AREA MINIMIZING CURRENTS MOD $2Q$:
LINEAR REGULARITY THEORY

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ABSTRACT. We establish a theory of Q-valued functions minimizing a suitable generalization of the Dirichlet integral. In a second paper the theory will be used to approximate efficiently area minimizing currents $\mathrm{mod}(p)$ when $p = 2Q$, and to establish a first general partial regularity theorem for every p in any dimension and codimension.

KEYWORDS: multiple valued functions, Dirichlet integral, regularity theory, area minimizing currents $\mathrm{mod}(p)$, minimal surfaces, linearization.

AMS MATH SUBJECT CLASSIFICATION (2010): 49Q15, 49Q05, 49N60, 35B65, 35J47.

1. INTRODUCTION

The aim of this work and its companion paper [5] is to give a proof of the following partial regularity theorem (for the definition of area minimizing currents $\mathrm{mod}(p)$ and the relevant terminology and notation we refer to [5]):

Theorem 1.1. *Assume $p \in \mathbb{N} \setminus \{0, 1\}$ and $a_0 > 0$, $\Sigma \subset \mathbb{R}^{m+n}$ is a complete C^{3,a_0} submanifold without boundary of dimension $m + \bar{n}$, $\Omega \subset \mathbb{R}^{m+n}$ is open and T is an m-dimensional integer rectifiable current supported in Σ which is area minimizing $\mathrm{mod}(p)$ in $\Omega \cap \Sigma$. Then, the interior singular set $\mathrm{Sing}(T)$ of T has Hausdorff dimension at most $m - 1$. If p in addition is odd, then the singular set is countably $(m - 1)$-rectifiable.*

The above result provides an affirmative answer in full generality to a question of B. White; see [1, Problem 4.20]. Prior to our work, some of the conclusions above were only known in some special cases. More precisely, in general codimension $\bar{n} > 1$:

(a) For $m = 1$ it is elementary that $\mathrm{Sing}(T)$ is discrete (and empty when $p = 2$);

(b) In general, Allard's interior regularity theory for stationary varifolds, cf. [2], implies that $\mathrm{Sing}(T)$ is a closed meager set in $(\mathrm{spt}^p(T) \cap \Omega) \setminus \mathrm{spt}^p(\partial T)$;

(c) For $p = 2$ $\mathrm{Sing}(T)$ has Hausdorff dimension at most $m - 2$ by Federer's classical work [12]; moreover the same reference shows that such set is in fact discrete when $m = 2$; for $m > 2$ its $(m - 2)$-rectifiability was first proved in [16], and the recent work [14] implies in addition that it has locally finite $\mathcal{H}^{m-2}$ measure.

In the case of codimension $\bar{n} = 1$ it was additionally known that:

(d) When $p = 2$, the singular set has $(m - 2)$-dimensional Hausdorff measure zero even in the case of minimizers of general uniformly elliptic integrands, see [15]; for the area functional, using [14], one can conclude additionally that it is $(m - 3)$-rectifiable and has locally finite $\mathcal{H}^{m-3}$ measure;

(e) When $p = 3$ and $m = 2$, [19] gives a complete description of the singularities, which consist of $C^{1,\alpha}$ arcs where three regular sheets meet at equal angles;

(f) When p is odd, [21] shows that the singular set has vanishing $\mathcal{H}^m$-Hausdorff measure for minimizers of a uniformly elliptic integrand, and that it has Hausdorff dimension at most $m-1$ for minimizers of the area functional;

(g) When $p = 4$, [20] shows that minimizers of uniformly elliptic integrands are represented by *immersed manifolds* outside of a closed set of zero $\mathcal{H}^{m-2}$ measure.

Our proof of Theorem 1.1 follows the blueprint of Almgren's partial regularity theory for area minimizing currents as worked out in the papers [10, 7, 6, 8, 9]. First of all, thanks to the general stratification theorem of the singular set, for every $\alpha > 0$ we know that at $\mathcal{H}^{m-1+\alpha}$-a.e. $x \in \operatorname{spt}^p(T) \setminus \operatorname{spt}^p(\partial T)$ there is at least one tangent cone which is flat, namely an integer multiple of an m-dimensional plane. If we call such points "flat", the main dimension estimate in Theorem 1.1 is achieved by showing that, for every $\alpha > 0$, $\mathcal{H}^{m-1+\alpha}$-a.e. flat point x is in fact regular. Every flat points x where the density of T is 1 is indeed regular by Allard's celebrated theorem. The problem arises when the multiplicity is higher than 1, because there are examples of singular flat points. For area minimizing *integral* currents such examples exist only in codimension $\bar{n} \geq 2$, whereas for area minimizing currents $\operatorname{mod}(p)$ such examples can be found also in codimension $\bar{n} = 1$ if p is even and larger than 2, see for instance Example 1.2 below.

An essential step in Almgren's theory is the approximation of the area minimizing currents, in regions where they are sufficiently close to an integer multiple of a plane, with multivalued functions which almost minimize an appropriate generalization of the Dirichlet energy. We will call "*linear theory*" the corresponding existence and regularity theory for those objects. In the case of integral currents a typical example where multivalued functions are needed is in the approximation of the current $[\![\Lambda]\!]$ induced by the holomorphic curve

$$\Lambda = \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$$

in a neighborhood of the origin (which is indeed a singular flat point of multiplicity 2). One way of understanding multiple-valued functions which take a fixed number Q of values is to model them as maps into the space of atomic measures with positive integral coefficients and total mass Q. For instance, slicing the the current Λ with (real) two-dimensional planes orthogonal to $\{(z, 0) : z \in \mathbb{C}\}$, for each $z \in \mathbb{C} \setminus \{0\}$ we find an integral 0-dimensional current which is the sum of two positive atoms:

$$\sum_{w^3=z^2} [\![(z, w)]\!] \ .$$

Such maps can be efficiently used to approximate area-minimizing currents $T \operatorname{mod}(p)$ in a neighborhood of a flat point x when

- either p is odd;
- or p is even and the density Q of T is strictly smaller than $\frac{p}{2}$.

When studying area-minimizing currents $\operatorname{mod}(p)$ for an even modulus $p = 2Q$ in a neighborhood of a flat point of density Q, the "classical" multivalued functions are not anymore the appropriate maps, as it is witnessed by the following example, taken from [20].

Example 1.2. Consider an open subset $\Omega \subset \mathbb{R}^2$ and two smooth functions $f, g : \Omega \to \mathbb{R}$ which solve the minimal surfaces equation in Ω. Assume in addition that the set $\{f = g\}$ contains a curve γ which divides Ω into two regions Ω_+ and Ω_-. Two explicit f and g are easy to find. The reader could take Ω to be a suitable ball B centered at the origin, $f \equiv 0$, and let g be the graph of the function which describes Enneper's minimal surface in a neighborhood of 0.

The set $\{f = g\}$ is then given by $\{(x, y) : x = \pm y\} \cap B$ and γ can be taken to be the segment $\{x = y\} \cap B$ while Ω_+ and Ω_- would then be $B \cap \{x > y\}$ and $B \cap \{x < y\}$, respectively.

We then define the following integral current T. Its support is the union of the graphs of f and g. However, while the portions of such graphs lying over Ω_+ will be taken with the standard orientation induced by Ω, the portions lying over Ω_- will be taken with the opposite orientation. In $\Omega \times \mathbb{R}$, the boundary of T is $4 \llbracket \gamma \rrbracket$. Moreover, by the structure theorem [20], the current is area minimizing   mod (4), because the graphs of f and g are both area minimizing currents   mod (2) (this could be proved using, for instance, the maximum principle).

The origin is a flat point of multiplicity $Q = 2$ for the current T above. By a simple rescaling procedure a good approximation of T in a neighborhood of the origin is given by the graphs of the second order Taylor polynomials of f and g at the origin. For the specific case described above where $f = 0$ and the graph of g is Enneper's surface, such functions are $f_0(x, y) = 0$ and $g_0(x, y) = 3(x^2 - y^2)$. This gives an obvious set-theoretic approximation of the support of the current T. In the approach that we outline in the rest of the paper, we will give to this set a structure of "special 2-valued function" h, where we consider the value $h(x, y)$ to be the sum of the two positive atoms $\llbracket f_0(x, y) \rrbracket + \llbracket g_0(x, y) \rrbracket$ on $\Omega_+ = B \cap \{x > y\}$ and the sum of two negative atoms $- \llbracket f_0(x, y) \rrbracket - \llbracket g_0(x, y) \rrbracket$ on $\Omega_- = B \cap \{x < y\}$. Such a choice is natural in view of the fact that the slices of the current T with lines orthogonal to the plane $\{(x, y, 0) : x, y \in \mathbb{R}\}$ are given by $\llbracket f(x, y) \rrbracket + \llbracket g(x, y) \rrbracket$ for $(x, y) \in \Omega_+$ and $- \llbracket f(x, y) \rrbracket - \llbracket g(x, y) \rrbracket$ for $(x, y) \in \Omega_-$.

Motivated by the above example, roughly speaking "special 2-valued functions" will be maps from Ω into the space of atomic measures with mass 2 satisfying the following requirements (cf. Definition 2.2 and Definition 2.7):

- The value of the map at any point in Ω is always either the sum of two positive atoms or the sum of two negative atoms;
- The domain Ω is subdivided by each map into three regions, the "positive region" where the values are two positive *disinct atoms*, the "negative region" where the values are two *distinct* negative atoms and the "interface", or "collapsed region", where the values are atoms counted with multiplicity 2: whether with a plus or minus sign, this will be of no relevance, because we will identify $-2 \llbracket z \rrbracket$ and $2 \llbracket z \rrbracket$ (which are equivalent 0-dimensional currents   mod (4)).

Roughly speaking, if the special 2-valued map is continuous, then the collapsed region disconnects the "positive" and the "negative" ones.

A natural Dirichlet energy, which comes out of Taylor expanding the area functional on the original current, is the sum of the Dirichlet energies of the various sheets: with such definition, the special 2-valued function h considered above is a minimizer of the Dirichlet energy, namely any competitor which coincides with it outside a compact set $K \subset\subset B$ has at least the same energy. This could be proved in an elementary way in our specific example, but it is also a general fact.

The reader might wonder why we introduce such complicated objects, rather than simply considering the union of the two graphs of f_0 and g_0 as a classical 2-valued function (namely, always taking positive atomic measures as values) as in [10]). The point is that with the latter choice, the resulting 2-valued function would not be a minimizer of the Dirichlet energy. A better competitor could be easily constructed by considering the following functions $\bar{f}$ and $\bar{g}$: both are harmonic in $B_1(0)$ and their values on $\partial B_1(0)$ are, respectively:

$$\bar{f}(x, y) = \begin{cases} 3(x^2 - y^2) & \text{if } |x| \geq |y| \\ 0 & \text{if } |x| \leq |y| \end{cases} \tag{1.1}$$

$$\bar{g}(x,y) = \begin{cases} 0 & \text{if } |x| \geq |y| \\ 3(x^2 - y^2) & \text{if } |x| \leq |y| \end{cases} \qquad (1.2)$$

The example above also shows that the regularity theory for Dirichlet-minimizing special Q-valued functions must necessarily allow for a larger set of singularities than its classical counterpart: indeed, for the special 2-valued map h constructed above any reasonable definition of the singular set $\mathrm{Sing}(h)$ must be such that $\{x = \pm y\} \cap B \subset \mathrm{Sing}(h)$, thus implying that the standard result $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq m - 2$ valid for a classical Q-valued map u defined on an m-dimensional domain and minimizing the Dirichlet energy (or even natural perturbations of the Dirichlet energy, see e.g. [18]) cannot hold true in our context.

Before proceeding with our analysis, let us remark that, in the paper [3], F. Almgren seems to initiate the investigation of a class of objects which are conceptually analogous to our special multiple valued functions. More precisely, Almgren's "multi-functions mod(p)" are defined as mappings taking values in the space of 0-dimensional integral polyhedral chains mod(p). The theory outlined in [3] may have some points in common with the content of Sections 2 and 4 of the present work, as well as Section 10 of [5]. The Dirichlet energy and the corresponding regularity theory, on the other hand, are not mentioned in [3], which rather seems to focus on describing the geometric properties of a class of *piecewise affine* multi-functions, which have the property to induce, via push-forward, dimension-preserving homomorphisms of the space of polyhedral chains. Since Almgren did not pursue this line of research anymore in later works, we don't know whether his ultimate goal was to seek a regularity theory for minimizing currents mod(p) along the lines of his Big Regularity Paper [4].

1.1. **Plan of the paper.** The first part of the paper aims at establishing the optimal partial regularity result for special Q-valued functions minimizing the Dirichlet energy. After providing the precise definition of the space $\mathscr{A}_Q(\mathbb{R}^n)$ of special Q-points in $\mathbb{R}^n$ and introducing the corresponding Sobolev spaces of $\mathscr{A}_Q(\mathbb{R}^n)$-valued maps, we show that any Dir-minimizing special Q-valued function u is Hölder continuous with respect to the natural metric space structure of $\mathscr{A}_Q(\mathbb{R}^n)$, and then that the - suitably defined - set $\mathrm{Sing}(u)$ of singular points of u is a closed subset of the m-dimensional domain of u having Hausdorff dimension $\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq m - 1$. We will then conclude the paper with some results concerning the geometry of (the currents associated to) the graphs of special multiple-valued functions, which will be crucial for the analysis to be carried out in [5].

2. Definition of $\mathscr{A}_Q(\mathbb{R}^n)$ and metric properties

For the classical Q-valued maps in $\mathbb{R}^n$, denoted $\mathcal{A}_Q(\mathbb{R}^n)$, we follow the terminology, notation and definitions of [10]. We first introduce the disjoint union $\mathcal{A}_Q(\mathbb{R}^n) \sqcup \mathcal{A}_Q(\mathbb{R}^n)$, which we identify with $\mathcal{A}_Q(\mathbb{R}^n) \times \{-1, 1\}$. Hence, an element in $\mathcal{A}_Q(\mathbb{R}^n) \sqcup \mathcal{A}_Q(\mathbb{R}^n)$ will be denoted by (S, ε), where S is an element of the space $\mathcal{A}_Q(\mathbb{R}^n)$ of atomic measures with positive integer coefficients and mass Q (namely $S = \sum_{i=1}^{Q} [\![P_i]\!]$ for $P_i \in \mathbb{R}^n$) and ε equals either 1 or -1.

Moreover, it is convenient to introduce the following notation. Recall that $\mathcal{G}(\cdot,\cdot)$ denotes the distance function in $\mathcal{A}_Q(\mathbb{R}^n)$.

Definition 2.1. If $S = \sum_i [\![S_i]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$, then $|S|^2 := \mathcal{G}(S, Q [\![0]\!])^2$ and

$$S \oplus v := \sum_i [\![S_i + v]\!]$$

$$S \ominus v := S \oplus (-v) = \sum_i [\![S_i - v]\!].$$

Note that, using $\boldsymbol{\eta}(S) := \frac{1}{Q} \sum_i S_i$, we get

$$|S|^2 = |S \ominus \boldsymbol{\eta}(S)|^2 + Q|\boldsymbol{\eta}(S)|^2 \tag{2.1}$$

$$\mathcal{G}(A,B)^2 = \mathcal{G}(A \ominus \boldsymbol{\eta}(A), B \ominus \boldsymbol{\eta}(B))^2 + Q|\boldsymbol{\eta}(A) - \boldsymbol{\eta}(B)|^2 \tag{2.2}$$

Definition 2.2. We denote by $\mathscr{A}_Q(\mathbb{R}^n)$ the quotient space

$$\mathscr{A}_Q(\mathbb{R}^n) := \mathcal{A}_Q(\mathbb{R}^n) \sqcup \mathcal{A}_Q(\mathbb{R}^n) / \sim$$

where $\sim$ is the equivalence relation given by

$$(S,-1) \sim (T,1) \iff \exists p \in \mathbb{R}^n \text{ with } S = Q [\![p]\!] = T, \tag{2.3}$$

$$(S,1) \sim (T,1) \iff S = T, \tag{2.4}$$

$$(S,-1) \sim (T,-1) \iff S = T. \tag{2.5}$$

We endow $\mathscr{A}_Q(\mathbb{R}^n)$ with the metric

$$\mathcal{G}_s((S,\alpha),(T,\beta))^2 = \begin{cases} \mathcal{G}(S,T)^2 & \text{if } \alpha = \beta \\[2mm] |S \ominus \boldsymbol{\eta}(S)|^2 + |T \ominus \boldsymbol{\eta}(T)|^2 + Q|\boldsymbol{\eta}(S) - \boldsymbol{\eta}(T)|^2 & \text{if } \alpha \neq \beta. \end{cases} \tag{2.6}$$

Remark 2.3. We can consider $\mathcal{G}_s$ as a pseudometric in $\mathcal{A}_Q(\mathbb{R}^n) \sqcup \mathcal{A}_Q(\mathbb{R}^n)$: $\mathscr{A}_Q(\mathbb{R}^n)$ results then from quotienting the corresponding pseudometric space to a metric space. It is hence straightforward to check that the quotient space topology coincides with the metric topology generated by $\mathcal{G}_s$.

Furthermore, for each $\alpha \in \{-1,1\}$ the injection $i_\alpha : \mathcal{A}_Q(\mathbb{R}^n) \ni S \mapsto (S,\alpha) \in \mathscr{A}_Q(\mathbb{R}^n)$ is an isometry.

Given the identification of $(Q [\![p]\!], 1)$ with $(Q [\![p]\!], -1)$, in the sequel we will often use the simplified notation $Q [\![p]\!]$ to denote both points in $\mathscr{A}_Q(\mathbb{R}^n)$.

Since working with the above definition of $\mathscr{A}_Q(\mathbb{R}^n)$ is sometimes inconvenient, we will next provide a useful characterization. We start by introducing the convention that, if (X,d) and (Y,δ) are two metric spaces, then, unless otherwise specified, we endow the product space $X \times Y$ with the product metric

$$d \times \delta((x,y),(v,w)) := \sqrt{d(x,v)^2 + \delta(y,w)^2}.$$

Definition 2.4. We denote by

- $\mathring{\mathcal{A}}_Q(\mathbb{R}^n)$ the space $\{T \in \mathcal{A}_Q(\mathbb{R}^n) : \boldsymbol{\eta}(T) = 0\} \subset \mathcal{A}_Q(\mathbb{R}^n)$ endowed with the metric $\mathcal{G}$;
- $\mathring{\mathscr{A}}_Q(\mathbb{R}^n)$ the space $\{(T,S) \in \mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \mathring{\mathcal{A}}_Q(\mathbb{R}^n) : \min\{|T|,|S|\} = 0\}$ endowed with the metric $\mathcal{G} \times \mathcal{G}$.

