\Sigma$ is the boundary of a convex domain in $\mathbb{R}^{n+1}$
and
(c) $\|\AA\|_{L^{p}(\Sigma)}=\left(\int_{\Sigma}|\AA|^{p}\right)^{\frac{1}{p}} \leq c_{0}$,
then

$$
\min _{\lambda \in \mathbb{R}}\|A-\lambda g\|_{L^{p}(\Sigma)} \leq C\|\AA\|_{L^{p}(\Sigma)}
$$

The idea of its proof is exactly the same as the one in the proof of Theorem 1.1 of the previous chapter. However, there is one difference. Then, in order to apply the technical Lemma 1.7 that yields a suitable covering of the hypersurface with geodesic balls, we had to obtain a uniform lower bound on the radii of the balls over which the hypersurface can locally be represented as a Lipschitz graph with given Lipschitz constant. We did this by invoking a Morrey-type estimate (Lemma A.1), which, obviously, does not apply here. It turns out, though, that we can apply Lemma 1.7 directly, thanks to convexity. In fact, as shall be sufficient, we will prove that a hypersurface of the type considered in Theorem 2.3 above is contained in an $(n+1)$-dimensional ring (or spherical shell) where we have (some) control over the inner and the outer radius (namely that they depend only on the data given in the assumptions of the theorem).

We thereby generalise a result given by A. Pogorelov in [Pog73, §VII.9, p.493], who proves a theorem stating in a quantitative manner that a convex two-dimensional surface, for which the ratio of the two principal radii of curvature is sufficiently close to one in each point, must be close to a round sphere, in the sense that it lies between two concentric spheres such that the ratio of their radii is also close to one. More precisely, we prove

Proposition 2.4. Let $n \geq 2, p \in(1, n]$ and $c_{0} \in(0,+\infty)$ be given. Then there exist $R>r>0$, depending only on $n, p$ and $c_{0}$ such that:
if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$
\operatorname{vol}_{n}(\partial U)=1 \quad \text { and } \quad \int_{\partial U}|\AA|^{p} \leq c_{0}
$$

then there exists an $x \in \mathbb{R}^{n+1}$ such that $B_{r}(x) \subset U \subset B_{R}(x)$.
In the next section, we quickly demonstrate how this proposition is used to obtain Theorem 2.3, whereas in Section 3, we prove Corollary 2.5 below, which concludes qualitative $C^{0}$-closeness to a sphere from the main estimate. The rest of the chapter will then be devoted to proving Proposition 2.4.

Corollary 2.5 (to Theorem $2.1^{\prime}$ and Proposition 2.4). Let $n \geq 2, p \in(1, n]$ and $\epsilon>0$ be given. Then there is a constant $\delta>0$, depending only on $n, p$ and $\epsilon$, such that:
if $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth, closed $n$-dimensional hypersurface such that
(a) $\operatorname{vol}_{n}(\Sigma)=1$,
(b) $\Sigma$ bounds a convex domain in $\mathbb{R}^{n+1}$
and
(c) $\|\AA\|_{L^{p}(\Sigma)}<\delta$,
then

$$
d_{\mathrm{HD}}\left(\Sigma, \partial B_{\rho_{0}}(x)\right)<\epsilon, \quad \text { for some } x \in \mathbb{R}^{n+1}
$$

where $\rho_{0}=\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{-\frac{1}{n}}$ and $d_{\mathrm{HD}}$ denotes the Hausdorff distance in Euclidean $\mathbb{R}^{n+1}$.

Remark 2.6. The avid reader might notice while working through this chapter, that the chain of implications may easily be adapted to the convex super-critical case to obtain the analogue of the (weaker) Theorem 2.3. Since we do not need (and do not get) any control on the deviation from one of the ratio of the radii found in Proposition 2.4, we can just use Hölder's inequality for the assumed bound on $\|\AA\|_{L^{p}(\Sigma)}^{p}$, to establish that $\Sigma$ is contained in a spherical shell whose radii $R>$ $r>0$ depend only on $n$ and $c_{0}$. Afterwards, we use these radii as in the proof of Theorem 2.3 to cover $\Sigma$ appropriately. By its qualitative nature, Corollary 2.5 then also extends to all exponents p. However, we do not see how to obtain the equivalent of the (stronger) Theorem 2.1' in that situation, nor how the above reasoning would help when seeking quantitative $C^{0}$-closeness in the convex super-critical case.

Proof of Lemma 2.2. For all $q \in \Sigma$, let $0<\lambda_{1}(q) \leq \lambda_{2}(q) \leq \cdots \leq \lambda_{n}(q)$ denote the eigenvalues of the second fundamental form $A$ (i.e. the principal curvatures) in $q$. We then have, for all $i, j \in\{1, \ldots, n\}$,

$$
\begin{aligned}
\left(\int_{\Sigma}\left|\lambda_{i}-\lambda_{j}\right|^{p}\right)^{\frac{1}{p}} & \leq\left(\int_{\Sigma}\left|\lambda_{i}-\frac{1}{n} H\right|^{p}\right)^{\frac{1}{p}}+\left(\int_{\Sigma}\left|\lambda_{j}-\frac{1}{n} H\right|^{p}\right)^{\frac{1}{p}} \\
& \leq 2\left(\int_{\Sigma}|\AA|^{p}\right)^{\frac{1}{p}}
\end{aligned}
$$

On the other hand $(1 \leq p \leq n)$,

$$
\int_{\Sigma} \lambda_{1}^{p} \leq \underbrace{\operatorname{vol}_{n}(\Sigma)^{\frac{n-p}{n}}}_{=1}\left(\int_{\Sigma} \lambda_{1}^{n}\right)^{\frac{p}{n}} \leq\left(\int_{\Sigma} \lambda_{1} \cdots \lambda_{n}\right)^{\frac{p}{n}}=\left(\int_{\Sigma} \operatorname{det} A\right)^{\frac{p}{n}}
$$

But, since $\Sigma$ bounds a convex region (cf., e.g., [Sch93, eqn.(2.5.29), p.112] or [CL57, theorems 3 or 4]),

$$
\int_{\Sigma} \operatorname{det} A=\int_{\Sigma} \operatorname{det} d \nu=\int_{S^{n}} 1=\operatorname{vol}_{n}\left(S^{n}\right)
$$

where we denoted by $\nu$ the outer unit normal of $\Sigma$ (i.e. its Gauss map). We conclude

$$
\begin{aligned}
\left(\int_{\Sigma}|A|^{p}\right)^{\frac{1}{p}} & =\left(\int_{\Sigma}\left(\lambda_{1}^{2}+\cdots+\lambda_{n}^{2}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq\left(\int_{\Sigma}\left(\lambda_{1}+\cdots+\lambda_{n}\right)^{p}\right)^{\frac{1}{p}} \\
& =\left(\int_{\Sigma}\left(\left(\lambda_{1}-\lambda_{1}\right)+\cdots+\left(\lambda_{n}-\lambda_{1}\right)+n \lambda_{1}\right)^{p}\right)^{\frac{1}{p}} \\
& \leq \sum_{i=2}^{n}\left(\int_{\Sigma}\left(\lambda_{i}-\lambda_{1}\right)^{p}\right)^{\frac{1}{p}}+n\left(\int_{\Sigma} \lambda_{1}^{p}\right)^{\frac{1}{p}} \\
& \leq 2(n-1)\left(\int_{\Sigma} \mid \AA \AA^{p}\right)^{\frac{1}{p}}+n\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{\frac{p}{n}}
\end{aligned}
$$

from which the desired estimate follows taking, e.g.,

$$
C=2^{p} \max \left\{2^{p}(n-1)^{p}, n^{p}\left(\operatorname{vol}_{n}\left(S^{n}\right)\right)^{\frac{p^{2}}{n}}\right\}
$$

## 2. Proof of Theorem 2.3

Let $\Sigma$ be as in Theorem 2.3. The only thing we need to check is how Proposition 2.4 implies the assumptions of Lemma 1.7, the rest of the proof being exactly as in Section 2 of Chapter 1. More precisely, we need to ensure that, for each point $q \in \Sigma$, there is a smooth Lipschitz function $u: D_{r_{0}}(0) \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$ on a ball of radius $r_{0}>0$ and with Lipschitz constant $L>0$, such that $r_{0}$ and $L$ do not depend on the point $q$ under consideration (nor, indeed, on $\Sigma$ ), and such that a portion of $\Sigma$ containing $q$ can be parametrised as the graph of $u$.

So fix $q \in \Sigma$, denote by $\Omega \subset \mathbb{R}^{n+1}$ the open convex domain enclosed by $\Sigma$ and let $e_{i}, i \in\{1, \ldots, n+1\}$, refer to the standard basis vectors in $\mathbb{R}^{n+1}$. By Proposition 2.4, there are $R>r>0$ such that $B_{r}(x) \subset \Omega \subset B_{R}(x)$ for some $x \in \Omega$. Without loss of generality, we may assume that $x=0$ and that $q=-|q| e_{n+1}$. We then have $r \leq|q| \leq R$. Now denote by $\pi_{-|q|}^{1}=\left\{y \in \mathbb{R}^{n+1}\left|\left\langle y, e_{n+1}\right\rangle=-|q|\right\}\right.$ the $n$-dimensional hyperplane parallel to the span of $\left\{e_{1}, \ldots, e_{n}\right\}$ and passing through $q$. Then a portion of $\Sigma$ containing $q$ can be written as the graph of a smooth convex function $u$ on $D_{r}=\overline{B_{r}(q)} \cap \pi_{-|q|}^{1}$. Moreover, we have $\|u\|_{L^{\infty}\left(D_{r}\right)} \leq R$, since $B_{r}(0) \subset \Omega \subset B_{R}(0)$. It is then easy to see that $u$ is $L$-Lipschitz on $D_{\rho}$, for all $\rho \in\left(0, \frac{r}{2}\right]$, with $L=\frac{4 R}{r}$ (see, for example, $[\mathbf{R V 7 4}$, Theorem A]). Since $q$ was arbitrary, the assumptions of Lemma 1.7 are met.

As a result, we can cover $\Sigma$ with $N$ geodesic balls of radius $2 r_{0}=r / 8$, where $N$ depends only on $n, p$ and (through $L$ and $r_{0}$ ) on $c_{0}$, and such that the local estimate (1.5) holds for all $r \leq r_{0}$. Moreover, the elements of this cover will have large enough overlap, in the sense that the intersection of two neighbouring balls of the cover will contain a geodesic ball of radius $r_{0}$. This then enables us to argue, once again, that the $\lambda \mathrm{s}$ in (1.5) differ at most by $C^{\prime}\|\AA\|_{L^{p}(\Sigma)}$, where $C^{\prime}$ depends only on $n, p$ and
$c_{0}$ (compare with (1.9)). The global estimate then follows. For more details, review the end of the proof laid out in Section 2 of Chapter 1.

## 3. Proof of Corollary 2.5

As was the case for the proof of Theorem 2.3 in the last section, we want to follow as closely as possible the argument of the super-critical case (cf. Section 3 of Chapter 1).
3.1. Preliminaries. Assume, by contradiction, that Corollary 2.5 were false. Then we would find a sequence $\left(U_{k}\right)_{k \in \mathbb{N}}$ of open convex subsets of $\mathbb{R}^{n+1}$ with smooth boundaries, satisfying, for each $k \in \mathbb{N}$, $\operatorname{vol}_{n}\left(\partial U_{k}\right)=1$ and $\int_{\partial U_{k}}|\AA|^{p} \leq c_{0}$ for some $c_{0}>0$ independent of $k$, and such that

$$
\lim _{k \rightarrow \infty}\|\AA\|_{L^{p}\left(\partial U_{k}\right)}=0
$$

Modulo translating each set $U_{k}$, Proposition 2.4 together with Lemma 2.2 implies the existence of $R>r>0$ such that

$$
B_{r}(0) \subset U_{k} \subset B_{R}(0) \quad(\forall k)
$$

Picking a subsequence, if necessary, we can without loss of generality assume that the closures $\overline{U_{k}}$ converge (in the Hausdorff topology) to a closed convex set $V \subset \mathbb{R}^{n+1}$ (see Blaschke's selection theorem, as in, e.g., [Sch93, Thm.1.8.6, p.50]). Clearly, we will have

$$
B_{r}(0) \subset V \subset \overline{B_{R}(0)}
$$

so that $V$ is non-degenerate. We shall prove that our assumptions imply that $\partial V$ is a sphere.

Before we begin, we make similar observations as in the proof of Corollary 1.2. Applying Theorem $2.1^{\prime}$ to each $\partial U_{k}$, we get, for every $k \in \mathbb{N}$, a $\lambda_{k} \in \mathbb{R}$ such that

$$
\left\|A-\lambda_{k} g\right\|_{L^{p}\left(\partial U_{k}\right)} \leq C\|\AA\|_{L^{p}\left(\partial U_{k}\right)}
$$

where $C>0$ depends only on $n$ and $p$. Then, using Theorem $2.1^{\prime}$ and Lemma 2.2, we obtain (cf. eq. (1.10))

$$
\left|\lambda_{k}\right| \leq \frac{\left(C^{\prime}\left(1+c_{0}^{p}\right)\right)^{\frac{1}{p}}}{\sqrt{n}}+\frac{C c_{0}}{\sqrt{n}}
$$

where $C^{\prime}>0$ is the constant from Lemma 2.2 that depends only on $n$ and $p$. Thus, $\left(\lambda_{k}\right)_{k \in \mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, and, modulo picking a subsequence, we may without loss of generality assume that $\lim _{k \rightarrow \infty} \lambda_{k}=\bar{\lambda} \in \mathbb{R}$.

Also, as we will see in the proof of Proposition 2.4 in the next section, there is a constant $\delta>0$, depending only on $n$ and $p$, such that, for each $k \in \mathbb{N},\|A\|_{L^{p}\left(\partial U_{k}\right)} \geq \delta$ (we apply Corollary 2.8, proved in Section 5, after using Lemma 2.2). With Theorem $2.1^{\prime}$, it then follows that (cf. eq. (1.11))

$$
\left|\lambda_{k}\right| \geq \frac{\delta}{\sqrt{n}}-\frac{C}{\sqrt{n}}\|\AA\|_{L^{p}\left(\partial U_{k}\right)}
$$

whence $|\bar{\lambda}| \geq \delta / \sqrt{n}>0$.

We now start by establishing the convergence to a sphere locally.
3.2. Local convergence. We proceed as in the proof of Corollary 1.2 and construct charts in which portions of $\partial U_{k}$ are represented by graphs of Lipschitz maps.

Consider any orthonormal system $\left(x^{1}, \ldots, x^{n}, x^{n+1}\right)$ at the origin of $\mathbb{R}^{n+1}$, and let $C_{-}=\left\{x^{n+1} \leq 0,\left(x^{1}\right)^{2}+\cdots+\left(x^{n}\right)^{2} \leq r^{2}\right\}$ be the half-infinite cylinder of radius $r$ pointing into the negative $x^{n+1}$-direction. Since $B_{r}(0) \subset U_{k}$, it follows that $\partial U_{k} \cap$ $C_{-}$must be the graph of a function

$$
u_{k}: \overline{D_{r}(0)} \rightarrow \mathbb{R},
$$

where $\overline{D_{r}(0)}=\left\{y \in \mathbb{R}^{n}| | y \mid \leq r\right\}$. Obviously, the $u_{k}$ will be convex, and $U_{k} \subset$ $B_{R}(0)$ implies that $\left\|u_{k}\right\|_{L^{\infty}\left(\overline{D_{r}(0)}\right)} \leq R$. It is easy to show that this forces the $u_{k}$ to be $\frac{4 R}{r}$-Lipschitz on $\overline{D_{\frac{r}{2}}(0)}$ (see, e.g., [RV74, Thm.A]).

We now argue verbatim as in Section 3, replacing the Lipschitz constant there by $\frac{4 R}{r}$ and the radius of the ball by $\frac{r}{2}$, to conclude that, locally, a subsequence of $\left(\partial U_{k}\right)_{k \in \mathbb{N}}$ converges to portions of spheres. More precisely, in the present situation (using that we already know that the $\bar{U}_{k}$ converge) we establish that $\partial V \cap C_{-}^{\prime}$, $C_{-}^{\prime}=\left\{x^{n+1} \leq 0,\left(x^{1}\right)^{2}+\cdots+\left(x^{n}\right)^{2} \leq r^{2} / 2\right\}$, is the portion of a sphere of radius $|\bar{\lambda}|^{-1}$. We now use this to obtain the global statement.
3.3. Local to global. If we consider a rotation $\Phi$ of $\mathbb{R}^{n+1}$, we can argue in exactly the same way as above to conclude that also $\partial V \cap \Phi\left(C_{-}^{\prime}\right)$ is the portion of a sphere of radius $|\bar{\lambda}|^{-1}$. Choosing $\Phi$ close enough to the identity, we obtain that $\partial V \cap C_{-}^{\prime} \cap \Phi\left(C_{-}^{\prime}\right)$ has large enough overlap to establish that the centres of the spheres containing $\partial V \cap C_{-}^{\prime}$ and $\partial V \cap \Phi\left(C_{-}^{\prime}\right)$ must coincide. We then conclude immediately that $V$ is a ball of radius $|\bar{\lambda}|^{-1}$, which contradicts our assumption. Hence the corollary holds.

## 4. Proof of Propostion 2.4

We will, in fact, prove the slightly weaker (notice the bound on $\int_{\partial U}|A|^{p}$ replacing the bound on $\int_{\partial U}\left|\AA^{p}\right|^{p}$ )

Proposition 2.7. Let $n \geq 2, p \in(1, n]$ and $c_{0} \in(0,+\infty)$ be given. Then there exist $R>r>0$, depending only on $n, p$ and $c_{0}$ such that: if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$
\operatorname{vol}_{n}(\partial U)=1 \quad \text { and } \quad \int_{\partial U}|A|^{p} \leq c_{0}
$$

then there exists an $x \in \mathbb{R}^{n+1}$ such that $B_{r}(x) \subset U \subset B_{R}(x)$.
In view of Lemma 2.2 in the first section, this is, indeed, sufficient for obtaining Proposition 2.4. The proof of Proposition 2.7, on the other hand, will be carried out by induction over $n \geq 2$. At the induction step, the following corollary, giving a lower bound on $\int_{\partial U}|A|^{p}$, will play a crucial role

Corollary 2.8 (to Proposition 2.7). Let $n \geq 2$ and $p \in(1, n]$ be given. Then there is a constant $\delta>0$ depending only on $n$ and $p$ such that: if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies $\operatorname{vol}_{n}(\partial U)=1$, then

$$
\int_{\partial U}|A|^{p} \geq \delta
$$

We prove it in Section 5. Also, we show the two sought-after inclusions of Proposition 2.7 in two separate lemmas, the proofs of which are deferred to sections 6 and 7 , respectively.

Lemma 2.9. Let $n \geq 2, p \in(1, n]$ and $c_{0} \in(0,+\infty)$ be given. Then there exists a constant $D>0$, depending only on $n, p$ and $c_{0}$, such that: if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$
\operatorname{vol}_{n}(\partial U)=1 \quad \text { and } \quad \int_{\partial U}|A|^{p} \leq c_{0}
$$

then $\operatorname{diam} U \leq D$ (where $\operatorname{diam} U$ denotes the diameter of $U$ in $\mathbb{R}^{n+1}$ ).
Lemma 2.10. Let $n \geq 2, p \in(1, n]$ and $c_{0} \in(0,+\infty)$ be given. Then there exists a constant $r>0$, depending only on $n, p$ and $c_{0}$, such that: if $U \subset \mathbb{R}^{n+1}$ is open, convex, has smooth boundary and satisfies

$$
\operatorname{vol}_{n}(\partial U)=1 \quad \text { and } \quad \int_{\partial U}|A|^{p} \leq c_{0}
$$

then there is an $x \in \mathbb{R}^{n+1}$ such that $B_{r}(x) \subset U$.
Clearly, both lemmas together imply Proposition 2.7. Also, Lemma 2.10 will be a consequence of Lemma 2.9. The proof of Lemma 2.9 in dimension $n$, on the other hand, will rely on us having proved Corollary 2.8 (and thus Proposition 2.7) for all dimensions $n^{\prime} \in\{2, \ldots, n-1\}$, except for when $n=2$. The reason why we did not split off the induction basis to a separate statement, is that the method used for proving the two-dimensional case is also useful in some $n$-dimensional cases. For an easier understanding of how our induction argument works, we give the following overview on the chains of implications.

- Induction base $(n=2)$ :

$$
\underbrace{\text { Lem.2.9 }\left.\right|_{2}} \Longrightarrow \Longrightarrow \text { Lem.2.10 }\left.\right|_{2}
$$

- Induction step $((n-1) \mapsto n)$

$$
\begin{aligned}
& \left\{\text { Prop.2.7 }\left.\right|_{n^{\prime}}\right\}_{n^{\prime} \in\{2, \ldots, n-1\}} \\
& \quad \Longrightarrow\left\{\text { Cor.2.8 }\left.\right|_{n^{\prime}}\right\}_{n^{\prime} \in\{2, \ldots, n-1\}} \Longrightarrow \underbrace{\Longrightarrow \text { Prop.2.7 }\left.\right|_{n}}
\end{aligned}
$$

## 5. Proof of Corollary 2.8

The following proof is a variant of the one of Corollary 2.5 (cf. Section 3). Nevertheless, we expose it in full detail to accommodate those readers who eagerly skipped the qualitative $C^{0}$-closeness in order to learn how to prove Theorem 2.1', first.
5.1. Preliminaries. Assume, by contradiction, that the Corollary were not true. Then there must be a sequence $\left(U_{j}\right)_{j \in \mathbb{N}}$ of open, convex subsets of $\mathbb{R}^{n+1}$ with smooth boundaries, satisfying, for all $j \in \mathbb{N}$, $\operatorname{vol}_{n}\left(\partial U_{j}\right)=1$ and $\int_{\partial U_{j}}|A|^{p} \leq c_{0}$ for some $c_{0}>0$ independent of $j$, and such that

$$
\lim _{j \rightarrow \infty} \int_{\partial U_{j}}|A|^{p}=0
$$

Modulo translating each set $U_{j}$, Proposition 2.7 then implies the existence of $R>$ $r>0$ such that

$$
B_{r}(0) \subset U_{j} \subset B_{R}(0) \quad(\forall j)
$$

Picking a subsequence, if necessary, we can then assume that the closures $\overline{U_{j}}$ converge (in the Hausdorff topology) to a closed convex set $V \subset \mathbb{R}^{n+1}$ (see Blaschke's selection theorem, Theorem 1.8.6 on p.50, in [Sch93]). Clearly, we will have

$$
B_{r}(0) \subset V \subset \overline{B_{R}(0)}
$$

i.e., $V$ is non-degenerate. We shall prove that our assumptions imply that $\partial V$ is contained in an affine subspace of $\mathbb{R}^{n+1}$, contradicting the above inclusions because of the convexity of $V$. We first argue locally, showing that, for every $q \in \partial V$, there is a neighbourhood $W$ of $q$ and an affine space $E \subset \mathbb{R}^{n+1}$, such that $\partial V \cap W \subset E$.
5.2. Local convergence. Consider any orthonormal system $\left(x^{1}, \ldots, x^{n}, x^{n+1}\right)$ at the origin $0 \in \mathbb{R}^{n+1}$, and let $C_{-}=\left\{x^{n+1} \leq 0,\left(x^{1}\right)^{2}+\cdots+\left(x^{n}\right)^{2} \leq r^{2}\right\}$ be the half-infinite cylinder of radius $r$ pointing into the negative $x^{n+1}$-direction. Since $B_{r}(0) \subset U_{j}$, it follows that $\partial U_{j} \cap C_{-}$must be the graph of a function

$$
u_{j}: \overline{D_{r}(0)} \rightarrow \mathbb{R},
$$

where $\overline{D_{r}(0)}=\left\{y \in \mathbb{R}^{n}| | y \mid \leq r\right\}$. Obviously, the $u_{j}$ will be convex, and $U_{j} \subset$ $B_{R}(0)$ implies that $\left.\left\|u_{j}\right\|_{L^{\infty}\left(\overline{D_{r}(0)}\right.}\right) \leq R$. It is easy to show that this forces the $u_{j}$ to be $\frac{4 R}{r}$-Lipschitz on $\overline{D_{\frac{r}{2}}(0)}$ (see, for example, [RV74, Theorem A]), and hence, by smoothness, $\left\|D u_{j}\right\|_{L^{\infty}\left(\overline{D_{r / 2}(0)}\right)} \leq \frac{4 R}{r}$. Now remember that, for the graph of a function $\varphi$, the second fundamental form $A_{\varphi}$ is given by (see also the proof of Lemma 1.3 in Section 4 of Chapter 1)

$$
A_{\varphi}=\frac{\operatorname{Hess} \varphi}{\sqrt{1+|D \varphi|^{2}}}
$$

so that

$$
|\operatorname{Hess} \varphi| \leq\left|A_{\varphi}\right| \sqrt{1+|D \varphi|^{2}}
$$

As a consequence, $\left\|D u_{j}\right\|_{L^{\infty}\left(\overline{D_{r / 2}(0)}\right)} \leq \frac{4 R}{r}$ and $\lim _{j \rightarrow \infty} \int_{\partial U_{j}}|A|^{p}=0$ imply that

$$
\left\|\operatorname{Hess} u_{j}\right\|_{L^{p}\left(\overline{D_{r / 2}(0)}\right)} \leq\left\|A_{u_{j}}\right\|_{L^{p}\left(\overline{D_{r / 2}(0)}\right)} \sqrt{1+16 \frac{R^{2}}{r^{2}}} \xrightarrow{j \rightarrow \infty} 0 .
$$

Since all $u_{j}$ are smooth, $\frac{4 R}{r}$-Lipschitz and bounded by $R$ on $\overline{D_{r / 2}(0)},\left(u_{j}\right)_{j \in \mathbb{N}}$ is a pointwise bounded, equicontinuous sequence in the space of continuous, realvalued functions on the compact domain $\overline{D_{r / 2}(0)}$. Then the Ascoli-Arzelà-Theorem (see, e.g., [Rud91, Thm.A5, p.394]) implies the existence of a subsequence $\left(u_{j_{k}}\right)_{k \in \mathbb{N}}$ that converges uniformly on $\overline{D_{r / 2}(0)}$ to a continuous function $\bar{u}$. Since the $u_{j_{k}}$ are Lipschitz and $u_{j_{k}} \longrightarrow \bar{u}(k \rightarrow \infty)$ uniformly, $\bar{u}$ will be Lipschitz with the same constant $\left(\frac{4 R}{r}\right)$. Now, for any $i \in\{1, \ldots, n\}$, we have

$$
\left\|D_{i} u_{j_{k}}\right\|_{L^{\infty}\left(\overline{D_{r / 2}(0)}\right)} \leq \frac{4 R}{r}
$$

and

$$
\left.\left\|D\left(D_{i} u_{j_{k}}\right)\right\|_{L^{1}\left(\overline{D_{r / 2}(0)}\right.}\right) \leq\left(\operatorname{vol}_{n}\left(\overline{D_{r / 2}(0)}\right)\right)^{1-1 / p}\left\|D\left(D_{i} u_{j_{k}}\right)\right\|_{L^{p}\left(\overline{D_{r / 2}(0)}\right.} \xrightarrow{k \rightarrow \infty} 0
$$

Consequently, by Rellich-Kondrachov (see, e.g., [Eva98, Thm.1, §5.7, p.272]), there is a subsequence $\left(u_{j_{k_{l}}}\right)_{l \in \mathbb{N}} \subset\left(u_{j_{k}}\right)_{k \in \mathbb{N}}$ and an $L^{1}$-function $v_{i}$ such that $D_{i} u_{j_{k_{l}}} \xrightarrow{l \rightarrow \infty} v_{i}$ in $L^{1}\left(\overline{D_{r / 2}(0)}\right)$. Moreover, since $D\left(D_{i} u_{j_{k_{l}}}\right) \xrightarrow{l \rightarrow \infty} 0$ in $L^{1}$, we have that $D v_{i}=0$ in the sense of distributions (see, e.g., [GT01, Thm.7.4, p.150]), implying that $v_{i}=c_{i}$ almost everywhere for some constant $c_{i} \in \mathbb{R}$ (see, e.g., [LL97, Thm.6.11, p.138]). But since $u_{j_{k_{l}}} \xrightarrow{l \rightarrow \infty} \bar{u}$ uniformly, and hence in $L^{1}$, as well as $D_{i} u_{j_{k_{l}}} \xrightarrow{l \rightarrow \infty} v_{i}$ in $L^{1}$, we conclude that $D_{i} \bar{u}=c_{i}$ in the sense of distributions. As a consequence, $D\left(\bar{u}(x)-\sum_{i=1}^{n} c_{i} x^{i}\right)=0$ in the sense of distributions, and we conclude that $\bar{u}(x)-$ $\sum_{i=1}^{n} c_{i} x^{i}=b$ almost everywhere for some constant $b \in \mathbb{R}$. By the continuity of $\bar{u}$, we see that $\bar{u}(x)=b+\sum_{i=1}^{n} c_{i} x^{i}$ everywhere, so that $\bar{u}$ is, in fact, an affine function. Writing $C_{-}^{\prime}=\left\{x^{n+1} \leq 0,\left(x^{1}\right)^{2}+\cdots+\left(x^{n}\right)^{2} \leq r^{2} / 4\right\}$, it then follows that $\partial V \cap C_{-}^{\prime}$ is contained in an affine hyperplane $E$. In the next step, we show how to conclude the global statement.
5.3. Local to global. If we consider a rotation $\Phi$ of $\mathbb{R}^{n+1}$, we can argue in the same way as above to show that $\partial V \cap \Phi\left(C_{-}^{\prime}\right)$ is contained in an affine hyperplane $F$. But if $\Phi$ is sufficiently close to the identity, $\partial V \cap C_{-}^{\prime} \cap \Phi\left(C_{-}^{\prime}\right)$ will have positive area, from which we conclude that $E=F$. It is then immediate to see that $\partial V$ is, as a whole, contained in the affine hyperplane $E$, which is exactly what we claimed.

As already mentioned, this, together with the convexity of $V$, is incompatible with the inclusions

$$
B_{r}(0) \subset V \subset \overline{B_{R}(0)}
$$

thus finishing the proof of the corollary.

## 6. Proof of Lemma 2.9

6.1. Some notation. Before starting the proof, we introduce some notations. Assume $\left(x^{1}, \ldots, x^{n+1}\right)$ is any orthonormal system in $\mathbb{R}^{n+1}$. For $m \in\{1, \ldots, n+1\}$, let $\mathrm{P}^{m}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{m},\left(x^{1}, \ldots, x^{n+1}\right) \mapsto\left(x^{n-m+2}, \ldots, x^{n+1}\right)$, denote the projection onto the last $m$ coordinates. For $r \in \mathbb{R}^{m}$, we define $\pi_{r}^{m}$ as the codimension $m$ hyperplane $\left\{x \in \mathbb{R}^{n+1} \mid \mathrm{P}^{m}(x)=r\right\}$, and $\omega_{0}^{m}$ as the orthogonal complement $\left(\pi_{0}^{m}\right)^{\perp}$ of $\pi_{0}^{m}$ in $\mathbb{R}^{n+1} . \omega_{0}^{m}$ is thus the $m$-dimensional hyperplane passing through the origin with the first $n-m+1$ coordinates that vanish. For $\theta>0$, let

$$
R^{m}(\theta)=\left\{\left(\theta \lambda_{1}, \ldots, \theta \lambda_{m}\right)\left|\sum_{i=1}^{m}\right| \lambda_{i} \mid \leq 1\right\}
$$

denote the standard $m$-dimensional cross-polytope (or hyperrhombus) of length $\theta$ in $\mathbb{R}^{m}$ (it is the convex hull of the $2 m$ points given by $\pm \theta$ times the standard basis vectors). We have $\operatorname{vol}_{m}\left(R^{m}(\theta)\right)=\frac{2^{m}}{m!} \theta^{m}$. Notice that, for all $y \in \mathbb{R}^{m} \backslash R^{m}(\theta)$, we have that $|y|>\frac{\theta}{\sqrt{m}}$. Finally, denote by $\mathcal{R}^{m}(\theta)$ the embedding $\left\{x \in \omega_{0}^{m} \mid \mathrm{P}^{m}(x) \in R^{m}(\theta)\right\}$ of $R^{m}(\theta)$ into $\mathbb{R}^{n+1}$.
6.2. Preliminaries. Let $n \geq 2, p \in(1, n]$ and $c_{0}>0$ be given. If $n \geq 3$, we shall assume that Proposition 2.7 has already been proved for all dimensions $n^{\prime} \in\{2, \ldots, n-1\}$. Since, for all $\tilde{p} \in(1, \min \{2, p\}]$, we have by Hölder's inequality that

$$
\left(\int_{\partial U}|A|^{\tilde{p}}\right)^{\frac{1}{\bar{p}}} \leq\left(\int_{\partial U}|A|^{p}\right)^{\frac{1}{p}}
$$

for any open convex set $U \subset \mathbb{R}^{n+1}$ with smooth boundary and $\operatorname{vol}_{n}(\partial U)=1$, we may as well assume without loss of generality that $p \in(1,2]$.

We prove Lemma 2.9 by contradiction. So assume the statement were false. Then there must be a sequence $\left(U_{k}\right)_{k \in \mathbb{N}}$ of open, convex sets with smooth boundary satisfying
(a) $\operatorname{vol}_{n}\left(\partial U_{k}\right)=1$,
(b) $\int_{\partial U_{k}}|A|^{p} \leq c_{0}$,
(c) $\operatorname{diam} U_{k} \xrightarrow{k \rightarrow \infty}+\infty$.

For all $k$, let $d_{k}=\operatorname{diam} U_{k}$. Modulo translations and rotations in $\mathbb{R}^{n+1}$, we can without loss of generality assume that $\left(0, \ldots, 0, \pm \frac{d_{k}}{2}\right) \in \overline{U_{k}}$. Let $R_{k}^{1}=R^{1}\left(\frac{d_{k}}{2}\right)=$ $\left[-\frac{d_{k}}{2}, \frac{d_{k}}{2}\right]$ and $\mathcal{R}_{k}^{1}=\mathcal{R}^{1}\left(\frac{d_{k}}{2}\right)$. Thus we assume that, $\forall k \in \mathbb{N}, \mathcal{R}_{k}^{1} \subset \overline{U_{k}}$, since $U_{k}$ is convex.

If $n-1 \geq 2$, let

$$
\delta_{k}^{1}=\max _{r \in R_{k}^{1}} \operatorname{diam}\left(U_{k} \cap \pi_{r}^{1}\right) .
$$

Then there are two possibilities:
(i) either $\quad \limsup _{k \rightarrow \infty} \frac{\delta_{k}^{1}}{d_{k}}=0$,
(ii) or $\quad \limsup _{k \rightarrow \infty} \frac{\delta_{k}^{1}}{d_{k}}>0$.

But since $\delta_{k}^{1} \leq d_{k}$, we may in fact assume that, after perhaps picking a subsequence, we are presented with one of the following two alternatives
(i) either $\lim _{k \rightarrow \infty} \frac{\delta_{k}^{1}}{d_{k}}=0$,
(ii) or there is a constant $\sigma_{2} \in(0,1]$ such that $\lim _{k \rightarrow \infty} \frac{\delta_{k}^{1}}{d_{k}}=\sigma_{2}$.

Now let us assume we are in the second case. Then, for $k$ large enough, we have $\frac{\sigma_{2}}{2} d_{k} \leq \delta_{k}^{1} \leq d_{k}$. Let $r_{k} \in R_{k}^{1}$ be such that $\operatorname{diam}\left(U_{k} \cap \pi_{r_{k}}^{1}\right)=\delta_{k}^{1}$ ( $R_{k}^{1}$ is compact). Then, modulo rotations (in $\mathbb{R}^{n+1}$ that leave the ( $n+1$ )st component invariant) and a restriction to the tail of the sequence, we can without loss of generality assume that $\left(0, \ldots, 0, \frac{\sigma_{2}}{4} d_{k}, r_{k}\right) \in \overline{U_{k}}, \forall k \in \mathbb{N}$. Since $\left(0, \ldots, 0, \pm \frac{d_{k}}{2}\right) \in \overline{U_{k}}$, the convexity of $U_{k}$ then implies that $\left(0, \ldots, 0, \frac{\sigma_{2}}{8} d_{k}, 0\right) \in \overline{U_{k}}$ (for $r_{k} \in R_{k}^{1}$ implicates $\left|r_{k}\right| \leq \frac{d_{k}}{2}$ ), and thus also that $\left(0, \ldots, 0, \frac{\sigma_{2}}{16} d_{k}, \pm \frac{d_{k}}{4}\right) \in \overline{U_{k}}$ (cf. Figure 2.1). As a result, modulo translations (in $\mathbb{R}^{n+1}$ that leave the last component invariant), we can without loss of generality assume that, $\forall k \in \mathbb{N},\left(0, \ldots, 0, \pm \frac{\sigma_{2}}{16} d_{k}, 0\right) \in \overline{U_{k}}$ and $\left(0, \ldots, 0,0, \pm \frac{d_{k}}{4}\right) \in \overline{U_{k}}$. By convexity, the convex hull of these four points is then also contained in $\overline{U_{k}}$, and we obtain that there must be a constant $c_{2} \in(0,1]$ such that $\mathcal{R}^{2}\left(c_{2} d_{k}\right) \subset \overline{U_{k}}$ for all $k$ (take, e.g., $\left.c_{2}=\min \left\{\frac{\sigma_{2}}{16}, \frac{1}{4}\right\}\right)$. Let $R_{k}^{2}=R^{2}\left(c_{2} d_{k}\right)$ and $\mathcal{R}_{k}^{2}=\mathcal{R}^{2}\left(c_{2} d_{k}\right)$. We thus assume that $\mathcal{R}_{k}^{2} \subset \overline{U_{k}}, \forall k \in \mathbb{N}$.

If $n-2 \geq 2$, let

$$
\delta_{k}^{2}=\max _{r \in R_{k}^{2}} \operatorname{diam}\left(U_{k} \cap \pi_{r}^{2}\right) .
$$

Then there are, again, two possibilities:
(i) either $\quad \limsup _{k \rightarrow \infty} \frac{\delta_{k}^{2}}{d_{k}}=0$,
(ii) or $\quad \limsup _{k \rightarrow \infty} \frac{\delta_{k}^{2}}{d_{k}}>0$.

In the second case, we can argue in an analogous manner to obtain that, without loss of generality, we may assume the existence of a constant $c_{3} \in(0,1]$ such that $\mathcal{R}_{k}^{3}=\mathcal{R}^{3}\left(c_{3} d_{k}\right) \subset \overline{U_{k}}, \forall k \in \mathbb{N}$.

Continuing this argument inductively, it is easy to see that, after an appropriate application of translations and rotations, as well as after picking a convenient subsequence, we may without loss of generality assume that
there exists an $m \in\{1, \ldots, n-1\}$ and a constant $c_{m} \in(0,1]$ such that, for all $k \in \mathbb{N}$,

$$
\mathcal{R}^{m}\left(c_{m} d_{k}\right) \subset \overline{U_{k}}
$$

moreover, if $m \leq n-2$, then

$$
\limsup _{k \rightarrow \infty} \frac{\max _{r \in R^{m}\left(c_{m} d_{k}\right)} \operatorname{diam}\left(U_{k} \cap \pi_{r}^{m}\right)}{d_{k}}=0
$$



Figure 2.1.

As we shall see shortly, even if $m=n-1$, the second statement still holds true.
6.3. The actual proof. Let $R_{k}^{m}=R^{m}\left(c_{m} d_{k}\right)$ and $\mathcal{R}_{k}^{m}=\mathcal{R}^{m}\left(c_{m} d_{k}\right)$.

