Anisotropic Energies in Geometric Measure Theory

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

ANTONIO DE ROSA

aus

Italien

Promotionskommission

Prof. Dr. Camillo De Lellis (Vorsitz)

Prof. Dr. Guido De Philippis

Prof. Dr. Thomas Kappeler

Prof. Dr. Benjamin Schlein

Zürich, 2017

ABSTRACT

In this thesis we focus on different problems in the Calculus of Variations and Geometric Measure Theory, with the common peculiarity of dealing with anisotropic energies. We can group them in two big topics:

1. The anisotropic Plateau problem:

   Recently in [37], De Lellis, Maggi and Ghiraldin have proposed a direct approach to the isotropic Plateau problem in codimension one, based on the "elementary" theory of Radon measures and on a deep result of Preiss concerning rectifiable measures.

   In the joint works [44],[38],[43] we extend the results of [37] respectively to any codimension, to the anisotropic setting in codimension one and to the anisotropic setting in any codimension.

   For the latter result, we exploit the anisotropic counterpart of Allard's rectifiability Theorem, [2], which we prove in [42]. It asserts that every d-varifold in $\mathbb{R}^n$ with locally bounded anisotropic first variation is d-rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive lower d-dimensional density. In particular we identify a necessary and sufficient condition on the Lagrangian for the validity of the Allard type rectifiability result. We are also able to prove that in codimension one this condition is equivalent to the strict convexity of the integrand with respect to the tangent plane.

   In the paper [45], we apply the main theorem of [42] to the minimization of anisotropic energies in classes of rectifiable varifolds. We prove that the limit of a minimizing sequence of varifolds with density uniformly bounded from below is rectifiable. Moreover, with the further assumption that all the elements of the minimizing sequence are integral varifolds with uniformly locally bounded anisotropic first variation, we show that the limiting varifold is also integral.

2. Stability in branched transport:

   Models involving branched structures are employed to describe several supply-demand systems such as the structure of the nerves of a leaf, the system of roots of a tree and the nervous or cardiovascular systems. Given a flow (traffic path) that transports a given measure μ^- onto a target measure μ^+, along a 1-dimensional network, the transportation cost per unit length is supposed in these models to be proportional to a concave power $\alpha \in (0,1)$ of the intensity of the flow. The transportation cost is called α-mass.

   In the paper [27] we address an open problem in the book [15] and we improve the stability for optimal traffic paths in the Euclidean space $\mathbb{R}^n$ with respect to variations of the given measures (μ^-, μ^+), which was known up to now only for $\alpha > 1 - \frac{1}{n}$. We prove it for exponents $\alpha > 1 - \frac{1}{n-1}$ (in particular, for every $\alpha \in (0,1)$ when $n = 2$), for a fairly large class of measures μ^+ and μ^-.

The α-mass is a particular case of more general energies induced by even, subadditive, and lower semicontinuous functions $H : \mathbb{R} \to [0, \infty)$ satisfying $H(0) = 0$. In the paper [28], we prove that the lower semicontinuous envelope of these energy functionals defined on polyhedral chains coincides on rectifiable currents with the H-mass.

ZUSAMMENFASSUNG

In dieser Arbeit behandeln wir verschiedene Probleme der Variationsrechnung und der Geometrischen Masstheorie, wobei wir jeweils die anisotropen Energien betrachten. Wir können die Probleme in zwei grosse Themen unterteilen:

1. Das anisotrope Plateau Problem:

   Vor kurzer Zeit haben De Lellis, Maggi und Ghiraldin in [37] eine direkte Betrachtungsweise des isotropen Plateau Problems in der Kodimension eins eingeführt, basierend auf der "elementaren" Theorie der Radonmasse und einem tiefgreifenden Ergebnis von Preiss über rektifizierbare Masse.

   In den gemeinsamen Arbeiten [44],[38],[43] erweitern wir die Ergebnisse von [37] jeweils auf eine beliebige Kodimension, auf den anisotropen Fall in der Kodimension eins und auf den anisotropen Fall in jeder Kodimension.

   Für das letztere Resultat nutzen wir das anisotrope Gegenstück von Allards Rektifizierbarkeitstheorem, [2], das wir in [42] beweisen. Es sagt aus, dass jeder d-Varifold in $\mathbb{R}^n$ mit lokal beschränkter anisotroper erster Variation d-rektifizierbar ist, wenn er auf die Menge von Punkten in $\mathbb{R}^n$ mit positiver niedriger d-dimensionaler Dichte beschränkt wird. Insbesondere identifizieren wir eine notwendige und hinreichende Bedingung an die Anisotropie für die Gültigkeit des Rektifizierbarkeitstheorem heraus. Wir beweisen ausserdem, dass diese Bedingung in Kodimension eins gleichbedeutend zu der strengen Konvexität des Integrands in Bezug auf die Tangentialebene ist.

   In der Arbeit [45] wenden wir den Hauptsatz von [42] auf die Minimierung anisotroper Energien in Klassen von rektifizierbaren Varifolds an. Wir beweisen, dass der Grenzwert einer Minimierungsfolge von Varifolds mit gleichmässig von unten begrenzter Dichte rektifizierbar ist. Darüber hinaus zeigen wir unter der weiteren Annahme, dass alle Elemente der Minimierungsfolge integrale Varifolds mit gleichmässig lokal beschränkter anisotroper erster Variation sind, und wir zeigen, dass der Grenzwert der Folge auch integral ist.

2. Stabilität im verzweigten Transport:

   Modelle mit verzweigten Strukturen werden eingesetzt, um mehrere Supply-Demand-Systeme wie die Struktur der Nerven eines Blattes, das System der Wurzeln eines Baumes oder Nerven- oder Herz-Kreislauf-Systeme zu beschreiben. Bei gegebenem Fluss, der ein bestimmtes Mass μ^- über ein 1-dimensionales Strom auf ein Zielmass μ^+ transportiert, werden die Transportkosten pro Längeneinheit in diesen Modellen proportional zu einer konkaven Potenzfunktion der Potenz $\alpha \in (0, 1)$ der Intensität des Flusses sein. Die Transportkosten nennen wir α-Masse.

   In der Arbeit [27] beziehen wir uns auf ein offenes Problem im Buch [15] und verbessern die Stabilität für optimale Transporte im euklidischen Raum $\mathbb{R}^n$, bezogen auf Variationen der gegebene Masse (μ^-, μ^+), die bisher nur für $\alpha > 1 - \frac{1}{n}$

bekannt war. Wir beweisen es für Exponenten $\alpha > 1 - \frac{1}{n-1}$ (insbesondere für jedes $\alpha \in (0,1)$ wenn $n = 2$) und für eine grosse Klasse von Massen μ^+ und μ^-.

Die α-Masse ist ein besonderer Fall von allgemeineren Energien, die durch gerade, subadditive und unterhalbstetige Funktionen $H : \mathbb{R} \to [0, \infty)$ mit $H(0) = 0$ generiert werden. In der Arbeit [28] beweisen wir, dass die unterhalbstetige Hülle dieser auf polyedrischen Strömen definierten Energiefunktionen auf rektifizierbaren Strömen dasselbe ist wie die H-Masse.

To strive, to seek, to find, and not to yield — A. L. Tennyson, Ulysses

ACKNOWLEDGMENTS

In the first place, I would like to thank my two advisors Prof. Dr. Camillo De Lellis and Prof. Dr. Guido De Philippis, great examples and sources of inspiration for my research. They have been brilliant teachers and mentors, but especially friends. I have enjoyed the time spent together, the discussions on several topics, the running training sessions, the basketball matches and the great number of actvities done together.

I thank all the friends and colleagues who accompanied me along these three years. I have always felt at home with their friendship and support. Particular affection goes to the friends of my research group, who gave me a great time in Zurich and contributed to my mathematical growth with many stimulating discussions.

I wish to thank my girlfriend Marika, who has filled up my days with love, bearing the distance with patience.

My best gratitude goes to my family. My brother Giovanni has been a constant support, a friend and an inspiration. My parents Giuseppe and Concita made everything possible with their scarifices and their never-ending love.

CONTENTS

1 INTRODUCTION

1.1 THE ANISOTROPIC PLATEAU PROBLEM

The famous Plateau Problem can be roughly stated as follows:

Given a $(d-1)$-dimensional boundary in $\mathbb{R}^n$, find a d-dimensional surface with least area among all the surfaces spanning the given boundary.

This problem stands at the crossroads of several mathematical disciplines, such as Calculus of Variations, Geometric Measure Theory, Analysis of PDE and Differential Geometry, and its study has produced many beautiful ideas which have found applications in all of these subjects.

To give a precise meaning to it, one needs to formalize mathematically the concepts of "surface", "spanning a given boundary" and "area".

The interest to the problem began in the XVIII century, when Minimal Surfaces where introduced. An immersed surface is said to be minimal if its mean curvature vector is constantly zero. The problem of finding a surface of least area stretched across a given closed contour has been posed first in [62] by Lagrange in 1762. This question is nowadays known as Plateau Problem from the physicist who investigated empirically the singularity of soap bubbles in $\mathbb{R}^3$. He deduced the so-called Plateau laws: soap films always meet in threes along an edge called a Plateau border, and they do so at an angle of $\arccos(-1/2) = 120°$. These Plateau borders meet in fours at a vertex, and they do so at an angle of $\arccos(-1/3)$ (the tetrahedral angle). These are the only possible singularities of soap bubbles.

The Plateau problem has been answered in a reasonable way only around 1930 from the independent works of Douglas, [48], and Rado, [78]. These results rely on the celebrated Lichtenstein theorem, namely in dimension two one can always find conformal parametrizations. This tool is crucial in their approach. Indeed they minimize the Dirichlet energy, which has better functional analytic properties than the area functional. Since for conformally parametrized surfaces the two functionals are equal, they recover a minimum also for the area.

Lichtenstein theorem is false in general for higher dimensions, posing the new issue of finding another formulation or proof for the higher dimension case. During the last sixty years a considerable amount of effort in Geometric Measure Theory has been devoted to this task. In particular we recall the notions of sets of finite perimeter [33, 34], of currents [51] and of varifolds [2, 4, 7], introduced respectively by De Giorgi, Federer, Fleming, Almgren and Allard. The classes of the aforementioned objects were endowed with appropriate topologies, in order to guarantee good compactness properties and the semicontinuity of the functionals to minimize, ensuring the existence of a minimizers through the use of the direct method of calculus of variations.

The need of a Regularity Theory for minimal surfaces has been first encountered in connection with these existence results. The minimizers are a-priori just "generalized surfaces". Nevertheless they come out to be more regular than expected. For instance a codimension one minimizing current in $\mathbb{R}^n$ is smooth and embedded for $n \leqslant 7$, has isolated singularities for $n = 8$ and has a relative-codimension 7 singular set for $n \geqslant 8$.

We immidiately realize that these regularity properties are too strong for the modelization of soap bubbles, which can have lines and points of singularity when $n = 3$, as already observed in the Plateau laws. In order to better model this natural phenomenon, several mathematicians have started to address the Plateau problem in a set-theoretical formulation: the competitor surfaces are just closed sets and the area functional is simply the Hausdorff measure. This theory was introduced by Reifenberg with homological spanning conditions and further developed by Harrison, David and others (cf. [77, 56, 55, 31, 30]).

