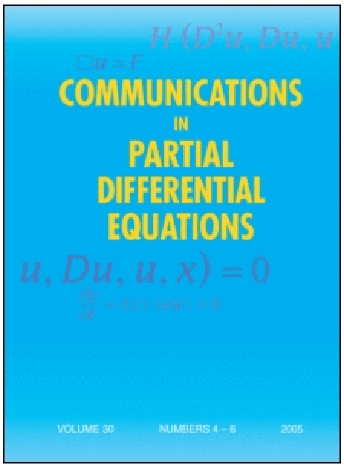

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Low-Regularity Solutions of the Periodic Camassa–Holm Equation

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We prove local existence and uniqueness of weak solutions of the Camassa–Holm equation with periodic boundary conditions in various spaces of low-regularity which include the periodic peakons. The proof uses the connection of the Camassa–Holm equation with the geodesic flow on the diffeomorphism group of the circle with respect to the L^2 metric.

Keywords Camassa–Holm equation; Low-regularity solutions; Periodic peakons.

Mathematics Subject Classification 35Q51; 35D05; 76B15.

1. Introduction

In this article we study the initial value problem of the Camassa–Holm equation (CH)

$$u_t + uu_x = -\partial_x\big(1-\partial_x^2\big)^{-1}\left(u^2 + \frac{u_x^2}{2}\right) \tag{1}$$

$$u(0,\cdot) = u_0, \tag{2}$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For smooth functions, equation (1) can be written in the more familiar form

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0.$$

CH is a one-dimensional nonlinear dispersive equation, which has been derived by Camassa and Holm as a model equation for water waves in Camassa and Holm (1993) (see also Fokas and Fuchssteiner, 1981). The function $u(t,x)$ in (1) stands for the fluid velocity at time t and in the x direction.

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1.1. *Statement of the Result*

Our results concern the existence and the uniqueness of solutions of (1), (2). In order to state them, we will first introduce some notation. As usual, $H^s = H^s(\mathbb{T}, \mathbb{R})$ denotes the Sobolev space

$$H^s := \left\{ f = \sum_k \hat{f}_k e^{2k\pi i x} : \|f\|_s < \infty \right\},$$

where

$$\|f\|_s := \left(\sum_k (1+k^2)^s |\hat{f}_k|^2 \right)^{1/2}.$$

For $s = 0$, we often write L^2 instead of H^0. More generally, for any $s \geq 0$, we define the operator Λ^s as

$$\Lambda^s f(x) = \sum_{k \in \mathbb{Z}} \widehat{\Lambda^s f}(k) e^{2k\pi i x}$$

where

$$\widehat{\Lambda^s f}(k) = \langle k \rangle^s \hat{f}(k), \quad \langle k \rangle := (1+k^2)^{1/2}.$$

Then for any $1 \leq p < \infty$ and $s \geq 0$, we define the $\|f\|_{s,p}$ norms as

$$\|f\|_{s,p} = \|\Lambda^s f\|_{L^p} := \left(\int_{\mathbb{T}} |\Lambda^s f(x)|^p dx \right)^{1/p},$$

and the spaces $H_p^s = H_p^s(\mathbb{T}, \mathbb{R})$ as

$$H_p^s := \left\{ f = \sum_k \hat{f}_k e^{2k\pi i x} : \|f\|_{s,p} < \infty \right\}.$$

Further, we introduce for a continuous function $f : \mathbb{T} \to \mathbb{R}$ the norm $\|f\|_{\mathrm{Lip}} := \|f\|_\infty + \mathrm{Lip}(f)$ and the space $\mathrm{Lip} \equiv \mathrm{Lip}(\mathbb{T})$

$$\mathrm{Lip}(\mathbb{T}) := \left\{ f \in C(\mathbb{R}) \text{ 1-periodic} : \mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

Let $0 < \rho, r < \infty$, and $1 \leq s < 2$, $1 \leq p < \infty$.

Definition 1. We define the set $B_{\rho,r}$ as

$$B_{\rho,r} := \left\{ f \in \mathrm{Lip} : \|f\|_\infty \leq \rho, \|f_x\|_\infty \leq r \right\}$$

and denote by $B_{\rho,r} \cap H_p^s$ and by $L^\infty \cap H_p^s$ the corresponding subsets of H_p^s endowed with the topology induced by the norm of H_p^s.

Definition 2. Let $u_0 \in H^1$ and $T > 0$ be given. A function

$$u \in C^0((-T, T), H^1) \cap C^1((-T, T), L^2)$$

is a weak solution of (1), (2) if $u(0) = u_0$ and (1) is satisfied for any $-T < t < T$ pointwise in x a.e. A function $u \in L^\infty((-T, T), H^1)$ is called a *distributional solution* of (1), (2) if for any $\varphi \in C_c^\infty((-T, T) \times \mathbb{T})$ we have

$$\int_0^T \int_{\mathbb{T}} \left\{ u\varphi_t + \left(\frac{u^2}{2} + (1 - \partial_x^2)^{-1} \left(u^2 + \frac{u_x^2}{2} \right) \right) \varphi_x \right\} dt\, dx + \int_{\mathbb{T}} u_0(x)\varphi(0, x) dx = 0$$

and

$$\int_{-T}^0 \int_{\mathbb{T}} \left\{ u\varphi_t + \left(\frac{u^2}{2} + (1 - \partial_x^2)^{-1} \left(u^2 + \frac{u_x^2}{2} \right) \right) \varphi_x \right\} dt\, dx - \int_{\mathbb{T}} u_0(x)\varphi(0, x) = 0.$$

Note that any weak solution of (1), (2) is a distributional solution.

We are now ready to state the main result of this article.

Theorem 1. *Let* $0 < \rho$, $r < \infty$, $1 \leq s < 2$, $1 < p < \infty$.

(i) *There exists* $T > 0$ *such that for any* $u_0 \in B_{\rho/2, r/2} \cap H_p^s$, (1), (2) *admits a unique weak solution u in the space*

$$X_{s,p,\rho,r} := C^0\big((-T, T), B_{\rho,r} \cap H_p^s\big) \cap C^1\big((-T, T), L^\infty \cap H_p^{s-1}\big).$$

For these solutions, $\int_{\mathbb{T}} u(t, x) dx$ *is a conserved quantity, i.e., is independent of* $-T < t < T$.

(ii) *The equation* (1), (2) *is locally in time* C^0*-well posed in the sense that the map*

$$B_{\rho/2, r/2} \cap H_p^s \ni u_0 \to u \in X_{s,p,\rho,r}$$

is continuous.

(iii) *Any distributional solution* $v \in L^\infty((-T, T), \mathrm{Lip})$ *of* (1), (2) *coincides* (*up to a set of measure zero*) *with the weak solution u constructed in* (i).

Remark 1. We expect that the same arguments used to prove Theorem 1 can be applied to show a similar result for the initial value problem of CH on the real line.

Remark 2. Note that for $1 \leq s < 2$ and $1 < p < \infty$ with $s > 1 + \frac{1}{p}$ the Sobolev space H_p^s can be (compactly) embedded into Lip. Hence by Moser's estimate, H_p^s is an algebra (see Section 3 for a more detailed review of these results). For such s and p (at least in the case $s > 3/2$, $p = 2$), Theorem 1 is well known—see e.g., Constantin and Escher (1998a), Constantin and Molinet (2000), Danchin (2001), Himonas and Misiolek (2000), Li and Olver (2000), and Misiolek (2002). (Similar results have also been obtained for the initial value problem on the line—in addition to the aforementioned articles, see in particular Rodriguez-Blanco, 2001 and Xin and Zhong, 2000.)

Remark 3. It has been shown by Constantin and Molinet that for initial data u_0 in H^1 with $(u_0)'' - u_0$ a non-negative Radon measure, (1), (2), has a global in time weak solution u such that $u(t,\cdot)_{xx} - u(t,\cdot)$ is a non-negative measure for any $t \geq 0$ (see the Proposition of p. 60 in Constantin and McKean, 1999). Further, they show that any other weak solution v with the property that $v(t,\cdot)_{xx} - v(t,\cdot)$ is a non-negative measure for any $t \geq 0$ coincides with u. Since

$$\|u(t,\cdot)\|_{\text{Lip}} \leq C\|u(t,\cdot)_{xx} - u(t,\cdot)\|_{\mathcal{M}},$$

it follows that such a weak solution is Lipschitz (here $\|u(t,\cdot)_{xx} - u(t,\cdot)\|_{\mathcal{M}}$ denotes the total variation of $u(t,\cdot)_{xx} - u(t,\cdot)$).

Therefore, Theorem 1 improves on their uniqueness result.

Remark 4. Among the previous works on the well-posedness of the periodic initial value problem for weak solutions of CH we mention Constantin and Strauss (2000) and Constantin and Escher (1998a) (cf. comments above), Danchin (2001, 2003), Himonas and Misiolek (2000), and Misiolek (2002). Our approach is most closely related to Misiolek (2002) where local (in time) existence, uniqueness and well-posedness results for C^1-solutions of CH for initial data in C^1 are obtained (classical solutions).

1.2. *Historical Comments*

Like the well known Korteweg–de Vries equation (KdV), the Camassa–Holm equation models the propagation of waves at the free surface of shallow water under the influence of gravity. In fact, CH and KdV have many features in common: They are both bi-Hamiltonian, integrable—hence, in particular, they have infinitely many conserved quantities—and admit soliton solutions (cf. Beals et al., 1998, 2000; Camassa and Holm, 1993; Camassa et al., 1994; Constantin and McKean, 1999; Constantin and Strauss, 2000; Fokas and Fuchssteiner, 1981, and references therein). Further they both come up in the description of the geodesic flow on the Bott–Virasoro group with respect to certain (weak) right invariant Riemannian metrics (cf. Constantin, 2000; Constantin and Kolev, 2002, 2003; Holm et al., 1999; Khesin and Misiolek, 2003; Kouranbaeva, 1999; McKean, 2000; Michor and Ratiu, 1998; Misiolek, 1998; Shkoller, 1998, as well as Arnold, 1966; Arnold and Khesin, 1998; Ebin and Marsden, 1970; Ovsienko and Khesin, 1987, and references therein).

Distinctive features of CH are that it admits solutions whose x-derivative gets unbounded in finite time, referred to as *wave-breaking solutions* (cf. Camassa and Holm, 1993; Constantin, 2000; Constantin and Escher, 1998b,c; Constantin and McKean, 1999; Danchin, 2003; Li and Olver, 2000, and references therein) and that the soliton solutions do not evolve in C^1-they are peaked solutions with a corner at their crests and referred to as *peakons*.

In the last five years the initial value problem (1), (2) of CH has been studied extensively (cf. Constantin, 2000; Constantin and Escher, 1998a,b,c; Constantin and McKean, 1999; Constantin and Molinet, 2000; Danchin, 2001; Himonas and Misiolek, 2000; Li and Olver, 2000; McKean, 1998; Misiolek, 2002, and references therein). We point out that in Xin and Zhong (2000) the authors proved existence of *global* weak solutions for initial data in $H^1(\mathbb{R})$. More precisely they approximate CH by adding a higher order parabolic term which depends on a small parameter ε (vanishing viscosity approximation). For any fixed initial data $u_0 \in H^1$ and any $\varepsilon > 0$ they solve the corresponding initial value problem. Then they show that when

$\varepsilon \downarrow 0$ these solutions converge, up to subsequences, to a weak solution of CH. In the recent article (Coclite et al., 2005), the authors proved that there is no need of extracting subsequences and that the unique limit of this approximation provides a strongly continuous semigroup.

Further results in the H^1 setting have been recently proved in Bressan and Constantin (2005) and Bressan and Fonte (2005).

1.3. *Plan of the Article*

In Section 2 we illustrate with more details Theorem 1: In particular, we give a rough outline of its proof and a detailed account of how the periodic peakons fit into our framework.

In Section 3 we give some preliminaries on Sobolev spaces and compositions, which are needed in the Sections 4, 5, 6, and 7, where we give the proof of the various parts of Theorem 1.

2. Theorem 1 and Periodic Peakons

To prove Theorem 1 we follow an approach which has been used previously by several authors, in particular in connection with relating (1) with the geodesic flow on the diffeomorphism group of $\mathbb{T}$ or on the Bott–Virasoro group (cf. Constantin, 2000; Constantin and Kolev, 2002, 2003; Holm et al., 1999; Khesin and Misiolek, 2003; Kouranbaeva, 1999; Michor and Ratiu, 1998; Misiolek, 1998; Shkoller, 1998). It is based on the observation that solutions u of equation (1) can be found by setting $u(t) = v(t) \circ \xi(t)^{-1}$—or more explicitly $u(t, x) = v(t, \xi(t)^{-1}(x))$ $(\forall x \in \mathbb{T})$—where:

- $t \to \xi(t)$ is a curve of homeomorphisms of $\mathbb{T}$, evolving in $\mathrm{Lip} \cap H^s_p$;
- $t \to v(t)$ is a curve in $\mathrm{Lip} \cap H^s_p$;
- $(\xi(t), v(t))$ solves

$$\dot{\xi}(t) = v(t) \quad \dot{v}(t) = F(\xi(t), v(t)) \tag{3}$$

$$\xi(0) = \mathrm{id} \quad v(0) = u_0 \tag{4}$$

where the map F is defined by

$$F(\zeta, w) := -\partial_x\big(1 - \partial_x^2\big)^{-1}\left[\big(w \circ \zeta^{-1}\big)^2 + \frac{1}{2}\big(w \circ \zeta^{-1}\big)_x^2\right] \circ \zeta.$$

It turns out that F is a C^1 vector field on $\mathscr{D}^s_p \times \big(\mathrm{Lip} \cap H^s_p\big)$, where $\mathscr{D}^s_p$ denotes the space of homeomorphism $\zeta \in \mathrm{Lip} \cap H^s_p$ with Lipschitz inverse (see Section 4). Hence (3), (4) can be viewed as a classical ODE and we can show that a solution (ξ, v) exists locally. The function $u(t) := v(t) \circ \xi(t)^{-1}$ is a local (in time) weak solution of (1), (2) with all the properties stated in Theorem 1 (see Section 5).

Conversely, we show that any local in time weak solution of (1), (2) is of the form $u(t) = v(t) \circ \xi(t)^{-1}$, where $t \to (\xi(t), v(t))$ is a C^1 curve in $\mathscr{D}^s_p \times \big(\mathrm{Lip} \cap H^s_p\big)$ which solves (3), (4). More precisely we show that this representation holds for every distributional solution $u \in L^\infty([-T, T] \times \mathrm{Lip})$, provided that we change u on a set of measure zero (in space and time). Note that the curve $t \to (\xi(t), v(t))$ has better regularity properties than $t \to u(t)$, which is continuous in H^s_p, but in general, it is *not* continuous in Lip (see Section 6).

Thanks to this representation, we then obtain the uniqueness of the local weak solutions of (1), (2) stated in Theorem 1 from the uniqueness of the C^1 solutions of the ODE 3, 4.

2.1. *The Periodic Peakons*

We illustrate Theorem 1 by considering the periodic peakons. According to Camassa and Holm (1993), the CH, considered on the line, admits for any given $c \in \mathbb{R}\setminus\{0\}$ the traveling wave solution $ce^{|x-ct|}$ referred to as *peakon*. Its periodic version is given by

$$u_c(t, x) := \gamma \sum_{n=-\infty}^{\infty} e^{-|x+n-ct|}, \tag{5}$$

which is a weak solution of (1) provided γ is chosen appropriately. Such solutions are called *periodic peakons*. Note that $u_c(t)$ has a crest with positive angle α_c at any $x \in \mathbb{R}$ with $x - ct \in \mathbb{Z}$ and hence $u_c(t) \notin C^1$. By an explicit computation of the Fourier coefficients of $u_c(t)$, one sees that $u_c(t) \in H^s \cap \mathrm{Lip}$ for any $c \neq 0$, any $t \in \mathbb{R}$ and any $s < 3/2$ (see e.g., Himonas and Misiolek, 2000).

