THE NASH–KUIPER THEOREM AND THE ONSAGER CONJECTURE

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ABSTRACT. We give an account of the analogies between the Nash–Kuiper C^1 solutions of the isometric embedding problem and the weak solutions of the incompressible Euler equations which violate the energy conservation. Such analogies have lead to the recent resolution of a well-known conjecture of Lars Onsager in the theory of fully developed turbulence.

1. THE NASH-KUIPER THEOREM

Let (Σ^n, g) be a smooth n-dimensional Riemannian manifold. A map $u : \Sigma \to \mathbb{R}^N$ is *isometric* if it preserves the length of curves, i.e. if

$$\ell_g(\gamma) = \ell_e(u \circ \gamma) \qquad \text{for any } C^1 \text{ curve } \gamma \subset \Sigma, \tag{1}$$

where $\ell_g(\gamma)$ denotes the length of γ with respect to the metric g:

$$\ell_g(\gamma) = \int \sqrt{g(\gamma(t))[\dot\gamma(t), \dot\gamma(t)]}\, dt\,. \tag{2}$$

As customary, in local coordinates we can express the metric tensor g as[1] $g = g_{ij} dx_i \otimes dx_j$. For a C^1 map u, condition (1) is equivalent to the system of partial differential equations

$$\partial_i u \cdot \partial_j u = g_{ij}\,. \tag{3}$$

In the usual language of Riemannian geometry, (3) means that g is the pullback of the Euclidean metric through the map u.

The existence of isometric immersions (resp. embeddings) of Riemannian manifolds into some Euclidean space is a classical problem, explicitly formulated for the first time by Schläfli, see [46]: in the latter Schläfli conjectured that the system is solvable *locally* if the dimension N of the target is at least $s_n := \frac{n(n+1)}{2}$. Such conjecture stands to reason because (3) consists precisely of s_n equations in N unknowns. In the first half of the twentieth

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[1]Here and in the rest of this note we follow Einstein's summation convention.

century Janet [36], Cartan [14] and Burstin [13] proved Schläfli's conjecture for analytic metrics.

For the very particular case of 2-dimensional spheres endowed with metrics of positive Gauss curvature, Weyl in [51] raised the question of the existence of (global!) isometric embeddings in $\mathbb{R}^3$. The Weyl's problem was solved by Lewy in [39] for analytic metrics and Nirenberg settled the case of smooth metrics in his PhD thesis in 1949; a different proof was given independently by Pogorelov [43] around the same time, building upon the work of Alexandrov [1] (see also [44]).

An important aspect of the Weyl's problem is the rigidity of the solutions found by Lewy, Nirenberg and Pogorelov. Indeed, already before the work of Lewy, Cohn–Vossen and Herglotz proved independently that C^2 isometric immersions of positively curved spheres are uniquely determined up to rigid motions, cf. [17, 33] and see also [49] for a thorough discussion.

Before the appearance of Nash's celebrated works, it was natural to expect that the assumption of C^2 regularity in the works of Cohn–Vossen and Herglotz was just of technical nature. But in his 1954 note [41] Nash astonished the geometry world and proved that the only true obstructions to the existence of isometric immersions are topological and that as soon as $N \geq n+1$ and there are no such obstructions, then there are in fact plenty of such immersions. Nash's Theorem was therefore in stark contrast with the intuition that codimension 1 smooth isometric immersions are rather rigid for $n = 2$ and that for $n > 2$, given that the system (3) is heavily overdetermined, existence of solutions should occur rarely.

In order to state Nash's Theorem we need some terminology.

Definition 1.1. *Let (Σ, g) be a Riemannian manifold. An immersion $v : \Sigma \to \mathbb{R}^N$ is short if we have it "shrinks" the length of curves. For C^1 immersions and in local coordinates such condition is equivalent to the inequality $(\partial_i v \cdot \partial_j v) w^i w^j \leq g_{ij} w^i w^j$ for any tangent vector w.*

Theorem 1.2. *Let (Σ, g) be a smooth closed n-dimensional Riemannian manifold and $v : \Sigma \to \mathbb{R}^N$ a C^∞ short immersion with $N \geq n+1$. Then, for any $\varepsilon > 0$ there exists a C^1 isometric immersion $u : \Sigma \to \mathbb{R}^N$ such that $\|u - v\|_{C^0} \leq \varepsilon$. If v is, in addition, an embedding, then u can be assumed to be an embedding as well.*

Indeed Nash gave a proof of Theorem 1.2 for $N \geq n+2$ and just remarked that it could be proved for $N \geq n+1$ with some additional work; the details were then given in two subsequent notes by Kuiper, [38]. For this reason Theorem 1.2 is called nowadays the Nash–Kuiper Theorem on C^1 isometric embeddings.

2. Rigidity versus Flexibility

Isometries of Riemannian manifolds behave then in rather different ways depending on their smoothness: from the one hand we have the rigidity of C^2 isometries, witnessed in the classical result of Cohn-Vossen and Herglotz, and on the other hand we have the flexibility of C^1 isometries stated in the Nash–Kuiper Theorem. A natural question is whether there is a threshold regularity which distinguishes between the two behaviors.

The Hölder spaces give a classical way to measure intermediate smoothness between C^1 and C^2: a C^1 map v belongs to the Hölder space $C^{1,\alpha}$ (with $0 < \alpha \leq 1$) if

$$|Dv(x) - Dv(y)| \leq C|x - y|^\alpha$$

for some constant C independent of $x, y \in \Sigma$. It is thus natural to look at $C^{1,\alpha}$ isometries of Riemannian manifolds and ask whether there is an α_0 for which such isometries display flexibility phenomena à la Nash–Kuiper for $\alpha < \alpha_0$ and rigidity phenomena à la Cohn-Vossen–Herglotz for $\alpha > \alpha_0$. The first mathematician who tackled such problem is Borisov, who published a series of works on the topic in the late fifties (see below). Later such question is mentioned by Gromov in [31] and by Yau in [52] and in the recent work [32] Gromov advanced the conjecture that the threshold α_0 is in fact $\frac{1}{2}$.