Remark 2.5. Observe that

$$\mathring{\mathscr{A}}_Q(\mathbb{R}^n) = \left(\mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \{Q\,[\![0]\!]\} \right) \cup \left(\{Q\,[\![0]\!]\} \times \mathring{\mathcal{A}}_Q(\mathbb{R}^n) \right) \subset \mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \mathring{\mathcal{A}}_Q(\mathbb{R}^n)\,.$$

Proposition 2.6. *Consider the metric spaces* $(\mathring{\mathscr{A}}_Q(\mathbb{R}^n), \mathcal{G} \times \mathcal{G})$ *and* $(\mathbb{R}^n, d)$ *where*

$$d(x,y) = \sqrt{Q}|x - y|\,.$$

Endow the product $\mathring{\mathscr{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ *with the corresponding product metric* $(\mathcal{G} \times \mathcal{G}) \times d$. *Then the map* $\boldsymbol{\iota} : \mathscr{A}_Q(\mathbb{R}^n) \to \mathring{\mathscr{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ *given by*

$$\boldsymbol{\iota}(T, \varepsilon) := \begin{cases} (T \ominus \boldsymbol{\eta}(T), Q\,[\![0]\!]\,, \boldsymbol{\eta}(T)) & \text{if } \varepsilon = 1\,, \\[2mm] (Q\,[\![0]\!]\,, T \ominus \boldsymbol{\eta}(T), \boldsymbol{\eta}(T)) & \text{if } \varepsilon = -1 \end{cases}$$

is an isometry with inverse

$$\boldsymbol{\iota}^{-1}(A, B, p) = \begin{cases} (A \oplus p, 1) & \text{if } |B| = 0 \\[2mm] (B \oplus p, -1) & \text{otherwise.} \end{cases}$$

In view of the previous proposition the metric $\mathcal{G} \times \mathcal{G}$ will be denoted by $\mathcal{G}_s$ when restricted to $\mathring{\mathscr{A}}_Q(\mathbb{R}^n)$.

Proof. It is clear that the maps $\boldsymbol{\iota}$ and $\boldsymbol{\iota}^{-1}$ are well defined, and it is also obvious that $\boldsymbol{\iota} \circ \boldsymbol{\iota}^{-1}$ and $\boldsymbol{\iota}^{-1} \circ \boldsymbol{\iota}$ are the identity map of the appropriate spaces.

Next, if we endow $\mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ with the product metric $\mathcal{G} \times d$, by (2.2) it is obvious that the map $\mathcal{A}_Q(\mathbb{R}^n) \ni A \mapsto (A \ominus \boldsymbol{\eta}(A), \boldsymbol{\eta}(A)) \in \mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ is an isometry with inverse $(A, v) \mapsto A \oplus v$. In particular this shows that, for any fixed $\varepsilon \in \{-1, 1\}$, the following holds

$$((\mathcal{G} \times \mathcal{G}) \times d)(\boldsymbol{\iota}(T, \varepsilon), \boldsymbol{\iota}(S, \varepsilon)) = \mathcal{G}_s((T, \varepsilon), (S, \varepsilon))\,.$$

On the other hand the identity $((\mathcal{G} \times \mathcal{G}) \times d)(\boldsymbol{\iota}(T, 1), \boldsymbol{\iota}(S, -1)) = \mathcal{G}_s((T, 1), (S, -1))$ is obvious from the definition of $\mathcal{G}_s$. $\qquad\square$

For further use, it is very convenient to introduce the following notation:

Definition 2.7. Let $u : E \to \mathscr{A}_Q(\mathbb{R}^n)$ be a Borel map, and consider the map $(v, w, z) = \boldsymbol{\iota} \circ u$. We then define:

$$\boldsymbol{\eta} \circ u := z \tag{2.7}$$
$$u^+ := v \oplus z \tag{2.8}$$
$$u^- := w \oplus z \tag{2.9}$$
$$E_+ := \{|v| > 0\} \tag{2.10}$$
$$E_- := \{|w| > 0\} \tag{2.11}$$
$$E_0 := \{|v| = |w| = 0\}\,. \tag{2.12}$$

Note in particular that E_+, E_- and E_0 are pairwise disjoint and their union is E: E_+, E_- and E_0 will be called the *canonical decomposition* of E induced by the map u. These sets are

those loosely described as positive, negative and collapsed regions in the example discussed in the introduction.

Similarly, consider a point $P = (R, S, z) \in \mathring{\mathscr{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ and a vector $z' \in \mathbb{R}^n$. We denote by $P \oplus z'$, resp. $P \ominus z'$, the points $(R, S, z + z')$ and $(R, S, z - z')$.

The following is thus an obvious corollary of Proposition 2.6.

Corollary 2.8. *Let $u : E \to \mathscr{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then $E_+, E_- \subset E$ are relatively open and $E_0 \subset E$ is relatively closed. Moreover $\boldsymbol{\eta} \circ u, u^+$ and u^- are all Lipschitz and their Lipschitz constants are at most $\mathrm{Lip}(u)$. More generally, if u is merely continuous, then $\boldsymbol{\eta} \circ u, u^+$ and u^- are also continuous and their modulus of continuity is at most that of u.*

Recall that any Lipschitz map $F : \mathbb{R}^{\bar{n}} \to \mathbb{R}^n$ induces a natural map $F : \mathcal{A}_Q(\mathbb{R}^{\bar{n}}) \to \mathcal{A}_Q(\mathbb{R}^n)$ via

$$F\left(\sum_i [\![T_i]\!]\right) := \sum_i [\![F(T_i)]\!] \,,$$

and hence a natural map $F : \mathscr{A}_Q(\mathbb{R}^{\bar{n}}) \to \mathscr{A}_Q(\mathbb{R}^n)$ by

$$F((T, \alpha)) := (F(T), \alpha) = \left(\sum_i [\![F(T_i)]\!], \alpha\right) \qquad \text{if } T = \sum_i [\![T_i]\!] \,.$$

In terms of the identification above we have

$$\left(\boldsymbol{\iota} \circ F \circ \boldsymbol{\iota}^{-1}\right)((R, S, z)) = \begin{cases} (F(R \oplus z) \ominus \boldsymbol{\eta}(F(R \oplus z)), Q[\![0]\!], \boldsymbol{\eta}(F(R \oplus z))) & \text{if } S = Q[\![0]\!] \\ (Q[\![0]\!], F(S \oplus z) \ominus \boldsymbol{\eta}(F(S \oplus z)), \boldsymbol{\eta}(F(S \oplus z))) & \text{if } R = Q[\![0]\!] \,. \end{cases}$$

3. Sobolev spaces, differentiability and Dirichlet energy

The embedding $\boldsymbol{\iota}$ allows to provide a straightforward definition of the Sobolev spaces $W^{1,p}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ using the theory developed in [10]. Similarly, we shall define the Dirichlet energy and its density.

Definition 3.1. Let Ω be an open subset of a C^1 manifold. We say that the function $u : \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$ belongs to the Sobolev space $W^{1,p}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ if each of the maps v, w, z given by $\boldsymbol{\iota}(u) = (v, w, z)$ belongs to the respective $W^{1,p}$ space.

If $u \in W^{1,2}$ we then define $|Du|^2 := |Dv|^2 + |Dw|^2 + Q|Dz|^2$ and the corresponding Dirichlet energy

$$\mathrm{Dir}(u, \Omega) := \int_\Omega |Du|^2 = \mathrm{Dir}(v, \Omega) + \mathrm{Dir}(w, \Omega) + Q\,\mathrm{Dir}(z, \Omega)\,.$$

Observe the validity of the identity (which holds as well for the "classical" Q-valued $W^{1,p}$ spaces)

$$\mathrm{Dir}(u, \Omega) = \mathrm{Dir}(u \ominus \boldsymbol{\eta} \circ u, \Omega) + Q\,\mathrm{Dir}(\boldsymbol{\eta} \circ u, \Omega)\,. \tag{3.1}$$

Using the definition above, one concludes obviously the analogues of

- The Lipschitz extension theorem, cf. [10, Theorem 1.7];
- The trace theorem, cf. [10, Proposition 2.10];
- The Sobolev embedding theorem, cf. [10, Proposition 2.11];
- The Poincaré inequality, cf. [10, Proposition 2.12];
- The Campanato-Morrey estimate of [10, Proposition 2.14].

From now on we will use all the results above referring to the corresponding statements in [10].

Next, it is useful to gain a local description of $|Du|$ in terms of the differentials of the maps u^+, u^- and $\boldsymbol{\eta} \circ u$. In particular this will allow us to apply the calculus tools of [10] making several computations straightforward.

Proposition 3.2. *Assume $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$. The maps u^+, u^- and $\boldsymbol{\eta} \circ u$ are approximately differentiable at a.e. point $x \in \Omega$. In particular, if we denote by $Du^+ = \sum_i [\![Du_i^+]\!]$, $Du^- = \sum_i [\![Du_i^-]\!]$ and $D(\boldsymbol{\eta} \circ u)$ their approximate differentials (using the conventions of [10, Section 1.3 & Section 2.2.1]), then we have*

$$|Du|^2(x) = \begin{cases} |Du^+|^2(x) = \sum_i |Du_i^+|^2 & \text{for a.e. } x \in \Omega_+ \cup \Omega_0 \\ |Du^-|^2(x) = \sum_i |Du_i^-|^2 & \text{for a.e. } x \in \Omega_- \cup \Omega_0 \\ Q|D(\boldsymbol{\eta} \circ u)|^2(x) & \text{for a.e. } x \in \Omega_0 \,. \end{cases} \tag{3.2}$$

Proof. Let $\boldsymbol{\iota}(u) = (v, w, z)$. From the very definition we know that $\boldsymbol{\eta} \circ u = z$ belongs to $W^{1,2}(\Omega, \mathbb{R}^n)$. Next observe that, if $a \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $b \in W^{1,2}(\Omega, \mathbb{R}^n)$, then $a \oplus b \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$, as one can easily check from [10, Definition 0.5]. Hence, u^+, u^- belong to $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. Thus, the approximate differentiability of $\boldsymbol{\eta} \circ u$, u^+ and u^- follows from [10, Corollary 2.7].

The approximate differentiability of v, w and the fact that they are identically $Q [\![0]\!]$ on Ω_0 implies easily that indeed $|Dv| = |Dw| = 0$ a.e. on Ω_0. This shows, therefore, the third case of (3.2). We now come to the other two cases and, by symmetry, we focus on the first one. Clearly, on $\Omega_+ \cup \Omega_0$ we have $|Dw| = 0$ and thus by definition

$$|Du|^2 = |Dv|^2 + Q|D(\boldsymbol{\eta} \circ u)|^2 \,.$$

On the other hand, on $\Omega_+ \cup \Omega_0$ we also have that $\boldsymbol{\eta} \circ u = \boldsymbol{\eta} \circ u^+$ and that

$$v = \sum_i [\![u_i^+ - \boldsymbol{\eta} \circ u^+]\!] = u^+ \ominus \boldsymbol{\eta} \circ u^+ \,.$$

Now, at every point of approximate differentiability x we readily check from [10, Definition 1.9 & Definition 2.6] that $D(\boldsymbol{\eta} \circ u^+)(x) = \frac{1}{Q} \sum_i Du_i^+(x)$ and that $Dv_i(x) = Du_i^+(x) - D(\boldsymbol{\eta} \circ u^+)(x)$. Recalling [10, Proposition 2.17] we have thus

$$|Du^+|^2(x) = \sum_i |Du_i^+(x)|^2 = \sum_i |Dv_i|^2 + Q|D(\boldsymbol{\eta} \circ u^+)(x)|^2 = |Dv|^2(x) + Q|D(\boldsymbol{\eta} \circ u)(x)|^2 \,.$$

The latter identity completes the proof. $\square$

4. Currents mod $(2Q)$ and $\mathscr{A}_Q(\mathbb{R}^n)$-valued maps

In this section we link the notion of special Q-valued maps to that of currents modulo $2Q$. This will not only be very useful in the proof of Theorem 1.1 given in [5], but it also highlights the intuition behind the definition of $\mathscr{A}_Q(\mathbb{R}^n)$ as described in the introduction. Consider a k-dimensional rectifiable set $E \subset \mathbb{R}^m$ with finite $\mathcal{H}^k$ measure and a proper Lipschitz map $u : E \to \mathscr{A}_Q(\mathbb{R}^n)$. We can use Definition 2.7, Corollary 2.8 and the theory presented in [7] to define a suitable notion of "graph" of u and correspondingly associate a rectifiable current to it.

Definition 4.1. Let $E \subset \mathbb{R}^m$ be countably k-rectifiable with finite $\mathcal{H}^k$ measure and let $u : E \to \mathscr{A}_Q(\mathbb{R}^n)$ be Lipschitz and proper (i.e. $\boldsymbol{\eta} \circ u$, u^+ and u^- are proper, see [7, Definition

1.2] for the definition of proper $\mathcal{A}_Q(\mathbb{R}^n)$-valued maps). Using the terminology of [7] we denote by

(i) $\mathrm{Gr}(u)$ the set

$$\mathrm{Gr}(u) := (\mathrm{Gr}(u^+) \cap (E_+ \times \mathbb{R}^n)) \cup (\mathrm{Gr}(u^-) \cap (E_- \times \mathbb{R}^n)) \cup (\mathrm{Gr}(\boldsymbol{\eta} \circ u) \cap (E_0 \times \mathbb{R}^n)) \, ;$$

(ii) $\mathbf{G}_u$ the integer rectifiable k-dimensional current

$$\mathbf{G}_u := \mathbf{G}_{u^+} \llcorner E_+ \times \mathbb{R}^n - \mathbf{G}_{u^-} \llcorner E_- \times \mathbb{R}^n + Q\, \mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner E_0 \times \mathbb{R}^n \, .$$

Remark 4.2. Even though [7] only defines multi-valued push-forwards and graphs over a Lipschitz k-dimensional submanifold, the theory can be easily extended to treat the case when the domain of the map is a countably k-rectifiable set; see [17] for details.

It is also not difficult to see that, if E is closed, then $\mathrm{spt}(\mathbf{G}_u) \subset \mathrm{Gr}(u)$. In fact, under some additional assumptions, for instance when E is a Lipschitz submanifold, we can easily conclude that $\mathrm{spt}(\mathbf{G}_u) = \mathrm{Gr}(u)$.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz set and $u : \overline{\Omega} \to \mathscr{A}_Q(\mathbb{R}^n)$ a Lipschitz map. Then, for $p = 2Q$,*

(i) *$\partial \mathbf{G}_u = \mathbf{G}_{u|_{\partial\Omega}} \bmod(p)$;*
(ii) *$\mathbf{G}_u$ is a representative $\bmod(p)$ (in fact, for every measurable $F \subset \Omega_0$, the current $\mathbf{G}_u - 2Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner F \times \mathbb{R}^n$ is also a representative $\bmod(p)$).*

Moreover, there are positive geometric constants $c(m,n,Q)$ and $C(m,n,Q)$ such that, if $E \subset \Omega$ is Borel measurable and $\mathrm{Lip}(u) \leq c$, then

$$\left| \|\mathbf{G}_u\|(E \times \mathbb{R}^n) - Q|E| - \tfrac{1}{2}\mathrm{Dir}(u, E) \right| \leq C \int_E |Du|^4 \, . \tag{4.1}$$

Proof. Recall that, by [11], an integer rectifiable current T is a representative $\bmod(p)$ if and only if its density is at most $\frac{p}{2}$ at $\|T\|$-a.e. point. Since this is obviously the case for the current $\mathbf{G}_u - 2Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner F \times \mathbb{R}^n$ for every measurable subset $F \subset \Omega_0$, the second point is trivial. Observe that

$$\mathbf{G}_{u^+} = \mathbf{G}_{u^+} \llcorner \Omega_+ \times \mathbb{R}^n + Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner (\Omega_0 \cup \Omega_-) \times \mathbb{R}^n$$
$$\mathbf{G}_{u^-} = \mathbf{G}_{u^-} \llcorner \Omega_- \times \mathbb{R}^n + Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner (\Omega_0 \cup \Omega_+) \times \mathbb{R}^n \, .$$

Therefore we conclude

$$\mathbf{G}_u = \mathbf{G}_{u^+} - \mathbf{G}_{u^-} + Q\mathbf{G}_{\boldsymbol{\eta} \circ u} - 2Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \llcorner \Omega_- \times \mathbb{R}^n \, .$$

In particular

$$\mathbf{G}_u = \mathbf{G}_{u^+} - \mathbf{G}_{u^-} + Q\mathbf{G}_{\boldsymbol{\eta} \circ u} \bmod(p) \, .$$

Furthermore, by applying the boundary operator $\bmod(p)$ to the above equation we see that

$$\partial \mathbf{G}_u = \partial \mathbf{G}_{u^+} - \partial \mathbf{G}_{u^-} + Q\partial \mathbf{G}_{\boldsymbol{\eta} \circ u} \bmod(p) \, .$$

We can now use the relation $\partial \mathbf{G}_f = \mathbf{G}_{f|_{\partial\Omega}}$ valid for classical graphs and multivalued graphs (cf. [7]) to conclude

$$\partial \mathbf{G}_u = \mathbf{G}_{u^+|_{\partial\Omega}} - \mathbf{G}_{u^-|_{\partial\Omega}} + Q\mathbf{G}_{\boldsymbol{\eta} \circ u|_{\partial\Omega}} \bmod(p) \, .$$

Now, using the same argument above we get as well

$$\mathbf{G}_{u^+|_{\partial\Omega}} - \mathbf{G}_{u^-|_{\partial\Omega}} + Q\mathbf{G}_{\boldsymbol{\eta} \circ u|_{\partial\Omega}} - 2Q\mathbf{G}_{\boldsymbol{\eta} \circ u|_{\partial\Omega}} \llcorner (\partial\Omega)_- \times \mathbb{R}^n = \mathbf{G}_{u|_{\partial\Omega}} \, ,$$

hence concluding the proof of the first point.

We now come to (4.1). First of all, by the obvious additivity in the set E of the various quantities involved in the inequality, it suffices to show it for subsets E of, respectively, Ω_+, Ω_- and Ω_0. For subsets of Ω_0 the inequality is the standard Taylor expansion of the area functional for Lipschitz graphs. Next, recall that, by [7, Corollary 3.3], the inequality in (4.1) holds for $\mathbf{G}_{u^+}$ and $\mathbf{G}_{u^-}$ (in fact, note that [7, Corollary 3.3] is stated for Lipschitz open domains E, rather than for Borel sets E; however, since for any Borel set we can find a sequence $E_k \supset E$ of Lipschitz open domains with $|E_k \setminus E| \to 0$, it is straightforward to infer the validity of [7, Corollary 3.3] for a general Borel E). If we take $E \subset \Omega_+$, from [7, Corollary 3.3] and Proposition 3.2 we then immediately conclude

$$\left| \|\mathbf{G}_u\|(E \times \mathbb{R}^n) - Q|E| - \tfrac{1}{2}\mathrm{Dir}(u, E) \right| = \left| \|\mathbf{G}_{u^+}\|(E \times \mathbb{R}^n) - Q|E| - \tfrac{1}{2}\mathrm{Dir}(u^+, E) \right|$$

$$\leq C \int_E |Du^+|^4 = C \int_E |Du|^4.$$

The case $E \subset \Omega_-$ can be proved in a similar fashion since

$$\| - \mathbf{G}_{u^-}\|(E \times \mathbb{R}^n) = \|\mathbf{G}_{u^-}\|(E \times \mathbb{R}^n).$$

$\square$

5. BiLipschitz embeddings and retractions, Lipschitz extensions

In this section we show that, as it is the case for $\mathcal{A}_Q(\mathbb{R}^n)$, there are a suitable biLipschitz embedding of $\mathscr{A}_Q(\mathbb{R}^n)$ into a sufficiently large Euclidean space and a corresponding retraction map of the ambient onto the embedding.

