GEOMETRIC STRUCTURE OF MIGDAL'S CONSTRAINED VORTEX SURFACES

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Abstract. This short note derives some geometric conditions that Migdal's constrained vortex surfaces have to satisfy, from which corresponding rigidity results can be drawn. The note will appear as appendix in a forthcoming article by Migdal, cf. [2].

In this note we prove a series of restrictions that Migdal's constrained vortex surfaces have to satisfy. These restrictions come from elementary computations of differential geometric nature: combined with some standard facts in topology and in the theory of harmonic functions, they imply a number of restrictions on the behavior of these surfaces, which can be interpreted as suitable "rigidity theorems". The note is mathematically self-contained, while for the origin of the concept, its relevance in the theory of turbulent flows, and several other (mathematical and physical) aspects, which are due to the work of Migdal in the last decades, we refer the reader to the forthcoming report [2]. The present note, which is seperately available on the webpages of the authors, will in fact appear as an appendix to the latter work.

1. CVS conditions: geometric interpretation and instability in the compact case

Let $\mathcal{S} \subset \mathbb{R}^3$ be a (sufficiently smooth) complete connected surface. We denote by $\mathcal{S}_-$ the region enclosed by $\mathcal{S}$, and set $\mathcal{S}_+ := \mathbb{R}^3 \setminus (\mathcal{S} \cup \mathcal{S}_-)$.

According to Migdal's definition, a CVS (constrained vortex surface) solution to the Euler equations is a pair of (sufficiently smooth) velocity fields (v_-, v_+) defined on $\mathcal{S}_-$ and $\mathcal{S}_+$, respectively, such that

$$\operatorname{div} v_+ = \operatorname{div} v_- = 0 \tag{1}$$

$$\operatorname{curl} v_+ = \operatorname{curl} v_- = 0 \tag{2}$$

$$v_+ , v_- \quad \text{are tangent to } \mathcal{S} \tag{3}$$

$$(Dv_+ + Dv_-) \cdot (v_+ - v_-) = 0 \tag{4}$$

$$|v_+|^2 - |v_-|^2 \text{ is locally constant on } \mathcal{S}. \tag{5}$$

Moreover, the constrained vortex surface has to satisfy the following stability condition

$$\langle D(v_+ + v_-) \cdot n \, , n \rangle < 0 \, , \tag{6}$$

where n is the exterior normal to $\mathcal{S}$ and $\langle a, b\rangle$ denotes the scalar product of the vectors a and b.

We have the following geometric characterization of (3), (4) and (5).

Theorem 1.1. *Conditions* (3), (4), (5) *are satisfied iff*

$$\begin{cases} A(v_-, v_-) = A(v_+, v_+) \\ [v_+, v_-] = 0\,, \end{cases} \tag{7}$$

where A is the second fundamental form of $\mathcal{S}$, and $[v_+, v_-]$ is the Lie bracket, i.e.

$$[v_+, v_-] = Dv_+ \cdot v_- - Dv_- \cdot v_+\,.$$

Remark 1.2. *Recall that the Lie Bracket $[v_+, v_-] = (Dv_+ \cdot v_- - Dv_- \cdot v_+)|_{\mathcal{S}}$ is tangent to $\mathcal{S}$ because v_+ and v_- are tangent to $\mathcal{S}$.*

Proof. Condition (5) is equivalent to

$$(Dv_+ \cdot v_+ - Dv_- \cdot v_-)|_{\mathcal{S}} \quad \text{is parallel to } n\,. \tag{8}$$

We rewrite (4) as $I + II = 0$, where

$$I = Dv_+ \cdot v_+ - Dv_- \cdot v_-$$
$$II = -Dv_+ \cdot v_- + Dv_- \cdot v_+\,,$$

and observe that I is normal to $\mathcal{S}$, as a consequence of (8), while II is tangent to $\mathcal{S}$. Indeed,

$$\langle II, n\rangle = A(v_+, v_-) - A(v_-, v_+) = 0\,.$$

In particular $I + II = 0$ iff $I = 0$ and $II = 0$. It is immediate to rewrite the two conditions as (7).

□

Regarding the stability condition (6), we can prove the following rigidity result.

Theorem 1.3. *If $\mathcal{S} \subset \mathbb{R}^3$ is a smooth compact closed surface and v_+, v_- a pair of vector fields that satisfy the conditions* (1) *and* (3), *then*

$$\int_{\mathcal{S}} \langle D(v_+ + v_-) \cdot n\,, n\rangle = 0\,. \tag{9}$$

In particular, (6) *cannot hold.*

The same argument allows for a similar conclusion when $\mathcal{S}$ is a smooth compact surface with boundary $\partial\mathcal{S}$ and $v^+ + v^-$ is tangent to the boundary.