In [37], De Lellis, Maggi and Ghiraldin have proved a compactness result for general classes of closed sets of codimension one. This approach turns out to have application to several formulations of the Plateau problem.

1.1.1 *Personal contributions and content of Part I*

The content of the first part of the thesis is the following:

Content of Chapter 3

In Chapter 3 we present our joint works [44, 38, 43].

In joint work with De Philippis and Ghiraldin in [44], we extend the results in [37] to any codimension. More precisely, we prove that every time the class of competitors for the Plateau problem consists of d-rectifiable sets and it is closed by Lipschitz deformations, then the infimum is achieved by a set K which is, away from the "boundary", an analytic manifold outside a closed set of Hausdorff dimension at most $(d-1)$.

The idea of the proof is to associate to a minimizing sequence of sets (K_j) the sequence of measures $\mu_j := \mathcal{H}^d \llcorner K_j$. By standard functional analysis we get that μ_j converges weakly* (up to subsequences) to a measure μ. As a first step, via deformation theorem, monotonicity formula and Preiss' theorem, we show that $\mu = \theta\mathcal{H}^d \llcorner K$, where K is a d-rectifiable set. As a second step, for $\mathcal{H}^d$-a.e. x, we "project" a diagonal sequence of blow-ups $\mu_j^{x,r}$ of μ_j on the approximate tangent space T_xK of K at x, proving that $\theta(x) = 1$. Finally, we prove that K is a stationary varifold and that it is smooth outside a closed set of relative codimension one (this follows by Allard's regularity theorem and a stratification argument). Simple examples show that this regularity is actually optimal.

In joint work with De Lellis and Ghiraldin [38], we extend the result [37] to the minimization of anisotropic energies in codimension one. Namely, given a Lagrangian

$$F : \mathbb{R}^n \times G(n-1, n) \ni (x, T) \mapsto F(x, T) \in (0, \infty),$$

the anisotropic energy associated to an $(n-1)$-rectifiable set K becomes

$$\mathbf{F}(K) := \int_K F(x, T_xK)\, d\mathcal{H}^{n-1}(x).$$

The Lagrangians are required to satisfy an ellipticity condition (which can be thought as a convexity property of F). We remark that these *elliptic integrands* have useful applications in many fields, for instance in Finsler Geometry or in the study of crystal structures (and more in general in Materials science).

The new strategy adopted in [38] does not need to implement Preiss' theorem, obtaining an argument conceptually easier even in the isotropic case. The idea is to get the rectifiability of the limiting measure μ using the theory of Caccioppoli sets. Indeed the existence of a non-trivial purely unrectifiable part of μ would allow to construct, via isoperimetric inequality on the sphere, some local competitors with vanishing energy, which would violate a proven uniform lower density bound.

In joint work with De Philippis and Ghiraldin, in [43] we extend the aforementioned results to the minimization of an elliptic integrand in any codimension. We cannot rely on [38], since the theory of Caccioppoli sets and the isoperimetric inequality are strictly linked to the codimension one case. We use instead the new anisotropic counterpart of Allard's rectifiability theorem, that we prove in [42], see Chapter 4.

Three applications of these results are an easy solution to the formulation of the Plateau problem proposed by Harrison and Pugh in [56], an easy proof of the existence in any codimension of sliding minimizers, introduced by David in [31, 30] and an easier solution to Reifenberg's homological formulation of the Plateau problem.

Content of Chapter 4

As already observed in the previous chapter, an important tool in the proof of [43] is a rectifiability result, which we prove in [42] and we present in Chapter 4. This is the sharp anisotropic counterpart of Allard's rectifiability theorem, [2], which asserts that every d-varifold in $\mathbb{R}^n$ with locally bounded (isotropic) first variation is d-rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive lower d-dimensional density.

It is a natural question whether the aforementioned result holds for varifolds whose first variation with respect to an anisotropic integrand is locally bounded. In joint work with De Philippis and Ghiraldin [42], we answer positively to this question. In particular we identify a necessary and sufficient condition on the Lagrangian for the validity of the Allard type rectifiability result. We are also able to prove that in codimension one this condition is equivalent to the strict convexity of the integrand with respect to the tangent plane.

The original proof of Allard in [2] for the area integrand heavily relies on the monotonicity formula, which is strongly linked to the isotropy of the area integrand, [3]. A completely different strategy must hence be used. In particular we provide a new independent proof of Allard's rectifiability theorem.

We briefly describe the main idea. Assume for simplicity V has positive lower d-dimensional density at $\|V\|$-almost every point. We use the notion of *tangent measure* introduced by Preiss, [76], in order to understand the local behavior of a varifold V with locally bounded anisotropic first variation. Indeed, at $\|V\|$-almost every point, we show that every tangent measure is translation invariant along *at least* d (fixed) directions, while the positivity of the lower d-dimensional density ensures that there exists at least one tangent measure that is invariant along *at most* d directions. The combination of these facts allows to show that the "Grassmannian part" of the varifold V at x is a Dirac delta δ_{T_x} on a fixed plane T_x. A key step is then to show that $\|V\| \ll \mathcal{H}^d$: this is achieved by using ideas borrowed from [5] and [41]. Once this is obtained, a simple rectifiability criterion, based on the results in [76], allows to show that V is d-rectifiable. This result can have many applications and further developments. See for instance Chapter 5.

Content of Chapter 5

In Chapter 5 we present our paper [45], where we apply the main theorem of [42] to the minimization of anisotropic energies in classes of rectifiable varifolds. We prove that the limit of a minimizing sequence of varifolds with density uniformly bounded from below is rectifiable. Moreover, with the further assumption that the minimizing sequence is made of integral varifolds with uniformly locally bounded anisotropic first variation, we show that the limiting varifold is also integral. We remark that every sequence of integral varifolds enjoying a uniform bound on the mass and on the isotropic first variation is precompact in the space of integral varifolds. This has been proved by Allard in [2, Section 6.4]. One of the main theorems of our work [45] is indeed an anisotropic counterpart of the aforementioned compactness result, under the assumption that the limiting varifold has positive lower density. The idea is to blow-up every varifold of the converging sequence in a point in which the limiting varifold V has Grassmannian part supported on a single d-plane S (note that this property holds $\|V\|$-a.e. by [42]). Along a diagonal sequence we get that the projections on S converge in total variation to an L^1 function on S. This function is integer valued thanks to the integrality assumption on the sequence and coincides with the density of the limiting varifold in the blow-up point, which is consequently an integer. Since the argument holds true for $\|V\|$-a.e. point, the limiting varifold turns out to be integral.

1.2 Stability in branched transport

The transport problem consists in finding an optimal way to transport a measure μ^- to a measure μ^+. Sometimes one is interested to have a mathematical model of transport in which it is better to carry the mass in a grouped way rather than in a separate way. The first deep analysis of this natural phenomena is due to D'Arcy Thompson, in his work [29]. Recently, this approach has been used for branched networks, which are very common in nature: one may just think to the way plants and trees absorb solar energy, or to the way the oxygen is irrigated to the blood and to how this is distributed to the human body throw the ramified bronchial and cardiovascular systems. To translate this principle in mathematical terms, one can consider costs which are proportional to a power $\alpha \in (0,1)$ of the flow and use the concavity of $x \to x^\alpha$. Obviously the bigger is α and the less powerful is the grouping effect, and in particular, in the limit case $\alpha = 1$, there is absolutely no benefit to group mass.

Different formulations have been introduced to model the branched transport problem: one of the first proposal came by Gilbert in [53], who considered finite directed weighted graph G with straight edges $e \in E(G)$ connecting two discrete measures, and a weight function $w : E(G) \to (0,\infty)$. The cost of G is defined to be:

$$\sum_{e \in E(G)} w(e)^\alpha \mathcal{H}^1(e). \tag{1.1}$$

Later Xia has extended this model to a continuous framework using Radon vector measures (namely 1-dimensional currents), see [85].

In [64, 13], new objects called traffic plans have been introduced and studied. Roughly speaking, a traffic plan is a measure on the set of 1-Lipschitz paths, where each path represents the trajectory of a single particle. These formulations were proved to be equivalent (see [15] and references therein) and in particular the link is encoded in a deep result, due to Smirnov, on the structure of acyclic, normal 1-dimensional currents (see Theorem 7.9).

As usual for a minimization problem, one of the main tasks is the existence of a minimizer. In the ambient space $\mathbb{R}^n$, the optimal transports have been proven to have finite cost for all α strictly bigger than the critical exponent $1-\frac{1}{n}$. A natural immediately related question, of special relevance in view of numerical simulations, is whether the optima are stable with respect to variations of the initial and final distribution of mass. For $\alpha > 1-\frac{1}{n}$, the stability is already known in the literature to be true (see [15]).

1.2.1 *Personal contributions and content of Part II*

The content of the second part of the thesis is the following:

Content of Chapter 7

In joint work with Colombo and Marchese [27], we improve the stability of the optimal transports for the aforementioned concave costs to $\alpha > 1-\frac{1}{n-1}$, for a fairly general class of measures to be transported, positively answering to the open problem formulated in [15, Problem 15.1]. The proof is based on slicing techniques, which allow to bring the problem in the good regime for the exponent.

The result is remarkable since it fully answers, in particular, to the stability in $\mathbb{R}^2$ (since $1-\frac{1}{n-1}=0$). This is fairly satisfactory since many applications in engineering and numerical modeling are precisely in $\mathbb{R}^2$. We address the stability in the framework of 1-currents. The original definition of "cost" of a traffic path slightly differs from the α-mass defined in Chapter 7. Indeed in [85, Definition 3.1] the author defines the cost of a traffic path as the lower semi-continuous relaxation on the space of normal currents of the functional (1.1) defined on *polyhedral chains*. In [86, Section 3], the author notices that, in the class of rectifiable currents, his definition of cost coincides with the α-mass. The proof of this fact is only sketched in [82, Section 6] and we prove it in detail in [28], see Chapter 8. In Chapter 7, we don't need to rely on this fact, but we stick to the notion of cost given by the α-mass. We prove independently that the α-mass is lower semi-continuous.

Content of Chapter 8

In joint work with Colombo, Marchese and Stuvard [28], we prove an explicit formula for the lower semicontinuous envelope of some functionals defined on real polyhedral chains. More precisely, denoting by $H:\mathbb{R}\to[0,\infty)$ an even, subadditive, and lower semicontinuous function with $H(0)=0$, and by Φ_H the functional induced by H on polyhedral k-chains, we prove that the lower semicontinuous envelope of Φ_H coincides on rectifiable k-currents with the H-mass. The validity of such a representation has recently attracted some attention.

For instance, as already observed in the previous chapter, it is clearly assumed in [85] for the choice $H(x) = |x|^\alpha$, with $\alpha \in (0,1)$ (see also [73], [15], [75]) and in [25] in order to define suitable approximations of the Steiner problem, with the choice $H(x) = (1 + \beta|x|)\mathbf{1}_{\mathbb{R}\setminus\{0\}}$, where $\beta > 0$ and $\mathbf{1}_A$ denotes the indicator function of the Borel set A.

We finally remark that in the last section of [82] the author sketches a strategy to prove an analogous version of our result in the framework of flat chains with coefficients in a normed abelian group G. Motivated by the relevance of such theorem for real valued flat chains, the ultimate aim of our note [28] is to present a self-contained complete proof of it when $G = \mathbb{R}$.