Theorem 5.1. *For every Q and n there is $\bar{N}(n, Q)$ and constants $C(n, Q), \delta_0(n, Q) > 0$ with the following properties.*

(i) *There is an injective map $\boldsymbol{\zeta} : \mathscr{A}_Q(\mathbb{R}^n) \to \mathbb{R}^{\bar{N}}$ such that*
  (a) $\mathrm{Lip}(\boldsymbol{\zeta}), \mathrm{Lip}(\boldsymbol{\zeta}^{-1}) \leq C$, *where $\boldsymbol{\zeta}^{-1}$ denotes the inverse of $\boldsymbol{\zeta}$ on $\mathcal{Q} := \boldsymbol{\zeta}(\mathscr{A}_Q(\mathbb{R}^n))$;*
  (b) $\mathrm{Dir}(u, M) = \int_M |D(\boldsymbol{\zeta} \circ u)|^2$ *for every Lipschitz submanifold M of any Euclidean space and for every $u \in W^{1,2}(M, \mathscr{A}_Q(\mathbb{R}^n))$;*
  (c) $|\boldsymbol{\zeta}(P)| = |P|$ *for every $P \in \mathscr{A}_Q(\mathbb{R}^n)$.*

(ii) *There is a map $\boldsymbol{\varrho} : \mathbb{R}^{\bar{N}} \to \mathcal{Q}$ with $\mathrm{Lip}(\boldsymbol{\varrho}) \leq C$ and $\boldsymbol{\varrho}(x) = x$ for every $x \in \mathcal{Q}$.*

(iii) *For every positive $\delta < \delta_0$ there is a map $\boldsymbol{\varrho}^\star_\delta : \mathbb{R}^{\bar{N}} \to \mathcal{Q}$ such that $|\boldsymbol{\varrho}^\star_\delta(P) - P| \leq C\delta^{8^{-nQ}}$ for every $P \in \mathcal{Q}$ and such that the following estimate holds for every $u \in W^{1,2}(M, \mathbb{R}^{\bar{N}})$:*

$$\int_M |D(\boldsymbol{\varrho}^\star_\delta \circ u)|^2 \leq \left(1 + C\delta^{8^{-nQ-1}}\right) \int_{\{\mathrm{dist}(u, \mathcal{Q}) \leq \delta^{nQ+1}\}} |Du|^2 + C \int_{\{\mathrm{dist}(u, \mathcal{Q}) > \delta^{nQ+1}\}} |Du|^2. \tag{5.1}$$

Remark 5.2. Observe that, in the proof given below, if we identify $\mathscr{A}_Q(\mathbb{R}^n)$ with $\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$, then:

  • the map $\boldsymbol{\zeta}$ takes the form $\boldsymbol{\zeta}(P, v) = (\boldsymbol{\zeta}_0(P), v)$ for a suitable $\boldsymbol{\zeta}_0 : \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n) \to \mathbb{R}^{\bar{N}-n}$;

  • the map $\boldsymbol{\varrho}$ takes the form $(w, v) \mapsto (\boldsymbol{\varrho}_0(w), v)$ for a $\boldsymbol{\varrho}_0 : \mathbb{R}^{\bar{N}-n} \to \boldsymbol{\zeta}_0(\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$;

  • the map $\boldsymbol{\varrho}^\star_\delta$ takes the form $(w, v) \mapsto (\boldsymbol{\varrho}^\star_{0,\delta}(w), v)$ for a $\boldsymbol{\varrho}^\star_{0,\delta} : \mathbb{R}^{\bar{N}-n} \to \boldsymbol{\zeta}_0(\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$.

Clearly the maps $\boldsymbol{\zeta}_0, \boldsymbol{\varrho}_0$ and $\boldsymbol{\varrho}^\star_{0,\delta}$ enjoy all the properties and estimates claimed in Theorem 5.1 with $\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n)$ replacing $\mathscr{A}_Q(\mathbb{R}^n)$ and $\mathbb{R}^{\bar{N}-n}$ replacing $\mathbb{R}^{\bar{N}}$.

Proof. In the whole proof we identify $\mathscr{A}_Q(\mathbb{R}^n)$ with

$$((\mathring{\mathcal{A}}_Q(\mathbb{R}^n) \times \{Q[\![0]\!]\}) \cup (\{Q[\![0]\!]\} \times \mathring{\mathcal{A}}_Q(\mathbb{R}^n))) \times \mathbb{R}^n \subset \mathcal{A}_Q(\mathbb{R}^n) \times \mathcal{A}_Q(\mathbb{R}^n) \times \mathbb{R}^n \,.$$

Proof of (i). Consider the restriction of the map $\boldsymbol{\xi}_{BW}$ of [10, Corollary 2.2] to $\mathring{\mathcal{A}}_Q(\mathbb{R}^n)$, which takes values in $\mathbb{R}^N$ for some $N = N(Q,n)$, and denote by id : $\mathbb{R}^n \to \mathbb{R}^n$ the identity map. We then see that (a) and (b) for $\boldsymbol{\zeta} = \boldsymbol{\xi}_{BW} \times \boldsymbol{\xi}_{BW} \times$ id follow directly from [10, Corollary 2.2] and the fact that $\mathcal{G}_s = \mathcal{G} \times \mathcal{G} \times \sqrt{Q}d$. For point (c) we need the fact that $|\boldsymbol{\xi}_{BW}(P)| = |P|$ for every $P \in \mathcal{A}_Q(\mathbb{R}^n)$: although this is not claimed in the statement of [10, Corollary 2.2], it follows easily from [10, Eq. (2.1)] and the fact that

$$\boldsymbol{\xi}_{BW}\Big(\sum_i [\![\lambda P_i]\!]\Big) = \lambda \boldsymbol{\xi}_{BW}\Big(\sum_i [\![P_i]\!]\Big) \,, \tag{5.2}$$

which in turn is an obvious outcome of the definition of $\boldsymbol{\xi}_{BW}$ given in [10, Section 2.1.3].

Proof of (ii). We would like to define the map $\boldsymbol{\varrho}$ as $\boldsymbol{\rho} \times \boldsymbol{\rho} \times$ id, where $\boldsymbol{\rho}$ is the map of [10, Theorem 2.1]. Note that the $\boldsymbol{\xi}$ of [10, Theorem 2.1] can be taken to be $\boldsymbol{\xi}_{BW}$, as it is obvious from the discussion in [10, Section 2.1]). In order to simplify the notation, from now on we drop the subscript $_{BW}$.

The first issue is that $\boldsymbol{\rho}$ is a retraction of $\mathbb{R}^N$ onto $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ rather than onto $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$. In order to deal with it, take $r : \mathcal{A}_Q(\mathbb{R}^n) \to \mathring{\mathcal{A}}_Q(\mathbb{R}^n)$ as $r(P) := P \ominus \boldsymbol{\eta}(P)$ and substitute $\boldsymbol{\rho}$ with $\boldsymbol{\rho}' := \boldsymbol{\xi} \circ r \circ \boldsymbol{\xi}^{-1} \circ \boldsymbol{\rho}$. The second issue is that $\boldsymbol{\rho}' \times \boldsymbol{\rho}'$ is a retraction of $\mathbb{R}^N \times \mathbb{R}^N$ onto $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$, so that our next goal is to find a retraction of $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$ onto $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \{0\} \cup \{0\} \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$. We first define the map $R : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ as

$$R(x,y) := \begin{cases} \left(x - \frac{|y|}{|x|}x, 0\right) & \text{if } |x| > |y| \\ \left(0, y - \frac{|x|}{|y|}y\right) & \text{if } |y| > |x| \\ (0,0) & \text{if } |y| = |x|. \end{cases}$$

Clearly R maps $\mathbb{R}^N \times \mathbb{R}^N$ onto $\mathbb{R}^N \times \{0\} \cup \{0\} \times \mathbb{R}^N$ and it is the identity on $\mathbb{R}^N \times \{0\} \cup \{0\} \times \mathbb{R}^N$. It can be checked in an elementary way that R is Lipschitz. A quick method to see it is the following. First observe that R is obviously locally Lipschitz on $(\mathbb{R}^N \times \mathbb{R}^N) \setminus \{(0,0)\}$. By Rademacher's theorem we can compute its differential, which we can do separately on the two relevant open regions $\{|x| > |y|\}$ and $\{|y| > |x|\}$. On the first region the differential is

$$DR = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

where

$$A = \left(1 - \frac{|y|}{|x|}\right)\text{Id} + \frac{|y|}{|x|^3}x \otimes x$$

$$B = -\frac{1}{|x||y|}x \otimes y \,.$$

Using the fact that $|y| < |x|$, we easily estimate the operator norm of the differential by $\|DR\|_o \leq \sqrt{2}$. Similarly in the region $\{|y| > |x|\}$. We have just concluded that the map R is locally Lipschitz with constant $\sqrt{2}$ on the open set $\{|y| \neq |x|\}$. Since it is continuous and it is

constant on the closed set $\{|y| = |x|\}$, it is elementary to see that it is globally Lipschitz with constant $\sqrt{2}$.

Now, observe that, by (5.2), R maps $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$ into $\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$, and hence into

$$(\boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))) \cap (\mathbb{R}^N \times \{0\} \cup \{0\} \times \mathbb{R}^N) = \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)) \times \{0\} \cup \{0\} \times \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)).$$

We can thus finally define our map $\boldsymbol{\varrho}$ as $\boldsymbol{\varrho} = (R \circ (\boldsymbol{\rho}' \times \boldsymbol{\rho}')) \times \mathrm{id}$.

Proof of (iii). We first consider the map $\boldsymbol{\rho}_\delta^\star$ of [6, Proposition 7.2]. As above, a first candidate for the map $\boldsymbol{\varrho}_\delta^\star$ would be $\boldsymbol{\rho}_\delta^\star \times \boldsymbol{\rho}_\delta^\star \times \mathrm{id}$. Again we start replacing $\boldsymbol{\rho}_\delta^\star$ with $\boldsymbol{\rho}_\delta' := \boldsymbol{\xi} \circ r \circ \boldsymbol{\xi}^{-1} \circ \boldsymbol{\rho}_\delta^\star$. Fix $P \in \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))$. Recall that, by [6, Proposition 7.2], $|\boldsymbol{\rho}_\delta^\star(P) - P| \le C\delta^{8^{-nQ}}$. Next,

$$\begin{aligned}
|\boldsymbol{\rho}_\delta'(P) - P| &\le C\mathcal{G}(r(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P)), \boldsymbol{\xi}^{-1}(P)) \\
&\le C\left(\mathcal{G}(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P)), \boldsymbol{\xi}^{-1}(P)) + \sqrt{Q}|\boldsymbol{\eta}(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P)))|\right) \\
&= C\left(\mathcal{G}(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P)), \boldsymbol{\xi}^{-1}(P)) + \sqrt{Q}|\boldsymbol{\eta}(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P))) - \boldsymbol{\eta}(\boldsymbol{\xi}^{-1}(P))|\right) \\
&\le 2C\mathcal{G}(\boldsymbol{\xi}^{-1}(\boldsymbol{\rho}_\delta^\star(P)), \boldsymbol{\xi}^{-1}(P)) \le 2C^2|\boldsymbol{\rho}_\delta^\star(P) - P|.
\end{aligned}$$

We thus conclude the estimate

$$|\boldsymbol{\rho}_\delta'(P) - P| \le C\delta^{8^{-nQ}} \qquad \forall P \in \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n)). \tag{5.3}$$

Furthermore, recall the elementary observation that $\mathrm{Dir}(f \ominus (\boldsymbol{\eta} \circ f)) \le \mathrm{Dir}(f)$, valid for every $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. In particular, combining it with [6, Proposition 7.2] and with part (i) of the theorem, we achieve

$$\begin{aligned}
\int_M |D(\boldsymbol{\rho}_\delta' \circ f)|^2 &\le \left(1 + C\delta^{8^{-nQ-1}}\right) \int_{\{\mathrm{dist}(f, \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))) \le \delta^{nQ+1}\}} |Df|^2 \\
&\quad + C \int_{\{\mathrm{dist}(f, \boldsymbol{\xi}(\mathring{\mathcal{A}}_Q(\mathbb{R}^n))) > \delta^{nQ+1}\}} |Df|^2
\end{aligned} \tag{5.4}$$

for every $f \in W^{1,2}(M, \mathbb{R}^n)$.

Our map $\boldsymbol{\varrho}_\delta^\star$ will be defined as $(R_\delta \circ (\boldsymbol{\rho}_\delta' \times \boldsymbol{\rho}_\delta')) \times \mathrm{id}$, where

$$R_\delta : \mathbb{R}^N \times \mathbb{R}^N \to (\mathbb{R}^N \times \{0\} \cup \{0\} \times \mathbb{R}^N)$$

is an appropriate "almost retraction" map which we will construct as follows. First introduce the function $\chi_\delta : [0, \infty[\to [0, \infty[$ as

$$\chi_\delta(s) = \begin{cases} 0 & \text{if } s \in [0, \delta] \\ (1 - \delta)^{-1}(s - \delta) & \text{if } s \in [\delta, 1] \\ 1 & \text{otherwise.} \end{cases}$$

We then define

$$R_\delta(x, y) = \begin{cases} \left(\chi_\delta(|x|)\frac{x}{|x|}, 0\right) & \text{if } |y| \le \delta^2 \\ \left(0, \chi_\delta(|y|)\frac{y}{|y|}\right) & \text{if } |x| \le \delta^2. \end{cases}$$

It is easy to see that R_δ is well defined, since on the intersection $\{\max\{|y|, |x|\} \ge \delta^2\}$ the map is identically 0. Moreover:

- the restriction of R_δ to $\{|y| \le \delta^2\}$ takes values into $\mathbb{R}^N \times \{0\}$ and has Lipschitz constant bounded by $1 + C\delta$;
- the restriction of R_δ to $\{|x| \le \delta^2\}$ takes values into $\{0\} \times \mathbb{R}^N$ and has also Lipschitz constant bounded by $1 + C\delta$.

Its global Lipschitz constant is controlled independently of δ and, finally, we can extend it to the whole $\mathbb{R}^N \times \mathbb{R}^N$ by first choosing a Lipschitz extension taking values in $\mathbb{R}^N \times \mathbb{R}^N$ and then composing it with the retraction map R of the proof of (ii)

We will now show that $\boldsymbol{\varrho}_\delta^\star := (R_\delta \circ (\boldsymbol{\rho}_\delta' \times \boldsymbol{\rho}_\delta')) \times \mathrm{id}$ has the desired properties. First observe that, if a point $P = (p, q, v) \in \mathbb{R}^{2N+1}$ belongs to Q, then either $p = 0$ or $q = 0$. Without loss of generality, assume that the second alternative holds. Then $R_\delta(p, 0) = (p', 0)$ with $|p - p'| \le C\delta$ and moreover p' is a positive multiple of p, which by (5.2) implies that $p' \in \boldsymbol{\xi}(\overset{\circ}{\mathcal{A}}_Q(\mathbb{R}^n))$. We therefore find that

$$|\boldsymbol{\varrho}_\delta^\star(P) - P| = |\boldsymbol{\rho}_\delta'(p') - p| \le |\boldsymbol{\rho}_\delta'(p') - p'| + |p' - p| \le C\delta^{8^{-nQ-1}} + C\delta\,.$$

We next come to (5.1). Without loss of generality observe that we can prove the estimate for a generic Lipschitz map $u = (v, w, z)$ on a bounded domain. Consider next the set $F := \{\mathrm{dist}(u, Q) \le \delta^{nQ+1}\}$. Let $u = (v, w, z)$ and let $(v', w') = R_\delta(v, w)$. If $z \in F$, we then have two cases

- $w'(z) = 0$ and $\mathrm{dist}(v'(z), \boldsymbol{\xi}(\overset{\circ}{\mathcal{A}}_Q(\mathbb{R}^n))) \le \mathrm{dist}(u(z), Q) \le \delta^{nQ+1}$;
- $v'(z) = 0$ and $\mathrm{dist}(w'(z), \boldsymbol{\xi}(\overset{\circ}{\mathcal{A}}_Q(\mathbb{R}^n))) \le \mathrm{dist}(u(z), Q) \le \delta^{nQ+1}$.

In the first case we have $\boldsymbol{\varrho}_\delta^\star \circ u(x) = (\boldsymbol{\rho}_\delta'(v'(x)), 0, z(x))$, whereas in the second case we have $\boldsymbol{\varrho}_\delta^\star \circ u(x) = (0, \boldsymbol{\rho}_\delta'(w'(x)), z(x))$. Using (5.4) we then can easily estimate

$$\int_M |D(\boldsymbol{\varrho}_\delta^\star \circ u)|^2 \le \left(1 + C\delta^{8^{-nQ-1}}\right) \int_F |D(R_\delta \circ (v, w))|^2 + C \int_{M \setminus F} |D(R_\delta \circ (v, w))|^2 + \int_M |Dz|^2\,. \tag{5.5}$$

Observe also that $(v, w)(F)$ is contained in $\{(x, y) : \min\{|x|, |y|\} \le \delta^{nQ+1} \le \delta^2\}$. On this set we easily compute $|DR_\delta| \le 1 + C\delta$. Moreover recall that $\|DR_\delta\|_\infty \le C$ for some constant C independent of δ. Thus we can write

$$\int_M |D(\boldsymbol{\varrho}_\delta^\star \circ u)|^2 \le \left(1 + C\delta^{8^{-nQ-1}}\right) \int_F (|Dv|^2 + |Dw|^2) + C \int_{M \setminus F} (|Dv|^2 + |Dw|^2) + \int_M |Dz|^2\,. \tag{5.6}$$

Considering that $|Du|^2 = |Dv|^2 + |Dw|^2 + |Dz|^2$, we then conclude the desired estimate (5.1). $\qquad\square$

We conclude this section by remarking that a simple corollary of the parts (i) and (ii) of the above theorem is the following analogue of [10, Theorem 1.7], recorded as Corollary 5.3 here below. In turn using the corollary, a simple inspection of the proof of the Lipschitz approximation theorem in [10, Proposition 2.5] shows that the same result is valid for Sobolev maps with values in $\mathscr{A}_Q(\mathbb{R}^n)$ (in fact it is possible to give an "intrinsic" proof of the Lipschitz extension property, in the spirit of [10, Section 1.2], and hence of the Lipschitz approximation theorem as well; however since Theorem 5.1 is necessary for some later arguments, for the sake of brevity we omit the intrinsic proofs).