Theorem 1.4. *Assume $\mathcal{S} \subset \mathbb{R}^3$ is a smooth compact surface with smooth boundary $\partial\mathcal{S}$ and v_+, v_- a pair of vector fields that satisfy the conditions* (1) *and* (3). *If in addition $v^+ + v^-$ is tangent to $\partial\mathcal{S}$, then*

$$\int_{\mathcal{S}} \langle D(v_+ + v_-) \cdot n\,, n\rangle = 0\,. \tag{10}$$

In particular, (6) *cannot hold.*

Remark 1.5. *Clearly Theorem 1.3 can be thought as the particular case of Theorem 1.4 has empty boundary. The smoothness needed by the proof given below is that the surface and its boundary are both C^1 (i.e. they have a tangent at every point, which in turn varies continuosly), and that the vector fields are C^1, i.e. continuously differentiable.*

Proof of Theorem 1.4. Fix a point $p \in \mathcal{S}$ and choose a local orthonormal frame around p consisting of e_1, e_2, n, where the vectors e_1 and e_2 are tangent to $\mathcal{S}$. Note that, since both v_+ and v_- are divergence free we have that $D(v_+ + v_-)$ is a trace-free matrix, and thus

$$\langle D(v_+ + v_-)\cdot n, n\rangle = -\langle D(v_+ + v_-)\cdot e_1, e_1\rangle - \langle D(v_+ + v_-)\cdot e_2, e_2\rangle\,. \quad (11)$$

Let g be the Riemannian scalar product induced on $\mathcal{S}$ by the Euclidean one, and denote by $\nabla^{\mathcal{S}}$ the corresponding Levi-Civita connection. Recall that the latter coincides with the connection induced by the Euclidean connection. This amounts to say that, if X, Y and Z are tangent vector fields to $\mathcal{S}$, then

$$g(\nabla^{\mathcal{S}}_Y X, Z) = \langle DX \cdot Y, Z\rangle\,.$$

If we apply this to $Y = Z = e_i$ and $X = v_+ + v_-$, (11) can be rewritten as

$$g(\nabla^{\mathcal{S}}_{e_1}(v_+ + v_-), e_1) + g(\nabla^{\mathcal{S}}_{e_2}(v_+ + v_-), e_2) = -\langle D(v_+ + v_-)\cdot n\,, n\rangle\,. \quad (12)$$

Notice now that the left hand side of (12) is the divergence in $\mathcal{S}$ of the tangent vector field $v_+ + v_-$, namely we have

$$\mathrm{div}_{\mathcal{S}}\,(v_+ + v_-) = -\langle D(v_+ + v_-)\cdot n\,, n\rangle\,. \quad (13)$$

Next, by Gauss theorem, the fact that $\mathcal{S}$ is has a smooth boundary implies that

$$\int_{\mathcal{S}} \mathrm{div}_{\mathcal{S}}\,(v_+ + v_-) = -\int_{\partial\mathcal{S}} (v_+ + v_-)\cdot \nu\,, \quad (14)$$

where we denote by ν the smooth unit vector field on $\partial\mathcal{S}$ which is tangent to $\mathcal{S}$, orthogonal to $\partial\mathcal{S}$ and points "inwards", i.e. towards $\mathcal{S}$. Since $v^+ + v^-$ is parallel to $\partial\mathcal{S}$, the integrand in the right hand side of (14) vanishes identically, which allows us to conclude (10). □

2. Rigidity of the CVS condition for closed surfaces

We prove rigidity results for closed connected CVS surfaces without imposing the stability condition (6). We prove that

(A) there are no CVS solutions with $\mathcal{S}$ homeomorphic to the sphere;
(B) if $\mathcal{S}$ has genus bigger than 1, then $\mathcal{S}$ cannot be real analytic;
(C) if $\mathcal{S}$ has genus 1, then there are no axisymmetric solutions.

The precise formulation of (A) is the following:

Theorem 2.1. *If (v_-, v_+) satisfies the CVS conditions* (1)*,* (2)*,* (3)*,* (4)*,* (5)*, and $\mathcal{S}$ is homeomorphic to the sphere, then $v_+ = 0$ and $v_- = 0$.*

Point (B) is a consequence of a more general fact which we state in the following theorem.