The only choice of γ which makes u_c a distributional solution is the one which ensures that $u_c(t, ct) = c$. (In the literature, γ has been computed incorrectly as being equal to c—see for instance Constantin and Escher, 1998a.) In other words, the height of the crest of the peakon must be the same as the speed of the crest, a fact which is crucial in our discussion below.

Lemma 1. *u_c is a distributional solution of* (1) *iff γ in* (5) *is chosen in such a way that $u_c(t, ct) = c$.*

Proof. To simplify the notation we drop the index c in u_c. Note that for u being a distributional solution of (1) we must have

$$u_t - u_{txx} + 3\left(\frac{u^2}{2}\right)_x + \left(\frac{u_x^2}{2}\right)_x - \left(\frac{u^2}{2}\right)_{xxx} = 0. \tag{6}$$

Denote by T the distribution on the left-hand side of (6). On the open set

$$\Omega := \left\{(t, x) \in \mathbb{R}^2 : x - ct \notin \mathbb{Z}\right\}$$

u_c is smooth with each of its derivatives bounded. Therefore on Ω the distribution T is given by

$$\partial_t(u - u_{xx}) + 2u_x(u - u_{xx}) + u\partial_x(u - u_{xx})$$

After simple calculations one has (see Constantin and Escher, 1998a, p. 502)

$$u_c(t, x) = \eta \cosh\left(x - ct - [x - ct] - \frac{1}{2}\right),$$

where η is a constant which depends on γ and c and where $[x-ct]$ denotes the integer part of $x-ct$. Therefore on Ω we have $u_{xx}-u=0$ and the distribution T is supported on $L:=\{(t,x): x-ct\in\mathbb{Z}\}$.

Write $T=T_1-T_2+3T_3+T_4-T_5$ where

$$T_1:=u_t \quad T_2=u_{xxt} \quad T_3:=\left(\frac{u^2}{2}\right)_x$$

$$T_4:=\left(\frac{u_x^2}{2}\right)_x \quad T_5:=\left(\frac{u^2}{2}\right)_{xxx}.$$

By the same considerations as above, each T_i can be decomposed as S_i+Z_i, where the distribution Z_i is represented by integration on Ω against a bounded function and S_i is a singular distribution supported on L. Therefore we will call S_i the singular part of the distribution T_i and a necessary and sufficient condition for u to be a distributional solution is that $S_1-S_2+3S_3+S_4-S_5=0$.

Since u is Lipschitz, $S_1=S_3=0$. Moreover, note that $(u_x)^2$ is a Lipschitz function on the whole $\mathbb{R}^2$ as one sees easily by computing u_x—see below. Therefore $S_4=0$ as well. Hence we conclude that $S_2+S_5=0$ is a necessary and sufficient condition for u to be a distributional solution of (1).

As $u(0,0)=u(t,ct)$ for any $t\in\mathbb{R}$, it suffices to check that $S_2+S_5=0$ in a neighborhood of $(0,0)$ if and only if $u(0,0)=c$. Note that in a neighborhood of $(0,0)$ we have

$$u(t,x)=\begin{cases}\eta\cosh\left(x-ct-\frac{1}{2}\right) & \text{for } x-ct>0\\ \eta\cosh\left(ct-x-\frac{1}{2}\right) & \text{for } x-ct<0\end{cases}$$

$$u_x(t,x)=\begin{cases}\eta\sinh\left(x-ct-\frac{1}{2}\right) & \text{for } x-ct>0\\ -\eta\sinh\left(ct-x-\frac{1}{2}\right) & \text{for } x-ct<0,\end{cases}$$

and

$$\left(\frac{u^2}{2}\right)_x(t,x)=\begin{cases}\eta^2\cosh\left(x-ct-\frac{1}{2}\right)\sinh\left(x-ct-\frac{1}{2}\right) & \text{for } x-ct>0\\ -\eta^2\cosh\left(ct-x-\frac{1}{2}\right)\sinh\left(ct-x-\frac{1}{2}\right) & \text{for } x-ct<0.\end{cases}$$

Therefore u_{xx} is the distribution

$$-2\eta\sinh\left(\frac{1}{2}\right)\cdot\delta_{x-ct}+g,$$

where

$$g(t,x)=\begin{cases}\eta\cosh\left(x-ct-\dfrac{1}{2}\right) & \text{for } x-ct>0\\ \eta\cosh\left(ct-x-\dfrac{1}{2}\right) & \text{for } x-ct<0.\end{cases}$$

Note that g is a Lipschitz function and $g=u_c$. Hence the singular part of the distribution u_{xxt} is given by

$$S_2=\partial_t\left(-2\eta\sinh\left(\frac{1}{2}\right)\delta_{x-ct}\right)=2c\eta\sinh\left(\frac{1}{2}\right)\delta^{(1)}_{x-ct}.$$

Similarly, $\left(\frac{u^2}{2}\right)_{xx}$ is given by

$$-2\eta^2\cosh\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}\right)\delta_{x-ct}+\tilde{g},$$

where $\tilde{g}$ can be shown to be a Lipschitz function. Therefore the singular part of the distribution $\left(\frac{u^2}{2}\right)_{xxx}$ is given by

$$S_5=-2\eta^2\cosh\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}\right)\delta^{(1)}_{x-ct}.$$

Hence $S_5+S_2=0$ if and only if

$$2\eta\sinh\left(\frac{1}{2}\right)\left[\eta\cosh\left(\frac{1}{2}\right)-c\right]=0,$$

that is if and only if

$$c=\eta\cosh\left(\frac{1}{2}\right)=u(0,0)=u(t,ct).\qquad\square$$

Theorem 1 says that $u_c(t)$ is the unique solution of (1) with initial data $u_c(0)$ and with the property that it belongs to the space $C^0(\mathbb{R},\mathrm{Lip}\cap H^s_p)\cap C^1(\mathbb{R},L^\infty\cap H^{s-1}_p)$. Note that for any $t>0$ the function $w(t):=u_c(t)-u_c(0)$ has two crests: one located at $x=0$ and the other located at $x=ct$. In both points, the derivative $w(t)_x$ has a jump discontinuity of the same fixed size (in other words the angle of both crests is α_c). Therefore the Lipschitz constant of $w(t)$ is bounded from below by a fixed constant r_c. This implies that $t\to u_c(t)$ is not continuous in the norm topology of Lip. Indeed, we prove in Theorem 1 that $t\to u(t)$ is continuous if we endow $\mathrm{Lip}\cap H^s_p$ with the topology given by the norm $\|\cdot\|_{H^s_p}$. Since with the peakons we are below the critical exponent for the Sobolev embedding, the inclusion $H^s_p\cap\mathrm{Lip}\hookrightarrow\mathrm{Lip}$ is *not* continuous.

Now let us examine the regularity of ξ and v. By definition, ξ is constructed in the following way: For any $x\in\mathbb{R}$ we denote by $\eta^{(x)}$ the unique solution of the ODE

$$\begin{cases}\dot{\eta}^{(x)}(t)=u_c(x,\eta^{(x)}(t))\\ \eta^{(x)}(0)=x\end{cases}\tag{7}$$

and set $\xi(t,x)=\eta^{(x)}(t)$.

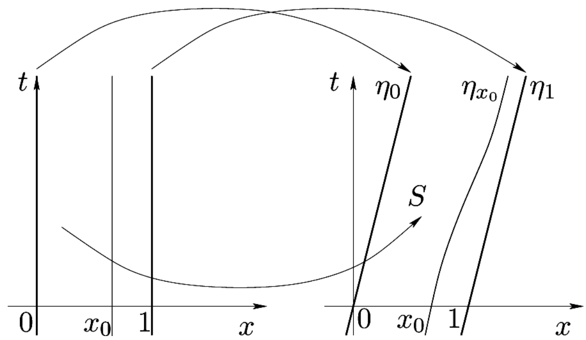

Figure 1. ξ smoothly maps $\mathbb{R} \times [0, 1]$ onto the oblique strip S.

Note that for every $k \in \mathbb{Z}$, $\eta^{(k)}(t) = k + ct$. Since u_c is globally Lipschitz, the solutions to (7) exist for all times and they are unique. Therefore, if we start with an initial datum $x \in]0, 1[$, the trajectory $\eta^{(x)}(t)$ is confined in the oblique strip

$$S := \{(t, x) \in \mathbb{R}^2 : x - ct \in [0, 1]\},$$

because the two lines bounding S are given by $\eta^{(0)}$ and $\eta^{(1)}$; cf. Figure 1. Since the restriction of u_c to S is smooth, we conclude that ξ smoothly maps $\mathbb{R} \times [0, 1]$ onto the strip S and has a smooth inverse. Summarizing, for any $k \in \mathbb{Z}$, we have:

(a) ξ is globally Lipschitz and $\xi(t, x + k) = \xi(t, x) + k$;
(b) $\xi(t, k) = k + ct$;
(c) ξ is smooth on the closed strip $\{(t, x) \in \mathbb{R}^2 : x - ct \in [k, k + 1]\}$.

Therefore, we can view $t \to \xi(t)$ as a curve of bi-Lipschitz homeomorphisms of $\mathbb{T}$. In this way, ξ is smooth on the open set $\mathbb{R} \times (\mathbb{T}\backslash\{0\})$ and on the line $\mathbb{R} \times \{0\}$ has smooth left and right traces.

By definition $v(t, x) = u(t, \xi(t, x))$. From the properties of ξ listed above we conclude:

(i) v is smooth on the open set $\mathbb{R} \times (\mathbb{T}\backslash\{0\})$;
(ii) v has smooth right and left traces on $\mathbb{R} \times \{0\}$.

In other words, the "moving crest" of u located at (t, ct) has been translated by ξ to $(t, 0)$, and outside $\mathbb{R} \times \{0\}$, v is very regular.

The final outcome of this analysis is that both $\xi_x(t, \cdot)$ and $v_x(t, \cdot)$ are piecewise smooth functions which are singular at the point $x = 0$ and that

$$\lim_{\tau \to t} \left(\|v_x(\tau, \cdot) - v_x(t, \cdot)\|_\infty + \|\xi_x(\tau, \cdot) - \xi_x(t, \cdot)\|_\infty \right) = 0.$$

Therefore the curves $t \to \xi(t)$ and $v \to v(t)$ are continuous curves in Lip, though $t \to u(t)$ is not.

2.2. *Critical Exponent*

One of the main aims of our article has been to improve the results of Misiolek (2002) and Danchin (2001) so to include the periodic peakons. These improvements might come as a surprise as both papers, Misiolek (2002) and Danchin (2001), suggest that the exponent 3/2 of the Sobolev space $H^{3/2}$ is critical for uniqueness/well-posedness. However, the notion of (uniform) well-posedness used in the example of Himonas and Misiolek (2000) (cf. also Constantin and Kolev, 2002) is unusually strong.

More precisely, the following identities for the periodic peakons $u_c(t)$ are obtained in Himonas and Misiolek (2000, p. 824) for periodic peakons with period 1 and $1 \leq s \leq 3/2$.

$$\|u_{c'}(t) - u_c(t)\|_{H^s}^2 = \|u_{c'}(0) - u_c(0)\|_{H^s}^2 + 8cc' \sum_{k\in\mathbb{Z}} \frac{1-\cos((c'-c)tk)}{(1+k^2)^{2-s}} \tag{8}$$

$$\|u_{c'}(0) - u_c(0)\|_{H^s}^2 = 4(c'-c)^2 \sum_{k\in\mathbb{Z}} (1+k^2)^{s-2} \tag{9}$$

and

$$\|u_c(0)\|_{H^s}^2 = 4c^2 \sum_{k\in\mathbb{Z}} (1+k^2)^{s-2}. \tag{10}$$

These identities are used in Himonas and Misiolek (2000) to provide two sequences $c'_n \geq c_n$ ($n \geq 1$) with the following properties:

(i) $c_n \uparrow \infty$, and hence $\|u_{c_n}(0)\|_{H^s} \uparrow \infty$ by (10);
(ii) $(c'_n - c_n) \downarrow 0$ and hence, by (9),

$$\lim_{n\uparrow\infty} \|u_{c_n}(0) - u_{c'_n}(0)\|_{H^s} = 0;$$

(iii) $\|u_{c'_n}(1) - u_{c_n}(1)\|_{H^s}^2 \geq K_s n^{3s}$, where $K_s > 0$ only depends on s.

As $\|u_{c_n}(0)\|_{H^s} \to \infty$ and $\|u_{c'_n}(0)\|_{H^s} \to \infty$, the sequences $(u_{c_n}(0))_{n\geq 1}$ and $(u_{c'_n}(0))_{n\geq 1}$ are *not* bounded in H^s. Rather than interpreting the latter inequality as a violation of well-posedness we propose to look at (8), (9) from a dynamical system point of view, saying that *none* of the periodic peakons is Lyapunov stable. Indeed, for any $1 \leq s \leq 3/2$, $c > 0$, and $\varepsilon > 0$ we let $\delta = \delta(\varepsilon, c, s) > 0$ be given by

$$\delta^2 := \min\left\{ \frac{|c|}{2}, \frac{\varepsilon^2}{8} \left(\sum_k (1+k^2)^{s-2} \right)^{-1} \right\}$$

and $t_\delta := \frac{\pi}{3\delta}$, so that we have

$$\|u_{c+\delta}(0) - u_c(0)\|_{H^s}^2 < \varepsilon^2 \quad \text{by (9);}$$

$$\|u_{c+\delta}(t_\delta) - u_c(t_\delta)\|_{H^s}^2 \geq 8c^2 \frac{1-\cos(\pi/3)}{2^{2-s}} \geq 2c^2 \quad \text{by (8).}$$

3. Preliminaries

In this section we collect some preliminary results needed for the proof of Theorem 1.

For $s, p \geq 1$ consider the linear space $E_p^s := H_p^s \cap \mathrm{Lip}$ supplied with the norm $\|u\|_{E_p^s} := \|u\|_{s,p} + \|u\|_{\mathrm{Lip}}$.

Lemma 2. *E_p^s is a Banach space.*

Proof. Let $(u_k)_{k\geq 1}$ be a Cauchy sequence in E_p^s. As $(u_k)_{k\geq 1}$ is a Cauchy sequence in Lip it is bounded in Lip and hence there exists a constant $C > 0$ such that, for any $k \geq 1$,

$$|u_k(x)| \leq C \quad \text{and} \quad |u_k'(x)| \leq C \tag{11}$$

uniformly in $x \in \mathbb{T}$ (up to a set of measure zero). As H_p^s is complete there exists $u \in H_p^s$ such that $u_k \to u$ in H_p^s. In particular $u_k \to u$ and $u_k' \to u'$ in L^1 as $p \geq 1$. Hence there exists a subsequence of $(u_k)_{k\geq 1}$, denoted for simplicity by the same letters, such that $u_k \to u$ and $u_k' \to u'$ pointwise. Passing to the limits in (11) we get that $u \in$ Lip. □

It will be useful to consider the intrinsic norms of H_p^s. According to Adams (1975, Theorem 7.48), the norm $\|f\|_{s,p}$ in H_p^s for $1 \leq s := 1 + \sigma < 2$ and $1 \leq p < \infty$ is equivalent to

$$\left(\|f\|_{1,p}^p + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f'(x) - f'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy \right)^{1/p}.$$

Lemma 3 (Moser's Estimate)**.** *For $s > 0$, $1 < p < \infty$, we have*

$$\|uv\|_{s,p} \leq C\|u\|_{L^\infty}\|v\|_{s,p} + C\|u\|_{s,p}\|v\|_{L^\infty} \tag{12}$$

(*cf. Taylor*, 1996, *Ch.* 13, *§*10, *Corollary* 10.6).