Corollary 5.3. *Let $B \subset \mathbb{R}^m$ and $f : B \to \mathscr{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then there exists an extension $\bar{f} : \mathbb{R}^m \to \mathscr{A}_Q(\mathbb{R}^n)$ of f, with $\mathrm{Lip}(\bar{f}) \le C(m, Q)\,\mathrm{Lip}(f)$. Moreover, if f is bounded*

then

$$\sup_{x \in \mathbb{R}^m} |\bar{f}(x)| \leq \sup_{x \in B} |f(x)|, \tag{5.7}$$

and for any $q \in \mathbb{R}^n$ it holds

$$\sup_{x \in \mathbb{R}^m} \mathcal{G}_s(\bar{f}(x), Q \llbracket q \rrbracket) \leq C(m, Q) \sup_{x \in B} \mathcal{G}_s(f(x), Q \llbracket q \rrbracket). \tag{5.8}$$

Proof. In order to get the Lipschitz extension it suffices to first extend $\boldsymbol{\zeta} \circ f$ and then compose the extension with $\boldsymbol{\zeta}^{-1} \circ \boldsymbol{\varrho}$. Next, assume that $M := \sup_{x \in B} |f(x)| < \infty$. Observe that $\mathscr{A}_Q(\mathbb{R}^n)$ is a cone, namely for every $\lambda \in [0, \infty[$ we can define $\lambda(T, \varepsilon) = (\sum_i \llbracket \lambda T_i \rrbracket, \varepsilon)$. We therefore introduce the projection of $\mathscr{A}_Q(\mathbb{R}^n)$ onto $\{S \in \mathscr{A}_Q(\mathbb{R}^n) : |S| \leq M\}$ by keeping S fixed if $|S| \leq M$ and mapping it to $\frac{M}{|S|} S$ if $|S| > M$. Such projection is 1-Lipschitz and it suffices to compose it with any Lipschitz extension of f to obtain a new extension with no larger Lipschitz constant and which satisfies the bound (5.7). Next, we prove (5.8). First, let us define the *oscillation* of f by

$$\operatorname{osc}(f) := \inf_{q \in \mathbb{R}^n} \sup_{x \in B} \mathcal{G}_s(f(x), Q \llbracket q \rrbracket), \tag{5.9}$$

and observe that since f is bounded the infimum in (5.9) is achieved. Let us then call $q_0 \in \mathbb{R}^n$ a value which realizes the oscillation, so that

$$R := \operatorname{osc}(f) = \sup_{x \in B} \mathcal{G}_s(f(x), Q \llbracket q_0 \rrbracket).$$

Of course, if $R = 0$ then f is identically equal to $Q \llbracket q_0 \rrbracket$ on B, and thus (5.8) is trivially true for the natural extension $\bar{f}(x) = Q \llbracket q_0 \rrbracket$ for every $x \in \mathbb{R}^m$. Thus, we can assume $R > 0$. We also set $L := \operatorname{Lip}(f)$. Then, we introduce the map

$$\tilde{f} \colon (B \times \{0\}) \cup (\mathbb{R}^m \times \{\tfrac{R}{L}\}) \subset \mathbb{R}^{m+1} \to \mathscr{A}_Q(\mathbb{R}^n) \tag{5.10}$$

which extends f, and which takes value $\tilde{f}(z) := Q \llbracket q_0 \rrbracket$ at every point $z = (x, R/L)$ with $x \in \mathbb{R}^m$. Since for any given $(x, 0) \in B \times \{0\}$ and $z \in \mathbb{R}^m \times \{\frac{R}{L}\}$ we have

$$\mathcal{G}_s(\tilde{f}((x,0)), \tilde{f}(z)) = \mathcal{G}_s(f(x), Q \llbracket q_0 \rrbracket) \leq R = L \left(\frac{R}{L} \right) \leq L |(x,0) - z|,$$

it is clear that $\operatorname{Lip}(\tilde{f}) = \operatorname{Lip}(f) = L$. We can now use the argument in the first part of the proof to extend $\tilde{f}$ to a function $F \colon \mathbb{R}^{m+1} \to \mathscr{A}_Q(\mathbb{R}^n)$, and then define $\bar{f}(x) := F((x,0))$ for all $x \in \mathbb{R}^m$. It is clear that $\bar{f}$ is an extension of f, and that both $\operatorname{Lip}(\bar{f}) \leq C(m, Q) \operatorname{Lip}(f)$ and (5.7) hold. We claim that this extension $\bar{f}$ also satisfies (5.8). To this aim, let $q \in \mathbb{R}^n$, and set

$$\delta_q := \mathcal{G}_s(Q \llbracket q \rrbracket, Q \llbracket q_0 \rrbracket) = \sqrt{Q} |q - q_0|. \tag{5.11}$$

We shall distinguish two cases. Set $C = C(m, Q)$ the constant above, and assume first that

$$\delta_q \leq (C + 1) R. \tag{5.12}$$

Then, for any $x \in \mathbb{R}^m$ it holds

$$\mathcal{G}_s(\bar{f}(x), Q \llbracket q \rrbracket) \leq \delta_q + R \leq (C + 2) R \leq (C + 2) \sup_{x \in B} \mathcal{G}_s(f(x), Q \llbracket q \rrbracket),$$

where in the last inequality we have used the definition of R. This proves the validity of (5.8) when (5.12) holds. Let us then suppose that (5.12) fails, so that

$$(C + 1) R < \delta_q. \tag{5.13}$$

By triangle inequality we have, for every $x \in B$:

$$\mathcal{G}_s(f(x), Q\,[\![q]\!]) \geq \delta_q - R \geq \frac{C}{C+1}\,\delta_q \geq \frac{\delta_q}{2}\,. \tag{5.14}$$

On the other hand, for any $y \in \mathbb{R}^m$ it holds

$$\mathcal{G}_s(\bar{f}(y), Q\,[\![q_0]\!]) = \mathcal{G}_s(F((y,0)), F(y, R/L)) \leq C\,R\,, \tag{5.15}$$

so that if we combine (5.14) and (5.15) we obtain

$$\mathcal{G}_s(\bar{f}(y), Q\,[\![q]\!]) \leq \delta_q + C\,R \overset{(5.13)}{\leq} 2\,\delta_q \leq 4\,\mathcal{G}_s(f(x), Q\,[\![q]\!]) \tag{5.16}$$

for every $x \in B$, for every $y \in \mathbb{R}^m$. This is stronger than (5.8), and thus it concludes the proof. $\square$

Finally, we record here another simple consequence of the existence of the embeddings and of the retraction (for which, again, an intrinsic proof in the spirit of [10, Section 4.3.1] is also possible).

Lemma 5.4 (Luckhaus lemma). *There is a constant $C(m,n,Q)$ with the following property. Assume $f, g \in W^{1,2}(\mathbb{S}^{m-1}, \mathscr{A}_Q(\mathbb{R}^n))$ (resp. $f, g \in W^{1,2}(\mathbb{S}^{m-1}, \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$ and let $\lambda < \frac{1}{2}$ be a given positive number. Then there is a $u \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \mathscr{A}_Q(\mathbb{R}^n))$ (resp. $u \in W^{1,2}(B_1 \setminus B_{1-\lambda}, \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$) such that*

$$u_{\partial B_1} = f \qquad and \qquad u|_{\partial B_{1-\lambda}} = g\left(\frac{\cdot}{1-\lambda}\right) \tag{5.17}$$

$$\mathrm{Dir}(u) \leq C\lambda\left(\mathrm{Dir}(f, \mathbb{S}^{m-1}) + \mathrm{Dir}(g, \mathbb{S}^{m-1})\right) + C\lambda^{-1}\int_{\mathbb{S}^{m-1}} \mathcal{G}_s(f,g)^2\,. \tag{5.18}$$

If f, g are, in addition, Lipschitz continuous, then the interpolating function u can be chosen such that

$$\mathrm{Lip}(u) \leq C\left(\mathrm{Lip}(f) + \mathrm{Lip}(g)\right) + C\lambda^{-1}\,\|\mathcal{G}_s(f,g)\|_\infty\,. \tag{5.19}$$

Proof. Consider the case $\mathscr{A}_Q(\mathbb{R}^n)$. The map u can be explicitly defined via

$$u(x) = (\boldsymbol{\zeta}^{-1} \circ \boldsymbol{\varrho})\left(\frac{|x| - (1-\lambda)}{\lambda}\boldsymbol{\zeta}\left(f\left(\frac{x}{|x|}\right)\right) + \frac{1 - |x|}{\lambda}\boldsymbol{\zeta}\left(g\left(\frac{x}{|x|}\right)\right)\right)\,.$$

In the case $\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n)$ we use the maps $\boldsymbol{\zeta}_0$ and $\boldsymbol{\varrho}_0$ of Remark 5.2 in place of $\boldsymbol{\zeta}$ and $\boldsymbol{\varrho}$. $\square$

Another useful tool will be the following approximation lemma. It is the $\mathscr{A}_Q(\mathbb{R}^n)$ version of [6, Lemma 4.5].

Lemma 5.5. *Let f be a map in $W^{1,2}(B_r, \mathscr{A}_Q(\mathbb{R}^n))$, where $B_r \subset \mathbb{R}^m$. Then for every ε there exists an approximating map $f_\varepsilon \in W^{1,2}(B_r, \mathscr{A}_Q(\mathbb{R}^n))$ such that*

(a) *f_ε is Lipschitz continuous;*
(b) *The following estimate holds:*

$$\int_{B_r} \mathcal{G}_s(f, f_\varepsilon)^2 + \int_{B_r}\left(|Df| - |Df_\varepsilon|\right)^2 + \int_{B_r}|D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ f_\varepsilon)|^2 \leq \varepsilon. \tag{5.20}$$

If $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathscr{A}_Q(\mathbb{R}^n))$, then f_ε can be chosen to satisfy also

$$\int_{\partial B_r} \mathcal{G}(f, f_\varepsilon)^2 + \int_{\partial B_r}\left(|Df| - |Df_\varepsilon|\right)^2 \leq \varepsilon. \tag{5.21}$$

The proof is the very same as given in [6, Lemma 4.5], only using the Lipschitz extension theorem for $\mathscr{A}_Q(\mathbb{R}^n)$.

6. Existence and compactness of Dir-minimizers

The following existence theorem is a simple consequence of the fact that we can identify $\mathscr{A}_Q(\mathbb{R}^n)$ (resp. $\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n)$) with a closed subset of $\mathcal{A}_Q(\mathbb{R}^n) \times \mathcal{A}_Q(\mathbb{R}^n) \times \mathbb{R}^n$ (resp. $\mathcal{A}_Q(\mathbb{R}^n) \times \mathcal{A}_Q(\mathbb{R}^n)$), and that the Dirichlet energy of a $\mathscr{A}_Q(\mathbb{R}^n)$-valued map is the sum of the Dirichlet energies of the corresponding factors. Therefore we leave the proof to the reader.

Theorem 6.1. *Assume $\Omega \subset \mathbb{R}^m$ is a bounded Lipschitz set and let $f \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ (resp. $f \in W^{1,2}(\Omega, \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$. Then there is a map $g \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ (resp. $g \in W^{1,2}(\Omega, \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$) such that $(g-f)|_{\partial\Omega} = 0$ (in the sense of the trace theorem [10, Proposition 2.10]) and which minimizes the Dirichlet energy over all maps with the same trace property.*

Note that if $f(x) = (\tilde{f}(x), 1)$ for a.e. $x \in \Omega$ then $g(x) = (\tilde{g}(x), 1)$ for a.e. x and $\tilde{g}$ is minimizing in $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

Definition 6.2. A map g as in Theorem 6.1 will be called a Dir-minimizer (or Dir-minimizing) in Ω.

Moreover, the following is another obvious consequence of the "factorization" of $\mathscr{A}_Q(\mathbb{R}^n)$ into $\overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n) \times \mathbb{R}^n$, in particular of (3.1).

Proposition 6.3. *A map $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ is Dir-minimizing in Ω if and only if both $u \ominus \boldsymbol{\eta} \circ u$ and $\boldsymbol{\eta} \circ u$ are Dir-minimizing in Ω. Moreover $u \ominus \boldsymbol{\eta} \circ u$ is a Dir-minimizer in $W^{1,2}(\Omega, \overset{\circ}{\mathscr{A}}_Q(\mathbb{R}^n))$ if and only if it is a Dir-minimizer in $\mathscr{A}_Q(\mathbb{R}^n)$.*

We close this section by the following compactness property of Dir-minimizers

Proposition 6.4. *Let $\{g_k\} \subset W^{1,2}(B_r, \mathscr{A}_Q(\mathbb{R}^n))$ be a sequence of maps which are Dir-minimizing in B_r and which converge weakly to some g. Then, for every $s < r$, the sequence converges strongly in $W^{1,2}(B_s, \mathscr{A}_Q(\mathbb{R}^n))$ and moreover the limiting g is Dir-minimizing in B_s. If $\limsup \mathrm{Dir}(g_k|_{\partial B_r}) < \infty$, then the same conclusion holds in B_r.*

Proof. First of all, using Fatou's lemma we get

$$\int_s^r \liminf_{k\to\infty} \mathrm{Dir}(g_k|_{\partial B_\sigma}) \, d\sigma < \infty$$

and thus we can reduce the first statement to the second. We assume therefore, without loss of generality, that $r = 1$ and

$$\sup_k \mathrm{Dir}(g_k|_{\partial B_1}) < \infty. \tag{6.1}$$

Observe next that, by weak convergence and trace theorems in the Sobolev spaces, we know:

$$\lim_{k\to\infty} \int_{B_1} \mathcal{G}_s(g_k, g)^2 = 0 \tag{6.2}$$

$$\lim_{k\to\infty} \int_{\partial B_1} \mathcal{G}_s(g_k, g)^2 = 0 \tag{6.3}$$

$$\liminf_{k\to\infty} \mathrm{Dir}(g_k, B_1) \geq \mathrm{Dir}(g, B_1) \tag{6.4}$$

$$\liminf_{k\to\infty} \mathrm{Dir}(g_k|_{\partial B_1}, \mathbb{S}^{m-1}) \geq \mathrm{Dir}(g|_{\partial B_1}, \mathbb{S}^{m-1}) . \tag{6.5}$$

Given any $\lambda \in]0, \frac{1}{2}]$, we can thus apply the Luckhaus Lemma 5.4 to find a sequence of maps h_k on $B_1 \setminus B_{1-\lambda}$ such that

- $h_k(x) = g_k(x)$ for every $x \in \partial B_1$ and $h_k(x) = g(\frac{x}{1-\lambda})$ for any $x \in \partial B_{1-\lambda}$;
- the following estimate holds

$$\limsup_{k\to\infty} \mathrm{Dir}(h_k, B_1 \setminus \overline{B}_{1-\lambda}) \leq C\lambda K , \tag{6.6}$$

  where C is a geometric constant depending on m, n, Q and

$$K = \limsup_{k} \mathrm{Dir}(g_k|_{\partial B_1}, \mathbb{S}^{m-1}) .$$

Assume now by contradiction that either g is not Dir-minimizing or that

$$\mathrm{Dir}(g, B_1) < \limsup_{k\to\infty} \mathrm{Dir}(g_k, B_1) .$$

For a subsequence of $\{g_k\}$, not relabeled, we then have that there is a map $\hat{g}$ with $\hat{g}|_{\partial B_1} = g|_{\partial B_1}$ and

$$\lim_{k\to\infty} (\mathrm{Dir}(g_k, B_1) - \mathrm{Dir}(\hat{g}, B_1)) = L > 0 . \tag{6.7}$$

Consider then the function

$$\hat{g}_k(x) = \begin{cases} h_k(x) & \text{if } 1 - \lambda < |x| < 1 \\ \hat{g}\left(\frac{x}{1-\lambda}\right) & \text{if } |x| \leq 1 - \lambda. \end{cases}$$

Since $\mathrm{Dir}(\hat{g}_k, B_{1-\lambda}) = (1-\lambda)^{m-2}\mathrm{Dir}(\hat{g}, B_1) \leq \mathrm{Dir}(\hat{g}, B_1)$, combining (6.7) and (6.6) we achieve

$$\liminf_{k\to\infty}(\mathrm{Dir}(g_k, B_1) - \mathrm{Dir}(\hat{g}_k, B_1) \geq L - CK\lambda .$$

In particular, the right hand side of the last inequality can be made positive by choosing λ appropriately small. Since however $\hat{g}_k|_{\partial B_1} = g_k|_{\partial B_1}$, for k large enough we would contradict the minimality of g_k. $\qquad\square$

7. First variations

7.1. **Notation for $\mathscr{A}_Q(\mathbb{R}^n)$-calculus.** In this section we derive some key identities for Dir-minimizers u defined over a bounded domain Ω, which come from computing first variations of the functional. We distinguish two types of variations: inner variations and outer variations. Given the decomposition of Ω as in Definition 2.7 in each of the domains Ω_+, Ω_- and Ω_0, we can regard u as an $\mathcal{A}_Q(\mathbb{R}^n)$-valued function, coinciding respectively with u^+, u^- and $Q[\![\boldsymbol{\eta} \circ u]\!]$. By Proposition 3.2, in each of these domains the respective map is approximately differentiable and we can use the chain rules of [10, Proposition 2.8]. When we deal with integrals over the whole domain we would then have rather cumbersome formulas where we break the integral in the

respective domains Ω_+, Ω_- and Ω_0, in spite of the fact that such formulas would nonetheless be rather straightforward. In order to simplify our notation we will then use the convention that $\sum_i [\![u_i(x)]\!]$, resp. $\sum_i [\![Du_i(x)]\!]$, will denote the multivalued maps $\sum_i [\![u_i^+(x)]\!]$, $\sum_i [\![u_i^-(x)]\!]$ and $Q[\![\boldsymbol{\eta} \circ u(x)]\!]$ (resp. $\sum_i [\![Du_i^+(x)]\!]$, $\sum_i [\![Du_i^-(x)]\!]$ and $Q[\![D(\boldsymbol{\eta} \circ u)(x)]\!]$) depending on whether x belongs to Ω_+, Ω_- or Ω_0.

7.2. **Inner variations.** Inner variations are generated by composing u with one-parameter families of diffeomorphisms Φ_t of Ω which are the identity on $\partial\Omega$. More specifically we consider a vector field $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, we let $\Phi_t(x) = x + t\varphi(x)$ and we observe that, whenever $|t|$ is sufficiently small, $u \circ \Phi_t$ is well defined, $u \circ \Phi_t \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ and $u \circ \Phi_t|_{\partial\Omega} = u|_{\partial\Omega}$. We therefore conclude that, if u is Dir-minimizing, then $\mathrm{Dir}(u \circ \Phi_t) \geq \mathrm{Dir}(u)$ for all sufficiently small t, and thus

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Dir}(u \circ \Phi_t). \tag{7.1}$$

Using the discussion above we can break the domain Ω into the pieces Ω_+, Ω_- and Ω_0 where we use the chain rules of [10, Proposition 2.8] to prove the following proposition (which corresponds to the first part of [10, Proposition 3.1]). Note that, since Φ_t is a diffeomorphism, the partition of the domain Ω induced by the map $u \circ \Phi_t$ is given by $\{\Phi_t^{-1}(\Omega_+), \Phi_t^{-1}(\Omega_-), \Phi_t^{-1}(\Omega_0)\}$.

Proposition 7.1. *Let Ω be a bounded open set and $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ a Dir-minimizer. Then for every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$ we have*

$$\int_\Omega \left(\sum_i \langle Du_i : Du_i \cdot D\varphi \rangle - |Du|^2 \mathrm{div}\, \varphi \right) = 0, \tag{7.2}$$

where $\langle A : B \rangle$ denotes the Hilbert-Schmidt scalar product between $n \times m$ matrices (i.e. $\langle A : B \rangle = \sum_{i,j} A_{ij} B_{ij}$).

7.3. **Outer variations.** Next consider a map $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\psi(x, u) = 0$ in a neighborhood of $\partial\Omega \times \mathbb{R}^n$ and which satisfies the growth conditions

$$|D_u \psi| \leq C < \infty \qquad \text{and} \qquad |\psi(x, u)| + |D_x \psi(x, u)| \leq C(1 + |u|) \tag{7.3}$$

for some constant C. For each fixed x, consider the map

$$\mathscr{A}_Q(\mathbb{R}^n) \ni P = \sum_i [\![P_i]\!] \mapsto \Psi(x, P) = \sum_i [\![P_i + \psi(x, P_i)]\!].$$

Observe that if we consider the obvious induced map on $\mathcal{A}_Q(\mathbb{R}^n) \sqcup \mathcal{A}_Q(\mathbb{R}^n)$, the latter commutes with the equivalence relation defining $\mathscr{A}_Q(\mathbb{R}^n)$ and thus induces a corresponding map on $\mathscr{A}_Q(\mathbb{R}^n)$ through $(P, \varepsilon) \to \Psi(x, (P, \varepsilon)) := (\Psi(x, P), \varepsilon)$.