Theorem 2.2. *The following holds:*

(i) *If $\mathcal{S}$ is a smooth closed connected CVS surface with genus different than* 1*, then $|v_+|^2 = |v_-|^2$.*
(ii) *If $\mathcal{S}$ is any smooth CVS surface and $|v_+|^2 = |v_-|^2$, then for any point $q \in \mathcal{S}$ one of the following conditions must necessarily hold:*
  (a) *The Gauss curvature of $\mathcal{S}$ is* 0 *at q;*
  (b) $v_+(q) = v_-(q)$*;*
  (c) $v_+(q) = -v_-(q)$*.*

From the above theorem we draw the following simple

Corollary 2.3. *Assume $\mathcal{S}$ is a closed smooth connected CVS surface with genus strictly higher than* 1*. Then either $v_+ = v_-$ on some nontrivial open subset of $\mathcal{S}$, or $v_+ = -v_-$ on some nontrivial open subset of $\mathcal{S}$.*

Proof. In fact, if the sets $\{v_+ = v_-\}$ and $\{v_+ = -v_-\}$ contain no interior points, then the Gauss curvature would vanish on a dense set, and by continuity it must vanish anywhere. But it is well known that a surface with vanishing Gauss curvature cannot be closed. □

The condition is even more stringent if $\mathcal{S}$ is real analytic, i.e. it can be described as the graph of a function with a converging Taylor series around any point, up to rotation of the coordinates.

Corollary 2.4. *Assume $\mathcal{S}$ is a closed real analytic connected CVS surface with genus strictly higher than* 1*. Then either $v_+ = v_-$ on $\mathcal{S}$, or $v_+ = -v_-$ on $\mathcal{S}$.*

Proof. Since both v_+ and v_- are locally gradients of harmonic functions which satisfy the Neumann boundary condition, it turns out that their restrictions to $\mathcal{S}$ are real analytic as well. But real analytic functions which vanish on a nontrivial open set, must vanish identically because $\mathcal{S}$ is connected. □

We finally detail the non existence result (C) for axisymmetric solutions of genus 1. Let us consider a closed simple curve $[0,1] \ni t \to \gamma(t) = (\gamma_r(t), \gamma_z(t)) \in (0,\infty)^2$, and the associated torus of rotation $\mathcal{S} \subset \mathbb{R}^3$, parametrized by

$$(t,\theta) \to (\gamma_r(t)\cos\theta, \gamma_r(t)\sin\theta, \gamma_z(t))\,. \tag{15}$$

We use polar coordinates $(x,y,z) = (r\cos\theta, r\sin\theta, z)$. Recall that $\frac{\partial}{\partial r}$, $\frac{1}{r}\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial z}$ is an orthonormal frame.

We look for CVS solutions with axisymmetry. The general Ansatz is the following.

$$v_- := \alpha \frac{1}{r^2}\frac{\partial}{\partial\theta} = \nabla\theta$$
$$v_+ := a(r,z)\frac{\partial}{\partial r} + b(r,z)\frac{1}{r}\frac{\partial}{\partial\theta} + c(r,z)\frac{\partial}{\partial z}\,,$$

for some $\alpha \in \mathbb{R}$ and C^1 functions a, b and c.

Theorem 2.5. *Let $\mathcal{S}$ be given by* (15)*, for some simple closed curve $\gamma = (\gamma_r, \gamma_z)$ of class C^2. If (v_-, v_+) is an axisymmetric pair that satisfies the CVS conditions* (1), (2), (3), (4), (5)*, then either $v_+ = v_-$ on $\mathcal{S}$, or $v_+ = -v_-$ on $\mathcal{S}$.*

2.1. **Proof of Theorem 2.1.** Since $\mathcal{S}_-$ is simply connected we can write $v_- = \nabla\Phi_-$ for some harmonic function Φ_-, whose gradient is tangent to $\mathcal{S}$. A simple integration by parts gives

$$\int_{\mathcal{S}_-} |v_-|^2\,dx = \int_{\mathcal{S}_-} |\nabla\Phi_-|^2\,dx = \int_{\mathcal{S}} \Phi_-\nabla\Phi_- \cdot n\,dx = 0\,, \tag{16}$$

where n denotes the exterior normal to $\mathcal{S}$. We deduce that $v_- = 0$. Thanks to Theorem 2.2(i) we deduce that $v_+ = 0$ on $\mathcal{S}$. Since v_+ is harmonic, the unique continuation principle implies that v_+ vanishes identically.