Part I

THE ANISOTROPIC PLATEAU PROBLEM

2 NOTATION OF PART I

In this chapter we summarize the notation that will be used in Part I. We will always work in $\mathbb{R}^n$ and $1 \leqslant d \leqslant n$ will always be an integer number. For any subset $X \subseteq \mathbb{R}^n$, we denote $\overline{X}$ its closure, $\mathrm{Int}(X)$ its interior and $X^c := \mathbb{R}^n \setminus X$ its complementary set.

We are going to use the following notation: $Q_{x,l}$ denotes the closed cube centered in the point $x \in \mathbb{R}^n$, with edge length l and we set

$$R_{x,a,b} := x + \left[-\frac{a}{2}, \frac{a}{2}\right]^d \times \left[-\frac{b}{2}, \frac{b}{2}\right]^{n-d} \quad \text{and} \quad B_r(x) = B_{x,r} := \{y \in \mathbb{R}^n \;:\; |y-x| < r\}. \tag{2.1}$$

When cubes, rectangles and balls are centered in the origin, we will simply write Q_l, $R_{a,b}$ and B_r and in particular we will call $B := B_1$ the unitary ball. Cubes and balls in the subspace $\mathbb{R}^d \times \{0\}^{n-d}$ are denoted with $Q^d_{x,l}$ and $B^d_{x,r}$ respectively.

For a matrix $A \in \mathbb{R}^n \otimes \mathbb{R}^n$, A^* denotes its transpose. Given $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$ we define $A : B := \mathrm{tr}\, A^*B = \sum_{ij} A_{ij}B_{ij}$, so that $|A|^2 = A : A$.

2.0.1 *Measures and rectifiable sets*

Given a locally compact metric space Y, we denote by $\mathcal{M}_+(Y)$ the set of positive Radon measures in Y, namely the set of measure on the σ-algebra of Borel sets of Y that are locally finite and inner regular. In particular we consider the subset of Borel probability measures $\mathcal{P}(Y) \subset \mathcal{M}_+(Y)$, namely $\mu \in \mathcal{P}(Y)$ if $\mu \in \mathcal{M}_+(Y)$ and $\mu(Y) = 1$.

For a Borel set $E \subset Y$, $\mu \llcorner E$ is the restriction of μ to E, i.e. the measure defined by $[\mu \llcorner E](A) = \mu(E \cap A)$ for every Borel set $A \subset Y$.

Consider an open set $\Omega \subset \mathbb{R}^n$. For an $\mathbb{R}^m$-valued Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ we denote by $|\mu| \in \mathcal{M}_+(\Omega)$ its total variation and we recall that, for all open sets $U \subset \Omega$,

$$|\mu|(U) = \sup \left\{ \int \big\langle \varphi(x), d\mu(x) \big\rangle \;:\quad \varphi \in C^\infty_c(U, \mathbb{R}^m), \quad \|\varphi\|_\infty \leqslant 1 \right\}.$$

Eventually, we denote by $\mathcal{H}^d$ the d-dimensional Hausdorff measure and for a d-dimensional vector space $T \subset \mathbb{R}^n$ we will often identify $\mathcal{H}^d \llcorner T$ with the d-dimensional Lebesgue measure $\mathcal{L}^d$ on $T \approx \mathbb{R}^d$.

A set $K \subset \mathbb{R}^n$ is said to be $\mathcal{H}^d$-rectifiable or simply d-rectifiable if it can be covered, up to an $\mathcal{H}^d$-negligible set, by countably many C^1 d-dimensional submanifolds. In the following we will only consider $\mathcal{H}^d$-measurable sets. Given a d-rectifiable set K, we denote T_xK the approximate tangent space of K at x, which exists for $\mathcal{H}^d$-almost every point $x \in K$, [79, Chapter 3]. A positive Radon measure $\mu \in \mathcal{M}_+(\Omega)$ is said to be d-rectifiable if there exists a d-rectifiable set $K \subset \Omega$ such that $\mu = \theta \mathcal{H}^d \llcorner K$ for some Borel function $\theta : \mathbb{R}^n \to \mathbb{R}^+$.

For $\mu \in \mathcal{M}_+(\Omega)$ we consider its lower and upper d-dimensional densities at x:

$$\Theta^d_*(x, \mu) = \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_d r^d}, \qquad \Theta^{d*}(x, \mu) = \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_d r^d},$$

where $\omega_d = \mathcal{H}^d(B^d)$ is the measure of the d-dimensional unit ball in $\mathbb{R}^d$. In case these two limits are equal, we denote by $\Theta^d(x,\mu)$ their common value. Note that if $\mu = \theta\mathcal{H}^d \llcorner K$ with K rectifiable, then $\theta(x) = \Theta^d_*(x,\mu) = \Theta^{d*}(x,\mu)$ for μ-a.e. x, see [79, Chapter 3].

If $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a Borel map and μ is a Radon measure, we let $\eta_\# \mu = \mu \circ \eta^{-1}$ be the push-forward of μ through η. Let $\eta^{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ be the dilation map, $\eta^{x,r}(y) = (y-x)/r$. For a positive Radon measure $\mu \in \mathcal{M}_+(\Omega)$, $x \in \operatorname{spt}\mu \cap \Omega$ and $r \ll 1$, we define

$$\mu_{x,r} = \frac{1}{\mu(B_r(x))}(\eta^{x,r}_\# \mu) \llcorner B. \tag{2.2}$$

The normalization in (2.2) implies, by the Banach-Alaoglu Theorem, that for every sequence $r_i \to 0$ there exists a subsequence $r_{i_j} \to 0$ and a Radon measure $\sigma \in \mathcal{M}_+(B)$, called *tangent measure* to μ at x, such that

$$\mu_{x,r_{i_j}} \overset{*}{\rightharpoonup} \sigma.$$

We collect all tangent measures to μ at x into $\operatorname{Tan}(x,\mu) \subset \mathcal{M}_+(B)$.

2.0.2 *Varifolds and integrands*

We denote by $G(n,d)$ the Grassmannian of (un-oriented) d-dimensional linear subspaces in $\mathbb{R}^n$ (often referred to as d-planes) and given any set $E \subset \mathbb{R}^n$ we denote by $G(E) = E \times G(n,d)$ the Grassmannian bundle over E. We will often identify a d-dimensional plane $T \in G(n,d)$ with the matrix $T \in (\mathbb{R}^n \otimes \mathbb{R}^n)_{sym}$ representing the *orthogonal projection* onto T.

Consider an open set $\Omega \subset \mathbb{R}^n$. A d-varifold on Ω is a positive Radon measure V on $G(\Omega)$ and we will denote with $\mathbf{V}_d(\Omega)$ the set of all d-varifolds on Ω.

Given a diffeomorphism $\psi \in C^1(\Omega, \mathbb{R}^n)$, we define the push-forward of $V \in \mathbf{V}_d(\Omega)$ with respect to ψ as the varifold $\psi^\# V \in \mathbf{V}_d(\psi(\Omega))$ such that

$$\int_{G(\psi(\Omega))} \Phi(x,S)\, d(\psi^\# V)(x,S) = \int_{G(\Omega)} \Phi(\psi(x), d_x\psi(S)) J\psi(x,S)\, dV(x,S),$$

for every $\Phi \in C^0_c(G(\psi(\Omega)))$. Here $d_x\psi(S)$ is the image of S under the linear map $d_x\psi(x)$ and

$$J\psi(x,S) := \sqrt{\det\left(\left(d_x\psi\big|_S\right)^* \circ d_x\psi\big|_S \right)}$$

denotes the d-Jacobian determinant of the differential $d_x\psi$ restricted to the d-plane S, see [79, Chapter 8]. Note that the push-forward of a varifold V is *not* the same as the push-forward of the Radon measure V through a map ψ defined on $G(\Omega)$ (the latter being denoted with an expressly different notation: $\psi_\# V$).

To a varifold $V \in \mathbf{V}_d(\Omega)$, we associate the measure $\|V\| \in \mathcal{M}_+(\Omega)$ defined by

$$\|V\|(A) = V(G(A)) \qquad \text{for all } A \subset \Omega \text{ Borel.}$$

Hence $\|V\| = \pi_\# V$, where $\pi : \Omega \times G(n,d) \to \Omega$ is the projection onto the first factor and the push-forward is intended in the sense of Radon measures. By the disintegration theorem for measures, see for instance [12, Theorem 2.28], we can write

$$V(dx, dT) = \|V\|(dx) \otimes \mu_x(dT),$$

where $\mu_x \in \mathcal{P}(G(n,d))$ is a (measurable) family of parametrized non-negative measures on the Grassmannian such that $\mu_x(G(n,d)) = 1$.

A d-dimensional varifold $V \in \mathbf{V}_d(\Omega)$ is said d-rectifiable if there exist a d-rectifiable set K and a Borel function $\theta : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$V = \theta \mathcal{H}^d \llcorner (K \cap \Omega) \otimes \delta_{T_x K}. \tag{2.3}$$

We denote with $\mathbf{R}_d(\Omega) \subseteq \mathbf{V}_d(\Omega)$ the subset of the d-rectifiable varifolds.

Moreover we say that a d-rectifiable varifold V is integral, or equivalently $V \in \mathbf{I}_d(\Omega)$, if in the representation (2.3), the density function θ is also integer valued.

If $V \in \mathbf{R}_d(\Omega)$, we represent it as in (2.3) and we can extend the notion of push forward with respect to maps $\psi : \mathbb{R}^n \to \mathbb{R}^n$ which are merely Lipschitz as follows (see [79, Section 15]):

$$\psi^\# V := \tilde{\theta} \mathcal{H}^d \llcorner \psi(K) \otimes \delta_{T_x \psi(K)}, \qquad \text{where } \tilde{\theta}(x) := \int_{\psi^{-1}(x) \cap K} \theta \, d\mathcal{H}^0.$$

We remark that $\tilde{\theta}$ is defined for $\mathcal{H}^d$-a.e. point of $\psi(K)$ and we observe that the following equality holds (see [79, Section 15]):

$$\|\psi^\# V\|(A) = \int_{\psi(K) \cap A} \tilde{\theta} \, d\mathcal{H}^d = \int_{K \cap \psi^{-1}(A)} \theta J_K \psi \, d\mathcal{H}^d, \quad \forall \text{ Borel set } A \subseteq \mathbb{R}^n, \tag{2.4}$$

where $J_K \psi(y)$ denotes the Jacobian determinant of the tangential differential $d_K \psi_y : T_y K \to \mathbb{R}^n$, see [79, Sections 12 and 15].

We will use the notation

$$\Theta^d_*(x, V) = \Theta^d_*(x, \|V\|) \qquad \text{and} \qquad \Theta^{d*}(x, V) = \Theta^{d*}(x, \|V\|)$$

for the upper and lower d-dimensional densities of $\|V\|$. In case $\Theta^d_*(x, V) = \Theta^{d*}(x, V)$, we denote their common value $\Theta^d(x, V)$.