Claim 1. For all $k \in \mathbb{N}$, we have

$$
\max _{r \in R_{k}^{m}} \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{r}^{m}\right) \leq \frac{2(m!)}{\left(c_{m}\right)^{m}\left(d_{k}\right)^{m}}
$$

Proof. For each $k \in \mathbb{N}$, let

$$
\mu_{k}=\max _{r \in R_{k}^{m}} \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{r}^{m}\right)
$$

and $\rho_{k}$ be the point in $R_{k}^{m}$ where this maximum is attained ( $R_{k}^{m}$ is compact). Let $\widetilde{R}_{k}^{m}$ denote the translation of $R^{m}\left(\frac{c_{m}}{2} d_{k}\right)$ into the point $\frac{\rho_{k}}{2}$, and let $\widetilde{\mathcal{R}}_{k}^{m}=$
$\left\{x \in \omega_{0}^{m} \mid \mathrm{P}^{m}(x) \in \widetilde{R}_{k}^{m}\right\}$. Notice that $\widetilde{R}_{k}^{m} \subset R_{k}^{m}$ and $\widetilde{\mathcal{R}}_{k}^{m} \subset \mathcal{R}_{k}^{m} \subset \overline{U_{k}}$. Observe also that, for any $y \in \widetilde{\mathcal{R}}_{k}^{m}$, we have $y_{k}+2\left(y-y_{k}\right) \in \mathcal{R}_{k}^{m}$, where $y_{k} \in \widetilde{\mathcal{R}}_{k}^{m}$ is such that $\mathrm{P}^{m}\left(y_{k}\right)=\rho_{k}$ (cf. Figure 2.2). Now, for any $y \in \widetilde{\mathcal{R}}_{k}^{m} \backslash\left\{y_{k}\right\}$, denote by $C_{k}(y)$ the cone with base $\partial U_{k} \cap \pi_{\rho_{k}}^{m}$ and tip $2 y-y_{k}$. Then, by convexity, $C_{k}(y) \subset \overline{U_{k}}$. Moreover, since $U_{k} \cap \pi_{\rho_{k}}^{m}$ is convex, $C_{k}(y) \cap \pi_{\mathrm{P}^{m}(y)}^{m}$ bounds a convex region and we must have

$$
\operatorname{vol}_{n-m}\left(C_{k}(y) \cap \pi_{P^{m}(y)}^{m}\right) \leq \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{P^{m}(y)}^{m}\right) .
$$



Figure 2.2.

Remark 2.11. The fact that $\operatorname{vol}(\partial M) \leq \operatorname{vol}(\partial N)$, whenever $M$ is a convex subset of the open set $N$ (and assuming both have smooth boundary) follows from the definition of the Hausdorff measure and the fact that the nearest point projection onto $M$ is norm-non-increasing - see, e.g., [BH99, Prop.2.4(4), p.177], and also [Cha06, Ex.III.12(i), p.161].

On the other hand, we have by construction

$$
\operatorname{vol}_{n-m}\left(C_{k}(y) \cap \pi_{\mathrm{P}^{m}(y)}^{m}\right)=\frac{1}{2} \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{\rho_{k}}^{m}\right)=\frac{\mu_{k}}{2} .
$$

As a consequence, we get for all $y \in \widetilde{\mathcal{R}}_{k}^{m}$ :

$$
\operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{\mathrm{P} m}^{m}(y)\right) \geq \frac{\mu_{k}}{2}
$$

But then the coarea formula (see, e.g., [Cha06, §III.8] or [Fed69, §3.2]) yields

$$
\begin{aligned}
1=\operatorname{vol}_{n}\left(\partial U_{k}\right) & \geq \operatorname{vol}_{n}\left(\left\{y \in \partial U_{k} \mid \mathrm{P}^{m}(y) \in \widetilde{\mathcal{R}}_{k}^{m}\right\}\right) \\
& =\int_{r \in \widetilde{R}_{k}^{m}} \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{r}^{m}\right) d r \geq \frac{\mu_{k}}{2} \operatorname{vol}_{m}\left(\widetilde{R}_{k}^{m}\right) .
\end{aligned}
$$

And since

$$
\operatorname{vol}_{m}\left(\widetilde{R}_{k}^{m}\right)=\operatorname{vol}_{m}\left(R^{m}\left(\frac{c_{m}}{2} d_{k}\right)\right)=\frac{2^{m}}{m!}\left(\frac{c_{m}}{2} d_{k}\right)^{m}=\frac{\left(c_{m}\right)^{m}\left(d_{k}\right)^{m}}{m!}
$$

the claim follows immediately.
Define

$$
\delta_{k}=\max _{r \in R^{m}\left(c_{m} d_{k}\right)} \operatorname{diam}\left(U_{k} \cap \pi_{r}^{m}\right),
$$

and let $r_{k} \in R_{k}^{m}=R^{m}\left(c_{m} d_{k}\right)$ be such that $\operatorname{diam}\left(U_{k} \cap \pi_{r_{k}}^{m}\right)=\delta_{k}$ ( $R_{k}^{m}$ is compact). If $m \leq n-2$, we have $\lim _{k \rightarrow \infty} \frac{\delta_{k}}{d_{k}}=0$ by assumption. If, however, $m=n-1$ (as is necessarily the case when $n=2$ ), then Claim 1 yields

$$
\lim _{k \rightarrow \infty} \operatorname{vol}_{1}\left(\partial U_{k} \cap \pi_{r}^{n-1}\right)=0, \quad \forall r \in R_{k}^{n-1}
$$

But, for each $r \in R_{k}^{n-1}, \partial U_{k} \cap \pi_{r}^{n-1}$ is a simple closed $C^{\infty}$-curve in the twodimensional hyperplane $\pi_{r}^{n-1}$, whence

$$
\operatorname{diam}\left(U_{k} \cap \pi_{r}^{n-1}\right) \leq \frac{1}{2} \operatorname{vol}_{1}\left(\partial U_{k} \cap \pi_{r}^{n-1}\right)
$$

It follows that $\lim _{k \rightarrow \infty} \frac{\delta_{k}}{d_{k}}=0$ also in the case $m=n-1$.
Now let

$$
\eta_{k}=\sqrt{m} \delta_{k}
$$

Since $\lim _{k \rightarrow \infty} \frac{\eta_{k}}{d_{k}}=0$, we may, modulo picking a subsequence that contains only the tail, without loss of generality assume that $\eta_{k}<\frac{c_{m}}{2} d_{k}, \forall k \in \mathbb{N}$. Define

$$
\widehat{R}_{k}^{m}=R^{m}\left(c_{m} d_{k}-\eta_{k}\right) \quad \text { and } \quad \widehat{\mathcal{R}}_{k}^{m}=\mathcal{R}^{m}\left(c_{m} d_{k}-\eta_{k}\right)
$$

Then we have
Claim 2. For all $k \in \mathbb{N}, r \in \widehat{R}_{k}^{m}$ and $q \in \partial U_{k} \cap \pi_{r}^{m}$, the angle $\angle\left(\nu(q), \pi_{0}^{m}\right)$ between the outer unit normal $\nu(q)$ to $\partial U_{k}$ in $q$ and the hyperplane $\pi_{0}^{m}$ is less than or equal to $\frac{\pi}{4}$.

Proof. Fix $r \in \widehat{R}_{k}^{m}$ and $q \in \partial U_{k} \cap \pi_{r}^{m}$, and let $y \in \widehat{\mathcal{R}}_{k}^{m}$ be such that $\mathrm{P}^{m}(y)=r$. The outer unit normal $\nu(q)$ to $\partial U_{k}$ in $q$ then decomposes as $\nu(q)=\nu^{\prime}+\nu^{\prime \prime}$, where $\nu^{\prime} \in \pi_{0}^{m}$ and $\nu^{\prime \prime} \in\left(\pi_{0}^{m}\right)^{\perp}=\omega_{0}^{m}$. Clearly,

$$
\left|\nu^{\prime}\right|=\cos \left(\angle\left(\nu(q), \pi_{0}^{m}\right)\right)
$$

and

$$
\left|\nu^{\prime \prime}\right|=\sin \left(\angle\left(\nu(q), \pi_{0}^{m}\right)\right)=\sqrt{1-\left|\nu^{\prime}\right|^{2}}
$$

If $\nu^{\prime \prime}=0$, then there is nothing to prove, so assume $\nu^{\prime \prime} \neq 0$. Let $y^{*} \in \mathcal{R}_{k}^{m}$ be such that $r^{*}=\mathrm{P}^{m}\left(y^{*}\right) \in \partial R_{k}^{m}$ and $\frac{y^{*}-y}{\left|y^{*}-y\right|}=\frac{\nu^{\prime \prime}}{\left|\nu^{\prime \prime}\right\rangle}$ (this is possible, since $R_{k}^{m}$ contains a ball of radius $\frac{\eta_{k}}{\sqrt{m}}$ around every $\widetilde{r} \in \widehat{R}_{k}^{m}$ ) - cf. Figure 2.3. Notice that $\left|y^{*}-y\right| \geq \frac{\eta_{k}}{\sqrt{m}}$, since $y \in \widehat{\mathcal{R}}_{k}^{m}$.


Figure 2.3.

Now, since $U_{k}$ is convex, we have $\overline{U_{k}} \subset\left\{z \in \mathbb{R}^{n+1} \mid\langle z-q, \nu(q)\rangle \leq 0\right\}$, where, once more, $\langle\cdot, \cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Hence,

$$
0 \geq\left\langle y^{*}-q, \nu(q)\right\rangle=\left\langle y^{*}-y, \nu(q)\right\rangle+\langle y-q, \nu(q)\rangle
$$

since $y^{*} \in \overline{U_{k}}$. But, by construction, we have

$$
\left\langle y^{*}-y, \nu(q)\right\rangle=\frac{\left|y^{*}-y\right|}{\left|\nu^{\prime \prime}\right|}\left\langle\nu^{\prime \prime}, \nu^{\prime \prime}\right\rangle=\left|y^{*}-y\right| \sqrt{1-\left|\nu^{\prime}\right|^{2}}
$$

Moreover, noticing that $q-y \in \pi_{0}^{m}$, we have (observe that $\langle q-y, \nu(q)\rangle \geq 0$, since $\left.y \in \overline{U_{k}}\right)$

$$
\langle q-y, \nu(q)\rangle=\left\langle q-y, \nu^{\prime}\right\rangle \leq|q-y|\left|\nu^{\prime}\right|
$$

We therefore get

$$
\operatorname{cotan}\left(\angle\left(\nu(q), \pi_{0}^{m}\right)\right)=\frac{\left|\nu^{\prime}\right|}{\sqrt{1-\left|\nu^{\prime}\right|^{2}}} \geq \frac{\left|y^{*}-y\right|}{|q-y|} \geq \frac{\eta_{k}}{\sqrt{m}|q-y|}=\frac{\delta_{k}}{|q-y|}
$$

But given that $q, y \in \overline{U_{k}} \cap \pi_{r}^{m}$ and $\operatorname{diam}\left(U_{k} \cap \pi_{r}^{m}\right) \leq \delta_{k}$, we obtain the desired inequality $\operatorname{cotan}\left(\angle\left(\nu(q), \pi_{0}^{m}\right)\right) \geq 1$.

We now wish to prove:

## Claim 3.

$$
\liminf _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}\right)>0
$$

Proof. Assume first that

$$
\lim _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \notin \widehat{\mathcal{R}}_{k}^{m}\right\}\right)=0
$$

Then there is nothing to prove, since, in that case,

$$
\begin{aligned}
& \liminf _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}\right) \\
& \quad=\liminf _{k \rightarrow \infty} \underbrace{\operatorname{vol}_{n}\left(\partial U_{k}\right)}_{=1}-\limsup _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \notin \widehat{\mathcal{R}}_{k}^{m}\right\}\right)=1
\end{aligned}
$$

So suppose that $\lim _{\sup }^{k \rightarrow \infty} \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \notin \widehat{\mathcal{R}}_{k}^{m}\right\}\right)>0$. Modulo taking a subsequence, we may without loss of generality assume that there is a $v_{c} \in(0,1)$ such that, for every $k \in \mathbb{N}$, we have

$$
\operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \notin \widehat{\mathcal{R}}_{k}^{m}\right\}\right) \geq v_{c}
$$

From the coarea formula, we then have

$$
\begin{aligned}
v_{c} & \leq \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \notin \widehat{\mathcal{R}}_{k}^{m}\right\}\right)=\int_{r \notin \widehat{R}_{k}^{m}} \operatorname{vol}_{n-m}\left(\partial U_{k} \cap \pi_{r}^{m}\right) d r \\
& \leq \sum_{i=n-m+2}^{n+1} \int_{|\rho| \geq \frac{c_{m} d_{k}-\eta_{k}}{\sqrt{m}}} \operatorname{vol}_{n-1}\left(\partial U_{k} \cap\left\{y=\left(y^{1}, \ldots, y^{n+1}\right) \in \mathbb{R}^{n+1} \mid y^{i}=\rho\right\}\right) d \rho
\end{aligned}
$$

since $R^{m}(\theta)$ contains the $m$-cube of length $\frac{\theta}{\sqrt{m}}$. Notice that all the integrals are well-defined, since the integrands vanish for arguments of integration with length greater than $d_{k}$. Since the right-hand side of the above inequality is invariant under renumbering of the coordinates, we may, modulo reflections, without loss of generality assume that

$$
\int_{-d_{k}}^{-\frac{c_{m} d_{k}-\eta_{k}}{\sqrt{m}}} \operatorname{vol}_{n-1}\left(\partial U_{k} \cap \pi_{\rho}^{1}\right) d \rho \geq \frac{v_{c}}{2 m}
$$

Consequently, for each $k \in \mathbb{N}$, there must be a $\rho_{k} \in\left[-d_{k},-\frac{c_{m} d_{k}-\eta_{k}}{\sqrt{m}}\right]$ such that

$$
\begin{equation*}
\operatorname{vol}_{n-1}\left(\partial U_{k} \cap \pi_{\rho_{k}}^{1}\right) \geq \frac{v_{c}}{2 \sqrt{m}\left(\left(\sqrt{m}-c_{m}\right) d_{k}+\eta_{k}\right)} \geq \frac{v_{c}}{\sqrt{m}\left(2 \sqrt{m}-c_{m}\right) d_{k}} \tag{2.2}
\end{equation*}
$$

since we had assumed that $\eta_{k} \leq \frac{c_{m}}{2} d_{k}$.
Now remember that, by assumption, $\left(0, \ldots, 0, c_{m} d_{k}\right) \in \overline{U_{k}}$, since $\mathcal{R}_{k}^{m} \subset \overline{U_{k}}$. Thus, by the convexity of $U_{k}$, the whole cone $C_{k}$ with base $\partial U_{k} \cap \pi_{\rho_{k}}^{1}$ and tip $\left(0, \ldots, 0, c_{m} d_{k}\right)$ must be contained in $\overline{U_{k}}$. Moreover, since $\partial U_{k} \cap \pi_{\rho_{k}}^{1}$ is convex, we must have

$$
\begin{equation*}
\operatorname{vol}_{n-1}\left(\partial U_{k} \cap \pi_{\rho}^{1}\right) \geq \operatorname{vol}_{n-1}\left(C_{k} \cap \pi_{\rho}^{1}\right)=\frac{c_{m} d_{k}-\rho}{c_{m} d_{k}-\rho_{k}} \operatorname{vol}_{n-1}\left(\partial U_{k} \cap \pi_{\rho_{k}}^{1}\right) \tag{2.3}
\end{equation*}
$$

for all $\rho \in\left(\rho_{k}, c_{m} d_{k}\right]$. But then the coarea formula yields (since $\left.\mathcal{R}^{1}\left(c_{m} d_{k}-\eta_{k}\right) \subset \widehat{\mathcal{R}}_{k}^{m}\right)$

$$
\begin{aligned}
& \operatorname{vol}_{n}\left(\partial U_{k}\right.\left.\cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}\right) \\
& \geq \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \mathcal{R}^{1}\left(c_{m} d_{k}-\eta_{k}\right)\right\}\right) \\
& \geq \operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y=\left(y^{1}, \ldots, y^{n+1}\right) \in \mathbb{R}^{n+1} \mid y^{n+1} \in\left[0, c_{m} d_{k}-\eta_{k}\right]\right\}\right) \\
&=\int_{0}^{c_{m} d_{k}-\eta_{k}} \operatorname{vol}_{n-1}\left(\partial U_{k} \cap \pi_{\rho}^{1}\right) d \rho \\
&\left(\begin{array}{c}
(2.2) \&(2.3) \\
\end{array} \int_{0}^{c_{m} d_{k}-\eta_{k}} \frac{c_{m} d_{k}-\rho}{c_{m} d_{k}-\rho_{k}} \cdot \frac{v_{c}}{\sqrt{m}\left(2 \sqrt{m}-c_{m}\right) d_{k}} d \rho\right. \\
&=\frac{v_{c}}{2 \sqrt{m}\left(2 \sqrt{m}-c_{m}\right)\left(c_{m} d_{k}-\rho_{k}\right) d_{k}}\left(\left(c_{m} d_{k}\right)^{2}-\left(\eta_{k}\right)^{2}\right) \\
& \geq \frac{v_{c}}{2 \sqrt{m}\left(2 \sqrt{m}-c_{m}\right)\left(1+c_{m}\right)}-\frac{v_{c}\left(c_{m}\right)^{2}}{2 \sqrt{m}\left(2 \sqrt{m}-c_{m}\right)\left(1+c_{m}\right)}\left(\frac{\eta_{k}}{d_{k}}\right)^{2},
\end{aligned}
$$

where the last line follows from $\rho_{k} \geq-d_{k}$. Remembering that $\lim \sup _{k \rightarrow \infty} \frac{\eta_{k}}{d_{k}}=0$, we see that the claim holds.

Henceforth we shall, modulo picking a subsequence, without loss of generality assume that $v_{0} \in(0,1)$ is such that

$$
\operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}\right) \geq v_{0} \quad \forall k \in \mathbb{N}
$$

The following final claim then yields a contradiction to the assumption

$$
\int_{\partial U_{k}}|A|^{p} \leq c_{0}
$$

## Claim 4.

$$
\liminf _{k \rightarrow \infty} \int_{\partial U_{k}}|A|^{p}=+\infty
$$

Proof. Fix $k \in \mathbb{N}$. For all $r \in \widehat{R}_{k}^{m}$, let $\Gamma_{k, r}^{n-m}=\partial U_{k} \cap \pi_{r}^{m}$. In the following, denote by $\bar{A}$ the second fundamental form of $\Gamma_{k, r}^{n-m}$ in $\pi_{r}^{m}$, and by $\bar{\nu}$ the Gauss map of $\Gamma_{k, r}^{n-m}$ in $\pi_{r}^{m}$. We have to distinguish the two cases: $m=n-1$ and $m \leq n-2$.

If $m=n-1$ (which is necessarily the case when $n=2$ ), $\Gamma_{k, r}^{1}$ is a simple closed $C^{\infty}$-curve in $\pi_{r}^{n-1}$. Consequently, there are two points $q_{1}$ and $q_{2}$ in $\Gamma_{k, r}^{1}$ such that $\bar{\nu}\left(q_{1}\right)=-\bar{\nu}\left(q_{2}\right)$. Corollary A.4, proved in the appendix, then gives

$$
\int_{\gamma}|\bar{A}| \geq\left|\bar{\nu}\left(q_{1}\right)-\bar{\nu}\left(q_{2}\right)\right|=2
$$

for each of the two arcs $\gamma \subset \Gamma_{k, r}^{1}$ joining $q_{1}$ and $q_{2}$. As a consequence,

$$
\int_{\Gamma_{k, r}^{1}}|\bar{A}| \geq 2 .
$$

Using Lemma A. 2 of the appendix, together with Claim 2, then yields

$$
\int_{\Gamma_{k, r}^{1}}|A| \geq \frac{1}{\sqrt{2}} \int_{\Gamma_{k, r}^{1}}|\bar{A}| \geq \sqrt{2}
$$

It thus follows from the coarea formula that

$$
\begin{aligned}
\int_{\partial U_{k}}|A| & \geq \int_{\partial U_{k} \cap\left\{\left.y \in \mathbb{R}^{n+1}\right|_{\left.y \in \widehat{\mathcal{R}}_{k}^{n-1}\right\}}|A|=\int_{\widehat{R}_{k}^{n-1}}\left(\int_{\Gamma_{k, r}^{1}}|A|\right) d r\right.} \\
& \geq \sqrt{2} \operatorname{vol}_{n-1}\left(\widehat{R}_{k}^{n-1}\right)
\end{aligned}
$$

Since $k \in \mathbb{N}$ was arbitrary and

$$
\operatorname{vol}_{n-1}\left(\widehat{R}_{k}^{n-1}\right)=\operatorname{vol}_{n-1}\left(R^{n-1}\left(c_{n-1} d_{k}-\eta_{k}\right)\right) \quad \xrightarrow{k \rightarrow \infty}+\infty
$$

we conclude that, indeed,

$$
\liminf _{k \rightarrow \infty} \int_{\partial U_{k}}|A|^{p} \geq \liminf _{k \rightarrow \infty}((\underbrace{\operatorname{vol}_{n}\left(\partial U_{k}\right)}_{=1})^{1-p}\left(\int_{\partial U_{k}}|A|\right)^{p})=+\infty
$$

if $m=n-1$.

Now assume that $m \leq n-2$, and denote by $\widetilde{\Gamma}_{k, r}^{n-m}$ the rescaling of $\Gamma_{k, r}^{n-m}$ such that $\operatorname{vol}_{n-m}\left(\widetilde{\Gamma}_{k, r}^{n-m}\right)=1$. Also, let $\widetilde{A}$ be the second fundamental form of $\widetilde{\Gamma}_{k, r}^{n-m}$ in $\pi_{r}^{m}$. Clearly, we have

$$
|\widetilde{A}|=\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{\frac{1}{n-m}}|\bar{A}|
$$

As a consequence, using Claim 2 with Lemma A.2, we obtain

$$
\begin{aligned}
\int_{\widetilde{\Gamma}_{k, r}^{n-m}}|\widetilde{A}|^{p} & =\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{\frac{p}{n-m}-1} \int_{\Gamma_{k, r}^{n-m}}|\bar{A}|^{p} \\
& \leq(\sqrt{2})^{p}\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{\frac{p}{n-m}-1} \int_{\Gamma_{k, r}^{n-m}}|A|^{p}
\end{aligned}
$$

Remember that we had assumed $p \in(1,2] \subset(1, n-m]$, and that Proposition 2.7 is already proved for every $n^{\prime} \in\{2, \ldots, n-1\}$. We may thus apply Corollary 2.8 to $\widetilde{\Gamma}_{k, r}^{n-m}$, yielding

$$
\begin{aligned}
\int_{\Gamma_{k, r}^{n-m}}|A|^{p} & \geq 2^{-\frac{p}{2}}\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{1-\frac{p}{n-m}} \int_{\widetilde{\Gamma}_{k, r}^{n-m}}|\widetilde{A}|^{p} \\
& \geq 2^{-\frac{p}{2}}\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{1-\frac{p}{n-m}} \delta
\end{aligned}
$$

for some $\delta>0$ depending only on $(n-m) \in\{2, \ldots, n-1\}$ and $p$. The coarea formula then yields

$$
\begin{aligned}
& \int_{\partial U_{k}}|A|^{p} \geq \int_{\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}}|A|^{p}=\int_{r \in \widehat{R}_{k}^{m}}\left(\int_{\Gamma_{k, r}^{n-m}}|A|^{p}\right) d r \\
& \geq 2^{-\frac{p}{2}} \delta \int_{r \in \widehat{R}_{k}^{m}}\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\left(\operatorname{vol}_{n-m}\left(\Gamma_{k, r}^{n-m}\right)\right)^{-\frac{p}{n-m}}\right) d r \\
& \quad \begin{array}{l}
\text { Claim 1 } \\
\\
\\
\geq 2^{-\frac{p}{2}} \delta\left(\frac{2(m!)}{\left(c_{m}\right)^{m}\left(d_{k}\right)^{m}}\right)^{-\frac{p}{n-m}} \underbrace{\operatorname{vol}_{n}\left(\partial U_{k} \cap\left\{y \in \mathbb{R}^{n+1} \mid y \in \widehat{\mathcal{R}}_{k}^{m}\right\}\right)}_{\substack{\text { Claim } 3 \\
\geq v_{0}>0}} \\
\\
\end{array} \quad \geq 2^{-p\left(\frac{1}{2}+\frac{1}{n-m}\right)}\left(\frac{\left(c_{m}\right)^{m}}{m!}\right)^{\frac{p}{n-m}} \delta v_{0}\left(d_{k}\right)^{\frac{p m}{n-m}} \xrightarrow[\substack{k \rightarrow \infty}]{\longrightarrow}+\infty,
\end{aligned}
$$

from which

$$
\liminf _{k \rightarrow \infty} \int_{\partial U_{k}}|A|^{p}=+\infty
$$

This proves the claim also in the case $m \leq n-2$, and Lemma 2.9 is shown.

## 7. Proof of Lemma 2.10

Let $n \geq 2, p \in(1, n]$ and $c_{0}>0$ be given, and assume the Lemma were false. Then there exists a sequence $\left(U_{k}\right)_{k \in \mathbb{N}}$ of open, convex subsets of $\mathbb{R}^{n+1}$ having smooth boundary, containing the origin, satisfying, for all $k \in \mathbb{N}$,
(a) $\operatorname{vol}_{n}\left(\partial U_{k}\right)=1$,
(b) $\int_{\partial U_{k}}|A|^{p} \leq c_{0}$,
(c) $\overline{U_{k}} \subset \overline{B_{R}(0)} \quad$ (for some $R>0$, depending only on $n, p$ and $c_{0}$, by virtue of Lemma 2.9),
and with the property that
(d) the $\overline{U_{k}}$ converge, in the sense of Hausdorff, to a compact convex set $V$ contained in an $n$-dimensional hyperplane in $\mathbb{R}^{n+1}$.
(As in the proof of Corollary 2.8, the fact that we can assume that the $\overline{U_{k}}-$ or, at least, a subsequence thereof - converge is a consequence of Blaschke's selection theorem ([Sch93, Thm.1.8.6, p.50]), whereas the fact that the limit must be contained in a hyperplane follows from our contradiction assumption, namely that

$$
\left.\lim _{k \rightarrow \infty} \sup _{x \in U_{k}} \sup \left\{\rho>0 \mid B_{\rho}(x) \subset U_{k}\right\}=0 .\right)
$$

Claim 1. $\operatorname{dim}(V)=n$
Proof. Assume, by contradiction, that $\operatorname{dim}(V) \leq n-1$, and consider, for $\epsilon>0$, the tubular neighbourhood $V_{\epsilon}=\left\{x \in \mathbb{R}^{n+1} \mid \operatorname{dist}(x, V)<\epsilon\right\}$ of $V . V_{\epsilon}$ is an open, convex subset of $\mathbb{R}^{n+1}$ which, by convergence, contains $\overline{U_{k}}$ for $k$ large enough. On the other hand, since we assumed that $\operatorname{dim}(V) \leq n-1$, we must have

$$
\lim _{\epsilon \searrow 0} \operatorname{vol}_{n}\left(\partial V_{\epsilon}\right)=0,
$$

for $V$ is bounded. Hence, choosing $\epsilon$ small enough, we can assume that $\operatorname{vol}_{n}\left(\partial V_{\epsilon}\right) \leq \frac{1}{2}$. But given that $\overline{U_{k}} \subset V_{\epsilon}$, the convexity of $U_{k}$ implies
$\operatorname{vol}_{n}\left(\partial U_{k}\right) \leq \operatorname{vol}_{n}\left(\partial V_{\epsilon}\right) \leq \frac{1}{2}, \quad$ whenever $\epsilon$ is small enough and $k$ large enough, (cf. Remark 2.11 on p. 32). This, however, contradicts $\operatorname{vol}_{n}\left(\partial U_{k}\right)=1(\forall k \in \mathbb{N})$, and the claim is proved.

Without loss of generality, we may assume that $V \subset\left\{(z, 0) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^{n}\right\}=$ $\pi_{0}^{1}$. We define

$$
I=\left\{x \in \pi_{0}^{1} \mid B_{\eta}(x) \cap \pi_{0}^{1} \subset V \text { for some } \eta>0\right\}
$$

the "interior", and

$$
B=V \backslash I
$$

the "boundary" of $V$ in $\pi_{0}^{1}$. For $x=(z, 0) \in \pi_{0}^{1}\left(z \in \mathbb{R}^{n}\right)$, let

$$
l_{x}=\{(z, \rho) \mid \rho \in \mathbb{R}\}
$$

denote the "vertical" line passing through $x$. Then we have
Claim 2. For any compact $K \subset I$, any $x=(z, 0) \in K$ and any $k \in \mathbb{N}$ large enough, $l_{x} \cap \overline{U_{k}}$ is a closed, non-degenerate segment joining the points $(z, a(z))$ and $(z, b(z))$, where $a(z)<b(z)$.

Proof. Fix a compact set $K \subset I$. Clearly, by convexity, $l_{x} \cap \partial U_{k}$ will never consist of more than two points, regardless of the $x \in \pi_{0}^{1}$ we choose. But assume that, for each $k$, there were an $x_{k}=\left(z_{k}, 0\right) \in K$ such that $l_{x_{k}} \cap \partial U_{k}$ consists of at most one point. Then
(i) either $l_{x_{k}} \cap \overline{U_{k}}=\emptyset$,
(ii) or $\quad l_{x_{k}}$ is tangent to $\partial U_{k}$.

Consider, for each $k \in \mathbb{N}$, the projection $V_{k}=\mathrm{P}^{m}\left(U_{k}\right)$ of $U_{k}$ onto $\pi_{0}^{1}$. Then $V_{k}$ is a convex subset of $\pi_{0}^{1}$ which is relatively open in $\pi_{0}^{1}$, and, in both of the cases above, $x_{k}=\left(z_{k}, 0\right) \notin V_{k}$ (but, possibly, $x_{k} \in \overline{V_{k}}$ ). Then the theorem of Hahn-Banach (see, e.g., Theorem I. 6 on p. 5 in [Bre83]) ensures the existence of a unit vector $e_{k} \in S^{n-1}$ such that

$$
V_{k} \subset\left\{(w, 0) \in \mathbb{R}^{n+1} \mid\left\langle\left(w-z_{k}\right), e_{k}\right\rangle_{\mathbb{R}^{n}} \leq 0\right\}
$$

where $\langle\cdot, \cdot\rangle_{\mathbb{R}^{n}}$ denotes the standard scalar product in $\mathbb{R}^{n}$. Defining the half-spaces

$$
S_{k}=\left\{(w, r) \in \mathbb{R}^{n+1} \mid r \in \mathbb{R} \text { and }\left\langle\left(w-z_{k}\right), e_{k}\right\rangle_{\mathbb{R}^{n}} \leq 0\right\}
$$

we conclude that $\overline{U_{k}} \subset S_{k}$ for each $k$. Modulo picking a subsequence, we may without loss of generality assume that there is a unit vector $e \in S^{n-1}$ and a point $x=(z, 0) \in K$ such that $e_{k} \xrightarrow{k \rightarrow \infty} e$ and $z_{k} \xrightarrow{k \rightarrow \infty} z\left(S^{n-1}\right.$ and $K$ are compact). Then, since the $\overline{U_{k}}$ converge to $V$, it follows that

$$
V \subset\left\{(w, 0) \in \mathbb{R}^{n+1} \mid\langle(w-z), e\rangle_{\mathbb{R}^{n}} \leq 0\right\}
$$

But since $x=(z, 0) \in K \subset I$, there is an $\eta>0$ such that $B_{\eta}(x) \cap \pi_{0}^{1} \subset V$, which contradicts the inclusion above (e.g., $\left(z+\frac{\eta}{2} e, 0\right) \in V$, but $\left\langle\left(\left(z+\frac{\eta}{2} e\right)-z\right), e\right\rangle_{\mathbb{R}^{n}}=$ $\frac{\eta}{2}>0$ ).

Now let $K \subset I$ be compact. For all $x=(z, 0) \in K$, let

$$
\nu_{k}^{+}(x) \text { be the outer unit normal to } \partial U_{k} \text { in }(z, b(z))
$$

and

$$
\nu_{k}^{-}(x) \text { be the outer unit normal to } \partial U_{k} \text { in }(z, a(z)) .
$$

Then we have

## Claim 3.

$$
\lim _{k \rightarrow \infty} \max _{x \in K}\left\{\left|\nu_{k}^{+}(x)-(0, \ldots, 0,1)\right|+\left|\nu_{k}^{-}(x)-(0, \ldots, 0,-1)\right|\right\}=0
$$

Proof. We restrict ourselves to showing that

$$
\lim _{k \rightarrow \infty} \max _{x \in K}\left\{\left|\nu_{k}^{+}(x)-(0, \ldots, 0,1)\right|\right\}=0
$$

the other limit following completely analogously. For each $k \in \mathbb{N}$, let $x_{k}=\left(z_{k}, 0\right) \in K$ be such that

$$
\left|\nu_{k}^{+}\left(x_{k}\right)-(0, \ldots, 0,1)\right|=\max _{x \in K}\left\{\left|\nu_{k}^{+}(x)-(0, \ldots, 0,1)\right|\right\}
$$

( $K$ is compact). By the convexity of $U_{k}$, we have that

$$
\overline{U_{k}} \subset T_{k}=\left\{y \in \mathbb{R}^{n+1} \mid\left\langle y-\left(z_{k}, b\left(z_{k}\right)\right), \nu_{k}^{+}\left(x_{k}\right)\right\rangle \leq 0\right\} .
$$

But since $\left(z_{k}, a\left(z_{k}\right)\right) \in \overline{U_{k}}$ and $a\left(z_{k}\right)<b\left(z_{k}\right)$, this implies that

$$
\begin{equation*}
\left\langle\nu_{k}^{+}\left(x_{k}\right),(0, \ldots, 0,1)\right\rangle \geq 0 \quad \forall k \in \mathbb{N} . \tag{2.4}
\end{equation*}
$$

Now assume $\nu^{+}$is the limit of a subsequence of $\left(\nu_{k}^{+}\left(z_{k}\right)\right)_{k \in \mathbb{N}}\left(S^{n-1}\right.$ is compact $)$, and $x=(z, 0) \in K$ is the limit of a further subsequence of $\left(x_{k}\right)_{k \in \mathbb{N}}=\left(\left(z_{k}, 0\right)\right)_{k \in \mathbb{N}}(K$ is compact). It then follows that

$$
V \subset T=\left\{y \in \mathbb{R}^{n+1} \mid\left\langle y-(z, 0), \nu^{+}\right\rangle \leq 0\right\}
$$

But, since $(z, 0) \in K \subset I$, there is a $\eta>0$ such that $(z+\eta e, 0) \in I, \forall e \in S^{n-1}$. Consequently, $\nu^{+}$must be orthogonal to $\pi_{0}^{1}$, from which

$$
\text { either } \quad \nu^{+}=(0, \ldots, 0,1) \quad \text { or } \quad \nu^{+}=(0, \ldots, 0,-1) \text {. }
$$

(2.4) then yields $\nu^{+}=(0, \ldots, 0,1)$, which is precisely what we wanted to show.

Now consider, for every $\epsilon \in(0,1)$, the sets

$$
\begin{aligned}
B_{\epsilon} & =\{((1-\epsilon) z, 0) \mid(z, 0) \in B\} \\
I_{\epsilon} & =\{(\rho z, 0) \mid(z, 0) \in B, 0 \leq \rho \leq 1-\epsilon\}
\end{aligned}
$$

and

$$
C_{\epsilon}=\left\{(z, r) \mid(z, 0) \in I_{\epsilon}, r \in \mathbb{R}\right\}
$$

(remember that we had assumed $0 \in V$ ). Set

$$
\Sigma_{k, \epsilon}^{i}=\partial U_{k} \cap C_{\epsilon} \quad \text { and } \quad \Sigma_{k, \epsilon}^{e}=\partial U_{k} \backslash \Sigma_{k, \epsilon}^{i} \quad(\epsilon \in(0,1), k \in \mathbb{N})
$$

Then we have
Claim 4.

$$
\lim _{\epsilon \searrow 0} \lim _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\Sigma_{k, \epsilon}^{e}\right)=0 .
$$

Proof. From claims 2 and 3, we conclude that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\Sigma_{k, \epsilon}^{i}\right)=2 \operatorname{vol}_{n}\left(I_{\epsilon}\right) \tag{2.5}
\end{equation*}
$$

This can be seen as follows. Claim 2 tells us that $\Sigma_{k, \epsilon}^{i}$ is the union of two graphs over the compact set $I_{\epsilon} \subset I$, namely those of the functions $a(z)$ and $b(z)\left((z, 0) \in I_{\epsilon}\right)$. By Claim 3, both of these functions converge, as $k \rightarrow \infty$, to a constant (in fact, both converge to 0 ) on all of $I_{\epsilon}$. This implies the above assertion.
Considering then, as in the proof of Claim 1, the tubular neighbourhood $V_{\delta}=$ $\left\{x \in \mathbb{R}^{n+1} \mid \operatorname{dist}(x, V)<\delta\right\}(\delta>0)$ of $V$, we know that $U_{k} \subset V_{\delta}$ for $k$ large enough (with respect to $\delta$ ), whence ( $U_{k}$ is convex)

$$
\operatorname{vol}_{n}\left(\partial U_{k}\right) \leq \operatorname{vol}_{n}\left(\partial V_{\delta}\right) \quad(k \text { large enough })
$$

But since we also have

$$
\lim _{\delta \searrow 0} \operatorname{vol}_{n}\left(\partial V_{\delta}\right)=2 \operatorname{vol}_{n}(V),
$$

we conclude that

$$
\begin{equation*}
1=\limsup _{k \rightarrow \infty} \underbrace{\operatorname{vol}_{n}\left(\partial U_{k}\right)}_{=1} \leq 2 \operatorname{vol}_{n}(V) \tag{2.6}
\end{equation*}
$$

Taking into account that $\lim _{\epsilon \searrow 0} \operatorname{vol}_{n}\left(I_{\epsilon}\right)=\operatorname{vol}_{n}(V)$, as well as that $\operatorname{vol}_{n}\left(\Sigma_{k, \epsilon}^{i}\right) \leq$ $\operatorname{vol}_{n}\left(\partial U_{k}\right)=1$, the combination of (2.6) with (2.5) yields, on one hand, that

$$
\operatorname{vol}_{n}(V)=\frac{1}{2}
$$

and, on the other hand, that

$$
\lim _{\epsilon \searrow 0} \lim _{k \rightarrow \infty} \operatorname{vol}_{n}\left(\Sigma_{k, \epsilon}^{i}\right)=1
$$

from which the claim follows.
Fix $\epsilon \in(0,1)$ and $k \in \mathbb{N}$. For every $x=(z, 0) \in B_{\epsilon}$, pick a unit normal $\nu(x)$ to $B_{\epsilon}$ in $\pi_{0}^{1}$.

Remark 2.12. Of course, $B_{\epsilon}$ might not be smooth, but since it is the boundary (in $\pi_{0}^{1}$ ) of a convex set $\left(I_{\epsilon}\right)$, we know from [Roc70, Thm.25.5, p.246], that $\nu(x)$ will be uniquely defined except for a set of zero ( $n-1$ )-dimensional Hausdorff measure.