2.2. **Proof of Theorem 2.2.** Let us begin by proving (i). A well-known theorem in topology implies that any tangent vector field to a closed surface $\mathcal{S}$ with genus different than 1 must necessarily vanish at some point. This excludes that the constant in (5) is positive (because then v_+ would never vanish) or negative (because then v_- would never vanish).

Let us pass to the proof of (ii). The key ingredient is the following lemma, whose proof is postponed at the end of this section.

Lemma 2.6. *Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface and let $U \subset \Sigma$ be an open set. Assume the existence of smooth velocity fields u, w defined on U and tangent to Σ. If*

(i) *$u(p), w(p) \neq 0$ and $g_p(v, w) = 0$ for any $p \in U$, where g is the metric induced by the ambient space $\mathbb{R}^3$;*
(ii) *$[u, w] = 0$ in U;*
(iii) *u and v are gradients in U, i.e. there exist $\alpha, \beta : U \to \mathbb{R}$ such that $g_p(u, \cdot) = \mathrm{d}_p\alpha$ and $g_p(v, \cdot) = \mathrm{d}_p\beta$ for any $p \in U$.*

Then, the Gaussian curvature of Σ is zero in U.

Let us explain how to prove Theorem 2.2(ii) given Lemma 2.6. Fix q such that $v_+(q) \neq v_-(q)$ and $v_+(q) \neq v_-(q)$ and let U be a neighborhood of q where $v_+ - v_-$ and $v_+ + v_-$ never vanish. Since $|v_+|^2 = |v_-|^2$, we have $g_p(u, w) = 0$ for any $p \in U$. Theorem 1.1 implies that

$[v_+, v_-] = 0$, hence $[u, w] = 0$ in U. Moreover, (2) imply that v_+ and v_- are irrotational in $\mathcal{S}_+$ and $\mathcal{S}_-$, respectively, and (3) says that v_+ and v_- are tangent to $\mathcal{S}$. This implies that, v_+ and v_- are locally gradients in $\mathcal{S}$. In particular u and w are locally gradients in U. We are in position to apply Lemma 2.6 with $\Sigma = \mathcal{S}$, which implies that $K = 0$ in U, where K denotes the Gaussian curvature of $\mathcal{S}$. This implies in particular that the Gauss curvature of $\mathcal{S}$ vanishes at q.

2.2.1. *Proof of Lemma 2.6.* Fix $p \in U$. We claim that there exists local coordinates around p

$$X : W \subset \mathbb{R}^2 \to U\,, \quad (x, y) \to X(x, y) \in U\,,$$

such that

$$\frac{\partial}{\partial x} = u\,, \quad \frac{\partial}{\partial y} = w\,.$$

The claim follows from (ii), indeed we can define

$$X(x, y) = \phi_u^x \circ \phi_w^y(p) \in U\,,$$

where ϕ_u and ϕ_w are the flow maps associated to u and v, respectively. Observe that $X(0, 0) = p$. Condition (ii) implies that the flow maps commute, hence

$$\frac{\partial}{\partial x}X(x, y) = u \circ X(x, y)\,, \quad \frac{\partial}{\partial y}X(x, y) = w \circ X(x, y)\,, \tag{17}$$

in particular (i), along with the implicit function theorem, says that X is a local parametrization in a neighborhood of p.

Let h be the metric in the new coordinates (x, y). It turns out that

$$h_{x,y} = h_{y,x} = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = g(u, w) = 0 \tag{18}$$

as a consequence of the previous claim and condition (i). We claim that

$$\frac{\partial}{\partial y}h_{x,x} = \frac{\partial}{\partial x}h_{y,y} = 0\,. \tag{19}$$

It immediately implies that Σ is flat in a neighborhood of p. Indeed, as a consequence of (19) we can write the metric as

$$h = a(x)dx^2 + b(y)dy^2\,,$$

which is clearly flat.

Let us prove (19). As a consequence of (iii) we get

$$\begin{aligned}\frac{\partial}{\partial y}h_{x,x} = \frac{\partial}{\partial y}g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= 2g\left(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)\\ &= 2\mathrm{Hess}\,\alpha\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\end{aligned} \tag{20}$$

the symmetry of the Hessian implies

$$\operatorname{Hess}\alpha\left(\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right)=\operatorname{Hess}\alpha\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=g\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right), \tag{21}$$

on the other hand we know that $g\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=0$, hence

$$\begin{aligned} g\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)&=\frac{\partial}{\partial x}g\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)-g\left(\frac{\partial}{\partial x},\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\right)\\ &=-g\left(\frac{\partial}{\partial x},\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}\right), \end{aligned} \tag{22}$$

where in the last step we used that $\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]=0$. By collection (20), (21), and (22) we deduce

$$\frac{\partial}{\partial y}h_{x,x}=-\frac{\partial}{\partial y}h_{x,x},$$

which implies our claim. We argue in the same way to show that $\frac{\partial}{\partial x}h_{y,y}=0$.