In a series of papers in the 1950s, cf. [2, 3, 4, 5], Borisov showed that the rigidity of the Weyl problem can in fact be extended to $C^{1,\theta}$ immersions provided θ is sufficiently large.

Theorem 2.1. *Let $(\mathbb{S}^2, g)$ be a surface with C^2 metric and positive Gauss curvature, and let $u \in C^{1,\theta}(\mathbb{S}^2; \mathbb{R}^3)$ be an isometric immersion with $\theta > 2/3$. Then $u(\mathbb{S}^2)$ is the boundary of an open convex set.*

Borisov's Theorem is more general, but his statement needs the introduction of Pogorelov's concept of bounded extrinsic curvature, cf. [20]:

Theorem 2.2. *If (Σ, g) is a surface with C^2 metric and positive Gauss curvature and $u \in C^{1,\theta}(\mathbb{S}^2; \mathbb{R}^3)$ is an isometric immersion with $\theta > 2/3$, then $u(\Sigma)$ has bounded extrinsic curvature in the sense of Pogorelov.*

Building upon the work of Pogorelov, [44], Theorem 2.1 can be immediately derived from Theorem 2.2.

The concept of Pogorelov's bounded extrinsic curvature can be easily explained as follows. When u is a smooth immersion, Gauss' Theorema Egregium asserts that the determinant of the differential dN of the Gauss map N of $u(\Sigma)$ equals the Gauss curvature κ. In particular, by the area formula, for any measurable subset A of $u(\Sigma)$ the surface area $|N(A)|$ of $N(A) \subset \mathbb{S}^2$ can be computed with the area formula and it is bounded by a constant times the area of A: such constant is simply the maximum of the absolute value of κ. On the other hand $N(A)$ and A are well defined as soon as u is a C^1 map and thus if an inequality of the form $|N(A)| \leq C|A|$ holds for every measurable A, we can assert that $u(\Sigma)$ has bounded curvature in a generalized sense.

Ultimately Theorem 2.2 states that a (suitable form of) Gauss' Theorema Egregium holds for $C^{1,\theta}$ immersions as soon as $\theta > \frac{2}{3}$. In [20] we discovered a very short proof of Borisov's Theorem, which exploits the same key computation of Constantin-E-Titi's proof, see [18], of part (a) of Onsager's conjecture, cf. Conjecture 3.1 below.

For sufficiently small Hölder exponents, instead, the Nash-Kuiper construction remains valid:

Theorem 2.3. *Let (Σ, g) be a C^2 Riemannian manifold of dimension n. Any short immersion $u : \Sigma \to \mathbb{R}^{n+1}$ can be uniformly approximated with $C^{1,\theta}$ isometric immersions, where:*

(a) $\theta < \frac{1}{1+n(n+1)}$ *when Σ is a closed ball;*
(b) $\theta < \frac{1}{1+n(n+1)^2}$ *when Σ is a general compact n-manifold;*
(c) $\theta < \frac{1}{5}$ *if Σ is a 2-dimensional disk.*

The maps can be chosen to be embeddings if u is an embedding.

Case (a) of this theorem was announced in [6] by Borisov, based on his habilitation thesis, under the additional assumption that g is analytic. A proof with $n = 2$ appeared more than 40 years later, cf. [7]. The general statements (a) and (b) of Theorem 2.3 have been proved in [20], whereas the improved bound for 2-dimensional disks, namely statement (c), has been shown rather recently in [23].

As a byproduct of the above constructions, there is no way of making sense of Gauss' Theorema Egregium for $C^{1,\theta}$ immersions when θ is sufficiently small. A natural question is then to ask whether these two behaviors are distinguished by a sharp threshold. In what follows we will call such question "Borisov–Gromov problem".

3. Onsager's Conjecture

Around 10 years ago László Székelyhidi and I pointed out a striking analogy between the Borisov–Gromov problem and a well-known conjecture in the theory of mathematical fluid dynamics, proposed in 1949 by the celebrated Norwegian theoretical physicist Lars Onsager (we refer to the survey articles [26] and [29] for a throrough discussion of this and several other points mentioned below). The unveiling of such analogy was a consequence of our work [24]: in that paper we applied methods which are reminiscent of those used by Nash in [41] in order to explain the existence of weak solutions to the incompressible Euler equations which do not preserve the total kinetic energy.

The incompressible Euler equations describe the motion of a perfect incompressible fluid. Written down by L. Euler over 250 years ago, these are the continuum equations corresponding to the conservation of momentum and mass of arbitrary fluid regions. In Eulerian variables they can be written as

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p = 0 \\ \\ \operatorname{div} v = 0, \end{cases} \tag{4}$$

where $v = v(x,t)$ is the velocity and $p = p(x,t)$ is the pressure. We will focus on the 3-dimensional case with periodic boundary conditions. In other words we take the spatial domain to be the flat 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. A classical solution on a given time interval $[0,T]$ is defined to be a pair $(v,p) \in C^1(\mathbb{T}^3 \times [0,T])$ which solves (4) pointwise.

As far as weak solutions are concerned, there are several notions (see for instance the survey article [26] and the lecture notes [50]). One commonly considered in the literature consists of pairs $(v,p) : \mathbb{T}^3 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}$ which solve (4) in the sense of distributions[2].

