We denote by e the Euclidean metric on $\mathbb{R}^n$.

Theorem 0.1. *For every $n \geq 2$ and every $\alpha \in (0,1]$ there is a constant $C = C(n,\alpha)$ with the following property. Let $u \in H^1(B_2, \mathbb{R}^n)$ be such that*

$$\|u^\sharp e - e\|_{L^\infty(B_1)} \leq \frac{1}{2}\,. \tag{0.2}$$

and

$$\det Du \geq 0 \qquad a.e.\,. \tag{0.3}$$

Then there is $A \in SO(n)$ such that

$$\|Du - A\|_{C^\alpha(B_1)} \leq C\|u^\sharp e - e\|_{C^\alpha(B_2)}\,. \tag{0.4}$$

Remark 0.5. From now on, $|A|$ will denote the Hilbert-Schmidt norm of the matrix $A \in \mathbb{R}^{n\times n}$, namely $|A| = \sqrt{\operatorname{tr} A^\top A}$ induced by the Hilbert-Schmidt scalar product $\langle A, B\rangle := \operatorname{tr} A^\top B$. In particular, using standard coordinates in B_2 and identifying $u^\sharp e$ with the corresponding $n\times n$ matrix, we have $|Du|^2 = \operatorname{tr} u^\sharp e$. Under the assumption (0.2) we thus conclude immediately that $\|Du\|_{L^\infty(B_2)} \leq C(n)$, namely that the map u is Lipschitz.

Remark 0.6. Note that an assumption like (0.3) is needed because it is easy to give examples of Lipschitz maps whose derivative belong to $O(n)$ almost everywhere but which are not affine. In fact such maps are "abundant" in an appropriate sense: in particular they form a residual set in the space $X := \{u \in \operatorname{Lip}(B_2, \mathbb{R}^n) : u^\sharp e \leq e\}$ endowed with the the L^∞ distance, which makes X a compact metric space (cf. [2] for the latter and more subtle results).

The two authors of the paper asked me while preparing their manuscript whether I could provide a reference or a proof for Theorem 0.1. While I felt that this should be a well-known "classical fact", I was unable to find a reference for it. I therefore suggested a simple argument which reduces (0.4) to an important work of [1] which essentially handles a corresponding "L^2-estimate". The reduction is given in this appendix. It uses some elementary facts from Linear Algebra (which are well known and I just include for the reader's convenience) and a Morrey-type decay. In what follows we denote by Id the identity matrix in $\mathbb{R}^{n\times n}$.

Lemma 0.7. *We have*

$$\operatorname{dist}(A, SO(n)) = \operatorname{dist}(A, O(n)) \qquad \textit{for all } A \in \mathbb{R}^{n\times n} \textit{ with } \det A \geq 0 \tag{0.8}$$

and

$$\operatorname{dist}(A, O(n)) \leq |A^\top A - \operatorname{Id}| \qquad \forall A \in \mathbb{R}^{n\times n}\,. \tag{0.9}$$

Proof. In order to show (0.8) fix first an arbitrary matrix A with $\det A \geq 0$. Recalling the polar decomposition of matrices there is a symmetric S and a $O_1 \in SO(n)$ such that $A = O_1 S$. Next, recalling that every symmetric matrix is diagonalizable, there is $O_2 \in SO(n)$ such that $A = O_1 O_2^\top D O_2$ for some diagonal matrix D. Next recall that if O is a diagonal matrix with an even number of entries equal to -1 and the remaining equal to 1, then $O \in SO(n)$. If one of the diagonal enties of D is zero, we can then assume without loss of generality that all enties of A are nonnegative. Otherwise, if no diagonal entry is 0, we can assume without loss of generality that they are all positive but at most 1. Since $\det A > 0$, we can exclude that one diagonal entry of D is negative and the others are all positive. Summarizing the arguments in the two cases, we can assume that all diagonal entries of D are nonnegative. Since $\mathrm{dist}\,(A, O(n)) = \mathrm{dist}\,(OA, O(n)) = \mathrm{dist}\,(AO, O(n))$ and $\mathrm{dist}\,(A, SO(n)) = \mathrm{dist}\,(OA, SO(n)) = \mathrm{dist}\,(AO, SO(n))$ for every $O \in SO(n)$, we conclude that it suffices to prove (0.8) for a diagonal matrix A which has all nonnegative entries. Denote them by $\lambda_1, \ldots, \lambda_n$. For any $O \in O(n)$ we can then compute explicitely

$$|A - O|^2 = \sum_i \lambda_i^2 + n - 2\sum_i \lambda_i O_{ii}\,.$$

Observe that $-1 \leq O_{ii} \leq 1$ because O is orthogonal. Since $\lambda_i \geq 0$ for every i we then conclude $|A - O|^2 \geq \sum_i \lambda_i^2 + n - 2\sum_i \lambda_i = |A - \mathrm{Id}\,|^2$. This however shows that $\mathrm{dist}\,(A, O(n))^2 = |A - \mathrm{Id}\,|^2 = \mathrm{dist}\,(A, SO(n))^2$.