Consider, for each $x=(z, 0) \in B_{\epsilon}$, the two-dimensional half-plane

$$
\tau^{+}(x)=\{x+(0, \ldots, 0, s)+t \nu(x) \mid s, t \geq 0\}
$$

Then the intersection

$$
\gamma_{k, x}=\tau^{+}(x) \cap \partial U_{k}
$$

of $\tau^{+}(x)$ with $\partial U_{k}$ is, by Claim 2, a curve in $\tau^{+}(x)$ joining $(z, a(z))$ with $(z, b(z))$ $(x=(z, 0))$. Thus, by the coarea formula (see, again, [Cha06, §III.8] or [Fed69,
§3.2]), and taking into account Remark 2.12, we have

$$
\begin{aligned}
\liminf _{k \rightarrow \infty} \int_{\Sigma_{k, \epsilon}^{e}}|A| & =\liminf _{k \rightarrow \infty} \int_{B_{\epsilon}}\left(\int_{\gamma_{k, x}}|A|\right) d x \geq \liminf _{k \rightarrow \infty} \int_{B_{\epsilon}}\left|\nu_{k}^{+}(x)-\nu_{k}^{-}(x)\right| d x \\
& \geq \operatorname{vol}_{n-1}\left(B_{\epsilon}\right)=(1-\epsilon)^{n-1} \operatorname{vol}_{n-1}(B),
\end{aligned}
$$

where the first inequality follows from Corollary A. 4 of the appendix, whereas the second is a consequence of Claim 3. We conclude

$$
\begin{equation*}
\liminf _{\epsilon \searrow 0} \liminf _{k \rightarrow \infty} \int_{\Sigma_{k, \epsilon}^{e}}|A| \geq \operatorname{vol}_{n-1}(B)>0 \tag{2.7}
\end{equation*}
$$

On the other hand, given that $\int_{\partial U_{k}}|A|^{p} \leq c_{0}$ and $p>1$ by assumption, Hölder's inequality yields

$$
\int_{\Sigma_{k, \epsilon}^{e}}|A| \leq\left(\operatorname{vol}_{n-1}\left(\Sigma_{k, \epsilon}^{e}\right)\right)^{1-\frac{1}{p}}\left(\int_{\Sigma_{k, \epsilon}^{e}}|A|^{p}\right)^{\frac{1}{p}} \leq c_{0}^{\frac{1}{p}}\left(\operatorname{vol}_{n-1}\left(\Sigma_{k, \epsilon}^{e}\right)\right)^{1-\frac{1}{p}}
$$

Using Claim 4, we then conclude

$$
\underset{\epsilon \searrow 0}{\lim \sup } \limsup _{k \rightarrow \infty} \int_{\Sigma_{k, c}^{e}}|A|=0,
$$

which contradicts (2.7). Our assumption at the beginning of the proof must therefore be wrong, and the lemma is proved.

## CHAPTER 3

## The $L^{2}$-theory

In this chapter we prove our main estimate in the $L^{2}$-case for $n$-dimensional hypersurfaces of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature (which is equivalent to being convex). The method thereby used mimics an argument in [DLT10]. Afterwards, we give an alternative proof due to G. Huisken of this estimate in the two-dimensional case under the assumption that the surface is mean convex and constitutes the boundary of a star-shaped domain. Finally, we exhibit how that last proof lends itself to generalisation to the $n$-dimensional case.

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## 1. The case Ric $\geq 0$

We first prove the $L^{2}$-estimate for hypersurfaces that have non-negative Ricci curvature, with the constant $C=\sqrt{\frac{n}{n-1}}$ on the right-hand side. As we shall see in the next chapter, that constant is optimal. There, we will also prove that the assumption Ric $\geq 0$ is optimal whenever $n \geq 3$ (i.e. in the sub-critical case).

Theorem 3.1. Let $n \geq 2$ be given and set $C=\sqrt{\frac{n}{n-1}}$. Then we have: if $\Sigma$ is a smooth, closed, connected hypersurface in $\mathbb{R}^{n+1}$ with induced Riemannian metric $g$ and non-negative Ricci curvature, then

$$
\begin{equation*}
\left(\int_{\Sigma}\left|A-\frac{1}{n}\left(\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H\right) g\right|^{2}\right)^{\frac{1}{2}} \leq C\left(\int_{\Sigma}\left|A-\frac{H}{n} g\right|^{2}\right)^{\frac{1}{2}} \tag{3.1}
\end{equation*}
$$

In particular, the above estimate holds for smooth, closed hypersurfaces which are the boundary of a convex set in $\mathbb{R}^{n+1}$.

Proof. The following argument is an adaptation of the proof of Theorem 0.1 in [DLT10] (see also [CLN06, §B.3, pp.517-519]).

Let

$$
\AA=A-\frac{1}{n} H g \quad \text { and } \quad \bar{H}=\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H
$$

and write the square of the left-hand side of (3.1) as

$$
\begin{align*}
\int_{\Sigma}\left|A-\frac{\bar{H}}{n} g\right|^{2} & =\int_{\Sigma}\left|\left(A-\frac{H}{n} g\right)+\left(\frac{H-\bar{H}}{n} g\right)\right|^{2}  \tag{3.2}\\
& =\int_{\Sigma}|\AA|^{2}+\frac{2}{n} \int_{\Sigma}(H-\bar{H}) \sum_{i, j=1}^{n} g^{i j} \AA_{i j}+\frac{1}{n} \int_{\Sigma}|H-\bar{H}|^{2} \\
& =\int_{\Sigma}|\AA|^{2}+\frac{1}{n} \int_{\Sigma}|H-\bar{H}|^{2}
\end{align*}
$$

Let $\varphi$ be the unique smooth solution of the following Poisson problem on $\Sigma$ (for existence, uniqueness and regularity see, e.g., [Aub98, Thm.4.7, p.104]):

$$
\left\{\begin{array}{l}
\Delta \varphi=H-\bar{H} \\
\int_{\Sigma} \varphi=0
\end{array}\right.
$$

We then have

$$
\int_{\Sigma}|H-\bar{H}|^{2}=\int_{\Sigma}(H-\bar{H}) \Delta \varphi=\int_{\Sigma}(H-\bar{H}) \sum_{i=1}^{n} \nabla_{i} \nabla^{i} \varphi=-\int_{\Sigma} \sum_{i=1}^{n} \nabla_{i} H \nabla^{i} \varphi
$$

By virtue of the Codazzi equations we find

$$
\nabla_{i} H=\sum_{l=1}^{n} \nabla_{i} A_{l}^{l}=\sum_{l=1}^{n} \nabla_{l} A_{i}^{l}=\sum_{l=1}^{n} \nabla_{l} \stackrel{\circ}{A}_{i}+\frac{1}{n} \sum_{l=1}^{n} \nabla_{l} H \delta_{i}^{l}=\frac{n}{n-1} \sum_{l=1}^{n} \nabla_{l} \stackrel{\circ}{A}_{i}^{l}
$$

Thus

$$
\int_{\Sigma}|H-\bar{H}|^{2}=-\frac{n}{n-1} \int_{\Sigma} \sum_{i, l=1}^{n} \nabla_{l} \stackrel{\circ}{A}_{i}^{l} \nabla^{i} \varphi=\frac{n}{n-1} \int_{\Sigma} \stackrel{\circ}{A}: \operatorname{Hess} \varphi .
$$

Since $\AA$ is trace-free, we have

$$
\AA: \operatorname{Hess} \varphi=\AA:\left(\operatorname{Hess} \varphi-\frac{1}{n} \Delta \varphi g\right),
$$

and thus, by Cauchy-Schwarz,

$$
\begin{aligned}
\int_{\Sigma}|H-\bar{H}|^{2} & =\frac{n}{n-1} \int_{\Sigma} \AA:\left(\operatorname{Hess} \varphi-\frac{1}{n} \Delta \varphi g\right) \\
& \leq \frac{n}{n-1}\left(\int_{\Sigma} \mid \AA \AA^{2}\right)^{\frac{1}{2}}\left(\int_{\Sigma}\left|\operatorname{Hess} \varphi-\frac{1}{n} \Delta \varphi g\right|^{2}\right)^{\frac{1}{2}} \\
& =\frac{n}{n-1}\left(\int_{\Sigma} \mid \AA \AA^{2}\right)^{\frac{1}{2}}\left(\int_{\Sigma}|\operatorname{Hess} \varphi|^{2}-\frac{1}{n} \int_{\Sigma}|\Delta \varphi|^{2}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Since

$$
\begin{align*}
\int_{\Sigma}|\operatorname{Hess} \varphi|^{2} & =\int_{\Sigma} \sum_{i, j=1}^{n} \nabla^{i} \nabla^{j} \varphi \nabla_{i} \nabla_{j} \varphi=-\int_{\Sigma} \sum_{i, j=1}^{n} \nabla^{j} \varphi \nabla^{i} \nabla_{i} \nabla_{j} \varphi  \tag{3.3}\\
& =-\int_{\Sigma} \sum_{i, j=1}^{n} \nabla^{j} \varphi \nabla^{i} \nabla_{j} \nabla_{i} \varphi \\
& =-\int_{\Sigma} \sum_{i, j=1}^{n} \nabla^{j} \varphi \nabla_{j} \nabla^{i} \nabla_{i} \varphi-\int_{\Sigma} \sum_{i, j=1}^{n} \nabla^{j} \varphi \operatorname{Ric}_{j}^{i} \nabla_{i} \varphi \\
& =\int_{\Sigma}|\Delta \varphi|^{2}-\int_{\Sigma} \operatorname{Ric}(\nabla \varphi, \nabla \varphi)
\end{align*}
$$

we have

$$
\begin{aligned}
\int_{\Sigma}|H-\bar{H}|^{2} & \leq \frac{n}{n-1}\left(\int_{\Sigma}|\AA|^{2}\right)^{\frac{1}{2}}\left(\frac{n-1}{n} \int_{\Sigma}|\Delta \varphi|^{2}-\int_{\Sigma} \operatorname{Ric}(\nabla \varphi, \nabla \varphi)\right)^{\frac{1}{2}} \\
& \leq \frac{n}{n-1}\left(\int_{\Sigma}|\AA|^{2}\right)^{\frac{1}{2}}\left(\frac{n-1}{n} \int_{\Sigma}|\Delta \varphi|^{2}\right)^{\frac{1}{2}} \\
& =\left(\frac{n}{n-1}\right)^{\frac{1}{2}}\left(\int_{\Sigma}|\AA|^{2}\right)^{\frac{1}{2}}\left(\int_{\Sigma}|\Delta \varphi|^{2}\right)^{\frac{1}{2}} \\
& =\left(\frac{n}{n-1}\right)^{\frac{1}{2}}\left(\int_{\Sigma}|\AA|^{2}\right)^{\frac{1}{2}}\left(\int_{\Sigma}|H-\bar{H}|^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

where we have used the assumption Ric $\geq 0$. Therefore

$$
\int_{\Sigma}|H-\bar{H}|^{2} \leq \frac{n}{n-1} \int_{\Sigma}|\AA|^{2}
$$

and so

$$
\int_{\Sigma}\left|A-\frac{\bar{H}}{n} g\right|^{2} \leq \int_{\Sigma}|\AA|^{2}+\frac{1}{n-1} \int_{\Sigma}|\AA|^{2}=\frac{n}{n-1} \int_{\Sigma}|\AA|^{2}
$$

as claimed.

## 2. Why Ric $\geq 0$ and convexity are the same

In this section we present an argument, recently brought to the author's attention by C. De Lellis, which concludes that every smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature must be convex. Although this seems to be a well-known fact (cf., e.g., [Des92]), we could not yet find a proof in the literature, and expose C. De Lellis' argument for the sake of completeness.

Proposition 3.2. Let $n \geq 2$ and suppose $\Sigma$ a smooth, closed and connected hypersurface of $\mathbb{R}^{n+1}$. Then the following are equivalent:
(i) Ric $\geq 0$ everywhere on $\Sigma$;
(ii) $A \geq 0$ everywhere on $\Sigma$.

In particular, if either of the above conditions hold, $\Sigma$ is convex.
Proof (by C. De Lellis). Denote by $\lambda_{1}, \ldots, \lambda_{n}$ the eigenvalues of $A$. In view of the contracted Gauss equations, Ric $=H A-A^{2}$, it is immediate to see that condition (ii) implies condition (i). The interesting question is therefore the converse implication.

We first argue on a pointwise level. Ric $\geq 0$ implies that, for every $i \in\{1, \ldots, n\}$, we have

$$
\left(\sum_{j=1}^{n} \lambda_{j}\right) \lambda_{i} \geq \lambda_{i}^{2} \geq 0
$$

Thus, each principal curvature $\lambda_{i}$ needs to be of the same sign than the mean curvature $H$. We conclude that, in each point of the hypersurface, Ric $\geq 0 \mathrm{im}-$ plies semi-definiteness of $A$. In other words, for each $q \in \Sigma$, one of the following three situations hold:
(a) $A(q)>0$,
(b) $A(q)<0$
or
(c) $A(q)=0$.

We now want to obtain the global statement by arguing that $A$ cannot change sign. Suppose this were not true, i.e., assume there were points $q_{+} \in \Sigma$ and $q_{-} \in \Sigma$, such that $A\left(q_{-}\right)<0<A\left(q_{+}\right)$. Clearly, by continuity, it follows that

$$
\operatorname{vol}_{n}\left(U_{+}\right)>0, \quad \text { where } U_{+}=\{q \in \Sigma \mid A(q)>0\} \subset \Sigma
$$

and

$$
\operatorname{vol}_{n}\left(U_{-}\right)>0, \quad \text { where } U_{-}=\{q \in \Sigma \mid A(q)<0\} \subset \Sigma
$$

For $e \in S^{n}$ and $c \in \mathbb{R}$, consider the hyperplane

$$
T_{e}(c)=\left\{x \in \mathbb{R}^{n+1} \mid\langle x, e\rangle=c\right\}
$$

Furthermore, let

$$
c_{e}=\min \left\{c \in \mathbb{R} \mid T_{e}(c) \cap \Sigma \neq \emptyset\right\}
$$

denote the smallest value $c \in \mathbb{R}$ for which $T_{e}(c)$ intersects $\Sigma$. Clearly, $\Sigma$ will then lie on one side of $T_{e}\left(c_{e}\right)$, and $\Sigma \cap T_{e}\left(c_{e}\right) \neq \emptyset$. Geometrically, $T_{e}\left(c_{e}\right)$ is a supporting hyperplane of $\Sigma$, and we must have that $A(q) \geq 0$, whenever $q \in T_{e}\left(c_{e}\right) \cap \Sigma$. Since $e$ was arbitrary, we conclude

$$
\nu(\{q \in \Sigma \mid A(q) \geq 0\})=S^{n}
$$

But, since for all $q \in \Sigma$, we have $T_{q} \Sigma=T_{\nu(q)} S^{n}$, the maps $A(\cdot, \cdot)$ and $\langle d \nu(\cdot), \cdot\rangle$ are equal, and Sard's theorem (see, e.g., [DFN85, Thm.10.2.1, p.79]) applied to $\nu: \Sigma \rightarrow S^{n}$ ensures that

$$
\operatorname{vol}_{n}(\nu(\{q \in \Sigma \mid A(q)=0\}))=0
$$

whence

$$
\nu\left(U_{+}\right)=S^{n} \backslash N
$$

for some null-set $N \subset S^{n}$.
Moreover, since almost every value of $\nu$ is regular and $\Sigma$ is compact, the number $\operatorname{vol}_{0}\left(\nu^{-1}(\cdot)\right)$ of pre-images under $\nu$ is finite for almost every point in $S^{n}$, as well as locally constant. In view of the area formula (see, e.g., [Fed69, §3.2]),

$$
0<\int_{U_{-}}|\operatorname{det} A|=\int_{\nu_{\left(U_{-}\right)}} \operatorname{vol}_{0}\left(\nu^{-1}(\cdot) \cap U_{-}\right)
$$

we then conclude that

$$
\operatorname{vol}_{n}\left(\nu\left(U_{-}\right)\right)>0
$$

From these considerations follows that the set

$$
B=\left\{e \in S^{n} \mid \exists q_{+} \in U_{+}, q_{-} \in U_{-} \text {such that } \nu\left(q_{+}\right)=\nu\left(q_{-}\right)=e\right\} \subset S^{n}
$$

has strictly positive measure. Now, for $e \in S^{n}$, consider the map $f_{e}: \Sigma \rightarrow \mathbb{R}$, $q \mapsto\langle q, e\rangle$. Then the critical points of $f_{e}$ are given by the set

$$
C_{e}=\{q \in \Sigma \mid \nu(q)= \pm e\} \subset \Sigma
$$

We show that $f_{e}$ is a Morse function for almost every $e \in S^{n}$. To do this, we have to check that Hess $f_{e}(q)$ has full rank whenever $q \in C_{e}$. But there, we have

$$
\text { Hess } f_{e}(q)=\mp A(q) \quad\left(q \in C_{e}\right)
$$

and we conclude that $f_{e}$ is a Morse function whenever

$$
C_{e} \subset\{q \in \Sigma \mid \operatorname{det} A(q) \neq 0\}
$$

i.e., whenever $e$ and $-e$ are regular values of $\nu$. But since the set of singular values of $\nu$ has measure zero, we conclude that, for almost all $e \in S^{n}$, the map $f_{e}$ is a Morse function. In particular, since $\operatorname{vol}_{n}(B)>0$, there is an $\bar{e} \in B$ for which $f_{\bar{e}}$ is Morse, and we get by construction

$$
\exists \bar{q}_{+} \in \Sigma \quad \text { such that } \quad \nu\left(\bar{q}_{+}\right)=\bar{e} \quad \text { and } \quad \text { Hess } f_{\bar{e}}\left(\bar{q}_{+}\right)>0
$$

and

$$
\exists \bar{q}_{-} \in \Sigma \quad \text { such that } \quad \nu\left(\bar{q}_{-}\right)=\bar{e} \quad \text { and } \quad \text { Hess } f_{\bar{e}}\left(\bar{q}_{-}\right)<0
$$

On the other hand, we saw that $f_{\bar{e}}$ must have the absolute minimum $c_{\bar{e}}$ in a point $\bar{q}_{0} \in T_{\bar{e}}\left(c_{\bar{e}}\right) \cap \Sigma$ for which we have Hess $f_{\bar{e}}\left(\bar{q}_{0}\right)>0$. Moreover, $\nu\left(\bar{q}_{0}\right)=-\bar{e}$, whence $\bar{q}_{0} \neq \bar{q}_{+}$. By construction, then, $f_{\bar{e}}$ has two distinct local minima.

But since $\Sigma$ is connected, one of the Morse inequalities yields (see, e.g., [DFN90, §16] or [Nic07a, §2.3])

$$
\mu_{1}-2 \geq \mu_{1}-\mu_{0} \geq b_{1}-b_{0}=b_{1}-1
$$

from which $\mu_{1} \geq 1$. Here, $b_{i}$ and $\mu_{i}$ denote the Betti numbers and the Morse numbers, respectively, and we have used that $b_{1} \geq 0, b_{0}=1$ (by connectedness) and $\mu_{0} \geq 2$ (since $f_{\bar{e}}$ has, at least, two local minima). Consequently, $f_{\bar{e}}$ needs to have at least one saddle point of index 1 . This, however, is impossible, since all the critical points of $f_{\bar{e}}$ have either index 0 (minima) or index $n$ (maxima), by construction. Thus our assumption that there is a point in which $A<0$ was false, and the proposition is proved.

## 3. G. Huisken's proof for two-dimensional mean-convex surfaces that bound a star-shaped domain

In this section we prove the $L^{2}$-estimate (3.1) for two-dimensional surfaces that are mean convex and bound a star-shaped domain in $\mathbb{R}^{3}$. The method hereby used was suggested by G. Huisken and uses inverse mean curvature flow. The additional assumptions recover the optimal constant $C=\sqrt{2}$ (cf. Proposition 4.1 of the next chapter). However, as we are also going to show in Chapter 4, the constant $C=\sqrt{2}$ does not work for generic surfaces (compare with [DLM05]). We wish to stress here that the following proof takes care of a more general situation than the corresponding result in Section 1 ( $H>0$ and star-shaped versus Ric $\geq 0$, i.e. convex), except for the fact that our requirement is weak (non-strict inequality). However, it seems an easy, albeit tedious matter to generalise Theorem 3.3 below to the case $H \geq 0$ by approximation with $(H>0)$-surfaces (compare also with [HI08, Theorem 2.5] for weakening of $H>0$ to $H \geq 0$ in the case of strict star-shapedness).

Theorem 3.3 (Huisken). If $\Sigma$ is the smooth, closed boundary of a star-shaped domain in $\mathbb{R}^{3}$ with induced Riemannian metric $g$, and has everywhere strictly positive mean curvature $H$, then

$$
\begin{equation*}
\left(\int_{\Sigma}\left|A-\frac{1}{2}\left(\frac{1}{\operatorname{vol}_{2}(\Sigma)} \int_{\Sigma} H\right) g\right|^{2}\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\int_{\Sigma}\left|A-\frac{H}{2} g\right|^{2}\right)^{\frac{1}{2}} \tag{3.4}
\end{equation*}
$$

The proof presented below differs only slightly from the one proposed by G. Huisken to C. De Lellis at a summer school in Rome (Italy) in 2005.
3.1. Preliminaries. For $M$ an $n$-dimensional, smooth, closed manifold and $T>0$, a solution to the inverse mean curvature flow (IMCF) is given by a smooth family of embeddings $F: M \times[0, T] \rightarrow \mathbb{R}^{n+1}$, such that

$$
\partial_{t} F(x, t)=\frac{1}{H(x, t)} \nu(x, t), \quad x \in M, \quad t \in[0, T]
$$

where $H(x, t)>0$ and $\nu(x, t)$ are the mean curvature and the exterior unit normal of $F(M, t)$ in the point $F(x, t)$. In 1990, Gerhardt [Ger90] and Urbas [Urb90] proved independently that, if the initial data $F(M, 0)=\Sigma$ is the smooth boundary of a star-shaped domain, then the IMCF has a smooth solution for all times $t>0$ which approaches a homothetically expanding spherical solution as $t \rightarrow+\infty$.

Consequently, the following approach to proving Theorem 3.3 seems promising. Consider the functional

$$
\mathcal{F}(\Sigma)=\int_{\Sigma}|\AA|^{2}-\frac{1}{2} \int_{\Sigma}\left(H-\frac{1}{\operatorname{vol}_{2}(\Sigma)} \int_{\Sigma} H\right)^{2}
$$

on the set of smooth, closed surfaces of $\mathbb{R}^{3}$. Notice that it is scale-invariant, and that the positivity of $\mathcal{F}(\Sigma)$ is equivalent to inequality (3.4). Also, $\mathcal{F}\left(S^{2}\right)=0$. In view of the results about IMCF mentioned above, it is then sufficient to prove that $\mathcal{F}$ is non-increasing along the flow starting at the surface $\Sigma$ which bounds a star-shaped domain.
3.2. Proof of Theorem 3.3. If $\Sigma$ is the smooth, closed boundary of a starshaped (with respect to, say, the origin) domain in $\mathbb{R}^{3}$ with $H>0$, let $\Sigma_{t}$ denote the smooth, global solution constructed in [Ger90] or [Urb90] of the IMCF starting at $\Sigma_{0}=\Sigma$. Then the rescaled surfaces $\frac{4 \pi}{\operatorname{vol}_{2}\left(\Sigma_{t}\right)} \Sigma_{t}$ converge to the round sphere $S^{2}(0)$ as $t \rightarrow+\infty$. If we introduce, for any smooth function $\varphi: \Sigma \times[0,+\infty) \rightarrow \mathbb{R}$, the notation

$$
\bar{\varphi}=\frac{1}{\operatorname{vol}_{2}\left(\Sigma_{t}\right)} \int_{\Sigma_{t}} \varphi
$$

then the following lemma immediately implies the theorem.
Lemma 3.4. For all $t \geq 0$ we have

$$
\frac{d}{d t} \mathcal{F}\left(\Sigma_{t}\right)=-\bar{H} \int_{\Sigma_{t}} \frac{|\AA|^{2}}{H}
$$

Indeed, $\mathcal{F}\left(\Sigma_{t}\right)$ is then non-increasing along the IMCF. And since $\mathcal{F}$ is scaleinvariant and $\lim _{t \rightarrow+\infty} \mathcal{F}\left(\Sigma_{t}\right)=\mathcal{F}\left(S^{2}\right)=0$, we conclude that $\mathcal{F}\left(\Sigma_{t}\right) \geq 0$ for all $t \geq 0$ and, in particular, that $\mathcal{F}(\Sigma) \geq 0$. This proves Theorem 3.3.
3.3. Proof of Lemma 3.4. We wish to remark that the functional $\mathcal{F}$ considered here is a special case of the functional given by (4.2), to be studied in Section 1 of Chapter 4. The calculations for its first variation performed there are, of course, valid also in the setting at hand, and we recover from (4.3s) with $f=1 / H$,
$n=2$ and $C=\sqrt{2}$ that, for all $t \geq 0$,

$$
\begin{align*}
\frac{d}{d t} \mathcal{F}(\Sigma)= & \int_{\Sigma} \left\lvert\, \AA \AA^{2}-2 \int_{\Sigma} \operatorname{Hess} \frac{1}{H}\right.: \AA-2 \int_{\Sigma} \frac{A^{2}: \AA}{H}  \tag{3.5}\\
& -\frac{1}{2} \int_{\Sigma}(H-\bar{H})^{2}+\int_{\Sigma}(H-\bar{H}) \Delta \frac{1}{H}+\int_{\Sigma}|A|^{2} \frac{H-\bar{H}}{H} \\
= & \int_{\Sigma}|\AA|^{2}-2 \int_{\Sigma} \operatorname{Hess} \frac{1}{H}: A+\int_{\Sigma} H \Delta \frac{1}{H}-2 \int_{\Sigma} \frac{A^{2}: \AA}{H} \\
& -\frac{1}{2} \int_{\Sigma} H^{2}+\frac{1}{2} \bar{H} \int_{\Sigma} H+\int_{\Sigma} H \Delta \frac{1}{H} \\
& -\bar{H} \int_{\Sigma} \Delta \frac{1}{H}+\int_{\Sigma}|A|^{2}-\bar{H} \int_{\Sigma} \frac{|A|^{2}}{H} \\
= & 2 \int_{\Sigma}|\AA|^{2}-2 \int_{\Sigma} \frac{A^{2}: \AA}{H}-\bar{H} \int_{\Sigma} \frac{|\AA|^{2}}{H} \\
& -2 \int_{\Sigma} \operatorname{Hess} \frac{1}{H}: A+2 \int_{\Sigma} H \Delta \frac{1}{H}-\bar{H} \int_{\Sigma} \Delta \frac{1}{H} .
\end{align*}
$$

Now, since $\Sigma$ is closed, $\int_{\Sigma} \Delta \varphi$ vanishes for any $C^{2}$-function $\varphi$ on $\Sigma$. Also, in view of the Codazzi equations, we have

$$
\begin{aligned}
\int_{\Sigma} \operatorname{Hess} \varphi: A & =\int_{\Sigma} \sum_{i, j=1}^{2} \nabla_{i} \nabla^{j} \varphi A_{j}^{i}=-\int_{\Sigma} \sum_{i, j=1}^{2} \nabla^{j} \varphi \nabla_{i} A_{j}^{i} \\
& =-\int_{\Sigma} \sum_{i, j=1}^{2} \nabla^{j} \varphi \nabla_{j} A_{i}^{i}=\int_{\Sigma} \sum_{i, j=1}^{2} \nabla_{j} \nabla^{j} \varphi A_{i}^{i}=\int_{\Sigma} H \Delta \varphi .
\end{aligned}
$$

Finally, recalling that every two-dimensional Riemannian manifold is Einstein (i.e. its Ricci curvature is a multiple of its metric, Ric $=\frac{\text { Scal }}{2} g$ ), we see that Ric $: \AA=0$. But since the (once contracted) Gauss equations tell us that Ric $=H A-A^{2}$, we conclude that

$$
\frac{A^{2}: \AA}{H}=A: \AA=|\AA|^{2}
$$

Putting these three observations together simplifies (3.5) to

$$
\frac{d}{d t} \mathcal{F}(\Sigma)=-\bar{H} \int_{\Sigma} \frac{|\AA|^{2}}{H}
$$

as required.

## 4. The flow approach to $n$ dimensions

The purpose of this section is to investigate how G. Huisken's method explained in Section 3 can be generalised to the $n$-dimensional case. As it turns out, requiring merely $H>0$ for the boundary of a star-shaped domain is not enough, but we have to assume that the domain is strictly convex. Moreover, we cannot recover the optimal constant $C=\sqrt{\frac{n}{n-1}}$ (cf. Proposition 4.1), but only get the result for
$C=\sqrt{n}$, and the reason for this remains unclear until now. Otherwise, the method generalises quite straightforwardly, although a much more careful inspection of the rate of change of the considered functional is necessary.
4.1. The setup. Our goal will be to prove

Theorem 3.5. Let $n \geq 3$ be given. Then we have:
if $\Sigma$ is the smooth, closed boundary of a strictly convex domain in $\mathbb{R}^{n+1}$ with induced Riemannian metric $g$, then

$$
\begin{equation*}
\left(\int_{\Sigma}\left|A-\frac{1}{n}\left(\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H\right) g\right|^{2}\right)^{\frac{1}{2}} \leq \sqrt{n}\left(\int_{\Sigma}\left|A-\frac{H}{n} g\right|^{2}\right)^{\frac{1}{2}} \tag{3.6}
\end{equation*}
$$

According to [Urb91, Theorem 1.1], the inverse mean curvature flow with $\Sigma$ as initial data has a smooth, global (i.e. which exists for all $t \geq 0$ ) solution $\Sigma_{t}$ which converges (as $t \rightarrow+\infty$ ) to a round sphere after rescaling to constant volume. As in the previous section, we want to consider a scale-invariant functional which vanishes on spheres and represents the sought-after inequality (3.6). We then show that this functional is monotone along the IMCF.

More precisely, consider the functional

$$
\mathcal{F}(\Sigma)=(n-1) \int_{\Sigma} \left\lvert\, \AA \AA^{2}-\frac{1}{n} \int_{\Sigma}(H-\bar{H})^{2}\right.
$$

on the set of smooth, closed hypersurfaces of $\mathbb{R}^{n+1}$, where, again, $\bar{H}=\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H$. It is immediate to see that $\mathcal{F}(\Sigma)$ is non-negative if and only if inequality (3.6) holds (use (3.2)). Notice also that $\mathcal{F}\left(S^{n}\right)=0$. We rescale $\mathcal{F}$ to the scale-invariant quantity

$$
\mathcal{H}(\Sigma)=\operatorname{vol}_{n}^{-\frac{n-2}{n}}(\Sigma) \mathcal{F}(\Sigma)=\operatorname{vol}_{n}^{-\frac{n-2}{n}}(\Sigma)\left((n-1) \int_{\Sigma} \left\lvert\, \AA \AA^{2}-\frac{1}{n} \int_{\Sigma}(H-\bar{H})^{2}\right.\right)
$$

By the same arguments as in the proof of Theorem 3.3, the following proposition then immediately implies Theorem 3.5.

Proposition 3.6. For $\Sigma, \Sigma_{t}$ an $\mathcal{H}$ as above, we have

$$
\frac{d}{d t} \mathcal{H}\left(\Sigma_{t}\right) \leq 0
$$

Proof of Proposition 3.6. In the next subsection, we shall show the following lemma giving the rate of change of $\mathcal{H}\left(\Sigma_{t}\right)$ along the IMCF.

Lemma 3.7. For $\Sigma, \Sigma_{t}$ and $\mathcal{H}$ as above, we have

$$
\begin{aligned}
\frac{d}{d t} \mathcal{H}\left(\Sigma_{t}\right)=-2 \operatorname{vol}_{n}^{-\frac{n-2}{n}}\left(\Sigma_{t}\right) & \left(\frac{1}{n} \bar{H} \int_{\Sigma_{t}} \frac{|\AA|}{H}+(n-2) \int_{\Sigma_{t}} \frac{|\nabla H|^{2}}{H^{2}}\right. \\
& \left.+(n-1) \int_{\Sigma_{t}} \frac{\operatorname{tr}_{g} A^{3}}{H}-\frac{2 n-1}{n} \int_{\Sigma_{t}}|A|^{2}+\frac{1}{n} \int_{\Sigma_{t}} H^{2}\right)
\end{aligned}
$$

The first two terms in parentheses in the above expression being obviously nonnegative, we have to focus our attention on the three terms on the second line to see that $\mathcal{H}\left(\Sigma_{t}\right)$ is non-increasing. If we denote by $\lambda_{1}, \ldots, \lambda_{n}$ the eigenvalues of $A$ (i.e. the principal curvatures of $\Sigma_{t}$ ), and introduce

$$
\mu_{i}=\frac{\lambda_{i}}{H}=\frac{\lambda_{i}}{\sum_{j=1}^{n} \lambda_{j}} \quad(i \in\{1, \ldots, n\})
$$

then the sign of $\frac{d}{d t} \mathcal{H}\left(\Sigma_{t}\right)$ immediately follows from the lemma below (the proof of which will be performed in Subsection 4.3), since it implies

$$
(n-1) \frac{\operatorname{tr}_{g} A^{3}}{H}-\frac{2 n-1}{n}|A|^{2}+\frac{1}{n} H^{2} \geq 0
$$

Lemma 3.8. If $n \geq 3$, then the function

$$
g(\mu)=(n-1) \sum_{i=1}^{n} \mu_{i}^{3}-\frac{2 n-1}{n} \sum_{i=1}^{n} \mu_{i}^{2}+\frac{1}{n}
$$

is non-negative on the domain $\pi=\left\{\mu \in \mathbb{R}^{n} \mid \sum_{i=1}^{n} \mu_{i}=1, \mu_{i} \geq 0\right\}$ and vanishes only if all the $\mu_{i}$ equal $\frac{1}{n}$ or one $\mu_{i}$ vanishes whereas the others all equal $\frac{1}{n-1}$.

Clearly, then, Lemma 3.7 and Lemma 3.8 imply the proposition.
4.2. Proof of Lemma 3.7. As in the proof of Lemma 3.4, the calculations of Section 1, Chapter 4, can be used also here. From (4.3c) (with $f=1 / H$ ) we recover the well-known fact that

$$
\frac{d}{d t} \operatorname{vol}_{n}\left(\Sigma_{t}\right)=\operatorname{vol}_{n}\left(\Sigma_{t}\right)
$$

under inverse mean curvature flow, whence

$$
\begin{equation*}
\frac{d}{d t} \mathcal{H}\left(\Sigma_{t}\right)=\operatorname{vol}_{n}^{-\frac{n-2}{n}}\left(\Sigma_{t}\right)\left(\frac{d}{d t} \mathcal{F}\left(\Sigma_{t}\right)-\frac{n-2}{n} \mathcal{F}\left(\Sigma_{t}\right)\right) . \tag{3.7}
\end{equation*}
$$

Using (4.3s) (with $C=\sqrt{n}$ and $f=1 / H$ ) we get

$$
\begin{aligned}
\frac{d}{d t} \mathcal{F}\left(\Sigma_{t}\right)-\frac{n-2}{n} \mathcal{F}\left(\Sigma_{t}\right)= & (n-1) \int_{\Sigma_{t}}|\AA|^{2}-2(n-1) \int_{\Sigma_{t}} \operatorname{Hess} \frac{1}{H}: \AA \\
& -2(n-1) \int_{\Sigma_{t}} \frac{A^{2}: \AA}{H}-\frac{1}{n} \int_{\Sigma_{t}}(H-\bar{H})^{2} \\
& +\frac{2}{n} \int_{\Sigma_{t}}(H-\bar{H}) \Delta \frac{1}{H}+\frac{2}{n} \int_{\Sigma_{t}}|A|^{2}-\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \frac{|A|^{2}}{H} \\
& -\frac{(n-1)(n-2)}{n} \int_{\Sigma_{t}}|\AA|^{2}+\frac{n-2}{n^{2}} \int_{\Sigma_{t}}(H-\bar{H})^{2}
\end{aligned}
$$

$$
\begin{aligned}
= & (n-1)\left(1-\frac{n-2}{n}\right) \int_{\Sigma_{t}}|\AA|^{2}-\frac{1}{n}\left(1-\frac{n-2}{n}\right) \int_{\Sigma_{t}}(H-\bar{H})^{2} \\
& +\frac{2}{n} \int_{\Sigma_{t}}|A|^{2}-2(n-1) \int_{\Sigma_{t}} \frac{A^{2}: \AA}{H}-\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \frac{|A|^{2}}{H} \\
& -2(n-1) \int_{\Sigma_{t}} \operatorname{Hess} \frac{1}{H}: \AA+\frac{2}{n} \int_{\Sigma_{t}}(H-\bar{H}) \Delta \frac{1}{H} \\
= & 2 \frac{n-1}{n} \int_{\Sigma_{t}}|A|^{2}-2 \frac{n-1}{n^{2}} \int_{\Sigma_{t}} H^{2}-\frac{2}{n^{2}} \int_{\Sigma_{t}} H^{2}+\frac{2}{n^{2}} \bar{H} \int_{\Sigma_{t}} H \\
& +\frac{2}{n} \int_{\Sigma_{t}}|A|^{2}-2(n-1) \int_{\Sigma_{t}} \frac{\operatorname{tr}_{g} A^{3}}{H}+2 \frac{n-1}{n} \int_{\Sigma_{t}}|A|^{2} \\
& -\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \frac{|A|^{2}}{H}-2(n-1) \int_{\Sigma_{t}} \operatorname{Hess} \frac{1}{H}: A \\
& 2 \frac{n-1}{n} \int_{\Sigma_{t}} H \Delta \frac{1}{H}+\frac{2}{n} \int_{\Sigma_{t}} H \Delta \frac{1}{H}-\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \Delta \frac{1}{H} \\
= & -2(n-1) \int_{\Sigma_{t}} \frac{\operatorname{tr}_{g} A^{3}}{H}+2 \frac{2 n-1}{n} \int_{\Sigma_{t}}|A|^{2}-2 \frac{1}{n} \int_{\Sigma_{t}} H^{2} \\
& -\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \frac{|\AA|^{2}}{H}-2(n-1) \int_{\Sigma_{t}} H e s s \frac{1}{H}: A+2 \int_{\Sigma_{t}} H \Delta \frac{1}{H} \\
& -\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \Delta \frac{1}{H} .
\end{aligned}
$$

Using that $\Sigma_{t}$ is closed, we recover, as in the proof of Lemma 3.4, that $\int_{\Sigma_{t}} \Delta \varphi$ vanishes for any $C^{2}$-function $\varphi$ on $\Sigma_{t}$ and that

$$
\begin{aligned}
\int_{\Sigma_{t}} \operatorname{Hess} \varphi: A & =\int_{\Sigma_{t}} \sum_{i, j=1}^{n} \nabla_{i} \nabla^{j} \varphi A_{j}^{i}=-\int_{\Sigma_{t}} \sum_{i, j=1}^{n} \nabla^{j} \varphi \nabla_{i} A_{j}^{i} \\
& =-\int_{\Sigma_{t}} \sum_{i, j=1}^{n} \nabla^{j} \varphi \nabla_{j} A_{i}^{i}=\int_{\Sigma_{t}} \sum_{i, j=1}^{n} \nabla_{j} \nabla^{j} \varphi A_{i}^{i}=\int_{\Sigma_{t}} H \Delta \varphi,
\end{aligned}
$$

where the Codazzi equations have been used. Similarly, we compute

$$
\int_{\Sigma_{t}} H \Delta \frac{1}{H}=-\int_{\Sigma_{t}} \sum_{i=1}^{n} \nabla_{i} H \nabla_{i} \frac{1}{H}=\int_{\Sigma_{t}} \sum_{i=1}^{n} \nabla_{i} H \frac{\nabla_{i} H}{H^{2}}=\int_{\Sigma_{t}} \frac{|\nabla H|^{2}}{H^{2}}
$$

so that we finally arrive at

$$
\begin{aligned}
\frac{d}{d t} \mathcal{F}\left(\Sigma_{t}\right)-\frac{n-2}{n} \mathcal{F}\left(\Sigma_{t}\right)= & -2\left((n-1) \int_{\Sigma_{t}} \frac{\operatorname{tr}_{g} A^{3}}{H}-\frac{2 n-1}{n} \int_{\Sigma_{t}}|A|^{2}+\frac{1}{n} \int_{\Sigma_{t}} H^{2}\right) \\
& -\frac{2}{n} \bar{H} \int_{\Sigma_{t}} \frac{\mid \AA \AA^{2}}{H}-2(n-2) \int_{\Sigma_{t}} \frac{|\nabla H|^{2}}{H^{2}}
\end{aligned}
$$

which, together with (3.7), immediately implies Lemma 3.7.
4.3. Proof of Lemma 3.8. Let $\alpha=\min _{i} \mu_{i}$ denote the smallest $\mu_{i}$ and define

$$
\beta_{i}=\mu_{i}-\alpha \quad(i \in\{1, \ldots, n\})
$$

which are all non-negative. Notice that at least one $\beta_{i}$ has to vanish. Since $\mu \in \pi$, we have that

$$
\sum_{i=1}^{n} \beta_{i}=1-n \alpha \quad \text { and } \quad \alpha \in\left[0, \frac{1}{n}\right]
$$

Now write

$$
\begin{aligned}
g(\mu)= & (n-1) \sum_{i=1}^{n}\left(\alpha+\beta_{i}\right)^{3}-\frac{2 n-1}{n} \sum_{i=1}^{n}\left(\alpha+\beta_{i}\right)^{2}+\frac{1}{n} \\
= & n(n-1) \alpha^{3}-(2 n-1) \alpha^{2}+\frac{1}{n}+(n-1) \sum_{i=1}^{n} \beta_{i}^{3} \\
& +3(n-1) \alpha \sum_{i=1}^{n} \beta_{i}^{2}+3(n-1) \alpha^{2} \sum_{i=1}^{n} \beta_{i} \\
& -\frac{2 n-1}{n} \sum_{i=1}^{n} \beta_{i}^{2}-2 \alpha \frac{2 n-1}{n} \sum_{i=1}^{n} \beta_{i} \\
= & (1-n \alpha)\left(-(n-1) \alpha^{2}+\alpha+\frac{1}{n}\right) \\
& +(1-n \alpha)\left(3(n-1) \alpha^{2}-2 \alpha \frac{2 n-1}{n}\right) \\
& +(n-1) \sum_{i=1}^{n} \beta_{i}^{3}+\left(3(n-1) \alpha-\frac{2 n-1}{n}\right) \sum_{i=1}^{n} \beta_{i}^{2} \\
= & \frac{1}{n}(1-n \alpha)^{2}(1-2(n-1) \alpha) \\
& +(n-1) \sum_{i=1}^{n} \beta_{i}^{3}+\left(3(n-1) \alpha-\frac{2 n-1}{n}\right) \sum_{i=1}^{n} \beta_{i}^{2} .
\end{aligned}
$$

Using that

$$
\sum_{i=1}^{n} \beta_{i}^{2}=\sum_{i=1}^{n} \beta_{i}^{3 / 2} \beta_{i}^{1 / 2} \leq \sqrt{\sum_{i=1}^{n} \beta_{i}^{3}} \sqrt{\sum_{i=1}^{n} \beta_{i}}=\sqrt{1-n \alpha} \sqrt{\sum_{i=1}^{n} \beta_{i}^{3}}
$$

i.e.

$$
\sum_{i=1}^{n} \beta_{i}^{3} \geq \frac{1}{1-n \alpha}\left(\sum_{i=1}^{n} \beta_{i}^{2}\right)^{2}
$$

yields

$$
\begin{aligned}
g(\mu) \geq & \frac{1}{n}(1-n \alpha)^{2}(1-2(n-1) \alpha) \\
& +\frac{n-1}{1-n \alpha}\left(\sum_{i=1}^{n} \beta_{i}^{2}\right)^{2}+\left(3(n-1) \alpha-\frac{2 n-1}{n}\right) \sum_{i=1}^{n} \beta_{i}^{2}
\end{aligned}
$$

where equality holds only if all $\beta_{i}$ vanish, or all non-zero ones are identical.
Consider now the function

$$
\phi: x \mapsto \frac{n-1}{1-n \alpha} x^{2}+\left(3(n-1) \alpha-\frac{2 n-1}{n}\right) x
$$

which is minimal when

$$
x=x_{\text {crit }}=(1-n \alpha)\left(\frac{2 n-1}{2 n(n-1)}-\frac{3}{2} \alpha\right)
$$

and monotonically increasing on $\left[x_{\text {crit }},+\infty\right)$. We then have

$$
g(\mu) \geq \frac{1}{n}(1-n \alpha)^{2}(1-2(n-1) \alpha)+\phi\left(\sum_{i=1}^{n} \beta_{i}^{2}\right)
$$

Using that at least one $\beta_{i}$ vanishes, we can estimate

$$
\sum_{i=1}^{n} \beta_{i} \leq \sqrt{n-1} \sqrt{\sum_{i=1}^{n} \beta_{i}^{2}}
$$

so that

$$
\sum_{i=1}^{n} \beta_{i}^{2} \geq \frac{1}{n-1}\left(\sum_{i=1}^{n} \beta_{i}\right)^{2}=\frac{(1-n \alpha)^{2}}{n-1}
$$

Notice that equality holds only if all the $\beta_{i}$ vanish, or $(n-1)$ of them are non-zero and identical.