We will associate to any d-varifold V, its "at most d-dimensional" part V_* defined as

$$V_* := V \llcorner \{x \in \Omega : \Theta^d_*(x, V) > 0\} \times G(n, d). \tag{2.5}$$

Note that

$$\|V_*\| = \|V\| \llcorner \{x \in \Omega : \Theta^d_*(x, V) > 0\}$$

and thus, by the Lebesgue-Besicovitch differentiation Theorem [12, Theorem 2.22], for $\|V_*\|$ almost every point (or equivalently for $\|V\|$ almost every x with $\Theta^d_*(x, V) > 0$)

$$\lim_{r \to 0} \frac{\|V_*\|(B_r(x))}{\|V\|(B_r(x))} = 1. \tag{2.6}$$

In particular,

$$\Theta^d_*(x, V_*) > 0 \qquad \text{for } \|V_*\|\text{-a.e. } x. \tag{2.7}$$

We will call concentration set of $V \in \mathbf{R}_d(\Omega)$ the set

$$\operatorname{conc}(V) := \{x \in \Omega \; : \; \Theta^d_*(x, V) > 0\},$$

and we will equivalently say that V is concentrated on $\mathrm{conc}(V)$.

Let $\eta^{x,r}(y) = (y-x)/r$, as in (2.2) we define

$$V_{x,r} := \frac{r^d}{\|V\|(B_r(x))}\big((\eta^{x,r})^{\#}V\big) \llcorner G(B), \tag{2.8}$$

where the additional factor r^d is due to the presence of the d-Jacobian determinant of the differential $d\eta^{x,r}$ in the definition of push-forward of varifolds. Note that, with the notation of (2.2):

$$\|V_{x,r}\| = \|V\|_{x,r}.$$

The normalization in (2.8) implies, by the Banach-Alaoglu Theorem, that for every sequence $r_i \to 0$ there exists a subsequence $r_{i_j} \to 0$ and a varifold $V^\infty \in \mathbb{V}_d(B)$, called *tangent varifold* to V at x, such that

$$V_{x,r_{i_j}} \stackrel{*}{\rightharpoonup} V^\infty.$$

We collect all tangent varifold to V at x into $\mathrm{Tan}(x, V)$.

The anisotropic integrand that we consider is a C^1 function

$$F : G(\Omega) \longrightarrow (0, +\infty). \tag{2.9}$$

We assume the existence of two positive constants λ, Λ such that

$$0 < \lambda \leqslant F(x,T) \leqslant \Lambda < \infty \qquad \text{for all } (x,T) \in G(\Omega). \tag{2.10}$$

Given $x \in \Omega$, we will also consider the "frozen" integrand

$$F_x : G(n,d) \to (0,+\infty), \qquad F_x(T) := F(x,T). \tag{2.11}$$

Given a d-rectifiable set $K \subset \Omega$ and an open subset $U \subset \mathbb{R}^n$, we define:

$$\mathbf{F}(K,U) := \int_{K\cap U} F(x, T_xK)\, d\mathcal{H}^{n-1}(x) \quad \text{and} \quad \mathbf{F}(K) := \mathbf{F}(K, \mathbb{R}^n). \tag{2.12}$$

It will be also convenient to look at the frozen energy: for $y \in \Omega$, we let

$$\mathbf{F}^y(K,U) := \int_{K\cap U} F_y(T_xK)\, d\mathcal{H}^{n-1}(x).$$

Given a d-varifold $V \in \mathbf{V}_d(\Omega)$ and an open subset $U \subset \mathbb{R}^n$, we define its *anisotropic energy* as

$$\mathbf{F}(V,U) := \int_{G(\Omega\cap U)} F(x,T)\, dV(x,T) \quad \text{and} \quad \mathbf{F}(V) := \mathbf{F}(V, \mathbb{R}^n).$$

For a vector field $g \in C^1_c(\Omega, \mathbb{R}^n)$, we consider the family of functions $\varphi_t(x) = x + tg(x)$, and we note that they are diffeomorphisms of Ω into itself for t small enough. The *anisotropic first variation* of $V \in \mathbf{V}_d(\Omega)$ is defined as

$$\delta_F V(g) := \frac{d}{dt}\mathbf{F}\big(\varphi_t^{\#}V, \Omega\big)\Big|_{t=0}.$$

It can be easily shown, see Appendix 4.6, that

$$\delta_F V(g) = \int_{G(\Omega)} \Big[\langle d_x F(x,T), g(x)\rangle + B_F(x,T) : Dg(x)\Big] dV(x,T), \tag{2.13}$$

where the matrix $B_F(x,T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is uniquely defined by

$$\begin{aligned} B_F(x,T) : L &:= F(x,T)(T:L) + \big\langle d_T F(x,T),\, T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \big\rangle \\ &=: F(x,T)(T:L) + C_F(x,T) : L \qquad \text{for all } L \in \mathbb{R}^n \otimes \mathbb{R}^n. \end{aligned} \tag{2.14}$$

Note that, via the identification of a d-plane T with the orthogonal projection onto it, $G(n,d)$ can be thought as a subset of $\mathbb{R}^n \otimes \mathbb{R}^n$ and this gives the natural identification:

$$\mathrm{Tan}_T G(n,d) = \big\{S \in \mathbb{R}^n \otimes \mathbb{R}^n : S^* = S, \quad T \circ S \circ T = 0, \quad T^\perp \circ S \circ T^\perp = 0\big\},$$

see Appendix 4.6 for more details. We are going to use the following properties of $B_F(x,T)$ and $C_F(x,T)$, which immediately follow from (2.14):

$$|B_F(x,T) - B_F(x,S)| \leqslant C(d,n,\|F\|_{C^1})\big(|S-T| + \omega(|S-T|)\big), \tag{2.15}$$

$$C_F(x,T) : \nu \otimes w = 0 \quad \text{for all } \nu, w \in T, \tag{2.16}$$

where ω is the modulus of continuity of $T \mapsto d_T F(x,T)$ (i.e.: a concave increasing function with $\omega(0^+) = 0$). We also note that trivially

$$|\delta_F V(g)| \leqslant \|F\|_{C^1(\mathrm{spt} g)} \|g\|_{C^1} \|V\|(\mathrm{spt}(g)), \tag{2.17}$$

and that, if F_x is the frozen integrand (2.11), then (2.13) reduces to

$$\delta_{F_x} V(g) = \int_{G(\Omega)} B_F(x,T) : Dg(y)\, dV(y,T).$$

Moreover, if we define the Lagrangian $F_{x,r}(z,T) := F(x + rz, T)$, then

$$B_F(x + rz, T) = B_{F_{x,r}}(z,T) \qquad \text{for all } (z,T) \in G(\mathbb{R}^n). \tag{2.18}$$

We say that a varifold V has *locally bounded anisotropic first variation* if $\delta_F V$ is a Radon measure on Ω, i.e. if

$$|\delta_F V(g)| \leqslant C(K)\|g\|_\infty \quad \text{for all } g \in C^1_c(\Omega, \mathbb{R}^n) \text{ with } \mathrm{spt}(g) \subset K \subset\subset \Omega.$$

Furthermore, we will say that V is F-*stationary* if $\delta_F V = 0$.

3 EXISTENCE AND REGULARITY RESULTS FOR THE ANISOTROPIC PLATEAU PROBLEM

3.1 introduction

Plateau problem consists in looking for a surface of minimal area among those surfaces spanning a given boundary. A considerable amount of effort in Geometric Measure Theory during the last sixty years has been devoted to provide generalized concepts of surface, area and of "spanning a given boundary", in order to apply the direct methods of the calculus of variations to the Plateau problem. In particular we recall the notions of sets of finite perimeter [33, 34], of currents [51] and of varifolds [2, 4, 7], introduced respectively by De Giorgi, Federer, Fleming, Almgren and Allard. A more "geometric" approach was proposed by Reifenberg in [77], where Plateau problem was set as the minimization of Hausdorff d-dimensional measure among compact sets and the notion of spanning a given boundary was given in term of inclusions of homology groups.

Any of these approach has some drawbacks: in particular, not all the "reasonable" boundaries can be obtained by the above notions and not always the solutions are allowed to have the type of singularities observed in soap bubbles (see the Plateau laws mentioned in the Introduction 1.1). Recently in [56] Harrison and Pugh, see also [55], proposed a new notion of spanning a boundary, which seems to include reasonable physical boundaries and they have been able to show existence of least area surfaces spanning a given boundary.

In the recent paper [37], De Lellis, Maggi and Ghiraldin have introduced a more general framework to solve the Plateau problem, using a deep result of Preiss concerning rectifiable measures. Roughly speaking they showed, in the codimension one case, that every time one has a class which contains "enough" competitors (namely the cone and the cup competitors, see [37, Definition 1]) it is always possible to prove that the infimum of the Plateau problem is achieved by the area of a rectifiable set. They then applied this result to provide a new proof of Harrison and Pugh theorem as well as to show the existence of sliding minimizers, a new notion of minimal sets proposed by David in [31, 30] and inspired by Almgren's $(\mathbf{M}, 0, \infty)$, [8].

In the following sections 3.3,3.4 and 3.5, we extend the result [37] respectively to any codimension, to the anisotropic setting in codimension one and to the full generality of the anisotropic problem in any codimension. Moreover, we use our results to get easy solutions to the formulations introduced by Harrison and Pugh in [56], by David in [31, 30] and by Reifenberg in [77]. We present them respectively in the sections 3.6,3.7 and 3.8.

3.2 setting and preliminaries

In the entire Chapter 3, we will asume that the anisotropy introduced in (2.9) is defined on the whole $G(\mathbb{R}^n)$, i.e. $\Omega = \mathbb{R}^n$.

In order to precisely state the main results of Chapter 3, let us introduce some notations and definitions.

Definition 3.1 (Lipschitz deformations)**.** Given a ball $B_{x,r}$, we let $\mathfrak{D}(x,r)$ be the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(z) = z$ in $\mathbb{R}^n \setminus B_{x,r}$ and which are smoothly isotopic to the identity inside $B_{x,r}$, namely those for which there exists an isotopy $\lambda \in C^\infty([0,1] \times \mathbb{R}^n; \mathbb{R}^n)$ such that

$$\lambda(0,\cdot) = \mathrm{Id}, \quad \lambda(1,\cdot) = \varphi, \quad \lambda(t,h) = h \quad \forall\,(t,h) \in [0,1] \times (\mathbb{R}^n \setminus B_{x,r}) \quad \text{and}$$

$$\lambda(t,\cdot) \text{ is a diffeomorphism of } \mathbb{R}^n\ \forall t \in [0,1].$$

We finally set $D(x,r) := \overline{\mathfrak{D}(x,r)}^{C^0} \cap \mathrm{Lip}(\mathbb{R}^n)$, the intersection of the Lipschitz maps with the closure of $\mathfrak{D}(x,r)$ with respect to the uniform topology.

Observe that, in the definition of $D(x,r)$, it is equivalent to require any C^k regularity on the isotopy λ, for $k \geqslant 1$, as C^k isotopies supported in $B_{x,r}$ can be approximated in C^k by smooth ones also supported in the same set.

The following definition describes the properties required on comparison sets: the key property for K' to be a competitor of K is that K' is close in energy to sets obtained from K via deformation maps as in Definition 3.1. This allows a larger flexibility on the choice of the admissible sets, since a priori K' might not belong to the competition class.