2.3. **Proof of Theorem 2.5.** Without loss of generality we assume that $\gamma:[0,L]\to(0,\infty)^2$ is parametrized by arclength, where L is the length of γ. Recall that we look for solutions of the form

$$\begin{aligned} v_- &:= \alpha\frac{1}{r^2}\frac{\partial}{\partial\theta}\\ v_+ &:= a(r,z)\frac{\partial}{\partial r}+b(r,z)\frac{1}{r}\frac{\partial}{\partial\theta}+c(r,z)\frac{\partial}{\partial z}, \end{aligned}$$

where $\alpha\in\mathbb{R}$ and a,b,c are C^1 functions.

Lemma 2.7. *We have that* $\operatorname{curl}v_+=0$ *if and only if* $b(r,z)r$ *is constant and*

$$\operatorname{curl}\left(a(r,z)\frac{\partial}{\partial r}+c(r,z)\frac{\partial}{\partial z}\right)=0. \tag{23}$$

Proof. We compute

$$\operatorname{curl}v_+=\left(\frac{1}{r}\frac{\partial c}{\partial\theta}-\frac{\partial b}{\partial z}\right)\frac{\partial}{\partial r}+\left(\frac{\partial a}{\partial z}-\frac{\partial c}{\partial r}\right)\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\left(\frac{\partial(rb)}{\partial r}-\frac{\partial a}{\partial\theta}\right)\frac{\partial}{\partial z}.$$

Since v_+ is axisymmetric, we have that $\operatorname{curl}v_+=0$ if and only if

$$\begin{cases}\frac{\partial b}{\partial z}=0\\ \frac{\partial(rb)}{\partial r}=0\\ \operatorname{curl}\left(a\frac{\partial}{\partial r}+c\frac{\partial}{\partial z}\right)=0.\end{cases} \tag{24}$$

The first two conditions amount to $br=C$, for some constant $C\in\mathbb{R}$. The latter amounts to (23). ∎

Let us begin by considering the case $\alpha \neq 0$. Without loss of generality we can assume $\alpha = 1$. After imposing $\operatorname{curl} v_+ = 0$, we can write

$$v_+ = a(r,z)\frac{\partial}{\partial r} + c(r,z)\frac{\partial}{\partial z} + Cv_- =: w_+ + Cv_- \,. \tag{25}$$

To satisfy the CVS conditions we need to impose the following properties on w_+:

(1) $\operatorname{div} w_+ = \operatorname{curl} w_+ = 0$
(2) w_+ is tangent to $\mathcal{S}$
(3) $|w_+|^2 + \frac{C^2-1}{r^2} = \ell$ in $\mathcal{S}$, for some $\ell \in \mathbb{R}$
(4) $[w_+, v_-] = 0$ in $\mathcal{S}$
(5) $A(w_+ + Cv_-, w_+ + Cv_-) = A(v_-, v_-)$

We show that (2), (3) and (4) imply that γ_r is constant.

Condition (4). We compute

$$\begin{aligned}
[w_+, v_-] &= \nabla_{w_+} v_- - \nabla_{v_-} w_+ \\
&= \nabla_{a\frac{\partial}{\partial r} + c\frac{\partial}{\partial z}} \left(\frac{1}{r^2}\frac{\partial}{\partial \theta}\right) - \frac{1}{r^2}\nabla_{\frac{\partial}{\partial \theta}} \left(a\frac{\partial}{\partial r} + c\frac{\partial}{\partial z}\right) = -\frac{2}{r^3}a\frac{\partial}{\partial \theta}\,.
\end{aligned}$$

Hence, we have to impose

$$a(\gamma_r(t), \gamma_z(t)) = 0\,, \quad \text{for any } t \in [0, L]\,. \tag{26}$$