Corollary 1. *For $s \geq 0$, $1 < p < \infty$, $H_p^s \cap L^\infty$ is an algebra with respect to the pointwise multiplication of functions.*

Proof. For $s > 0$, the Corollary follows from Lemma 3. If $s = 0$, the statement follows from the inequality $\|uv\|_{L^p} \leq \|u\|_{L^p}\|v\|_{L^\infty}$. □

Corollary 2. *For $s \geq 0$, $1 < p < \infty$, E_p^s is an algebra with respect to the pointwise multiplication of functions.*

Proof. By the Leibnitz rule in Lip one gets that

$$\|uv\|_{\text{Lip}} \leq \|u\|_{L^\infty}\|v\|_{\text{Lip}} + \|v\|_{L^\infty}\|u\|_{\text{Lip}}$$

for any $u, v \in$ Lip. With Lemma 3 one then concludes the proof. □

For $s, p \geq 1$, denote by $\mathcal{D}_p^s$ the Banach manifold of transformations $\mathbb{R} \to \mathbb{R}$ with manifold structure given by the following two Banach charts

$$\mathcal{U}_0 := \left\{ \xi(x) = x + f(x) : f \in E_p^s, |f(0)| < 1/2, \operatorname{essinf} f' > -1 \right\}$$
$$\mathcal{U}_1 := \left\{ \xi(x) = x + f(x) : f \in E_p^s, 0 < f(0) < 1, \operatorname{essinf} f' > -1 \right\}.$$

Lemma 4. *Let $1 \leq s < 2$ and $1 \leq p < \infty$. Then $\xi \in \mathcal{D}_p^s$ defines a homeomorphism $\xi : \mathbb{R} \to \mathbb{R}$ with inverse ξ^{-1} in $\mathcal{D}_p^s$.*

Remark 5. As the element $\xi \in \mathscr{D}_p^s$ satisfies $\xi'(x) > 0$ and $\xi(x+1) = \xi(x) + 1$ it induces an orientation preserving homeomorphism of the circle $\mathbb{T}(= \mathbb{R}/\mathbb{Z}) \to \mathbb{T}$, $x \bmod 1 \mapsto \xi(x) \bmod 1$.

Proof. As the periodic function f in the definition of $\xi \in \mathscr{D}_p^s$ belongs to Lip, ξ is locally absolutely continuous and therefore for any $x < y$

$$\xi(y) - \xi(x) = \int_x^y \xi'(s)ds \geq (\operatorname{essinf} \xi')(y - x) > 0. \tag{13}$$

Hence, ξ is strictly increasing and therefore injective. As $\xi(x+1) = \xi(x) + 1$ and ξ is continuous, ξ is onto and therefore defines a homeomorphism of $\mathbb{R}$. Moreover, according to (13), for any $x < y$

$$y - x \leq \frac{1}{\operatorname{essinf} \xi'}(\xi(y) - \xi(x))$$

which shows that ξ^{-1} is Lipschitz. To prove that $\xi^{-1} \in \mathscr{D}_p^s$ it thus remains to show that $g(x) := \xi^{-1}(x) - x \in H_p^s$. We will use the intrinsic norm in H_p^s. One has, with $s = 1 + \sigma$, $0 \leq \sigma < 1$

$$\begin{aligned} I_{\sigma,p}(g') &:= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|g'(x) - g'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy \\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\xi'(\xi^{-1}(x)) - \xi'(\xi^{-1}(y))|^p}{|x-y|^{1+\sigma p}} \frac{dx\, dy}{|\xi'(\xi^{-1}(x))\xi'(\xi^{-1}(y))|^p} \\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\xi'(x_1) - \xi'(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}} \left| \frac{x_1 - y_1}{\xi(x_1) - \xi(y_1)} \right|^{1+\sigma p} \frac{dx_1\, dy_1}{|\xi'(x_1)\xi'(y_1)|^{p-1}} \\ &\leq I_{\sigma,p}(f') \frac{1}{(\operatorname{essinf} \xi')^{1+\sigma p}} \frac{1}{(\operatorname{essinf} \xi')^{2(p-1)}} < \infty. \end{aligned}$$

□

Lemma 5. *Let $1 \leq s < 2$, $1 \leq p < \infty$. Then $u \circ \xi \in E_p^s$ for any $u \in E_p^s$ and $\xi \in \mathscr{D}_p^s$.*

Proof. First we show that $u \circ \xi \in \text{Lip}$. Clearly $u \circ \xi : \mathbb{R} \to \mathbb{R}$ is 1-periodic. As the locally Lipschitz functions in $\mathbb{R}$ are locally absolutely continuous we get from the chain rule (still valid for locally absolutely continuous functions) that $(u \circ \xi)' = u' \circ \xi \cdot \xi'$. In particular, $\|u \circ \xi\|_{\text{Lip}} \leq \|u\|_{\text{Lip}}(1 + \|f\|_{\text{Lip}}) < \infty$ where $\xi(x) = x + f(x)$, $f \in E_p^s$.

In order to prove that $u \circ \xi \in H_p^s$ we will estimate the intrinsic norm of $u \circ \xi$. Using the Hölder inequality we have, with $s = 1 + \sigma$,

$$\begin{aligned} I_{\sigma,p}(u' \circ \xi \cdot \xi') &:= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u'(\xi(x))\xi'(x) - u'(\xi(y))\xi'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy \\ &\leq 2^p \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u'(\xi(x)) - u'(\xi(y))|^p}{|x-y|^{1+\sigma p}} |\xi'(x)|^p dx\, dy \\ &\quad + 2^p \int_{\mathbb{T}}\int_{\mathbb{T}} |u'(\xi(y))|^p \frac{|\xi'(x) - \xi'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy. \end{aligned} \tag{14}$$

The two terms in (14) are estimated separately. With $\xi(x) = x + f(x)$, one gets

$$\int_{\mathbb{T}}\int_{\mathbb{T}} |u'(\xi(y))|^p \frac{|\xi'(x) - \xi'(y)|^p}{|x-y|^{1+\sigma p}} dx\,dy \le \|u\|^p_{\mathrm{Lip}} I_{\sigma,p}(f') < \infty \tag{15}$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u'(\xi(x)) - u'(\xi(y))|^p}{|x-y|^{1+\sigma p}} |\xi'(x)|^p dx\,dy$$

$$\le (1 + \|f\|_{\mathrm{Lip}})^p \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u'(\xi(x)) - u'(\xi(y))|^p}{|\xi(x) - \xi(y)|^{1+\sigma p}} \left|\frac{\xi(x) - \xi(y)}{x - y}\right|^{1+\sigma p} dx\,dy$$

$$\le (1 + \|f\|_{\mathrm{Lip}})^{1+p(\sigma+1)} \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u'(x_1) - u'(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}} (\xi^{-1})'(x_1)(\xi^{-1})'(y_1) dx_1\,dy_1$$

$$\le (1 + \|f\|_{\mathrm{Lip}})^{1+p(\sigma+1)} (1 + \|g\|_{\mathrm{Lip}})^2 I_{\sigma,p}(u') < \infty,$$

where as before $g(x) := \xi^{-1}(x) - x$. Hence, $u \circ \xi \in H^s_p$. □

Lemma 6. *Let* $1 \le s < 2$, $1 < p < \infty$. *Then for any* $\xi \in \mathscr{D}^s_p$, $0 < \rho \le \infty$, *and* $0 < r < \infty$ *the right translation*

$$B_{\rho,r} \cap H^s_p \to H^s_p, \quad u \mapsto u \circ \xi$$

is continuous.

Proof. First we prove the statement for $s = 1$: Let $u_k \to u$ in H^1_p. Then one has

$$\int_{\mathbb{T}} |(u_k - u) \circ \xi(x)|^p dx = \int_{\mathbb{T}} |u_k(y) - u(y)|^p (\xi^{-1})'(y) dy$$
$$\le \|\xi^{-1}\|_{\mathrm{Lip}} \|u_k - u\|^p_{L^p} \to 0, \quad k \to \infty \tag{16}$$

as $\|\xi^{-1}\|_{\mathrm{Lip}} < \infty$. In the same way we estimate

$$\int_{\mathbb{T}} |(u'_k - u') \circ \xi(x)|^p |\xi'(x)|^p dx \le \int_{\mathbb{T}} |u'_k(y) - u'(y)|^p |\xi'(\xi^{-1})|^p (\xi^{-1})'(y) dy$$
$$\le \|\xi\|^p_{\mathrm{Lip}} \|\xi^{-1}\|_{\mathrm{Lip}} \|u'_k - u'\|^p_{L^p} \to 0, \quad k \to 0.$$

Let us now consider the case $s = 1 + \sigma$ with $0 < \sigma < 1$. Let $(u_k)_{k\ge 1}$, be a sequence in $B_{\rho,r} \cap H^s_p$ which converges to u in H^s_p. Introduce

$$f_k := (u'_k - u') \circ \xi \quad \text{and} \quad q := \xi'.$$

Taking the case $s = 1$ into account it remains to show that $\|f_k q\|_{\sigma,p} \to 0$ for $k \to \infty$. By the Kenig–Ponce–Vega inequality (Taylor, 2000, Formula (2.1) on p. 106)

$$\|\Lambda^\sigma(f_k q) - f_k \Lambda(q)\|_{L^p} \le C\|q\|_{L^\infty} \|f_k\|_{\sigma,p} \tag{17}$$

we get

$$\begin{aligned}\|f_k q\|_{\sigma,p} &= \|\Lambda^\sigma(f_k q)\|_{L^p}\\ &\le \|\Lambda^\sigma(f_k q) - f_k\Lambda^\sigma(q)\|_{L^p} + \|f_k\Lambda^\sigma(q)\|_{L^p}\\ &\le C\|q\|_{L^\infty}\|f_k\|_{\sigma,p} + \|f_k\Lambda^\sigma(q)\|_{L^p}. \end{aligned} \tag{18}$$

We estimate the last two terms in (18) separately: Using the intrinsic norm for H^σ_p,

$$\|f_k\|^p_{\sigma,p} = \|(u'_k - u')\circ\xi\|^p_{\sigma,p} \le C\|(u'_k - u')\circ\xi\|^p_{L^p} + CI_{\sigma,p}((u'_k - u')\circ\xi).$$

By (16), one obtains

$$\|(u'_k - u')\circ\xi\|^p_{L^p} \le \|\xi^{-1}\|^p_{\mathrm{Lip}}\|u'_k - u'\|^p_{L^p} \to 0$$

and $I_{\sigma,p}((u'_k - u')\circ\xi)$ is bounded by

$$\begin{aligned}&\|\xi^{-1}\|^2_{\mathrm{Lip}}\|\xi\|^{1+\sigma p}_{\mathrm{Lip}}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|(u'_k - u')(x_1) - (u'_k - u')(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}}dx_1\,dy_1\\ &\le \|\xi^{-1}\|^2_{\mathrm{Lip}}\|\xi\|^{1+\sigma p}_{\mathrm{Lip}}I_{\sigma,p}(u'_k - u') \to 0\end{aligned}$$

as $k\to\infty$. Now, consider the term $\|f_k\Lambda^\sigma(q)\|_{L^p}$. As $\|f_k\|_{L^\infty} \le 2r < \infty$[a] and $\Lambda^\sigma(q)\in L^p$ we get that $|f_k\Lambda^\sigma(q)| \le 2r|\Lambda^\sigma(q)| \in L^p$. As any subsequence of $(f_k)_{k\ge1}$ contains a subsequence that converges to 0 a.e. the Lebesgue convergence theorem implies that $\|f_k\Lambda^\sigma(q)\|_{L^p}\to 0$ as $k\to\infty$. □

Lemma 7. *Let* $1\le s<2$, $1\le p<\infty$. *For any sequence* $(u_k)_{k\ge1}\subseteq E^s_p$ *with* $u_k\to u$ *in* E^s_p *and any sequence* $(\xi_k)_{k\ge1}$ *in* $\mathscr{D}^s_p$ *so that* $\xi_k\to\xi$ *in* H^s_p *with*

$$\sup_{k\ge1}\left(\|\xi_k\|_{\mathrm{Lip}} + \|\xi_k^{-1}\|_{\mathrm{Lip}}\right) < \infty \tag{19}$$

and $\xi\in\mathscr{D}^s_p$ *one has*

$$u_k\circ\xi_k \to u\circ\xi$$

in H^s_p.

Before proving the lemma let us state the following corollary.

Corollary 3. *Let* $1\le s<2$, $1\le p<\infty$. *Then the following map is continuous*

$$E^s_p\times\mathscr{D}^s_p\to H^s_p,\ (u,\xi)\mapsto u\circ\xi.$$

Proof of Lemma 7. Assume that $u_k\to u$ in E^s_p and and let $(\xi_k)_{k\ge1}$ be as in the statement of the lemma.

[a]Note that $\|u'\|_{L^\infty}\le r$ as $\|u'_k\|_{L^\infty}\le r$ and there exists a subsequence of $(u'_k)_{k\ge1}$ that converges pointwise a.e. to u'.

Claim 1. $u_k \circ \xi \to u \circ \xi$ in H^s_p uniformly on subsets $B \subseteq \mathscr{D}^s_p$ satisfying $\sup_{\xi \in B} \left(\|\xi\|_{\mathrm{Lip}} + \|\xi^{-1}\|_{\mathrm{Lip}} + \|\xi\|_{s,p} \right) < \infty$.

Proof of Claim 1. The statement for $s = 1$ has already been proved in Lemma 6. In fact, it has been shown that

$$\|(u_k - u) \circ \xi\|_{L^p} \le \|\xi^{-1}\|^{1/p}_{\mathrm{Lip}} \|u_k - u\|_{L^p}$$

and

$$\|(u'_k - u') \circ \xi \cdot \|_{L^p} \le \|\xi\|_{\mathrm{Lip}} \|\xi^{-1}\|^{1/p}_{\mathrm{Lip}} \|u'_k - u'\|^p_{L^p}.$$

Consider the case $1 < s = 1 + \sigma < 2$. It follows from (14), (15), and (16) that

$$\begin{aligned} I_{\sigma,p}((u'_k - u') \circ \xi \cdot \xi') &\le I_{\sigma,p}(f') C \|u_k - u\|^p_{\mathrm{Lip}} + \|\xi\|^{1+ps}_{\mathrm{Lip}} \|\xi^{-1}\|^2_{\mathrm{Lip}} I_{\sigma,p}(u' - u'_k) \\ &\le C\|u - u_k\|^p_{E^s_p} \end{aligned} \tag{20}$$

where $C > 0$ only depends on $\sup_{\xi \in B} \left(\|\xi\|_{\mathrm{Lip}} + \|\xi^{-1}\|_{\mathrm{Lip}} + \|\xi\|_{s,p} \right) < \infty$.

Claim 2. Assume that $u \circ \xi_k \to u \circ \xi$ in H^s_p for any u from a dense subset $A \subseteq E^s_p$ and for any $(\xi_n)_{n \ge 1} \subseteq \mathscr{D}^s_p$, satisfying condition (19) and $\xi_n \to \xi$ in H^s_p. Then $u \circ \xi_n \to u \circ \xi$ in H^s_p for any $u \in E^s_p$.