Since

$$
\frac{(1-n \alpha)^{2}}{n-1} \geq x_{\text {crit }} \quad \forall \alpha \geq-\frac{1}{n(n-3)}
$$

and $\alpha \geq 0$ by assumption, we see that

$$
\begin{aligned}
g(\mu) & \geq \frac{1}{n}(1-n \alpha)^{2}(1-2(n-1) \alpha)+\phi\left(\frac{(1-n \alpha)^{2}}{n-1}\right) \\
& =\frac{n-2}{n(n-1)} \alpha(1-n \alpha)^{2} \\
& \geq 0
\end{aligned}
$$

where the second equality holds only if $\alpha \in\{0,1 / n\}$ and the first, as already mentioned, only if all $\beta_{i}$ vanish, or all except one of them are non-zero and equal. In view of the definitions of $\alpha$ and $\beta_{i}$, this leaves precisely the two asserted possibilities and thus concludes the proof.

## CHAPTER 4

## About the optimality of some of our results

This chapter is dedicated to producing a few optimality results around the matters discussed so far. We first show that the constant $C=\sqrt{\frac{n}{n-1}}$ on the right-hand side of (3.1) is, in fact, optimal among Ricci-positive (i.e. convex) hypersurfaces. We then argue that the condition Ric $\geq 0$ is optimal whenever we are in the subcritical setting (i.e. when $p<n$ ). Finally, we establish that the constant $C=\sqrt{2}$ is not the appropriate one for all two-dimensional surfaces, thereby demonstrating the importance of the additional assumptions in, both, Theorem 3.1 and Theorem 3.3.

Notice that the given counterexamples do not assume unit $n$-volume. However, this is irrelevant, since both sides of the main estimate scale identically.

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## 1. The optimality of the constant $C=\sqrt{\frac{n}{n-1}}$ in Theorem 3.1

Proposition 4.1. Let $n \geq 2$ and $C<\sqrt{\frac{n}{n-1}}$ be given. Then there is a deformation $\Sigma$ of the standard sphere $S^{n}$ such that

$$
\begin{equation*}
\left(\int_{\Sigma}\left|A-\frac{1}{n}\left(\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H\right) g\right|^{2}\right)^{\frac{1}{2}}>C\left(\int_{\Sigma}\left|A-\frac{H}{n} g\right|^{2}\right)^{\frac{1}{2}} \tag{4.1}
\end{equation*}
$$

Moreover, $\Sigma$ can be chosen arbitrarily close to $S^{n}$, ensuring that $\operatorname{Ric}_{\Sigma}>0$.

We will prove this proposition by a geometric flow technique.
1.1. Preliminaries. On the set of smooth closed hypersurfaces in $\mathbb{R}^{n+1}$ we introduce the functional

$$
\begin{equation*}
\mathcal{F}(\Sigma)=\left(C^{2}-1\right) \int_{\Sigma}|\AA|^{2}-\frac{1}{n} \int_{\Sigma}\left(H-\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} H\right)^{2} \tag{4.2}
\end{equation*}
$$

It is immediate to see that $\mathcal{F}(\Sigma)$ is negative if and only if inequality (4.1) holds (compare with (3.2) of the previous chapter). Notice also that $\mathcal{F}\left(S^{n}\right)=0$.

Let $F_{0}: S^{n} \rightarrow \mathbb{R}^{n+1}$ denote the canonical embedding of $S^{n}$ into $\mathbb{R}^{n+1}$. Given a smooth, real-valued function $f$ on $S^{n}$, we consider the family of hypersurfaces $\Sigma_{t}$ given by the embeddings $F: S^{n} \times[0, T] \rightarrow \mathbb{R}^{n+1}$ for some $T>0$ and such that

$$
\left\{\begin{array}{l}
F\left(S^{n}, 0\right)=F_{0}\left(S^{n}\right) \\
\partial_{t} F(x, t)=f(x) \nu_{t}(x) \quad\left(x \in S^{n}\right)
\end{array}\right.
$$

where $\nu_{t}(x)$ denotes the outer unit normal of $\Sigma_{t}$ in $F(x, t)$ (the existence of such an $F$ should follow from, for instance, [Car65, §§35-48]).

In what follows, we will calculate the second derivative of $\mathcal{F}\left(\Sigma_{t}\right)$ at $t=0$ and show that we can choose an $f: S^{n} \rightarrow \mathbb{R}$ such that

$$
\begin{cases}\left.\frac{d}{d t} \mathcal{F}\left(\Sigma_{t}\right)\right|_{t=0} & =0 \\ \left.\frac{d^{2}}{d t^{2}} \mathcal{F}\left(\Sigma_{t}\right)\right|_{t=0} & <0\end{cases}
$$

This shows that we can deform $S^{n}$ slightly to obtain a surface that satisfies (4.1), which proves the optimality of (3.1) (obviously, by continuity, the deformation of $S^{n}$ thus obtained will have non-negative Ricci curvature for $t$ small enough).

In the sequel, we shall omit the subscript $t$ for the sake of readability.
1.2. The first variation of $\mathcal{F}$. We choose any coordinate patch and compute

$$
\begin{align*}
& \partial_{t} g_{i j}=\partial_{t}\left\langle\partial_{i} F, \partial_{j} F\right\rangle=\left\langle\partial_{i} \partial_{t} F, \partial_{j} F\right\rangle+\left\langle\partial_{i} F, \partial_{j} \partial_{t} F\right\rangle=2 f A_{i j}  \tag{4.3a}\\
& \partial_{t} g^{i j}=-\sum_{k, l=1}^{n} g^{i k} g^{j l} \partial_{t} g_{k l}^{(4.3 \mathrm{a})}=-2 f A^{i j} \tag{4.3b}
\end{align*}
$$

$$
\begin{align*}
\partial_{t} \sqrt{\operatorname{det} g} & =\frac{1}{2 \sqrt{\operatorname{det} g}} \partial_{t} \operatorname{det} g=\frac{1}{2 \sqrt{\operatorname{det} g}} \operatorname{det} g \operatorname{tr}_{g}\left(\partial_{t} g\right)=f \operatorname{tr}_{g}(A) \sqrt{\operatorname{det} g}  \tag{4.3c}\\
& =f H \sqrt{\operatorname{det} g}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
\partial_{t} \partial_{i} F & =\left(\partial_{i} f\right) \nu+f\left(\partial_{i} \nu\right)=\left(\partial_{i} f\right) \nu+f \sum_{k=1}^{n} A^{k}{ }_{i} \partial_{k} F  \tag{4.3d}\\
\partial_{t} \nu & =\sum_{i, j=1}^{n}\left\langle\partial_{t} \nu, \partial_{i} F\right\rangle g^{i j} \partial_{j} F=-\sum_{i, j=1}^{n}\left\langle\nu, \partial_{t} \partial_{i} F\right\rangle g^{i j} \partial_{j} F=-\nabla f
\end{align*}
$$

Consequently,

$$
\begin{align*}
& \partial_{t} A_{i j}=\partial_{t}\left\langle\partial_{i} \nu, \partial_{j} F\right\rangle=\left\langle\partial_{i} \partial_{t} \nu, \partial_{j} F\right\rangle+\left\langle\partial_{i} \nu, \partial_{t} \partial_{j} F\right\rangle  \tag{4.3f}\\
& \quad \begin{aligned}
&(4.3 \mathrm{da)} \&(4.3 \mathrm{e}) \\
&=-\left\langle\partial_{i} \nabla f, \partial_{j} F\right\rangle+f \sum_{k=1}^{n} A^{k}{ }_{j} A_{k i}=-\operatorname{Hess}_{i j} f+f\left(A^{2}\right)_{i j}
\end{aligned} .
\end{align*}
$$

and

$$
\begin{equation*}
\partial_{t} H=\sum_{i, j=1}^{n}\left(\partial_{t} g^{i j}\right) A_{i j}+\sum_{i, j=1}^{n} g^{i j}\left(\partial_{t} A_{i j}^{(4.3 \mathrm{~b}) \&(4.3 \mathrm{f})}=-\Delta f-f|A|^{2},\right. \tag{4.3~g}
\end{equation*}
$$

from which

$$
\begin{align*}
& \partial_{t} \stackrel{\circ}{A}_{i j}= \partial_{t} A_{i j}-\frac{1}{n}\left(\partial_{t} H\right) g_{i j}-\frac{1}{n} H\left(\partial_{t} g_{i j}\right)  \tag{4.3h}\\
& \begin{aligned}
&(4.3 \mathrm{a}),(4.3 \mathrm{f}) \&(4.3 \mathrm{~g}) \\
&=-\operatorname{Hess}_{i j} f+\frac{1}{n} \Delta f g_{i j}+f\left(\left(A^{2}\right)_{i j}-\frac{1}{n} H A_{i j}\right) \\
&+\frac{1}{n} f\left(|A|^{2} g_{i j}-H A_{i j}\right) \\
&=-\operatorname{Hess}_{i j} f+\frac{1}{n} \Delta f g_{i j}+f(A \AA)_{i j} \\
&+\frac{1}{n} f\left(\left(\left\lvert\, \AA \AA^{2}+\frac{H^{2}}{n}\right.\right) g_{i j}-H\left(\AA_{i j}+\frac{1}{n} H g_{i j}\right)\right), \\
&=-\operatorname{Hess}_{i j} f+\frac{1}{n} \Delta f g_{i j}+f(A \AA)_{i j}+\frac{1}{n} f|\AA|^{2} g_{i j}-\frac{1}{n} f H \AA_{i j},
\end{aligned}
\end{align*}
$$

$$
\begin{align*}
& \partial_{t}|A|^{2}=2 \sum_{i, j, k, l=1}^{n} g^{i k} A_{k l}\left(\left(\partial_{t} g^{j l}\right) A_{i j}+g^{j l}\left(\partial_{t} A_{i j}\right)\right)  \tag{4.3i}\\
&(4.3 \mathrm{~b}) \&(4.3 \mathrm{f}) \\
&=-4 f \sum_{i, j, l=1}^{n} A^{i}{ }_{l} A^{l j} A_{j i}-2 \text { Hess } f: A+2 \sum_{i, j, k=1}^{n} f A^{i j} A_{j}{ }^{k} A_{k i} \\
&=-2 \operatorname{Hess} f: A-2 f \operatorname{tr}_{g}\left(A^{3}\right)
\end{align*}
$$

and

$$
\begin{align*}
& \partial_{t}|\AA|^{2}=\partial_{t}\left(|A|^{2}-\frac{1}{n} H^{2}\right)=\partial_{t}|A|^{2}-\frac{2}{n} H\left(\partial_{t} H\right)  \tag{4.3j}\\
&(4.3 \mathrm{~g}) \&(4.3 \mathrm{i}) \\
&=-2 \text { Hess } f: A-2 f \operatorname{tr}_{g}\left(A^{3}\right)+\frac{2}{n} H \Delta f+\frac{2}{n} f H|A|^{2} \\
&=-2 \text { Hess } f: \AA-2 f A^{2}: \AA .
\end{align*}
$$

By (4.3c), we have for any smooth function $\varphi: S^{n} \times[0, T] \rightarrow \mathbb{R}$

$$
\begin{equation*}
\frac{d}{d t} \int_{\Sigma} \varphi=\int_{\Sigma}^{(4.3 \mathrm{c})} f H \varphi+\int_{\Sigma} \partial_{t} \varphi \tag{4.3k}
\end{equation*}
$$

As a consequence, we get

$$
\frac{d}{d t} \int_{\Sigma}|\stackrel{\circ}{A}|^{2}=\int_{\Sigma}^{(4.3 \mathrm{j}) \&(4.3 \mathrm{k})} f H|\stackrel{\circ}{A}|^{2}-2 \int_{\Sigma} \operatorname{Hess} f: \AA-2 \int_{\Sigma} f A^{2}: \AA
$$

On the other hand, if we introduce the notation

$$
\bar{\varphi}=\frac{1}{\operatorname{vol}_{n}(\Sigma)} \int_{\Sigma} \varphi
$$

we have

$$
\begin{equation*}
\frac{d}{d t} \bar{\varphi}=\frac{1}{\operatorname{vol}_{n}(\Sigma)}\left(\int_{\Sigma} f H(\varphi-\bar{\varphi})+\int_{\Sigma} \partial_{t} \varphi\right) \tag{4.3~m}
\end{equation*}
$$

since

$$
\begin{equation*}
\frac{d}{d t} \frac{1}{\operatorname{vol}_{n}(\Sigma)}=-\frac{1}{\operatorname{vol}_{n}^{2}(\Sigma)} \frac{d}{d t} \int_{\Sigma} 1 \stackrel{(4.3 \mathrm{k})}{=}-\frac{1}{\operatorname{vol}_{n}^{2}(\Sigma)} \int_{\Sigma} f H \tag{4.3n}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\partial_{t}(\varphi-\bar{\varphi}) \stackrel{(4.3 \mathrm{~m})}{=} \partial_{t} \varphi-\overline{\left(\partial_{t} \varphi\right)}-\overline{(f H(\varphi-\bar{\varphi}))} \tag{4.3o}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{d}{d t} \int_{\Sigma}(\varphi-\bar{\varphi})^{2} & =\int_{\Sigma}^{(4.3 \mathrm{k}) \&(4.30)} f H(\varphi-\bar{\varphi})^{2}+2 \int_{\Sigma}(\varphi-\bar{\varphi})\left(\partial_{t}(\varphi-\bar{\varphi})\right)  \tag{4.3p}\\
& =\int_{\Sigma} f H(\varphi-\bar{\varphi})^{2}+2 \int_{\Sigma}(\varphi-\bar{\varphi}) \partial_{t} \varphi
\end{align*}
$$

where we have used that $\int_{\Sigma}(\varphi-\bar{\varphi})=0$. It follows that

$$
\begin{equation*}
\partial_{t}\left(H-\frac{(4.3 \mathrm{H}) \&(4.3 \mathrm{H})}{=}\right)-\Delta f-f|A|^{2}+\overline{\left(\Delta f+f|A|^{2}\right)}-\overline{(f H(H-\bar{H}))} \tag{4.3q}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d t} \int_{\Sigma}(H-\bar{H})^{2}=\int_{\Sigma}^{(4.3 \mathrm{~g}) \&(4.3 \mathrm{p})} f H(H-\bar{H})^{2}-2 \int_{\Sigma}(H-\bar{H}) \Delta f-2 \int_{\Sigma} f|A|^{2}(H-\bar{H}) \tag{4.3r}
\end{equation*}
$$

Putting all this together finally yields
(4.3s)

$$
\begin{aligned}
\frac{d}{d t} \mathcal{F}(\Sigma)= & \left(C^{(4.31) \&(4.3 \mathrm{r})}-1\right)\left(\int_{\Sigma} f H|\AA|^{2}-2 \int_{\Sigma} \operatorname{Hess} f: \AA-2 \int_{\Sigma} f A^{2}: \AA\right) \\
& -\frac{1}{n}\left(\int_{\Sigma} f H(H-\bar{H})^{2}-2 \int_{\Sigma}(H-\bar{H}) \Delta f-2 \int_{\Sigma} f|A|^{2}(H-\bar{H})\right) .
\end{aligned}
$$

Since $\AA$ vanishes on $S^{n}=\left.\Sigma\right|_{t=0}$, whereas $H$ is constant there, we immediately see that, for any $f$,

$$
\left.\frac{d}{d t}\right|_{t=0} \mathcal{F}(\Sigma)=0
$$

Consequently, $S^{n}$ is a critical point of $\mathcal{F}$ and it makes sense to study the second order variation of $\mathcal{F}$ at $t=0$. This is precisely what we wish to do next.
1.3. The second variation of $\mathcal{F}$. If we denote by $\eta$ the canonical metric on $S^{n}$, we have at $t=0$

$$
\begin{aligned}
\left.g\right|_{t=0} & =\eta \\
\left.A\right|_{t=0} & =\eta \\
\left.H\right|_{t=0} & =n, \\
\left.\AA\right|_{t=0} & =0
\end{aligned}
$$

and

$$
\left.(H-\bar{H})\right|_{t=0}=0 .
$$

Omitting any subscript $S^{n}$ on quantities or operators which are now evaluated on $S^{n}=\left.\Sigma_{t}\right|_{t=0}$ rather than on $\Sigma_{t}$, we obtain from (4.3h), (4.3j) and (4.3q)

$$
\begin{equation*}
\left.\partial_{t}\right|_{t=0} \stackrel{(4.3 \mathrm{~h})}{\stackrel{\circ}{=}=-}-\operatorname{Hess} f+\frac{1}{n}(\Delta f) \eta \tag{4.4a}
\end{equation*}
$$

(4.3j)
and

$$
\begin{equation*}
\left.\partial_{t}\right|_{t=0}(H-\bar{H})^{2}=\left.2\left((H-\bar{H})\left(\partial_{t}(H-\bar{H})\right)\right)\right|_{t=0}=0 . \tag{4.4d}
\end{equation*}
$$

In view of the vanishing of $\AA$ and $(H-\bar{H})$ at $t=0$, these are sufficient to compute the second variation of $\mathcal{F}$. Indeed, from (4.3s) we obtain using the identities above

$$
\begin{array}{r}
\left.\frac{d^{2}}{d t^{2}}\right|_{t=0} \mathcal{F}(\Sigma)=\left(C^{2}-1\right)\left(-2 \int_{S^{n}} \operatorname{Hess} f:\left(-\operatorname{Hess} f+\frac{1}{n} \eta \Delta f\right)\right.  \tag{4.4e}\\
\left.-2 \int_{S^{n}} f \eta^{2}:\left(-\operatorname{Hess} f+\frac{1}{n} \eta \Delta f\right)\right) \\
-\frac{1}{n}\left(-2 \int_{S^{n}} \Delta f\left(-\Delta f-f n^{2}+\overline{\left(\Delta f+f n^{2}\right)}\right)\right. \\
\\
\left.-2 \int_{S^{n}} f n^{2}\left(-\Delta f-f n^{2}+\overline{\left(\Delta f+f n^{2}\right)}\right)\right)
\end{array}
$$

$$
\begin{aligned}
= & \left(C^{2}-1\right)\left(2 \int_{S^{n}} \operatorname{Hess} f: \operatorname{Hess} f+2 \int_{S^{n}} f \Delta f-2 \int_{S^{n}} \frac{n}{n} f \Delta f\right) \\
& -\frac{2}{n}\left(\int_{S^{n}} \Delta f(\Delta f+n f-\overline{(\Delta f+n f)})\right. \\
& \left.\quad+\int_{S^{n}} n f(\Delta f+n f-\overline{(\Delta f+n f)})\right) \\
= & 2\left(C^{2}-1\right) \int_{S^{n}}\left|\operatorname{Hess}^{\circ} f\right|^{2}-\frac{2}{n} \int_{S^{n}}(\Delta f+n f-\overline{(\Delta f+n f)})^{2}
\end{aligned}
$$

where Hess $f=$ Hess $f-\frac{1}{n}(\Delta f) g$ is the traceless part of Hess $f$, and we have used that $\int_{S^{n}}(\varphi-\bar{\varphi})=0$ for any smooth function $\varphi$ on $S^{n}$.
1.4. Proof of Proposition 4.1. By the same calculation as in (3.3) of Section 1, Chapter 3, we have

$$
\int_{S^{n}}|\operatorname{Hess} f|^{2}=\int_{S^{n}}(\Delta f)^{2}-\int_{S^{n}} \operatorname{Ric}(\nabla f, \nabla f)=\int_{S^{n}}(\Delta f)^{2}-(n-1) \int_{S^{n}}|\nabla f|^{2}
$$

since Ric $=(n-1) \eta$ on $S^{n}$. Consequently,

$$
\int_{S^{n}}|\operatorname{Hess} f|^{2}=\frac{n-1}{n} \int_{S^{n}}(\Delta f)^{2}-(n-1) \int_{S^{n}}|\nabla f|^{2} .
$$

On the other hand, if we assume that $\bar{f}=0$, we obtain through partial integration

$$
\begin{aligned}
\int_{S^{n}}(\Delta f+n f-\overline{(\Delta f+n f)})^{2}= & \int_{S^{n}}(\Delta f)^{2}+2 n \int_{S^{n}} f \Delta f+n^{2} \int_{S^{n}} f^{2} \\
& -\frac{1}{\operatorname{vol}_{n}\left(S^{n}\right)}\left(\int_{S^{n}} \Delta f+n \int_{S^{n}} f\right)^{2} \\
= & \int_{S^{n}}(\Delta f)^{2}-2 n \int_{S^{n}}|\nabla f|^{2}+n^{2} \int_{S^{n}} f^{2}
\end{aligned}
$$

Hence, (4.4e) is equivalent to

$$
\begin{align*}
\left.\frac{d^{2}}{d t^{2}}\right|_{t=0} \mathcal{F}(\Sigma)= & 2\left(C^{2} \frac{n-1}{n}-1\right) \int_{S^{n}}(\Delta f)^{2}-2 n\left(C^{2} \frac{n-1}{n}-1\right) \int_{S^{n}}|\nabla f|^{2}  \tag{4.5}\\
& +2\left(\int_{S^{n}}|\nabla f|^{2}-n \int_{S^{n}} f^{2}\right) \\
= & 2\left(C^{2} \frac{n-1}{n}-1\right) \int_{S^{n}}\left((\Delta f)^{2}-n|\nabla f|^{2}\right)+2 \int_{S^{n}}\left(|\nabla f|^{2}-n f^{2}\right)
\end{align*}
$$

as long as we require $\int_{S^{n}} f=0$.

Now let $f$ be a spherical harmonic of order $k$, where $k \geq 1$ will be chosen below, and set $\alpha=\left(C^{2} \frac{n-1}{n}-1\right) . f$ satisfies $-\Delta f=k(k+n-1) f, \bar{f}=0$, and (4.5) becomes

$$
\begin{align*}
\left.\frac{d^{2}}{d t^{2}}\right|_{t=0} \mathcal{F}(\Sigma)= & 2 \alpha\left(k^{2}(k+n-1)^{2}-n k(k+n-1)\right) \int_{S^{n}} f^{2}  \tag{4.6}\\
& +2(k(k+n-1)-n) \int_{S^{n}} f^{2} \\
= & 2\left(\int_{S^{n}} f^{2}\right)(\alpha k(k+n-1)(k(k+n-1)-n) \\
& +(k(k+n-1)-n)) \\
= & 2(k-1)(k+n)\left(\int_{S^{n}} f^{2}\right)(\alpha k(k+n-1)+1) .
\end{align*}
$$

For $k>1$, this quantity is negative whenever

$$
\alpha k(k+n-1)+1<0,
$$

which can be achieved as soon as

$$
k>\sqrt{\frac{(n-1)^{2}}{4}+\frac{n}{n-C^{2}(n-1)}}-\frac{n-1}{2},
$$

since we had assumed that $\alpha=\left(C^{2} \frac{n-1}{n}-1\right)<0$. As a result, there exists a function on $S^{n}$ such that $\left.\frac{d^{2}}{d t^{2}}\right|_{t=0} \mathcal{F}(\Sigma)<0$ and the Proposition is proved.

## 2. The optimality of the assumption $R i c \geq 0$ for the general sub-critical estimate

In this section we prove that, for $p \in[1, n)$, there are surfaces for which estimate (3.1) fails (for any constant) if we don't assume the Ricci curvature to be nonnegative. This will be a direct consequence of the following proposition which is an easy generalisation of Proposition 7.1 in [DLM05].

Proposition 4.2 (De Lellis, Müller, P.). Let $n \geq 2$ be given. There exists a family of smooth, closed, connected hypersurfaces $\Sigma_{\epsilon} \subset \mathbb{R}^{n+1}$ such that:

$$
\begin{align*}
& C \geq \operatorname{vol}_{n}\left(\Sigma_{\epsilon}\right) \geq c>0, \quad \text { for every } \epsilon>0  \tag{4.7a}\\
& \lim _{\epsilon \searrow 0} \operatorname{vol}_{n}\left(\left\{q \in \Sigma_{\epsilon} \mid \operatorname{Ric}(q)<0\right\}\right)=0  \tag{4.7b}\\
& \lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}}|\AA|^{p}=0, \quad \text { for every } p \in[1, n) \tag{4.7c}
\end{align*}
$$

$\Sigma_{\epsilon}$ converges in the Hausdorff topology to the union of two round spheres;
and

$$
\begin{equation*}
\lim _{\epsilon \searrow 0}\left(\inf _{\lambda} \int_{\Sigma_{\epsilon}}|A-\lambda g|^{p}\right)>0, \quad \text { for every } p \in[1, n) \tag{4.7e}
\end{equation*}
$$

In particular, this immediately implies
Corollary 4.3. Assume $n>2$. Then, for every $C>0$ and every $\delta>0$ we can find a smooth, closed hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ such that

$$
\left(\int_{\Sigma}\left|A-\frac{\bar{H}}{n} g\right|^{2}\right)^{\frac{1}{2}}>C\left(\int_{\Sigma}|\AA|^{2}\right)^{\frac{1}{2}}
$$

and where the portion of $\Sigma$ on which the Ricci curvature is negative has n-dimensional volume smaller than $\delta$.

The proof of Proposition 4.2 presented below follows closely the construction in [DLM05]. The idea is to consider two round spheres of radii 1 and $1 / 2$, respectively, and glue them together with a small hyperbolic neck so that the $L^{p}$-norm of the second fundamental form on that neck becomes arbitrarily small. As in [DLM05], we choose a catenoidal neck to simplify the computations.
2.1. Preliminaries. Let $I$ be a closed interval. We call a hypersurface $\Sigma \subset$ $\mathbb{R}^{n+1}$ hypersurface of revolution (around the $x^{n+1}$-axis) if there exist two smooth functions $f$ and $h$ on $I$, with $f>0$ and $\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}>0$ on the interior int $I$ of $I$, and such that $\Sigma$ is parametrised by the map

$$
\begin{aligned}
F: S^{n-1} \times I & \rightarrow \mathbb{R}^{n+1} \\
(x, t) & \mapsto(f(t) \Phi(x), h(t))
\end{aligned}
$$

where $\Phi$ denotes the canonical embedding of $S^{n-1}$ into $\mathbb{R}^{n}$. We call the curve $\phi$ : $I \rightarrow \mathbb{R}^{2}, t \mapsto(f(t), h(t))$ the generating curve of $\Sigma$, and assume it is injective.

In the coordinates $(x, t)$, the metric of $\Sigma$ and its inverse are given by

$$
g=\left(\begin{array}{cc}
f^{2} \eta & 0 \\
0 & \left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}
\end{array}\right) \quad \text { and } \quad g^{-1}=\left(\begin{array}{cc}
f^{-2} \eta^{-1} & 0 \\
0 & \frac{1}{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}}
\end{array}\right)
$$

where $\eta$ denotes the canonical metric on $S^{n-1}$. Consequently,

$$
\begin{equation*}
\sqrt{\operatorname{det} g}=f^{n-1} \sqrt{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}} \sqrt{\operatorname{det} \eta} . \tag{4.8a}
\end{equation*}
$$

One easily checks that the outward unit normal on $\Sigma$ is given by

$$
\nu=\frac{1}{\sqrt{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}}}\left(h^{\prime} \Phi,-f^{\prime}\right)
$$

Then one computes immediately

$$
A=\frac{1}{\sqrt{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}}}\left(\begin{array}{cc}
f h^{\prime} \eta & 0  \tag{4.8b}\\
0 & f^{\prime} h^{\prime \prime}-f^{\prime \prime} h^{\prime}
\end{array}\right)
$$

and

$$
\begin{equation*}
H=\frac{(n-1) \frac{h^{\prime}}{f}+\frac{f^{\prime} h^{\prime \prime}-f^{\prime \prime} h^{\prime}}{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}}}{\sqrt{\left(f^{\prime}\right)^{2}+\left(h^{\prime}\right)^{2}}} . \tag{4.8c}
\end{equation*}
$$

2.2. Detailed construction. For $\epsilon \in\left(0,2^{-(n-1)}\right]$, consider the three families $S_{\epsilon}^{u}, C_{\epsilon}$ and $S_{\epsilon}^{l}$ of hypersurfaces of revolution given by the parametrisations

$$
\begin{aligned}
F_{S_{\epsilon}^{u}}: & S^{n-1} \times\left[z_{\epsilon}^{u}, c_{\epsilon}^{u}+\frac{1}{2}\right], \quad(x, t) \mapsto\left(\sqrt{\frac{1}{4}-\left(c_{\epsilon}^{u}-t\right)^{2}} \Phi(x), t\right) \\
F_{C_{\epsilon}}: & S^{n-1} \times\left[t_{\epsilon}^{l}, t_{\epsilon}^{u}\right] \\
& (x, t) \mapsto\left(\sqrt[n-1]{\epsilon \cosh \left(\frac{(n-1) t}{\epsilon}\right)} \Phi(x), \int_{0}^{t}\left(\epsilon \cosh \left(\frac{(n-1) s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} d s\right),
\end{aligned}
$$

and

$$
F_{S_{\epsilon}^{l}}: S^{n-1} \times\left[c_{\epsilon}^{l}-1, z_{\epsilon}^{l}\right], \quad(x, t) \mapsto\left(\sqrt{1-\left(c_{\epsilon}^{l}-t\right)^{2}} \Phi(x), t\right)
$$

respectively, where

$$
\begin{array}{rlrl}
t_{\epsilon}^{u} & =\frac{\epsilon}{n-1} \operatorname{arccosh}\left(\frac{\sqrt[n]{2}}{2 \sqrt[n]{\epsilon}}\right), & t_{\epsilon}^{l} & =-\frac{\epsilon}{n-1} \operatorname{arccosh}\left(\frac{1}{\sqrt[n]{\epsilon}}\right) \\
z_{\epsilon}^{u} & =\int_{0}^{t_{\epsilon}^{u}}\left(\epsilon \cosh \left(\frac{(n-1) s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} d s, & z_{\epsilon}^{l}=\int_{0}^{t_{\epsilon}^{l}}\left(\epsilon \cosh \left(\frac{(n-1) s}{\epsilon}\right)\right)^{-\frac{n-2}{n-1}} d s
\end{array}
$$

and
$c_{\epsilon}^{u}=z_{\epsilon}^{u}+\sqrt{\frac{1}{4}-\frac{\epsilon^{2 / n}}{2^{2 / n}}}$,

$$
c_{\epsilon}^{l}=z_{\epsilon}^{l}-\sqrt{1-\epsilon^{2 / n}} .
$$

The parameters $c_{\epsilon}^{u / l}, z_{\epsilon}^{u / l}$ and $t_{\epsilon}^{u / l}$ were chosen such that, for each $\epsilon \in\left(0,2^{-(n-1)}\right]$, $\Sigma_{\epsilon}=S_{\epsilon}^{u} \cup C_{\epsilon} \cup S_{\epsilon}^{l}$ is a closed hypersurface of revolution, generated by a $C^{1}$-curve which is piecewise $C^{\infty}$. Its constituents are a portion $S_{\epsilon}^{u}$ of a sphere of radius $1 / 2$ and a portion $S_{\epsilon}^{l}$ of a sphere of radius 1 (so that $\left.\AA\right|_{S_{\epsilon}^{u / l}}=0$ ), connected by a catenoidal neck $C_{\epsilon}$ (so that $\left.H\right|_{C_{\epsilon}}=0$, as is immediately verified using (4.8c)). The sets $\gamma_{\epsilon}^{u}=S_{\epsilon}^{u} \cap C_{\epsilon}$ and $\gamma_{\epsilon}^{l}=S_{\epsilon}^{l} \cap C_{\epsilon}$ on which the constituents touch are $(n-1)-$ dimensional spheres of radius $\sqrt[n]{\epsilon}$ and $\sqrt[n]{\epsilon / 2}$, respectively (see Figure 4.1).
Remark 4.4. It might not be completely obvious why the resulting surface should be $C^{1}$. At the top and bottom it is clear that only a coordinate singularity occurs. At the two junctions $\gamma_{\epsilon}^{u / l}$, however, a short computation shows that the tangent spaces on both sides coincide. Hence we could re-parametrise the three generating curves to get a single $C^{1}$-curve (for instance as the graph over $\left[c_{\epsilon}^{l}-1, c_{\epsilon}^{u}+1 / 2\right]$ in the variable $x^{n+1}$ ).


Figure 4.1.
2.3. Proof of Proposition 4.2 . We use the construction of the previous subsection. One immediately sees that, as $\epsilon \searrow 0$,

$$
z_{\epsilon}^{u} \searrow 0, \quad z_{\epsilon}^{l} \nearrow 0, \quad c_{\epsilon}^{u} \searrow \frac{1}{2} \quad \text { and } \quad c_{\epsilon}^{l} \nearrow-1
$$

Therefore, since the radii of $\gamma_{\epsilon}^{u}$ and $\gamma_{\epsilon}^{l}$ also converge to zero, we conclude that (4.9) $S_{\epsilon}^{u}$ and $S_{\epsilon}^{l}$ converge, respectively, to a sphere $S_{0}^{u}$ of radius $\frac{1}{2}$ and to a sphere $S_{0}^{l}$ of radius 1 , which are tangent at the origin in $\mathbb{R}^{n+1}$.

Now observe that

$$
\begin{equation*}
\cosh \left(\frac{(n-1) t}{\epsilon}\right) \in\left[1, \max \left\{2^{-\frac{n-1}{n}} \epsilon^{-\frac{1}{n}}, \epsilon^{-\frac{1}{n}}\right\}\right]=\left[1, \epsilon^{-\frac{1}{n}}\right] \quad \text { on }\left[t_{\epsilon}^{l}, t_{\epsilon}^{u}\right] . \tag{4.10a}
\end{equation*}
$$

Moreover,

$$
\begin{align*}
t_{\epsilon}^{u}-t_{\epsilon}^{l} & =\frac{\epsilon}{n-1}\left(\operatorname{arccosh}\left(2^{-\frac{n-1}{n}} \epsilon^{-\frac{1}{n}}\right)+\operatorname{arccosh}\left(\epsilon^{-\frac{1}{n}}\right)\right) \\
& \in\left[\frac{\epsilon}{n(n-1)} \ln \left(2^{-(n-1)} \epsilon^{-2}\right), \frac{\epsilon}{n(n-1)} \ln \left(2^{n+1} \epsilon^{-2}\right)\right] \tag{4.10b}
\end{align*}
$$

which can be seen using the trivial estimate

$$
\ln t \leq \operatorname{arccosh} t=\ln \left(t+\sqrt{t^{2}-1}\right) \leq \ln (2 t) \quad \forall t \geq 1
$$

As a consequence, setting $\omega_{n-1}=\operatorname{vol}_{n-1}\left(S^{n-1}\right)$,
$\operatorname{vol}_{n}\left(C_{\epsilon}\right)=\int_{C_{\epsilon}} 1=\omega_{n-1} \int_{t_{\epsilon}^{l}}^{t_{\epsilon}^{u}} \sqrt{\operatorname{det} g} d t \stackrel{(4.8 \mathrm{a})}{=} \omega_{n-1} \int_{t_{\epsilon}^{l}}^{t_{\epsilon}^{u}} \epsilon^{\frac{1}{n-1}} \cosh ^{\frac{n}{n-1}}\left(\frac{(n-1) t}{\epsilon}\right) d t$
(4.10a) \& (4.10b)

$$
\begin{equation*}
\leq \frac{\omega_{n-1}}{n(n-1)} \epsilon \ln \left(2^{n+1} \epsilon^{-2}\right) \xrightarrow{\epsilon \searrow 0} 0, \tag{4.11}
\end{equation*}
$$

which implies (4.7b) and, together with (4.9), (4.7d). Also, we immediately get (4.7a). Thus it remains to show (4.7c) and (4.7e).