Condition (2) and (3). Recall that

$$n = (\dot\gamma_z(t)\cos\theta, \dot\gamma_z(t)\sin\theta, -\dot\gamma_r(t))\,,$$

where n denotes the exterior normal to $\mathcal{S}$. By using (26), we deduce

$$w_+ = c(r,z)\frac{\partial}{\partial z}\,, \quad \text{on } \mathcal{S}\,, \tag{27}$$

hence

$$0 = w_+ \cdot n = -c(\gamma_r(t), \gamma_z(t))\,\dot\gamma_r(t)\,.$$

In particular, by employing (27) and (3) we get

$$\begin{aligned}
0 &= |c(\gamma_r(t), \gamma_z(t))|^2 |\dot\gamma_r(r)|^2 = |w_+|^2 |\dot\gamma_r(t)|^2 \\
&= \left(\ell + \frac{1 - C^2}{\gamma_r^2(t)}\right) |\dot\gamma_r(t)|^2\,.
\end{aligned}$$

If either $C^2 \neq 1$ or $\ell \neq 0$, then $\gamma_r = \text{const}$. This is impossible because γ is a closed simple curved.

If $C^2 = 1$ and $\ell = 0$, then (3) implies that $|w_+|^2 = 0$ on $\mathcal{S}$. It amounts to $v_+ = v_-$ when $C = 1$ and $v_+ = -v_-$ when $C = -1$.

The case $\alpha = 0$. Let us now assume that $\alpha = 0$. By applying Lemma 2.7 we deduce

$$v_+ = a(r,z)\frac{\partial}{\partial r} + c(r,z)\frac{\partial}{\partial z} + C\frac{1}{r^2}\frac{\partial}{\partial \theta} =: w_+ + C\frac{1}{r^2}\frac{\partial}{\partial \theta}\,.$$

In this case, to satisfy the CVS conditions we need to impose the following properties on w_+:

(1') $\operatorname{div} w_+ = \operatorname{curl} w_+ = 0$
(2') w_+ is tangent to $\mathcal{S}$
(3') $|w_+|^2 + \frac{C^2}{r^2} = \ell$ in $\mathcal{S}$, for some $\ell \in \mathbb{R}$
(4') $A(w_+ + C\frac{1}{r^2}\frac{\partial}{\partial\theta}, w_+ + C\frac{1}{r^2}\frac{\partial}{\partial\theta}) = 0$.

We show that (4') forces $C = 0$. Since w_+ is tangent to $\mathcal{S}$, there exists $\lambda : \mathcal{S} \to \mathbb{R}$ such that

$$w_+(\gamma) = \lambda\left(\dot\gamma_r\frac{\partial}{\partial r} + \dot\gamma_z\frac{\partial}{\partial z}\right). \tag{28}$$

We use the well-known identities

$$A\left(\dot\gamma_r\frac{\partial}{\partial r} + \dot\gamma_z\frac{\partial}{\partial z}, \dot\gamma_r\frac{\partial}{\partial r} + \dot\gamma_z\frac{\partial}{\partial z}\right) = \kappa$$
$$A\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = \gamma_r\dot\gamma_z$$
$$A\left(\dot\gamma_r\frac{\partial}{\partial r} + \dot\gamma_z\frac{\partial}{\partial z}, \frac{\partial}{\partial\theta}\right) = 0\,,$$

where $\kappa = \ddot\gamma_z\dot\gamma_r - \ddot\gamma_r\dot\gamma_z$ is the curvature of γ. We deduce

$$0 = A\left(w_+ + C\frac{1}{r^2}\frac{\partial}{\partial\theta}, w_+ + C\frac{1}{r^2}\frac{\partial}{\partial\theta}\right) = \lambda^2\kappa + \frac{C^2}{\gamma_r^3}\dot\gamma_z \tag{29}$$

Let us consider $t_0 \in [0, L]$, a minimum point for γ_r. Since $\dot\gamma_r(t_0) = 0$ and γ is parametrized by arclength, we deduce that $\dot\gamma_z(t_0) \neq 0$. Hence, (29) implies that $\lambda(t_0) \neq 0$. So, in a small neighborhood of t_0 we can rewrite (29) as

$$\ddot\gamma_z\dot\gamma_r - \ddot\gamma_r\dot\gamma_z = \kappa = \frac{C^2}{\lambda^2\gamma_r^3}\dot\gamma_z\,.$$

We multiply the latter by $\dot\gamma_z$, and use the identity $\ddot\gamma_z\dot\gamma_z = -\ddot\gamma_r\dot\gamma_r$ (which is a consequence of $(\dot\gamma_r)^2 + (\dot\gamma_z)^2 = 1$), to get

$$\ddot\gamma_r = -\frac{C^2}{\lambda^2\gamma_r^3}(1 - (\dot\gamma_r)^2)\,. \tag{30}$$

Using that $\ddot\gamma_r(t_0) \geq 0$ and $\dot\gamma_r(t_0) = 0$ we conclude that $C = 0$.