Proof of Claim 2. Take an arbitrary $u \in E^s_p$. By assumption, there exists $(u_k)_{k \ge 1} \subseteq E^s_p$ such that $u_k \to u$ in E^s_p. Then

$$u \circ \xi_n - u \circ \xi = (u \circ \xi_n - u_k \circ \xi_n) + (u_k \circ \xi_n - u_k \circ \xi) + (u_k \circ \xi - u \circ \xi). \tag{21}$$

It follows from condition (19) and Claim 1 that $u_k \circ \xi_n \to u \circ \xi_n$, $k \to \infty$, in H^s_p uniformly in $n \ge 1$. In particular, for any $\epsilon > 0$ one can find $k = k(\epsilon) \ge 1$ such that $\|u_k \circ \xi_n - u \circ \xi_n\|_{s,p} < \epsilon/3$ and $\|u_k \circ \xi - u \circ \xi\|_{s,p} < \epsilon/3$ uniformly in $n \ge 1$. As $u_k \in A$, it follows from our assumption that there exists $n_\epsilon \ge 1$ such that $\forall n \ge n_\epsilon$, $\|u_k \circ \xi_n - u_k \circ \xi\|_{s,p} < \epsilon/3$. Finally, (21) implies that $\forall n \ge n_\epsilon$, $\|u \circ \xi_n - u \circ \xi\|_{s,p} < \epsilon$.

Claim 3. For any $u \in C^\infty$ and for any $(\xi_k)_{k \ge 1} \subseteq \mathscr{D}^s_p$ satisfying condition (19) and $\xi_k \to \xi$ in H^s_p one has $u \circ \xi_k \to u \circ \xi$ in H^s_p.

Proof of Claim 3. As $u \circ \xi_k - u \circ \xi = (u \circ \zeta_k - u) \circ \xi$ where $\zeta_k := \xi_k \circ \xi^{-1}$, Lemma 6 shows that it is enough to prove $\|u' \circ \zeta_k - u\|_{s,p} \to 0$ as $k \to \infty$.

As $u \in C^\infty$ and as the inclusion $H^s_p \hookrightarrow C(\mathbb{T})$ is continuous, one gets by Lebesgue's convergence theorem that $\|u \circ \zeta_k - u\|_{L^p} \to 0$ for $k \to \infty$. By the triangle inequality,

$$\begin{aligned} \|u' \circ \zeta_k \cdot \zeta'_k - u'\|_{L^p} &\le \|(u' \circ \zeta_k - u') \cdot \zeta'_k\|_{L^p} + \|u'(\zeta'_k - 1)\|_{L^p} \\ &\le \|\zeta_k\|_{\mathrm{Lip}} \|u' \circ \zeta_k - u'\|_{L^p} + \|u'\|_{L^\infty} \|\zeta'_k - 1\|_{L^p}. \end{aligned}$$

By Lebesgue's convergence theorem, one gets $\|u' \circ \zeta_k - u'\|_{L^p} \to 0$ as $k \to \infty$. As $\|\zeta_k\|_{\mathrm{Lip}}$ is uniformly bounded by (19) it follows that $u \circ \zeta_k \to u$ in H^1_p.

It remains to show that

$$I_{\sigma,p}(u' \circ \zeta_k \cdot \zeta_k' - u') \to 0 \quad \text{as } k \to \infty \tag{22}$$

where $s = 1 + \sigma$. We estimate the quantity

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|(u'(\zeta_k(x)) \cdot \zeta_k'(x) - u'(x)) - (u'(\zeta_k(y)) \cdot \zeta_k'(y) - u'(y))|^p}{|x-y|^{1+\sigma p}} dx\, dy. \tag{23}$$

by $I + II$ where

$$I := C \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|u'(\zeta_k(x)) \cdot (\zeta_k'(x) - 1) - u'(\zeta_k(y)) \cdot (\zeta_k'(y) - 1)|^p}{|x-y|^{1+\sigma p}} dx\, dy$$

and

$$II := C \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|(u'(\zeta_k(x)) - u'(\zeta_k(y))) - (u'(x) - u'(y))|^p}{|x-y|^{1+\sigma p}} dx\, dy.$$

Let us estimate I and II separately. Using that for any $0 \le \sigma < 1$

$$\left(\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(x) - f(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy \right)^{1/p} \le C \|f\|_{\sigma,p}$$

we get that

$$I \le C \|(u' \circ \zeta_k)(\zeta_k' - 1)\|_{\sigma,p}^p. \tag{24}$$

Denote $g_k := u' \circ \zeta_k \in L^\infty$ and $f_k := \zeta_k' - 1 \in H_p^\sigma$. It follows from estimate (17) that

$$\begin{aligned} \|(u' \circ \zeta_k)(\zeta_k' - 1)\|_{\sigma,p} &= \|f_k g_k\|_{\sigma,p} = \|\Lambda^\sigma(f_k g_k)\|_{L^p} \\ &\le \|\Lambda^\sigma(f_k g_k) - f_k \Lambda^\sigma(g_k)\|_{L^p} + \|f_k \Lambda^\sigma(g_k)\|_{L^p} \\ &\le C \|g_k\|_{L^\infty} \|f_k\|_{\sigma,p} + \|f_k \Lambda^\sigma(g_k)\|_{L^p}. \end{aligned} \tag{25}$$

The first term in the latter inequality converges to zero as $\|g_k\|_{L^\infty} \le \|u\|_{\text{Lip}} < \infty$. Consider the second term. We have

$$\begin{aligned} \|f_k \Lambda^\sigma(g_k)\|_{L^p}^p &= \int_0^1 |\zeta_k'(x) - 1|^p |\Lambda^\sigma(u' \circ \zeta_k)|^p dx \\ &\le \|\Lambda^\sigma(u' \circ \zeta_k)\|_{L^\infty}^p \|\zeta_k' - 1\|_{L^p}^p \end{aligned} \tag{26}$$

By the Sobolev embedding theorem, $\|\Lambda^\sigma(u' \circ \zeta_k)\|_{L^\infty}^p \le \|\Lambda^\sigma(u' \circ \zeta_k)\|_{1,p}$. Next we prove that $\|\Lambda^\sigma(u' \circ \zeta_k)\|_{1,p}$ is uniformly bounded for $k \ge 1$. For $\sigma = 0$, the statement follows easily from the Lebesgue convergence theorem and the uniform boundednes for $k \ge 1$ of the norms $\|\zeta_k\|_{\text{Lip}}$. If $0 < \sigma < 1$ we estimate $\|\Lambda^\sigma(u' \circ \zeta_k)\|_{1,p}$ by $\|u' \circ \zeta_k\|_{s,p}$. Using the intrinsic norms it suffices to prove that $\sup_{k \ge 1} I_{\sigma,p}((u' \circ \zeta_k)')$ is bounded

$$I_{\sigma,p}((u' \circ \zeta_k)') := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|u''(\zeta_k(x))\zeta_k'(x) - u''(\zeta_k(y))\zeta_k'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy$$

$$\leq 2^p \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u''(\zeta_k(x))|^p |\zeta_k'(x) - \zeta_k'(y)|^p}{|x-y|^{1+\sigma p}} dx\, dy$$
$$+ 2^p \int_{\mathbb{T}}\int_{\mathbb{T}} |\zeta_k'(y)|^p \frac{|u''(\zeta_k(x)) - u''(\zeta_k(x))|^p}{|x-y|^{1+\sigma p}} dx\, dy$$
$$\leq C\|u''\|_{L^\infty}^p \|\zeta_k - \mathrm{id}\,\|_{s,p}^p + C\|\zeta_k\|_{\mathrm{Lip}}^p \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u''(\zeta_k(x)) - u''(\zeta_k(y))|^p}{|x-y|^{1+\sigma p}} dx\, dy$$
$$\leq C\|u''\|_{L^\infty}^p \|\zeta_k - \mathrm{id}\,\|_{s,p}^p + C\|\zeta_k\|_{\mathrm{Lip}}^{1+(\sigma+1)p} \|\zeta_k^{-1}\|_{\mathrm{Lip}}^2 \|u'\|_{s,p}^p.$$

Hence

$$I \leq C\|\zeta_k - \mathrm{id}\,\|_{s,p}^p \to 0, \quad k \to \infty$$

by Lemma 6. Now let us estimate II: As ζ_k is an absolutely continuous function and $u \in C^\infty$ we have

$$\zeta_k(y) - \zeta_k(x) = (y-x)\int_0^1 \zeta_k'(x + s(y-x))ds,$$
$$u'(y) - u'(x) = (y-x)\int_0^1 u''(x + s(y-x))ds$$

and

$$u'(\zeta_k(y)) - u'(\zeta_k(x)) = (\zeta_k(y) - \zeta_k(x))\int_0^1 u''\big(\zeta_k(x) + s(\zeta_k(y) - \zeta_k(x))\big)ds.$$

Hence

$$u'(\zeta_k(y)) - u'(\zeta_k(x)) = (y-x)a_k(x,y)$$

where

$$a_k(x,y) := \int_0^1 u''\big(\zeta_k(x) + s(\zeta_k(y) - \zeta_k(x))\big)ds \cdot \int_0^1 \zeta_k'(x + s(y-x))ds.$$

Hence $II \leq III$, where

$$III := \int_{\mathbb{T}}\int_{\mathbb{T}} |a_k(x,y) - b(x,y)|^p \frac{|y-x|^p}{|x-y|^{1+\sigma p}} dx\, dy \tag{27}$$

and

$$b(x,y) := \int_0^1 u''(x + s(y-x))ds.$$

Observe that $\frac{|x-y|^p}{|x-y|^{1+\sigma p}} = |x-y|^{(1-\sigma)p-1}$ and as $0 \leq \sigma < 1$ we conclude that $(1-\sigma)p - 1 > -1$. Note that the function x^α is integrable on $[0,1]$ for $\alpha > -1$. As the sequence $(\zeta_k')_{k\geq 1}$ is bounded in L^∞ (condition (19)) we conclude that $(a_k - b)_{k\geq 1}$

is bounded in $L^\infty(\mathbb{T}\times\mathbb{T})$. Further, by the Sobolev embedding theorem, $\zeta_k \to \mathrm{id}$ uniformly, $\zeta_k' \to 1$ in L^p, and we conclude by Lebesgue's convergence theorem that

$$a_n \to b \quad \text{a.e. on } \mathbb{T}\times\mathbb{T}.$$

Using once again Lebesgue's convergence theorem we get from (27) that $\lim_{k\to\infty} III = 0$.

Finally, the statement of Lemma 7 follows from Claims 1, 2, and 3. □

Lemma 8. *For any sequence* $(\xi_k)_{k\geq 1}$ *in* $\mathscr{D}_p^s$ *and any* $\xi \in \mathscr{D}_p^s$ *with* $\xi_k \to \xi$ *in* $\mathscr{D}_p^s$, *it follows that*

$$\xi_k^{-1} \to \xi^{-1} \quad \textit{in } H_p^s$$

and

$$\sup_{k\geq 1}(\|\xi_k\|_{\mathrm{Lip}} + \|\xi_k^{-1}\|_{\mathrm{Lip}}) < \infty. \tag{28}$$

Remark 6. It can be checked that, for some s and p in the range considered above, the map $\xi \mapsto \xi^{-1}$ is not continuous in $\mathscr{D}_p^s$.

Proof. Since $\xi \in \mathscr{D}_p^s$, we have $\operatorname{essinf} \xi' > 0$. Therefore, $\|\xi_k - \xi\|_{\mathrm{Lip}} \to 0$ implies the existence of $\varepsilon > 0$ such that $\operatorname{essinf} \xi_k' \geq \varepsilon > 0$ for every k. This gives $\|(\xi^{-1})'\|_{L^\infty} \leq \varepsilon^{-1}$, which implies (28).

Next we consider the case $s = 1$. One has

$$\int_0^1 |\xi_k^{-1}(x) - \xi^{-1}(x)|^p dx = \|\xi_k^{-1}\|_{\mathrm{Lip}} \int_0^1 |y - \xi^{-1}(\xi_k(y))|^p dy \to 0 \quad \text{as } k \uparrow \infty,$$

by Lemma 7. Next we prove that $(\xi_k^{-1})' \to (\xi^{-1})'$ in L^p. One has

$$\begin{aligned}
\|(\xi_k^{-1})' - (\xi^{-1})'\|_{L^p} &\leq \left(\int_{\mathbb{T}} \left|\frac{1}{\xi_k'(\xi_k^{-1}(x))} - \frac{1}{\xi'(\xi^{-1}(x))}\right|^p dx\right)^{1/p} \\
&\leq \|\xi_k^{-1}\|_{\mathrm{Lip}}\|\xi^{-1}\|_{\mathrm{Lip}}\left(\int_{\mathbb{T}} \left|\xi'(\xi^{-1}(x)) - \xi_k'(\xi_k^{-1}(x))\right|^p dx\right)^{1/p} \\
&\leq \|\xi_k^{-1}\|_{\mathrm{Lip}}\|\xi^{-1}\|_{\mathrm{Lip}}\left(\int_{\mathbb{T}} \left|\xi'(\xi_k^{-1}(x)) - \xi_k'(\xi_k^{-1}(x))\right|^p dx\right)^{1/p} \\
&\quad + \|\xi_k^{-1}\|_{\mathrm{Lip}}\|\xi^{-1}\|_{\mathrm{Lip}}\left(\int_{\mathbb{T}} \left|\xi'(\xi^{-1}(x)) - \xi'(\xi_k^{-1}(x))\right|^p dx\right)^{1/p} \\
&\leq \|\xi_k^{-1}\|_{\mathrm{Lip}}\|\xi^{-1}\|_{\mathrm{Lip}}\|\xi_k\|_{\mathrm{Lip}}^{1/p}\left(\int_{\mathbb{T}} \left|\xi'(y) - \xi_k'(y)\right|^p dy\right)^{1/p} \\
&\quad + \|\xi_k^{-1}\|_{\mathrm{Lip}}\|\xi_k\|_{\mathrm{Lip}}^{1/p}\|\xi^{-1}\|_{\mathrm{Lip}}\left(\int_{\mathbb{T}} \left|\xi'(\xi^{-1}\circ\xi_k(x)) - \xi'(x)\right|^p dx\right)^{1/p}.
\end{aligned}$$

The first term in the latter expression converges to zero as $\xi_k \to \xi$ in H_p^s (even in $\mathscr{D}_p^s$). The second term converges to zero by Remark 12 (see Appendix A) and Lemma 7.