For (4.7c), we first use (4.8b) to obtain

$$
\left.|A|^{p}\right|_{C_{\epsilon}}=n^{\frac{p}{2}}(n-1)^{\frac{p}{2}} \epsilon^{-\frac{p}{n-1}} \cosh ^{-\frac{n p}{n-1}}\left(\frac{(n-1) t}{\epsilon}\right)
$$

Then, with (4.8a), we calculate

$$
\begin{aligned}
\int_{C_{\epsilon}}|A|^{p} & =\omega_{n-1} \int_{t_{\epsilon}^{l}}^{t_{\epsilon}^{u}}|A|^{p} \sqrt{\operatorname{det} g} d t \\
& =\omega_{n-1} \int_{t_{\epsilon}^{l}}^{t_{\epsilon}^{u}} n^{\frac{p}{2}}(n-1)^{\frac{p}{2}} \epsilon^{-\frac{p-1}{n-1}} \cosh ^{-\frac{n(p-1)}{n-1}}\left(\frac{(n-1) t}{\epsilon}\right) d t .
\end{aligned}
$$

Finally, we use (4.10a) and (4.10b) to conclude

$$
\int_{C_{\epsilon}}|A|^{p} \in\left[(n(n-1))^{\frac{p-2}{2}} \epsilon \ln \left(2^{-(n-1)} \epsilon^{-2}\right),(n(n-1))^{\frac{p-2}{2}} \epsilon^{\frac{n-p}{n-1}} \ln \left(2^{-(n-1)} \epsilon^{-2}\right)\right]
$$

Hence

$$
\begin{equation*}
\lim _{\epsilon \searrow 0} \int_{C_{\epsilon}}|A|^{p}=0 \tag{4.12}
\end{equation*}
$$

and the fact that $H$ vanishes on $C_{\epsilon}$, whereas $\AA$ vanishes on $S_{\epsilon}^{l}$ and $S_{\epsilon}^{u}$, implies immediately (4.7c).

For (4.7e), we use (4.9), (4.11) and (4.12) to get

$$
\begin{aligned}
\lim _{\epsilon \searrow 0}\left(\inf _{\lambda} \int_{\Sigma_{\epsilon}}|A-\lambda g|^{p}\right) & =\inf _{\lambda}\left(\int_{S_{0}^{l}}|A-\lambda g|^{p}+\int_{S_{0}^{u}}|A-\lambda g|^{p}\right) \\
& =\inf _{\lambda}\left(\omega_{n} n^{\frac{p}{2}}|1-\lambda|^{p}+\frac{\omega_{n} n^{\frac{p}{2}}}{2^{n}}\left|\frac{1}{2}-\lambda\right|^{p}\right)>0 .
\end{aligned}
$$

As already mentioned in the previous subsection, the hypersurfaces $\Sigma_{\epsilon}$ are only $C^{1}$. They are, however, hypersurfaces of revolution. The curves generating them are $C^{1}$ and piecewise $C^{\infty}$ (see Remark 4.4), bearing two jump discontinuities in their higher derivatives. A standard smoothing argument therefore yields a family
of hypersurfaces of revolution satisfying the requirements of Proposition 4.2. This finishes the proof.

Remark 4.5. The interested reader might wonder whether the surfaces just constructed fulfil (in the case $n=2$ ) the assumptions of Theorem 3.3. In view of our comments at the beginning of Section 3 in Chapter 3, we only need to check whether these surfaces are star-shaped. However, an easy calculation shows that they are not, as soon as

$$
\left.\left(\sqrt{1-\frac{\epsilon}{r}} \operatorname{arccosh} \sqrt{\frac{r}{\epsilon}}\right)\right|_{r=\frac{1}{2}}>1
$$

But this is always true for $\epsilon$ small enough.
3. Generic two-dimensional surfaces fail to satisfy the $L^{2}$-estimate with

$$
C=\sqrt{2}
$$

In this section we want to show that there are surfaces in $\mathbb{R}^{3}$ for which estimate (3.4) fails, thereby demonstrating that additional assumptions, as in theorems 3.1 and 3.3, are essential. More precisely, we wish to prove the following proposition.

Proposition 4.6 (De Lellis, Topping, P.). There exists a family of smooth, closed, connected surfaces $\Sigma_{\epsilon} \subset \mathbb{R}^{3}$ such that

$$
\begin{gather*}
C \geq \operatorname{vol}_{n}\left(\Sigma_{\epsilon}\right) \geq c>0, \quad \text { for every } \epsilon>0  \tag{4.13a}\\
\Sigma_{\epsilon} \text { converges in the Hausdorff topology }  \tag{4.13b}\\
\text { to a double copy of a round sphere; }
\end{gather*}
$$

and

$$
\begin{equation*}
\lim _{\epsilon \searrow 0} \frac{\int_{\Sigma_{\epsilon}}\left|A-\frac{1}{2 \operatorname{vol}_{2}\left(\Sigma_{\epsilon}\right)}\left(\int_{\Sigma_{\epsilon}} H\right) g\right|^{2}}{\int_{\Sigma_{\epsilon}}|\AA|^{2}}=3 \tag{4.13c}
\end{equation*}
$$

The idea is very similar to the one in Section 2, the only difference being that, this time, we attach two concentric spheres of almost the same radius by a catenoidal neck. Of course, to do this smoothly enough (i.e. at least $C^{1}$ ), there will be a transition zone to take into account. Below we give the construction in detail.
3.1. Detailed construction. Let $r>0$. We use the notations and calculations of Section 2. For $\epsilon \in(0, r)$, consider the four families $S_{\epsilon}^{o}, P_{\epsilon}, C_{\epsilon}$ and $S_{\epsilon}^{i}$ of surfaces
of revolution given by the parametrisations

$$
\begin{array}{rlr}
F_{S_{\epsilon}^{o}}: S^{1} \times\left[c_{\epsilon}-r-\delta_{\epsilon}, z_{\epsilon}\right], & (x, t) \mapsto\left(\sqrt{\left(r+\delta_{\epsilon}\right)^{2}-\left(c_{\epsilon}-t\right)^{2}} \Phi(x), t\right), \\
F_{P_{\epsilon}}: S^{1} \times\left[-\rho_{\epsilon}-\sqrt{\frac{\epsilon}{r}} \delta_{\epsilon},-\rho_{\epsilon}\right], \\
& (x, t) \mapsto\left(-t \Phi(x), z_{\epsilon}+\alpha_{\epsilon}\left(\left(\sqrt{\frac{\epsilon}{r}} \frac{\delta_{\epsilon}}{2}\right)^{2}-\left(t+\rho_{\epsilon}+\sqrt{\frac{\epsilon}{r}} \frac{\delta_{\epsilon}}{2}\right)^{2}\right)\right), \\
F_{C_{\epsilon}}: & S^{1} \times\left[-z_{\epsilon}, z_{\epsilon}\right], & (x, t) \mapsto\left(\epsilon \cosh \left(\frac{t}{\epsilon}\right) \Phi(x),-t\right),
\end{array}
$$

and

$$
F_{S_{\epsilon}^{i}}: S^{1} \times\left[z_{\epsilon}, r-c_{\epsilon}\right], \quad(x, t) \mapsto\left(\sqrt{r^{2}-\left(c_{\epsilon}+t\right)^{2}} \Phi(x),-t\right)
$$

respectively, where the parameters

$$
\begin{array}{ll}
z_{\epsilon}=\epsilon \operatorname{arccosh}\left(\sqrt{\frac{r}{\epsilon}}\right), & c_{\epsilon}=-r \sqrt{1-\frac{\epsilon}{r}}-z_{\epsilon} \\
\delta_{\epsilon}=\frac{2 z_{\epsilon}}{\sqrt{1-\frac{\epsilon}{r}}}, & \alpha_{\epsilon}=\frac{1}{2 z_{\epsilon}}
\end{array}
$$

and

$$
\rho_{\epsilon}=\sqrt{r \epsilon}
$$

were chosen such that, for each $\epsilon \in(0, r), \Sigma_{\epsilon}=S_{\epsilon}^{i} \cup C_{\epsilon} \cup P_{\epsilon} \cup S_{\epsilon}^{o}$ is a closed surface of revolution, generated by a $C^{1}$-curve which is piecewise $C^{\infty}$. Its constituents are a portion $S_{\epsilon}^{i}$ of a sphere of radius $r$ inside a portion $S_{\epsilon}^{o}$ of a concentric sphere of radius $r+\delta_{\epsilon}\left(\right.$ so that $\left.\left.\AA\right|_{S_{\epsilon}^{i / o}}=0\right)$, connected by a catenoidal neck $C_{\epsilon}$ (so that $\left.H\right|_{C_{\epsilon}}=0$ ) and a transitional region $P_{\epsilon}$ the cross-section of which is a piece of parabola. The sets $\gamma_{\epsilon}^{i}=S_{\epsilon}^{i} \cap C_{\epsilon}, \gamma_{\epsilon}^{m}=C_{\epsilon} \cap P_{\epsilon}$ and $\gamma_{\epsilon}^{o}=S_{\epsilon}^{o} \cap P_{\epsilon}$ on which the constituents touch are circles of radius $\rho_{\epsilon}, \rho_{\epsilon}$ and $\rho_{\epsilon}+\sqrt{\frac{\epsilon}{r}}$, respectively (see Figure 4.2). Notice that a remark analogous to Remark 4.4 holds here also.
3.2. Proof of Proposition 4.6. In the construction of the previous subsection, letting $\epsilon \searrow 0$, one immediately sees that

$$
z_{\epsilon} \searrow 0, \quad c_{\epsilon} \nearrow-r \quad \text { and } \quad \delta_{\epsilon} \searrow 0
$$

Since the radii of $\gamma_{\epsilon}^{i}, \gamma_{\epsilon}^{m}$ and $\gamma_{\epsilon}^{o}$ also converge to zero, we conclude that
$S_{\epsilon}^{i}$ and $S_{\epsilon}^{o}$ converge each to a sphere $S_{0}$ of radius $r$, with opposite orientations.
We now prove that the areas of $P_{\epsilon}$ and $C_{\epsilon}$ converge to zero as $\epsilon \searrow 0$. Let $\tau_{\epsilon}(t)$ denote the derivative with respect to $t$ of the second component of $F_{P_{\epsilon}}$, i.e.

$$
\tau_{\epsilon}(t)=-\frac{t+\rho_{\epsilon}}{z_{\epsilon}}-\frac{1}{\sqrt{\frac{r}{\epsilon}-1}}
$$



Figure 4.2.

Then

$$
\tau_{\epsilon}(t) \in\left[-\frac{1}{\sqrt{\frac{r}{\epsilon}-1}}, \frac{1}{\sqrt{\frac{r}{\epsilon}-1}}\right] \quad \text { for } t \in\left[-\rho_{\epsilon}-\sqrt{\frac{\epsilon}{r}} \delta_{\epsilon},-\rho_{\epsilon}\right]
$$

Consequently, in view of (4.8a),
$\left.\sqrt{\operatorname{det} g}\right|_{P_{\epsilon}}(t)=-t \sqrt{1+\tau_{\epsilon}^{2}(t)} \leq \frac{r}{\sqrt{\frac{r}{\epsilon}-1}}+\frac{2 z_{\epsilon} \sqrt{\frac{\epsilon}{r}}}{1-\frac{\epsilon}{r}} \quad$ for $t \in\left[-\rho_{\epsilon}-\sqrt{\frac{\epsilon}{r}} \delta_{\epsilon},-\rho_{\epsilon}\right]$.
Thus

$$
\begin{aligned}
\operatorname{vol}_{2}\left(P_{\epsilon}\right)=\int_{P_{\epsilon}} 1 & \leq 2 \pi\left(\frac{r}{\sqrt{\frac{r}{\epsilon}-1}}+\frac{2 z_{\epsilon} \sqrt{\frac{\epsilon}{r}}}{1-\frac{\epsilon}{r}}\right) \sqrt{\frac{\epsilon}{r}} \frac{2 z_{\epsilon}}{\sqrt{1-\frac{\epsilon}{r}}} \\
& =\frac{4 \pi r z_{\epsilon}}{\frac{r}{\epsilon}-1}+\frac{8 \pi \epsilon z_{\epsilon}^{2}}{r\left(1-\frac{\epsilon}{r}\right)^{3 / 2}} \xrightarrow{\epsilon \searrow 0} 0 .
\end{aligned}
$$

Similarly,

$$
\left.\sqrt{\operatorname{det} g}\right|_{C_{\epsilon}}(t)=\epsilon \cosh ^{2}\left(\frac{t}{\epsilon}\right) \in[\epsilon, r] \quad \text { for } t \in\left[-z_{\epsilon}, z_{\epsilon}\right]
$$

and so

$$
\operatorname{vol}_{2}\left(C_{\epsilon}\right)=\int_{C_{\epsilon}} 1 \leq 4 \pi r \epsilon \operatorname{arccosh} \sqrt{\frac{r}{\epsilon}} \leq 4 \pi r \epsilon \ln \left(2 \sqrt{\frac{r}{\epsilon}}\right)=2 \pi r \epsilon \ln \left(\frac{4 r}{\epsilon}\right) \xrightarrow{\epsilon \searrow 0} 0,
$$

where we have used, once again,

$$
\ln t \leq \operatorname{arccosh} t=\ln \left(t+\sqrt{t^{2}-1}\right) \leq \ln (2 t) \quad \forall t \geq 1
$$

With (4.14), we conclude (4.13b) and (4.13a). It remains to show (4.13c).

We begin with showing that $\lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}}|\AA|^{2}=8 \pi$. Since $\left.\AA{ }^{A}\right|_{S_{\epsilon}^{i / o}}=0$, we infer from (4.14) that $\lim _{\epsilon \searrow 0} \int_{S_{\epsilon}^{i / o}}|\AA|^{2}=0$. On $C_{\epsilon}$, (4.8b) and (4.8c) imply that

$$
\left.|\AA|^{2}\right|_{C_{\epsilon}}(t)=\frac{2}{\epsilon^{2} \cosh ^{4}\left(\frac{t}{\epsilon}\right)}, \quad t \in\left[-z_{\epsilon}, z_{\epsilon}\right]
$$

Consequently, since $\left.\sqrt{\operatorname{det} g}\right|_{C_{\epsilon}}(t)=\epsilon \cosh ^{2}\left(\frac{t}{\epsilon}\right)$,

$$
\begin{aligned}
\int_{C_{\epsilon}}|\AA|^{2} & =4 \pi \int_{-\epsilon \operatorname{arccosh}}^{\epsilon \operatorname{arccosh} \sqrt{\frac{r}{\epsilon}}} \frac{d t}{\epsilon \cosh ^{2}\left(\frac{t}{\epsilon}\right)}=8 \pi \int_{1}^{\sqrt{\frac{r}{\epsilon}}} \frac{d s}{s^{2} \sqrt{s^{2}-1}} \\
& =8 \pi\left[\sqrt{1-\frac{1}{s}}\right]_{s=1}^{\sqrt{\frac{\epsilon}{r}}}=8 \pi \sqrt{1-\frac{\epsilon}{r}} \xrightarrow{\epsilon \searrow 0} 8 \pi
\end{aligned}
$$

Similarly, in view of (4.8a), (4.8b) and (4.8c), we have on $P_{\epsilon}$,

$$
\begin{aligned}
\left|\AA \AA^{2} \sqrt{\operatorname{det} g}\right|_{P_{\epsilon}}(t)=-\frac{t}{2 \sqrt{1+\tau_{\epsilon}^{2}(t)}}\left(\frac{\tau_{\epsilon}(t)}{t}+\frac{1}{z_{\epsilon}\left(1+\tau_{\epsilon}^{2}(t)\right)}\right)^{2} \\
t \in\left[-\rho_{\epsilon}-\sqrt{\frac{\epsilon}{r}} \delta_{\epsilon},-\rho_{\epsilon}\right]
\end{aligned}
$$

where, again, $\tau_{\epsilon}(t)$ is given by

$$
\tau_{\epsilon}(t)=-\frac{t+\rho_{\epsilon}}{z_{\epsilon}}-\frac{1}{\sqrt{\frac{r}{\epsilon}-1}}
$$

For $t \in\left[-\rho_{\epsilon}-\sqrt{\frac{\epsilon}{r}} \delta_{\epsilon},-\rho_{\epsilon}\right]$,

$$
\frac{-t}{2 \sqrt{1+\tau_{\epsilon}^{2}(t)}} \in\left[\frac{\sqrt{\epsilon} \sqrt{r-\epsilon}}{2}, \frac{\sqrt{r \epsilon}}{2}+\frac{z_{\epsilon}}{\sqrt{\frac{r}{\epsilon}-1}}\right]
$$

Moreover,

$$
\frac{\tau_{\epsilon}(t)}{t} \in\left[-\frac{1}{2 z_{\epsilon}+\sqrt{r^{2}-r \epsilon}}, \frac{1}{\sqrt{r^{2}-r \epsilon}}\right] \quad \text { and } \quad \frac{1}{z_{\epsilon}\left(1+\tau_{\epsilon}^{2}(t)\right)} \in\left[\frac{1-\frac{\epsilon}{r}}{z_{\epsilon}}, \frac{1}{z_{\epsilon}}\right]
$$

so that

$$
\frac{\tau_{\epsilon}(t)}{t}+\frac{1}{z_{\epsilon}\left(1+\tau_{\epsilon}^{2}(t)\right)} \in\left[\frac{1-\frac{\epsilon}{r}}{z_{\epsilon}}-\frac{1}{2 z_{\epsilon}+\sqrt{r^{2}-r \epsilon}}, \frac{1}{z_{\epsilon}}+\frac{1}{\sqrt{r^{2}-r \epsilon}}\right] .
$$

For $\epsilon$ small enough, the lower bound is non-negative, and we can estimate

$$
\left|\AA \AA^{2} \sqrt{\operatorname{det} g}\right|_{P_{\epsilon}} \leq \frac{\sqrt{r \epsilon}}{2 z_{\epsilon}^{2}}+\frac{2}{z_{\epsilon} \sqrt{\frac{r}{\epsilon}-1}}+\frac{5}{2 \sqrt{r \epsilon}\left(\frac{r}{\epsilon}-1\right)}+\frac{z_{\epsilon}}{r \epsilon\left(\frac{r}{\epsilon}-1\right)^{3 / 2}}
$$

Consequently,

$$
\int_{P_{\epsilon}}|\AA|^{2} \leq \frac{2 \pi r \epsilon}{\sqrt{r^{2}-r \epsilon} z_{\epsilon}}+\frac{8 \pi r \epsilon}{r^{2}-r \epsilon}+\frac{10 \pi r \epsilon z_{\epsilon}}{\left(r^{2}-r \epsilon\right)^{3 / 2}}+\frac{4 \pi r \epsilon z_{\epsilon}^{2}}{\left(r^{2}-r \epsilon\right)^{2}} \xrightarrow{\epsilon \searrow 0} 0
$$

and we conclude that, indeed,

$$
\lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}}|\AA|^{2}=8 \pi
$$

In order to establish that

$$
\lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}}\left|A-\frac{\bar{H}}{2} g\right|^{2}=24 \pi
$$

where $\bar{H}=\frac{1}{\operatorname{vol}_{2}\left(\Sigma_{\epsilon}\right)} \int_{\Sigma_{\epsilon}} H$, we proceed in a completely analogous manner. First, on $P_{\epsilon}$, we get

$$
\begin{aligned}
& \int_{P_{\epsilon}} H^{2}= 2 \pi \int_{-\sqrt{r \epsilon}}^{-\sqrt{r \epsilon}} \frac{2 z_{\epsilon}}{\sqrt{\frac{r}{\epsilon}-1}} \\
& \sqrt{1+\tau_{\epsilon}^{2}(t)}\left(-\frac{\tau_{\epsilon}(t)}{t}+\frac{1}{z_{\epsilon}\left(1+\tau_{\epsilon}^{2}(t)\right)}\right)^{2} d t \\
& \leq \frac{4 \pi \epsilon \sqrt{r}}{\sqrt{r-\epsilon}}\left(\frac{1}{z_{\epsilon}}+\frac{2}{\sqrt{r^{2}-r \epsilon}+2 z_{\epsilon}}+\frac{1}{\sqrt{r^{2}-r \epsilon}}+\frac{z_{\epsilon}}{\left(\sqrt{r^{2}-r \epsilon}+2 z_{\epsilon}\right)^{2}}\right. \\
&\left.+\frac{2 z_{\epsilon}}{\left(\sqrt{r^{2}-r \epsilon}+2 z_{\epsilon}\right) \sqrt{r^{2}-r \epsilon}}+\frac{z_{\epsilon}^{2}}{\left(\sqrt{r^{2}-r \epsilon}+2 z_{\epsilon}\right)^{2} \sqrt{r^{2}-r \epsilon}}\right)
\end{aligned}
$$

$$
\xrightarrow{\epsilon \searrow 0} 0 .
$$

Since $\lim _{\epsilon \searrow 0} \operatorname{vol}_{2}\left(P_{\epsilon}\right)=0$, this also implies

$$
\lim _{\epsilon \searrow 0}\left|\int_{P_{\epsilon}} H\right| \leq \lim _{\epsilon \searrow 0}\left(\operatorname{vol}_{2}\left(P_{\epsilon}\right) \int_{P_{\epsilon}} H^{2}\right)^{1 / 2}=0
$$

Next, on $S_{\epsilon}^{i / o}$, we calculate

$$
\begin{aligned}
\int_{S_{\epsilon}^{i}} H & =-4 \pi\left(z_{\epsilon}-\left(c_{\epsilon}-r\right)\right)=-4 \pi\left(2 z_{\epsilon}+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right) \xrightarrow{\epsilon \geq_{0}}-8 \pi r, \\
\int_{S_{\epsilon}^{i}} H^{2} & =\frac{8 \pi}{r}\left(z_{\epsilon}-\left(c_{\epsilon}-r\right)\right)=\frac{8 \pi}{r}\left(2 z_{\epsilon}+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right) \xrightarrow{\epsilon \searrow 0} 16 \pi, \\
\int_{S_{\epsilon}^{O}} H & =4 \pi\left(z_{\epsilon}-\left(c_{\epsilon}-r-\delta_{\epsilon}\right)\right) \\
& =4 \pi\left(2 z_{\epsilon}\left(1+\frac{1}{\sqrt{1-\frac{\epsilon}{r}}}\right)+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right) \xrightarrow{\epsilon \geq 0} 8 \pi r
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{S_{\epsilon}^{O}} H^{2} & =\frac{8 \pi}{r+\delta_{\epsilon}}\left(z_{\epsilon}-\left(c_{\epsilon}-r-\delta_{\epsilon}\right)\right) \\
& =\frac{8 \pi}{r+\frac{2 z_{\epsilon}}{\sqrt{1-\frac{\epsilon}{r}}}}\left(2 z_{\epsilon}\left(1+\frac{1}{\sqrt{1-\frac{\epsilon}{r}}}\right)+r\left(1+\sqrt{1-\frac{\epsilon}{r}}\right)\right) \xrightarrow{\epsilon \searrow 0} 16 \pi
\end{aligned}
$$

where we have used that, on a sphere of radius $R, H=2 / R$ and $\sqrt{\operatorname{det} g}=R$. Finally, in view of $\left.H\right|_{C_{\epsilon}}=0$, we conclude that

$$
\lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}} H=0 \quad \text { and } \quad \lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}} H^{2}=32 \pi
$$

Thus

$$
\begin{aligned}
\lim _{\epsilon \searrow 0} \int_{\Sigma_{\epsilon}}\left|A-\frac{\bar{H}}{2} g\right|^{2} & =\lim _{\epsilon \searrow 0}\left(\int_{\Sigma_{\epsilon}}|\stackrel{\circ}{A}|^{2}+\frac{1}{2} \int_{\Sigma_{\epsilon}} H^{2}-\frac{1}{2} \bar{H} \int_{\Sigma_{\epsilon}} H\right) \\
& =24 \pi
\end{aligned}
$$

as claimed. This establishes (4.13c). As in Section 2, a standard smoothing argument on the generating curves of the family $\Sigma_{\epsilon}$ yields the full statement of Proposition 4.6.

Remark 4.7. One might be tempted to think that, by the same technique, we could attach even more spheres inside the ones just constructed in order to obtain a bigger quotient in (4.13c). However, a quick inspection reveals that

$$
\lim _{\epsilon \searrow 0} \frac{\int_{\Sigma_{\epsilon}^{N}}\left|A-\frac{1}{2 \operatorname{vol}_{2}\left(\Sigma_{\epsilon}^{N}\right)}\left(\int_{\Sigma_{\epsilon}^{N}} H\right) g\right|^{2}}{\int_{\Sigma_{\epsilon}^{N}}|\AA|^{2}}= \begin{cases}2, & N \text { odd } \\ 2+\frac{1}{N-1}, & N \text { even }\end{cases}
$$

if we arrange that $\Sigma_{\epsilon}^{N}$ converges to $N \geq 2$ copies of a sphere $S_{0}$ of radius $r$.

## APPENDIX A

## A few small lemmas

This appendix contains three little results that were used in the thesis but did not really fit anywhere else. In particular, we feel that stating and proving them at the locations where they were used would have broken too much the current train of thoughts. Also, we think that they might be interesting on their own right.

## Contents

1. A Morrey-type estimate
2. On the restriction of the second fundamental form to a linear subspace
3. On the variation of the Gauss map along a curve in a convex hypersurface

## 1. A Morrey-type estimate

In this section we want to show the following variation of Morrey's embedding theorem, valid for $W^{1, p}$-functions $(p>n)$ on the open ball $B_{R}(0)$ of radius $R>0$ around the origin in $\mathbb{R}^{n}(n \geq 1)$.
Lemma A.1. If $R>0, n<p \leq \infty, u \in W^{1, p}\left(B_{R}(0)\right)$ and $\alpha=1-n / p$, then there is a constant $C$, depending only on $n$ and $p$, such that

$$
\sup _{\substack{x, y \in B_{R}(0) \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C\|D u\|_{L^{p}\left(B_{R}(0)\right)}
$$

Proof. Since $W^{1, p} \cap C^{\infty}$ is dense in $W^{1, p}$ for all $p$ and on any domain (cf., e.g., [Eva98, Thm.2, §5.3.2, p.251]), we will henceforth assume that $u \in W^{1, p}\left(B_{R}(0)\right) \cap$ $C^{\infty}\left(B_{R}(0)\right)$. The usual Morrey estimate is as follows (see, e.g., Corollary IX. 14 on p. 168 in [Bre83]):

$$
\sup _{x \in B_{R}(0)}|u(x)|+\sup _{\substack{x, y \in B_{R}(0) \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq \bar{C}_{R}\left(\|u\|_{L^{p}\left(B_{R}(0)\right)}+\|D u\|_{L^{p}\left(B_{R}(0)\right)}\right)
$$

where $\bar{C}_{R}$ is a constant independent of $u$, but depending on the radius $R$ of the ball under consideration (as well as on $n$ and $p$ ). The Lemma therefore states that the Hölder semi-norm of $u$ can be bounded only by the $L^{p}$ norm of its derivative, and
that this can be done independently of the size of the ball $u$ is defined on. The proof of this is done by a scaling argument.

Set $v(x)=u(R x)$. Then $v \in W^{1, p}\left(B_{1}(0)\right) \cap C^{\infty}\left(B_{1}(0)\right)$. Moreover, we have that

$$
\begin{aligned}
\sup _{x \in B_{1}(0)}|v(x)| & =\sup _{x \in B_{R}(0)}|u(x)|, \\
\sup _{\substack{x, y \in B_{1}(0) \\
x \neq y}} \frac{|v(x)-v(y)|}{|x-y|^{\alpha}} & =\sup _{\substack{x, y \in B_{1}(0) \\
x \neq y}} \frac{|u(R x)-u(R y)|}{|R x-R y|^{\alpha}} R^{\alpha} \\
& =R^{\alpha} \sup _{\substack{x, y \in B_{R}(0) \\
x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}, \\
\|v\|_{L^{p}\left(B_{1}(0)\right)} & =\left(\int_{B_{1}(0)}|v(x)|^{p} d x\right)^{\frac{1}{p}} \\
& =\left(\frac{1}{R^{n}} \int_{B_{1}(0)}|u(R x)|^{p} d(R x)\right)^{\frac{1}{p}} \\
& =R^{-\frac{n}{p}}\|u\|_{L^{p}\left(B_{R}(0)\right)}, \\
\|D v\|_{L^{p}\left(B_{1}(0)\right)} & =\left(\int_{B_{1}(0)}^{\left.|D v(x)|^{p} d x\right)^{\frac{1}{p}}}\right. \\
& =(\frac{1}{R^{n}} \int_{B_{1}(0)} \underbrace{|D(u(R x))|^{p}} d(R x))^{\frac{1}{p}} \\
& =\left(R^{p-n} \int_{B_{R}(0)}|D u(x)|^{p} d x\right)^{\frac{1}{p}} \\
& =R^{1-\frac{n}{p}}\|D u\|_{L^{p}\left(B_{R}(0)\right)}
\end{aligned}
$$

Now, denoting by $\widetilde{\omega}_{n}=\operatorname{vol}_{n}\left(B_{1}(0)\right)$, we let

$$
\begin{aligned}
& \widetilde{u}(x)=u(x)-\frac{1}{\widetilde{\omega}_{n} R^{n}} \int_{B_{R}(0)} u\left(x^{\prime}\right) d x^{\prime} \\
& \widetilde{v}(x)=v(x)-\frac{1}{\widetilde{\omega}_{n}} \int_{B_{1}(0)} v\left(x^{\prime}\right) d x^{\prime}
\end{aligned}
$$

Notice that $\widetilde{u} \in W^{1, p}\left(B_{R}(0)\right) \cap C^{\infty}\left(B_{R}(0)\right)$ and $\widetilde{v} \in W^{1, p}\left(B_{1}(0)\right) \cap C^{\infty}\left(B_{1}(0)\right)$. Thus, the usual Morrey embedding yields:

$$
\sup _{\substack{x, y \in B_{1}(0) \\ x \neq y}} \frac{|\widetilde{v}(x)-\widetilde{v}(y)|}{|x-y|^{\alpha}} \leq \bar{C}_{1}\left(\|\widetilde{v}\|_{L^{p}\left(B_{1}(0)\right)}+\|D \widetilde{v}\|_{L^{p}\left(B_{1}(0)\right)}\right)
$$

where the constant $\bar{C}_{1}$ is independent of $\widetilde{v}$ and of $R$. By construction, $\widetilde{v}(x)-\widetilde{v}(y)=$ $v(x)-v(y)$ and $D \widetilde{v}=D v$. Moreover, by our considerations above, $\|\widetilde{v}\|_{L^{p}\left(B_{1}(0)\right)}=$ $R^{-n / p}\|\widetilde{u}\|_{L^{p}\left(B_{R}(0)\right)}$, since $\widetilde{v}(x)=\widetilde{u}(R x)$. Applying the Poincaré Lemma for balls (cf., e.g., [Eva98, Thm.2, §5.8.1, p.276]), we obtain $\|\widetilde{u}\|_{L^{p}\left(B_{R}(0)\right)} \leq C_{P} R\|D u\|_{L^{p}\left(B_{R}(0)\right)}$, where $C_{P}$ is independent of $u$ and $R$. Putting all this together, we finally arrive at:

$$
R^{\alpha} \sup _{\substack{x, y \in B_{R}(0) \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq \bar{C}_{1}\left(1+C_{P}\right) R^{1-\frac{n}{p}}\|D u\|_{L^{p}\left(B_{R}(0)\right)}
$$

which is the desired inequality, since $\alpha=1-n / p$.

## 2. On the restriction of the second fundamental form to a linear subspace

In this section we wish to prove that the second fundamental form of the intersection of a convex hypersurface with a linear subspace is controlled in each point by the second fundamental form of the hypersurface and the angle between the Gauss map of the hypersurface and the linear subspace.

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set with smooth boundary $\Sigma=$ $\partial \Omega$. Assume we are given a $k$-dimensional $(k \in\{2, \ldots, n\})$ linear subspace $\Pi_{x}^{k}$ of $\mathbb{R}^{n+1}$ passing through $x \in \Omega$. Set $\bar{\Sigma}=\Sigma \cap \Pi_{x}^{k}$, and let $\nu$ and $\bar{\nu}$ denote the Gauss-maps of $\Sigma$ in $\mathbb{R}^{n+1}$ and $\bar{\Sigma}$ in $\Pi_{x}^{k}$, respectively. Also, let $A$ and $\bar{A}$ be the second fundamental forms of $\Sigma$ in $\mathbb{R}^{n+1}$ and $\bar{\Sigma}$ in $\Pi_{x}^{k}$, respectively. Finally, for each $q \in \bar{\Sigma} \subset \Sigma$, let $\alpha(q) \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ denote the angle between $\nu$ in $q$ and $\Pi_{x}^{k}$. We have $\cos \alpha(q)=\langle\nu(q), \bar{\nu}(q)\rangle_{\mathbb{R}^{n+1}}$, where $\langle\cdot, \cdot\rangle_{\mathbb{R}^{n+1}}$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Moreover, $\cos \alpha(q) \neq 0$ (i.e., $\alpha(q) \notin\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ ), since $x$ was an inner point of $\Omega$. The following holds.

Lemma A.2. In every point $q \in \bar{\Sigma}$, we have

$$
\left|\bar{A}_{q}\right| \leq \frac{1}{\cos \alpha(q)}\left|A_{q}\right|
$$

Proof. Without loss of generality, we may assume that $x=0$. Since all the considerations that follow are valid pointwise, we will drop subscripts or other references to the point at hand. Notice that, in each point $q \in \bar{\Sigma}=\Sigma \cap \Pi_{0}^{k}$, we can write

$$
\nu=\cos \alpha \bar{\nu}+\nu^{\perp}
$$

where $\nu^{\perp} \in\left(\Pi_{0}^{k}\right)^{\perp}$ is a vector in the orthogonal complement of $\Pi_{0}^{k}$. Let $\bar{X}$ and $\bar{Y}$ be two arbitrary vector fields tangent to $\bar{\Sigma}$ and extended, first to vector fields tangent to $\Sigma$, and then to vector fields in $\mathbb{R}^{n+1}$, each time in a relative neighbourhood of $\bar{\Sigma}$. Denoting by $D$ the standard derivation in $\mathbb{R}^{n+1}$, we have, in every point of $\bar{\Sigma}$ :

$$
\begin{aligned}
A(\bar{X}, \bar{Y}) & =-\left\langle D_{\bar{X}} \bar{Y}, \nu\right\rangle_{\mathbb{R}^{n+1}}=-\cos \alpha\left\langle D_{\bar{X}} \bar{Y}, \bar{\nu}\right\rangle_{\mathbb{R}^{n+1}}-\left\langle D_{\bar{X}} \bar{Y}, \nu^{\perp}\right\rangle_{\mathbb{R}^{n+1}} \\
& =-\cos \alpha\left\langle D_{\bar{X}} \bar{Y}, \bar{\nu}\right\rangle_{\mathbb{R}^{n+1}}=\cos \alpha \bar{A}(\bar{X}, \bar{Y})
\end{aligned}
$$

where we have used that $D_{\bar{X}} \bar{Y} \in \Pi_{0}^{k}$ if $\bar{X}, \bar{Y} \in \Pi_{0}^{k}$.
Now fix $q \in \bar{\Sigma}$. If we choose an orthonormal basis $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$ in $T_{q} \Sigma$ such that $\left(\epsilon_{1}, \ldots, \epsilon_{k}\right)$ is an orthonormal basis in $T_{q} \bar{\Sigma}$ that diagonalises $\bar{A}$, we have

$$
\begin{aligned}
|\bar{A}|^{2} & =\sum_{i=1}^{k}\left(\bar{A}\left(\epsilon_{i}, \epsilon_{i}\right)\right)^{2}=\frac{1}{\cos ^{2} \alpha} \sum_{i=1}^{k}\left(A\left(\epsilon_{i}, \epsilon_{i}\right)\right)^{2} \\
& \leq \frac{1}{\cos ^{2} \alpha} \sum_{j=1}^{n} \sum_{i=1}^{k}\left(A\left(\epsilon_{i}, \epsilon_{j}\right)\right)^{2} \leq \frac{1}{\cos ^{2} \alpha} \sum_{i, j=1}^{n}\left(A\left(\epsilon_{i}, \epsilon_{j}\right)\right)^{2}=\frac{1}{\cos ^{2} \alpha}|A|^{2}
\end{aligned}
$$

The lemma follows.

## 3. On the variation of the Gauss map along a curve in a convex hypersurface

In this section we establish that the difference of the Gauss map of a convex hypersurface $\Sigma$ between two points can be estimated in magnitude from above by the integral of the largest principal curvature of $\Sigma$ along any curve in $\Sigma$ joining those two points.

The following considerations are valid for any dimension $n \geq 1 .\langle\cdot, \cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$, and Greek indices refer to components in the usual basis of $\mathbb{R}^{n+1}$. For an immersion $f: U \subset \mathbb{R}^{n} \rightarrow \mathbb{R}^{n+1}$, Latin indices refer to components in the induced coordinate basis and $D$ denotes differentiation with respect to this basis.