Definition 3.2 (Deformed competitors and deformation class)**.** Let $K \subset \mathbb{R}^n \setminus H$ be relatively closed and $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$. A *deformed competitor* for K in $B_{x,r}$ is any set of the form

$$\varphi\,(K) \quad \text{where} \quad \varphi \in D(x,r).$$

A family $\mathscr{P}(\mathbf{F},H)$ of relatively closed d-rectifiable subsets $K \subset \mathbb{R}^n \setminus H$ is called a *deformation class* if for every $K \in \mathscr{P}(\mathbf{F},H)$, for every $x \in K$ and for a.e. $r \in (0, \mathrm{dist}(x,H))$

$$\inf\left\{\mathbf{F}(J) : J \in \mathscr{P}(\mathbf{F},H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\right\} \leqslant \mathbf{F}(L) \tag{3.1}$$

whenever L is any deformed competitor for K in $B_{x,r}$.

Once we fix a closed set H, we can formulate Plateau problem in the class $\mathscr{P}(\mathbf{F},H)$:

$$m_0 := \inf\left\{\mathbf{F}(K) : K \in \mathscr{P}(\mathbf{F},H)\right\}. \tag{3.2}$$

We will say that a sequence $(K_j) \subset \mathscr{P}(\mathbf{F},H)$ is a *minimizing sequence* if $\mathbf{F}(K_j) \downarrow m_0$.

3.3 ISOTROPIC PLATEAU PROBLEM IN HIGHER CODIMENSION

In this section we present our paper [44] in joint work with De Philippis and Ghiraldin, where we extend the isotropic result [37] to any codimension. More precisely we fix $F \equiv 1$ and we prove that the infimum m_0 in (3.2) for a deformation class is achieved by a compact set K which is, away from the "boundary", an analytic manifold outside a closed set of Hausdorff dimension at most $(d-1)$, see Theorem 3.3 below for the precise statement.

Although the general strategy of the proof is the same of [37], some non-trivial modifications have to be done in order to deal with sets of any co-dimension. In particular, with respect to [37], we use the notion of deformation class introduced in Definition 3.2, the main reason being the following: one of the key steps of the proof consists in showing a precise density lower bound for the measure obtained as limit of the sequence of Radon measures naturally associated to a minimizing sequence (K_j), see Steps 1 and 4 in the proof of Theorem 3.3. In order to obtain such a lower bound, instead of relying on relative isopermetric inequalities on the sphere as in [37] (which are peculiar of the codimension one case), we use the deformation theorem of David and Semmes in [32] to obtain suitable competitors, following a strategy already introduced by Federer and Fleming for rectifiable currents, see [51] and [8]. Moreover, since our class is essentially closed by Lipschitz deformations, we are actually able to prove that any set achieving the infimum is a stationary varifold and that, in addition, it is smooth outside a closed set of relative codimension one (this does not directly follows by Allard's regularity theorem, see Step 7 in the proof of Theorem 3.3). Simple examples show that this regularity is actually optimal.

The following theorem is the main result of Section 3.3 and establishes the behavior of minimizing sequences for the isotropic Plateau problem in any codimension. Notice that since $F \equiv 1$, just through this section we will denote $\mathscr{P}(H) := \mathscr{P}(\mathbf{F}, H)$. The Definifion 3.2 and the Plateau problem (3.2) are obviously understood with $\mathbf{F}(K) = \mathcal{H}^d(K)$.

Theorem 3.3. *Let $H \subset \mathbb{R}^n$ be closed and $\mathscr{P}(H)$ be a deformation class in the sense of Definition 3.2. Assume the infimum in Plateau problem (3.2) is finite and let $(K_j) \subset \mathscr{P}(H)$ be a minimizing sequence. Then, up to subsequences, the measures $\mu_j := \mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to a measure $\mu = \mathcal{H}^d \llcorner K$, where $K = \operatorname{spt} \mu$ is a countably d-rectifiable set. Furthermore:*

(a) the integral varifold naturally associated to μ is stationary in $\mathbb{R}^n \setminus H$;

(b) K is a real analytic submanifold outside a relatively closed set $\Sigma \subset K$ with $\dim_{\mathcal{H}}(\Sigma) \leqslant d-1$.

In particular, $\liminf_j \mathcal{H}^d(K_j) \geqslant \mathcal{H}^d(K)$ and if $K \in \mathscr{P}(H)$, then K is a minimum for (3.2).

3.3.1 *Preliminary results*

Let us recall the following deep structure result for Radon measures due to Preiss [76, 35], which will play a key role in the proof of Theorem 3.3.

Theorem 3.4. *Let d be an integer and μ a locally finite measure on $\mathbb{R}^n$ such that the d-density $\Theta(x,\mu)$ exists and satisfies $0 < \Theta(x,\mu) < +\infty$ for μ-a.e. x. Then $\mu = \Theta(\cdot,\mu)\mathcal{H}^d \llcorner K$, where K is a countably $\mathcal{H}^d$-rectifiable set.*

In order to apply Preiss' Theorem, we will rely on the monotonicity formula for minimal surfaces, which roughly speaking can be obtained by comparing the given minimizer with a cone. To this aim let us introduce the following definition:

Definition 3.5 (Cone competitors)**.** For a set $K \subset \mathbb{R}^n$ we define the cone competitor in $B_{x,r}$ as the following set

$$\mathbf{C}_{x,r}(K) = \big(K \setminus B_{x,r}\big) \cup \big\{\lambda x + (1-\lambda)z : z \in K \cap \partial B_{x,r}\,, \lambda \in [0,1]\big\}\,. \tag{3.3}$$

Let us note that in general a cone competitor in $B_{x,r}$ is not a deformed competitor in $B_{x,r}$. On the other hand, as in [37], we can show that:

Lemma 3.6. *Given a deformation class $\mathscr{P}(H)$ in the sense of Definition 3.2, for any $K \in \mathscr{P}(H)$ countably $\mathcal{H}^d$-rectifiable and for every $x \in K$, the set K verifies the following inequality for a.e. $r \in (0, \operatorname{dist}(x, H))$:*

$$\inf\left\{\mathcal{H}^d(J) : J \in \mathscr{P}(H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\right\} \leqslant \mathcal{H}^d(\mathbf{C}_{x,r}(K)).$$

Proof. Without loss of generality, let us consider balls B_r centered at 0 with $B_r \subset\subset \mathbb{R}^n \setminus H$. We assume in addition that $K \cap \partial B_r$ is $\mathcal{H}^{d-1}$-rectifiable with $\mathcal{H}^{d-1}(K \cap \partial B_r) < \infty$ and that r is a Lebesgue point of $t \in (0,\infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t)$. All these conditions are fulfilled for a.e. r and, again by scaling, we can assume that $r = 1$ and use B instead of B_1. For $s \in (0,1)$ let us set

$$\varphi_s(r) = \begin{cases} 0\,, & r \in [0, 1-s)\,,\\ \frac{r-(1-s)}{s}\,, & r \in [1-s, 1]\,,\\ r\,, & r \geqslant 1\,,\end{cases}$$

and $\phi_s(x) = \varphi_s(|x|)\frac{x}{|x|}$ for $x \in \mathbb{R}^n$. In this way, one easily checks that $\phi_s : \mathbb{R}^n \to \mathbb{R}^n \in D(0,1)$.

Since $\phi_s(K \cap \overline{B_{1-s}}) = \{0\}$, we need to show that

$$\limsup_{s\to 0^+} \mathcal{H}^d\big(\phi_s(K \cap (B \setminus B_{1-s}))\big) \leqslant \frac{\mathcal{H}^{d-1}(K \cap \partial B)}{d} = \mathcal{H}^d(\mathbf{C}_{x,r}(K))\,.$$

Let $x_0 \in K \cap \partial B_t$ and let us fix an orthonormal base $\nu_1, \dots, \nu_d$ of the approximate tangent space $T_{x_0}K$ such that $\nu_i \in T_{x_0}K \cap T_{x_0}\partial B_t$ for $i \leqslant d-1$. Let

$$J_d^K\phi_s = \Big|(\bigwedge^d D\phi_s)(T_{x_0}K)\Big| = |D\phi_s(\nu_1) \wedge \cdots \wedge D\phi_s(\nu_d)|$$

be the d-dimensional tangential Jacobian of ϕ_s with respect to K. A simple computation shows that

$$\begin{aligned} J_d^K\phi_s(x) &\leqslant \left(\frac{\varphi_s(|x|)}{|x|}\right)^d + |\nu_d \cdot \hat{x}|\, \varphi_s'(|x|) \left(\frac{\varphi_s(|x|)}{|x|}\right)^{d-1} \\ &\leqslant 1 + |\nu_d \cdot \hat{x}|\, \varphi_s'(|x|)\left(\frac{\varphi_s(|x|)}{|x|}\right)^{d-1}\,, \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in K\,. \end{aligned} \tag{3.4}$$

Here $\hat{x} = x/|x|$ and in the last inequality we have exploited that $\varphi(r) \leqslant r$ for $r \in [1-s, 1]$. Using that $|\nu_d \cdot \hat{x}|$ is the tangential co-area factor of the map $f(x) = |x|$, we find with the aid of the area and co-area formulas,

$$\begin{aligned} &\mathcal{H}^d\big(\phi_s(K \cap (B \setminus B_{1-s}))\big) = \int_{K\cap(B\setminus B_{1-s})} J_d^K\phi_s \, d\mathcal{H}^d \\ &= \int_{K\cap(B\setminus B_{1-s})\cap\{|\nu_d\cdot\hat{x}|\neq 0\}} J_d^K\phi_s \, d\mathcal{H}^d + \int_{K\cap(B\setminus B_{1-s})\cap\{|\nu_d\cdot\hat{x}|=0\}} J_d^K\phi_s \, d\mathcal{H}^d \\ &\leqslant \int_{1-s}^1 dt \int_{K\cap\partial B_t} \frac{J_d^K\phi_s}{|\nu_d \cdot \hat{x}|}\, d\mathcal{H}^{d-1} + \mathcal{H}^d\big(K \cap (B \setminus B_{1-s}) \cap \{|\nu_d \cdot \hat{x}| = 0\}\big), \end{aligned} \tag{3.5}$$

since $|J_d^K \phi_s| \leqslant 1$ where $|\nu_d \cdot \hat{x}| = 0$. Using

$$\lim_{s\to 0} \mathcal{H}^d\big(K \cap (B \setminus B_{1-s})\big) = 0\,,$$

the second term in (3.5) can be ignored. Moreover, being $t = 1$ a Lebesgue point of $t \in (0,\infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t)$, we have

$$\lim_{s\to 0} \frac{1}{s}\int_{1-s}^{1} |\mathcal{H}^{d-1}(K\cap \partial B_t) - \mathcal{H}^{d-1}(K \cap \partial B)|\, dt = 0\,.$$

Thanks to this and to the estimate (3.4), we infer from (3.5) that

$$\limsup_{s\to 0^+} \mathcal{H}^d(\varphi_s(K\cap B)) \leqslant \mathcal{H}^{d-1}(K\cap\partial B) \limsup_{s\to 0^+} \frac{1}{s}\int_{1-s}^{1}\left(\frac{\varphi_s(t)}{t}\right)^{d-1} dt = \frac{\mathcal{H}^{d-1}(K\cap \partial B)}{d}\,,$$

as required. □

Another key result we are going to use is a deformation theorem for closed sets due to David and Semmes [32, Proposition3.1], analogous to the one for rectifiable currents [79, 50]. We provide a slightly extended statement for the sake of forthcoming proofs.