As for (0.9) fix $A \in \mathbb{R}^{n\times n}$ and let $O \in O(n)$ be such that $\mathrm{dist}\,(A, O(n)) = |A - O|$. Since both sides of the inequality take the same value for A and $O^{-1}A$, we can assume that $O = \mathrm{Id}$. By the minimality condition of Id we must have that $A - \mathrm{Id}$ is orthogonal (in the Hilbert-Schmidt scalar product) to the tangent to $O(n)$ at Id, which is the space of skew-symmetric matrices. We therefore conclude that A is symmetric and, again applying the $O(n)$ invariance of the inequality, we can assume w.l.o.g. that it is diagonal. Let $\lambda_i = A_{ii}$ be the diagonal entries and observe that none of them can be negative: if $\lambda_k < 0$ then the matrix B which has $B_{ij} = 0$ for $i \neq j$, $B_{ii} = 1$ for $i \neq k$ and $B_{kk} = -1$ satisfies $B \in O(n)$ and $|A - B| < |A - \mathrm{Id}\,|$. (0.9) is thus reduced to proving

$$\sum_i (\lambda_i - 1)^2 \leq \sum_i (\lambda_i^2 - 1)^2$$

under the assumption that $\lambda_i \geq 0$ for every i. This is equivalent to prove $(x-1)^2 \leq (x^2-1)^2 = (x-1)^2(x+1)^2$ for $x \geq 0$, which is obvious. □

Proof of Theorem 0.1. Fix $x \in B_1$ and let $S(x)$ be the unique positive definite symmetric matrix such that $S(x)^2 = u^\sharp e(x)$. Observe that, by (0.2), we have

$$|S(x)| + |S(x)^{-1}| \leq C\,.$$

Let v be the map $v(y) := u(y)(S(x))^{-1}$ and observe further that

$$\begin{aligned}
\mathrm{dist}\,(Dv(y), SO(n)) \leq& \mathrm{dist}\,(Dv(y), O(n)) \leq |Dv^\top(y) Dv(y) - \mathrm{Id}\,| \\
\leq& |S(x)^{-1} Du^\top(y) Du(y) S(x)^{-1} - \mathrm{Id}\,| \\
=& |S(x)^{-1}(Du^\top(y) Du(y) - S(y)^2) S(x)^{-1}| \\
=& |S(x)^{-1}(u^\sharp e(y) - u^\sharp e(x)) S(x)^{-1}| \\
\leq& 4[u^\sharp e]_{\alpha, B_2} |x-y|^\alpha = 4[u^\sharp e - e]_{\alpha, B_2} |x-y|^\alpha
\end{aligned}$$

(where we use the standard notation $[f]_{\alpha,\Omega} := \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \Omega\}$). In particular, for every $r \leq 1$ we can apply the Friesecke-James-Müller inequality, namely [1, Theorem

3.1], to find a matrix $A(x,r) \in SO(n)$ with the property that

$$\fint_{B_r(x)} |Dv - A(x,r)|^2 \leq C \fint_{B_r(x)} \mathrm{dist}\,(Dv, SO(n))^2 \leq C[u^\sharp e - e]^2_{\alpha,B_2} r^{2\alpha} \tag{0.10}$$

(note that [1, Theorem 3.1] is stated for a general open set U in place of $B_r(x)$, with a constant depending on U; however an obvious scaling argument shows that the constant is the same for balls of arbitrary radii). Recalling that

$$\min_c \int |f - c|^2 = \int \left| f - \fint f \right|^2 ,$$

we conclude

$$\begin{aligned}
\fint_{B_r(x)} \left| Du - \fint_{B_r(x)} Du \right|^2 &\leq \fint_{B_r(x)} |Du - A(x,r)S(x)|^2 \\
&= \fint_{B_r(x)} |(Dv - A(x,r))S(x)|^2 \\
&\leq 4 \fint_{B_r(x)} |Dv - A(x,r)|^2 \leq C[u^\sharp e - e]^2_{\alpha,B_2} r^{2\alpha} .
\end{aligned}$$

Morrey's estimate then implies that $Du \in C^\alpha(B_1)$ and

$$[Du]_{\alpha,B_1} \leq C[u^\sharp e - e]_{\alpha,B_2} . \tag{0.11}$$

On the other hand, again by the Friesecke-James-Müller estimate, there is $A \in SO(n)$ such that

$$\fint_{B_1} |Du - A|^2 \leq C \fint_{B_1} \mathrm{dist}\,(Du, SO(n))^2 \leq C\|u^\sharp e - e\|^2_{C^0(B_1)} . \tag{0.12}$$

(0.11) and (0.12) immediately imply (0.4). □

References

[1] Gero Friesecke, Richard D. James, and Stefan Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461–1506, 2002.

[2] Bernd Kirchheim, Emanuele Spadaro, and László Székelyhidi, Jr. Equidimensional isometric maps. *Comment. Math. Helv.*, 90(4):761–798, 2015.