Assume $\Sigma=\partial \Omega \subset \mathbb{R}^{n+1}$ is a smooth hypersurface, where $\Omega$ is an open convex domain. Since $\Omega$ is convex, we can parametrise $\Sigma$ by $S^{n}$ (through projection). Moreover (using stereographic projection), we can pick a point $S \in \Sigma$ such that there is a set $U \subset \mathbb{R}^{n}$ and an embedding $f: U \rightarrow \mathbb{R}^{n+1}$ with $f(U)=\Sigma \backslash\{S\}$. Let $g$ denote the Riemannian metric of $\Sigma$. Let $A$ denote the second fundamental form of $\Sigma$ with eigenvalues $\lambda_{n} \geq \ldots \geq \lambda_{1} \geq 0$ and corresponding orthonormal eigenframe $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right)$ (i.e. $\left.\sum_{j=1}^{n} A^{i}{ }_{j}\left(\epsilon_{l}\right)^{j}=\lambda_{l}\left(\epsilon_{l}\right)^{i}\right)$. Finally, let $\nu: U \rightarrow \mathbb{R}^{n+1}$ be the Gauss map which associates to each coordinate value $x$ the outer unit normal vector to $\Sigma$ based at $f(x)$. Then the following is true:

Lemma A.3. For any points $q_{1}, q_{2} \in U$ and any path $\gamma:[0,1] \rightarrow U$ joining them (i.e. with $\gamma(0)=q_{1}$ and $\gamma(1)=q_{2}$ ), we have that

$$
\left|\nu\left(q_{2}\right)-\nu\left(q_{1}\right)\right| \leq \int_{\gamma} \lambda_{n}
$$

Taking into consideration the pointwise inequality

$$
\lambda_{n}=\sqrt{\lambda_{n}^{2}} \leq \sqrt{\sum_{j=1}^{n} \lambda_{j}^{2}}=|A|
$$

we also have

Corollary A.4. If $U, q_{1}, q_{2}$ and $\gamma$ are as in the Lemma, then

$$
\left|\nu\left(q_{2}\right)-\nu\left(q_{1}\right)\right| \leq \int_{\gamma}|A|
$$

Proof of Lemma A.3. For $q_{1}, q_{2}$ and $\gamma$ as above, we have

$$
\left(\nu\left(q_{2}\right)\right)^{\alpha}-\left(\nu\left(q_{1}\right)\right)^{\alpha}=\int_{0}^{1} \sum_{i=1}^{n}\left(\left(D_{i} \nu\right)^{\alpha} \circ \gamma(t)\right) \dot{\gamma}^{i}(t) d t
$$

Thus

$$
\left|\nu\left(q_{2}\right)-\nu\left(q_{1}\right)\right| \leq \int_{0}^{1}\left|\sum_{i=1}^{n}\left(\left(D_{i} \nu\right) \circ \gamma(t)\right) \dot{\gamma}^{i}(t)\right| d t .
$$

Consider for any $i, l \in\{1, \ldots, n\}$ the pointwise equality

$$
\begin{aligned}
\left\langle D_{i} f, \sum_{j=1}^{n}\left(\epsilon_{l}\right)^{j} D_{j} \nu\right\rangle & =-\sum_{j=1}^{n} A_{i j}\left(\epsilon_{l}\right)^{j}=-\sum_{j, k=1}^{n} g_{i k} A_{j}^{k}\left(\epsilon_{l}\right)^{j}=-\sum_{k=1}^{n} g_{i k}\left(\lambda_{l}\right)\left(\epsilon_{l}\right)^{k} \\
& =\left\langle D_{i} f,-\sum_{k=1}^{n}\left(\lambda_{l}\right)\left(\epsilon_{l}\right)^{k} D_{k} f\right\rangle
\end{aligned}
$$

Since the normal part of $D_{j} \nu$ vanishes $\left(0=D_{j} 1=D_{j}\langle\nu, \nu\rangle=2\left\langle\nu, D_{j} \nu\right\rangle\right)$, we obtain

$$
\sum_{j=1}^{n}\left(\epsilon_{l}\right)^{j} D_{j} \nu=\sum_{k=1}^{n}\left(-\lambda_{l}\right)\left(\epsilon_{l}\right)^{k} D_{k} f
$$

Expanding $\dot{\gamma}$ in the orthonormal eigenbasis $\left(\epsilon_{l}\right)_{l \in\{1, \ldots, n\}}$ of $A, \dot{\gamma}^{i}=\sum_{l, r, s=1}^{n} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{i}$, we obtain

$$
\begin{aligned}
\left|\sum_{i=1}^{n} \dot{\gamma}^{i} D_{i} \nu\right|^{2} & =\sum_{i, j=1}^{n}\left\langle\dot{\gamma}^{i} D_{i} \nu, \dot{\gamma}^{j} D_{j} \nu\right\rangle=\sum_{i, j, l, r, s=1}^{n} \dot{\gamma}^{j}\left\langle g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{i} D_{i} \nu, D_{j} \nu\right\rangle \\
& =\sum_{j, k, l, r, s=1}^{n} \dot{\gamma}^{j}\left\langle\left(-\lambda_{l}\right) g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{k} D_{k} f, D_{j} \nu\right\rangle \\
& =\sum_{j, k, l, r, s=1}^{n} \dot{\gamma}^{j}\left(\lambda_{l}\right) g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{k}\left(-\left\langle D_{k} f, D_{j} \nu\right\rangle\right) \\
& =\sum_{j, k, l, r, s=1}^{n} \dot{\gamma}^{j}\left(\lambda_{l}\right) g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{k} A_{j k} \\
& =\sum_{j, k, l, r, s=1}^{n} \dot{\gamma}^{j}\left(\lambda_{l}\right)^{2} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s}\left(\epsilon_{l}\right)^{k} g_{j k} \\
& =\sum_{j, k, l, r, s=1}^{n}\left(\lambda_{l}\right)^{2} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s} g_{j k} \dot{\gamma}^{j}\left(\epsilon_{l}\right)^{k}=\sum_{l=1}^{n}\left(\lambda_{l}\right)^{2}\left(g\left(\dot{\gamma}, \epsilon_{l}\right)\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \leq \sum_{l=1}^{n}\left(\lambda_{n}\right)^{2}\left(g\left(\dot{\gamma}, \epsilon_{l}\right)\right)^{2} \\
& =\left(\lambda_{n}\right)^{2} \sum_{j, k, l, r, s=1}^{n} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s} g_{j k} \dot{\gamma}^{j}\left(\epsilon_{l}\right)^{k} \\
& =\left(\lambda_{n}\right)^{2} \sum_{j, k, r, s=1}^{n} \sum_{l=1}^{n} \sum_{l^{\prime}=1}^{n} \delta_{l l^{\prime}} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s} g_{j k} \dot{\gamma}^{j}\left(\epsilon_{l^{\prime}}\right)^{k} \\
& =\left(\lambda_{n}\right)^{2} \sum_{j, k, r, s=1}^{n} \sum_{l=1}^{n} \sum_{l^{\prime}=1}^{n} g\left(\epsilon_{l}, \epsilon_{l^{\prime}}\right) g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s} g_{j k} \dot{\gamma}^{j}\left(\epsilon_{l^{\prime}}\right)^{k} \\
& =\left(\lambda_{n}\right)^{2} g\left(\sum_{l, r, s=1}^{n} g_{r s} \dot{\gamma}^{r}\left(\epsilon_{l}\right)^{s} \epsilon_{l}, \sum_{j, k, l^{\prime}=1}^{n} g_{j k} \dot{\gamma}^{j}\left(\epsilon_{l^{\prime}}\right)^{k} \epsilon_{l^{\prime}}\right) \\
& =\left(\lambda_{n}\right)^{2} g(\dot{\gamma}, \dot{\gamma}),
\end{aligned}
$$

where the inequality follows from $0 \leq \lambda_{1} \leq \ldots \leq \lambda_{n}$. Consequently,

$$
\left|\sum_{i=1}^{n} \dot{\gamma}^{i} D_{i} \nu\right| \leq \lambda_{n}\|\dot{\gamma}\|_{g}=\lambda_{n} \sqrt{g(\dot{\gamma}, \dot{\gamma})}
$$

This yields

$$
\left|\nu\left(q_{2}\right)-\nu\left(q_{1}\right)\right| \leq \int_{0}^{1}\left(\lambda_{n} \circ \gamma(t)\right)\|\dot{\gamma}(t)\|_{g \circ \gamma(t)} d t=\int_{\gamma} \lambda_{n}
$$

## APPENDIX B

## On the second variation around a spherical cap of some $L^{2}$-integral quantities along a volume-preserving flow

This appendix contains some computations made in a first attempt to combine the results of the present thesis with those of [DLT10]. More precisely, we consider a spherical cap $M$ with (totally-umbilical) boundary $\partial M$, and deform its metric in a volume preserving manner (thus deforming the induced metric on $\partial M$, as well). On $M$, we then consider the first and second variations of the $L^{2}-$ norms of the traceless Ricci tensor, as well as of the difference between the scalar curvature and its mean across $M$. Afterwards, we do the same on $\partial M$ with the $L^{2}$-norms of the traceless Ricci tensor, as well as of the induced quantities $A$ and $H-\bar{H}$ ( $\bar{H}$ denoting the mean of $H$ across $\partial M)$. We end by exposing a simpler situation.

The four involved quantities are precisely those relevant in [DLT10] and in this work.

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Consider a closed subset $M$ of the $(n+1)$-sphere $S_{R}^{n+1}$ with radius $R$, such that its boundary $\partial M$ coincides with an $n$-sphere $S_{r}^{n}$ of radius $r$. We want to look at metric deformations of $S_{R}^{n+1}$ which fix the volume of $M$. So, denoting by $\widehat{\eta}$ the canonical metric of $S_{R}^{n+1}$, we consider the one-parameter family $\widehat{g}_{t}$ of metrics on $S_{R}^{n+1}$ which satisfies (at least locally around $t=0$ )

$$
\left\{\begin{array}{l}
\partial_{t} \widehat{g}_{t}=\widehat{h}, \\
\widehat{g}_{0}=\widehat{\eta},
\end{array}\right.
$$

where $\widehat{h}$ denotes a symmetric bilinear form on $S_{R}^{n+1}$ such that

$$
\begin{equation*}
\int_{M} \operatorname{tr}_{\widehat{g_{t}}} \widehat{h} d \operatorname{vol}_{\widehat{g_{t}}}=0 \tag{B.1}
\end{equation*}
$$

Let $g_{t}, \eta$ and $h$ denote the restrictions of $\widehat{g}_{t}, \widehat{\eta}$ and $\widehat{h}$ to $M$, respectively. Let $\widetilde{g}_{t}$ and $\widetilde{\eta}$ denote the metrics induced by $g_{t}$ and $\eta$ on $\partial M$, respectively. In what follows, we wish to calculate the second variation with respect to the preceding flow of the following $L^{2}$-integral quantities on $M$ and $\partial M$ which are derived from the respective metrics $g_{t}$ and $\widetilde{g}_{t}$ and which are critical at $t=0$.

$$
\begin{array}{ll}
\int_{M}\|\mathrm{Ric}\|_{g}^{2} & \text { (see eqs. (B.19) and (B.42)), } \\
\int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2} & \text { (see eqs. (B.21) and (B.43)), } \\
\int_{\partial M}\|\AA\|_{\partial}^{2} & \text { (see eqs. (B.38)) and (B.44)), } \\
\int_{\partial M}(H-\bar{H})^{2} & \text { (see eqs. (B.40)) and (B.45)), } \\
\int_{\partial M}\|\mathrm{Ric}\|_{g}^{2} & \text { (see eqs. (B.41) and (B.46)). }
\end{array}
$$

We start by setting up our notations and conventions.

## 1. Notations and conventions

For simplicity, we will suppress any explicit reference to the dependence of a quantity on $t$. The following notations are used for some quantities derived from the metric $g$ :

| $\Gamma$ | Christoffel symbols of the second kind |
| ---: | :--- |
| Riem | Riemann tensor |
| Ric | Ricci tensor |
| Ric | Traceless part of Ric |
| Scal | Scalar curvature |
| Vol | Volume of $M$ with respect to $g$ |
| $\nabla$ | Levi-Civita connection |
| $\Delta$ | Laplace-Beltrami operator with respect to $\nabla$ |
| div | Covariant divergence operator acting on symmetric two-tensor fields |
|  | or vector fields |
| $\partial$ | Coordinate derivative or coordinate vector fields |

Here, we choose the following sign convention for the Riemannian curvature tensor:

$$
\operatorname{Riem}(X, Y) Z=\nabla_{Y} \nabla_{X} Z-\nabla_{X} \nabla_{Y} Z-\nabla_{\nabla_{Y} X-\nabla_{X} Y} Z
$$

where $X, Y$ and $Z$ are vector fields on $M$. This way, the $(n+1)$-sphere of radius $R$ has positive scalar curvature

$$
\mathrm{Scal}_{\hat{\eta}}=\frac{n(n+1)}{R^{2}}
$$

The boundary $\partial M$ of $M$ will be viewed as an embedded submanifold of $M$, its induced metric will be denoted by $\widetilde{g}$, and the quantities from above will be equipped
with a tilde to designate those respective to $(\partial M, \widetilde{g})$. Furthermore, whenever we work in local coordinates $x^{0}, \ldots, x^{n}$ near the boundary, we shall assume that $x_{0}$ vanishes on $\partial M$. Greek indices will run from 0 to $n$, whereas Latin ones will run from 1 to $n$. Indices will always be raised and lowered using the ambient metric $g$ and summation over repeated ones shall be understood implicitly. We introduce the following notation for the extrinsic quantities inferred from the embedding of $\partial M$ into $M$ :
$\nu \quad$ Outer unit normal on $\partial M$
$A \quad$ Second fundamental form
$\AA$ Traceless part of $A$
$H$ Mean curvature
Here, we choose the following sign convention for the second fundamental form:

$$
A(\widetilde{X}, \widetilde{Y})=-g\left(\nabla_{\widetilde{X}} \widetilde{Y}, \nu\right)
$$

where $\widetilde{X}$ and $\widetilde{Y}$ are vector fields on $\partial M$ (extended to a neighbourhood of $\partial M$ in $M$ ). (Notice that in the case of a subset $\Omega$ of $\mathbb{R}^{n+1}$ with smooth boundary $\partial \Omega$, this sign convention for the second fundamental form of the boundary ensures the equivalence of its positive definiteness with the convexity of the enclosed volume $\Omega$.)

Next, we introduce some notation for two-tensor fields $B$ and $C$ on $M$ :
$B: C \quad$ Full contraction of $B$ and $C$ with respect to $g$, i.e. $B: C=B^{\mu \nu} C_{\mu \nu}$
$\operatorname{tr}_{g} B \quad$ Trace of $B$ with respect to $g$, i.e. $\operatorname{tr}_{g} B=g: B$
$\|B\|_{g} \quad$ Norm of $B$ with respect to $g$, i.e. $\|B\|_{g}^{2}=B: B$
Similarly, we define for two-tensor fields $\widetilde{B}$ and $\widetilde{C}$ on $\partial M$ :
$\widetilde{B}: \widetilde{C} \quad$ Full contraction of $\widetilde{B}$ and $\widetilde{C}$ with respect to $\widetilde{g}$,
i.e. $\widetilde{B}: \widetilde{C}=\widetilde{B}_{i j}\left(\widetilde{g}^{-1}\right)^{i k}\left(\widetilde{g}^{-1}\right)^{j l} \widetilde{C}_{k l}$
$\operatorname{tr} \widetilde{q} \widetilde{B} \quad$ Trace of $\widetilde{B}$ with respect to $\widetilde{g}$, i.e. $\operatorname{tr} \widetilde{q} \widetilde{B}=\widetilde{g}: \widetilde{B}$
$\|\stackrel{g}{B}\|_{\tilde{g}} \quad$ Norm of $\widetilde{B}$ with respect to $\widetilde{g}$, i.e. $\|\stackrel{g}{B}\|_{\tilde{g}}^{2}=\widetilde{B}: \widetilde{B}$
Finally, the following integral notations will be used for the average of (smooth) functions $f$ over $(M, g)$ and $\varphi$ over $(\partial M, \widetilde{g})$, respectively:

$$
\begin{aligned}
& \bar{f}=f_{M} f d \mathrm{vol}_{g}=\frac{1}{\mathrm{Vol}} \int_{M} f d \mathrm{vol}_{g}, \\
& \bar{\varphi}=f_{\partial M} \varphi d \mathrm{vol}_{g}=\frac{1}{\widetilde{\mathrm{Vol}}} \int_{\partial M} \varphi d \mathrm{vol}_{g} .
\end{aligned}
$$

## 2. The first and second variations of the quantities on $M$

We start from the evolution of $g$ :

$$
\begin{equation*}
\partial_{t} g_{\mu \nu}=h_{\mu \nu} \tag{B.2}
\end{equation*}
$$

Then, denoting by $\delta$ the Kronecker delta,

$$
\begin{aligned}
0 & =\partial_{t} \delta_{\lambda}^{\mu}=\partial_{t}\left(g^{\mu \nu} g_{\nu \lambda}\right) \stackrel{(\mathrm{B} .2)}{=}\left(\partial_{t} g^{\mu \nu}\right) g_{\nu \lambda}+g^{\mu \nu} h_{\nu \lambda} \\
\text { (B.3) } \quad \Longrightarrow \quad \partial_{t} g^{\mu \nu} & =-h^{\mu \nu} .
\end{aligned}
$$

Also,

$$
\begin{align*}
& \partial_{t} \sqrt{\operatorname{det} g}=\frac{\partial_{t} \operatorname{det} g}{2 \sqrt{\operatorname{det} g}}=\frac{\operatorname{det} g}{2 \sqrt{\operatorname{det} g}} \operatorname{tr}_{g}\left(\partial_{t} g\right) \\
& \text { (B.2) } \\
&=\frac{1}{2} \operatorname{tr}_{g} h \sqrt{\operatorname{det} g} \tag{B.4}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\partial_{t} \mathrm{Vol}=\partial_{t} \int_{M} d \mathrm{vol}_{g} \stackrel{(\mathrm{~B} .4)}{=} \frac{1}{2} \int_{M} \operatorname{tr}_{g} h d \mathrm{vol}_{g} \stackrel{(\mathrm{~B} .1)}{=} 0 . \tag{B.5}
\end{equation*}
$$

Furthermore,

$$
\begin{aligned}
& \partial_{t} \Gamma_{\alpha \beta}^{\sigma}= \partial_{t}\left(\frac{1}{2} g^{\sigma \tau}\left(\partial_{\alpha} g_{\tau \beta}+\partial_{\beta} g_{\tau \alpha}-\partial_{\tau} g_{\alpha \beta}\right)\right) \\
& \text { (В.2),(В.3) } \\
&=-\frac{1}{2} h^{\sigma \tau}\left(\partial_{\alpha} g_{\tau \beta}+\partial_{\beta} g_{\tau \alpha}-\partial_{\tau} g_{\alpha \beta}\right)+\frac{1}{2} g^{\sigma \tau}\left(\partial_{\alpha} h_{\tau \beta}+\partial_{\beta} h_{\tau \alpha}-\partial_{\tau} h_{\alpha \beta}\right) \\
&= \frac{1}{2} g^{\sigma \tau}\left(\nabla_{\alpha} h_{\tau \beta}+\nabla_{\beta} h_{\tau \alpha}-\nabla_{\tau} h_{\alpha \beta}\right) \\
&+\frac{1}{2} g^{\sigma \tau}\left(\Gamma_{\alpha \tau}^{\rho} h_{\rho \beta}+\Gamma_{\alpha \beta}^{\rho} h_{\tau \rho}+\Gamma_{\beta \tau}^{\rho} h_{\rho \alpha}+\Gamma_{\beta \alpha}^{\rho} h_{\tau \rho}-\Gamma_{\tau \alpha}^{\rho} h_{\rho \beta}-\Gamma_{\tau \beta}^{\rho} h_{\alpha \rho}\right) \\
&-\frac{1}{2} h_{\rho}^{\sigma} g^{\rho \tau}\left(\partial_{\alpha} g_{\tau \beta}+\partial_{\beta} g_{\tau \alpha}-\partial_{\tau} g_{\alpha \beta}\right) \\
&= \frac{1}{2} g^{\sigma \tau}\left(\nabla_{\alpha} h_{\tau \beta}+\nabla_{\beta} h_{\tau \alpha}-\nabla_{\tau} h_{\alpha \beta}\right)+h^{\sigma}{ }_{\rho} \Gamma_{\alpha \beta}^{\rho}-h^{\sigma}{ }_{\rho} \Gamma_{\alpha \beta}^{\rho} \\
& \text { (B.6) }= \frac{1}{2} g^{\sigma \tau}\left(\nabla_{\alpha} h_{\tau \beta}+\nabla_{\beta} h_{\tau \alpha}-\nabla_{\tau} h_{\alpha \beta}\right) .
\end{aligned}
$$

From this,

$$
\begin{aligned}
& \partial_{t} \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\sigma}= \partial_{t}\left(\partial_{\beta} \Gamma_{\alpha \gamma}^{\sigma}-\partial_{\alpha} \Gamma_{\beta \gamma}^{\sigma}+\Gamma_{\beta \mu}^{\sigma} \Gamma_{\alpha \gamma}^{\mu}-\Gamma_{\alpha \mu}^{\sigma} \Gamma_{\beta \gamma}^{\mu}\right) \\
&= \partial_{\beta} \partial_{t} \Gamma_{\alpha \gamma}^{\sigma}-\partial_{\alpha} \partial_{t} \Gamma_{\beta \gamma}^{\sigma} \\
&+\left(\partial_{t} \Gamma_{\beta \mu}^{\sigma}\right) \Gamma_{\alpha \gamma}^{\mu}+\Gamma_{\beta \mu}^{\sigma}\left(\partial_{t} \Gamma_{\alpha \gamma}^{\mu}\right)-\left(\partial_{t} \Gamma_{\alpha \mu}^{\sigma}\right) \Gamma_{\beta \gamma}^{\mu}-\Gamma_{\alpha \mu}^{\sigma}\left(\partial_{t} \Gamma_{\beta \gamma}^{\mu}\right) \\
&= \nabla_{\beta} \partial_{t} \Gamma_{\alpha \gamma}^{\sigma}-\nabla_{\alpha} \partial_{t} \Gamma_{\beta \gamma}^{\sigma} \\
&+\left(\partial_{t} \Gamma_{\mu \gamma}^{\sigma}\right) \Gamma_{\beta \alpha}^{\mu}+\left(\partial_{t} \Gamma_{\alpha \mu}^{\sigma}\right) \Gamma_{\beta \gamma}^{\mu}-\left(\partial_{t} \Gamma_{\alpha \gamma}^{\mu}\right) \Gamma_{\beta \mu}^{\sigma} \\
&-\left(\partial_{t} \Gamma_{\mu \gamma}^{\sigma}\right) \Gamma_{\alpha \beta}^{\mu}-\left(\partial_{t} \Gamma_{\beta \mu}^{\sigma}\right) \Gamma_{\alpha \gamma}^{\mu}+\left(\partial_{t} \Gamma_{\beta \gamma}^{\mu}\right) \Gamma_{\alpha \mu}^{\sigma} \\
&+\left(\partial_{t} \Gamma_{\beta \mu}^{\sigma}\right) \Gamma_{\alpha \gamma}^{\mu}+\Gamma_{\beta \mu}^{\sigma}\left(\partial_{t} \Gamma_{\alpha \gamma}^{\mu}\right)-\left(\partial_{t} \Gamma_{\alpha \mu}^{\sigma}\right) \Gamma_{\beta \gamma}^{\mu}-\Gamma_{\alpha \mu}^{\sigma}\left(\partial_{t} \Gamma_{\beta \gamma}^{\mu}\right) \\
&= \nabla_{\beta} \partial_{t} \Gamma_{\alpha \gamma}^{\sigma}-\nabla_{\alpha} \partial_{t} \Gamma_{\beta \gamma}^{\sigma} \\
& \text { (в.6) } \\
&= \frac{1}{2} \nabla_{\beta}\left(g^{\sigma \tau}\left(\nabla_{\alpha} h_{\tau \gamma}+\nabla_{\gamma} h_{\tau \alpha}-\nabla_{\tau} h_{\alpha \gamma}\right)\right) \\
&-\frac{1}{2} \nabla_{\alpha}\left(g^{\sigma \tau}\left(\nabla_{\beta} h_{\tau \gamma}+\nabla_{\gamma} h_{\tau \beta}-\nabla_{\tau} h_{\beta \gamma}\right)\right) \\
&= \frac{1}{2} g^{\sigma \tau}\left(\nabla_{\beta} \nabla_{\alpha} h_{\tau \gamma}-\nabla_{\alpha} \nabla_{\beta} h_{\tau \gamma}+\nabla_{\beta} \nabla_{\gamma} h_{\alpha \tau}\right. \\
&\left.\quad-\nabla_{\alpha} \nabla_{\gamma} h_{\beta \tau}-\nabla_{\beta} \nabla_{\tau} h_{\alpha \gamma}+\nabla_{\alpha} \nabla_{\tau} h_{\beta \gamma}\right) .
\end{aligned}
$$

Applying the definition of the Riemann tensor along with the first Bianchi identities, we obtain:

$$
\begin{align*}
& \partial_{t} \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\sigma}= \frac{1}{2} g^{\sigma \tau}\left(-\operatorname{Riem}_{\alpha \beta \tau}{ }^{\mu} h_{\mu \gamma}-\operatorname{Riem}_{\alpha \beta \gamma}{ }^{\mu} h_{\mu \tau}\right. \\
&+\nabla_{\gamma} \nabla_{\beta} h_{\alpha \tau}-\operatorname{Riem}_{\gamma \beta \alpha}{ }^{\mu} h_{\mu \tau}-\operatorname{Riem}_{\gamma \beta \tau}{ }^{\mu} h_{\alpha \mu} \\
&\left.\quad-\nabla_{\alpha} \nabla_{\gamma} h_{\beta \tau}-\nabla_{\beta} \nabla_{\tau} h_{\alpha \gamma}+\nabla_{\alpha} \nabla_{\tau} h_{\beta \gamma}\right) \\
&=\frac{1}{2} g^{\sigma \tau}\left(\operatorname{Riem}_{\beta \tau \alpha}{ }^{\mu} h_{\mu \gamma}+\operatorname{Riem}_{\tau \alpha \beta}{ }^{\mu} h_{\mu \gamma}-\operatorname{Riem}_{\alpha \beta \gamma}{ }^{\mu} h_{\mu \tau}\right. \\
&+\operatorname{Riem}_{\beta \alpha \gamma}{ }^{\mu} h_{\mu \tau}+\operatorname{Riem}_{\alpha \gamma \beta}{ }^{\mu} h_{\mu \tau}+\operatorname{Riem}_{\beta \gamma \tau}{ }^{\mu} h_{\mu \alpha} \\
&\left.+\nabla_{\gamma} \nabla_{\beta} h_{\tau \alpha}-\nabla_{\alpha} \nabla_{\gamma} h_{\beta \tau}-\nabla_{\beta} \nabla_{\tau} h_{\alpha \gamma}+\nabla_{\alpha} \nabla_{\tau} h_{\beta \gamma}\right) \\
&=\frac{1}{2} g^{\sigma \tau}\left(-2 \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\mu} h_{\mu \tau}+\operatorname{Riem}_{\alpha \gamma \beta}{ }^{\mu} h_{\mu \tau}+\operatorname{Riem}_{\beta \tau \alpha}{ }^{\mu} h_{\mu \gamma}\right. \\
& \quad \operatorname{Riem}_{\alpha \tau \beta}{ }^{\mu} h_{\mu \gamma}-\operatorname{Riem}_{\gamma \beta \tau}{ }^{\mu} h_{\mu \alpha} \\
&\left.\quad-\nabla_{\alpha} \nabla_{\gamma} h_{\beta \tau}-\nabla_{\beta} \nabla_{\tau} h_{\alpha \gamma}+\nabla_{\alpha} \nabla_{\tau} h_{\beta \gamma}+\nabla_{\gamma} \nabla_{\beta} h_{\tau \alpha}\right) . \tag{B.7}
\end{align*}
$$

Consequently,

$$
\partial_{t} \operatorname{Riem}_{\alpha \beta \gamma \delta}=\left(\partial_{t} g_{\delta \sigma}\right) \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\sigma}+g_{\delta \sigma}\left(\partial_{t} \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\sigma}\right)
$$

(B.2),(B.7)

$$
=h_{\delta \sigma} \operatorname{Riem}_{\alpha \beta \gamma}{ }^{\sigma}
$$

$-\operatorname{Riem}_{\alpha \beta \gamma}{ }^{\mu} h_{\mu \delta}+\frac{1}{2} \operatorname{Riem}_{\alpha \gamma \beta}{ }^{\mu} h_{\mu \delta}+\frac{1}{2} \operatorname{Riem}_{\beta \delta \alpha}{ }^{\mu} h_{\mu \gamma}$
$-\frac{1}{2} \operatorname{Riem}_{\alpha \delta \beta}{ }^{\mu} h_{\mu \gamma}-\frac{1}{2} \operatorname{Riem}_{\gamma \beta \delta}{ }^{\mu} h_{\mu \alpha}$
$-\frac{1}{2} \nabla_{\alpha} \nabla_{\gamma} h_{\beta \delta}-\frac{1}{2} \nabla_{\beta} \nabla_{\delta} h_{\alpha \gamma}+\frac{1}{2} \nabla_{\alpha} \nabla_{\delta} h_{\beta \gamma}+\frac{1}{2} \nabla_{\gamma} \nabla_{\beta} h_{\delta \alpha}$ $=\frac{1}{2} \operatorname{Riem}_{\alpha \gamma \beta}{ }^{\mu} h_{\mu \delta}+\frac{1}{2} \operatorname{Riem}_{\beta \delta \alpha}{ }^{\mu} h_{\mu \gamma}$
$-\frac{1}{2} \operatorname{Riem}_{\alpha \delta \beta}{ }^{\mu} h_{\mu \gamma}-\frac{1}{2} \operatorname{Riem}_{\gamma \beta \delta}{ }^{\mu} h_{\mu \alpha}$
$-\frac{1}{2} \nabla_{\alpha} \nabla_{\gamma} h_{\beta \delta}-\frac{1}{2} \nabla_{\beta} \nabla_{\delta} h_{\alpha \gamma}$
$+\frac{1}{2} \nabla_{\alpha} \nabla_{\delta} h_{\beta \gamma}+\frac{1}{2} \nabla_{\gamma} \nabla_{\beta} h_{\delta \alpha}$.
Contracting over the second and fourth index, we get after relabelling:

$$
\begin{align*}
\partial_{t} \operatorname{Ric}_{\alpha \beta}= & \left(\partial_{t} g^{\sigma \tau}\right) \operatorname{Riem}_{\alpha \sigma \beta \tau}+g^{\sigma \tau}\left(\partial_{t} \operatorname{Riem}_{\alpha \sigma \beta \tau}\right) \\
& =(\text { B.3),(B.8) } \\
= & -\operatorname{Riem}_{\alpha \mu \beta \nu} h^{\mu \nu}+\frac{1}{2} \operatorname{Ric}_{\alpha \mu} h_{\beta}^{\mu}+\frac{1}{2} \operatorname{Ric}_{\beta \mu} h_{\alpha}^{\mu} \\
& -\frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \operatorname{tr}_{g} h-\frac{1}{2} \Delta h_{\alpha \beta}+\frac{1}{2} \nabla_{\alpha} \operatorname{div} h_{\beta}+\frac{1}{2} \nabla_{\beta} \operatorname{div} h_{\alpha} . \tag{B.9}
\end{align*}
$$

Contracting again yields:

$$
\begin{align*}
\partial_{t} \mathrm{Scal}= & \left(\partial_{t} g^{\alpha \beta}\right) \operatorname{Ric}_{\alpha \beta}+g^{\alpha \beta}\left(\partial_{t} \operatorname{Ric}_{\alpha \beta}\right) \\
\text { (в.3),(в.9) } & =-\operatorname{Ric}_{\alpha \beta} h^{\alpha \beta}-\operatorname{Ric}_{\mu \nu} h^{\mu \nu}+\operatorname{Ric}_{\alpha \mu} h^{\alpha \mu} \\
& -\Delta \operatorname{tr}_{g} h+\operatorname{div} \operatorname{div} h \\
= & -\operatorname{Ric}: h-\Delta \operatorname{tr}_{g} h+\operatorname{div} \operatorname{div} h
\end{align*}
$$

So we have,

$$
\begin{align*}
& \partial_{t} \operatorname{Ric}_{\alpha \beta}= \partial_{t} \operatorname{Ric}_{\alpha \beta}-\frac{1}{n+1}\left(\partial_{t} \operatorname{Scal}\right) g_{\alpha \beta}-\frac{1}{n+1} \operatorname{Scal}\left(\partial_{t} g_{\alpha \beta}\right) \\
&(\text { (B.2),(B.9),(B.10) } \\
&=-\operatorname{Riem}_{\alpha \mu \beta \nu} h^{\mu \nu}+\frac{1}{2} \operatorname{Ric}_{\alpha \mu} h_{\beta}^{\mu}+\frac{1}{2} \operatorname{Ric}_{\beta \mu} h_{\alpha}^{\mu} \\
&+\frac{1}{n+1} \operatorname{Ric}: h g_{\alpha \beta}-\frac{1}{n+1} \operatorname{Scal} h_{\alpha \beta} \\
&-\frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \operatorname{tr}_{g} h-\frac{1}{2} \Delta h_{\alpha \beta}+\frac{1}{n+1} \Delta \operatorname{tr}_{g} h g_{\alpha \beta} \\
&+\frac{1}{2} \nabla_{\alpha} \operatorname{div} h_{\beta}+\frac{1}{2} \nabla_{\beta} \operatorname{div} h_{\alpha}-\frac{1}{n+1} \operatorname{div} \operatorname{div} h g_{\alpha \beta} . \tag{B.11}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
& \partial_{t} \| \operatorname{Ric}_{\|}^{2}= 2 \operatorname{Ric}_{\mu \nu} g^{\mu \alpha}\left(\partial_{t} g^{\nu \beta}\right) \operatorname{Ric}_{\alpha \beta}+2 \operatorname{Ric}_{\mu \nu} g^{\mu \alpha} g^{\nu \beta}\left(\partial_{t} \operatorname{Ric}_{\alpha \beta}\right) \\
&(\mathrm{B} .3),(\mathrm{B} .9) \\
&=-2 \operatorname{Ric}_{\nu \mu} \operatorname{Ric}^{\mu}{ }_{\beta} h^{\nu \beta}-2 \operatorname{Riem}_{\alpha \mu \beta \nu} \operatorname{Ric}^{\alpha \beta} h^{\mu \nu}+\operatorname{Ric}_{\beta}{ }^{\alpha} \operatorname{Ric}_{\alpha \mu} h^{\beta \mu} \\
&+\operatorname{Ric}_{\alpha}{ }^{\beta} \operatorname{Ric}_{\beta \mu} h^{\alpha \mu}-\operatorname{Ric}^{\alpha \beta} \nabla_{\alpha} \nabla_{\beta} \operatorname{tr}_{g} h-\operatorname{Ric}^{\alpha \beta} \Delta h_{\alpha \beta} \\
&+\operatorname{Ric}^{\alpha \beta} \nabla_{\alpha} \operatorname{div} h_{\beta}+\operatorname{Ric}^{\alpha \beta} \nabla_{\beta} \operatorname{div} h_{\alpha} \\
&=-2 \operatorname{Riem}_{\alpha \mu \beta \nu} \operatorname{Ric}^{\alpha \beta} h^{\mu \nu} \tag{B.12}
\end{align*}
$$

$-\operatorname{Ric}: \nabla^{2} \operatorname{tr}_{g} h-\operatorname{Ric}: \Delta h+2$ Ric $: \nabla \operatorname{div} h$,
and

$$
\begin{align*}
& \partial_{t} \text { Scal }^{2}=2 \operatorname{Scal}\left(\partial_{t} \text { Scal }\right) \\
& \quad \stackrel{(\text { B. 10) }}{ } \\
& \quad=-2 \text { Scal Ric }: h-2 \text { Scal } \Delta \operatorname{tr}_{g} h+2 \text { Scal div div } h, \tag{B.13}
\end{align*}
$$

so that

$$
\begin{align*}
\partial_{t}\|\operatorname{Ric}\|_{g}^{2}= & \partial_{t}\left\|\operatorname{Ric}-\frac{1}{n+1} \operatorname{Scal} g\right\|_{g}^{2}=\partial_{t}\left(\|\operatorname{Ric}\|_{g}^{2}-\frac{1}{n+1} \text { Scal }^{2}\right) \\
\begin{array}{c}
(\mathrm{B} .12),(\text { (B.13) } \\
=
\end{array} & -2 \operatorname{Riem}_{\alpha \mu \beta \nu} \operatorname{Ric}^{\alpha \beta} h^{\mu \nu}+2 \frac{1}{n+1} \operatorname{Scal~Riem}_{\alpha \mu \beta \nu} g^{\alpha \beta} h^{\mu \nu} \\
& -\operatorname{Ric}: \nabla^{2} \operatorname{tr}_{g} h-\operatorname{Ric}: \Delta h+2 \frac{1}{n+1} \operatorname{Scal} \Delta \operatorname{tr}_{g} h \\
& +2 \operatorname{Ric}: \nabla \operatorname{div} h-2 \frac{1}{n+1} \text { Scal div div } h \\
= & -2 \operatorname{Riem}^{\alpha}{ }_{\mu}{ }^{\beta}{ }_{\nu} \operatorname{Ric}_{\alpha \beta} h^{\mu \nu} \\
& -\operatorname{Ric}_{\circ} \nabla^{2} \operatorname{tr}_{g} h-\operatorname{Ric}: \Delta h+2 \operatorname{Ric}: \nabla \operatorname{div} h .
\end{align*}
$$

As a result, omitting the volume element for simplicity,

$$
\begin{align*}
\partial_{t} \int_{M}\|\mathrm{Ric}\|_{g}^{2}= & \frac{1}{2} \int_{M}^{(\mathrm{B} .4)} \operatorname{tr}_{g} h\|\mathrm{Ric}\|_{g}^{2}+\int_{M} \partial_{t}\|\mathrm{Ric}\|_{g}^{2} \\
= & \frac{1}{2} \int_{M} \operatorname{tr}_{g} h\|\operatorname{Ric}\|_{g}^{2}-2 \int_{M} \operatorname{Riem}^{\alpha}{ }_{\mu}{ }^{\beta}{ }_{\nu} \operatorname{Ric}_{\alpha \beta} h^{\mu \nu} \\
& -\int_{M} \operatorname{Ric}: \nabla^{2} \operatorname{tr}_{g} h-\int_{M} \operatorname{Ric}: \Delta h+2 \int_{M} \operatorname{Ric}: \nabla \operatorname{div} h .
\end{align*}
$$

Also, we find that

$$
\partial_{t}(\text { Scal }-\overline{\mathrm{Scal}})=\partial_{t}\left(\text { Scal }-\frac{1}{\mathrm{Vol}} \int_{M} \text { Scal }\right)
$$

$$
\begin{align*}
& \stackrel{(\mathrm{B} .4),(\mathrm{B} .5)}{=} \partial_{t} \mathrm{Scal}-\frac{1}{2 \mathrm{Vol}} \int_{M} \operatorname{tr}_{g} h \mathrm{Scal}-\frac{1}{\mathrm{Vol}} \int_{M} \partial_{t} \mathrm{Scal} \\
& \quad \begin{array}{l}
\text { (B.10) } \\
= \\
\quad
\end{array} \quad \text { Ric }: h-\Delta \operatorname{tr}_{g} h+\operatorname{div} \operatorname{div} h-\frac{1}{2 \mathrm{Vol}} \int_{M} \operatorname{tr}_{g} h \mathrm{Scal} \\
& \quad+\frac{1}{\mathrm{Vol}} \int_{M} \operatorname{Ric}: h+\frac{1}{\mathrm{Vol}} \int_{M} \Delta \operatorname{tr}_{g} h-\frac{1}{\mathrm{Vol}} \int_{M} \operatorname{div} \operatorname{div} h, \tag{B.16}
\end{align*}
$$

yielding:

$$
\begin{align*}
& \partial_{t} \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2}= \frac{1}{2} \int_{M} \operatorname{tr}_{g} h(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2} \\
&+2 \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})\left(\partial_{t}(\mathrm{Scal}-\overline{\mathrm{Scal}})\right) \\
&(\mathrm{B} .16) \\
&= \frac{1}{2} \int_{M} \operatorname{tr}_{g} h(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2}-2 \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})(\text { Ric }: h) \\
&-2 \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})\left(\Delta \operatorname{tr}_{g} h\right) \\
&+2 \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})(\operatorname{div} \operatorname{div} h), \tag{B.17}
\end{align*}
$$

since $\int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})=0$.
Now, at $t=0,\left.(M, g)\right|_{t=0}$ is a subset of the standard $(n+1)$-sphere with radius $R$, and we have

$$
\begin{aligned}
\left.g_{\alpha \beta}\right|_{t=0} & =\eta_{\alpha \beta}, \\
\left.\operatorname{Riem}_{\alpha \beta \gamma \delta}\right|_{t=0} & =\frac{1}{R^{2}}\left(\eta_{\alpha \gamma} \eta_{\beta \delta}-\eta_{\alpha \delta} \eta_{\beta \gamma}\right), \\
\left.\operatorname{Riem}_{\mu \nu}^{\alpha}{ }_{\mu}\right|_{t=0} & =\frac{1}{R^{2}} \eta^{\alpha \beta} \eta_{\mu \nu}-\frac{1}{R^{2}} \delta^{\alpha}{ }_{\nu} \delta^{\beta}{ }_{\mu} \\
\left.\operatorname{Ric}_{\alpha \beta}\right|_{t=0} & =\frac{n}{R^{2}} \eta_{\alpha \beta}, \\
\operatorname{Scal}^{\left.\right|_{t=0}} & =\frac{n(n+1)}{R^{2}}, \\
\left.\operatorname{Ric}_{\alpha \beta}\right|_{t=0} & =0 .
\end{aligned}
$$