Hence if u takes values in $\mathscr{A}_Q(\mathbb{R}^n)$, then we have a well defined map $x \mapsto \Psi(x, u(x))$, which we will denote by $\Psi(x, u) = u + \psi(x, u)$ and which in a neighborhood of $\partial\Omega$ agrees with u. We wish to show that $x \mapsto \Psi(x, u) \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ when $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ (and Ω is bounded). A possible procedure is the following:

- When u is Lipschitz, we observe that $\Psi(x, u)$ is also Lipschitz. Using Definition 2.7 consider the sets Ω_+, Ω_- and Ω_0 and observe that $(\Psi(x, u))^+(x) = \Psi(x, u^+(x))$ and $(\Psi(x, u))^-(x) = \Psi(x, u^-(x))$. In particular

$$\mathrm{Dir}(\Psi(x, u)) = \int_{\Omega_+} |D\Psi(x, u^+(x))|^2 + \int_{\Omega_-} |D\Psi(x, u^-(x))|^2 + Q \int_{\Omega_0} |D\Psi(x, \boldsymbol{\eta} \circ u(x))|^2. \tag{7.4}$$

- Using the chain rules of [10, Proposition 1.12] we see then easily that there is a constant $\hat{C}$ (depending only on m, n, Q and C in (7.3)) such that, if u is Lipschitz, then

$$\mathrm{Dir}(\Psi(x,u)) \leq \hat{C}\left(|\Omega| + \|u\|_{L^2} + \mathrm{Dir}(u)\right) .$$

- Using the analogue of [10, Proposition 2.5], for a general map $u \in W^{1,2}$ we find a sequence of Lipschitz maps u_k converging to u in L^2 and with equibounded Dirichlet energy. The corresponding maps $x \mapsto \Psi(x, u_k)$ converge then to $\Psi(x,u)$ and have equibounded Dirichlet energy. We then conclude that $\Psi(x,u) \in W^{1,2}$. Next, considering [10, Equation (2.9)], we can also observe that the convergence is in fact strong in $W^{1,2}$ and thus (7.4) holds for a general $W^{1,2}$ map.

We are now ready to define outer variations. Consider indeed a smooth ψ which is supported in $\Omega' \times \mathbb{R}^n$ for some $\Omega' \Subset \Omega$ and has the same properties and growth conditions as above and let $\Psi_t(x,u) := u + t\,\psi(x,u)$. Then, if u is a Dir-minimizer, $\mathrm{Dir}(\Psi_t(x,u)) \geq \mathrm{Dir}(u)$ and we thus can write

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Dir}(\Psi_t(x,u)) . \tag{7.5}$$

Arguing as in the previous paragraph we then conclude the following analogue of the second part of [10, Proposition 3.1].

Proposition 7.2. *Let $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ be a Dir-minimizer and assume $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ is a vector field which vanishes identically in a neighborhood of $\partial\Omega \times \mathbb{R}^n$ and satisfies the growth conditions (7.3) for some constant C. Then we have*

$$\int \sum_i \left(\langle Du_i : D_x\psi(x,u_i)\rangle + \langle Du_i : D_u\psi(x,u_i) \cdot Du_i\rangle\right) = 0 . \tag{7.6}$$

7.4. **Key identities.** Arguing as in [10, Section 3.1.2] we test the identities (7.2) and (7.6) with φ and ψ of the following special form: $\varphi(x) = \phi(|x|)x$ and $\psi(x,u) = \phi(|x|)u$. If we let ϕ converge to the indicator function of the interval $[0, r[$ we then reach the following key identities.

Proposition 7.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and let $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ be a Dir-minimizer. Then for a.e. $r \in [0, \mathrm{dist}(x, \partial\Omega)[$ the following equalities hold:*

$$(m-2)\int_{B_r(x)} |Du|^2 = r\int_{\partial B_r(x)} |Du|^2 - 2r\int_{\partial B_r(x)} |\partial_\nu u|^2 , \tag{7.7}$$

$$\int_{B_r(x)} |Du|^2 = \int_{\partial B_r(x)} \sum_i \langle \partial_\nu u_i, u_i \rangle , \tag{7.8}$$

where ν denotes the outer unit normal to $\partial B_r(x)$ and $\sum_i [\![\partial_\nu u_i]\!]$ is the multivalued map $y \mapsto \sum_i [\![Du_i(u) \cdot \nu(y)]\!]$.

The proof of the proposition follows from the very same computations of [10, Section 3.1.2].

8. Hölder regularity of Dir-minimizers

In this section we show that Dir-minimizers are Hölder continuous. In particular we will prove the following

Theorem 8.1. *There are constants $\alpha_0(m,n,Q) > 0$ and $C(m,n,Q)$ with the following property. Assume $u \in W^{1,2}(B_{2r}(x), \mathscr{A}_Q(\mathbb{R}^n))$ is a Dir-minimizer. Then $u \in C_{loc}^{0,\alpha_0}(B_{2r}(x))$. Indeed we have the estimates*

$$[u]_{\alpha_0, B_r(x)} \leq C r^{1 - m/2 - \alpha_0} \left(\mathrm{Dir}(u, B_{2r}(x)) \right)^{\frac{1}{2}} \tag{8.1}$$

$$\rho^{2 - m - 2\alpha_0} \mathrm{Dir}(u, B_\rho(x)) \leq (2r)^{2 - m - 2\alpha_0} \mathrm{Dir}(u, B_{2r}(x)) \qquad \forall\, 0 < \rho < 2r. \tag{8.2}$$

The estimate (8.2) gives a corresponding estimate for $\boldsymbol{\zeta} \circ u$ and then (8.2) implies (8.1) through the classical theory of Campanato spaces, cf. [13, Proposition 3.7 & Theorem 2.9]. We therefore focus our attention on (8.2), which is a direct consequence of the following proposition.

Proposition 8.2. *There is a constant $\alpha_0(m,n,Q) > 0$ such that the following inequality holds for every $u \in W^{1,2}(B_1(x), \mathscr{A}_Q(\mathbb{R}^n))$:*

$$\mathrm{Dir}(u, B_1) \leq \frac{1}{m - 2 + 2\alpha_0} \mathrm{Dir}(u|_{\partial B_1}, \partial B_1). \tag{8.3}$$

Indeed, let u be as in Theorem 8.1 and set

$$I(\rho) := \mathrm{Dir}(u, B_\rho(x))$$
$$J(\rho) := \mathrm{Dir}(u|_{\partial B_\rho(x)}, \partial B_\rho(x)).$$

Notice that,

$$I(\rho) = \int_0^\rho \int_{\partial B_\sigma(x)} |Du|^2 \, d\sigma \geq \int_0^\rho J(\sigma) \, d\sigma.$$

Moreover, by rescaling and translating, (8.3) gives

$$I(\rho) \leq \frac{r}{m - 2 + 2\alpha_0} J(\rho).$$

We thus conclude easily that $(r^{2 - m - 2\alpha_0} I(r))' \geq 0$, which obviously implies (8.2).

We split the proof of Proposition 8.2 into two cases depending on the dimension of the domain, namely $m = 2$ and $m > 2$. In the case $m = 2$ it suffices to prove the existence of a constant C such that, if $\tilde{u} \in W^{1,2}(\partial B_1, \mathscr{A}_Q(\mathbb{R}^n))$, then we can find an extension u of $\tilde{u}$ to B_1 satisfying the inequality

$$\mathrm{Dir}(u, B_1) \leq C \mathrm{Dir}(\tilde{u}, \mathbb{S}^1).$$

The latter property is a classical fact for usual harmonic extensions of maps with values in the Euclidean space: in that case the constant C can be taken to be 1. For the $\mathscr{A}_Q(\mathbb{R}^n)$ case we consider $\boldsymbol{\zeta} \circ \tilde{u}$ and we then let h be its harmonic extension to B_1. Setting $u := \boldsymbol{\zeta}^{-1} \circ \boldsymbol{\varrho} \circ h$, the inequality is then an easy consequence of the estimate for the harmonic extension and the Lipschitz regularity of $\boldsymbol{\zeta}^{-1}$ and $\boldsymbol{\varrho}$. The case $m \geq 3$ is harder and we need one important auxiliary result.

8.1. 0-homogeneous minimizers.
The following lemma shows that 0-homogeneous minimizers are necessarily constant.

Lemma 8.3. *Let $m \geq 3$ and let $u \in W^{1,2}(B_1, \mathscr{A}_Q(\mathbb{R}^n))$ be a Dir-minimizer with the additional property that*

$$\mathrm{Dir}(u, B_1) \geq \frac{1}{m - 2} \mathrm{Dir}(u|_{\partial B_1}, \mathbb{S}^{m-1}). \tag{8.4}$$

Then u is constant.

Proof. Observe that

$$\text{Dir}(u|_{\partial B_1}, \mathbb{S}^{m-1}) = \int_{\partial B_1} \left(|Du|^2 - |\partial_\nu u|^2 \right) , \tag{8.5}$$

where ν denotes the outer unit normal to ∂B_1. Using (7.7), (8.4) and (8.5) we conclude

$$\int_{\partial B_1} |\partial_\nu u|^2 \leq 0 ,$$

namely that $\partial_\nu u$ vanishes identically on ∂B_1. But then (7.8) implies that $\text{Dir}(u, B_1) = 0$, which clearly gives the constancy of the function u. $\qquad\square$

8.2. **Proof of Proposition 8.2 for** $m \geq 3$. Consider first that, given the classical inequality for Euclidean valued harmonic functions, we can assume without loss of generality that the function u takes values in $\mathring{\mathscr{A}}_Q(\mathbb{R}^n)$. In this case both $\text{Dir}(u, B_1)$ and $\text{Dir}(u|_{\partial B_1}, \partial B_1)$ can be split as

$$\text{Dir}(u, B_1) = \text{Dir}(u^+, B_1) + \text{Dir}(u^-, B_1) ,$$
$$\text{Dir}(u|_{\partial B_1}, \partial B_1) = \text{Dir}(u^+|_{\partial B_1}, \partial B_1) + \text{Dir}(u^-|_{\partial B_1}, \partial B_1) .$$

Assume now that the proposition is false and find a sequence of Dir-minimizers $\{u_k\} \subset W^{1,2}(B_1, \mathring{\mathscr{A}}_Q(\mathbb{R}^n))$ such that

$$\text{Dir}(u_k, B_1) \geq \frac{1}{m - 2 + (k+1)^{-1}} \text{Dir}(u_k|_{\partial B_1}, \partial B_1) .$$

After normalizing the maps we can assume that

$$\text{Dir}(u_k, B_1) = 1 .$$

We consider further the numbers

$$\beta_k := \min\{|\{|u_k^+| = 0\}|, |\{|u_k^-| = 0\}|\}$$

and, up to subsequences, we distinguish two cases: $\liminf_{k\to\infty} \beta_k > 0$ and $\lim_{k\to\infty} \beta_k = 0$.

First case. In this case we have the existence of a constant $\beta > 0$ such that $|\{|u_k^+| = 0\}| \geq \beta$ and $|\{|u_k^-| = 0\}| \geq \beta$ for every k. Since $|D|v|| \leq |Dv|$ for any Q-valued map, we conclude from a classical variant of the Poincaré inequality that $\sup_k(\||u_k^+|\|_{L^2} + \||u_k^-|\|_{L^2}) < \infty$. Up to subsequences we can then assume that u_k converges weakly in $W^{1,2}$ to some u and the Proposition 6.4 would imply that:

- u is Dir-minimizing;
- $\text{Dir}(u, B_1) = \lim_{k\to\infty} \text{Dir}(u_k, B_1) = 1$.

On the other hand, the semicontinuity of the Dirichlet energy would also imply that

$$\text{Dir}(u|_{\partial B_1}, \partial B_1) \leq \liminf_{k\to\infty} \text{Dir}(u_k|_{\partial B_1}, \partial B_1) \leq m - 2 .$$

So, according to Lemma 8.3, u would have to be constant, which clearly is in contradiction with $\text{Dir}(u, B_1) = 1$.

Second case. In this case, again up to extraction of a subsequence, we can assume that $\lim_{k\to\infty} |\{|u_k^+| = 0\}| = 0$. In turn this implies that $\lim_{k\to\infty} |\{|u_k^-| > 0\}| = 0$. In particular,

since $\|D|u_k^-|\|_{L^2(B_1)} \leq 1$, we get $\lim_{k\to\infty} \||u_k^-|\|_{L^2(B_1)} = 0$. In turn this implies as well that $|u_k^-||_{\partial B_1}$ is bounded in $H^{1/2}$ and converges weakly to 0 distributionally. Thus

$$\lim_{k\to\infty} \int_{\partial B_1} |u_k^-|^2 = 0 \,.$$

Consider now the map $w_k : \partial B_1 \ni x \to (u_k^+(x), 1) \in \mathscr{A}_Q(\mathbb{R}^n)$ (where we have "eliminated the negative part" of u_k) and observe that

$$\lim_{k\to\infty} \int_{\partial B_1} \mathcal{G}_s(w_k, u_k)^2 = 0 \,.$$

In particular, for $\lambda > 0$ small (to be chosen later) use the Luckhaus Lemma 5.4 to construct a function $h_k : B_1 \setminus B_{1-\lambda} \to \mathscr{A}_Q(\mathbb{R}^n)$ with the properties that

- $h_k|_{\partial B_1} = u_k|_{\partial B_1}$;
- $h_k(x) = w_k((1-\lambda)^{-1}x)$ for every $x \in \partial B_{1-\lambda}$;
- The following estimate holds with a constant C independent of λ:

$$\limsup_{k\to\infty} \mathrm{Dir}(h_k, B_1 \setminus B_{1-\lambda}) \leq C\lambda \,.$$

Now, we use [10, Proposition 3.10] to find a map $z_k \in W^{1,2}(B_{1-\lambda}, \mathcal{A}_Q(\mathbb{R}^n))$ with the property that

$$\mathrm{Dir}(z_k, B_{1-\lambda}) \leq \frac{1-\lambda}{m-2+\gamma} \mathrm{Dir}(u_k^+|_{\partial B_{1-\lambda}}, \partial B_{1-\lambda}) \leq \frac{(1-\lambda)^{m-2}(m-2+(k+1)^{-1})}{m-2+\gamma} \,,$$

where $\gamma = \gamma(m, n, Q) > 0$. Clearly the map

$$\hat{u}_k(x) = \begin{cases} h_k(x) & \text{if } 1 \geq |x| \geq 1-\lambda \\ (z_k(x), 1) \in \mathscr{A}_Q(\mathbb{R}^n) & \text{if } |x| \leq 1-\lambda \end{cases}$$

is in $W^{1,2}(B_1, \mathscr{A}_Q(\mathbb{R}^n))$ and has the same trace as u_k on ∂B_1. By minimality

$$1 = \lim_{k\to\infty} \mathrm{Dir}(u_k, B_1) \leq \limsup_k \mathrm{Dir}(\hat{u}_k, B_1) \leq \frac{(1-\lambda)^{m-2}(m-2)}{m-2+\gamma} + C\lambda \,.$$

Observe that λ can be chosen arbitrarily small. On the other hand,

$$\lim_{\lambda\to 0} \left(\frac{(1-\lambda)^{m-2}(m-2)}{m-2+\gamma} + C\lambda \right) = \frac{m-2}{m-2+\gamma} < 1 \,,$$

which gives a contradiction, thus completing the proof of Proposition 8.2, and, in turn, of Theorem 8.1.

9. MONOTONICITY OF THE FREQUENCY FUNCTION

As in the case of "classical" Q-valued functions, we introduce a suitable frequency function for maps taking values in $\mathscr{A}_Q(\mathbb{R}^n)$.

Definition 9.1. Let $\Omega \subset \mathbb{R}^m$ be an open set and consider a map $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$. For every $x \in \Omega$ and every $r \in]0, \mathrm{dist}(x, \partial\Omega)[$ we define

$$D_{x,u}(r) := \mathrm{Dir}(u, B_r(x))$$

$$H_{x,u}(r) := \int_{\partial B_r(x)} |u|^2 \,.$$

Moreover, if $H_{x,u}(r) > 0$, we define the frequency function

$$I_{x,u}(r) := \frac{r D_{x,u}(r)}{H_{x,u}(r)} \, .$$

If the point x and the function u are clear from the context, we will drop the subscripts from the corresponding quantities. In our context the celebrated monotonicity theorem of Almgren for the frequency function remains valid. More precisely we have the following theorem.

Theorem 9.2. *Let Ω be a bounded open set and $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ a Dir-minimizing map. Fix a point $x \in \Omega$ and let $\rho := \mathrm{dist}(x, \partial\Omega)$. Then either $u^+ \equiv u^- \equiv Q \llbracket 0 \rrbracket$ on $B_\rho(x)$ or $H_{x,u}(r) > 0$ for every $r \in]0, \rho[$ and in particular $I_{x,u}(r)$ is well defined. Moreover, in the latter case:*

(a) *The function $r \mapsto I_{x,u}(r)$ is monotone nondecreasing and therefore*

$$I_0 := \lim_{r \to 0} I_{x,u}(r)$$

exists and is finite.

(b) *$I_0 = 0$ if and only if $\max\{|u^+(x)|, |u^-(x)|\} > 0$.*

(c) *There is a positive constant $c_0(m, n, Q)$ such that, if $u(x) = Q \llbracket 0 \rrbracket$, then $I_0 \geq c_0(m, n, Q)$.*

(d) *The function $r \mapsto I_{x,u}(r)$ is constant if and only if $u|_{B_r(x)}$ is I_0-homogeneous, i.e. for each $y \in \mathbb{S}^{m-1}$ one of the following alternative holds:*

$$u(ry) = (u^+(ry), 1) \qquad and \qquad u^+(ry) = \sum_i \llbracket r^{I_0} u_i^+(y) \rrbracket \qquad \forall r \in]0, \rho[, \qquad (9.1)$$

$$u(ry) = (u^-(ry), 1) \qquad and \qquad u^-(ry) = \sum_i \llbracket r^{I_0} u_i^-(y) \rrbracket \qquad \forall r \in]0, \rho[, \qquad (9.2)$$

$$u(ry) = \left(Q \llbracket r^{I_0} \boldsymbol{\eta} \circ u^+(y) \rrbracket, 1 \right) = \left(Q \llbracket r^{I_0} \boldsymbol{\eta} \circ u^-(y) \rrbracket, -1 \right) \qquad \forall r \in]0, \rho[. \qquad (9.3)$$

The theorem follows from some important identities which we summarize in the following proposition.

Proposition 9.3. *Let Ω, x, ρ and u be as in Theorem 9.2. Then the maps $r \mapsto H(r), D(r)$ are both absolutely continuous and the following identities hold for a.e. $r \in]0, \rho[$:*

$$D'(r) = \frac{m-2}{r} D(r) + 2 \int_{\partial B_r(x)} |\partial_\nu u|^2 \qquad (9.4)$$

$$H'(r) = \frac{m-1}{r} H(r) + 2D(r) \, . \qquad (9.5)$$

Moreover there is a constant $C_0(m, n, q)$ such that, if $u(x) = Q \llbracket 0 \rrbracket$, then

$$H(r) \leq C_0 r D(r) \qquad \forall r \in]0, \rho[. \qquad (9.6)$$

Proof. Without loss of generality we can assume $x = 0$. The absolute continuity of the map $r \mapsto D(r)$ is an obvious consequence of the absolute continuity of integrals. Passing in polar coordinates we easily see that

$$D'(r) = \int_{\partial B_r} |Du|^2$$

for a.e. $r \in]0, \rho[$. The identity (9.4) is then an obvious consequence of (7.7).