For classical solutions (i.e. if $v \in C^1$) the total energy

$$e(t) := \frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2\,dx$$

is conserved by the flow induced by (4), so that $e(t) = e(0)$. The proof is an easy computation. We can scalar multiply the first equation by v to derive:

$$\sum_j v_j \partial_t v_j + \sum_j v_j \sum_k v_k \partial_{x_k} v_j + \sum_j v_j \partial_{x_j} p = 0\,,$$

which we can rewrite as

$$\partial_t \frac{|v|^2}{2} + (v\cdot\nabla)\left(\frac{|v|^2}{2} + p\right) = 0\,.$$

Using the fact that v is divergence free we finally achieve

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) = 0\,,$$

which integrated in the space variable implies

$$\frac{d}{dt}\int_{\mathbb{T}^3} \frac{|v|^2(x,t)}{2}\,dx = 0\,.$$

Onsager in [42] was the first to suggest the existence of weak solutions which might dissipate the energy. Based on calculations in Fourier space, he formulated the following conjecture (in fact he had a non-rigorous argument for part (a)).

Conjecture 3.1. *Consider periodic 3-dimensional weak solutions of (4), where the velocity v satisfies the uniform Hölder condition*

$$|v(x,t) - v(x',t)| \le C|x - x'|^\theta, \tag{5}$$

[2]Recall the classical computation that $(v\cdot\nabla)v = \operatorname{div}(v\otimes v)$ if $\operatorname{div} v = 0$, so that distributional solutions are defined for any $v \in L^2(\mathbb{T}^3 \times [0,1])$.

for constants C and θ independent of x, x' and t.

(a) If $\theta > \frac{1}{3}$, then the total kinetic energy of v is constant;

(b) For any $\theta < \frac{1}{3}$ there are solutions v for which the total kinetic energy is not *constant.*

As already mentioned, Onsager in [42] actually suggested the existence of solutions for which the energy is strictly decreasing: in order to distinguish them from "non-conservative" weak solutions we will call them *dissipative solutions*. Moreover the threshold $\frac{1}{3}$ has a deep physical meaning, since it is related to the Kolmogorov's energy spectrum in his theory of fully developed turbulence, that Onsager rediscovered independently.

4. Energy conservation

The "positive part" of the Conjecture, namely statement (a), was proved by Constantin, E and Titi in [18] (a previous work of Eyink, [30], reached the critical threshold $\frac{1}{3}$ in a different scale of spaces). The argument of Constantin, E and Titi is very elegant and surprisingly simple. Fix a weak solutions (v, p) and consider a standard family of mollifiers φ_ε *in space*. As usual we will denote by v_ε the mollification of v with the kernel φ_ε. Clearly

$$\operatorname{div} v_\varepsilon = 0\,.$$

On the other hand the momentum balance in the Euler equations has a nonlinear term and for this reason v_ε is not a solution. We can however regard it as an "approximate solution":

$$\partial_t v_\varepsilon + \operatorname{div} v_\varepsilon \otimes v_\varepsilon + \nabla p_\varepsilon = \operatorname{div} (\underbrace{v_\varepsilon \otimes v_\varepsilon - (v\otimes v)_\varepsilon}_{=:T_\varepsilon})\,.$$

It is not difficult to show that v_ε has enough smoothness to carry on the computations of the previous section, which therefore show

$$\frac{d}{dt}\frac{1}{2}\int \frac{|v_\varepsilon|^2}{2}(x,t)\,dx = \int T_\varepsilon : Dv_\varepsilon\,. \tag{6}$$

Since the right hand side of such equation has a commutator structure, a clever, yet elementary, computation allows to show that it converges to 0 as $\varepsilon \downarrow 0$ as soon as $v \in C^{0,1/3+\varepsilon}$ (in fact one can allow for slightly less regularity, cf. [18] and [16]).

Theorem 2.2 is proved in [20] with a very similar strategy. The isometric immersion u is approximated with a standard mollification procedure by smooth immersions u_ε. After writing the area formula for the Gauss map N_ε of $u_\varepsilon(\Sigma)$, we can understand Gauss' Theorema Egregium as a suitable family of integral identities. Something similar can be done for the Euler equations as well: we can write a suitable local version of the energy conservation as an integral identity involving a smooth test function; the equation (6) corresponds to the particular case of choosing a test function identically equal to 1.

Theorem 2.2 is achieved in [20] by passing into the limit in such identities as $\varepsilon \downarrow 0$. The convergence follows from a commutator estimate which has a striking similarity with the one of [18].

5. Energy dissipation: L^2, L^∞ and C^0

Concerning part (b), the first construction ever of an L^2 solution that violates the energy conservation is due to Scheffer in [45]. A different argument was later given by Shnirelman in [47], who was also able, a few years later, to give the first construction of a solution which dissipates the energy, cf. [48]. In [24] we gave a rather simple proof of these results, constructing bounded weak solutions of the incompressible Euler equations which violate the usual conservation energy and the uniqueness of the Cauchy problem in several ways (see also [25]). The key was to regard solutions of the system (4) as divergence-free matrix fields satisfying a suitable algebraic constraint: in particular we realized that this point of view allowed to use well established techniques from the theory of differential inclusions, cf. [15, 8, 21, 40, 37].

In the latter field, the authors consider systems of partial differential equations which prescribe the values of the gradients of the solutions and thus clearly the system (3) is a differential inclusion. A couple of decades ago the groundbreaking paper [40] of Müller and Šverak established a fruitful connection between the techniques used in the theory of differential inclusions and Gromov's h-principle (more precisely his convex integration methods) of which the Nash–Kuiper theorem is a primary example, cf. [31].

Our intuition that a suitable approach à la Nash could provide a line of attack for part (b) of the Conjecture was confirmed by the following result, which we proved in [27], using a suitable "convex integration scheme".