Considering the two quantities $\int_{M}\|\mathrm{Ric}\|_{g}^{2}$ and $\int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2}$, the evolutions of which are given by (B.15) and (B.17), respectively, we see that both vanish at $t=0$. Moreover, since both are non-negative at all times, we infer from their evolutions
that they must be minimal initially. We are thus interested in their second variations at $t=0$. For this, we calculate:

$$
\begin{align*}
\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}= & -\frac{1}{R^{2}} \operatorname{tr}_{\eta} h \eta_{\alpha \beta}+\frac{1}{R^{2}} h_{\alpha \beta}+\frac{n}{R^{2}} h_{\alpha \beta}+\frac{n}{(n+1) R^{2}} \operatorname{tr}_{\eta} h \eta_{\alpha \beta}-\frac{n}{R^{2}} h_{\alpha \beta} \\
& -\frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \operatorname{tr}_{\eta} h-\frac{1}{2} \Delta h_{\alpha \beta}+\frac{1}{n+1} \Delta \operatorname{tr}_{\eta} h \eta_{\alpha \beta} \\
& +\frac{1}{2} \nabla_{\alpha} \operatorname{div} h_{\beta}+\frac{1}{2} \nabla_{\beta} \operatorname{div} h_{\alpha}-\frac{1}{n+1} \operatorname{div} \operatorname{div} h \eta_{\alpha \beta} \\
= & \frac{1}{R^{2}} \stackrel{\circ}{\alpha}_{\alpha \beta}-\frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \operatorname{tr}_{\eta} h-\frac{1}{2} \Delta h_{\alpha \beta}+\frac{1}{n+1} \Delta \operatorname{tr}_{\eta} h \eta_{\alpha \beta} \\
& +\frac{1}{2} \nabla_{\alpha} \operatorname{div} h_{\beta}+\frac{1}{2} \nabla_{\beta} \operatorname{div} h_{\alpha}-\frac{1}{n+1} \operatorname{div} \operatorname{div} h \eta_{\alpha \beta}, \tag{B.18}
\end{align*}
$$

(B.15)

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{M}\|\operatorname{Ric}\|_{g}^{2}= & -\left.2 \int_{M}\left(\operatorname{Riem}_{\mu}^{\alpha}{ }_{\nu}\right)\right|_{t=0}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) h^{\mu \nu} \\
& -\int_{M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \nabla^{\alpha} \nabla^{\beta} \operatorname{tr}_{\eta} h-\int_{M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \Delta h^{\alpha \beta} \\
& +2 \int_{M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \nabla^{\alpha} \operatorname{div} h^{\beta} \\
\begin{array}{c}
\text { (B.18) } \\
=
\end{array} & +\frac{2}{R^{4}} \int_{M} h: \stackrel{\circ}{h}-\frac{1}{R^{2}} \int_{M} h: \nabla^{2} \operatorname{tr}_{\eta} h-\frac{1}{R^{2}} \int_{M} h: \Delta h+\frac{2}{(n+1) R^{2}} \int_{M} \operatorname{tr}_{\eta} h\left(\Delta \operatorname{tr}_{\eta} h\right) \\
& +\frac{2}{R^{2}} \int_{M} h: \nabla \operatorname{div} h-\frac{2}{(n+1) R^{2}} \int_{M} \operatorname{tr}_{\eta} h(\operatorname{div} \operatorname{div} h) \\
& -\frac{1}{R^{2}} \int_{M} \stackrel{\circ}{h} \nabla^{2} \operatorname{tr}_{\eta} h+\frac{1}{2} \int_{M}\left\|\nabla^{2} \operatorname{tr}_{\eta} h\right\|_{\eta}^{2}+\frac{1}{2} \int_{M} \Delta h: \nabla^{2} \operatorname{tr}_{\eta} h \\
& -\frac{1}{n+1} \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2}-\int_{M} \nabla^{2} \operatorname{tr}_{\eta} h: \nabla \operatorname{div} h+\frac{1}{n+1} \int_{M}(\operatorname{div} \operatorname{div} h)\left(\Delta \operatorname{tr}_{\eta} h\right) \\
& -\frac{1}{R^{2}} \int_{M} \stackrel{\circ}{h}: \Delta h+\frac{1}{2} \int \Delta h: \nabla^{2} \operatorname{tr}_{\eta} h+\frac{1}{2} \int_{M}\|\Delta h\|_{\eta}^{2} \\
& -\frac{1}{n+1} \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2}-\int_{M} \Delta h: \nabla \operatorname{div} h+\frac{1}{n+1} \int_{M}(\operatorname{div} \operatorname{div} h)\left(\Delta \operatorname{tr}_{\eta} h\right) \\
& +\frac{2}{R^{2}} \int_{M} \stackrel{\circ}{h} \nabla \operatorname{div} h-\int_{M} \nabla^{2} \operatorname{tr}_{\eta} h: \nabla \operatorname{div} h-\int_{M} \Delta h: \nabla \operatorname{div} h \\
& +\frac{2}{n+1} \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)(\operatorname{div} \operatorname{div} h)+\int_{M}\|\nabla \operatorname{div} h\|_{\eta}^{2} \\
& +\int_{M} \nabla^{\beta} \operatorname{div}^{\alpha} \nabla_{\alpha} \operatorname{div} h_{\beta}-\frac{2}{n+1} \int_{M}(\operatorname{div} \operatorname{div} h)^{2}
\end{aligned}
$$

$$
\begin{align*}
= & \frac{2}{R^{4}} \int_{M} \stackrel{\circ}{h}: h-\frac{2}{R^{2}} \int_{M} \stackrel{\circ}{h}: \nabla^{2} \operatorname{tr}_{\eta} h \\
& -\frac{2}{R^{2}} \int_{M} \stackrel{\circ}{h}: \Delta h+\frac{4}{R^{2}} \int_{M} \stackrel{\circ}{h}: \nabla \operatorname{div} h \\
& +\frac{1}{2} \int_{M}\left\|\nabla^{2} \operatorname{tr}_{\eta} h\right\|_{\eta}^{2}+\frac{1}{2} \int_{M}\|\Delta h\|_{\eta}^{2}+\int_{M}\|\nabla \operatorname{div} h\|_{\eta}^{2} \\
& -\frac{2}{n+1} \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2}-\frac{2}{n+1} \int_{M}(\operatorname{div} \operatorname{div} h)^{2} \\
& +\int_{M} \nabla^{2} \operatorname{tr}_{\eta} h: \Delta h-2 \int_{M} \nabla^{2} \operatorname{tr}_{\eta} h: \nabla \operatorname{div} h \\
& -2 \int_{M} \Delta h: \nabla \operatorname{div} h+\int_{M} \nabla^{\beta} \operatorname{div} h^{\alpha} \nabla_{\alpha} \operatorname{div} h_{\beta} \\
& +\frac{4}{n+1} \int_{M}\left(\Delta \operatorname{tr}_{g} h\right)(\operatorname{div} \operatorname{div} h) . \tag{B.19}
\end{align*}
$$

Similarly, we calculate

$$
\begin{align*}
& \left.\partial_{t}\right|_{t=0}(\text { Scal }-\overline{\mathrm{Scal}}{ }^{(\mathrm{B} .16)}=-\Delta \operatorname{tr}_{\eta} h+\operatorname{div} \operatorname{div} h-\frac{n}{R^{2}} \operatorname{tr}_{\eta} h-\frac{n(n+1)}{2 R^{2} \operatorname{Vol}} \underbrace{=}_{(\mathrm{B} .1)} \int_{M}^{\int_{M}} \operatorname{tr}_{\eta} h \quad \\
& +\frac{1}{\mathrm{Vol}} \int_{M} \Delta \operatorname{tr}_{\eta} h-\frac{1}{\mathrm{Vol}} \int_{M} \operatorname{div} \operatorname{div} h+\frac{n}{R^{2} \operatorname{Vol}} \underbrace{\int_{M} \operatorname{tr}_{\eta} h}_{\stackrel{(\mathrm{B}, 1)}{=} 0} \\
& =-\Delta \operatorname{tr}_{\eta} h+\operatorname{div} \operatorname{div} h-\frac{n}{R^{2}} \operatorname{tr}_{\eta} h+\frac{1}{\operatorname{Vol}} \int_{M} \Delta \operatorname{tr}_{\eta} h \\
& -\frac{1}{\mathrm{Vol}} \int_{M} \operatorname{div} \operatorname{div} h, \tag{B.20}
\end{align*}
$$

so that (since Scal $\left.\right|_{t=0}$ is constant on $M$ )

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{M}(\mathrm{Scal}-\overline{\mathrm{Scal}})^{2}= & -\frac{2 n}{R^{2}} \int_{M}\left(\left.\partial_{t}\right|_{t=0}(\mathrm{Scal}-\overline{\mathrm{Scal}})\right) \operatorname{tr}_{\eta} h \\
& -2 \int_{M}\left(\left.\partial_{t}\right|_{t=0}(\mathrm{Scal}-\overline{\mathrm{Scal}})\right) \Delta \operatorname{tr}_{\eta} h \\
& +2 \int_{M}\left(\left.\partial_{t}\right|_{t=0}(\mathrm{Scal}-\overline{\mathrm{Scal}})\right) \operatorname{div} \operatorname{div} h
\end{aligned}
$$

$$
\begin{aligned}
& \text { (B.20) } \\
& =\frac{2 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \Delta \operatorname{tr}_{\eta} h-\frac{2 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h \\
& +\frac{2 n^{2}}{R^{4}} \int_{M}\left(\operatorname{tr}_{\eta} h\right)^{2}-\frac{2 n}{R^{2} \mathrm{Vol}}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right) \underbrace{\left(\int_{M} \operatorname{tr}_{\eta} h\right)}_{\stackrel{(\mathrm{B}, 1)}{=} 0} \\
& +\frac{2 n}{R^{2} \mathrm{Vol}}\left(\int_{M} \operatorname{div} \operatorname{div} h\right) \underbrace{\left(\int_{M} \operatorname{tr}_{\eta} h\right)}_{(\stackrel{(B, 1)}{=})_{0}} \\
& +2 \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2}-2 \int_{M} \Delta \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h+\frac{2 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \Delta \operatorname{tr}_{\eta} h \\
& -\frac{2}{\text { Vol }}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right)^{2}+\frac{2}{\text { Vol }}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right)\left(\int_{M} \operatorname{div} \operatorname{div} h\right) \\
& -2 \int_{M} \Delta \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h+2 \int_{M}(\operatorname{div} \operatorname{div} h)^{2}-\frac{2 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h \\
& +\frac{2}{\mathrm{Vol}}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right)\left(\int_{M} \operatorname{div} \operatorname{div} h\right)-\frac{2}{\mathrm{Vol}}\left(\int_{M} \operatorname{div} \operatorname{div} h\right)^{2} \\
& =\frac{2 n^{2}}{R^{4}} \int_{M}\left(\operatorname{tr}_{\eta} h\right)^{2}+\frac{4 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \Delta \operatorname{tr}_{\eta} h \\
& -\frac{4 n}{R^{2}} \int_{M} \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h+2 \int_{M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2} \\
& -\frac{2}{\mathrm{Vol}}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right)^{2}+2 \int_{M}(\operatorname{div} \operatorname{div} h)^{2} \\
& -\frac{2}{\mathrm{Vol}}\left(\int_{M} \operatorname{div} \operatorname{div} h\right)^{2}-4 \int \Delta \operatorname{tr}_{\eta} h \operatorname{div} \operatorname{div} h \\
& +\frac{4}{\mathrm{Vol}}\left(\int_{M} \Delta \operatorname{tr}_{\eta} h\right)\left(\int_{M} \operatorname{div} \operatorname{div} h\right)
\end{aligned}
$$

## 3. The first and second variations of the quantities on $\partial M$

We shall now focus our attention to the boundary $\partial M$ of $M$. Its induced metric $\widetilde{g}$ is given by

$$
\tilde{g}_{i j}=g_{i j} .
$$

Also, the inverse of $\widetilde{g}$ is quickly computed to be

$$
\left(\tilde{g}^{-1}\right)^{i j}=g^{i j}-\nu^{i} \nu^{j}
$$

where $\nu$ denotes the outer unit normal field on $\partial M$ given in our coordinates by

$$
\nu^{\alpha}=-\frac{g^{0 \alpha}}{\sqrt{g^{00}}}
$$

Notice that, for any two-tensor field $B$ on $M$, its restriction to $\partial M$ satisfies

$$
\underset{g}{\operatorname{tr} \sim} B=\left(\widetilde{g}^{-1}\right)^{i j} B_{i j}=g^{i j} B_{i j}-\nu^{i} \nu^{j} B_{i j}=g^{\sigma \tau} B_{\sigma \tau}-\nu^{\sigma} \nu^{\tau} B_{\sigma \tau}=\operatorname{tr}_{g} B-B(\nu, \nu)
$$

We start, again, by the evolution of the metric:

$$
\begin{equation*}
\partial_{t} \widetilde{g}_{i j} \stackrel{(\text { B.2) }}{=} h_{i j} . \tag{B.22}
\end{equation*}
$$

Then, similarly to the computation of (B.4),

$$
\begin{equation*}
\partial_{t} \sqrt{\operatorname{det} \widetilde{g}}=\frac{\sqrt{\operatorname{det} \widetilde{g}}}{2} \operatorname{tr}_{g}\left(\partial_{t} \widetilde{g}\right) \stackrel{(\mathrm{B} .22)}{=} \frac{1}{2}\left(\operatorname{tr}_{g} h-h(\nu, \nu)\right) \sqrt{\operatorname{det} \widetilde{g}}, \tag{B.23}
\end{equation*}
$$

Also, the outer unit normal evolves according to

$$
\begin{aligned}
& \partial_{t} \nu^{\alpha}=\frac{g^{0 \alpha}}{2\left(g^{00}\right)^{3 / 2}}\left(\partial_{t} g^{00}\right)-\frac{1}{\sqrt{g^{00}}}\left(\partial_{t} g^{0 \alpha}\right) \\
& \quad \begin{aligned}
\text { (В.3) } & g^{0 \alpha} \\
2\left(g^{00}\right)^{3 / 2} & \left(-g^{0 \sigma} g^{0 \tau} h_{\sigma \tau}\right)-\frac{1}{\sqrt{g^{00}}}\left(-g^{0 \sigma} g^{\alpha \tau} h_{\sigma \tau}\right)=\frac{1}{2} \nu^{\alpha} \nu^{\sigma} \nu^{\tau} h_{\sigma \tau}-\nu^{\sigma} h^{\alpha}{ }_{\sigma}
\end{aligned}
\end{aligned}
$$

$$
\begin{equation*}
=\frac{1}{2} h(\nu, \nu) \nu^{\alpha}-\left(h^{\sharp}(\nu)\right)^{\alpha}, \tag{B.24}
\end{equation*}
$$

where $\left(h^{\sharp}(\nu)\right)^{\alpha}=h^{\alpha}{ }_{\sigma} \nu^{\sigma}$. Consequently,

$$
\partial_{t}\left(\widetilde{g}^{-1}\right)^{i j}=\partial_{t} g^{i j}-\left(\partial_{t} \nu^{i}\right) \nu^{j}-\nu^{i}\left(\partial_{t} \nu^{j}\right)
$$

$$
\begin{equation*}
\stackrel{\text { 3),(B.24) }}{=}-h^{i j}-h(\nu, \nu) \nu^{i} \nu^{j}+\left(h^{\sharp}(\nu)\right)^{i} \nu^{j}+\nu^{i}\left(h^{\sharp}(\nu)\right)^{j} \tag{B.25}
\end{equation*}
$$

On the other hand, similarly to the computation of (B.3), we have

$$
\begin{equation*}
\partial_{t}\left(\widetilde{g}^{-1}\right)^{i j}=-\left(\widetilde{g}^{-1}\right)^{i k}\left(\widetilde{g}^{-1}\right)^{j l} h_{k l} \tag{B.26}
\end{equation*}
$$

Now, denoting by $\partial_{\alpha}$ the coordinate vector fields, we get for the evolution of the second fundamental form $A$ :

$$
\begin{aligned}
\partial_{t} A_{i j} & =\partial_{t}\left(-g\left(\nabla_{i} \partial_{j}, \nu\right)\right)=\partial_{t}\left(-\Gamma_{i j}^{\sigma} g\left(\partial_{\sigma}, \nu\right)\right)=\partial_{t}\left(\Gamma_{i j}^{\sigma} \frac{\delta_{\sigma}^{0}}{\sqrt{g^{00}}}\right) \\
& =\partial_{t}\left(\frac{\Gamma_{i j}^{0}}{\sqrt{g^{00}}}\right)=-\frac{\Gamma_{i j}^{0}}{2\left(g^{00}\right)^{3 / 2}}\left(\partial_{t} g^{00}\right)+\frac{1}{\sqrt{g^{00}}} \partial_{t} \Gamma_{i j}^{0}
\end{aligned}
$$

(B.3),(B.6)

$$
=\frac{1}{2} \Gamma_{i j}^{0} \frac{h^{00}}{\left(g^{00}\right)^{3 / 2}}+\frac{g^{0 \tau}}{2 \sqrt{g^{00}}}\left(\nabla_{i} h_{j \tau}+\nabla_{j} h_{i \tau}-\nabla_{\tau} h_{i j}\right)
$$

$$
=\frac{1}{2} A_{i j} \frac{h^{00}}{g^{00}}-\frac{1}{2}((\nabla h)(\nu))_{i j}-\frac{1}{2}((\nabla h)(\nu))_{j i}+\frac{1}{2}\left(\nabla_{\nu} h\right)_{i j}
$$

$$
\begin{equation*}
=\frac{1}{2} A_{i j} h(\nu, \nu)-\frac{1}{2}((\nabla h)(\nu))_{i j}-\frac{1}{2}((\nabla h)(\nu))_{j i}+\frac{1}{2}\left(\nabla_{\nu} h\right)_{i j} \tag{B.27}
\end{equation*}
$$

where $((\nabla h)(\nu))_{i j}=\nabla_{i} h_{\sigma j} \nu^{\sigma}$. Therefore,

As a consequence,

$$
\begin{align*}
& \partial_{t} \AA_{i j}= \partial_{t} A_{i j}-\frac{1}{n}\left(\partial_{t} H\right) \widetilde{g}_{i j}-\frac{1}{n} H\left(\partial_{t} \widetilde{g}_{i j}\right) \\
& \begin{aligned}
\text { (В.22),(В.27),(в.28) }
\end{aligned} \\
&= \frac{1}{2} h(\nu, \nu) \AA_{i j}+\frac{1}{n} \widetilde{A_{:}} h g_{i j}-\frac{1}{n} H h_{i j} \\
&-\frac{1}{2}((\nabla h)(\nu))_{i j}-\frac{1}{2}((\nabla h)(\nu))_{j i}+\frac{1}{n} \operatorname{div} h(\nu) g_{i j} \\
&+\frac{1}{2}\left(\nabla_{\nu} h\right)_{i j}-\frac{1}{2 n} \nabla_{\nu} \operatorname{tr}_{g} h g_{i j}-\frac{1}{2 n}\left(\nabla_{\nu} h\right)(\nu, \nu) g_{i j} . \tag{B.29}
\end{align*}
$$

Moreover,

$$
\begin{aligned}
\partial_{t}\|\AA\|_{\tilde{g}}^{2} & =\partial_{t}\left(\|A\|_{g}^{2}-\frac{1}{n} H^{2}\right) \\
& =2 A_{i k}\left(\widetilde{g}^{-1}\right)^{k l} A_{j l}\left(\partial_{t}\left(\widetilde{g}^{-1}\right)^{i j}\right)+2 \widetilde{A^{\prime}}\left(\partial_{t} A\right)-\frac{2}{n} H\left(\partial_{t} H\right)
\end{aligned}
$$

$$
\text { (B. 26), (B. } 27 \text { ),(B. } 28 \text { ) }
$$

$$
=-2 A^{2}: h+\|A\|_{g}^{2} h(\nu, \nu)-2 \widetilde{A:}((\nabla h)(\nu))+\widetilde{A:}\left(\nabla_{\nu} h\right)
$$

$$
+\frac{2}{n} H A: h-\frac{1}{n} h(\nu, \nu) H^{2}+\frac{2}{n} H \operatorname{div} h(\nu)
$$

$$
-\frac{1}{n} H\left(\nabla_{\nu} h\right)(\nu, \nu)-\frac{1}{n} H \nabla_{\nu} \operatorname{tr}_{g} h
$$

$$
=-2(A \AA) \widetilde{:} h+\|\AA\|_{g}^{2} h(\nu, \nu)
$$

$$
\begin{equation*}
-2 \AA \stackrel{\AA}{:}((\nabla h)(\nu))+\overparen{A} \overleftarrow{:}\left(\nabla_{\nu} h\right) \tag{B.30}
\end{equation*}
$$

$$
\begin{align*}
& \partial_{t} H=\left(\partial_{t}\left(\widetilde{g}^{-1}\right)^{i j}\right) A_{i j}+\left(\widetilde{g}^{-1}\right)^{i j} \partial_{t} A_{i j} \\
& \stackrel{\text { (В.26),(В. } 27)}{=} \widetilde{:} \widetilde{:} A+\frac{1}{2} h(\nu, \nu) H-\operatorname{tr}_{g}^{\sim}((\nabla h)(\nu))+\frac{1}{2} \operatorname{tr} \underset{g}{\sim}\left(\nabla_{\nu} h\right) \\
& =-\overparen{A: h}+\frac{1}{2} h(\nu, \nu) H-\operatorname{tr}_{g}((\nabla h)(\nu))+\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& +\frac{1}{2} \operatorname{tr}_{g}\left(\nabla_{\nu} h\right)-\frac{1}{2}\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& =-\overparen{A: h}+\frac{1}{2} h(\nu, \nu) H-(\operatorname{div} h)(\nu)+\frac{1}{2}\left(\nabla_{\nu} h\right)(\nu, \nu)+\frac{1}{2} \nabla_{\nu} \operatorname{tr}_{g} h . \tag{B.28}
\end{align*}
$$

where, for any two-tensor fields $\widetilde{B}$ and $\widetilde{C}$ on $\partial M, \widetilde{B} \widetilde{C}$ is defined as $(\widetilde{B} \widetilde{C})_{i j}=$ $\widetilde{B}_{i k}\left(\widetilde{g}^{-1}\right)^{k l} \widetilde{C}_{l j}$, and $\widetilde{B}^{2}=\widetilde{B} \widetilde{B}$. Hence,

$$
\begin{align*}
\partial_{t} \int_{\partial M}\|\AA\|_{g}^{(\mathrm{B} .23)}= & \frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} \operatorname{tr}_{g} h-\frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} h(\nu, \nu)+\int_{\partial M} \partial_{t}\|\AA\|_{g}^{2} \\
= & \frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} \operatorname{tr}_{g} h-\frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} h(\nu, \nu)-2 \int_{\partial M}(A \AA) \widetilde{: h} \\
& +\int_{\partial M}\|\AA\|_{g}^{2} h(\nu, \nu)-2 \int_{\partial M} \AA \cong((\nabla h)(\nu))+\int_{\partial M} \AA \cong\left(\nabla_{\nu} h\right) \\
= & \frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} \operatorname{tr}_{g} h+\frac{1}{2} \int_{\partial M}\|\AA\|_{g}^{2} h(\nu, \nu) \\
& -2 \int_{\partial M}(A \AA) \widetilde{:} h-2 \int_{\partial M} \AA \cong((\nabla h)(\nu))+\int_{\partial M} \AA \AA\left(\nabla_{\nu} h\right)
\end{align*}
$$

Also, since for any smooth function $f$ on $\partial M$,
(B.32)

$$
\begin{aligned}
\partial_{t} \bar{f} & =\partial_{t} f_{\partial M} f=\partial_{t}\left(\frac{1}{\widetilde{\mathrm{Vol}}} \int_{\partial M} f\right) \\
& =-\frac{1}{\widetilde{\mathrm{Vol}}^{2}}\left(\partial_{t} \int_{\partial M} 1\right) \int_{\partial M} f+\frac{1}{\widetilde{\mathrm{Vol}}} \partial_{t} \int_{\partial M} f
\end{aligned}
$$

$$
\stackrel{(\mathrm{B} .23)}{=}-\frac{1}{2} f_{\partial_{M}}\left(\operatorname{tr}_{g} h-h(\nu, \nu)\right) f_{\partial M} f+\frac{1}{2} f_{\partial M}\left(\operatorname{tr}_{g} h-h(\nu, \nu)\right) f+f_{\partial M} \partial_{t} f
$$

$$
\begin{equation*}
=\frac{1}{2} f_{\partial M}(f-\bar{f})\left(\operatorname{tr}_{g} h-h(\nu, \nu)\right)+f_{\partial M} \partial_{t} f, \tag{B.33}
\end{equation*}
$$

we have

$$
\begin{align*}
\partial_{t}(H-\bar{H}) \stackrel{(\mathrm{B} .33)}{=} & \partial_{t} H-f_{\partial M} \partial_{t} H-\frac{1}{2} f_{\partial M}(H-\bar{H})\left(\operatorname{tr}_{g} h-h(\nu, \nu)\right) \\
\stackrel{(\mathrm{B} .28)}{=} & -A: h+f_{\partial M} A: h+\frac{1}{2} h(\nu, \nu) H-\frac{1}{2} f_{\partial M} h(\nu, \nu) H \\
& -\operatorname{div} h(\nu)+f_{\partial M} \operatorname{div} h(\nu)+\frac{1}{2}\left(\nabla_{\nu} h\right)(\nu, \nu)-\frac{1}{2} f_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& +\frac{1}{2} \nabla_{\nu} \operatorname{tr}_{g} h-\frac{1}{2} f_{\partial M} \nabla_{\nu} \operatorname{tr}_{g} h \\
& \quad-\frac{1}{2} f_{\partial M}(H-\bar{H}) \operatorname{tr}_{g} h+\frac{1}{2} f_{\partial M}(H-\bar{H}) h(\nu, \nu) . \tag{B.34}
\end{align*}
$$

Thus,

$$
\begin{aligned}
\partial_{t} \int_{\partial M}(H-\bar{H})^{2}= & \frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} \operatorname{tr}_{g} h-\frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} h(\nu, \nu) \\
& +2 \int_{\partial M}(H-\bar{H}) \partial_{t}(H-\bar{H}) \\
= & \frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} \operatorname{tr}_{g} h-\frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} h(\nu, \nu) \\
& -2 \int_{\partial M}(H-\bar{H}) A: h+\int_{\partial M}(H-\bar{H}) H h(\nu, \nu) \\
& -2 \int_{\partial M}(H-\bar{H})(\operatorname{div} h)(\nu)+\int_{\partial M}(H-\bar{H})\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& +\int_{\partial M}(H-\bar{H}) \nabla_{\nu} \operatorname{tr}_{g} h \\
= & \frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} \operatorname{tr}_{g} h+\frac{1}{2} \int_{\partial M}(H-\bar{H})^{2} h(\nu, \nu) \\
& -2 \int_{\partial M}(H-\bar{H}) A: h+\bar{H} \int_{\partial M}(H-\bar{H}) h(\nu, \nu) \\
& -2 \int_{\partial M}(H-\bar{H})\left(\operatorname{div}^{\prime} h\right)(\nu)+\int_{\partial M}(H-\bar{H})\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& +\int_{\partial M}(H-\bar{H}) \nabla_{\nu} \operatorname{tr}_{g} h,
\end{aligned}
$$

where we have used that $\int_{\partial M}(H-\bar{H})=0$.
Finally, just in the same way that we established (B.15), we have

$$
\begin{align*}
\partial_{t} \int_{\partial M}\|\mathrm{Ric}\|_{g}^{2}= & \frac{1}{2} \int_{\partial M} \operatorname{tr}_{g} h\|\operatorname{Ric}\|_{g}^{2}-\frac{1}{2} \int_{\partial M} h(\nu, \nu)\|\operatorname{Ric}\|_{g}^{2}+\int_{\partial M} \partial_{t}\|\operatorname{Ric}\|_{g}^{2} \\
= & \frac{1}{2} \int_{\partial M} \operatorname{tr}_{g} h\|\operatorname{Ric}\|_{g}^{2}-\frac{1}{2} \int_{\partial M} h(\nu, \nu)\|\operatorname{Ric}\|_{g}^{2} \\
& -2 \int_{\partial M} \operatorname{Riem}^{\alpha}{ }_{\mu}{ }^{\beta}{ }_{\nu} \operatorname{Ric}_{\alpha \beta} h^{\mu \nu}-\int_{\partial M} \operatorname{Ric}: \nabla^{2} \operatorname{tr}_{g} h \\
36) \quad & \quad \int_{\partial M} \operatorname{Ric}: \Delta h+2 \int_{\partial M} \operatorname{Ric}: \nabla \operatorname{div} h .
\end{align*}
$$

Now, at $t=0,\left.(\partial M, \widetilde{g})\right|_{t=0}$ is an $n$-sphere with radius $r$, and we have by our choice of coordinates

$$
\begin{aligned}
\left.\widetilde{g}_{i j}\right|_{t=0} & =\widetilde{\eta}_{i j} \quad\left(=\eta_{i j}\right) \\
\left.\widetilde{\operatorname{Riem}}_{i j k l}\right|_{t=0} & =\frac{1}{r^{2}}\left(\widetilde{\eta}_{i k} \widetilde{\eta}_{j l}-\widetilde{\eta}_{i l} \widetilde{\eta}_{j k}\right), \\
\left.\widetilde{\operatorname{Ric}}_{i j}\right|_{t=0} & =\frac{n-1}{r^{2}} \widetilde{\eta}_{i j} \\
\left.\widetilde{\operatorname{Scal}}\right|_{t=0} & =\frac{n(n-1)}{r^{2}}, \\
\left.\widetilde{\operatorname{Ric}}_{i j}\right|_{t=0} & =0
\end{aligned}
$$

To recover the second fundamental form, remark that $\partial M$ is umbilical in $M$, i.e. $\AA=$ 0 , or $A=\frac{H}{n} \widetilde{\eta}$. Plugging this into the Gauss equations,

$$
\widetilde{\operatorname{Riem}}_{i j k l}=\operatorname{Riem}_{i j k l}+A_{i k} A_{j l}-A_{i l} A_{j k}
$$

we conclude that

$$
\begin{aligned}
& H=n \frac{\sqrt{R^{2}-r^{2}}}{R r} \\
& A=\frac{\sqrt{R^{2}-r^{2}}}{R r} \widetilde{\eta}
\end{aligned}
$$

Considering the three quantities $\int_{\partial M}\|\AA\|_{g}^{2}, \int_{\partial M}(H-\bar{H})^{2}$ and $\int_{\partial M} \|$ Ric $\|_{g}^{2}$, the evolutions of which are given by (B.31), (B.35) and (B.36), respectively, we see that all vanish at $t=0$. Moreover, since these quantities are non-negative at all times, we infer from their evolutions that they must be minimal initially. We are thus interested in their second variations at $t=0$. For this, we calculate:

$$
\begin{align*}
\left.\partial_{t}\right|_{t=0} \stackrel{\circ}{A_{i j}}= & 0 \\
& +\frac{\sqrt[(\mathrm{B} .29)]{R^{2}-r^{2}}}{n R r} \operatorname{tr}_{\eta}^{\sim} h \eta_{i j}-\frac{\sqrt{R^{2}-r^{2}}}{R r} h_{i j} \\
& +\frac{1}{2}((\nabla h)(\nu))_{i j}-\frac{1}{2}((\nabla h)(\nu))_{j i}+\frac{1}{n} \operatorname{div} h(\nu) \eta_{i j}-\frac{1}{2 n} \nabla_{\nu} \operatorname{tr}_{\eta} h \eta_{i j}-\frac{1}{2 n}\left(\nabla_{\nu} h\right)(\nu, \nu) \eta_{i j} \\
= & -\frac{\sqrt{R^{2}-r^{2}}}{R r} h_{i j}+\frac{\sqrt{R^{2}-r^{2}}}{n R r} \operatorname{tr}_{\eta} h \eta_{i j}-\frac{\sqrt{R^{2}-r^{2}}}{n R r} h(\nu, \nu) \eta_{i j} \\
& -\frac{1}{2}((\nabla h)(\nu))_{i j}-\frac{1}{2}((\nabla h)(\nu))_{j i}+\frac{1}{n} \operatorname{div} h(\nu) \eta_{i j} \\
& +\frac{1}{2}\left(\nabla_{\nu} h\right)_{i j}-\frac{1}{2 n} \nabla_{\nu} \operatorname{tr}_{\eta} h \eta_{i j}-\frac{1}{2 n}\left(\nabla_{\nu} h\right)(\nu, \nu) \eta_{i j} . \tag{B.37}
\end{align*}
$$

Then, in view of the fact that $\AA$ vanishes at $t=0$,

$$
\begin{aligned}
& \left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}\|\stackrel{\circ}{A}\|_{g}^{2}=-2 \frac{(\mathrm{~B} .31)}{R r} \int_{\partial M} \widetilde{\eta}_{k l}\left(\widetilde{\eta}^{-1}\right)^{l i}\left(\left.\partial_{t}\right|_{t=0} \stackrel{\circ}{A}_{i j}\right)\left(\widetilde{\eta}^{-1}\right)^{k s}\left(\widetilde{\eta}^{-1}\right)^{j t} h_{s t} \\
& \left.-2 \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0} \AA\right) \widetilde{A}\right)((\nabla h)(\nu))+\int_{\partial M}\left(\left.\partial_{t}\right|_{t=0} \stackrel{\circ}{A}\right) \widetilde{:}\left(\nabla_{\nu} h\right) \\
& \stackrel{\text { (B. } 37)}{=} 2 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} \tilde{h: h}-2 \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M} \operatorname{tr}_{\eta} h \operatorname{tr}_{\tilde{\eta}} h+2 \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M} h(\nu, \nu) \operatorname{tr} \underset{\eta}{\sim} h \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}((\nabla h)(\nu)): \%-2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M} \operatorname{div} h(\nu) \operatorname{tr} \sim{ }_{\eta}^{\sim} h \\
& -\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\nabla_{\nu} h\right) \widetilde{:} h+\frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right) \operatorname{tr} \sim \tilde{\eta}^{h} \\
& +\frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \operatorname{tr} \tilde{\eta}_{n}^{h} \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}((\nabla h)(\nu)) \overparen{: h}-2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\operatorname{tr}_{\tilde{\eta}}(\nabla h)(\nu)\right) \operatorname{tr}_{\eta} h \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\operatorname{tr}_{\tilde{\eta}}(\nabla h)(\nu)\right) h(\nu, \nu) \\
& +\int_{\partial M}\|(\nabla h)(\nu)\|_{\eta}^{2}+\int_{\partial M}((\nabla h)(\nu))_{j i}\left(\tilde{\eta}^{-1}\right)^{i k}\left(\tilde{\eta}^{-1}\right)^{j l}((\nabla h)(\nu))_{k l} \\
& -\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\operatorname{tri}_{\tilde{\eta}}(\nabla h)(\nu)\right)-\int_{\partial M}\left(\nabla_{\nu} h\right) \widetilde{\Im}((\nabla h)(\nu)) \\
& +\frac{1}{n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\operatorname{tr}_{\tilde{\eta}}(\nabla h)(\nu)\right)+\frac{1}{n} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\operatorname{tr}_{\tilde{\eta}}(\nabla h)(\nu)\right) \\
& -\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\nabla_{\nu} h\right) \widetilde{: h}+\frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\operatorname{tr}_{\tilde{\eta}} \nabla_{\nu} h\right) \operatorname{tr}_{\eta} h \\
& -\frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\operatorname{tr}_{\tilde{\eta}} \nabla_{\nu} h\right) h(\nu, \nu) \\
& -\int_{\partial M}((\nabla h)(\nu)) \widetilde{:}\left(\nabla_{\nu} h\right)+\frac{1}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\operatorname{tr}_{\tilde{\eta}} \nabla_{\nu} h\right)+\frac{1}{2} \int_{\partial M}\left\|\nabla_{\nu} h\right\|_{\tilde{\eta}}^{2} \\
& -\frac{1}{2 n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\operatorname{tr}_{\tilde{\eta}}\left(\nabla_{\nu} h\right)\right)-\frac{1}{2 n} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\operatorname{tr}_{\tilde{\eta}} \nabla_{\nu} h\right) .
\end{aligned}
$$

Remember that the restriction to $\partial M$ of any two-tensor field $B$ on $M$ satisfies $\operatorname{tr}_{\underset{g}{ }} B=\operatorname{tr}_{g} B-B(\nu, \nu)$. Also, it is easy to see that, for any two-tensor fields $B$ and $C$ on $M$ restricted to $\partial M$,

$$
B: C=B: C-\operatorname{tr}_{g}(B(\nu, \cdot) C(\nu, \cdot))-\operatorname{tr}_{g}(B(\cdot, \nu) C(\cdot, \nu))+B(\nu, \nu) C(\nu, \nu)
$$