Before stating the theorem, let us introduce some further notation. Given a closed cube $Q = Q_{x,l}$ and $\varepsilon > 0$, we cover Q with a grid of closed smaller cubes with edge length $\varepsilon \ll l$, with non empty intersection with $\mathrm{Int}(Q)$ and such that the decomposition is centered in x (i.e. one of the subcubes is centered in x). The family of this smaller cubes is denoted $\Lambda_\varepsilon(Q)$. We set

$$\begin{aligned} C_1 &:= \bigcup \{T \cap Q : T \in \Lambda_\varepsilon(Q), T \cap \partial Q \neq \emptyset\},\\ C_2 &:= \bigcup \{T \in \Lambda_\varepsilon(Q) : (T\cap Q) \not\subset C_1, T \cap \partial C_1 \neq \emptyset\},\\ Q^1 &:= \overline{Q \setminus (C_1 \cup C_2)} \end{aligned} \tag{3.6}$$

and consequently

$$\Lambda_\varepsilon(Q^1 \cup C_2) := \big\{T \in \Lambda_\varepsilon(Q) : T \subset (Q^1 \cup C_2)\big\}\,.$$

For each nonnegative integer $m \leqslant n$, let $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ denote the collection of all m-dimensional faces of cubes in $\Lambda_\varepsilon(Q^1 \cup C_2)$ and $\Lambda^*_{\varepsilon,m}(Q^1 \cup C_2)$ will be the set of the elements of $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ which are not contained in $\partial(Q^1 \cup C_2)$. We also let $S_{\varepsilon,m}(Q^1 \cup C_2) := \bigcup \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ be the m-skeleton of order ε in $Q^1 \cup C_2$.

Theorem 3.7. *Let $r > 0$ and E be a compact subset of Q such that $\mathcal{H}^d(E) < +\infty$ and $Q \subset B_{x_0,r}$. There exists a map $\Phi_{\varepsilon,E} \in D(x_0, r)$ satisfying the following properties:*

(1) $\Phi_{\varepsilon,E}(x) = x$ *for* $x \in \mathbb{R}^n \setminus (Q^1 \cup C_2)$*;*

(2) $\Phi_{\varepsilon,E}(x) = x$ *for* $x \in S_{\varepsilon,d-1}(Q^1 \cup C_2)$*;*

(3) $\Phi_{\varepsilon,E}(E \cap (Q^1 \cup C_2)) \subset S_{\varepsilon,d}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2)$*;*

(4) $\Phi_{\varepsilon,E}(T) \subset T$ *for every* $T \in \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$*, with* $m = d, ..., n$*;*

(5) either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = 0$ *or* $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = \mathcal{H}^d(T)$, *for every* $T \in \Lambda^*_{\varepsilon,d}(Q^1)$*;*

(6) $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) \leqslant k_1 \mathcal{H}^d(E \cap T)$ *for every* $T \in \Lambda_\varepsilon(Q^1 \cup C_2)$*;*

where k_1 *depends only on* n *and* d *(but not on* ε*).*

Proof. Proposition 3.1 in [32] provides a map $\widetilde{\Phi_{\varepsilon,E}} \in D(x_0, r)$ satisfying properties (1)-(4) and (6). We want to set

$$\Phi_{\varepsilon,E} := \Psi \circ \widetilde{\Phi_{\varepsilon,E}},$$

where Ψ will be defined below. We first define Ψ on every $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ distinguishing two cases

(a) if either $\mathcal{H}^d(\widetilde{\Phi_{\varepsilon,E}}(E) \cap T) = 0$ or $\mathcal{H}^d(\widetilde{\Phi_{\varepsilon,E}}(E) \cap T) = \mathcal{H}^d(T)$ or $T \notin \Lambda^*_{\varepsilon,d}(Q^1)$, then we set $\Psi_{|T} = \mathrm{Id}$;

(b) otherwise, since $\widetilde{\Phi_{\varepsilon,E}}(E)$ is compact, there exists $y_T \in T$ and $\delta_T > 0$ such that $B_{\delta_T}(y_T) \cap \widetilde{\Phi_{\varepsilon,E}}(E) = \emptyset$; we define

$$\Psi_{|T}(x) = x + \alpha(x - y_T) \min\left\{1, \frac{|x - y_T|}{\delta_T}\right\},$$

where $\alpha > 0$ such that the point $x + \alpha(x - y_T) \in (\partial T) \times \{0\}^{n-d}$.

The second step is to define Ψ on every $T' \in \Lambda_{\varepsilon,d+1}(Q^1 \cup C_2)$. Without loss of generality, we can assume T' centered in 0. We divide T' in pyramids $P_{T,T'}$ with base $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ and vertex 0. Assuming $T \subset \{x_{d+1} = -\frac{\varepsilon}{2}, x_{d+2}, ..., x_n = 0\}$ and $T' \subset \{x_{d+2}, ..., x_n = 0\}$, we set

$$\Psi_{|P_{T,T'}}(x) = -\frac{2x_{d+1}}{\varepsilon}\Psi_{|T}\left(-\frac{x}{x_{d+1}}\frac{\varepsilon}{2}\right).$$

We iterate this procedure on all the dimensions till to n, defining it well in $Q^1 \cup C_2$. Since $\Psi_{|\partial(Q^1 \cup C_2)} = \mathrm{Id}$, we can extend the map as the identity outside $Q^1 \cup C_2$. In addition, one can easily check that $\Psi \in D(x_0, r)$ and thus, since $\widetilde{\Phi_{\varepsilon,E}} \in D(x_0, r)$ and the class $D(x_0, r)$ is closed by composition, this concludes the proof. □

Later we will need to implement the above deformation of a set E on a rectangle rather than a cube. The deformation theorem can be proved for very general cubical complexes, [6]; however, for the sake of exposition, we limit ourselves to the simple case of a rectangular complex, which can be deduced by Theorem 3.7 through a bi-Lipschitz (linear) transformation of $\mathbb{R}^n$. More precisely, let us consider a closed rectangle

$$R := [0, \ell_1] \times \cdots \times [0, \ell_n] \qquad \ell_1 \leqslant \cdots \leqslant \ell_n$$

and a tiling of $\mathbb{R}^n$ made of rectangle ε-homothetic to R. Let $\Lambda^R_\varepsilon(R)$ denote the family of the translated and ε scaled copies of R and let us set

$$C^R_1 := \bigcup \left\{T \cap R : T \in \Lambda^R_\varepsilon(R), T \cap \partial R \neq \emptyset\right\},$$
$$C^R_2 := \bigcup \left\{T \in \Lambda^R_\varepsilon(R) : (T \cap R) \not\subset C^R_1, T \cap \partial C^R_1 \neq \emptyset\right\},$$
$$R^1 := \overline{R \setminus (C^R_1 \cup C^R_2)}.$$

As before, for each nonnegative integer $m \leqslant n$, we let $\Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ denote the collection of all m-dimensional faces of rectangles in $\Lambda^R_\varepsilon(R^1 \cup C^R_2)$ and $\Lambda^{R*}_{\varepsilon,m}(R^1 \cup C^R_2)$ will be the set of the elements of $\Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ which are not contained in $\partial(R^1 \cup C^R_2)$. We also let $S^R_{\varepsilon,m}(R^1 \cup C^R_2) := \bigcup \Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ be the m-skeleton of order ε in $R^1 \cup C^R_2$. Then the following theorem is an immediate consequence of Theorem 3.7:

Theorem 3.8. *Let $r > 0$ and E be a compact subset of R such that $\mathcal{H}^d(E) < +\infty$ and $R \subset B_{x_0,r}$. There exists a map $\Phi_{\varepsilon,E} \in D(x_0, r)$ satisfying the following properties:*

(1) $\Phi_{\varepsilon,E}(x) = x$ *for* $x \in \mathbb{R}^n \setminus (R^1 \cup C^R_2)$;

(2) $\Phi_{\varepsilon,E}(x) = x$ *for* $x \in S^R_{\varepsilon,d-1}(R^1 \cup C^R_2)$;

(3) $\Phi_{\varepsilon,E}(E \cap (R^1 \cup C^R_2)) \subset S^R_{\varepsilon,d}(R^1 \cup C^R_2) \cup \partial(R^1 \cup C^R_2)$;

(4) $\Phi_{\varepsilon,E}(T) \subset T$ *for every* $T \in \Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$, *with* $m = d, ..., n$;

(5) either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = 0$ *or* $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = \mathcal{H}^d(T)$, *for every* $T \in \Lambda^{R*}_{\varepsilon,d}(R^1)$;

(6) $\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) \leqslant k_1 \mathcal{H}^d(E \cap T)$ *for every* $T \in \Lambda^R_\varepsilon(R^1 \cup C^R_2)$;

where k_1 depends only on n, d and ℓ_n/ℓ_1 (but not on ε).

Note that this time the constant k_1 depends also from the ratio ℓ_n/ℓ_1. In the sequel we will apply this construction only to rectangles where this ratio is between 1 and 4: $1 \leqslant \ell_n/\ell_1 \leqslant 4$, thus obtaining a constant k_1 actually depending just on n and d.

3.3.2 *Proof of Theorem 3.3*

Proof of Theorem 3.3. Up to extracting subsequences we can assume the existence of a Radon measure μ on $\mathbb{R}^n \setminus H$ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu, \qquad \text{as Radon measures on } \mathbb{R}^n \setminus H, \tag{3.7}$$

where $\mu_j = \mathcal{H}^d \llcorner K_j$. We set $K = \operatorname{spt} \mu$ and we remark that $K \cap H = \emptyset$, since $\mu \in \mathcal{M}_+(\mathbb{R}^n \setminus H)$. We divide the argument in several steps.

Step one: We show the existence of $\theta_0 = \theta_0(n, d) > 0$ such that

$$\mu(B_{x,r}) \geqslant \theta_0\, \omega_d r^d, \qquad \forall x \in K \text{ and } r < d_x := \operatorname{dist}(x, H). \tag{3.8}$$

To this end, it is sufficient to prove the existence of $\beta = \beta(n, d) > 0$ such that

$$\mu(Q_{x,l}) \geqslant \beta\, l^d, \qquad x \in K \text{ and } l < 2d_x/\sqrt{n}.$$

Let us assume by contradiction that there exist $x \in \operatorname{spt} \mu$ and $l < 2d_x/\sqrt{n}$ such that

$$\frac{\mu(Q_{x,l})^{\frac{1}{d}}}{l} < \beta.$$

We claim that this assumption, for β chosen sufficiently small depending only on d and n, implies that for some $l_\infty \in (0, l)$

$$\mu(Q_{x,l_\infty}) = 0, \tag{3.9}$$

which is a contradiction with the property of x to be a point of $\operatorname{spt}\mu$. In order to prove (3.9), we assume that $\mu(\partial Q_{x,l}) = 0$, which is true for a.e. l.

To prove (3.9), we construct a sequence of nested cubes $Q_i = Q_{x,l_i}$ such that, if β is sufficiently small, the following holds:

(i) $Q_0 = Q_{x,l}$;

(ii) $\mu(\partial Q_{x,l_i}) = 0$;

(iii) setting $m_i := \mu(Q_i)$ then:
$$\frac{m_i^{\frac{1}{d}}}{l_i} < \beta;$$

(iv) $m_{i+1} \leqslant (1 - \frac{1}{k_1})m_i$, where k_1 is the constant in Theorem 3.7 (6);

(v) $(1 - 4\varepsilon_i)l_i \geqslant l_{i+1} \geqslant (1 - 6\varepsilon_i)l_i$, where
$$\varepsilon_i := \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} \tag{3.10}$$
and $k = \max\{6, 6/(1 - (\frac{k_1 - 1}{k_1})^{\frac{1}{d}})\}$ is a universal constant;

(vi) $\lim_i m_i = 0$ and $\lim_i l_i > 0$.