Next consider a classical Sobolev f and let us write

$$\int_{\partial B_r} f^2 = r^{m-1} \int_{\partial B_1} f^2(rx) \, dx$$

Differentiating in r we get the distributional identity

$$\frac{d}{dr} \int_{\partial B_r} f^2 = \frac{m-1}{r} \int_{\partial B_r} f^2 + \int_{\partial B_r} \langle \nabla f^2(x), r^{-1}x \rangle \, dx \,,$$

which easily shows the absolute continuity of the function. We apply the latter identity with $f = |u|$ and use the chain rule formulas analogous to [10, Section 1.3.1] to then derive

$$H'(r) = \frac{m-1}{r} H(r) + 2 \int_{\partial B_r} \sum_i \langle \partial_\nu u_i, u_i \rangle \,.$$

The identity (9.5) is then a consequence of (7.8).

Finally, in order to show (9.6) observe first that we can assume, without loss of generality, $r = 1$. We then use the interior Hölder regularity Theorem 8.1 to derive

$$\int_{\partial B_1} |u(sx)|^2 \, dx \leq C \mathrm{Dir}(u, B_1) \text{ for all } s \in \left[0, \frac{1}{2}\right] \,.$$

Next differentiating the function $s \mapsto \int_{\partial B_1} |u(sx)|^2 \, dx$ and integrating in $s \in \left[\frac{1}{2}, 1\right]$ we easily conclude

$$M := \max_{s \in [1/2,1]} \int_{\partial B_1} |u(sx)|^2 \leq C \int_{1/2}^1 \int_{\partial B_s} |u||Du| \, ds + C \mathrm{Dir}(u, B_1) \,.$$

In particular we derive

$$M \leq C M^{\frac{1}{2}} (\mathrm{Dir}(u, B_1))^{\frac{1}{2}} + C \mathrm{Dir}(u, B_1) \leq \frac{M}{2} + C \mathrm{Dir}(u, B_1) \,,$$

from which $M \leq C \mathrm{Dir}(u, B_1) = D(1)$ easily follows. Since $H(1) \leq M$, this gives the desired bound. $\qquad\square$

Proof of Theorem 9.2. Without loss of generality we can assume $x = 0$ and $\rho = 1$. First of all, if $H(1) = 0$, then clearly the map identically equal to $Q \llbracket 0 \rrbracket$ in B_1 is a competitor, hence by minimality it has to be $D(1) = 0$ and thus $u \equiv Q \llbracket 0 \rrbracket$ in B_1. Hence, let us consider the case $H(1) > 0$. By Proposition 9.3, H will be positive in a neighborhood of 1 and thus we can consider the smallest $r_0 < 1$ for which $H > 0$ on $]r_0, 1[$. On such interval we can differentiate in r and, using the identities (9.4), (9.5) and (7.8) compute

$$I'(r) = \frac{r}{H(r)^2} \left[\int_{\partial B_r} |\partial_\nu u|^2 \int_{\partial B_2} |u|^2 - \left(\int_{\partial B_r} \sum_i \langle \partial_\nu u_i, u_i \rangle \right)^2 \right] \geq 0 \,. \qquad (9.7)$$

In particular we conclude that I is monotone on $]r_0, 1[$ and so

$$H(r) \geq \frac{r}{I(1)} D(r) \,.$$

Now, if it were $r_0 > 0$, then we would have $H(r_0) = 0$ and, by (9.5)

$$H'(r) \leq \frac{m-1+I(1)}{r} H(r) \qquad \text{for a.e. } r \in]r_0, 1[.$$

But then the usual Gronwall's lemma would imply that H vanishes on $]r_0, 1[$, which is a contradiction.

We have thus proved the first claim of the theorem, namely that $H > 0$ in $]0, 1[$ under the assumption that u is nontrivial in B_1. Moreover (9.7) shows (a). (c) is now an obvious consequence of (9.6), which in turn shows that $I_0 = 0$ implies $u(0) \neq Q \llbracket 0 \rrbracket$. Now, if $u(0) \neq Q \llbracket 0 \rrbracket$, namely $|u(0)| > 0$, then by Theorem 8.1 we have that

$$\lim_{r \to 0} \frac{H(r)}{r^{m-1}} = \mathcal{H}^{m-1}(\partial B_1) |u(0)|^2 > 0 \,.$$

On the other hand by Theorem 8.1 we have $D(r) \leq r^{m-2+2\alpha_0} D(1)$. Combining these two facts we then discover that $\lim_{r \to 0} I(r) = 0$.

We finally come to (d). If u is I_0-homogeneous, then the usual chain rules imply that $\partial_\nu u_i(x) = I_0 |x|^{-1} u_i(x)$ for a.e. x and so we conclude that I' is identically 0. On the other hand if $I' \equiv 0$, recalling that $H(r) > 0$, we conclude the existence of a function $\lambda(r)$ such that then $\partial_\nu u_i(x) = \lambda(r) u_i(x)$ holds for a.e. r and a.e. $x \in \partial B_r$. On the other hand, this would imply

$$I_0 = I(r) \overset{(7.8)}{=} \frac{r \int_{\partial B_r} \sum_i \langle \partial_\nu u_i, u_i \rangle}{H(r)} = r \lambda(r) \,.$$

Hence we have

$$\partial_\nu u_i(x) = \frac{I_0}{|x|} u_i(x) \qquad \text{for a.e. } x \in B_1 \,.$$

In particular the same identity holds for u^+ a.e. on B_1^+, for u^- a.e. on B_1^- and for the classical function $\boldsymbol{\eta} \circ u$ everywhere. Since however $u^+ = Q \llbracket \boldsymbol{\eta} \circ u \rrbracket$ on $B_1 \setminus B_1^+$, and $u^-(x) = Q \llbracket \boldsymbol{\eta} \circ u \rrbracket$ on $B_1 \setminus B_1^-$, we conclude that the same identity actually holds a.e. on B_1 for all the three functions u^+, u^- and $\boldsymbol{\eta} \circ u$. In particular, by the argument given in [10, Section 3.4.2], we conclude that all of them are I_0-homogeneous. This in turn implies (d) and completes the proof. □

10. Blow-up and estimate of the singular set

Definition 10.1. Given a Dir-minimizer $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$, we say that a point $x \in \Omega$ is regular if there is a neighborhood U of x such that

   (a) u coincides with $(u^+, 1)$ in U and x is a regular point for the Dir-minimizer $u^+ \in W^{1,2}(U, \mathcal{A}_Q(\mathbb{R}^n))$;
   (b) Or u coincides with $(u^-, -1)$ in U and x is a regular point for the Dir minimizer $u^- \in W^{1,2}(U, \mathcal{A}_Q(\mathbb{R}^n))$.

The set of regular points will be denoted by $\mathrm{Reg}(u)$, whereas its complement, the set of singular points, will be denoted by $\mathrm{Sing}(u)$.

Note that (a) and (b) are not mutually exclusive: they can both hold, in which case both u^+ and u^- coincide with $Q \llbracket \boldsymbol{\eta} \circ u \rrbracket$ in U.

Theorem 10.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, and let $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ be a* Dir-*minimizer. Then the Hausdorff dimension of* $\mathrm{Sing}(u)$ *is at most $m - 1$.*

First of all observe that, by continuity, both Ω_+ and Ω_- are open sets. Moreover, in the respective sets u^+ and u^- are minimizers taking values in $\mathcal{A}_Q(\mathbb{R}^n)$. Since $\mathrm{Sing}(u) \cap \Omega^\pm = \mathrm{Sing}(u^\pm|_{\Omega^\pm})$, we easily conclude from [10, Theorem 0.11] that the dimension of $\mathrm{Sing}(u) \cap (\Omega_+ \cup \Omega_-)$ is at most $m - 2$. It remains to study $\mathrm{Sing}(u) \cap \Omega_0$. On the other hand since $u = Q \llbracket \boldsymbol{\eta} \circ u \rrbracket$ on Ω_0 it follows immediately that $\mathrm{Reg}(u) \cap \Omega_0$ consists of the interior of Ω_0. Thus the theorem will follow immediately from the following

Proposition 10.3. *Consider a connected bounded open set $\Omega \subset \mathbb{R}^m$ and let $u \in W^{1,2}(\Omega, \mathscr{A}_Q(\mathbb{R}^n))$ be a* Dir-*minimizer. If the dimension of Ω_0 is strictly larger than $m-1$, then $\Omega_0 = \Omega$.*

Proof. The proof is entirely analogous to the one of [10, Proposition 3.22] and we just sketch it here for the reader's convenience. First of all we observe that without loss of generality we can assume $\boldsymbol{\eta} \circ u \equiv 0$. In this case the statement of the proposition becomes then that either $\mathrm{Dir}(u) = 0$, or the Hausdorff dimension of Ω_0 is at most $m-1$. We argue by contradiction and assume that $\mathrm{Dir}(u) > 0$ and $\mathcal{H}^{m-1+\alpha}_\infty(\Omega_0) > 0$ for some $\alpha > 0$.

We then fix a point x where

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^{m-1+\alpha}_\infty(\Omega_0 \cap B_r(x))}{r^{m-1+\alpha}} > 0, \tag{10.1}$$

which for measure theoretic reasons occurs at $\mathcal{H}^{m-1+\alpha}$-a.e. $x \in \Omega_0$. For any $x \in \Omega$ we define $\rho(x) := \mathrm{dist}(x, \partial\Omega)$. We then claim that for at least one x where (10.1) holds we must have $\mathrm{Dir}(u, B_{\rho(x)}(x)) > 0$. Otherwise $\mathrm{Dir}(u, B_{\rho(x)}(x)) = 0$ for every $x \in \Omega_0$ by a simple density argument. This would imply that Ω_0 is open. Since it is also obviously closed and Ω is connected, we conclude that $\Omega_0 = \Omega$, which is a contradiction to $\mathrm{Dir}(u, \Omega) > 0$.

Fix then a point x where (10.1) holds and $\mathrm{Dir}(u, B_{\rho(x)}(x)) > 0$. We take advantage of Theorem 9.2 in order to consider $I_0 = \lim_{r \downarrow 0} I_{x,u}(r)$, and we define the rescaled functions

$$y \mapsto u_r(y),$$

where $u_r^\pm(y) = \sum_i \left[\!\left[r^{-I_0} u_i^\pm(ry + x)\right]\!\right]$. Using the compactness of Dir-minimizers and the monotonicity of the frequency function, we conclude that, up to subsequences, rescaled maps converge (locally strongly in $W^{1,2}(\mathbb{R}^m, \mathscr{A}_Q(\mathbb{R}^n))$) to tangent functions defined on $\mathbb{R}^m$ which are locally Dir-minimizers, take values in $\mathscr{A}_Q(\mathbb{R}^n)$ and are nontrivial. In turn, for an appropriate chosen subsequence, (10.1) is used with Theorem 8.1 and with the upper semicontinuity of the $\mathcal{H}^{m-1+\alpha}_\infty$ measure to conclude that at least one such tangent function v has the property that $\mathcal{H}^{m-1+\alpha}(\{|v| = 0\} \cap B_1) > 0$.

Observe that v is I_0 homogeneous. We can repeat the procedure and find a tangent function to v at some y with all the properties above. Such function turns out to be independent of the variable y. Repeating the construction m times we end up with a function w which has positive Dirichlet energy, is a local energy minimizer, is constant and for which the set $\{|w| = 0\}$ is nonempty. This is clearly a contradiction. $\qquad\square$

11. Currents associated to normal graphs on a submanifold

The remaining sections of this work are aimed at obtaining several additional results concerning the geometry of (the integer rectifiable currents associated to) graphs of $\mathscr{A}_Q(\mathbb{R}^n)$-valued functions, which will play a pivotal role in the approximation procedure of area minimizing currents modulo $p = 2Q$ at points of density Q carried out in [5].

From now on, we will often work under the following assumptions.

Assumption 11.1. We consider:

(M) an open submanifold $\mathcal{M} \subset \mathbb{R}^{m+n}$ of class C^3 and dimension m, with $\mathcal{H}^m(\mathcal{M}) < \infty$, which is the graph of a function $\boldsymbol{\varphi} \colon \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ with $\|\boldsymbol{\varphi}\|_{C^3} \leq c$. We will let A and H denote the second fundamental form and the mean curvature vector of $\mathcal{M}$ as a submanifold of $\mathbb{R}^{m+n}$, respectively;

(U) a regular tubular neighborhood $\mathbf{U}$ of $\mathcal{M}$ in $\mathbb{R}^{m+n}$, defined as

$$\mathbf{U} = \left\{ x + \mathrm{v} : \ x \in \mathcal{M} , \mathrm{v} \in T_x^\perp \mathcal{M} \text{ with } |\mathrm{v}| \le c_0 \right\} ,$$

where the constant c_0 is so small that a unique nearest point projection $\mathbf{p} \colon \mathbf{U} \to \mathcal{M}$ is well defined and of class C^2;

(N) a proper Lipschitz map $N \colon \mathcal{M} \to \mathscr{A}_Q(\mathbb{R}^{m+n})$ which satisfies $N_i^+(x), N_i^-(x), \boldsymbol{\eta} \circ N(x) \in T_x^\perp \mathcal{M} \ \forall i$ and $\forall x \in \mathcal{M}$; the map N induces an

$$F \colon \mathcal{M} \to \mathscr{A}_Q(\mathbb{R}^{m+n})$$

by setting

$$F(x) = \begin{cases} \left(\sum_i \llbracket x + N_i^-(x) \rrbracket , -1 \right) & \text{on } \mathcal{M}_- \\ \left(\sum_i \llbracket x + N_i^+(x) \rrbracket , +1 \right) & \text{on } \mathcal{M}_+ \cup \mathcal{M}_0 . \end{cases}$$

Observe that $F^\pm$ and $\boldsymbol{\eta} \circ F$ are proper maps, and they are Lipschitz continuous by Corollary 2.8. Hence, recalling [7], we have a natural way of pushing forward $\llbracket \mathcal{M} \rrbracket$ through the multivalued map $F^\pm$: the corresponding notation is $\mathbf{T}_{F^\pm}$ (in order to distinguish it from the classical "push-forward" via one-valued functions).

Definition 11.2. We introduce the notation $\mathbf{T}_F$ for the integer rectifiable current which is naturally induced by F and which is a representative $\mathrm{mod}(p)$. More precisely, we set

$$\mathbf{T}_F = \mathbf{T}_{F^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+) - \mathbf{T}_{F^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-) + Q(\boldsymbol{\eta} \circ F)_\sharp \llbracket \mathcal{M}_0 \rrbracket \tag{11.1}$$

and we introduce the notation

$$\mathbf{T}_F^+ := \mathbf{T}_{F^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+) \tag{11.2}$$

$$\mathbf{T}_F^- := -\mathbf{T}_{F^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-) \tag{11.3}$$

$$\mathbf{T}_F^0 := Q(\boldsymbol{\eta} \circ F)_\sharp \llbracket \mathcal{M}_0 \rrbracket . \tag{11.4}$$

Remark 11.3. Observe that $\|\mathbf{T}_F^0\|(\mathbf{U} \setminus \mathbf{p}^{-1}(\mathcal{M}_0)) = 0$. In particular, since the sets $\mathbf{p}^{-1}(\mathcal{M}_+), \mathbf{p}^{-1}(\mathcal{M}_-)$ and $\mathbf{p}^{-1}(\mathcal{M}_0)$ are pairwise disjoint, for every Borel $E \subset \mathbf{U}$ we have

$$\|\mathbf{T}_F\|(E) = \|\mathbf{T}_F^0\|(E) + \|\mathbf{T}_F^-\|(E) + \|\mathbf{T}_F^+\|(E) . \tag{11.5}$$

12. COMPATIBLE TRIPLES

Suppose (g^+, g^-, g) is a triple of Lipschitz continuous functions with $g^\pm \colon U \to \mathcal{A}_Q(\mathbb{R}^n)$ and $g \colon U \to \mathbb{R}^n$ with the additional property that they satisfy the following

Definition 12.1 (Compatibility conditions)**.**

(a) For any $x \in U$, either $\mathrm{sep}(g^+(x)) = 0$ or $\mathrm{sep}(g^-(x)) = 0$.

(b) $g(x) = \boldsymbol{\eta} \circ g^+(x)$ whenever $\mathrm{sep}(g^+(x)) = 0$ and $g(x) = \boldsymbol{\eta} \circ g^-(x)$ whenever $\mathrm{sep}(g^-(x)) = 0$.

Here, we have used the notation introduced in [10], according to which $\mathrm{sep}(T) = \min\{|t_i - t_j| : t_i \ne t_j, T = \sum_{l=1}^Q \llbracket t_l \rrbracket\}$ with the convention that $\min \emptyset = 0$. Also note that if (g^+, g^-, g) satisfies the compatibility conditions, and if for some $x \in U$ it holds $\mathrm{sep}(g^+(x)) = 0 = \mathrm{sep}(g^-(x))$ then necessarily $g^+(x) = g^-(x) = Q \llbracket v \rrbracket$ for some $v \in \mathbb{R}^n$, and $g(x) = v$.

To such a triple we can associate a Lipschitz map into $\mathscr{A}_Q(\mathbb{R}^n)$ by means of the following transformation. We first define

$$\mathbf{j}\colon (g^+, g^-, g) \mapsto (v, w, z) = \mathbf{j}(g^+, g^-, g) := (g^+ \ominus \boldsymbol{\eta} \circ g^+, g^- \ominus \boldsymbol{\eta} \circ g^-, \boldsymbol{\eta} \circ g^+ + \boldsymbol{\eta} \circ g^- - g).$$

Then, we map $\mathbf{j}(g^+, g^-, g)$ into $f := \boldsymbol{\iota}^{-1}(\mathbf{j}(g^+, g^-, g))$. In particular, the $\mathscr{A}_Q(\mathbb{R}^n)$-valued map f can be explicitly given as

$$f(x) = \begin{cases} (g^+(x), 1) & \text{if } \mathrm{sep}(g^-(x)) = 0 \\ (g^-(x), -1) & \text{otherwise.} \end{cases}$$

Consistently with the notation of the previous sections, since $\boldsymbol{\iota}$ is an isometry, we identify f and $(v, w, z) = \mathbf{j}(g^+, g^-, g)$ and use interchangeably both symbols, depending on which is most convenient at the moment. One readily checks that f is a Lipschitz map from U into $\mathscr{A}_Q(\mathbb{R}^n)$.

Note that moreover $\mathbf{j}(f^+, f^-, \boldsymbol{\eta} \circ f) = f$. We thus have a right inverse of the map $\mathbf{j}$. However, there is *not* a 1-to-1 correspondence between $\mathscr{A}_Q(\mathbb{R}^n)$-valued maps f and triples (g^+, g^-, g) satisfying (a) and (b). We therefore introduce the following terminology.

Definition 12.2. The triple $(f^+, f^-, \boldsymbol{\eta} \circ f)$ will be called the *canonical decomposition* of the map f.

Next note that the following lemma is a very simple consequence of the above definitions.

Lemma 12.3. *For any $f : U \to \mathscr{A}_Q(\mathbb{R}^n)$ the following holds.*

(i) *First of all we have the estimates*

$$\max\{\mathrm{Lip}(f^\pm), \mathrm{Lip}(\boldsymbol{\eta} \circ f)\} \le \mathrm{Lip}(f);\tag{12.1}$$

and

$$\mathrm{Lip}(f) \le \mathrm{Lip}(g^+) + \mathrm{Lip}(g^-).\tag{12.2}$$

for any (g^+, g^-, g) such that $\mathbf{j}(g^+, g^-, g) = f$.