We now use (3') to deduce that $v_+ = 0$. Indeed, $\lambda^2 = \ell$, since λ is continuous we deduce that λ is a constant. If $\lambda = 0$ then $v_+ = 0$. If $\lambda \neq 0$, then (30) gives $\kappa = 0$, which contradicts the fact that γ is closed.

3. Solutions with cylindrical symmetry

In [1] Migdal finds stable solutions to the CVS equations with linear growth and vorticity concentrated on a cylinder $\mathcal{S} \subset \mathbb{R}^3$. It turns out that the cross section of $\mathcal{S}$ is noncompact. Given the cylindrical symmetry and the linear growth at infinity, that is the best one can hope for. Below we show that there are *no* solutions to the CVS equations (irrespectively of the stability condition) with cylindrical symmetry, compact cross section and linear growth.

Consider:

(i) a smooth simple closed curve $\sigma \subset \mathbb{R}^2$;
(ii) the bounded simply connected domain Ω_- bounded by σ;
(iii) the unbounded domain $\Omega_+ := \mathbb{R}^2 \setminus (\sigma \cup \Omega_-)$;
(iv) the surface $\mathcal{S} \subset \mathbb{R}^3$ given by $\sigma \times \mathbb{R}$;
(v) the cylindrical domains $\mathcal{S}_\pm := \Omega_\pm \times \mathbb{R}$.

We are looking for two bounded vector fields $v_\pm : \Omega_\pm \to \mathbb{R}^2$ and a quadratic function $q : \mathbb{R}^3 \to \mathbb{R}$ with the following properties:

(a) q is harmonic;
(b) the maps $u_\pm := (v_\pm, 0) + \nabla q : \mathcal{S}_\pm \to \mathbb{R}^3$ are divergence free, curl-free and tangent to $\mathcal{S}$;
(c) for any point $p \in \mathcal{S}$, the vector $u_+(p) - u_-(p)$ belongs to the kernel of $Du_+(p) + Du_-(p)$.

We claim the following.

Theorem 3.1. *$v_\pm$ and q must vanish identically.*

We can in fact consider the following more general situation:

(i') σ_i, $i \in \{1, \ldots, N_0\}$, is an arbitrary finite collection of simple closed curves which are pairwise disjoint;
(ii') $\{\Omega_j\}$, $j \in \{1, \ldots, N_0 + 1\}$ are the connected components of $\mathbb{R}^2 \setminus \bigcup_i \sigma_i$;
(iii') The surface $\mathcal{S} \subset \mathbb{R}^3$ is the union of the cylinders $\mathcal{S}_i := \sigma_i \times \mathbb{R}$;
(iv') The cylindrical domains are given by $\Lambda_j := \Omega_j \times \mathbb{R}$.

Under these more general assumptions we are looking for N_0+1 bounded vector fields $v_j : \Omega_j \to \mathbb{R}^2$ and a quadratic function $q : \mathbb{R}^3 \to \mathbb{R}$ with the following properties:

(a') q is harmonic;
(b') The maps $u_j := (v_j, 0) + \nabla q : \Lambda_j \to \mathbb{R}^3$ are divergence-free, curl-free, and tangent to $\partial\Lambda_j \subset \mathcal{S}$;
(c') If Λ_i and Λ_j have a common boundary $\mathcal{S}_k$, then for any point $p \in \mathcal{S}_k$ the vector $u_i(p) - u_j(p)$ belongs to the kernel of $Du_i(p) + Du_j(p)$.

Under these assumptions Theorem 3.1 can be generalized to

Theorem 3.2. *v_i and q must vanish identically.*

Proof of Theorem 3.1. Consider the function $\varphi(x, y) = q(x, y, 0)$ and the vector field $\xi_-(x, y) = v_-(x, y) + \nabla\varphi(x, y)$. Observe that v_- is curl-free and, since Ω_- is simply connected, there is a potential $\zeta_- : \Omega_- \to \mathbb{R}$ for ξ_-. Now,

$$\frac{\partial \zeta_-}{\partial \nu} = 0 \qquad \text{on } \sigma = \partial\Omega_- . \tag{31}$$

On the other hand, since φ is quadratic, $\Delta\zeta_-$ is a constant. Observe that, therefore, from

$$\int_{\Omega_-} \Delta\zeta_- = \int_\sigma \frac{\partial\zeta_-}{\partial\nu} = 0\,, \tag{32}$$

it turns out that

$$\Delta\zeta_- = \Delta\varphi = 0\,. \tag{33}$$

But then we can integrate by parts to conclude

$$\int_{\Omega_-} |\nabla\zeta_-|^2 = \int_\sigma \zeta_- \frac{\partial\zeta_-}{\partial\nu} = 0\,. \tag{34}$$

In particular ξ_- vanishes identically.