Following [32], we are going to construct the sequence of cubes by induction: the cube Q_0 satisfies by construction hypotheses (i)-(iii). Suppose that cubes until step i are already defined.

Setting $m_i^j := \mathcal{H}^d(K_j \cap Q_i)$, we cover Q_i with the family $\Lambda_{\varepsilon_i l_i}(Q_i)$ of closed cubes with edge length $\varepsilon_i l_i$ as described in Section 3.3.1 and we set C_1^i and C_2^i for the corresponding sets defined in (3.6). We define Q_{i+1} to be the internal cube given by the construction, and we note that C_2^i and Q_{i+1} are non-empty if, for instance,

$$\varepsilon_i = \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} < \frac{1}{k} \leqslant \frac{1}{6},$$

which is guaranteed by our choice of k. Observe moreover that $C_1^i \cup C_2^i$ is a strip of width at most $2\varepsilon_i l_i$ around ∂Q_i, hence the side l_{i+1} of Q_{i+1} satisfies $(1 - 4\varepsilon_i)l_i \leqslant l_{i+1} < (1 - 2\varepsilon_i)l_i$.

Now we apply Theorem 3.7 to Q_i with $E = K_j$ and $\varepsilon = \varepsilon_i l_i$, obtaining the map $\Phi_{i,j} = \Phi_{\varepsilon_i l_i, K_j}$. We claim that, for every j sufficiently large,

$$m_i^j \leqslant k_1(m_i^j - m_{i+1}^j) + o_j(1). \tag{3.11}$$

Indeed, since (K_j) is a minimizing sequence, by the definition of deformation class we have that

$$\begin{aligned} m_i^j &\leqslant m_i + o_j(1) \leqslant \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_i\right)\right) + o_j(1) \\ &= \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) + \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap (C_1^i \cup C_2^i)\right)\right) + o_j(1) \\ &\leqslant k_1 \mathcal{H}^d\left(K_j \cap (C_1^i \cup C_2^i)\right) + o_j(1) = k_1(m_i^j - m_{i+1}^j) + o_j(1). \end{aligned}$$

The last inequality holds because $\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) = 0$ for j large enough: otherwise, by property (5) of Theorem 3.7, there would exist $T \in \Lambda^*_{\varepsilon_i l_i, d}(Q_{i+1})$ such that $\mathcal{H}^d(\Phi_{i,j}(K_j \cap T)) = \mathcal{H}^d(T)$. Together with property (ii), this would imply

$$l_i^d \varepsilon_i^d = \mathcal{H}^d(T) \leqslant \mathcal{H}^d\left(\Phi_{i,j}\left(K_j\right) \cap Q_i\right) \leqslant k_1 \mathcal{H}^d\left(K_j \cap Q_i\right) \leqslant k_1 m_i^j \to k_1 m_i$$

and therefore, substituting (3.10),

$$\frac{m_i}{k^d \beta^d} \leqslant k_1 m_i,$$

which is false if β is sufficiently small ($m_i > 0$ because $x \in \mathrm{spt}(\mu)$). Passing to the limit in j in (3.11) we obtain (iv):

$$m_{i+1} \leqslant \frac{k_1 - 1}{k_1} m_i. \tag{3.12}$$

Since $l_{i+1} \geqslant (1 - 4\varepsilon_i) l_i$, we can slightly shrink the cube Q_{i+1} to a concentric cube Q'_{i+1} with $l'_{i+1} \geqslant (1 - 6\varepsilon_i) l_i > 0$, $\mu(\partial Q'_{i+1}) = 0$ and for which (iv) still holds, just getting a lower value for m_{i+1}. With a slight abuse of notation, we rename this last cube Q'_{i+1} as Q_{i+1}.

We now show (iii). Using (3.12) and condition (iii) for Q_i, we obtain

$$\frac{m_{i+1}^{\frac{1}{d}}}{l_{i+1}} \leqslant \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \frac{m_i^{\frac{1}{d}}}{(1 - 6\varepsilon_i) l_i} < \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \frac{\beta}{1 - 6\varepsilon_i}.$$

The last quantity will be less than β if

$$\left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \leqslant 1 - 6\varepsilon_i = 1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}. \tag{3.13}$$

In turn, inequality (3.13) is true because (iii) holds for Q_i, provided we choose $k \geqslant 6/\big(1 - (1 - 1/k_1)^{\frac{1}{d}}\big)$. Furthermore, estimating $\varepsilon_0 < 1/k$ by (iii) and (v), we also have $\varepsilon_{i+1} \leqslant \varepsilon_i$.

We are left to prove (vi): $\lim_i m_i = 0$ follows directly from (iv); regarding the non degeneracy of the cubes, note that

$$\begin{aligned} \frac{l_\infty}{l_0} := \liminf_i \frac{l_i}{l_0} &\geqslant \prod_{i=0}^{\infty} (1 - 6\varepsilon_i) = \prod_{i=0}^{\infty} \left(1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}\right) \\ &\geqslant \prod_{i=0}^{\infty} \left(1 - \frac{6 m_0^{\frac{1}{d}}}{k\beta l_0 \prod_{h=0}^{i-1}(1 - 6\varepsilon_h)} \left(\frac{k_1 - 1}{k_1}\right)^{\frac{i}{d}}\right) \\ &\geqslant \prod_{i=0}^{\infty} \left(1 - \frac{6}{k(1 - 6\varepsilon_0)^i} \left(\frac{k_1 - 1}{k_1}\right)^{\frac{i}{d}}\right), \end{aligned}$$

where we used $\varepsilon_h \leqslant \varepsilon_0$ in the last inequality. Since $\varepsilon_0 < 1/k$, the last product is strictly positive, provided

$$k > \frac{6}{1 - \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}}},$$

which is guaranteed by our choice of k. We conclude that $l_\infty > 0$, which ensures claim (3.9).

Step two: We fix $x \in K$, and prove that

$$r \mapsto \frac{\mu(B_{x,r})}{r^d} \quad \text{is increasing on } (0, d_x). \tag{3.14}$$

The proof amounts to prove a differential inequality for the function $f(r) := \mu(B_{x,r})$. In turn, this inequality is obtained in a two step approximation: first one exploits the rectifiability of the minimizing sequence (K_j) and property (3.1) to compare K_j with the cone competitor $\mathbf{C}_{x,r}(K_j)$, see (3.3). The comparison, a priori, is only allowed with elements of $\mathscr{P}(H)$, so for almost every $r < d_x$ the following holds:

$$\begin{aligned} f_j(r) &= \mathcal{H}^d(K_j) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \leqslant m_0 + o_j(1) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \\ &\leqslant o_j(1) + \inf_{K' \in \mathscr{P}(H)} \mathcal{H}^d(K') - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \leqslant o_j(1) + \inf_{\substack{K' \in \mathscr{P}(H) \\ K' \setminus \overline{B_{x,r}} = K_j \setminus \overline{B_{x,r}}}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}), \end{aligned}$$

where $f_j(r) := \mathcal{H}^d(K_j \cap B_{x,r})$. Nevertheless, K_j can be compared with its cone competitor, up to an error infinitesimal in j, thanks to Lemma 3.6. We recover

$$\begin{aligned} f_j(r) &\leqslant \inf_{\substack{K' \in \mathscr{P}(H) \\ K' \setminus \overline{B_{x,r}} = K_j \setminus \overline{B_{x,r}}}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}) \leqslant o_j(1) + \mathcal{H}^d(\mathbf{C}_{x,r}(K_j) \cap \overline{B_{x,r}}) \\ &\leqslant o_j(1) + \frac{r}{d}\mathcal{H}^{d-1}(K_j \cap \partial B_{x,r}) = o_j(1) + \frac{r}{d} f_j'(r)\,. \end{aligned} \tag{3.15}$$

We want to pass to the limit in j in order to obtain the desired monotonicity formula. To this aim, we observe that by Fatou's lemma, if we set $g(t) := \liminf_j f_j'(t)$, then

$$f(r) - f(s) = \mu(B_{x,r} \setminus B_{x,s}) \geqslant \int_s^r g(t)\, dt\,, \qquad \text{provided } \mu(\partial B_{x,r}) = \mu(\partial B_{x,s}) = 0\,.$$

This shows that $Df \geqslant g\mathcal{L}^1$. On the other hand, using the differentiability a.e. of f and letting $s \uparrow r$, we also conclude $f' \geqslant g$ $\mathcal{L}^1$-a.e., whereas $Df \geqslant f'\mathcal{L}^1$ is a simple consequence of the fact that f is an increasing function.

We can now pass to the limit on j in (3.15) to yeld $f(r) \leqslant \frac{r}{d} g(r) \leqslant \frac{r}{d} f'(r)$ for a.e. $r < d_x$. The positivity of the measure $D\log(f)$ implies the claimed monotonicity formula.

Step three: By (3.8) and (3.14), the d-dimensional density of the measure μ, namely:

$$\theta(x) = \lim_{r \to 0^+} \frac{f(r)}{\omega_d r^d} \geqslant \theta_0\,,$$

exists, is finite and positive μ-almost everywhere. Preiss' Theorem 3.4 implies that $\mu = \theta \mathcal{H}^d \llcorner \tilde{K}$ for some countably $\mathcal{H}^d$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since K is the support of μ, then $\mathcal{H}^d(\tilde{K} \setminus K) = 0$. On the other hand, by differentiation of Hausdorff measures, (3.8) yields $\mathcal{H}^d(K \setminus \tilde{K}) = 0$. Hence K is d-rectifiable and $\mu = \theta \mathcal{H}^d \llcorner K$.

Step four: We prove that $\theta(x) \geqslant 1$ for every $x \in K$ such that the approximate tangent space to K exists (thus, $\mathcal{H}^d$-a.e. on K). For further use (see step 7 below) we actually prove a slightly more general results: $\theta(x) \geqslant 1$ for every $x \in K \setminus H$ such that there exists a sequence $r_k \downarrow 0$ for which

$$\frac{\mu_{x,r_k}}{r_k^d} \overset{*}{\rightharpoonup} \theta(x)\mathcal{H}^d \llcorner \pi, \qquad \text{as } k \to +\infty \tag{3.16}$$

where π is a d-dimensional plane. Here the measures $\mu_{x,r}$ are defined as $\mu_{x,r}(A) = \mu(x + rA)$ for every Borel set A, rather than the definition given in (2.2).