Now consider the case $1 < s = 1 + \sigma < 2$. As $\xi_k^{-1} - \xi^{-1} = (\xi_k^{-1} \circ \xi - \mathrm{id}) \circ \xi^{-1}$, by Lemma 6, it is enough to prove that $\xi_k^{-1} \circ \xi \to \mathrm{id}$ in $B_{\rho,r} \cap H_p^s$ for some $0 < \rho \leq \infty$ and $0 < r < \infty$. In order to prove that $\xi_k^{-1} \circ \xi \to \mathrm{id}$ in H_p^s we will prove that $\zeta_k^{-1} \to \mathrm{id}$ in H_p^s with $\zeta_k := (\xi_k^{-1} \circ \xi)^{-1}$. We have

$$
\begin{aligned}
&I_{\sigma,p}((\zeta_k^{-1})' - 1) \\
&:= \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{1}{\zeta_k' \circ \zeta_k^{-1}(x)} - \frac{1}{\zeta_k' \circ \zeta_k^{-1}(y)} \right|^p \frac{dx\,dy}{|x-y|^{1+\sigma p}} \\
&\leq \|\zeta_k^{-1}\|_{\mathrm{Lip}}^{2p} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\zeta_k' \circ \zeta_k^{-1}(x) - \zeta_k' \circ \zeta_k^{-1}(y)|^p}{|x-y|^{1+\sigma p}} dx\,dy \\
&\leq \|\zeta_k^{-1}\|_{\mathrm{Lip}}^{2p} \|\zeta_k\|_{\mathrm{Lip}}^2 \int_{\mathbb{T}} \int_{\mathbb{T}} \left\{ \frac{|\zeta_k'(x_1) - \zeta_k(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}} \left| \frac{\zeta_k^{-1}(x) - \zeta_k^{-1}(y)}{x-y} \right|^{1+\sigma p} \right\} dx_1\,dx_2 \\
&\leq C \|\zeta_k^{-1}\|_{\mathrm{Lip}}^{1+(2+\sigma)p} \|\zeta_k\|_{\mathrm{Lip}}^2 \|\zeta_k - \mathrm{id}\|_{s,p}^p. \qquad (29)
\end{aligned}
$$

By Lemma 7, $\|\zeta_k - \mathrm{id}\|_{s,p} \to 0$ for $k \to \infty$. □

4. The Vector Field *F*

For $(\xi, v) \in \mathscr{D}_p^s \times E_p^s$ $(1 \leq s < 2, 1 < p < \infty)$ define the vector field F by

$$(\xi, v) \mapsto F(\xi, v) := (v, f(\xi, v)) \in E_p^s \times E_p^s \qquad (30)$$

where

$$-f(\xi, v) := \left\{ (1 - \partial_x^2)^{-1} \partial_x \left((v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2 \right) \right\} \circ \xi. \qquad (31)$$

First note that according to Lemmas 4 and 5, $\xi^{-1} \in \mathscr{D}_p^s$ and $v \circ \xi^{-1} \in E_p^s$. As $(v \circ \xi^{-1})' \in H_p^{s-1} \cap L^\infty$ it follows from Corollaries 1 and 2 that

$$(v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2 \in H_p^{s-1} \cap L^\infty.$$

As the range of the operator $(1 - \partial_x^2)^{-1} \partial_x$ acting on $H_p^{s-1} \cap L^\infty$ is contained in E_p^s we get that

$$(1 - \partial_x^2)^{-1} \partial_x \left((v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2 \right) \in E_p^s$$

hence by Lemma 5

$$F(\xi, v) = (v, f(\xi, v)) \in E_p^s \times E_p^s.$$

In this section we prove that F is a C^1-vector field.

First, let us rewrite $f(\xi, v)$ in the following way

$$
\begin{aligned}
f(\xi, v) &= -R_\xi \circ \left\{ \Lambda^{-2} \partial_x \left((v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2 \right) \right\} \\
&= -(R_\xi \circ \Lambda^{-2} \partial_x \circ R_{\xi^{-1}}) h(\xi, v) \qquad (32)
\end{aligned}
$$

where $\Lambda^2 = 1 - \partial_x^2$ and

$$h(\xi, v) := R_\xi \circ \left((v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2 \right) \tag{33}$$

where $R_\xi v$ denotes the right translation $v \circ \xi$ of $v \in E_p^s$ by the element $\xi \in \mathscr{D}_p^s$. Write $P_\xi := R_\xi \circ P \circ R_{\xi^{-1}}$ for any given operator P acting on E_p^s or $H_p^s \cap L^\infty$. Note that $(\Lambda_\xi)^{-1} = (\Lambda^{-1})_\xi$ and $R_{\xi^{-1}} = (R_\xi)^{-1}$.

Let $t \to \xi_t \in \mathscr{D}_p^s$ be a C^1-path passing through $\xi_0 = \xi$ with $\frac{d}{dt}|_{t=0}\xi_t = w \in E_s^p$. The *directional derivative* of P_ξ in direction w, $D_w(P_\xi)(f)$, is defined (at least in a formal way) as

$$D_w(P_\xi)(f) := \frac{d}{dt}\Big|_{t=0}(P_{\xi_t}f).$$

Remark 7. The directional derivative $D_w(P_\xi)(f)$ is *not* an invariant quantity as it depends on the choice of the Banach chart in $\mathscr{D}_p^s$. Nevertheless, this notion is useful in the further calculations.

Following Misiolek (2002, p. 1086) we have the following lemma.

Lemma 9.

(i) $(\partial_x)_\xi = \frac{1}{\xi'(x)}\partial_x$.
(ii) $D_w(\partial_x)_\xi = -\frac{w'}{(\xi')^2}\partial_x$.

Proof. For any $f \in E_p^s$ and $\xi \in \mathscr{D}_p^s$ one has

$$(\partial_x)_\xi f := R_\xi \circ \partial_x \circ R_{\xi^{-1}} f = R_\xi((f' \circ \xi^{-1})/(\xi' \circ \xi^{-1})) = \frac{1}{\xi'(x)}\partial_x f. \tag{34}$$

Let $t \to \xi_t \in \mathscr{D}_p^s$ be a C^1-path in $\mathscr{D}_p^s$ such that $\xi_0 = \xi$ and $\frac{d}{dt}|_{t=0}\xi_t = w \in E_s^p$. Then, using (34) we get

$$D_w(\partial_x)_\xi := \frac{d}{dt}\Big|_{t=0}\left(\frac{1}{\xi_t'(x)}\partial_x\right) = -\frac{w'}{(\xi')^2}\partial_x. \qquad \square$$

Lemma 10. *The operators* $(1 \pm \partial_x)_\xi : E_p^s \to H_p^{s-1} \cap L^\infty$ *are bijective and*

$$(1 \pm \partial_x)_\xi^{-1} f = \pm\left(F_\pm(0) \pm \int_0^x \xi'(y)e^{\pm\xi(y)}f(y)dy\right)e^{\mp\xi(x)} \tag{35}$$

where

$$F_\pm(0) := \pm(e^{\pm 1} - 1)^{-1}\int_0^1 \xi'(y)e^{\pm\xi(y)}f(y)dy. \tag{36}$$

Moreover, for any $\xi \in \mathscr{D}_p^s$ *given, the linear mapping*

$$(1 \pm \partial_x)_\xi^{-1} : H_p^{s-1} \cap L^\infty \to E_p^s \tag{37}$$

is bounded.

Proof. It is straightforward to verify that $(1 \pm \partial_x)_\xi$ are 1-1. To see that these operators are onto we argue as follows. Given $f \in H_p^{s-1} \cap L^\infty$ we solve the equation $(1 \pm \partial_x)_\xi g = f$ or

$$g \pm \frac{1}{\xi'} g' = f \quad \text{or} \quad g' \pm \xi' g = \pm \xi' f. \tag{38}$$

The solution of the homogeneous equation $g' \pm \xi' g = 0$ is $g_\pm(x) = Ce^{\mp\xi(x)}$. By the method of variation of parameters and the substitution $g(x) = F_\pm(x)e^{\mp\xi(x)}$ into (38), one gets

$$F_\pm(x) = F_\pm(0) \pm \int_0^x \xi'(y)e^{\pm\xi(y)} f(y)dy. \tag{39}$$

The value of $F_\pm(0)$ is determined by the requirement that $g(0) = g(1)$. This leads to

$$F_\pm(0) = \pm(e^{\pm 1} - 1)^{-1} \int_0^1 \xi'(y)e^{\pm\xi(y)} f(y)dy.$$

that proves (35) and (36).

It remains to show that $g \in E_p^s$. By the formula for g, one has $g \in L^\infty$. Then $g \in \mathrm{Lip}$ as $g' = \mp\xi' g \pm \xi' f \in L^\infty$. Similarly, to see that $g' \in H_p^{s-1}$, we use that $H_p^{s-1} \cap L^\infty$ is an algebra (Corollary 1) and $g \in \mathrm{Lip} \subseteq H_p^1 \subseteq H_p^{s-1}$ (for $1 \leq s < 2$) to conclude that $\xi' f \in H_p^{s-1}$ and $\xi' g \in H_p^{s-1}$. The boundedness of the operator (37) follows in a straightforward way from (35) and (36). □

Corollary 4. *For any $w \in E_p^s$ and any $\xi \in \mathcal{D}_p^s$, the operator $D_w(1 \pm \partial_x)_\xi^{-1} : H_p^{s-1} \cap L^\infty \to E_p^s$ applied to $f \in H_p^{s-1} \cap L^\infty$ is given by the sum of the following operators*

$$I := \mp w(x)\left(F_\pm(0) \pm \int_0^x \xi'(y)e^{\pm\xi(y)} f(y)\right)e^{\mp\xi(x)} \tag{40}$$

$$II := \pm e^{\mp\xi(x)} \int_0^x (w'(y) \pm w(y)\xi'(y))e^{\pm\xi(y)} f(y)dy \tag{41}$$

$$III := \pm e^{\mp\xi(x)}(e^{\pm 1} - 1)^{-1} \int_0^1 (w'(y) \pm w(y)\xi'(y))e^{\pm\xi(y)} f(y)dy \tag{42}$$

where $F_\pm(0) := \pm(e^{\pm 1} - 1)^{-1} \int_0^1 \xi'(y)e^{\pm\xi(y)} f(y)dy$. For any $\xi \in \mathcal{D}_p^s$,

$$D_{(\cdot)}(1 \pm \partial_x)_\xi^{-1} : E_p^s \to \mathcal{L}\big(H_p^{s-1} \cap L^\infty, E_p^s\big)$$
$$w \mapsto D_w(1 \pm \partial_x)_\xi^{-1}$$

is a bounded linear map.

Proof. The claimed formula for $D_w(1 \pm \partial_x)_\xi^{-1}$ follows directly from Lemma 10. The formula shows that $D_w(1 \pm \partial_x)_\xi^{-1}(f)$ is linear in $w \in E_p^s$ and in $f \in H_p^{s-1} \cap L^\infty$. It remains to provide an appropriate bound for the norm of the bilinear functional $D_w(1 \pm \partial_x)_\xi^{-1}(f)$. The terms I, II, and III are estimated separately.

First, we estimate $\|D_w(1 \pm \partial_x)_\xi^{-1}(f)\|_{L^\infty}$. With g given by $\xi = \mathrm{id} + g$ one has

$$\|I\|_{L^\infty} \leq C\|w\|_{L^\infty}(1 + \|g\|_{\mathrm{Lip}})\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}$$

and

$$\|II\|_{L^\infty} + \|III\|_{L^\infty} \leq C(\|w\|_{\mathrm{Lip}} + \|w\|_{L^\infty}(1 + \|g\|_{\mathrm{Lip}}))\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}.$$

Next, we estimate $\|D_w(1 \pm \partial_x)_\xi^{-1}(f)\|_{\mathrm{Lip}}$. One has

$$\begin{aligned}\|I\|_{\mathrm{Lip}} \leq{}& C\|w\|_{\mathrm{Lip}}(1 + \|g\|_{\mathrm{Lip}})\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})} + \|w\|_{L^\infty}(1 + \|g\|_{\mathrm{Lip}})\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}\\ &+ \|w\|_{L^\infty}(1 + \|g\|_{\mathrm{Lip}})^2\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}.\end{aligned}$$

In the same way one estimates

$$\begin{aligned}&\|II\|_{\mathrm{Lip}} + \|III\|_{\mathrm{Lip}}\\ &\quad\leq (1 + \|g\|_{\mathrm{Lip}})(\|II\|_{L^\infty} + \|III\|_{L^\infty}) + (\|w\|_{\mathrm{Lip}} + \|w\|_{L^\infty}(1 + \|g\|_{\mathrm{Lip}}))\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}.\end{aligned}$$

Finally, using that $H_p^{s-1} \cap L^\infty$ is an algebra, we get by Moser's inequality

$$\begin{aligned}\|I'\|_{s-1,p} \leq{}& C(1 + \|g\|_{\mathrm{Lip}})\|f\|_{L^\infty} e^{2(1+\|g\|_{L^\infty})}(\|w'\|_{s-1,p} + (1 + \|g'\|_{s-1,p})\|w\|_\infty)\\ &+ C\|w\|_{L^\infty}((1 + \|g\|_{L^\infty})\|f\|_{s-1,p} + (1 + \|g\|_{s-1,p})\|f\|_{L^\infty}\\ &+ (1 + \|g\|_{\mathrm{Lip}})^2\|f\|_{L^\infty})e^{2(1+\|g\|_{L^\infty})}.\end{aligned}$$

In the same way one can estimate $\|II'\|_{s-1,p}$ and $\|III'\|_{s-1,p}$. □

Lemma 11. *The mapping*

$$\mathscr{D}_p^s \to \mathscr{L}\big(E_p^s, \mathscr{L}(H_p^{s-1} \cap L^\infty, E_p^s)\big), \quad \xi \mapsto D_{(\cdot)}(1 \pm \partial_x)_\xi^{-1}$$

is continuous.

Proof. Assume that $(\xi_n)_{n\geq 1} \subseteq \mathscr{D}_p^s$ converges to ξ in $\mathscr{D}_p^s$. We have to show that

$$D_{(\cdot)}(1 \pm \partial_x)_{\xi_n}^{-1} \to D_{(\cdot)}(1 \pm \partial_x)_\xi^{-1} \quad \text{as } n \to \infty.$$

The terms I, II, and III in Corollary 4 are treated separately. Using that $H_p^{s-1} \cap L^\infty$ is an algebra the claimed continuity can be verified in a straightforward way. □

From Lemma 11 we immediately obtain the following proposition.

Proposition 1. *The mapping*

$$\mathscr{D}_p^s \to \mathscr{L}\big(H^{s-1} \cap L^\infty, E_p^s\big), \quad \xi \mapsto (1 \pm \partial_x)_\xi^{-1}$$

is C^1.

Finally, we compute the directional derivative of $h(\xi, v)$—see Misiolek (2002, (3.7)–(3.8)). Recall from (33) that $h(\xi, v) = R_\xi \circ \big((v \circ \xi^{-1})^2 + \frac{1}{2}((v \circ \xi^{-1})')^2\big)$.

Let $t \to \xi_t \in \mathcal{D}_p^s$ be a C^1-path with $\xi_0 = \xi$ and $\frac{d}{dt}|_{t=0}\xi_t = w \in E_p^s$.

Lemma 12.

(i) $\frac{d}{dt}|_{t=0} h(\xi_t, v) = -\frac{(v')^2 w'}{(\xi')^3}$.
(ii) $\frac{d}{dt}|_{t=0} h(\xi, v + tw) = 2vw + \frac{v'w'}{(\xi')^2}$.

Proof. (i) From the definition of h one gets

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} h(\xi_t, v) &= \frac{d}{dt}\Big|_{t=0}\big((v \circ \xi_t^{-1})^2 \circ \xi_t\big) + \frac{d}{dt}\Big|_{t=0}\frac{1}{2}\big(((v \circ \xi_t^{-1})')^2 \circ \xi_t\big)\\ &= \frac{1}{2}(v')^2 \frac{d}{dt}\Big|_{t=0}(\xi_t')^{-2} = -\frac{(v')^2}{(\xi')^3} w'.\end{aligned}$$

(ii) By a straightforward computation

$$\frac{d}{dt}|_{t=0} h(\xi, v + tw) = \{2(v \circ \xi^{-1}) \cdot (w \circ \xi^{-1}) + (v \circ \xi^{-1})' \cdot (w \circ \xi^{-1})'\} \circ \xi. \qquad \square$$

From the formulas in Lemma 12 one immediately obtains, using that $H^{s-1} \cap L^\infty$ is an algebra (cf. Corollary 1), that the following result holds.

Proposition 2. *The map h, defined on $\mathcal{D}_p^s \times E_p^s$ takes values in $H^{s-1} \cap L^\infty$. Viewed as a map $h : \mathcal{D}_p^s \times E_p^s \to H^{s-1} \cap L^\infty$, h is C^1.*

We now apply the above results to study the vector field $F(\xi, v) = (v, f(\xi, v))$ where, when expressed in terms of the map h, f is given by $f = -(\Lambda^{-2}\partial_x)_\xi h(\xi, v)$—see (32). Note that

$$\begin{aligned}\Lambda^{-2}\partial_x &= \Lambda^{-2}(1 + \partial_x) - \Lambda^{-2}\\ &= (1 - \partial_x)^{-1}(1 + \partial_x)^{-1}(1 + \partial_x) - \Lambda^{-2}\\ &= (1 - \partial_x)^{-1} - (1 - \partial_x)^{-1}(1 + \partial_x)^{-1}.\end{aligned} \tag{43}$$

Hence

$$f = -(1 - \partial_x)_\xi^{-1} h + (1 - \partial_x)_\xi^{-1} \circ (1 + \partial_x)_\xi^{-1} h. \tag{44}$$

Proposition 3. *The mapping $F : \mathcal{D}_p^s \times E_p^s$, $(\xi, v) \mapsto (v, f(\xi, v))$ defines a C^1-vector field in a neighborhood of* $(\mathrm{id}, 0) \in \mathcal{D}_p^s \times E_p^s$.