Define next $\xi_+(x,y) = v_+(x,y) + \nabla\varphi(x,y)$. (c) implies that $\xi_+(x,y)$ is in the kernel of $D\xi_+(x,y)$ for every $(x,y) \in \sigma$. Assume ζ_+ is a potential for ξ_+ in some simply connected domain $U \cap \Omega_+$, where U is the neighborhood of some point $(x,y) \in \sigma$. Then the latter condition can be rewritten as

$$\frac{1}{2}\nabla|\nabla\zeta_+|^2 = 0 \quad \text{on } \sigma \cap U\,.$$

If $|\nabla\zeta_+| = 0$ on $\sigma\cap U$, then ζ_+ is a constant over $\sigma\cap U$ and if we extend ζ_+ to $\Omega_-\cap U$ by setting it equal to the latter constant, we immediately see that ζ_+ is C^1 on U and weakly harmonic, hence harmonic. So ζ_+ must vanish by unique continuation for harmonic functions. If $|\nabla\zeta_+| = c > 0$, we conclude that $D^2\zeta_+(p)$ has a nontrivial kernel for every $p \in \sigma \cap U$, but since the trace of the two-dimensional matrix $D^2\zeta_+(p)$ is zero, we must conclude that $D^2\zeta_+$ vanishes identically on $\sigma\cap U$. But this means that $\frac{\partial\zeta_+}{\partial x}$ and $\frac{\partial\zeta_+}{\partial y}$ are both locally constant over $\sigma \cap U$. The assumption that $\frac{\partial\zeta_+}{\partial\nu} = 0$ on $\sigma \cap U$ implies therefore that $\nabla\zeta_+$ must vanish identically on $\sigma \cap U$.

Having concluded that both ξ_- and ξ_+ vanish identically, we immediately conclude that actually $u_+ = u_-$ on $\mathcal{S}$ and that the corresponding function u given by defining $u = u_\pm$ on $\mathcal{S}_\pm$ is the gradient of a quadratic harmonic function. Since u must be tangent to $\mathcal{S}$, we see right away that u must vanish identically. □

Proof of Theorem 3.2. The proof is by induction over the number N_0 of curves. The start of the the induction, namely $N_0 = 1$, is in fact Theorem 3.1. Consider therefore an arbitrary $N_0 > 1$ and assume that the theorem is correct when N_0 is substituted by N_0-1. Fix a collection of curves σ_i as in (i') above. Each σ_i bounds a unique simply connected domain Ξ_i in $\mathbb{R}^2$, and given that the curves are pairwise disjoint, each σ_j with $j \neq i$ is either contained in Ξ_i or in the interior of its complement. Since the curves are finitely many it is obvious that one of them is an "innermost" curve, namely there is a Ξ_i which does not contain any curve σ_j. Without loss of generality we can assume $i = 1$ and observe that Ξ_1 must be one of the domains Ω_i: again without loss of generality we can assume it is Ω_1. We then denote by Ω_2 the only other connected component of $\mathbb{R}^2 \setminus \bigcup_i \sigma_i$ whose boundary intersects σ_1 (in fact we must have $\sigma_1 \subset \partial\Omega_2$, but observe that the inclusion might be strict). If we set $v_- := v_1$ and $v_+ := v_2$, we can now repeat the argument of Theorem 3.1. First of all the potential ζ_- exists in our case as well because Ω_1 is simply connected, and hence the conclusions (31)-(32) can all be drawn in our case as well. The subsequent argument leads to the conclusion that v^+ is in fact a smooth continuation of v^- across $\sigma_1 \cap U$ in any simply connected neighborhood U of $p \in \sigma = \sigma_1$: the argument can be taken verbatim in our case as long as U does not intersect any other curve σ_j with $j > 1$. In particular we conclude that σ_1 could actually be eliminated from the collection of curves because the function $\tilde{v}$ defined to be v_1 in Ω_1 and v_2 in Ω_2 is in fact smooth across σ_1. Having reduced the number of curves by 1 we can apply the inductive assumption and conclude the validity of the theorem. □

References

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