Then, defining $X \cdot Y=g_{\mu \nu} X^{\mu} Y^{\nu}$ and $|X|_{g}^{2}=X \cdot X$ for any vectorfields $X$ and $Y$ on $M$ at any time $t$, we further obtain

$$
\begin{aligned}
& \left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}\|\AA\|_{\tilde{g}}^{2}=2 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M}\left(\|h\|_{\eta}^{2}-2|h(\nu)|_{\eta}^{2}+(h(\nu, \nu))^{2}\right)-2 \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M}\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right)^{2} \\
& +4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left((\nabla h)(\nu): h-\left(\nabla_{\nu} h\right)(\nu) \cdot h(\nu)-(\nabla h)(\nu, \nu) \cdot h(\nu)+\left(\nabla_{\nu} h\right)(\nu, \nu) h(\nu, \nu)\right) \\
& -4 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}(\operatorname{div} h(\nu))\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right) \\
& -2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\left(\nabla_{\nu} h\right): h-2\left(\nabla_{\nu} h\right)(\nu) \cdot h(\nu)+\left(\nabla_{\nu} h\right)(\nu, \nu) h(\nu, \nu)\right) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right)+2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right) \\
& +\int_{\partial M}\left(\|(\nabla h)(\nu)\|_{\eta}^{2}-\left|\left(\nabla_{\nu} h\right)(\nu)\right|_{\eta}^{2}-|(\nabla h)(\nu, \nu)|_{\eta}^{2}+\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}\right) \\
& +\int_{\partial M}\left(((\nabla h)(\nu))^{\beta \alpha}((\nabla h)(\nu))_{\alpha \beta}-2\left(\nabla_{\nu} h\right)(\nu) \cdot(\nabla h)(\nu, \nu)+\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}\right) \\
& +\frac{1}{2} \int_{\partial M}\left(\left\|\nabla_{\nu} h\right\|_{\eta}^{2}-2\left|\left(\nabla_{\nu} h\right)(\nu)\right|_{\eta}^{2}+\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}\right)-\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))^{2} \\
& -\frac{1}{2 n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2}+\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)-\frac{1}{n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& -2 \int_{\partial M}\left((\nabla h)(\nu):\left(\nabla_{\nu} h\right)-\left|\left(\nabla_{\nu} h\right)(\nu)\right|_{\eta}^{2}-\left(\nabla_{\nu} h\right)(\nu) \cdot(\nabla h)(\nu, \nu)+\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}\right) \\
& -\frac{1}{2 n} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2} \\
& =2 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M}\|h\|_{\eta}^{2}-4 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M}|h(\nu)|_{\eta}^{2} \\
& +2(n-1) \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M}(h(\nu, \nu))^{2}+4 \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M}\left(\operatorname{tr}_{\eta} h\right)(h(\nu, \nu)) \\
& -2 \frac{R^{2}-r^{2}}{n R^{2} r^{2}} \int_{\partial M}\left(\operatorname{tr}_{\eta} h\right)^{2}+4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}(\nabla h)(\nu): h \\
& -2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\nabla_{\nu} h\right): h-4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}(\nabla h)(\nu, \nu) \cdot h(\nu) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu) h(\nu, \nu) \\
& -4 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}(\operatorname{div} h(\nu))\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{n R r} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\operatorname{tr}_{\eta} h-h(\nu, \nu)\right)
\end{aligned}
$$

$$
\begin{align*}
& +\int_{\partial M}\|(\nabla h)(\nu)\|_{\eta}^{2}+\frac{1}{2} \int_{\partial M}\left\|\nabla_{\nu} h\right\|_{\eta}^{2} \\
& +\int_{\partial M}((\nabla h)(\nu))^{\beta \alpha}((\nabla h)(\nu))_{\alpha \beta}-2 \int_{\partial M}((\nabla h)(\nu)):\left(\nabla_{\nu} h\right) \\
& -\int_{\partial M}|(\nabla h)(\nu, \nu)|_{\eta}^{2}-\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))^{2}-\frac{1}{2 n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2} \\
& +\frac{n-1}{2 n} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}+\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\frac{2}{n} \int_{\partial M}(\operatorname{div} h(\nu))\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)-\frac{1}{n} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) . \tag{B.38}
\end{align*}
$$

Similarly, we calculate (using that $\left.H\right|_{t=0}$ is constant on $\partial M$ )

$$
\begin{aligned}
\left.\partial_{t}\right|_{t=0}(H-\bar{H})= & -\frac{\sqrt{R^{2}-r^{2}}}{R r}\left(\operatorname{tr}_{\eta} h-f_{\partial M} \operatorname{tr}_{\eta} h\right)+\frac{\sqrt{R^{2}-r^{2}}}{R r}\left(h(\nu, \nu)-f_{\partial M} h(\nu, \nu)\right) \\
& +n \frac{\sqrt{R^{2}-r^{2}}}{2 R r}\left(h(\nu, \nu)-f_{\partial M} h(\nu, \nu)\right)-\operatorname{div} h(\nu)+f_{\partial M} \operatorname{div} h(\nu) \\
& +\frac{1}{2}\left(\nabla_{\nu} h\right)(\nu, \nu)-\frac{1}{2} f_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)+\frac{1}{2} \nabla_{\nu} \operatorname{tr}_{\eta} h-\frac{1}{2} f_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h \\
= & -\frac{\sqrt{R^{2}-r^{2}}}{R r} \operatorname{tr}_{\eta} h+\frac{\sqrt{R^{2}-r^{2}}}{R r} f_{\partial M} \operatorname{tr}_{\eta} h \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} h(\nu, \nu)-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} f_{\partial M} h(\nu, \nu) \\
& -\operatorname{div} h(\nu)+f_{\partial M} \operatorname{div} h(\nu)+\frac{1}{2}\left(\nabla_{\nu} h\right)(\nu, \nu) \\
\text { (B.39) } & -\frac{1}{2} f_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)+\frac{1}{2} \nabla_{\nu} \operatorname{tr}_{\eta} h-\frac{1}{2} f_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h
\end{aligned}
$$

to find

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}(H-\bar{H})^{2}= & -2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right) \operatorname{tr}_{\eta} h \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right) h(\nu, \nu) \\
& +n \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right) h(\nu, \nu)
\end{aligned}
$$

$$
\begin{aligned}
& -2 \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right)(\operatorname{div} h(\nu)) \\
& +\int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right)\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +\int_{\partial M}\left(\left.\partial_{t}\right|_{t=0}(H-\bar{H})\right)\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text { (B. 39) } \\
& \stackrel{\text { B. 39) }}{=2} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M}\left(\operatorname{tr}_{\eta} h\right)^{2}-2 \frac{R^{2}-r^{2}}{\widetilde{\mathrm{Vol}^{2} r^{2}}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)^{2}-(n+2) \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} h(\nu, \nu) \operatorname{tr}_{\eta} h \\
& +(n+2) \frac{R^{2}-r^{2}}{\operatorname{Vol}^{2} r^{2}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)+2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \operatorname{div} h(\nu) \\
& -2 \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right)-\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu) \operatorname{tr}_{\eta} h \\
& +\frac{\sqrt{R^{2}-r^{2}}}{\widehat{\mathrm{Vol} R r}}\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)-\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \nabla_{\nu} \operatorname{tr}_{\eta} h \\
& +\frac{\sqrt{R^{2}-r^{2}}}{\widehat{\mathrm{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -(n+2) \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} h(\nu, \nu) \operatorname{tr}_{\eta} h+(n+2) \frac{R^{2}-r^{2}}{\widehat{\operatorname{Vol}^{2} r^{2}}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)
\end{aligned}
$$

$$
\begin{aligned}
& -(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} h(\nu, \nu) \operatorname{div} h(\nu)+(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} \int_{\partial M} h(\nu, \nu)\left(\nabla_{\nu} h\right)(\nu, \nu)-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 \widehat{\mathrm{VolRr}}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} \int_{\partial M} h(\nu, \nu) \nabla_{\nu} \operatorname{tr}_{\eta} h-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 \widehat{\mathrm{VolR} R}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \operatorname{div} h(\nu)-2 \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right) \\
& -(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} h(\nu, \nu) \operatorname{div} h(\nu)+(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right) \\
& +2 \int_{\partial M}(\operatorname{div} h(\nu))^{2}-\frac{2}{\widehat{\operatorname{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)^{2}-\int_{\partial M}(\operatorname{div} h(\nu))\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\frac{1}{\widehat{\text { Vol }}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)-\int_{\partial M}(\operatorname{div} h(\nu))\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +\frac{1}{\widehat{\mathrm{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \nabla_{\nu} \operatorname{tr}_{\eta} h+\frac{\sqrt{R^{2}-r^{2}}}{\widehat{\text { Vol } R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} \int_{\partial M} h(\nu, \nu) \nabla_{\nu} \operatorname{tr}_{\eta} h-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 \widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -\int_{\partial M}(\operatorname{div} h(\nu))\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)+\frac{1}{\widehat{\text { Vol }}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +\frac{1}{2} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)-\frac{1}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2}-\frac{1}{2 \widehat{\text { Vol }}}\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2} \\
& -\frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h\left(\nabla_{\nu} h\right)(\nu, \nu)+\frac{\sqrt{R^{2}-r^{2}}}{\widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 R r} \int_{\partial M} h(\nu, \nu)\left(\nabla_{\nu} h\right)(\nu, \nu)-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{2 \widehat{\mathrm{VolRr}}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& -\int_{\partial M}(\operatorname{div} h(\nu))\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)+\frac{1}{\widehat{\operatorname{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\frac{1}{2} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}-\frac{1}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2} \\
& +\frac{1}{2} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)-\frac{1}{2 \widehat{\mathrm{Vol}}}\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& =2 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M}\left(\operatorname{tr}_{\eta} h\right)^{2}-2 \frac{R^{2}-r^{2}}{\widetilde{\operatorname{Vol}} R^{2} r^{2}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)^{2} \\
& -2(n+2) \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} h(\nu, \nu) \operatorname{tr}_{\eta} h \\
& +2(n+2) \frac{R^{2}-r^{2}}{\widetilde{\operatorname{Vol}} R^{2} r^{2}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right) \\
& +(n+2)^{2} \frac{R^{2}-r^{2}}{2 R^{2} r^{2}} \int_{\partial M}(h(\nu, \nu))^{2} \\
& -(n+2)^{2} \frac{R^{2}-r^{2}}{2 \widetilde{\mathrm{Vol}} R^{2} r^{2}}\left(\int_{\partial M} h(\nu, \nu)\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& +4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \operatorname{div} h(\nu) \\
& -4 \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right) \\
& -2(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} h(\nu, \nu) \operatorname{div} h(\nu) \\
& +2(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol}_{2 r}}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \operatorname{div} h(\nu)\right) \\
& -2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h \nabla_{\nu} \operatorname{tr}_{\eta} h \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol}_{2}}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} h(\nu, \nu) \nabla_{\nu} \operatorname{tr}_{\eta} h \\
& -(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol}_{2}}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} \operatorname{tr}_{\eta} h\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& +2 \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\mathrm{Vol} R r}}\left(\int_{\partial M} \operatorname{tr}_{\eta} h\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} h(\nu, \nu)\left(\nabla_{\nu} h\right)(\nu, \nu) \\
& -(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\operatorname{Vol} R r}}\left(\int_{\partial M} h(\nu, \nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +2 \int_{\partial M}(\operatorname{div} h(\nu))^{2}-\frac{2}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)^{2} \\
& +\frac{1}{2} \int_{\partial M}\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2}-\frac{1}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right)^{2} \\
& +\frac{1}{2} \int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2}-\frac{1}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)^{2} \\
& -2 \int_{\partial M}(\operatorname{div} h(\nu))\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& +\frac{2}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -2 \int_{\partial M}(\operatorname{div} h(\nu))\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\frac{2}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \operatorname{div} h(\nu)\right)\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right) \\
& +\int_{\partial M}\left(\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\nabla_{\nu} \operatorname{tr}_{\eta} h\right) \\
& -\frac{1}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M}\left(\nabla_{\nu} h\right)(\nu, \nu)\right)\left(\int_{\partial M} \nabla_{\nu} \operatorname{tr}_{\eta} h\right)
\end{aligned}
$$

Finally, and completely analogously to the derivation of (B.19), we obtain

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}\|\operatorname{Ric}\|_{g}^{2}= & -\left.2 \int_{\partial M}^{(\mathrm{B} .36)}\left(\operatorname{Riem}_{\mu}^{\alpha}{ }_{\mu}{ }_{\nu}\right)\right|_{t=0}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) h^{\mu \nu} \\
& -\int_{\partial M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \nabla^{\alpha} \nabla^{\beta} \operatorname{tr}_{\eta} h-\int_{\partial M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \Delta h^{\alpha \beta} \\
& +2 \int_{\partial M}\left(\left.\partial_{t}\right|_{t=0} \operatorname{Ric}_{\alpha \beta}\right) \nabla^{\alpha} \operatorname{div} h^{\beta} \\
= & \frac{2}{R^{4}} \int_{\partial M} \stackrel{\circ}{h}: h-\frac{2}{R^{2}} \int_{\partial M} \stackrel{\circ}{h:} \nabla^{2} \operatorname{tr}_{\eta} h \\
& -\frac{2}{R^{2}} \int_{\partial M} \stackrel{\circ}{h: \Delta h}+\frac{4}{R^{2}} \int_{\partial M} \stackrel{\circ}{h}: \nabla \operatorname{div} h \\
& +\frac{1}{2} \int_{\partial M}\left\|\nabla^{2} \operatorname{tr}_{\eta} h\right\|_{\eta}^{2}+\frac{1}{2} \int_{\partial M}\|\Delta h\|_{\eta}^{2}+\int_{\partial M}\|\nabla \operatorname{div} h\|_{\eta}^{2} \\
& -\frac{2}{n+1} \int_{\partial M}\left(\Delta \operatorname{tr}_{\eta} h\right)^{2}-\frac{2}{n+1} \int_{\partial M}(\operatorname{div} \operatorname{div} h)^{2} \\
& +\int_{\partial M} \nabla^{2} \operatorname{tr}_{\eta} h: \Delta h-2 \int_{\partial M} \nabla^{2} \operatorname{tr}_{\eta} h: \nabla \operatorname{div} h \\
& -2 \int_{\partial M} \Delta h: \nabla \operatorname{div} h+\int_{\partial M} \nabla^{\beta} \operatorname{div} h^{\alpha} \nabla_{\alpha} \operatorname{div} h_{\beta} \\
& +\frac{4}{n+1} \int_{\partial M}\left(\Delta \operatorname{tr}_{g} h\right)(\operatorname{div} \operatorname{div} h) .
\end{aligned}
$$

## 4. The special case $h=f g$

For $f$ a smooth function, the ansatz $h=f g$ implicates on $M$ :
$\operatorname{tr}_{g} h=(n+1) f, \quad \stackrel{\circ}{h}=0, \quad \nabla^{2} \operatorname{tr}_{g} h=(n+1)$ Hess $f, \quad \Delta \operatorname{tr}_{g} h=(n+1) \Delta f$,
$\Delta h=(\Delta f) g, \quad \operatorname{div} h=\nabla f, \quad \nabla \operatorname{div} h=\operatorname{Hess} f, \quad \operatorname{div} \operatorname{div} h=\Delta f$,
so that

$$
\begin{aligned}
\left\|\nabla^{2} \operatorname{tr}_{g} h\right\|_{g}^{2} & =(n+1)^{2}\|\operatorname{Hess} f\|_{g}^{2}, & \|\Delta h\|_{g}^{2} & =(n+1)(\Delta f)^{2} \\
\nabla^{2} \operatorname{tr}_{g} h: \Delta h & =(n+1)(\Delta f)^{2}, & \nabla^{2} \operatorname{tr}_{g} h: \nabla \operatorname{div} h & =(n+1)\|\operatorname{Hess} f\|_{g}^{2} \\
\Delta h: \nabla \operatorname{div} h & =(\Delta f)^{2}, & \nabla^{\beta} \operatorname{div} h^{\alpha} \nabla_{\alpha} \operatorname{div} h_{\beta} & =\| \text { Hess } f \|_{g}^{2}
\end{aligned}
$$

It follows that, in this case,

$$
\begin{align*}
\left.\partial_{t}^{2}\right|_{t=0} \int_{M}\|\mathrm{Ric}\|_{g}^{2}= & 0-0-0+0 \\
& +\frac{1}{2}(n+1)^{2} \int_{M}\|\operatorname{Hess} f\|_{\eta}^{2}+\frac{1}{2}(n+1) \int_{M}(\Delta f)^{2} \\
& +\int_{M}\|\operatorname{Hess} f\|_{\eta}^{2}-\frac{2}{n+1}(n+1)^{2} \int_{M}(\Delta f)^{2} \\
& -\frac{2}{n+1} \int_{M}(\Delta f)^{2}+(n+1) \int_{M}(\Delta f)^{2} \\
& -2(n+1) \int_{M}\|\operatorname{Hess} f\|_{\eta}^{2}-2 \int_{M}(\Delta f)^{2} \\
& +\int_{M}\|\operatorname{Hess} f\|_{\eta}^{2}+\frac{4}{n+1}(n+1) \int_{M}(\Delta f)^{2} \\
= & \frac{1}{2}(n-1)^{2}\left(\int_{M}\|\operatorname{Hess} f\|_{\eta}^{2}-\frac{1}{n+1} \int_{M}(\Delta f)^{2}\right) \\
= & \frac{1}{2}(n-1)^{2} \int_{M}\|\operatorname{Hess} f\|_{\eta}^{2},
\end{align*}
$$

where $\operatorname{Hess}^{\circ} f$ denotes the traceless part of Hess $f$. Similarly,

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{M}(\text { Scal }-\overline{\text { Scal }})^{2}= & \frac{2 n^{2}}{R^{4}}(n+1)^{2} \int_{M} f^{2}+\frac{4 n}{R^{2}}(n+1)^{2} \int_{M} f \Delta f \\
& -\frac{4 n}{R^{2}}(n+1) \int_{M} f \Delta f+2(n+1)^{2} \int_{M}(\Delta f)^{2} \\
& -\frac{2}{\mathrm{Vol}}(n+1)^{2}\left(\int_{M} \Delta f\right)^{2}+2 \int_{M}(\Delta f)^{2} \\
& -\frac{2}{\mathrm{Vol}}\left(\int_{M} \Delta f\right)^{2}-4(n+1) \int_{M}(\Delta f)^{2} \\
& +\frac{4}{\mathrm{Vol}}(n+1)\left(\int_{M} \Delta f\right)^{2} \\
= & \frac{2 n^{2}}{R^{4}}(n+1)^{2} \int_{M} f^{2}+\frac{4 n^{2}}{R^{2}}(n+1) \int_{M} f \Delta f \\
& +2 n^{2} \int_{M}(\Delta f)^{2}-\frac{2 n^{2}}{\mathrm{Vol}}\left(\int \Delta f\right)^{2} \\
\text { B. 43) } \quad & \left.=2 n^{2}\left(\left(\frac{n+1}{R} f+\Delta f\right)-\frac{n+1}{R} f+\Delta f\right)\right)^{2} .
\end{aligned}
$$

On the other hand, the ansatz $h=f g$ yields on the boundary $\partial M$ :

$$
\left.\begin{array}{rlrl}
\operatorname{tr}_{g} h & =(n+1) f, & \nabla_{\nu} \operatorname{tr}_{g} h & =(n+1) \nabla_{\nu} f, \\
h(\nu) & =f \nu^{b}, & h(\nu, \nu) & =f, \\
\nabla h & =(\nabla f) g, & (\nabla h)(\nu) & =(\nabla f) \nu^{b},
\end{array} r(\nabla h)(\nu, \nu)=\nabla f,\right\}
$$

where $\nu^{\mathrm{b}}$ is defined as $\left(\nu^{\mathrm{b}}\right)_{\alpha}=g_{\alpha \beta} \nu^{\beta}$. Consequently,

$$
\begin{aligned}
\|h\|_{g}^{2} & =(n+1) f^{2}, & |h(\nu)|_{g}^{2} & =f^{2}, \\
\|(\nabla h)(\nu)\|_{g}^{2} & =|\nabla f|_{g}^{2}, & (\nabla h(\nu))^{\beta \alpha}(\nabla h(\nu))_{\alpha \beta} & =\left(\nabla_{\nu} f\right)^{2}, \\
(\nabla h)(\nu): h & =f \nabla_{\nu} f, & (\nabla h)(\nu): \nabla_{\nu} h & =\left(\nabla_{\nu} f\right)^{2}, \\
(\nabla h)(\nu, \nu) \cdot h(\nu) & =f \nabla_{\nu} f, & |(\nabla h)(\nu, \nu)|_{g}^{2} & =|\nabla f|_{g}^{2}, \\
\left\|\nabla_{\nu} h\right\|_{g}^{2} & =(n+1)\left(\nabla_{\nu} f\right)^{2}, & \left(\nabla_{\nu} h\right): h & =(n+1) f \nabla_{\nu} f .
\end{aligned}
$$

It then follows that,

$$
\begin{align*}
\left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}\|\AA\|_{g}^{2}= & 2(n+1) \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}-4 \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2} \\
& +2 \frac{n-1}{n} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}+4 \frac{n+1}{n} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2} \\
& -2 \frac{(n+1)^{2}}{n} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2} \\
& +4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f-2(n+1) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& -4 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f+2 \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& -4 \frac{n}{n} \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f+2 \frac{n(n+1)}{n} \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& +2 \frac{n}{n} \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& +\int_{\partial M}^{|\nabla f|_{\eta}^{2}+\frac{1}{2}(n+1)} f_{\partial M}\left(\nabla_{\nu} f\right)^{2}+\int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
& -2 \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\int_{\partial M}|\nabla f|_{\eta}^{2}-\frac{2}{n} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
& -\frac{1}{2 n}(n+1)^{2} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}+\frac{n-1}{2 n} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}+\frac{2}{n} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
& +\frac{2}{n}(n+1) \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\frac{1}{n}(n+1) \int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
= & 0 .
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}(H-\bar{H})^{2}=2(n+1)^{2} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}-2(n+1)^{2} \frac{R^{2}-r^{2}}{\widehat{\operatorname{Vol}^{2} r^{2}}}\left(\int_{\partial M} f\right)^{2} \\
& -2(n+1)(n+2) \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}+2(n+1)(n+2) \frac{R^{2}-r^{2}}{\widetilde{\operatorname{Vol}} R^{2} r^{2}}\left(\int_{\partial M} f\right)^{2} \\
& +\frac{(n+2)^{2}}{2} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}-\frac{(n+2)^{2}}{2} \frac{R^{2}-r^{2}}{\widehat{\operatorname{Vol}^{2} r^{2}}}\left(\int_{\partial M} f\right)^{2} \\
& +4(n+1) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f-4(n+1) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\text { Vol } R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& -2(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f+2(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\operatorname{Vol} R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& -2(n+1)^{2} \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f+2(n+1)^{2} \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\mathrm{Vol} R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& +(n+1)(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& -(n+1)(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\text { Vol } R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& -2(n+1) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f+2(n+1) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\text { VolRr }}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& +(n+2) \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f-(n+2) \frac{\sqrt{R^{2}-r^{2}}}{\widehat{\mathrm{Vol} R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& +2 \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\frac{2}{\widehat{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2}+\frac{(n+1)^{2}}{2} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
& -\frac{(n+1)^{2}}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2}+\frac{1}{2} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\frac{1}{2 \widehat{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2} \\
& -2(n+1) \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}+2 \frac{n+1}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2}-2 \int_{\partial M}\left(\nabla_{\nu} f\right)^{2} \\
& +\frac{2}{\widehat{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2}+(n+1) \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\frac{n+1}{\widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2} \\
& =\frac{n^{2}}{2} \frac{R^{2}-r^{2}}{R^{2} r^{2}} \int_{\partial M} f^{2}-\frac{n^{2}}{2} \frac{R^{2}-r^{2}}{\widetilde{\mathrm{Vol} R^{2} r^{2}}\left(\int_{\partial M} f\right)^{2}, ~} \\
& -n^{2} \frac{\sqrt{R^{2}-r^{2}}}{R r} \int_{\partial M} f \nabla_{\nu} f \\
& +n^{2} \frac{\sqrt{R^{2}-r^{2}}}{\widetilde{\mathrm{Vol} R r}}\left(\int_{\partial M} f\right)\left(\int_{\partial M} \nabla_{\nu} f\right) \\
& +\frac{n^{2}}{2} \int_{\partial M}\left(\nabla_{\nu} f\right)^{2}-\frac{n^{2}}{2 \widetilde{\mathrm{Vol}}}\left(\int_{\partial M} \nabla_{\nu} f\right)^{2} \\
& =\frac{n^{2}}{2} \int_{\partial M}\left(\begin{array}{c}
\left(\frac{\sqrt{R^{2}-r^{2}}}{R r} f-\nabla_{\nu} f\right) \\
\left.-\frac{\left(\frac{\sqrt{R^{2}-r^{2}}}{R r} f-\nabla_{\nu} f\right)}{2}\right)^{2} . . . . . . ~ . . ~ . ~ . ~
\end{array}\right. \tag{B.45}
\end{align*}
$$

Finally, in exactly the same way we obtained (B.42), we have
(B.46)

$$
\begin{aligned}
\left.\partial_{t}^{2}\right|_{t=0} \int_{\partial M}\|\mathrm{Ric}\|_{g}^{2} & =\frac{1}{2}(n-1)^{2}\left(\int_{\partial M}\|\operatorname{Hess} f\|_{\eta}^{2}-\frac{1}{n+1} \int_{\partial M}(\Delta f)^{2}\right) \\
& =\frac{1}{2}(n-1)^{2} \int_{\partial M}\|\operatorname{Hess} f\|_{\eta}^{2} .
\end{aligned}
$$

## Bibliography

[And94] Ben Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differential Equations 2 (1994), no. 2, 151-171. MR 1385524 (97b:53012) (cited on page xii)
[Aub98] Thierry Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR MR1636569 (99i:58001) (cited on pages xix and 46)
[Bar95] Robert G. Bartle, The elements of integration and Lebesgue measure, Wiley Classics Library, John Wiley \& Sons Inc., New York, 1995, Containing a corrected reprint of the 1966 original [The elements of integration, Wiley, New York; MR0200398 (34 \# 293)], A Wiley-Interscience Publication. MR 1312157 (95k:28001) (cited on page xix)
[Ber03] Marcel Berger, A panoramic view of Riemannian geometry, Springer-Verlag, Berlin, 2003. MR 2002701 (2004h:53001) (cited on page xix)
[Bes87] Arthur L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR 867684 (88f:53087) (cited on page xix)
[BF36] Herbert Busemann and Willy Feller, Krümmungseigenschaften Konvexer Flächen, Acta Math. 66 (1936), no. 1, 1-47. MR 1555408 (cited on page xii)
[BG92] Marcel Berger and Bernard Gostiaux, Géométrie différentielle: variétés, courbes et surfaces, second ed., Mathématiques. [Mathematics], Presses Universitaires de France, Paris, 1992. MR 1207362 (93j:53001) (cited on page xix)
[BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038) (cited on page 32)
[Bor67] Yuriĭ E. Borovskiŭ, Convex surfaces with a quasiconformal spherical mapping, Sibirsk. Mat. Z. 8 (1967), 535-547. MR 0214759 ( $35 \# 5608$ ) (cited on page xii)
[Bor68] __, An estimate for convex surfaces with a quasiconformal spherical mapping, Sibirsk. Mat. Ž. 9 (1968), 530-535. MR 0227399 (37 \#2983) (cited on page xii)
[Bou90] Joseph V. Boussinesq, Cours d'analyse infinitésimale: à l'usage des personnes qui étudient cette science en vue de ses applications mécaniques et physiques, vol. I-II, Gauthier-Villard et fils, Paris, 1887-1890. (cited on page xiv)
[Bre83] Haïm Brezis, Analyse fonctionnelle, Collection Mathématiques Appliquées pour la Maitrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983, Théorie et applications. [Theory and applications]. MR 697382 (85a:46001) (cited on pages xix, 40 and 77 )
[Car65] Constantin Carathéodory, Calculus of variations and partial differential equations of the first order. Part I: Partial differential equations of the first order, Translated by Robert B. Dean and Julius J. Brandstatter, Holden-Day Inc., San Francisco, 1965. MR MR0192372 (33 \#597) (cited on page 60)
[Cha06] Isaac Chavel, Riemannian geometry, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction. MR 2229062 (2006m:53002) (cited on pages 32, 33 and 42)
[CL57] Shiing-shen Chern and Richard K. Lashof, On the total curvature of immersed manifolds, Amer. J. Math. 79 (1957), 306-318. MR 0084811 (18,927a) (cited on page 22)
[CLN06] Bennett Chow, Peng Lu, and Lei Ni, Hamilton's Ricci flow, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 2006. MR 2274812 (2008a:53068) (cited on pages xvii and 46)
[CZ56] Alberto P. Calderón and Antoni Zygmund, On singular integrals, Amer. J. Math. 78 (1956), 289-309. MR 0084633 (18,894a) (cited on page xv)
[Dar96] Gaston Darboux, Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, vol. 1-4, Gauthier-Villard, Paris, 1887-1896. (cited on page xiv)
[Dar12] , Notice historique sur le général Meusnier, Éloges académiques et discours: vol. publ. par le Comité du jubilé scientifique de M. Gaston Darboux, A. Hermann et fils, Paris, 1912, pp. 218-262. (cited on page xiv)
[dC76] Manfredo P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976, Translated from the Portuguese. MR 0394451 (52 \#15253) (cited on pages xi and xix)
[dC92] , Riemannian geometry, Mathematics: Theory \& Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. MR MR1138207 (92i:53001) (cited on page xix)
[Des92] Sharief Deshmukh, Hypersurfaces of nonnegative Ricci curvature in a Euclidean space, J. Geom. 45 (1992), no. 1-2, 48-50. MR 1188097 (93m:53056) (cited on page 48)
[DFN85] Boris A. Dubrovin, Anatolĭ̆ T. Fomenko, and Sergeй P. Novikov, Modern geometrymethods and applications. Part II, Graduate Texts in Mathematics, vol. 104, SpringerVerlag, New York, 1985, The geometry and topology of manifolds, Translated from the Russian by Robert G. Burns. MR 807945 ( $86 \mathrm{~m}: 53001$ ) (cited on page 49)
[DFN90] , Modern geometry-methods and applications. Part III, Graduate Texts in Mathematics, vol. 124, Springer-Verlag, New York, 1990, Introduction to homology theory, Translated from the Russian by Robert G. Burns. MR 1076994 (91j:55001) (cited on page 50)
[Dis71] V. I. Diskant, Certain estimates for convex surfaces with a bounded curvature function, Sibirsk. Mat. Ž. 12 (1971), 109-125. MR 0284954 (44 \#2178) (cited on page xii)
[DLM05] Camillo De Lellis and Stefan Müller, Optimal rigidity estimates for nearly umbilical surfaces, J. Differential Geom. 69 (2005), no. 1, 75-110. MR MR2169583 (2006e:53078) (cited on pages vi, vii, xiii, xiv, 50, 65 and 66)
[DLM06] _, A $C^{0}$ estimate for nearly umbilical surfaces, Calc. Var. Partial Differential Equations 26 (2006), no. 3, 283-296. MR 2232206 (2007d:53003) (cited on page xiii)
[DLT10] Camillo De Lellis and Peter M. Topping, Almost-Schur lemma, ArXiv e-prints (2010), arXiv:1003.3527v1 [math.DG]. (cited on pages xv, xvi, xvii, 45, 46 and 83)
[EG92] Lawrence C. Evans and Ronald F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660 (93f:28001) (cited on page xix)
[Eis20] Luther P. Eisenhart, Darboux's Anteil an der Geometrie, Acta Math. 42 (1920), no. 1, 275-284. MR 1555167 (cited on page xiv)
[Eul67] Leonhard Euler, Recherches sur la courbure des surfaces, Histoire de l'académie royale des sciences et belles lettres de Berlin 16 (1767), 119-143. (cited on page xiv)
[Eva98] Lawrence C. Evans, Partial differential equations, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR MR1625845 (99e:35001) (cited on pages xix, 9, 15, 28, 77 and 79)
[Fed69] Herbert Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR 0257325 (41 \#1976) (cited on pages 33, 42 and 49)
[Fet63] Abram I. Fet, Stability theorems for convex, almost spherical surfaces, Dokl. Akad. Nauk SSSR 153 (1963), 537-539. MR 0157292 (28 \#527) (cited on page xii)
[Ger90] Claus Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differential Geom. 32 (1990), no. 1, 299-314. MR 1064876 (91k:53016) (cited on page 51)
[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, Riemannian geometry, third ed., Universitext, Springer-Verlag, Berlin, 2004. MR 2088027 (2005e:53001) (cited on page xix)
[GT01] David Gilbarg and Neil S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR MR1814364 (2001k:35004) (cited on pages xix, 9 and 28)
[Gug69] Heinrich W. Guggenheimer, Nearly spherical surfaces, Aequationes Math. 3 (1969), 186-193. MR 0247595 ( $40 \# 859$ ) (cited on page xii)
[Har47] Philip Hartman, Systems of total differential equations and Liouville's theorem on conformal mappings, Amer. J. Math. 69 (1947), 327-332. MR 0021395 (9,59h) (cited on page xii)
[HI08] Gerhard Huisken and Tom Ilmanen, Higher regularity of the inverse mean curvature flow, J. Differential Geom. 80 (2008), no. 3, 433-451. MR 2472479 (2010c:53097) (cited on page 50)
[Hil20] David Hilbert, Gaston Darboux, Acta Math. 42 (1920), no. 1, 269-273, 1842-1917. MR 1555166 (cited on page xiv)
[Jos08] Jürgen Jost, Riemannian geometry and geometric analysis, fifth ed., Universitext, Springer-Verlag, Berlin, 2008. MR 2431897 (2009g:53036) (cited on page xix)
[Kou71] Dimitri Koutroufiotis, Ovaloids which are almost spheres, Comm. Pure Appl. Math. 24 (1971), 289-300. MR 0282318 ( 43 \#8030) (cited on page xii)
[Küh08] Wolfgang Kühnel, Differentialgeometrie, fourth ed., Vieweg Studium: Aufbaukurs Mathematik. [Vieweg Studies: Mathematics Course], Vieweg, Wiesbaden, 2008, Kurven-Flächen-Mannigfaltigkeiten. [Curves-surfaces-manifolds]. MR 2527182 (2010k:53001) (cited on page xi)
[Lee97] John M. Lee, Riemannian manifolds, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997, An introduction to curvature. MR 1468735 (98d:53001) (cited on page xix)
[Lei99] Kurt Leichtweiß, Nearly umbilical ovaloids in the n-space are close to spheres, Results Math. 36 (1999), no. 1-2, 102-109. MR 1706540 (2000h:52007) (cited on page xii)
[LL97] Elliott H. Lieb and Michael Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997. MR 1415616 (98b:00004) (cited on pages 9 and 28)
[LM10] Tobias Lamm and Jan Metzger, Small surfaces of Willmore type in Riemannian manifolds, Int. Math. Res. Not. IMRN (2010), no. 19, 3786-3813. MR 2725514 (cited on page xiii)
[LMS09] Tobias Lamm, Jan Metzger, and Felix Schulze, Foliations of asymptotically flat manifolds by surfaces of Willmore type, ArXiv e-prints (2009), arXiv:0903.1277v1 [math.DG]. (cited on page xiii)
[MdL85] Jean-Baptiste-Marie-Charles Meusnier de Laplace, Mémoire sur la courbure des surfaces, Mémoires de mathématique et de physique, presentés à l'Académie royale des sciences, par divers sçavans \& lûs dans ses assemblées, vol. 10, Académie royale des sciences (France), Paris, 1785, pp. 477-513. (cited on page xiv)
[Met07] Jan Metzger, Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature, J. Differential Geom. 77 (2007), no. 2, 201-236. MR 2355784 (2008j:53042) (cited on page xiii)
[Mil97] John W. Milnor, Topology from the differentiable viewpoint, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original. MR 1487640 (98h:57051) (cited on page xix)
[Mon00] Gaspard Monge, Feuilles d'analyse appliquée à la géométrie: à l'usage de l'école polytechnique, Thermidor, Paris, 1800, first published in 1795. (cited on page xiv)
[Mon50] __, Application de l'analyse à la géométrie, 5 ed., Bachelier, Paris, 1850. (cited on page xiv)
[Moo73] John D. Moore, Almost spherical convex hypersurfaces, Trans. Amer. Math. Soc. 180 (1973), 347-358. MR 0320964 ( 47 \#9497) (cited on page xii)
[Nev69] N. S. Nevmeržickií, Stability in the umbilical surface theorem. I, Vestnik Leningrad. Univ. 24 (1969), no. 7, 55-60. MR 0251672 (40 \#4899) (cited on page xii)
[Nic07a] Liviu I. Nicolaescu, An invitation to Morse theory, Universitext, Springer, New York, 2007. MR 2298610 (2009m:58023) (cited on page 50)
[Nic07b] , Lectures on the geometry of manifolds, second ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007, see also http://www.nd.edu/~lnicolae/. MR 2363924 (2008g:53001) (cited on page xix)
[O'N83] Barrett O'Neill, Semi-Riemannian geometry, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, With applications to relativity. MR MR719023 (85f:53002) (cited on page xix)
[Pau08] Arno Pauly, Flächen mit lauter Nabelpunkten, Elem. Math. 63 (2008), no. 3, 141-144. MR 2424898 (2009d:53007) (cited on page xii)
[Pog67] Aleksei V. Pogorelov, Nearly spherical surfaces, J. Analyse Math. 19 (1967), 313-321. MR 0215263 (35 \#6105) (cited on page xii)
[Pog73] , Extrinsic geometry of convex surfaces, American Mathematical Society, Providence, R.I., 1973, Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35. MR MR0346714 (49 \#11439) (cited on pages xii, 19 and 21)
[Pre10] Andrew Pressley, Elementary differential geometry, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2010. MR 2598317 (2011b:53003) (cited on page xi)
[Res68] Yuriĭ G. Reshetnyak, Certain estimates for almost umbilical surfaces, Sibirsk. Mat. Ž. 9 (1968), 903-917. MR 0235496 (38 \#3805) (cited on page xii)
[Res94] , Stability theorems in geometry and analysis, Mathematics and its Applications, vol. 304, Kluwer Academic Publishers Group, Dordrecht, 1994, Translated from the 1982 Russian original by N. S. Dairbekov and V. N. Dyatlov, and revised by the author, Translation edited and with a foreword by S. S. Kutateladze. MR MR1326375 (96i:30016) (cited on pages xii, xiii, xiv and xv)
[Roc70] R. Tyrrell Rockafellar, Convex analysis, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683 (43 \#445) (cited on pages xix and 42)
[Rud87] Walter Rudin, Real and complex analysis, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157 (88k:00002) (cited on pages xix and 9)
[Rud91] , Functional analysis, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR 1157815 (92k:46001) (cited on pages xix, 8 and 28)
[RV74] A. Wayne Roberts and Dale E. Varberg, Another proof that convex functions are locally Lipschitz, Amer. Math. Monthly 81 (1974), 1014-1016. MR 0352371 (50 \#4858) (cited on pages 23,25 and 27)
[Sch88] Rolf Schneider, Closed convex hypersurfaces with curvature restrictions, Proc. Amer. Math. Soc. 103 (1988), no. 4, 1201-1204. MR 955009 (90a:53010) (cited on page xii)
[Sch89] $\qquad$ , Stability in the Aleksandrov-Fenchel-Jessen theorem, Mathematika 36 (1989), no. 1, 50-59. MR 1014200 (90h:52015) (cited on page xii)
[Sch93] , Convex bodies: the Brunn-Minkowski theory, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR 1216521 (94d:52007) (cited on pages xix, 22, 24, 27 and 39)
[Spi65] Michael Spivak, Calculus on manifolds. A modern approach to classical theorems of advanced calculus, W. A. Benjamin, Inc., New York-Amsterdam, 1965. MR 0209411 (35 \#309) (cited on page xix)
[Spi99] , A comprehensive introduction to differential geometry. Vol. IV, third ed., Publish or Perish Inc., Huston, Texas, 1999. (cited on pages xi and xii)
[Ste70] Elias M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR MR0290095 (44 \#7280) (cited on page 14)
[Str33] Dirk J. Struik, Outline of a history of differential geometry: I, Isis 19 (1933), no. 1, pp. 92-120 (English). (cited on page xiv)
[Str88] - Lectures on classical differential geometry, second ed., Dover Publications Inc., New York, 1988. MR 939369 (89b:53002) (cited on page xi)
[Tru96] Clifford Truesdell, Jean-Baptiste-Marie Charles Meusnier de la Place [sic!] (17541793): an historical note, Meccanica 31 (1996), no. 5, 607-610. MR 1420154 (cited on page xiv)
[Urb90] John I. E. Urbas, On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures, Math. Z. 205 (1990), no. 3, 355-372. MR 1082861 (92c:53037) (cited on page 51)
[Urb91] , An expansion of convex hypersurfaces, J. Differential Geom. 33 (1991), no. 1, 91-125. MR 1085136 ( $91 \mathrm{j}: 58155$ ) (cited on page 53)
[Vod70] Sergej K. Vodop' yanov, Estimates of the deviation of quasi-umbilical surfaces from a sphere, Sibirsk. Mat. Ž. 11 (1970), 971-987, 1195. MR 0298603 (45 \#7655) (cited on page xii)
[Vol63] Yuriĭ A. Volkov, Stability of the solution of Minkowski's problem, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom 18 (1963), no. 1, 33-43. MR 0146738 (26 \#4258) (cited on page xii)

## Curriculum Vitae

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