Proof. By (44)

$$f(\xi, v) = -(1 - \partial_x)_\xi^{-1} h(\xi, v) + (1 - \partial_x)_\xi^{-1}(1 + \partial_x)_\xi^{-1} h(\xi, v).$$

By Proposition 2,

$$h : \mathcal{D}_p^s \times E_p^s \to H_p^{s-1} \cap L^\infty$$

is a C^1-map and by Proposition 1

$$\mathcal{D}_p^s \to \mathcal{L}\big(H_p^{s-1} \cap L^\infty, E_p^s\big), \quad \xi \mapsto (1-\partial_x)_\xi^{-1}$$

as well as

$$\mathcal{D}_p^s \to \mathcal{L}\big(H_p^{s-1} \cap L^\infty, E_p^s\big), \quad \xi \mapsto (1-\partial_x)_\xi^{-1}(1+\partial_x)_\xi^{-1}$$

are C^1-maps. By the considerations above we conclude that f and hence F is a C^1-map. □

5. ODE on $\mathcal{D}_p^s \times E_p^s$

Consider the following ODE on $\mathcal{D}_p^s \times E_p^s$

$$\begin{cases} \dot{\xi} = v \\ \dot{v} = f(\xi, v) \end{cases} \tag{45}$$

with initial data

$$\begin{cases} \xi(0) = \mathrm{id} \\ v(0) = v_0 \in E_p^s \end{cases} \tag{46}$$

where $1 \le s < 2$ and $1 < p < \infty$ and

$$f(\xi, v) := -\left\{(1-\partial_x^2)^{-1}\partial_x\left((v\circ\xi^{-1})^2 + \frac{1}{2}((v\circ\xi^{-1})')^2\right)\right\}\circ\xi.$$

By Proposition 3, $F(\xi, v) = (v, f(\xi, v))$ is a C^1-vector field in an open neighborhood of $(\mathrm{id}, 0)$ in $\mathcal{D}_p^s \times E_p^s$. Hence the standard existence and uniqueness theorem for ODEs can be applied (see, e.g., Lang, 1972, IV, § 1).

Theorem 2. *Let* $1 \le s < 2$ *and* $1 < p < \infty$. *Then there exists a neighborhood* $U(0)$ *of zero in* E_p^s *and* $T > 0$ *such that the initial value problem* (45), (46) *has a unique* C^1*-solution*

$$(\xi, v) : (-T, T) \to \mathcal{D}_p^s \times E_p^s, \quad t \mapsto (\xi(t), v(t)).$$

Moreover, the map

$$(-T, T) \times U(0) \to \mathcal{D}_p^s \times E_p^s, \quad (t, v_0) \mapsto (\xi(t, v_0), v(t, v_0))$$

is C^1.

With the notations of Theorem 2, we have the following corollary.

Corollary 5. *The map*

$$U(0) \to C^1((-T, T), \mathcal{D}_p^s \times E_p^s), \quad v_0 \mapsto (\xi(\cdot, v_0), v(\cdot, v_0))$$

is a C^1*-map.*

6. Initial Value Problem for the Camassa–Holm Equation

Recall that the Camassa–Holm equation (1), (2) with periodic boundary conditions ($x \in \mathbb{T}$, $t \in \mathbb{R}$) is given by

$$\begin{cases} u_t + uu_x = -(1-\partial_x^2)^{-1}\partial_x\left(u^2 + \frac{1}{2}u_x^2\right) \\ u|_{t=0} = u_0. \end{cases} \tag{47}$$

Let $1 \leq s < 2$ and $1 < p < \infty$. We will see that a solution $(\xi, v) \in C^1((-T,T), \mathscr{D}_p^s \times E_p^s)$

$$\begin{cases} \dot{\xi} = v \\ \dot{v} = f(\xi, v) \end{cases} \tag{48}$$

with initial data

$$\begin{cases} \xi(0) = \mathrm{id} \\ v(0) = u_0 \in E_p^s, \end{cases} \tag{49}$$

where $0 < T < \infty$ and

$$f(\xi, v) := -\left\{(1-\partial_x^2)^{-1}\partial_x\left((v\circ\xi^{-1})^2 + \frac{1}{2}((v\circ\xi^{-1})_x)^2\right)\right\}\circ\xi$$

gives rise to a solution of (47) in H_p^s in the sense explained below. Set

$$u(x,t) := v(\xi^{-1}(x,t), t).$$

By Lemmas 4 and 5, we have that

$$u(t) \in H_p^s \quad \forall t \in (-T,T).$$

Further, as $t \mapsto \xi(t)$ is continuous in $\mathscr{D}_p^s$, it follows by Lemma 8 that the curve $t \mapsto \xi(t)^{-1}$ is continuous in H_p^s as well and satisfies

$$\sup_{t\in(-T',T')} (\|\xi(t)\|_{\mathrm{Lip}} + \|\xi(t)^{-1}\|_{\mathrm{Lip}}) < \infty$$

for any $0 < T' < T$. According to Lemma 7, we get that

$$t \mapsto v(t)\circ\xi(t)^{-1} \in H_p^s$$

is continuous on $(-T,T)$. Clearly, the curve $t \mapsto v(t)\circ\xi(t)^{-1} \in H_p^s$ satisfies the initial value condition

$$v(0)\circ\xi(0)^{-1} = v(0) = u_0.$$

A direct computation shows that $t \mapsto u(t) := v(t) \circ \xi(t)^{-1} \in H^s_p$ satisfies the Camassa–Holm equation

$$u_t + uu_x = -(1-\partial_x^2)^{-1}\partial_x\left(u^2 + \frac{1}{2}u_x^2\right) \tag{50}$$

pointwise for any $t \in (-T, T)$ and any $x \in \mathbb{T}$. As $t \mapsto u(t)$ is continuous in H^s_p it follows from (50) that

$$t \mapsto u_t(t) \in H^{s-1}_p \cap L^\infty$$

is continuous in $H^{s-1}_p \cap L^\infty$. Hence we have shown that

$$u \in C^0((-T', T'), B_{\rho',r'} \cap H^s_p) \cap C^1((-T', T'), H^{s-1}_p \cap L^\infty),$$

where $\rho' > 0$ and $r' > 0$ are appropriately chosen positive constants (see Definition 1) depending on $0 < T' < T$.

Next, fix ρ and r as in Theorem 1, and consider the set $U(0) := B_{\rho,r} \cap E^s_p$, which is a neighborhood of the origin. Recall that the map

$$(-T, T) \times U(0) \to \mathcal{D}^s_p \times E^s_p, \quad (t, u_0) \to (\xi(t, u_0), v(t, u_0))$$

is continuous. Therefore, there exists a $T'' > 0$ such that

$$v(t, u_0) \in B_{3\rho/4, 3r/4} \cap H^s_p \quad \text{for every } (t, u_0) \in (-T'', T'') \times B_{\rho/2, r/2} \cap H^s_p. \tag{51}$$

Recall that $\xi(0, u_0) = \mathrm{id}$ for every u_0. Hence, for every $\varepsilon > 0$, there exists $0 < T''' < T''$ such that

$$\|\xi(t, u_0) - \mathrm{id}\|_{E^s_p} < \varepsilon \quad \text{for every } (t, u_0) \in (T''', T''') \times B_{\rho/2, r/2} \cap H^s_p. \tag{52}$$

If $\varepsilon > 0$ is sufficiently small, (51) and (52) imply that $u(t, \cdot) = v(t, \xi^{-1})$ belongs to $B_{\rho,r} \cap H^s_p$ for every $(t, u_0) \in (-T''', T''') \times B_{\rho,r} \cap H^s_p$. This proves statement (i) of Theorem 1. The C^0-wellposedness of (47), as stated in Theorem 1(ii), follows from Lemma 8 and the regularity of the map $(t, u_0) \to (v(t, u_0), \xi(t, u_0))$ proved in Theorem 2.

7. Uniqueness

In this section we prove item (iii) of Theorem 1.

Theorem 3. *Let* $u_0 \in \mathrm{Lip}$ *and let* $u^1, u^2 \in L^\infty([0, T], \mathrm{Lip})$ *be two distributional solutions of*

$$\begin{cases} u_t + uu_x = -\partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 - \frac{1}{2}u_x^2\right) \\ u(0, \cdot) = u_0. \end{cases} \tag{53}$$

Then $u^1 = u^2$ *almost everywhere on* $[0, T] \times \mathbb{T}$.

Remark 8. For $u \in L^\infty([0,T], \mathrm{Lip})$ we denote by g the unique solution of

$$g - g_{xx} = u^2 - \frac{1}{2}u_x^2. \tag{54}$$

Note that, since $u, u_x \in L^\infty([0,T], L^\infty)$, then $g \in L^\infty([0,T], W^{2,\infty})$, where $W^{2,\infty}$ denotes the space

$$\{u \in L^\infty : u_x, u_{xx} \in L^\infty\}.$$

Using this notation, u is a distributional solution of (53) if the following identity holds for all $\varphi \in C_c^\infty(]-T, T[\times \mathbb{T})$:

$$\begin{aligned}\int_{[0,T]\times\mathbb{T}} u(t,x)\varphi_t(t,x)\,dt\,dx + \int_{\mathbb{T}} u_0(x)\varphi(0,x)\,dx \\ = -\frac{1}{2}\int_{[0,T]\times\mathbb{T}} u^2(t,x)\varphi_x(t,x)\,dt\,dx - \int_{[0,T]\times\mathbb{T}} g(t,x)\varphi_x(t,x)\,dt\,dx.\end{aligned} \tag{55}$$

Theorem 3 is an easy corollary of the following proposition.

Proposition 4. *Let* $u \in L^\infty([0,T], \mathrm{Lip})$ *be a distributional solution of* (53). *Then there exist*:

(i) $v \in C^1([0,T], \mathrm{Lip})$;
(ii) *and* $\xi \in C^1([0,T], \mathcal{D}^1_\infty)$,

such that

$$u(t,x) = v(t, \xi(t)^{-1}(x)) \quad \textit{for a.e. } (t,x) \in [0,T] \times \mathbb{T} \tag{56}$$

and the maps $\xi : [0,T] \to \mathcal{D}^1_\infty$ *and* $v : [0,T] \to \mathrm{Lip}$ *solve the ODE*

$$\begin{cases} \dot{\xi}(t) = v(t) & \dot{v}(t) = F(\xi(t), v(t)) \\ \xi(0) = \mathrm{id} & v(0) = u_0. \end{cases} \tag{57}$$

Here F is the (*Lipschitz*) *vector field*

$$F : \mathcal{D}^1_\infty \times \mathrm{Lip} \ni (\zeta, w) \to F(\zeta, w) \in \mathrm{Lip}$$

defined in Section 4.

Remark 9. The main point of Proposition 4 is that the pair (ξ, v) is more regular with respect to the variable t than u (see the examples of Section 2). Moreover, note that the representation $u(t,\cdot) = v(t, \xi(t,\cdot))$ holds for the entire life span $[0,T]$ of the solution u.

Proof of Theorem 3. We apply Proposition 4 above. Theorem 3 follows from the uniqueness of C^1 solutions of the ODE (57). □

Before coming to the proof of Proposition 4, we state an elementary lemma which we will use several times. For the reader's convenience we include a proof of it.

Lemma 13. *Let* $z \in L^\infty([0,T] \times \mathbb{T})$ *be such that* z_t, z_x, z_{tx} *are in* $L^\infty([0,T] \times \mathbb{T})$. *Then* $z \in \mathrm{Lip}([0,T], \mathrm{Lip})$.

Proof. It suffices to show that

$$\|z(t,\cdot) - z(s,\cdot)\|_{\mathrm{Lip}} \le |t-s| \|z_{tx}\|_\infty \quad \text{for any } t, s \in]0,T[. \tag{58}$$

Indeed this gives $z \in \mathrm{Lip}(]0,T[, \mathrm{Lip})$. To get $z \in \mathrm{Lip}([0,T], \mathrm{Lip})$ it then suffices to show for any $0 < t < T$ the bounds

$$\|z(0,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} \le t \|z_{tx}\|_\infty$$
$$\|z(T,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} \le (T-t) \|z_{tx}\|_\infty.$$

These bounds follow from (58). Indeed fix $t \in]0,T[$ and consider 2 sequences $\sigma_n \downarrow 0$ and $\tau_n \uparrow T$. Since z is continuous, we have

$$\begin{aligned}
\|z(0,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} &= \sup_{y \neq x} \frac{|z(0,x) - z(0,y) - z(t,x) + z(t,y)|}{|y-x|} \\
&\le \liminf_{n\uparrow\infty} \sup_{y\neq x} \frac{|z(\sigma_n,x) - z(\sigma_n,y) - z(t,x) + z(t,y)|}{|y-x|} \\
&= \liminf_{n\uparrow\infty} \|z(\sigma_n,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} \\
&\le \lim_{n\uparrow\infty} |t-\sigma_n| \|z_{tx}\|_\infty = t\|z_{tx}\|_\infty.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\|z(T,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} &\le \liminf_{n\uparrow\infty} \|z(\tau_n,\cdot) - z(t,\cdot)\|_{\mathrm{Lip}} \le \lim_{n\uparrow\infty} |t-\tau_n| \|z_{tx}\|_\infty \\
&= (T-t)\|z_{tx}\|_\infty.
\end{aligned}$$

It remains to prove (58). Fix $[t,s] \subset]0,T[$ and take a standard family of smooth mollifiers $\{\rho_\varepsilon\}_\varepsilon$ in $\mathbb{R} \times \mathbb{T}$. Choose $\varepsilon > 0$ so small that $0 < t - \varepsilon < s + \varepsilon < T$ and set $z^\varepsilon := z * \rho_\varepsilon$. Since $z^\varepsilon \in C^\infty$ we can write

$$\begin{aligned}
|z^\varepsilon(t,x) - z^\varepsilon(s,x) - z^\varepsilon(t,y) + z^\varepsilon(s,y)| &= \left| \int_x^y \left(z_x^\varepsilon(t,\zeta) - z_x^\varepsilon(s,\zeta) \right) d\zeta \right| \\
&= \left| \int_x^y \int_s^t z_{tx}^\varepsilon(\sigma,\zeta) d\sigma \, d\zeta \right| \\
&\le |t-s| |x-y| \|z_{tx}^\varepsilon\|_\infty.
\end{aligned}$$

Therefore, since $\int_{\mathbb{R}\times\mathbb{T}} \rho(t, x)dt\,dx = 1$ and hence $\|z^{\varepsilon}_{tx}\|_{L^\infty([t,s]\times\mathbb{T})} \leq \|z_{tx}\|_{L^\infty([0,T]\times\mathbb{T})}$, it follows that

$$\sup_{x\neq y} \frac{\left|z^{\varepsilon}(t, x) - z^{\varepsilon}(s, x) - z^{\varepsilon}(t, y) + z^{\varepsilon}(s, y)\right|}{|x - y|} \leq |t - s|\|z_{tx}\|_{\infty}.$$

Note that

$$\begin{aligned}\|z(t, \cdot) - z(s, \cdot)\|_{\text{Lip}} &= \sup_{x\neq y} \frac{|z(t, x) - z(s, x) - z(t, y) + z(s, y)|}{|x - y|} \\ &\leq \liminf_{\varepsilon\downarrow 0} \sup_{x\neq y} \frac{|z^{\varepsilon}(t, x) - z^{\varepsilon}(s, x) - z^{\varepsilon}(t, y) + z^{\varepsilon}(s, y)|}{|x - y|} \\ &\leq |t - s|\|z_{tx}\|_{\infty}\end{aligned}$$

This concludes the proof of Lemma 13. □

Remark 10. Passing to a lift of z, Lemma 13 can be used to show that, if $z \in \text{Lip}([0, T] \times \mathbb{T}, \mathbb{T})$ and $z_{tx} \in L^\infty([0, T] \times \mathbb{T})$, then

$$z \in \text{Lip}([0, T], \text{Lip}(\mathbb{T}, \mathbb{T})).$$

Remark 11. For $\Omega = [0, T] \times \mathbb{T}$ and $\Omega = \mathbb{T}$ we define the space

$$W^{1,\infty}(\Omega) = \{u \in L^\infty(\Omega) : Du \in L^\infty\},$$

Then $\text{Lip}(\Omega) \subseteq W^{1,\infty}(\Omega)$ and $W^{1,\infty}(\Omega) \subseteq \text{Lip}(\Omega)$. The latter inclusion is understood in the following way:

> For every $u \in W^{1,\infty}(\Omega)$ there exists $v \in \text{Lip}(\Omega)$ such that $u = v$ almost everywhere.