Theorem 5.1. *Given any positive smooth function e on $[0,T]$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of continuous functions which solves* (4) *in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = e(t)$.*

The construction of continuous solutions of (4) follows the basic strategy of Nash in the sense that at each step of the iteration we add a highly oscillatory correction. Note that both (4) and the equation of isometries (3) are quadratic – the oscillatory perturbation is chosen in such a way as to minimize the linearization.

Indeed we construct a sequence of subsolutions (v_q, p_q, R_q), i.e. solutions of

$$\begin{cases} \partial_t v_q + \operatorname{div} v_q \otimes v_q + \nabla p_q = -\operatorname{div} R_q \\ \\ \operatorname{div} v_q = 0 \end{cases} \tag{7}$$

and iteratively remove the error R_q, which is a symmetric 3×3 matrix field. As a first observation note that if one is only interested in measuring the "distance" of a smooth pair (v_q, p_q) from being a solution of (4), then only the traceless part of R_q is relevant: we can write

$$R_q = \rho_q \mathrm{Id} + \mathring{R}_q,$$

where $\mathring{R}_q$ is a traceless 3×3 symmetric matrix, since $\operatorname{div}(\rho_q \mathrm{Id}) = \nabla \rho_q$. Hence if $\mathring{R}_q = 0$ then v_q is a solution of the Euler equations (perhaps with a different pressure).

Recall that we also aim in Theorem 5.1 at satisfying a certain energy profile for the total kinetic energy. We choose therefore a sequence $e_q = e_q(t)$ with $e_q(t) \to e(t)$ and set

$$\rho_q(t) := \frac{1}{3(2\pi)^3} \left(e_{q+1}(t) - \frac{1}{2} \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx \right),$$

$$R_q(x,t) := \rho_q(t)\mathrm{Id} + \mathring{R}_q(x,t).$$

In the next section we explain the key points of the iteration and also which kind of Hölder regularity one could expect for the final solution.

6. A Nash-type iteration

Our aim is to build a sequence of triples $(v_q, p_q, \mathring{R}_q)$ solving (7) which converge uniformly to a triple $(v, p, 0)$ (actually in what follows we will mostly focus on the velocity v). The sequence will be achieved iteratively by adding a suitable perturbation to v_q and p_q. We thus set

$$w_q = v_q - v_{q-1}.$$

The size of w_q will be controlled with two parameters. The *amplitude* δ_q bounds the C^0 norm:

$$\|w_q\|_0 \lesssim \delta_q^{1/2}. \tag{8}$$

Up to negligible errors the Fourier transform of the perturbation w_q will be localized in a shell centered around a given *frequency* λ_q. Hence

$$\|\nabla w_q\|_0 \lesssim \delta_q^{1/2} \lambda_q. \tag{9}$$

Along the iteration we will have $\delta_q \to 0$ and $\lambda_q \to \infty$ at a rate that is at least exponential. For the sake of definiteness we may think

$$\lambda_q := \lambda^q \quad \text{and } \delta_q := \lambda_q^{-2\theta_0} \tag{10}$$

for some $\lambda > 1$ (although in the actual proofs a slightly super-exponential growth is required). The positive number θ_0 is the threshold Hölder regularity which we are able to achieve through the iteration, since it can be easily shown by interpolation that $\|v_q - v_{q-1}\|_\alpha = \|w_q\|_\alpha \lesssim \delta_q^{1/2} \lambda_q^\alpha \lesssim \lambda_q^{\alpha - \theta_0}$ and thus $\{v_q\}_q$ is a Cauchy sequence in C^α whenever $\alpha < \theta_0$.

The perturbation w_{q+1} is added to "balance" the error R_q and indeed we will see that $R_q \sim w_{q+1} \otimes w_{q+1}$. For this reason we will have

$$\|\mathring{R}_q\|_0 \leq c_0 \delta_{q+1} \tag{11}$$

$$\|\nabla \mathring{R}_q\|_0 \lesssim \delta_{q+1}\lambda_q \tag{12}$$

The main part of the perturbation w_{q+1} satisfies (ideally, as we will see later) an Ansatz of the type

$$w_o(x,t) = W\Big(v_q(x,t), R_q(x,t), \lambda_{q+1}x, \lambda_{q+1}t\Big)\,, \tag{13}$$

where W is a function which we are going to specify next. The pressure p_{q+1} will be defined similarly as $p_{q+1} = p_q + P(v_q, R_q, \lambda_{q+1}x, \lambda_{q+1}t)$, but we will not enter into the details in our discussion, since its role is anyway secondary.

First of all, the oscillatory nature of the perturbation requires us to impose that W is periodic in the variable $\xi \in \mathbb{T}^3$. Next, observe that v_{q+1} must satisfy the divergence-free condition $\operatorname{div} v_{q+1} = 0$ and $v + w_o$ is not likely to fulfill this: we need to add a suitable correction w_c in order to satisfy it. Consider therefore a vector potential for v_q, namely write v_q as $\nabla \times z_q$ for some smooth z_q. Subsequently we would like to perturb z_q to a new

$$z_{q+1}(x,t) = z_q(x,t) + \frac{1}{\lambda_{q+1}} Z(v(x,t), R(x,t), \lambda_{q+1}x, \lambda_{q+1}t)\,.$$

Computing $v_{q+1} := \nabla \times z_{q+1}$ we get

$$v_{q+1}(x,t) = v_q(x,t) + \underbrace{(\nabla_\xi \times Z)(v(x,t), \tilde{R}(x,t), \lambda x, \lambda t)}_{(P)} + O\left(\frac{1}{\lambda}\right)\,.$$

The term (P) would correspond to w_o if we were able to find a vector potential Z for W which is *periodic in* ξ. This requires $\operatorname{div}_\xi W = 0$ and $\langle W \rangle = 0$, where we use the notation $\langle , \rangle$ to denote the average in the ξ variable.

