

AN EXAMPLE IN THE GRADIENT THEORY OF PHASE TRANSITIONS

CAMILLO DE LELLIS¹

Abstract. We prove by giving an example that when $n \geq 3$ the asymptotic behavior of functionals $\int_{\Omega} \varepsilon |\nabla^2 u|^2 + (1 - |\nabla u|^2)^2 / \varepsilon$ is quite different with respect to the planar case. In particular we show that the one-dimensional ansatz due to Aviles and Giga in the planar case (see [2]) is no longer true in higher dimensions.

Mathematics Subject Classification. 49J45, 74G65, 76M30.

Received December 4, 2001.

1. INTRODUCTION

This paper is devoted to the study of the asymptotic behavior of functionals

$$F_{\varepsilon}^{\Omega}(u) := \int_{\Omega} \left(\varepsilon |\nabla^2 u|^2 + \frac{(1 - |\nabla u|^2)^2}{\varepsilon} \right) \quad \Omega \subset \mathbf{R}^n \quad (1)$$

as $\varepsilon \downarrow 0$, where u maps Ω into \mathbf{R} . This problem was raised by Aviles and Giga in [2] in connection with the mathematical theory of liquid crystals and more recently by Gioia and Ortiz in [9] for modeling the behavior of thin film blisters. Recently many authors have studied the planar case giving strong evidences that, as conjectured by Aviles and Giga in [2], the sequence (F_{ε}) Γ -converge (in the strong topology of $W^{1,3}$: see [1] for a discussion of such a choice and a rigorous setting) to the functional

$$F_{\infty}^{\Omega}(u) := \begin{cases} \frac{1}{3} \int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 d\mathcal{H}^{n-1} & \text{if } |\nabla u| = 1, u \in W^{1,\infty} \\ +\infty & \text{otherwise.} \end{cases}$$

Here $J_{\nabla u}$ denotes the set of points where ∇u has a jump and $|\nabla u^+ - \nabla u^-|$ is the amount of this jump. Of course the first line of the previous definition makes sense only for particular choices of u , such as piecewise C^1 . For a rigorous setting the reader should think about a suitable function space S which contains piecewise C^1 functions and on which we can give a precise meaning to the above integral (for example a natural choice would be $\{u | \nabla u \in BV\}$; however this space turns out not to be the natural one: we refer again to [1] for a discussion of this topic).

Partial results in proving Aviles and Giga's conjecture (*i.e.* compactness of minimizers of F_{ε}^{Ω} , estimates from below on $F_{\varepsilon}^{\Omega}(u_{\varepsilon})$ and a suitable weak formulation for the problem of minimizing F subject to some boundary conditions) can be found in [1, 3, 5–8].

Keywords and phrases: Phase transitions, Γ -convergence, asymptotic analysis, singular perturbation, Ginzburg–Landau.

¹ Scuola Normale Superiore, P.zza dei Cavalieri 7, 56100 Pisa, Italy; e-mail: delellis@cibs.sns.it

In their first work Aviles and Giga based their conjecture on the following ansatz (which they made in the case $n = 2$):

Conjecture 1.1. *Let us choose a map $w : \Omega \rightarrow \mathbf{R}$ (with $\Omega \subset \mathbf{R}^n$ bounded open set containing 0) such that:*

- (a) w is Lipschitz and satisfies the eikonal equation $|\nabla w| = 1$;
- (b) ∇w is constant in $\{x_1 < 0\}$ and in $\{x_1 > 0\}$.

Let us define $E := \inf\{\liminf_{\varepsilon} F_{\varepsilon}^{\Omega}(u_{\varepsilon}) : \|u_{\varepsilon} - w\|_{W^{1,3}} \rightarrow 0\}$. Then there exists a family of functions w_{ε} such that:

- (i) the component of ∇w_{ε} perpendicular to $(1, 0, \dots, 0)$ is constant;
- (ii) $w_{\varepsilon} \rightarrow w$ in $W^{1,3}$;
- (iii) $\lim F_{\varepsilon}^{\Omega}(w_{\varepsilon}) = E$.