Proof of Proposition 4. The strategy of the proof is the following:

(i) First we prove the existence of a $\tilde{u} \in \text{Lip}([0, T] \times \mathbb{T})$ such that

$$\begin{aligned}\tilde{u}(t, x) &= u(t, x) \quad \text{for a.e. } (t, x) \\ \tilde{u}(0, x) &= u_0(x) \quad \text{for all } x.\end{aligned}$$

(ii) For every $x \in \mathbb{T}$ we let $\eta^{(x)} \in C^1([0, T], \mathbb{T})$ be the unique solution of

$$\begin{cases} \dfrac{d}{dt}\eta^{(x)}(t) = \tilde{u}(t, \eta^{(x)}(t)) \\ \eta^{(x)}(0) = x. \end{cases} \tag{59}$$

Since $\eta^{(x+1)}(0) = \eta^{(x)}(0) + 1$ and $\tilde{u}(t, x + 1) = \tilde{u}(t, x)$, by the uniqueness of solutions to 59 we have $\eta^{(x+1)}(t) = \eta^{(x)}(t) + 1$. Therefore we can define $\xi : [0, T] \times \mathbb{T} \to \mathbb{T}$ as

$\xi(t, x) := \eta^{(x)}(t)$. Then we prove the following properties of ξ:

(a) $\xi \in \mathrm{Lip}([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$;
(b) $\xi(t, \cdot)$ is a homeomorphism for every t and we denote by $\zeta(t, \cdot)$ its inverse;
(c) $\zeta \in \mathrm{Lip}([0, T] \times \mathbb{T}, \mathbb{T})$.

We remark that actually one can prove that

$$\zeta \in \mathrm{Lip}([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$$

but as we do not use this property we only prove the weaker one (c).

(iii) We define $v(t, x) := \tilde{u}(t, \xi(t, x))$. It will be useful to introduce the map Ψ: $[0, T] \times \mathbb{T} \to [0, T] \times \mathbb{T}$ given by

$$\Psi(t, x) := (t, \xi(t, x)).$$

Then we prove that, with g given as in Remark 8, we have

$$v_t = -g_x \circ \Psi. \tag{60}$$

The right-hand side of (60) is well defined because Ψ is Lipschitz with Lipschitz inverse. In particular, $\Psi^{-1}(E)$ is a null set for every null set E; cf. Appendix A.

(iv) We use equation (60) and Lemma 13 to show that

$$v \in \mathrm{Lip}([0, T], \mathrm{Lip}).$$

(v) We use (59), (60), and the regularity of (ξ, v) proved in (ii) and (iv) to show that:

(a) $\xi \in C^1([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$;
(b) $v \in C^1([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$;
(c) (ξ, v) solves 57.

We will prove these five steps in the five lemmas below. □

Lemma 14. *Let u and u_0 be as in Proposition 4. Then there exists $\tilde{u} \in \mathrm{Lip}([0, T] \times \mathbb{T})$ such that*

$$\begin{aligned} \tilde{u}(t, x) &= u(t, x) \quad \text{for a.e. } (t, x) \\ \tilde{u}(0, x) &= u_0(x) \quad \text{for all } x. \end{aligned} \tag{61}$$

Proof. Define g as in Remark 8. As u is a distributional solution of (53), we have the following identity in the sense of distributions

$$u_t = -uu_x - g_x.$$

As observed in Remark 8, we have $g \in L^\infty([0, T], W^{2,\infty})$, and hence $g_x \in L^\infty$. Therefore $u_t \in L^\infty$ and we conclude $u \in W^{1,\infty}([0, T] \times \mathbb{T})$. According to Remark 11, there exists $\tilde{u} \in \mathrm{Lip}([0, T] \times \mathbb{T})$ such that $u = \tilde{u}$ almost everywhere.

It remains to prove

$$\tilde{u}(0, x) = u_0(x) \quad \text{for every } x \in \mathbb{T}. \tag{62}$$

We fix $\psi \in C^\infty(\mathbb{T})$ and $\zeta \in C^\infty([-1, 1])$ such that $\zeta(0) = 1$ and $\zeta(t) = 0$ for $t \in [1/2, 1] \cup [-1, -1/2]$. We set for $0 \le t \le \varepsilon$

$$\zeta^\varepsilon(t) := \zeta\left(\frac{t}{\varepsilon}\right) \quad \varphi^\varepsilon(t, x) := \zeta^\varepsilon(t)\psi(x).$$

Note that for $\varepsilon > 0$ sufficiently small, φ^ε is an admissible test function in (55). Since $\tilde{u} = u$ almost everywhere, we can substitute u with $\tilde{u}$ in (55) to conclude

$$\begin{aligned} &\int_{[0,\varepsilon]\times\mathbb{T}} \tilde{u}(t, x)\zeta_t^\varepsilon(t)\psi(x)dt\,dx + \int_{\mathbb{T}} u_0(x)\psi(x)dx \\ &\quad = -\int_{[0,\varepsilon]\times\mathbb{T}} \left(\frac{\tilde{u}^2(t, x)}{2} + g(t, x)\right)\zeta^\varepsilon(t)\psi_x(x)dt\,dx\,. \end{aligned} \tag{63}$$

Since ψ_x, g, and ζ^ε are all uniformly bounded, the right-hand side of (63) converges to 0 for $\varepsilon \downarrow 0$. Moreover,

$$\begin{aligned} \int_{[0,\varepsilon]\times\mathbb{T}} \tilde{u}(t, x)\zeta_t^\varepsilon(t)\psi(x)dt\,dx &= \int_{[0,\varepsilon]\times\mathbb{T}} \tilde{u}(t, x)\frac{1}{\varepsilon}\zeta_t\left(\frac{t}{\varepsilon}\right)\psi(x)dt\,dx \\ &= \int_0^1 \int_{\mathbb{T}} \tilde{u}(\varepsilon\tau, x)\zeta_t(\tau)\psi(x)d\tau\,dx. \end{aligned} \tag{64}$$

As $\varepsilon \downarrow 0$, the functions $\tilde{u}(\varepsilon\tau, x)$ converge uniformly to $\tilde{u}(0, x)$. Therefore, passing to the limit $\varepsilon \downarrow 0$ in (64) and in (63), we get

$$\int_{\mathbb{T}} u_0(x)\psi(x)dx = -\int_0^1 \zeta_t(\tau)d\tau \int_{\mathbb{T}} \tilde{u}(0, x)\psi(x)dx = \int_{\mathbb{T}} \tilde{u}(0, x)\psi(x)dx. \tag{65}$$

Since (65) holds for every $\psi \in C^\infty$ and u_0 and $\tilde{u}(0, \cdot)$ are both continuous, we conclude that (62) holds. □

Lemma 15. *For every $x \in \mathbb{T}$ we let $\eta^{(x)} \in C^1([0, T], \mathbb{T})$ be the unique solution of* (59) *and set $\xi(t, x) := \eta^{(x)}(t)$. Then*:

(a) $\xi \in \mathrm{Lip}([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$;
(b) *$\xi(t, \cdot)$ is an homeomorphism for every t and we denote by $\zeta(t, \cdot)$ its inverse*;
(c) $\zeta \in \mathrm{Lip}([0, T] \times \mathbb{T}, \mathbb{T})$.

Proof. First of all we claim that $\xi \in \mathrm{Lip}([0, T] \times \mathbb{T}, \mathbb{T})$. Note that $\eta^{(x)} \in C^1([0, T], \mathbb{T})$ for every $x \in \mathbb{T}$, with $\|\eta^{(x)}\|_{C^1([0,T])}$ uniformly bounded. □

Therefore, it suffices to check that $\|\xi(t, \cdot)\|_{\mathrm{Lip}}$ is uniformly bounded for $t \in [0, T]$. In order to get this, for every x_1, x_2, we set

$$r(t) := \xi(t, x_1) - \xi(t, x_2) \pmod 1.$$

From (59), we get

$$\left|\frac{dr}{dt}\right| \leq \|u\|_{\mathrm{Lip}}|r(t)|.$$

From Gronwall's Lemma we conclude $|r(t)| \leq |r(0)|e^{Ct}$ with $C = \|u\|_{\mathrm{Lip}}$. As $r(0) = \xi(0, x_1) - \xi(0, x_2) = x_1 - x_2 \pmod 1$, it follows that

$$|\xi(t, x_1) - \xi(t, x_2)| \leq e^{Ct}|x_1 - x_2|, \tag{66}$$

which is the desired conclusion.

Proof of (a). Recall that the map $\Psi : [0, T] \times \mathbb{T} \to [0, T] \times \mathbb{T}$ is defined as $\Psi(t, x) = (t, \xi(t, x))$. We rewrite (59) as

$$\xi_t = \tilde{u} \circ \Psi.$$

By Lemma 14, $\tilde{u}$ is Lipschitz and by the considerations above so is Ψ. Hence ξ_t is Lipschitz, that is $\xi_t \in \mathrm{Lip}([0, T] \times \mathbb{T}, \mathbb{T})$. Therefore $\xi_{tx} \in L^\infty([0, T] \times \mathbb{T})$ and from Remark 10 we get $\xi \in \mathrm{Lip}([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$.

Proof of (b). The same argument which leads to (66) yields the estimate

$$|x_1 - x_2| = |\xi(0, x_1) - \xi(0, x_2)| \leq e^{Ct}|\xi(t, x_1) - \xi(t, x_2)|. \tag{67}$$

It follows that $\xi(t, \cdot)$ is injective. Hence $\xi(t, \cdot) : \mathbb{T} \to \mathbb{T}$ is onto. It then follows from (67) that the inverse of $\xi(t, \cdot)$ is Lipschitz continuous.

Proof of (c). Since $\xi(t, \cdot)$ is a Lipschitz homeomorphism and is homotopic to the identity map, we must have $\xi_x(t, \cdot) \geq 0$. Therefore, for any $t \in [0, T]$ (67) gives

$$\operatorname{essinf} \xi_x(t, \cdot) \geq e^{-CT} \quad \text{a.e.} \tag{68}$$

Fix a standard family of (non-negative) mollifiers $\{\rho_\varepsilon\}$ on $\mathbb{T}$ and define $\xi^\varepsilon(t, \cdot) = \xi(t, \cdot) * \rho_\varepsilon$. Note that $\xi^\varepsilon \in C^1([0, T] \times \mathbb{T}, \mathbb{T})$. Indeed:

- Since $\xi \in \mathrm{Lip}([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$, we have $\xi^\varepsilon \in \mathrm{Lip}([0, T], C^1(\mathbb{T}, \mathbb{T}))$. This implies that ξ^ε_x exists at every point and that it is continuous.
- By (59), $\xi(\cdot, x) \in C^1([0, T], \mathbb{T})$ for every $x \in \mathbb{T}$ and ξ_t is continuous in both time and space variables. This implies that ξ^ε_t exists at every point and is continuous.

As $\xi_x \geq 0$, $\rho_\varepsilon \geq 0$, and $\int_{\mathbb{T}} \rho_\varepsilon(x)dx = 1$, we get from (68)

$$\xi^\varepsilon_x(t, x) \geq e^{-CT} \quad \text{for every } (t, x) \in [0, T] \times \mathbb{T}. \tag{69}$$

Hence there exists $\zeta^\varepsilon \in C^1([0, T] \times \mathbb{T}, \mathbb{T})$ such that

$$\xi^\varepsilon\big(t, \zeta^\varepsilon(t, x)\big) = x. \tag{70}$$

From (69) we have

$$\|\zeta^\varepsilon_x\|_{L^\infty([0,T]\times\mathbb{T})} \le e^{CT}.$$

Differentiating (70) in t, we get

$$\xi^\varepsilon_t(t,\zeta^\varepsilon(t,x)) = -\xi^\varepsilon_x(t,\zeta^\varepsilon(t,x))\zeta^\varepsilon_t(t,x).$$

Therefore, we conclude that

$$\|\zeta^\varepsilon_t\|_{L^\infty\left([0,T]\times\mathbb{T}\right)} \le e^{CT}\|\xi^\varepsilon_t\|_{L^\infty([0,T]\times\mathbb{T})}.$$

Hence $\|\zeta^\varepsilon\|_{\mathrm{Lip}([0,T]\times\mathbb{T},\mathbb{T})}$ is uniformly bounded for $0<\varepsilon\le 1$. By the Ascoli–Arzelà Theorem we can extract a subsequence $\{\zeta^{\varepsilon_i}\}_i$ which converges uniformly to a map $\psi\in\mathrm{Lip}([0,T]\times\mathbb{T},\mathbb{T})$. For this map we have $\xi(t,\psi(t,x))=x$ for every $(t,x)\in[0,T]\times\mathbb{T}$. We conclude that $\psi(t,x)=\zeta(t,x)$ and this proves (c). □

Lemma 16. *Define* $v(t,x):=\tilde{u}(t,\xi(t,x))$ *and* $\Psi(t,x):=(t,\xi(t,x))$. *Then*

$$v_t(t,x) = -g_x(\Psi(t,x)) \quad \textit{for a.e. } (t,x), \tag{71}$$

where g *is the function given by* (54). *Note that identity* (71) *makes sense, since* $g_x\in L^\infty$ *and* Ψ *is Lipschitz with Lipschitz inverse* (*and therefore* $\Psi^{-1}(E)$ *is a null set for every null set* $E\subset[0,T]\times\mathbb{T}$; *see Appendix* A).

Proof. By the Rademacher Theorem (Evans and Gariepy, 1992), we have that $\tilde{u}$ and v are differentiable a.e. and their differential coincides with their distributional derivative. Let G be the set of points $(t,x)\in[0,T]\times\mathbb{T}$ where $\tilde{u}$ is differentiable. As Ψ is Lipschitz with Lipschitz inverse, $\Psi^{-1}(G)$ has full measure. Again by the Rademacher Theorem we conclude that the set

$$H := \{(t,x)\in G : \Psi \text{ and } v \text{ are differentiable at } (t,x)\}$$

has full measure.