Similar considerations (see for instance [50]) lead to the following set of conditions that we would like to impose on W:

- $\xi \mapsto W(v, R, \xi, \tau)$ is 2π-periodic with vanishing average, i.e.

$$\langle W \rangle := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} W(v, R, \xi, \tau)\, d\xi = 0; \tag{H1}$$

- The average stress is given by R, i.e.

$$\langle W \otimes W \rangle = R\,; \tag{H2}$$

- The "cell problem" is satisfied:

$$\begin{cases} \partial_\tau W + v \cdot \nabla_\xi W + \operatorname{div}_\xi (W \otimes W) + \nabla_\xi P = 0 \\ \\ \operatorname{div}_\xi W = 0\,, \end{cases} \tag{H3}$$

  where $P = P(v, R, \xi, \tau)$ is a suitable pressure;

- W is smooth in all its variables and satisfies the estimates

$$|W| \lesssim |R|^{1/2},\ |\partial_v W| \lesssim |R|^{1/2},\ |\partial_R W| \lesssim |R|^{-1/2}. \tag{H4}$$

As a consequence of (H1)-(H2) we obtain

$$\int_{\mathbb{T}^3} |v_{q+1}|^2\,dx \sim \int_{\mathbb{T}^3} |v_q|^2\,dx + \int_{\mathbb{T}^3} \langle |W|^2 \rangle\,dx = \int_{\mathbb{T}^3} |v_q|^2\,dx + 3(2\pi)^3 \rho_q(t)$$

and thus the total kinetic energy of the v_{q+1} is (up to small errors) $e_{q+1}(t)$.

Having defined the couple (v_{q+1}, p_{q+1}) we face the problem of finding a suitable stress tensor $\mathring{R}_{q+1}$. An important remark is that it is possible to select a good "elliptic operator" which solves the equations $\operatorname{div} \mathring{R} = f$. The relevant technical lemma is the following one.

Lemma 6.1 (The operator div^{-1}). *There exists a homogeneous Fourier-multiplier operator of order -1, denoted*

$$\operatorname{div}^{-1} : C^\infty(\mathbb{T}^3;\mathbb{R}^3) \to C^\infty(\mathbb{T}^3;\mathcal{S}_0^{3\times 3})$$

such that, for any $f \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ with average $\fint_{\mathbb{T}^3} f = 0$ we have

(a) $\operatorname{div}^{-1} f(x)$ *is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;*
(b) $\operatorname{div}\operatorname{div}^{-1} f = f$.

Assuming the existence of an ideal profile W, the next stress tensor $\mathring{R}_{q+1}$ would then be defined through

$$\begin{aligned}
\mathring{R}_{q+1} &= -\ \operatorname{div}^{-1}\Big[\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1}\Big] \\
&= -\ \underbrace{\operatorname{div}^{-1}\Big[\partial_t w_{q+1} + v_q \cdot \nabla w_{q+1}\Big]}_{=:\mathring{R}^{(1)}_{q+1}} \\
&\quad -\ \underbrace{\operatorname{div}^{-1}\Big[\operatorname{div}(w_{q+1} \otimes w_{q+1} - R_q) + \nabla(p_{q+1} - p_q)\Big]}_{=:\mathring{R}^{(2)}_{q+1}} \\
&\quad -\ \underbrace{\operatorname{div}^{-1}\Big[w_{q+1} \cdot \nabla v_q\Big]}_{=:\mathring{R}^{(3)}_{q+1}}
\end{aligned} \tag{14}$$

where div^{-1} is the operator of order -1 from Lemma 6.1. Since we are assuming that the size of the corrector w_c is negligible compared to w_o, we will discuss the corresponding terms where w_o replaces w_q.

The main issues are therefore

- to show that indeed it is possible to send δ_q to 0 as $q \uparrow \infty$ (so that the scheme converges)
- and to obtain a relation between δ_q and λ_q in the form of (10).

If we were able to find a "profile" W satisfying (H1)-(H2)-(H3)-(H4), then the iteration proposed so far would lead to a proof of the Onsager's conjecture. In order to see this first expand $W(v, R, \xi, \tau)$ as a Fourier series in ξ. We then could compute

$$\mathring{R}^{(3)} = \mathrm{div}^{-1}\Big[w_o \cdot \nabla v_q\Big] = \mathrm{div}^{-1} \sum_{k\in\mathbb{Z}^3, k\neq 0} c_k(x,t) e^{i\lambda_{q+1}k\cdot x}, \tag{15}$$

where the coefficients $c_k(x,t)$ vary much slower than the rapidly oscillating exponentials. When we apply the operator div^{-1} we can therefore treat the c_k as constants and gain a factor $\frac{1}{\lambda_{q+1}}$ in the outcome: a typically "stationary phase argument". Note that it is crucial that c_0 vanishes: this is in fact the content of condition (H1).

Using (H4) we can estimate the size of each term c_k as

$$\|c_k\|_0 \lesssim \|W\|_0 \|\nabla v_q\|_0 \lesssim \|R_q\|_0^{1/2} \|\nabla v_q\|_0 .$$

Applying (9) and (11) we arrive at

$$\|\mathring{R}^{(3)}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} . \tag{16}$$

In fact in our computations so far we are ignoring a lot of technical issues: the relevant estimates are much more complicated and affected by several other terms which we are neglecting.