This ansatz has been proved by Jin and Kohn in [8] for $n = 2$. It reduces the problem of finding E to a one dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the result $E = F_{\infty}^{\Omega}(w)$, which means that at w the Γ -limit of F_{ε}^{Ω} exists and coincides with $F_{\infty}^{\Omega}(w)$. With a standard cut and paste argument (see [4]) it can be proved that the same happens for every w which is piecewise affine. In the next section we will prove the following theorem:

Theorem 1.2. *Let u be the function $u(x_1, x_2, x_3) = |x_3|$ and C the cylinder $\{|x_1|^2 + |x_2|^2 < 1\}$. Then there exists (u_k) such that:*

- (a) every u_k is piecewise affine (being the union of a finite number of affine pieces) and satisfies the eikonal equation;
- (b) $\lim_k F_{\infty}^C(u_k) < F_{\infty}^C(u)$;
- (c) $u_k \rightarrow u$ strongly in $W^{1,p}$ for every $p < \infty$.

The proof can be easily generalized to every $n \geq 3$. As an easy corollary we get that the one-dimensional ansatz fails for $n \geq 3$. Moreover this failure means that F cannot be the Γ -limit of F_{ε}^{Ω} for $n \geq 3$.

Corollary 1.3. *The one-dimensional ansatz is not true for $n \geq 3$.*

Proof. As already observed, being every u_k piecewise affine, there is a family of functions $u_{k,\varepsilon}$ such that $u_{k,\varepsilon}$ converge to u_k in $W^{1,p}$ (for every $p < \infty$) and $\lim_{\varepsilon} F_{\varepsilon}^C(u_{k,\varepsilon}) = F_{\infty}^C(u_k)$. A standard diagonal argument gives a sequence $(u_{k,\varepsilon(k)})$ strongly converging to u in $W^{1,p}$ such that $\lim_k F_{\varepsilon(k)}^C(u_{k,\varepsilon(k)}) < F_{\infty}^C(u)$. □

2. THE EXAMPLE

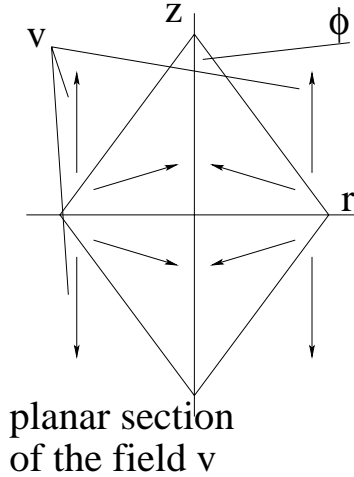
In this section we prove Theorem 1.2. First of all we recall the following fact:

(Curl) If $v : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a piecewise constant vector field, then v is a gradient if and only if for every hyperplane of discontinuity π the right trace and the left trace of v have same component parallel to π .

The building block of the construction of Theorem 1.2 is the following vector field, depending on a parameter $\phi \in (0, \pi/2)$. First of all we fix in \mathbf{R}^3 a system of cylindrical coordinates (r, θ, z) and then we call A the cone given by $\{z > 0, r < 1, (1 - r) > z \tan \phi\}$ and A' the reflection of A with respect to the plane $\{z = 0\}$. Hence we put

$$\begin{aligned}
 v(r, \theta, z) &= (0, 0, 1) && \text{if } z > 0 \text{ and } (r, \theta, z) \notin A \\
 v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, \cos(2\phi)) && \text{if } z > 0 \text{ and } z \in A \\
 v(r, \theta, z) &= (0, 0, -1) && \text{if } z < 0 \text{ and } (r, \theta, z) \notin A' \\
 v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, -\cos(2\phi)) && \text{if } z < 0 \text{ and } z \in A'.
 \end{aligned}$$

It is easy to see that v maps every plane $\{\theta = \alpha\} \cup \{\theta = \alpha + \pi\}$ into itself. Moreover the restrictions of v to these planes all look like as in the following picture



Lemma 2.1. *The vector field v is the gradient of a function w . Moreover there is a sequence of piecewise affine functions w_k such that:*

- (a) $w_k \rightarrow w$ strongly in $W_{loc}^{1,p}$ for every p ;
- (b) $F_{\infty}^{\Omega}(w_k) \rightarrow F_{\infty}^{\Omega}(w)$ for every open set $\Omega \subset \subset \mathbf{R}^3$.