(ii) *The canonical decomposition of the domain U of f can be determined using any triple (g^+, g^-, g) such that $\mathbf{j}(g^+, g^-, g) = f$. More precisely:*

$$U_\pm = \{\mathrm{sep}(g^\pm) > 0\}\tag{12.3}$$

$$U_0 = U \setminus (U_+ \cup U_-) = \{\mathrm{sep}(g^+) = \mathrm{sep}(g^-) = 0\}.\tag{12.4}$$

(iii) *The following identities hold whenever $\mathbf{j}(g^+, g^-, g) = f$:*

$$\sum_i [\![Dg_i^+]\!] = \sum_i [\![Df_i]\!] \qquad \text{a.e. on } U_+ \cup U_0,\tag{12.5}$$

$$\sum_i [\![Dg_i^-]\!] = \sum_i [\![Df_i]\!] \qquad \text{a.e. on } U_- \cup U_0,\tag{12.6}$$

$$|Df| = \begin{cases} |Dg^+| & \text{a.e. on } U_+ \\ |Dg^-| & \text{a.e. on } U_- \\ \sqrt{Q}|Dg| & \text{a.e. on } U_0. \end{cases}\tag{12.7}$$

13. Taylor expansion of area and excess

We start with a series of theorems which are focused on Taylor expansions of the mass of $\mathbf{T}_F$ and important variants. The first one, which corresponds to [7, Theorem 3.2] is the following.

Theorem 13.1 (Expansion of $\mathbf{M}(\mathbf{T}_F)$)**.** *If $\mathcal{M}$, N and F are as in Assumption 11.1 and $\bar{c}$ is smaller than a geometric constant, then*

$$\mathbf{M}(\mathbf{T}_F) = Q\,\mathcal{H}^m(\mathcal{M}) - Q \int_{\mathcal{M}} \langle H, \boldsymbol{\eta} \circ N \rangle + \frac{1}{2} \int_{\mathcal{M}} |DN|^2$$
$$+ \int_{\mathcal{M}} \sum_i \Big(P_2(x, N_i) + P_3(x, N_i, DN_i) + R_4(x, DN_i) \Big), \qquad (13.1)$$

where P_2, P_3 and R_4 are C^1 functions with the following properties:

(i) *$v \mapsto P_2(x, v)$ is a quadratic form on the normal bundle of $\mathcal{M}$ satisfying*

$$|P_2(x, v)| \le C|A(x)|^2|v|^2 \qquad \forall\, x \in \mathcal{M},\ \forall\, v \perp T_x\mathcal{M}; \qquad (13.2)$$

(ii) *$P_3(x, v, D) = \sum_i L_i(x, v) Q_i(x, D)$, where $v \mapsto L_i(x, v)$ are linear forms on the normal bundle of $\mathcal{M}$ and $D \mapsto Q_i(x, D)$ are quadratic forms on the space of $(m+n) \times (m+n)$-matrices, satisfying*

$$|L_i(x, v)| \le C|A(x)||v| \qquad\qquad \forall x \in \mathcal{M}, \forall v \perp T_x\mathcal{M},$$
$$|Q_i(x, D)| \le C|D|^2 \qquad\qquad \forall x \in \mathcal{M}, \forall D \in \mathbb{R}^{(m+n) \times (m+n)};$$

(iii) *$|R_4(x, D)| = |D|^3 L(x, D)$, for some function L with $\mathrm{Lip}(L) \le C$, which satisfies $L(x, 0) = 0$ for every $x \in \mathcal{M}$ and is independent of x when $A \equiv 0$.*

Moreover, for any Borel function $h : \mathbb{R}^{m+n} \to \mathbb{R}$,

$$\left| \int h\, d\|\mathbf{T}_F\| - \int_{\mathcal{M}} \sum_i h \circ F_i \right| \le C \int_{\mathcal{M}} \Big(\sum_i |A||h \circ F_i||N_i| + \|h\|_\infty (|DN|^2 + |A|^2|N|^2) \Big), \ (13.3)$$

and, if $h(q) = g(\mathbf{p}(q))$ for some g, we have

$$\left| \int h\, d\|\mathbf{T}_F\| - \int_{\mathcal{M}} (Q - Q\langle H, \boldsymbol{\eta} \circ N \rangle + \tfrac{1}{2}|DN|^2)\, g \right| \le C \int_{\mathcal{M}} (|A|^2|N|^2 + |DN|^4)|g|. \quad (13.4)$$

Proof. Observe that the first part of the statement is a simple consequence of (13.4). The latter one can be easily reduced to [7, Theorem 3.2] using (11.5). Indeed, if we introduce $g^+ := g\mathbf{1}_{\mathcal{M}_+}$, $g^- := g\mathbf{1}_{\mathcal{M}_-}$ and $g^0 := g\mathbf{1}_{\mathcal{M}_0}$ and the corresponding $h^\square(p) = g^\square(\mathbf{p}(p))$, it suffices to prove (13.4) for each pair $(h^\square, g^\square)$. In such cases, however, (13.4) can be concluded from [7, Theorem 3.2] (more specifically [7, Eq. (3.4)]) applied to each $\mathbf{T}_{F^\square}$.

As for (13.3) precisely the same argument reduces it to prove it for $h^+ := h\mathbf{1}_{\mathbf{p}^{-1}(\mathcal{M}_+)}$, $h^- := h\mathbf{1}_{\mathbf{p}^{-1}(\mathcal{M}_-)}$ and $h^0 := h\mathbf{1}_{\mathbf{p}^{-1}(\mathcal{M}_0)}$. As above, each such case can be inferred from [7, Theorem 3.2] (more specifically [7, Eq. (3.3)]) applied to the corresponding $\mathbf{T}_{F^\square}$. $\square$

An important corollary of the theorem above is the following.

Corollary 13.2 (Expansion of $\mathbf{M}(\mathbf{G}_f)$)**.** *Assume $\Omega \subset \mathbb{R}^m$ is an open set with bounded measure and $f : \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$ a Lipschitz map with $\mathrm{Lip}(f) \le \bar{c}$. Then,*

$$\mathbf{M}(\mathbf{G}_f) = Q|\Omega| + \frac{1}{2} \int_\Omega |Df|^2 + \int_\Omega \sum_i \bar{R}_4(Df_i), \qquad (13.5)$$

where $\bar{R}_4 \in C^1$ satisfies $|\bar{R}_4(D)| = |D|^3 \bar{L}(D)$ for $\bar{L}$ with $\mathrm{Lip}(\bar{L}) \leq C$ and $\bar{L}(0) = 0$.

Proof. The statement follows from Theorem 13.1 applied to the case in which $\mathcal{M}$ is flat: since $A = 0$ (and thus $H = 0$), the linear and third order terms in the expansion (13.1) vanish. $\square$

We next come to two further Taylor expansions.

Proposition 13.3 (Expansion of a curvilinear excess). *There exists a dimensional constant $C > 0$ such that, if $\mathcal{M}$, F and N are as in Assumption 11.1 with $\bar{c}$ small enough, then*

$$\left| \int |\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|_{no}^2 \, d\|\mathbf{T}_F\|(x) - \int_{\mathcal{M}} |DN|^2 \right| \leq C \int_{\mathcal{M}} \left(|A|^2 |N|^2 + |DN|^4 \right), \qquad (13.6)$$

where $\vec{\mathbf{T}}_F$ and $\vec{\mathcal{M}}$ are the unit m-vectors orienting $\mathbf{T}_F$ and $T\mathcal{M}$, respectively, and $|\cdot|_{no}$ is the non-oriented distance defined by

$$|\vec{\mathbf{T}}_F - \vec{\mathcal{M}}(\mathbf{p}(x))|_{no} := \min\{|\vec{\mathbf{T}}_F - \vec{\mathcal{M}}(\mathbf{p}(x))|, |\vec{\mathbf{T}}_F + \vec{\mathcal{M}}(\mathbf{p}(x))|\} \qquad (13.7)$$

Proof. Proceeding as in the argument leading to Theorem 13.1, we can reduce the statement to corresponding ones where $\mathbf{T}_F$ is replaced by $\mathbf{T}_{F\square}$ and $\mathcal{M}$ is replaced by $\mathcal{M}_{\pm}$ or $\mathcal{M}_0$, after observing that

$$|\vec{\mathbf{T}}_F(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|_{no}^2 = |\vec{\mathbf{T}}_{F\square}(x) - \vec{\mathcal{M}}(\mathbf{p}(x))|^2 \qquad \text{at } \|\mathbf{T}_F\|\text{--a.e. } x \,.$$

Each of these statements can then be concluded from [7, Proposition 3.4]: note indeed that, although [7, Proposition 3.4] is "global", a local version (where, given any Borel $E \subset \mathcal{M}$, in the right hand side of [7, Eq. (3.13)] $\mathbf{T}_F$ is substituted by $\mathbf{T}_F \llcorner \mathbf{p}^{-1}(E)$ and in the left hand side $\mathcal{M}$ is substituted by E) follows directly from the proof given there.

$\square$

The final Taylor expansion which we treat in this section is the one of a suitable cylindrical excess.

Theorem 13.4 (Expansion of a cylindrical excess). *There exist dimensional constants $C, c > 0$ with the following property. Let $f : \mathbb{R}^m \to \mathscr{A}_Q(\mathbb{R}^n)$ be a Lipschitz map with $\mathrm{Lip}(f) \leq c$. For any $0 < s$, set $L := \fint_{B_s} D(\boldsymbol{\eta} \circ f)$ and denote by $\vec{\tau}$ the m-dimensional simple unit vector orienting the graph of the linear map $y \mapsto L \cdot y$. Then, we have*

$$\left| \int_{\mathbf{C}_s} \left| \vec{\mathbf{G}}_f - \vec{\tau} \right|_{no}^2 \, d\|\mathbf{G}_f\| - \int_{B_s} \mathcal{G}_s(Df, Q[\![L]\!])^2 \right| \leq C \int_{B_s} |Df|^4 \,. \qquad (13.8)$$

Proof. Denote by E the quantity

$$E = \int_{\mathbf{C}_s} \left| \vec{\mathbf{G}}_f - \vec{\tau} \right|_{no}^2 \, d\|\mathbf{G}_f\| \,.$$

Observe that, if we set $U := B_s$ and introduce the triples $(f^+, f^-, \boldsymbol{\eta} \circ f)$ and (U_+, U_-, U_0), we easily conclude that

$$E = \int_{U_+ \times \mathbb{R}^n} |\vec{\mathbf{G}}_{f^+} - \vec{\tau}|^2 d\|\mathbf{G}_{f^+}\| + \int_{U_- \times \mathbb{R}^n} |\vec{\mathbf{G}}_{f^-} - \vec{\tau}|^2 d\|\mathbf{G}_{f^-}\|$$

$$+ Q \int_{U_0 \times \mathbb{R}^n} |\vec{\mathbf{G}}_{\boldsymbol{\eta} \circ f} - \vec{\tau}|^2 d\|\mathbf{G}_{\boldsymbol{\eta} \circ f}\| \,.$$

We next can apply the same computations of the proof of [7, Theorem 3.5] to arrive at

$$E = \int_{U_+} |Df^+|^2 + Q\,|U_+|\,|L|^2 - 2\int_{U_+} \sum_i Df_i^+ : L$$
$$+ \int_{U_-} |Df^-|^2 + Q\,|U_-|\,|L|^2 - 2\int_{U_-} \sum_i Df_i^- : L$$
$$+ Q\int_{U_0} |D\boldsymbol{\eta} \circ f|^2 + Q\,|U_0|\,|L|^2 - 2Q\int_{U_0} D\boldsymbol{\eta} \circ f : L$$
$$+ O\left(\int_{B_s} |Df|^4\right).$$

This easily gives

$$E = \int_{U_+} \mathcal{G}(Df^+, Q\,[\![L]\!])^2 + \int_{U_-} \mathcal{G}(Df^-, Q\,[\![L]\!])^2$$
$$+ Q\int_{U_0} |D\boldsymbol{\eta} \circ f - L|^2 + O\left(\int_{B_s} |Df|^4\right).$$

Using (12.5) we then conclude (13.8). $\qquad\square$

14. Taylor expansion of first variations

In this section we consider Taylor expansions of the first variations.

We begin with the expansion for the first variation of graphs. In the following theorem, $\mathrm{Lip}_c(\Omega \times \mathbb{R}^n, \mathbb{R}^d)$ denotes the space of functions $\zeta \in \mathrm{Lip}(\Omega \times \mathbb{R}^n, \mathbb{R}^d)$ for which there exists $\Omega' \subset\subset \Omega$ such that $f(x,y) = 0$ when $x \notin \Omega'$.

Theorem 14.1 (Expansion of $\delta \mathbf{G}_f(\chi)$)**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and $f : \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$ a map with $\mathrm{Lip}(f) \leq \bar{c}$. Consider a function $\zeta \in \mathrm{Lip}_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and the corresponding vector field $\chi \in \mathrm{Lip}_c(\Omega \times \mathbb{R}^n, \mathbb{R}^{m+n})$ given by $\chi(x,y) = (0, \zeta(x,y))$. Then,*

$$\left| \delta\mathbf{G}_f(\chi) - \int_\Omega \sum_i \left(D_x\zeta(x, f_i) + D_y\zeta(x, f_i) \cdot Df_i\right) : Df_i \right| \leq C\int_\Omega |D\zeta||Df|^3. \qquad (14.1)$$

The next two theorems deal with general $\mathbf{T}_F$ as in Assumption 11.1. We restrict our attention to "outer and inner variations". Outer variations result from deformations of the normal bundle of $\mathcal{M}$ which are the identity on $\mathcal{M}$ and map each fibre into itself, whereas inner variations result from composing the map F with isotopies of $\mathcal{M}$.

Theorem 14.2 (Expansion of outer variations)**.** *Let $\mathcal{M}$, $\mathbf{U}$, $\mathbf{p}$ and F be as in Assumption 11.1 with $\bar{c}$ sufficiently small. If $\varphi \in \mathrm{Lip}_c(\mathcal{M})$ and $X(q) := \varphi(\mathbf{p}(q))(q - \mathbf{p}(q))$, then*

$$\delta\mathbf{T}_F(X) = \int_{\mathcal{M}} \left(\varphi\,|DN|^2 + \sum_i (N_i \otimes D\varphi) : DN_i\right) - \underbrace{Q\int_{\mathcal{M}} \varphi\langle H, \boldsymbol{\eta} \circ N\rangle}_{\mathrm{Err}_1} + \sum_{i=2}^{3} \mathrm{Err}_i \qquad (14.2)$$

where

$$|\mathrm{Err}_2| \le C \int_{\mathcal{M}} |\varphi||A|^2|N|^2 \qquad (14.3)$$

$$|\mathrm{Err}_3| \le C \int_{\mathcal{M}} \left(|\varphi|(|DN|^2|N||A| + |DN|^4) + |D\varphi|(|DN|^3|N| + |DN||N|^2|A|) \right). \qquad (14.4)$$

Let Y be a Lipschitz vector field on $T\mathcal{M}$ with compact support, and define X on $\mathbf{U}$ setting $X(q) = Y(\mathbf{p}(q))$. Let $\{\Psi_\varepsilon\}_{\varepsilon \in]-\eta,\eta[}$ be any isotopy with $\Psi_0 = \mathrm{id}$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Psi_\varepsilon = Y$ and define the following isotopy of $\mathbf{U}$: $\Phi_\varepsilon(q) = \Psi_\varepsilon(\mathbf{p}(q)) + (q - \mathbf{p}(q))$. Clearly $X = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Phi_\varepsilon$.

Theorem 14.3 (Expansion of inner variations)**.** *Let* $\mathcal{M}$, $\mathbf{U}$ *and* F *be as in Assumption 11.1 with* $\bar{c}$ *sufficiently small. If* X *is as above, then*

$$\delta\mathbf{T}_F(X) = \int_{\mathcal{M}} \left(\frac{|DN|^2}{2}\mathrm{div}_{\mathcal{M}} Y - \sum_i DN_i : (DN_i \cdot D_{\mathcal{M}}Y) \right) + \sum_{i=1}^{3} \mathrm{Err}_i, \qquad (14.5)$$

where

$$\mathrm{Err}_1 = -Q \int_{\mathcal{M}} \left(\langle H, \boldsymbol{\eta} \circ N \rangle \mathrm{div}_{\mathcal{M}} Y + \langle D_Y H, \boldsymbol{\eta} \circ N \rangle \right), \qquad (14.6)$$

$$|\mathrm{Err}_2| \le C \int_{\mathcal{M}} |A|^2 \left(|DY||N|^2 + |Y||N||DN| \right), \qquad (14.7)$$

$$|\mathrm{Err}_3| \le C \int_{\mathcal{M}} \left(|Y||A||DN|^2(|N| + |DN|) + |DY|(|A||N|^2|DN| + |DN|^4) \right). \qquad (14.8)$$

The three theorems can all be proved appealing to the computations in [7, Section 4]. First of all, by a standard approximation procedure we can assume that the test vector fields are in fact smooth. Next consider the case of Theorem 14.2. Using the triple F^+, F^- and $\boldsymbol{\eta} \circ F$, and taking into account the fact that the currents $\mathbf{T}_{F^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+)$, $\mathbf{T}_{F^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-)$, and $\mathbf{T}_{\boldsymbol{\eta} \circ F} \llcorner \mathbf{p}^{-1}(\mathcal{M}_0)$ are supported on disjoint sets, we can compute

$$\delta\mathbf{T}_F(X) = \delta\mathbf{T}_{F^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+)(X) + \delta\mathbf{T}_{F^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-)(X) + Q\,\delta\mathbf{T}_{\boldsymbol{\eta} \circ F} \llcorner \mathbf{p}^{-1}(\mathcal{M}_0)(X). \qquad (14.9)$$

We can then appeal to [7, Theorem 4.2] to get the corresponding Taylor expansions of the three pieces separately and use (12.5), (12.6) and (12.7) to conclude the desired formulas. The proof of Theorem 14.1 is entirely analogous, using [7, Theorem 4.1]. In both cases there is only one thing to notice: although in the statements of [7, Theorem 4.1 & Theorem 4.2] the domain is assumed to be an open set (and the map φ in [7, Theorem 4.2] is assumed to have compact support), it can be easily seen that the proof given in [7] is not using any specific property of the domain of the map except for its Borel measurability (and the assumption on the support of the map φ in [7, Theorem 4.2] is also redundant).