Similar arguments for the two other error tensors $\mathring{R}^{(1)}_{q+1}$ and $\mathring{R}^{(2)}_{q+1}$ would lead to an estimate of type

$$\|\mathring{R}_{q+1}\|_0 \lesssim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} . \tag{17}$$

Of course, this is just one of the estimates for $(v_{q+1}, p_{q+1}, R_{q+1})$ and similar ones should be obtained for all the other quantities (and for other norms). However, (17) already implies a relation between δ_q and λ_q. Indeed, comparing it with (11), the inductive step requires

$$\delta_{q+2} \sim \frac{\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q}{\lambda_{q+1}} .$$

Assuming $\lambda_q \sim \lambda^q$ for some fixed $\lambda \gg 1$, this would lead to

$$\delta_q^{1/2} \sim \lambda^{-q/3} \sim \lambda_q^{-1/3}, \tag{18}$$

which gives $\theta_0 = 1/3$ as the critical Hölder regularity.

7. First Höder regularity and improvement

It tuns out that almost all conditions on the function $W = W(v, R, \xi, \tau)$ can be fulfilled, as shown in [27]. Let us first examine the simple case in which we set $v = 0$: it is then possible to construct a function $W_s(R, \xi) = W(0, R, \xi, \tau)$ satisfying the constraints (H1)-(H4). The basic building block is given by Beltrami flows. For the details we refer the reader to [27], but one important aspect of Beltrami flows is that one can construct several different W_s with the property that any linear combinations of them still satisfy (H1)-(H3)-(H4) (and a suitable version of (H2)). In fact W_s takes the form

$$W_s(R,\xi) = \sum_{k\in\Lambda} a_k(R) B_k e^{ik\cdot\xi}$$

where Λ is a subset of $\mathbb{Z}^3$ with the property that $|k|$ is a fixed constant for every $k \in \Lambda$ and B_k is related to k by a precise algebraic formula. Note in particular that two distinct profiles W_s^1 and W_s^2 whose linear combination is still a profile can be obtained by choosing two disjoint faimilies Λ's in the same sphere intersected with the lattice $\mathbb{Z}^3$.

Another aspect which is important about Beltrami flows is that the corresponding stationary profiles W_s are only defined for R in a suitably small cone $\mathcal{C}$ of tensors R, whose axis is the half-line $\{\lambda \mathrm{Id} : \lambda \in \mathbb{R}^+\}$.

Having obtained a profile $W(0,R,\xi,\tau) = W_s(R,\xi)$, it seems natural to extend W by imposing that $\partial_\tau W + v\cdot\nabla_\xi W = 0$, leading to the formula

$$W(v,R,\xi,\tau) = W_s(R,\xi - v\tau) = \sum_{k\in\Lambda} a_k(R) B_k e^{i(k-v\tau)\cdot\xi}\,. \tag{19}$$

However the latter fails to satisfy (H4), because $|\partial_v W(v,R,\xi,\tau)| \sim |R|^{1/2}|\tau|$. This is a serious problem: observing that τ is the "fast time" variable, in the construction (13) $\tau = \lambda_{q+1} t$, leading to an additional factor λ_{q+1} in the estimates for $\mathring{R}^{(1)}_{q+1}$ and $\mathring{R}^{(2)}_{q+1}$: this loss destroys any hope that the scheme might converge.

In [27] a "phase function" $\phi_k(v,\tau)$ was introduced to deal with the transport part of the cell problem. By considering W of the form

$$\sum_{|k|=\lambda_0} a_k(R)\phi_k(v,\tau) B_k e^{ik\cdot\xi} \tag{20}$$

the cell problem in (H3) leads to the equation

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = 0\,.$$

Since the exact solution $\phi_k(v,\tau) = e^{-i(v\cdot k)\tau}$ is incompatible with the requirement (H4), an approximation is used such that

$$\partial_\tau \phi_k + i(v\cdot k)\phi_k = O\left(\mu_q^{-1}\right), \qquad |\partial_v \phi_k| \lesssim \mu_q$$

for some new parameter μ_q. To be precise, the approximation involves a partition of unity over the space of velocities and the use of 8 distinct families $\Lambda^{(j)}$.

This leads to the following corrections to (H3) and (H4): (H3) is only satisfied approximately,

$$\partial_\tau W + v\cdot\nabla_\xi W + \mathrm{div}_\xi (W\otimes W) + \nabla_\xi P = O(\mu_q^{-1})$$

and in (H4) the second inequality is replaced by

$$|\partial_v W| \lesssim \mu_q |R|^{1/2}.$$

In [28] the approach above was subsequently used to show the first example of Hölder flows with prescribed energy profiles, more precisely:

Theorem 7.1. *Given any positive smooth function e on $[0,T]$ and any $\alpha < \frac{1}{10}$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of C^α functions which solves* (4) *in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\, dx = e(t)$.*

A further improvement was obtained in [12], following an idea first introduced by Isett in [34]. We change the Ansatz (20) on W and look for a perturbation w_o which has the form

$$w_o(x,t) = W_s(R_q(x,t), \lambda_{q+1}\Phi_q(x,t)) = \sum_{k\in\Lambda^{(1)}} a_k(R_q(x,t)) B_k e^{i\lambda_{q+1}\Phi_q(x,t)}\,, \tag{21}$$

where Φ_q solves the transport equation

$$\partial_t \Phi_q + (v_q \cdot \nabla_x)\Phi_q = 0\,. \tag{22}$$

With (21), we would have

$$\mathring{R}^{(1)}_{q+1} = \sum_{k\in\Lambda^{(1)}} \nabla a_k(R_q)(\partial_t R_q + (v_q\cdot\nabla)R_q) e^{i\lambda_{q+1}\Phi_q}\,. \tag{23}$$

Assuming that $D\Phi_q(x,t)$ is not too far from the identity, the stationary phase argument leads to

$$\|\mathring{R}^{(1)}\|_0 \lesssim \delta_{q+1}^{3/2}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{-1}\,. \tag{24}$$

In fact in the latter estimates we are also assuming that the advective derivative $\partial_t R_q + (v_q\cdot\nabla)R_q$ satisfies a better bound than the usual derivative DR_q. This is indeed correct, as first pointed out by Isett in [34], and intuitively it can be justified by observing that even the advective derivative $(\partial_t v_q + v_q\cdot\nabla)v_q$ satisfies a better bound than Dv_q.