Proof. We consider the restriction of v to the plane $P := \{\theta = 0\} \cup \{\theta = \pi\}$. As already noticed v maps this plane into itself. Moreover its restriction to it satisfies condition (Curl), hence on P v is the gradient of a scalar function w . Moreover we can find such a w so that it is identically zero on the line $\{z = 0\} \cap P$. Hence w is symmetric with respect to the z axis and so we can extend w to the whole three-dimensional space so to build a cylindrically symmetric function. It is easy to check that the gradient of such a function is equal to v .

We call this function w as well and we will prove that it satisfies conditions (a) and (b) written above.

(a) Our goal is approximating v with piecewise constant gradient fields. First of all we do it in the upper half-space $\{z > 0\}$. For every n we take a regular n -agon B_n which is inscribed to the circle of radius 1 and lies on the plane $\{z = 0\}$. The vertices of this n -agon are given by $V_i := (1, 2i\pi/n, 0)$.

Hence we construct the pyramid A^n with vertex $V := (0, 0, \cot \phi)$ and base B_n . In the pyramid we identify n different regions A_1^n, \dots, A_n^n , where every A_i^n is given by the tetrahedron with vertices $(0, 0, 0), V, V_i, V_{i+1}$. After this we put v_n equal to $(0, 0, 1)$ outside A^n and in every A_i^n we put

$$v_n(r, \theta, z) \equiv (\sin 2\phi, \pi + (2i + 1)\pi/n, \cos 2\phi).$$

It is easy to see that v_n satisfies condition (Curl), hence it is the gradient of some function w_n . Moreover we can choose w_n in such a way that it is identically 0 on $\{z = 0\}$. Then we extend w_n to the lower half space $\{z < 0\}$ just by imposing $w_n(r, \theta, -z) = w_n(r, \theta, z)$. It is not difficult to see that ∇w_n converges strongly to ∇w in L_{loc}^p for every p .

(b) Now we check that the previous construction satisfies also the second condition of the lemma. We fix an open set $\Omega \subset \subset \mathbf{R}^3$ and we observe that both w_k and w satisfy the eikonal equation in Ω . Moreover we call L_i^n the triangle with vertices V, V_i, V_{i+1} and L^n the union of L_i^n (so L^n is the “lateral surface” of the pyramid A^n). Finally we denote by L the lateral surface of the cone A , i.e. the set $\{(1 - r) = z \tan \phi\}$.

- (i) The amount of jump of v_n (i.e. $|v_n^+ - v_n^-|$) on L^n is constant and equal to the value of $|v^+ - v^-|$ on L . Moreover the area of L^n is converging to the area of L . The same happens on the symmetric sets in the lower half-space $\{z < 0\}$.

- (ii) Let us call B the base of the cone. The right and left traces of v_n coincides with those of v on $B_n \cup (\{z = 0\} \setminus B)$. Moreover the area of $B \setminus B_n$ is converging to zero.
- (iii) The vector fields v_n are discontinuous also on the triangles T_i^n joining V , $(0, 0, 0)$ and V_i (and on the symmetric triangles lying on $\{z < 0\}$). The amount of jump of v_n on each of these triangles is given by

$$|v_n^+ - v_n^-| = 2 \sin(\pi/n).$$

Moreover the area of everyone is given by $(\cot \phi)/2$. Hence

$$\int_{\cup_i T_i^n} |v_n^+ - v_n^-|^3 d\mathcal{H}^2 = 4n \cot \phi \sin^3 \pi/n.$$

The right hand side goes to zero as $n \rightarrow \infty$ and this completes the proof. □

Proof of Theorem 1.2. First of all we pass from the cartesian coordinates of the statement to the cylindrical coordinates (r, θ, z) given by $x_3 = z$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$ (and sometimes we will denote the elements of \mathbf{R}^3 with (y, z) , where $y \in \mathbf{R}^2$ and $z \in \mathbf{R}$).

We take w as in the previous lemma. First of all let us compute $F_\infty^C(w)$ where C is the cylinder $\{r < 1\}$. As in the previous proof we call L the lateral surface of the cone, that is the set $\{r - 1 = z \tan \phi\}$. The value of $|\nabla w^+ - \nabla w^-|$ on the surface L is given by $2 \sin \phi$ and the area of L is given by $\pi/\sin \phi$: the same happens for the symmetric of L lying on the half-space $\{z < 0\}$. On the base of the cylinder we have $|\nabla w^+ - \nabla w^-| = 2|\cos 2\phi|$. Hence

$$a(\phi) := F_\infty^C(u) - F_\infty^C(w) = \frac{\pi}{3}[8 - 8 \cos^3 2\phi - 16 \sin^2 \phi]$$

and it can be easily checked that for ϕ close enough to zero, $a(\phi)$ is positive.