Reducing Theorem 14.3 to the case of [7, Theorem 4.3] is however different, since in the final part of the proof one integration by parts is used to treat the linear error term and thus the assumption that the domain is open and that the vector field Y has compact support is crucial. In this case we proceed instead as follows. First of all we decompose the first variation of $\mathbf{T}_F$ as in (14.9) and we denote by N^+, N^- and $\boldsymbol{\eta} \circ N$ the triple corresponding to the $\mathscr{A}_Q(\mathbb{R}^n)$-valued map "normal part" N. For each of the three summands in (14.9) we then follow the proof of [7, Theorem 4.3] till [7, Eq. (4.13)]. Taking $\delta\mathbf{T}_{F^+}$ as an example, we get

the expansion

$$\delta \mathbf{T}_{F+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+)(X)$$

$$= \int_{\mathcal{M}_+} \Big(\frac{|DN^+|^2}{2} \mathrm{div}_{\mathcal{M}} Y - \sum_i DN_i^+ : (DN_i^+ \cdot D_{\mathcal{M}} Y) \Big) + J_2^+ + \mathrm{Err}_2^+ + \mathrm{Err}_3^+ , \qquad (14.10)$$

where

$$J_2^+ = Q \int_{\mathcal{M}_+} \sum_j \Big(\langle A(e_j, \nabla_{e_j} Y), \boldsymbol{\eta} \circ N^+ \rangle + \langle A(e_j, Y), D_{e_j} \boldsymbol{\eta} \circ N^+ \rangle \Big) \qquad (14.11)$$

$$\mathrm{Err}_2^+ \leq \int_{\mathcal{M}_+} |A|^2 \Big(|DY| |N^+|^2 + |Y| |N^+| |DN^+| \Big) , \qquad (14.12)$$

$$\mathrm{Err}_3^+ \leq \int_{\mathcal{M}} \Big(|Y| |A| |DN^+|^2 (|N^+| + |DN^+|) + |DY| (|A| |N^+|^2 |DN| + |DN^+|^4) \Big) . \qquad (14.13)$$

We next sum to (14.10) the corresponding expansions for the other two summands in the decomposition of $\delta \mathbf{T}_F(X)$ (namely $\delta \mathbf{T}_{F-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-)(X)$ and $Q\, \delta \mathbf{T}_{\boldsymbol{\eta} \circ F} \llcorner \mathbf{p}^{-1}(\mathcal{M}_0)(X)$). Using then (12.5), (12.6) and (12.7), we easily reach

$$\delta \mathbf{T}_F(X) = \int_{\mathcal{M}} \Big(\frac{|DN|^2}{2} \mathrm{div}_{\mathcal{M}} Y - \sum_i DN_i : (DN_i \cdot D_{\mathcal{M}} Y) \Big) + J_2 + \mathrm{Err}_2 + \mathrm{Err}_3 , \qquad (14.14)$$

where Err_2 and Err_3 satisfy the estimates claimed in Theorem 14.3 and

$$J_2 = Q \int_{\mathcal{M}} \sum_j \Big(\langle A(e_j, \nabla_{e_j} Y), \boldsymbol{\eta} \circ N \rangle + \langle A(e_j, Y), D_{e_j} \boldsymbol{\eta} \circ N \rangle \Big) . \qquad (14.15)$$

Note that at this stage the term J_2 corresponds to the term J_2 of [7, Eq. (4.17)]. Thus we can follow the remaining part of the proof of [7, Theorem 4.3] where an integration by parts transforms J_2 into the term Err_1 of the expansion (14.5).

15. Reparametrization theorem on normal bundles

In this section we state and prove the analogues of the results in [7, Section 5] in the context of $\mathscr{A}_Q(\mathbb{R}^n)$-valued maps.

Theorem 15.1 ($\mathscr{A}_Q(\mathbb{R}^n)$ parametrizations)**.** *Let $Q, m, n \in \mathbb{N}$ and $s < r < 1$. Then, there are constants $c_0, C > 0$ (depending on Q, m, n and $\frac{r}{s}$) with the following property. Let $\varphi, \mathcal{M}$ and $\mathbf{U}$ be as in Assumption 11.1 with $\Omega = B_s$ and let $f : B_r \to \mathscr{A}_Q(\mathbb{R}^n)$ be such that*

$$\|\varphi\|_{C^2} + \mathrm{Lip}(f) \leq c_0 \qquad and \qquad \|\varphi\|_{C^0} + \|f\|_{C^0} \leq c_0\, r. \qquad (15.1)$$

Set $\boldsymbol{\Phi}(x) := (x, \varphi(x))$. Then, there are maps F and N as in Assumption 11.1(N) such that $\mathbf{T}_F = \mathbf{G}_f \llcorner \mathbf{U}$ and

$$\mathrm{Lip}(N) \leq C \big(\|D^2 \varphi\|_{C^0} \|N\|_{C^0} + \|D\varphi\|_{C^0} + \mathrm{Lip}(f) \big) , \qquad (15.2)$$

$$\frac{1}{2\sqrt{Q}} |N(\boldsymbol{\Phi}(x))| \leq \mathcal{G}_s(f(x), Q\, [\![\varphi(x)]\!]) \leq 2\sqrt{Q} |N(\boldsymbol{\Phi}(x))| \qquad \forall x \in B_s , \qquad (15.3)$$

$$|\boldsymbol{\eta} \circ N(\boldsymbol{\Phi}(x))| \leq C |\boldsymbol{\eta} \circ f(x) - \varphi(x)| + C \mathrm{Lip}(f) |D\varphi(x)| |N(\boldsymbol{\Phi}(x))| \qquad \forall x \in B_s . \qquad (15.4)$$

Finally, assume $x \in B_s$ and $(x, \boldsymbol{\eta} \circ f(x)) = \xi + \mathrm{v}$ for some $\xi \in \mathcal{M}$ and $\mathrm{v} \perp T_\xi \mathcal{M}$. Then,

$$\mathcal{G}_s(N(\xi), Q\, [\![\mathrm{v}]\!]) \leq 2\sqrt{Q}\, \mathcal{G}_s(f(x), Q\, [\![\boldsymbol{\eta} \circ f(x)]\!]) . \qquad (15.5)$$

For further reference, we state the following immediate corollary of Theorem 15.1, corresponding to the case of a linear φ.

Proposition 15.2 (Q-valued graphical reparametrization)**.** *Let $Q, m, n \in \mathbb{N}$ and $s < r < 1$. There exist positive constants c, C (depending only on Q, m, n and $\frac{r}{s}$) with the following property. Let π_0 and π be m-planes with $|\pi - \pi_0| \leq c$ and $f : B_r(\pi_0) \to \mathscr{A}_Q(\mathbb{R}^n)(\pi_0^\perp)$ with $\mathrm{Lip}(f) \leq c$ and $|f| \leq cr$. Then, there is a Lipschitz map $g : B_s(\pi) \to \mathscr{A}_Q(\mathbb{R}^n)(\pi^\perp)$ with $\mathbf{G}_g = \mathbf{G}_f \llcorner \mathbf{C}_s(\pi)$ and such that the following estimates hold on $B_s(\pi)$:*

$$\|g\|_{C^0} \leq Cr|\pi - \pi_0| + C\|f\|_{C^0}, \tag{15.6}$$

$$\mathrm{Lip}(g) \leq C|\pi - \pi_0| + C\mathrm{Lip}(f). \tag{15.7}$$

Again Theorem 15.1 will be reduced to the corresponding [7, Theorem 5.1]. First of all we introduce the triple $(f^+, f^-, \boldsymbol{\eta} \circ f)$ and, setting $U := B_r$, consider U_+, U_- and U_0. For each of the maps $f^+, f^-, \boldsymbol{\eta} \circ f$ we apply [7, Theorem 5.1] and find the corresponding "parametrizations", which we denote G^+, G^-, g, so that

$$\mathbf{T}_{G^+} = \mathbf{G}_{f^+} \llcorner \mathbf{U} \tag{15.8}$$

$$\mathbf{T}_{G^-} = \mathbf{G}_{f^-} \llcorner \mathbf{U} \tag{15.9}$$

$$\mathbf{T}_g = \mathbf{G}_{\boldsymbol{\eta} \circ f} \llcorner \mathbf{U}. \tag{15.10}$$

We now wish to show two things, which we summarize in the following

Lemma 15.3. *The triple (G^+, G^-, g) satisfies the compatibility conditions of Definition 12.1, and the map $F = \mathbf{j}(G^+, G^-, g)$ satisfies $\mathbf{T}_F = \mathbf{G}_f \llcorner \mathbf{U}$. In fact the following stronger conclusion holds:*

$$\mathbf{T}_{G^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+) = \mathbf{T}_{F^+} \llcorner \mathbf{p}^{-1}(\mathcal{M}_+) = \mathbf{G}_{f^+} \llcorner \mathbf{U} \cap (U_+ \times \mathbb{R}^n) \tag{15.11}$$

$$\mathbf{T}_{G^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-) = \mathbf{T}_{F^-} \llcorner \mathbf{p}^{-1}(\mathcal{M}_-) = \mathbf{G}_{f^-} \llcorner \mathbf{U} \cap (U_- \times \mathbb{R}^n) \tag{15.12}$$

$$\mathbf{T}_g \llcorner \mathbf{p}^{-1}(\mathcal{M}_0) = \mathbf{T}_{\boldsymbol{\eta} \circ F} \llcorner \mathbf{p}^{-1}(\mathcal{M}_0) = \mathbf{G}_{\boldsymbol{\eta} \circ f} \llcorner \mathbf{U} \cap (U_0 \times \mathbb{R}^n). \tag{15.13}$$

Before coming to the proof of the lemma, we observe that, by virtue of [7, Theorem 5.1 & Lemma 5.4] it implies Theorem 15.1 and the following "geometric algorithm" to find the values of F; see also [17].

Lemma 15.4 (Geometric reparametrization)**.** *The values of F in Theorem 15.1 can be determined at any point $p \in \mathcal{M}$ as follows. Let $\varkappa$ be the orthogonal complement of $T_p\mathcal{M}$. Then $p + \varkappa$ intersects $\mathrm{Gr}(\boldsymbol{\eta} \circ f)$ at a unique point q and if $x := \mathbf{p}_{\pi_0}(q)$, then*

  (i) *$p \in \mathcal{M}_0$ if and only if $x \in U_0$;*
  (ii) *$p \in \mathcal{M}_+$ if and only if $x \in U_+$;*
  (iii) *$p \in \mathcal{M}_-$ if and only if $x \in U_-$.*

Furthermore:

  (iv) *If $p \in \mathcal{M}_0$, then $F(p) = Q \llbracket (x, \boldsymbol{\eta} \circ f(x)) \rrbracket = Q \llbracket q \rrbracket$;*
  (v) *If $p \in \mathcal{M}_+$, then $\mathrm{spt}(F(p)) = \mathrm{Gr}(f^+) \cap (p + \varkappa)$ and the multiplicity of every point q in the value $F(x)$ equals the multiplicity of the point $\mathbf{p}_{\pi_0}^\perp(q)$ in $f^+(\mathbf{p}_{\pi_0}(q))$;*
  (vi) *If $p \in \mathcal{M}_-$, then $\mathrm{spt}(F(p)) = \mathrm{Gr}(f^-) \cap (p + \varkappa)$ and the multiplicity of every point q in the value $F(x)$ equals the multiplicity of the point $\mathbf{p}_{\pi_0}^\perp(q)$ in $f^-(\mathbf{p}_{\pi_0}(q))$.*

Proof of Lemma 15.3. The lemma is an obvious consequence of the geometric algorithm given in [7, Lemma 5.4] to determine G^+, G^- and $\boldsymbol{\eta} \circ G$. Consider indeed a point $p \in \mathcal{M}$ where the

$\text{sep}(G^+(p)) = 0$ and let $q = \boldsymbol{\eta} \circ G^+(p)$. If $\varkappa = (T_p\mathcal{M})^\perp$, [7, Lemma 5.4] implies immediately that $p + \varkappa$ intersects $\text{Gr}(f^+)$ only in the point q and that, having set $x := \mathbf{p}_{\pi_0}(q)$ and $v := \mathbf{p}_{\pi_0}^\perp(q)$, $f^+(x) = Q\,[\![v]\!]$, so $v = \boldsymbol{\eta} \circ f(x)$. This means that $p + \varkappa$ intersects the graph of $\boldsymbol{\eta} \circ f$ in the point q, which in turn, again by [7, Lemma 5.4] must be precisely the value of g at p. We have thus proved that, if $\text{sep}(G^+(p)) = 0$, then $G^+(p) = Q\,[\![g(p)]\!]$. The same argument also shows that, if $\text{sep}(G^-(p)) = 0$, then $G^-(p) = Q\,[\![g(p)]\!]$, thus proving condition (b) in Definition 12.1. Next, we show the validity of condition (a), namely that $\min\{\text{sep}(G^+(p)), \text{sep}(G^-(p))\} = 0$ for every $p \in \mathcal{M}$. Fix $p \in \mathcal{M}$, and set again $\varkappa := (T_p\mathcal{M})^\perp$. By [7, Lemma 5.4], $(p + \varkappa) \cap \text{Gr}(\boldsymbol{\eta} \circ f) = \{q\}$. If we set $x := \mathbf{p}_{\pi_0}(q)$, then $x \in U_+$ or $x \in U_-$ or $x \in U_0$. If $x \in U_+$, then there is $v \in \mathbb{R}^n$ such that $f^-(x) = Q\,[\![v]\!]$, so that $\text{sep}(f^-(x)) = 0$ and $\boldsymbol{\eta} \circ f(x) = v$. Thus, $(p + \varkappa) \cap \text{Gr}(f^-) = \{q\}$, $G^-(p) = Q\,[\![q]\!]$, and $\text{sep}(G^-(p)) = 0$. Analogously, one proves that if $x \in U_-$ then $\text{sep}(G^+(p)) = 0$, and that if $x \in U_0$ then $\text{sep}(G^+(p)) = 0 = \text{sep}(G^-(p))$. Since $\{U_+, U_-, U_0\}$ is a partition of U, at each point $p \in \mathcal{M}$ we necessarily have that either $\text{sep}(G^+(p)) = 0$ or $\text{sep}(G^-(p)) = 0$, as we wanted.

Note that, not only the argument above implies the (G^+, G^-, g) satisfies the compatibility conditions of Definition 12.1 and hence they allow to get a well defined F, but they also imply immediately the conclusions (i), (ii) and (iii) of Lemma 15.4. Knowing the latter, the conclusions (iv), (v) and (vi) of Lemma 15.4 are again an obvious corollary of [7]. In turn they easily imply (15.11), (15.12) and (15.13). Finally, these three identities easily imply $\mathbf{T}_F = \mathbf{G}_f \llcorner \mathbf{U}$. $\qquad\qquad\square$

16. L^1 estimate on the separation over tilting planes

We conclude with the analogue of [8, Lemma 5.6].

Lemma 16.1. *Fix m, n, l and Q. There are geometric constants c_0, C_0 with the following property. Consider two triples of planes $(\pi, \varkappa, \varpi)$ and $(\bar{\pi}, \bar{\varkappa}, \bar{\varpi})$, where*

- *π and $\bar{\pi}$ are m-dimensional;*
- *$\varkappa$ and $\bar{\varkappa}$ are $\bar{n}$-dimensional and orthogonal, respectively, to π and $\bar{\pi}$;*
- *ϖ and $\bar{\varpi}$ are l-dimensional and orthogonal, respectively, to $\pi \times \varkappa$ and $\bar{\pi} \times \bar{\varkappa}$.*

Assume $\text{An} := |\pi - \bar{\pi}| + |\varkappa - \bar{\varkappa}| \le c_0$ and let $\Psi : \pi \times \varkappa \to \varpi$, $\bar{\Psi} : \bar{\pi} \times \bar{\varkappa} \to \bar{\varpi}$ be two maps whose graphs coincide and such that $|\bar{\Psi}(0)| \le c_0 r$ and $\|D\bar{\Psi}\|_{C^0} \le c_0$. Let $u : B_{8r}(0, \bar{\pi}) \to \mathscr{A}_Q(\bar{\varkappa})$ be a map with $\text{Lip}(u) \le c_0$ and $\|u\|_{C^0} \le c_0 r$ and set $f(x) := \sum_i [\![(u_i(x), \bar{\Psi}(x, u_i(x)))]\!]$ and $\mathbf{f}(x) := (\boldsymbol{\eta} \circ u(x), \bar{\Psi}(x, \boldsymbol{\eta} \circ u(x)))$. Then there are

- *a map $\hat{u} : B_{4r}(0, \pi) \to \mathscr{A}_Q(\varkappa)$ such that the map $\hat{f}(x) := \sum_i [\![(\hat{u}_i(x), \Psi(x, \hat{u}_i(x)))]\!]$ satisfies $\mathbf{G}_{\hat{f}} = \mathbf{G}_f \llcorner \mathbf{C}_{4r}(0, \pi)$*
- *and a map $\hat{\mathbf{f}} : B_{4r}(0, \pi) \to \varkappa \times \varpi$ defined by $\mathbf{G}_{\hat{\mathbf{f}}} = \mathbf{G}_{\mathbf{f}} \llcorner \mathbf{C}_{4r}(0, \pi)$.*

Finally, if $\mathbf{g}(x) := (\boldsymbol{\eta} \circ \hat{u}(x), \Psi(x, \boldsymbol{\eta} \circ \hat{u}(x)))$, then

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1} \le C_0 \left(\|f\|_{C^0} + r\text{An}\right)\left(\text{Dir}(f) + r^m(\|D\bar{\Psi}\|_{C^0}^2 + \text{An}^2)\right). \tag{16.1}$$

Proof. We start introducing the maps $f^\pm$ and $u^\pm$. We then apply the reparametrization theorem to determine maps $g^\pm$ and $v^\pm$ which satisfy $\mathbf{G}_{g^\pm} = \mathbf{G}_{f^\pm} \llcorner \mathbf{C}_{4r}(0, \pi)$ and $\mathbf{G}_{v^\pm} = \mathbf{G}_{u^\pm} \llcorner \mathbf{C}_{4r}(0, \pi)$. Recall that $\boldsymbol{\eta} \circ f^+ = \boldsymbol{\eta} \circ f^- = \boldsymbol{\eta} \circ f$ and $\boldsymbol{\eta} \circ u^+ = \boldsymbol{\eta} \circ u^- = \boldsymbol{\eta} \circ u$. By Lemma 15.3 and Lemma 15.4, we can next decompose $U = B_{4r}(0, \pi)$ into disjoint sets U_+, U_- and U_0 by setting $U_\pm = \{x \in U : \text{sep}(v^\pm(x)) \ne 0\}$. Then, we define:

(a) $\hat{u}(x) := v^+(x)$ for $x \in U_+ \cup U_0$, so that $\hat{f}(x) = g^+(x) = (v^+(x), \Psi(x, v^+(x)))$ for $x \in U_+ \cup U_0$;

(b) $\hat{u}(x) := v^-(x)$ for $x \in U_-$, so that $\hat{f}(x) = g^-(x) = (v^-(x), \Psi(x, v^-(x)))$ for $x \in U_-$;

(c) $\hat{f}(x) = f^-(x) = f^+(x) = Q \llbracket \mathbf{g}(x) \rrbracket = Q \left\llbracket \hat{\mathbf{f}}(x) \right\rrbracket$ for $x \in U_0$.

Hence, if we introduce

$$\mathbf{g}^+ := (\boldsymbol{\eta} \circ v^+, \Psi(\cdot, \boldsymbol{\eta} \circ v^+)), \tag{16.2}$$

$$\mathbf{g}^- := (\boldsymbol{\eta} \circ v^-, \Psi(\cdot, \boldsymbol{\eta} \circ v^-)), \tag{16.3}$$

we easily conclude that

$$\|\hat{\mathbf{f}} - \mathbf{g}\|_{L^1(B_{4r}(0,\pi))} = \|\hat{\mathbf{f}} - \mathbf{g}^+\|_{L^1(U_+)} + \|\hat{\mathbf{f}} - \mathbf{g}^-\|_{L^1(U_-)}. \tag{16.4}$$

Now we apply [8, Lemma 5.6] to each $f^\pm$ in order to infer

$$\|\hat{\mathbf{f}} - \mathbf{g}^\pm\|_{L^1} \leq C_0 \left(\|f^\pm\|_{C^0} + r\mathrm{An}\right) \left(\mathrm{Dir}(f^\pm) + r^m (\|D\bar{\Psi}\|_{C^0}^2 + \mathrm{An}^2)\right). \tag{16.5}$$

Considering Lemma 12.3 we have $\|f^\pm\|_{C^0} \leq \|f\|_{C^0}$ and $\mathrm{Dir}(f^\pm) \leq \mathrm{Dir}(f)$. Hence (16.1) is an obvious consequence of (16.4) and (16.5). □

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