However, since $\|Dv_q\|_0 \to \infty$, we expect the deformation matrix $D\Phi_q$ to be controllable only for short times. More precisely, by a well-known elementary estimate on ODEs, if $\Phi_q(x,t_0) = x$, then

$$\|D\Phi_q(\cdot,t) - \mathrm{Id}\|_0 \lesssim \|\nabla v_q\|_0 |t-t_0| \lesssim \delta_q^{1/2}\lambda_q|t-t_0| \tag{25}$$

for $|t-t_0| \lesssim (\delta_q^{1/2}\lambda_q)^{-1}$. The latter is a typical "CFL condition", cf. [19].

To handle this problem we proceed as in [12] and consider a partition of unity $(\chi_j)_j$ on the time interval $[0,T]$ such that the support of each χ_j is an interval I_j of size $\frac{1}{\mu_q}$ for some $\mu_q \gg 1$. In each time interval I_j we set $\Phi_{q,j}$ to be the solution of the transport equation (22) which satisfies

$$\Phi_{q,j}(x,t_j) = x,$$

where t_j is the center of the interval I_j. Recalling that $\|Dv_q\|_0 \lesssim \delta_q^{1/2}\lambda_q$, (25) leads to

$$\|D\Phi_{q,j}\|_0 = O(1) \quad \text{and} \quad \|D\Phi_{q,j} - \mathrm{Id}\|_0 \lesssim \frac{\delta_q^{1/2}\lambda_q}{\mu_q} \tag{26}$$

provided

$$\mu_q \geq \delta_q^{1/2}\lambda_q, \tag{27}$$

an estimate we will henceforth assume. Observe also that $|\partial_t \chi_j| \lesssim \mu_q$.

The new fluctuation will take the form

$$w_o = \sum_j \chi_j(t) \sum_{k\in\Lambda^{(i(j))}} a_k(R_q) B_k e^{i\lambda_{q+1}k\cdot\Phi_{q,j}} \tag{28}$$

where:

- $i(j)$ equals 1 if j is odd and 2 if j is even;
- $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are two disjoint families.

The new Ansatz leads then to the following estimate

$$\|\mathring{R}_{q+1}\|_0 \lesssim \delta_{q+1}^{1/2}\mu_q\lambda_{q+1}^{-1} + \delta_{q+1}\delta_q^{1/2}\lambda_q\mu_q^{-1} \tag{29}$$

Optimizing in μ_q we then reach

$$\|\mathring{R}_{q+1}\|_0 \lesssim \delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{-1/2}, \tag{30}$$

namely

$$\delta_{q+2} \sim \delta_{q+1}^{3/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{-1/2}.$$

The latter relation leads to a threshold $\theta_0 = \frac{1}{5}$ and hence to the following theorem

Theorem 7.2. *Given any positive smooth function e on $[0,T]$ and any $\alpha < \frac{1}{5}$ there is a pair $(v,p) : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3 \times \mathbb{R}$ of C^α functions which solves* (4) *in the distributional sense and satisfies $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = e(t)$.*

8. First Onsager-critical construction

In [11] Buckmaster observed that, by choosing the cut-off functions χ_i appropriately in (28) it is possible to show that the solution produced in the proof of Theorem 7.2 enjoys $C^{1/3-\varepsilon}$ regularity at almost every time-slice. The idea is to make the cut-off flat on large portions of their supports while paying very steep time derivatives on small portions. The price to pay is thatthe "global" Hölder control gets much weaker: the solutions is just slightly better than continuous (i.e. it has a very small Hölder exponent, depending on ε). In [9], jointly with Buckmaster and Székelyhidi we exploited a quantitative version of the latter idea to reach the first nonconservative solutions up Onsager's threshold $1/3$, albeit in a weaker form than as stated in his conjecture.

Theorem 8.1. *For every $\alpha < \frac{1}{3}$ there are a nontrivial continuous compactly supported solution $(v, p) : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ of (4) and an L^1 function $C : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|v(x,t) - v(y,t)| \leq C(t)|x-y|^\alpha \qquad \forall t \in \mathbb{R}, \forall x, y \in \mathbb{T}^3 .$$

9. h-PRINCIPLE

The Beltrami flows together with the transport Ansatz explained in the previous sections settle the issue of convergence (at least for Hölder exponents $\theta < 1/5$), but are not sufficient to conclude an h-principle statement which is a satisfactory counterpart of Theorem 1.2. The reason is that the stationary profiles W_s defined through Beltrami flows are only defined for R's belonging to a suitably small cone of tensors.

Nevertheless, there is a very simple set of stationary flows (which we will call "Mikado flows") based on pipe flow, which can generate all R. These flows were introduced by Daneri and Székelyhidi in [22].