Therefore let us fix an α for which $a(\alpha) > 0$ and let us agree that w is constructed as in the previous lemma by choosing $\phi = \alpha$. Given $\rho > 0$ and $x \in \mathbf{R}^2$ we define $w_{x,\rho}$ in the cylinder $C_{x,\rho} := \{(y, z) : |y - x| \leq \rho\} \subset \mathbf{R}^3$ as $w_{x,\rho}(y, z) = \rho w((y - x)/\rho, z/\rho)$. It is easy to see that

$$F_\infty^{C_{x,\rho}}(u) - F_\infty^{C_{x,\rho}}(w_{x,\rho}) = a(\alpha)\rho^2. \tag{2}$$

Let us fix ε and take ρ such that $\rho \cot \alpha < \varepsilon$. Thanks to Besicovitch Covering lemma we can cover \mathcal{H}^2 almost all $D := \{z = 0, r \leq 1\}$ with a disjoint countable family of closed discs D_i such that every D_i has radius $r_i < \rho$, center x_i and is contained in D . We construct u_ε by putting $u_\varepsilon \equiv w_{x_i,\rho_i}$ in the cylinder C_{x_i,ρ_i} .

Since ∇u_ε coincides with ∇u in $\{z \geq \varepsilon\}$ and satisfies the eikonal equation, it is easy to see that $u_\varepsilon \rightarrow u$ locally in the strong topology of $W^{1,p}$. Moreover equation (2) implies that

$$F_\infty^C(u) - F_\infty^C(u_\varepsilon) = \sum_i a(\alpha)r_i^2 = a(\alpha).$$

At this point, using the previous lemma we can approximate the function u_ε in the cylinders C_{x_i,ρ_i} with piecewise affine functions in such a way that their traces coincide with the trace of u_ε on the boundary of C_{x_i,ρ_i} . Using standard diagonal arguments for every ε we can find a sequence of piecewise affine functions u_ε^k which converge in $W^{1,p}$ to u_ε and such that $F_\infty^C(u_\varepsilon^k) \rightarrow F_\infty^C(u_\varepsilon)$. Moreover, again using diagonal arguments, we can construct the sequence u_ε^k so that each one is a finite union of affine pieces.

Finally, one last diagonal argument, gives a sequence \tilde{u}_k such that:

- (a) \tilde{u}_k is a finite union of affine pieces;
- (b) $\lim_k F_\infty^C(\tilde{u}_k) < F_\infty^C(u)$;
- (c) $\tilde{u}_k \rightarrow u$ strongly in $W^{1,p}$ for every $p < \infty$.

□

REFERENCES

- [1] L. Ambrosio, C. De Lellis and C. Mantegazza, Line energies for gradient vector fields in the plane. *Calc. Var. Partial Differential Equations* **9** (1999) 327-355.
- [2] P. Aviles and Y. Giga, A mathematical problem related to the physical theory of liquid crystal configurations. *Proc. Centre Math. Anal. Austral. Nat. Univ.* **12** (1987) 1-16.
- [3] P. Aviles and Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg–Landau type energy for gradient fields. *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999) 1-17.
- [4] C. De Lellis, *Energie di linea per campi di gradienti*, Ba. D. Thesis. University of Pisa (1999).
- [5] A. De Simone, R.W. Kohn, S. Müller and F. Otto, A compactness result in the gradient theory of phase transition. *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001) 833-844.
- [6] P.-E. Jabin and B. Perthame, Compactness in Ginzburg–Landau energy by kinetic averaging. *Comm. Pure Appl. Math.* **54** (2001) 1096-1109.
- [7] W. Jin, *Singular perturbation and the energy of folds*, Ph.D. Thesis. Courant Insitute, New York (1999).
- [8] W. Jin and R.V. Kohn, Singular perturbation and the energy of folds. *J. Nonlinear Sci.* **10** (2000) 355-390.
- [9] M. Ortiz and G. Gioia, The morphology and folding patterns of buckling driven thin-film blisters. *J. Mech. Phys. Solids* **42** (1994) 531-559.