For every $(t,x)\in H$ we can apply the chain rule for differentiable functions

$$v_t(t,x) = \tilde{u}_t(\Psi(t,x)) + \tilde{u}_x(\Psi(t,x))\xi_t(t,x).$$

In view of the definition of ξ in Lemma 15, $\xi_t(t,x)=\tilde{u}(t,\xi(t,x))$, hence

$$v_t(t,x) = \tilde{u}_t(\Psi(t,x)) + \tilde{u}_x(\Psi(t,x))\tilde{u}(\Psi(t,x)). \tag{72}$$

Therefore, we get that (72) holds for a.e. (t,x).

Since $\tilde{u}$ is a distributional solution of (53), we have

$$\tilde{u}_t + \tilde{u}\tilde{u}_x = -g_x. \tag{73}$$

Note that both sides of this equation are given by L^∞ functions. For $(t, x) \in H$ we then conclude

$$\tilde{u}_t(t, x) + \tilde{u}(t, x)\tilde{u}_x(t, x) = -g_x(t, x), \tag{74}$$

i.e., (74) holds a.e. Again, the fact that Ψ is Lipschitz with Lipschitz inverse implies that

$$\tilde{u}_t(\Psi(t, x)) + \tilde{u}(\Psi(t, x))\tilde{u}_x(\Psi(t, x)) = -g_x(\Psi(t, x)), \tag{75}$$

for a.e. (t, x). Since both (72) and (75) hold for a.e. (t, x), we get (71). □

Lemma 17. *Let v be as in Lemma* 16. *Then* $v \in \mathrm{Lip}([0, T], \mathrm{Lip})$.

Proof. By (71),

$$v_t = -g_x(\Psi) \tag{76}$$

holds in the sense of distributions. Set $k := g_x$ and recall that $k_x \in L^\infty$, since $g \in L^\infty\big([0, T], W^{2,\infty}\big)$ by Remark 8.

Let ρ_ε be a standard family of smooth mollifiers in the x variable and set $k^\varepsilon := k * \rho_\varepsilon$. Then $k^\varepsilon \to k$ pointwise almost everywhere. Since the map Ψ is Lipschitz with Lipschitz inverse, we conclude that

$$k^\varepsilon(\Psi(t, x)) \to k(\Psi(t, x)) \quad \text{for a.e. } (t, x).$$

Note that, since k^ε is smooth in the x variable, we can apply the chain rule to get

$$\partial_x\big(k^\varepsilon \circ \Psi\big) = \xi_x k_x^\varepsilon \circ \Psi.$$

Therefore

$$\|\partial_x(k^\varepsilon \circ \Psi)\|_\infty \leq \|k_x\|_\infty \|\xi_x\|_\infty.$$

Letting $\varepsilon \downarrow 0$ we conclude that

$$\partial_x(k \circ \Psi) \in L^\infty.$$

Therefore, differentiating (76) we conclude that $v_{tx} \in L^\infty$. By Lemma 13, this implies $v \in \mathrm{Lip}([0, T], \mathrm{Lip})$. □

Lemma 18. *Let ξ be as in Lemma* 15 *and v as in Lemma* 16. *Then*:

(i) $\xi \in C^1([0, T], \mathrm{Lip}(\mathbb{T}, \mathbb{T}))$;
(ii) $v \in C^1([0, T], \mathrm{Lip})$;
(iii) (ξ, v) *solves* (57).

Proof. First of all note that ξ is defined in such a way that $\xi(0, x) = x$ for every $x \in \mathbb{T}$. By the definition of v (Lemma 16) and by Lemma 14 it follows that $v(0, x) = u_0(x)$ for every $x \in \mathbb{T}$.

Let g be defined as in Remark 8 and Ψ be as in Lemma 16. Note that

$$-g_x \circ \Psi = [F(\xi(t, \cdot), v(t, \cdot))](x), \tag{77}$$

where F is defined as in Section 4. From now on, in order to simplify the notation, we use $v(t)$ and $\xi(t)$ to denote the maps $v(t, \cdot) \in \mathrm{Lip}$ and $\xi(t, \cdot) \in \mathrm{Lip}(\mathbb{T}, \mathbb{T})$.

By Lemmas 15 and 17, both curves, $t \to v(t)$ and $t \to \xi(t)$, are Lipschitz, respectively in the Banach space Lip and $\mathrm{Lip}(\mathbb{T}, \mathbb{T})$. Since the vector field F: $\mathrm{Lip}(\mathbb{T}, \mathbb{T}) \times \mathrm{Lip} \to \mathrm{Lip}$ is Lipschitz (see Section 4), it follows that the curve

$$t \to F\big(\xi(t), v(t)\big)$$

is a Lipschitz curve on Lip.

Therefore to conclude the proof we need to show that the curves $t \to \xi(t)$ and $t \to v(t)$ are differentiable at any $t_0 \in [0, T]$ and that their derivatives coincide with $v(t_0)$ and $F(\xi(t_0), v(t_0))$.

Differentiability of $t \to \xi(t)$. Fix $t_0 \in [0, T[$ and set $\varepsilon := T - t_0$. Consider the function

$$r(\tau, x) := \frac{\xi(t_0 + \tau, x) - \xi(t_0, x)}{\tau} \quad \text{for } \tau \in]0, \varepsilon] \text{ and } x \in \mathbb{T}$$

and the corresponding continuous curve $]0, \varepsilon] \ni \tau \to r(\tau)$ in Lip. Since $\xi_t = v$ in the sense of distributions, the following equality holds for almost every $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$

$$r(\tau, x) = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} v(t, x) dt. \tag{78}$$

Since v is Lipschitz, the right-hand side of (78) is well defined for every $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$ and yields a continuous function. Since r is continuous on $]0, \varepsilon] \times \mathbb{T}$, we conclude that (78) holds for every $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$. With $v(t_0) = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} v(t_0) d\tau$ we then get

$$\|r(\tau) - v(t_0)\|_{\mathrm{Lip}} \leq \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \|v(t) - v(t_0)\|_{\mathrm{Lip}} d\tau. \tag{79}$$

The continuity of the curve $t \to v(t)$ in Lip gives that

$$\lim_{\tau \downarrow 0} \|r(\tau) - v(t_0)\|_{\mathrm{Lip}} = 0. \tag{80}$$

This means that $t \to \xi(t)$ is differentiable from the right at every $t_0 \in [0, T[$ and its right derivative is $v(t_0)$. With a similar argument we conclude that it is differentiable from the left at every $t_0 \in]0, T]$ and its left derivative is $v(t_0)$.

Differentiability of $t \to v(t)$. Fix again $t_0 \in [0, T[$ and set $\varepsilon := T - t_0$. Consider

$$r(\tau, x) := \frac{v(t_0 + \tau, x) - v(t_0, x)}{\tau}$$

and the corresponding continuous curve $]0, \varepsilon] \ni \tau \to r(\tau)$ in Lip.

From Lemma 16, we get the following identity for a.e. $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$:

$$r(\tau, x) = -\frac{1}{\tau}\int_{t_0}^{t_0+\tau} g_x(\Psi(t, x))dt. \tag{81}$$

From (77), we get

$$r(\tau, x) = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} [F(\xi(t), v(t))](x)dt \tag{82}$$

for a.e. $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$.

Since both the curves $\tau \to r(\tau)$ and $t \to F(\xi(t), v(t))$ are continuous curves in Lip, we conclude that (82) holds for every $(\tau, x) \in]0, \varepsilon] \times \mathbb{T}$. Hence

$$\|r(\tau) - F(\xi(t_0), v(t_0))\|_{\mathrm{Lip}} \leq \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \|F(\xi(t), v(t)) - F(\xi(t_0), v(t_0))\|_{\mathrm{Lip}}dt. \tag{83}$$

Since the map $t \to F(\xi(t), v(t))$ is continuous in Lip, we conclude that

$$\lim_{\tau\downarrow 0} \|r(\tau) - F(\xi(t_0), v(t_0))\|_{\mathrm{Lip}} = 0. \tag{84}$$

This means that $t \to v(t)$ is differentiable from the right at every $t_0 \in [0, T[$ and its right differential is $F(\xi(t_0), v(t_0))$. With a similar argument we conclude that it is differentiable from the left at every $t_0 \in]0, T]$ and its left differential is $F(\xi(t_0), v(t_0))$. □

Appendix A. Regularity of Composition

Proposition 5. *For any* $1 \leq p < \infty$ *and for any* $s \geq 1$, *the composition*

$$L^p \times \mathscr{D}^s_p \to L^p, \quad (u, \xi) \mapsto u \circ \xi$$

is continuous.

Remark 12. Indeed we prove that for any $(u_k)_{k\geq 1} \subseteq L^p$ with $u_k \to u$ in L^p and $(\xi_k)_{k\geq 1} \subseteq \mathscr{D}^s_p$, $\xi_k \to \xi \in \mathscr{D}^s_p$ in H^s_p with

$$\sup_{k\geq 1}\left(\|\xi_k\|_{W^{1,\infty}} + \|\xi_k^{-1}\|_{W^{1,\infty}}\right) < \infty$$

one has that $u_k \circ \xi_k \to u \circ \xi$ in L^p.

Proof of Remark 12. Let $(u_k)_{k\geq 1}$ be a sequence in L^p with $u_k \to u$ in L^p and $(\xi_k)_{k\geq 1}$ be a sequence in $\mathscr{D}^s_p$ with $\xi_k \to \xi \in \mathscr{D}^s_p$ in H^s_p. Then

$$u_k \circ \xi_k - u \circ \xi = (u_k \circ \xi_k - u \circ \xi_k) + (u \circ \xi_k - u \circ \xi)$$

and we estimate the 2 terms in the latter sum separately: We have

$$\|u_k \circ \xi_k - u \circ \xi_k\|^p_{L^p} = \int_{\mathbb{T}} |u_k(\xi_k(x)) - u(\xi_k(x))|^p\, dx$$
$$= \int_{\mathbb{T}} |u_k(y) - u(y)|^p (\xi_k^{-1})'\, dy \leq K\|u_k - u\|^p_{L^p}$$

where $K := \sup_{k\geq 1} \|\xi_k^{-1}\|_{W^{1,\infty}} < \infty$. For $u \in L^p$ arbitrary, approximate u by a sequence of step functions $(v_k)_{k\geq 1} \subseteq L^\infty$, $u = \lim_{k\to\infty} v_k$ in L^p (see the arguments below). Then

$$\|u \circ \xi_k - u \circ \xi\|_{L^p} \leq \|u \circ \xi_k - v_n \circ \xi_k\|_{L^p} + \|v_n \circ \xi_k - v_n \circ \xi\|_{L^p} + \|v_n \circ \xi - u \circ \xi\|_{L^p}.$$

By the considerations above,

$$\|u \circ \xi_k - v_n \circ \xi_k\|_{L^p} + \|v_n \circ \xi - u \circ \xi\|_{L^p} \leq 2K^{1/p}\|u - v_n\|_{L^p}$$

with the same constant $K = \sup_{k\geq 1} \|\xi_k^{-1}\|_{W^{1,\infty}} < \infty$. Hence it remains to estimate the norm $\|w \circ \xi_k - w \circ \xi\|_{L^p}$ where $w \in L^\infty$. For $n \geq 1$, let

$$w_n := \sum_{j=1}^{2^n} \alpha_{n,j} 1_{I_{n,j}}$$

where $I_{n,j} := \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right]$ $(1 \leq j \leq 2^n)$ is a decomposition of $\mathbb{T} \cong (0, 1]$ and $\alpha_{n,j} := \frac{1}{2^n}\int_{I_{n,j}} w(x)dx$. Note that $(I_{n,j})_{n\geq 1, 1\leq j\leq 2^n}$ is a net in the sense of Shilov and Gurevich (1977, p. 208). According to De Possel's theorem (Shilov and Gurevich, 1977, p. 215, Theorem 2) this net is a Vitali system, hence, by Shilov and Gurevich (1977, p. 220, Theorem 1),

$$w = \lim_{n\to\infty} w_n \quad \text{a.e on } \mathbb{T}.$$

As $w \in L^\infty$, $\|w_n\|_{L^\infty} \leq \|w\|_{L^\infty}$ $\forall n \geq 1$, and by Lebesgue's convergence theorem $w_n \to w$ in L^p. Hence we have

$$\|w \circ \xi_k - w \circ \xi\|_{L^p} \leq \|w \circ \xi_k - w_n \circ \xi_k\|_{L^p} + \|w_n \circ \xi_k - w_n \circ \xi\|_{L^p} + \|w_n \circ \xi - w \circ \xi\|_{L^p}.$$

As above

$$\|w \circ \xi_k - w_n \circ \xi_k\|_{L^p} + \|w_n \circ \xi - w \circ \xi\|_{L^p} \leq 2K^{1/p}\|w_n - w\|_{L^p}$$

with $K = \sup_{k\geq 1} \|\xi_k^{-1}\|_{W^{1,\infty}} < \infty$. Further, as for any $n \geq 1$ and $1 \leq j \leq 2^n$,

$$|\alpha_{n,j}| \leq \|w\|_{L^\infty},$$

we get

$$\|w_n \circ \xi_k - w_n \circ \xi\|_{L^p} \leq \|w\|_{L^\infty} \sum_{j=1}^{2^n} \|1_{I_{n,j}} \circ \xi_k - 1_{I_{n,j}} \circ \xi\|_{L^p}.$$

For $0 < \epsilon < 1/2$, let $J_{n,j}^\epsilon := \left(\frac{j}{2^n} - \frac{\epsilon}{2^{np}}, \frac{j}{2^n} + \frac{\epsilon}{2^{np}}\right)$. As $\xi_k \to \xi$ in H_p^s and $s \geq 1$, there exists $k_\epsilon \geq 1$ so that $\|\xi - \xi_k\|_{L^\infty} < \frac{\epsilon}{2^{np}}$ $\forall k \geq k_\epsilon$. Then, for any $k \geq k_\epsilon$,

$$1_{I_{n,j}}(\xi_k(x)) - 1_{I_{n,j}}(\xi(x)) = 0 \quad \text{if } \xi(x) \notin J_{n,j-1}^\epsilon \cup J_{n,j}^\epsilon$$

and

$$0 \leq |1_{I_{n,j}}(\xi_k(x)) - 1_{I_{n,j}}(\xi(x))| \leq 1 \quad \text{if } \xi(x) \in J^{\epsilon}_{n,j-1} \cup J^{\epsilon}_{n,j}.$$

Hence for any $k \geq k_\epsilon$

$$\int_0^1 |1_{I_{n,j}}(\xi_k(x)) - 1_{I_{n,j}}(\xi(x))|^p \, dx \leq \int_{\{x\in\mathbb{T}|\xi(x)\in J^{\epsilon}_{n,j-1}\cup J^{\epsilon}_{n,j}\}} dx$$
$$= \int_{J^{\epsilon}_{n,j-1}\cup J^{\epsilon}_{n,j}} (\xi^{-1})'(y)dy \leq \|\xi^{-1}\|_{W^{1,\infty}} \frac{4\epsilon}{2^{np}}$$

and therefore

$$\sum_{j=1}^{2^n} \|1_{I_{n,j}} \circ \xi_k - 1_{I_{n,j}} \circ \xi\|_{L^p} \leq \|\xi^{-1}\|^{1/p}_{W^{1,\infty}} \sum_{j=1}^{2^n} \frac{(4\epsilon)^{1/p}}{2^n} = \|\xi^{-1}\|^{1/p}_{W^{1,\infty}}(4\epsilon)^{1/p}. \qquad \square$$

Remark 13. It can be checked that the map

$$L^\infty \times \mathcal{D}^s_p \to L^\infty, \quad (u, \xi) \to u \circ \xi$$

is not continuous.

The proof of the following lemma is straightforward.

Lemma 19. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is measurable and $\xi : \mathbb{R} \to \mathbb{R}$ is a bi-Lipschitz homeomorphism. Then the composition $f \circ \xi$ is measurable and, if $g = f$ a.e., then $g \circ \xi = f \circ \xi$ a.e.*

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