Lemma 9.1. *For any compact subset $\mathcal{N}$ consisting of positive definite 3×3 matrices there exists a smooth vector field*

$$W_s : \mathcal{N} \times \mathbb{T}^3 \to \mathbb{R}^3, \quad i = 1, 2$$

such that, for every $R \in \mathcal{N}$

$$\begin{cases} \operatorname{div}_\xi (W_s(R,\xi) \otimes W(R,\xi)) = 0, \\ \\ \operatorname{div}_\xi W_s(R,\xi) = 0, \end{cases} \tag{31}$$

and

$$\langle W_s \rangle = 0, \tag{32}$$

$$\langle W_s \otimes W_s \rangle = R. \tag{33}$$

In particular, in [22] the authors could prove the following h-principle result

Theorem 9.2. *Let $(\bar v, \bar p, \bar R)$ be a smooth solution of*

$$\begin{cases} \partial_t \bar v + \operatorname{div}(\bar v \otimes \bar v) + \nabla \bar p = -\operatorname{div} \bar R \\ \\ \operatorname{div} \bar v = 0 \end{cases} \tag{34}$$

on $\mathbb{T}^3 \times [0,T]$ such that $\bar R(x,t)$ is positive definite for all x, t. Then for any $\alpha < 1/5$ there exists a sequence $\{(v_k, p_k)\} \subset C^\alpha$ of weak solutions of (4) such that

$$v_k \overset{*}{\rightharpoonup} \bar v \quad \textit{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar v \otimes \bar v + \bar R \quad \textit{in } L^\infty$$

uniformly in time and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} v_k \otimes v_k \, dx = \int_{\mathbb{T}^3} (\bar v \otimes \bar v + \bar R) \, dx.$$

10. Isett's proof of Onsager's conjecture

The stationary profile W_s reached through the Mikado flows in [22] have not only the feature of being defined on any compact subset of positive definite R's, but they also yield improved estimates in several error terms in the iteration. On the other hand they are not compatible with the "patching in time" used in (28) and indeed in [22] they are only used for finitely many steps of the iteration, whereas the "tail" of the series $\sum_q w_q$ still consists of oscillatory perturbations whose building blocks are Beltrami flows.

In [35] Isett has been able to overcome this last obstruction by introducing a different "patching strategy". Isett's key idea can be easily explained as follows. Considered a given triple $(v_q, p_q, \mathring{R}_q)$ reached at a certain step of the iteration, satisfying all the estimates outlined in the previous subsection. The obstruction to using Mikado flows could be overcome if $\mathring{R}_q$ were supported in a union of disjoint time-stripes of the form $\mathbb{T}^3 \times [a_k, b_k]$, where $b_k - a_k \sim (\delta_q^{1/2}\lambda_q)^{-1}$, compatibly with the CFL condition (25). In this case there would be no need of "patching" the oscillatory perturbations, since they would be supported on disjoint time-stripes where the CFL condition holds and the flows $\Phi_{q,j}$ of the previous subsections are close to the identity.

In order to reach this ideal situation, Isett in [35] partitions the whole time interval in smaller intervals $[c_k, c_{k+1}]$ with size $\sim (\delta_q^{1/2}\lambda_q)^{-1}$. In intervals of comparable size it is possible to find exact solutions (z_k, r_k) of the incompressible Euler equations with $z_k(\cdot, c_k) = v_q(\cdot, c_k)$. Patching such solutions with a partition of unity one can obtain new velocity and pressure fields $(\tilde{v}_q, \tilde{p}_q)$ together with "separate" time-stripes where they are exact solutions of the Euler equations. The remaining regions consist of time-stripes where we have to find a new stress tensor $\tilde{R}_q$. Isett shows that such tensor can be found so that its size is not much larger than $\mathring{R}_q$: in fact it essentially satisfies the same estimates with worse constants.

Now there are no obstructions to apply the oscillatory perturbations of [22] to the new triple $(\tilde{v}_q, \tilde{p}_q, \tilde{R}_q)$ and therefore one can reach the following statement

Theorem 10.1. *For every $\alpha < \frac{1}{3}$ there is a nontrivial continuous compactly supported solution $(v,p) \in C^\alpha(\mathbb{T}^3 \times \mathbb{R})$ of (4).*

11. h-principle and Onsager's conjecture with dissipative solutions

In the previous "patching" of exact solutions of the Euler equations a canonical choice of the stress tensor $\tilde{R}_q$ would be

$$\tilde{R}_q := \operatorname{div}^{-1}[\partial_t \tilde{v}_q + (\tilde{v}_q \cdot \nabla)\tilde{v}_q + \nabla \tilde{p}_q]\,. \tag{35}$$

However [35] generates $\tilde{R}_q$ with a different, more complicated, procedure, since the author is not able to reach the desired estimate through the operator div^{-1}. A suboptimal outcome is that Theorem 10.1 does not produce "dissipative solutions".

This has been instead accomplished in [10], where in a joint work with Buckmaster, Székelyhidi and Vicol we derive appropriate estimates for the "canonical" $\tilde{R}_q$ as defined in (35). We can therefore derive the existence of dissipative solutions in the whole range of Hölder exponents of the second part of Onsager's conjecture. Indeed such a statement is obtained as a corollary of the exact counterpart of the h-principle result in [22]:

Theorem 11.1. *Let $(\bar{v}, \bar{p}, \bar{R})$ be a smooth solution of (34) on $\mathbb{T}^3 \times [0,T]$ such that $\bar{R}(x,t)$ is positive definite for all x, t. Then for any $\alpha < 1/3$ there exists a sequence $\{(v_k, p_k)\} \subset C^\alpha$ of weak solutions of (4) such that*

$$v_k \overset{*}{\rightharpoonup} \bar{v} \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \bar{v} \otimes \bar{v} + \bar{R} \quad \text{in } L^\infty$$

uniformly in time and furthermore for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} v_k \otimes v_k \, dx = \int_{\mathbb{T}^3} (\bar{v} \otimes \bar{v} + \bar{R})\, dx.$$

Corollary 11.2. *For every $\alpha < \frac{1}{3}$ and every positive smooth $e : [0,T] \to \mathbb{R}$ there exists a solution $(v,p) \in C^{1/3}(\mathbb{T}^3 \times [0,T])$ of (4) such that*

$$\frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx = e(t)\,.$$

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