Let us assume without loss of generality that $x = 0$ and $\pi = \{x_{d+1} = ... = x_n = 0\}$. Note that $\mu_{x,r}$ are supported on $(K - x)/r$ and that (3.16) and the lower density estimates (3.8) imply that the support of μ_{x,r_k} has to converge in the Kuratowski sense to the support of $\mathcal{H}^d \llcorner \pi$. In particular, for every $\varepsilon > 0$, there are infinitely many small $\rho > 0$ such that

$$K \cap B_\rho \subset \left\{ y \in \mathbb{R}^n : |y_{d+1}|, ..., |y_n| < \frac{\varepsilon}{100}\rho \right\}. \tag{3.17}$$

Let us now assume, by contradiction, that $\theta(0) < 1$. Thanks to (3.14) and (3.17) we can slightly tilt ρ to find $r > 0$ and $\alpha < 1$ such that $\mu(\partial Q_r) = 0$ and

$$\frac{\mu(Q_r)}{r^d} \leqslant \alpha < 1, \quad K \cap (Q_r \setminus R_{r,\varepsilon r}) = \emptyset, \tag{3.18}$$

where $R_{r,\varepsilon r}$ is defined as in (2.1). In particular, since μ_j are weakly converging to μ, we get that for $j \geqslant j(r)$

$$\frac{\mu_j(Q_r)}{r^d} \leqslant \alpha < 1 \quad \text{and} \quad \mu_j(Q_r \setminus R_{r,\varepsilon r}) = o_j(1), \tag{3.19}$$

We now wish to clear the small amount of mass appearing in the complement of $R_{r,\varepsilon r}$: we achieve this by repeatedly applying Theorem 3.8. We set $Q_r \cap \{x_{d+1} \geqslant \frac{\varepsilon}{2} r\} =: R$, and we apply Theorem 3.8 to this rectangle with $E = K_j^0 := K_j$, obtaining the map $\varphi_{1,j}$. We recall that the obtained constant k_1 for the area bound is universal, since it depends on the side ratio of R, which is bounded from below by 1 and from above by 4, provided ε small enough. We set $K_j^1 := \varphi_{1,j}(K_j^0)$ and repeat the argument with $Q_r \cap \{x_{d+1} \leqslant -\frac{\varepsilon}{2} r\} =: R$ and $E := K_j^1$, obtaining the map $\varphi_{2,j}$. We again set $K_j^2 := \varphi_{2,j}(K_j^1)$ and iterate this procedure to the rectangles $Q_r \cap \{x_{d+2} \geqslant \frac{\varepsilon}{2} r\}, ..., Q_r \cap \{x_n \leqslant -\frac{\varepsilon}{2} r\}$. After $2(n-d)$ iteration, we set

$$K_j^{2(n-d)} := \varphi_{2(n-d),j} \circ ... \circ \varphi_{1,j}(K_j).$$

We are going to use the cube $Q_{r(1-\sqrt{\varepsilon})}$ because, taking ε small enough, then $\sqrt{\varepsilon} > 4\overline{C}\varepsilon$, where $\overline{C} > 1$ is the side ratio considered before. This allows us to claim that

$$\mathcal{H}^d(K_j^{2(n-d)} \cap (Q_{r(1-\sqrt{\varepsilon})} \setminus R_{r(1-\sqrt{\varepsilon}),6\varepsilon r})) = 0. \tag{3.20}$$

Otherwise there would exist a d-face of a smaller rectangle $T \subset (Q_r \setminus R_{r,\varepsilon r})$ such that

$$\mathcal{H}^d(K_j^{2(n-d)} \cap T) = \mathcal{H}^d(T) \geqslant \varepsilon^d r^d ,$$

which would lead to the following contradiction for j large:

$$\varepsilon^d r^d \leqslant \mathcal{H}^d(T) \leqslant \mathcal{H}^d\left(K_j^{2(n-d)} \cap (Q_r \setminus R_{r,\varepsilon r})\right) \leqslant k_1^{2(n-d)}\mathcal{H}^d\left(K_j \cap (Q_r \setminus R_{r,\varepsilon r})\right) = o_j(1).$$

In particular, we cleared any measure on every slab

$$\bigcup_{i=d+1}^{n} \left\{3\varepsilon r < |x_i| < (1-\sqrt{\varepsilon})\frac{r}{2}\right\} \cap Q_{r(1-\sqrt{\varepsilon})}.$$

We want now to construct a map $P \in D(0,r)$, collapsing $R_{r(1-\sqrt{\varepsilon}),6\varepsilon r}$ onto the tangent plane. To this end, for $x \in \mathbb{R}^n$, $x = (x', x'')$ with $x' \in \mathbb{R}^d$ and $x'' \in \mathbb{R}^{n-d}$, we set

$$\|x'\| := \max\{|x_i| : i = 1,\dots,d\} \qquad \|x''\| := \max\{|x_i| : i = d+1,\dots,n\} \tag{3.21}$$

and we define P as follows:

$$P(x) = \begin{cases} \left(x', g(\|x'\|)\frac{(\|x''\|-3\varepsilon r)_+}{1-6\varepsilon}\frac{x''}{\|x''\|} + (1-g(\|x'\|))x''\right) & \text{if } \max\{\|x'\|, \|x''\|\} \leqslant r/2 \\ \mathrm{Id} & \text{otherwise,} \end{cases} \tag{3.22}$$

where $g : [0, r/2] \to [0, 1]$ is a compactly supported cut off function such that

$$g \equiv 1 \quad \text{on } [0, r(1-\sqrt{\varepsilon})/2] \qquad \text{and} \qquad |g'| \leqslant 10/r\sqrt{\varepsilon}\,.$$

It is not difficult to check that $P \in D(0,r)$ and that $\operatorname{Lip} P \leqslant 1 + C\sqrt{\varepsilon}$, for some dimensional constant C.

We now set $\widetilde{K_j} := P(K_j^{2(n-d)})$, which verifies, thanks to (3.20),

$$\mathcal{H}^d\left(\widetilde{K_j} \cap \left(Q_{(1-\sqrt{\varepsilon})r} \setminus Q^d_{(1-\sqrt{\varepsilon})r}\right)\right) = 0 \tag{3.23}$$

and

$$\begin{aligned} &\mathcal{H}^d\big(\widetilde{K_j} \cap \big(Q_r \setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big) \leqslant (1+C\sqrt{\varepsilon})^d \mathcal{H}^d\big(K_j^{2(n-d)} \cap \big(Q_r \setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big) \\ &\leqslant (1+C\sqrt{\varepsilon})k_1^{2(n-d)}\mathcal{H}^d\big(K_j \cap \big(Q_r \setminus (Q_{r(1-\sqrt{\varepsilon})} \cup R_{r,\varepsilon r})\big)\big) \\ &\quad + (1+C\sqrt{\varepsilon})\mathcal{H}^d\big(K_j \cap \big(R_{r,\varepsilon r} \setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big) \\ &\leqslant o_j(1) + (1+C\sqrt{\varepsilon})\mathcal{H}^d\big(K_j \cap \big(R_{r,\varepsilon r} \setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big)\,, \end{aligned} \tag{3.24}$$

where in the last inequality we have used (3.19). Moreover, by using (3.18), (3.19) and (3.23), we also have that, for ε small and j large:

$$\begin{aligned} \frac{\mathcal{H}^d(\widetilde{K_j} \cap Q^d_{r(1-\sqrt{\varepsilon})})}{r^d(1-\sqrt{\varepsilon})^d} = \frac{\mathcal{H}^d(\widetilde{K_j} \cap Q_{r(1-\sqrt{\varepsilon})})}{r^d(1-\sqrt{\varepsilon})^d} &\leqslant (1+C\sqrt{\varepsilon})\frac{\mathcal{H}^d(K_j^{2(n-d)} \cap Q_r)}{r^d} \\ &\leqslant (1+C\sqrt{\varepsilon})\frac{\mathcal{H}^d(K_j \cap Q_r) + o_j(1)}{r^d} \\ &\leqslant \alpha + o_j(1) < 1. \end{aligned} \tag{3.25}$$

As a consequence of (3.25) and the compactness of $\widetilde{K_j}$, there exist $y_j' \in Q^d_{(1-\sqrt{\varepsilon})r}$ and $\delta_j > 0$ such that, if we set $y_j := (y_j', 0)$, then

$$\widetilde{K_j} \cap B^d_{y_j,\delta_j} = \emptyset \quad \text{and} \quad B^d_{y_j,\delta_j} \subset Q^d_{(1-\sqrt{\varepsilon})r}. \tag{3.26}$$

After the last deformation, our set $\widetilde{K_j} \cap Q_{r(1-\sqrt{\varepsilon})}$ is contained in the tangent plane and we want to use the property (3.26) to collapse $\widetilde{K_j} \cap Q_{r(1-\sqrt{\varepsilon})}$ into $\left(\partial Q^d_{(1-\sqrt{\varepsilon})r}\right) \times \{0\}^{n-d}$. To this end, for every $j \in \mathbb{N}$ let us define the following Lipschitz map:

$$\varphi_j(x) = \begin{cases} \left(x' + z'_{j,x}, x''\right) & \text{if } x \in R_{r(1-\sqrt{\varepsilon}),r} \\ x & \text{otherwise,} \end{cases}$$

with

$$z'_{j,x} := \min\left\{1, \frac{\left|x' - y_j'\right|}{\delta_j}\right\} \frac{(r - 4\|x''\|)_+}{r} \gamma_{j,x}(x' - y_j'),$$

where $\gamma_{j,x} > 0$ is such that $x' + \gamma_{j,x}(x' - y_j') \in \partial Q^d_{(1-\sqrt{\varepsilon})r} \times \{0\}^{n-d}$ and $\|x''\|$ is defined in (3.21). One can easily check that $\varphi_j \in D(0, r)$. Moreover, setting $\varphi_j(\widetilde{K_j}) =: K_j'$, we have that

$$K_j' \setminus Q_r = K_j \setminus Q_r$$

and

$$\mathcal{H}^d(K_j' \cap Q_{r(1-\sqrt{\varepsilon})}) = 0, \tag{3.27}$$

thanks to (3.23), since

$$\mathcal{H}^d\left(\partial Q^d_{(1-\sqrt{\varepsilon})r} \times \{0\}^{n-d}\right) = 0.$$

Since $\mathscr{P}(H)$ is a deformation class, by (3.1) there exists a sequence of competitors $(J_j)_{j\in\mathbb{N}} \subset \mathscr{P}(H)$ such that $J_j \setminus \overline{B}_{0,r} = K_j \setminus \overline{B}_{0,r}$ and $\mathcal{H}^d(J_j) = \mathcal{H}^d(K_j') + o_j(1)$. Hence, thanks to (3.24) and (3.27), we get

$$\begin{aligned}
\mathcal{H}^d(K_j) - \mathcal{H}^d(J_j) &\geqslant \mathcal{H}^d(K_j) - \mathcal{H}^d(K_j') - o_j(1) = \mathcal{H}^d(K_j \cap Q_r) - \mathcal{H}^d(K_j' \cap Q_r) - o_j(1) \\
&\geqslant \mathcal{H}^d\left(K_j \cap Q_{r(1-\sqrt{\varepsilon})}\right) + \mathcal{H}^d\left(K_j \cap (R_{r,\varepsilon r} \setminus Q_{r(1-\sqrt{\varepsilon})})\right) + \\
&\quad - o_j(1) - (1 + C\sqrt{\varepsilon})\mathcal{H}^d\left(K_j \cap (R_{r,\varepsilon r} \setminus Q_{r(1-\sqrt{\varepsilon})})\right) \\
&\geqslant \mathcal{H}^d\left(K_j \cap Q_{r(1-\sqrt{\varepsilon})}\right) - C\sqrt{\varepsilon}\mathcal{H}^d\left(K_j \cap (R_{r,\varepsilon r} \setminus Q_{r(1-\sqrt{\varepsilon})})\right) - o_j(1).
\end{aligned}$$

Passing to the limit as $j \to \infty$ and using (3.7), (3.8) and (3.18), we get

$$\begin{aligned}
\liminf_j \mathcal{H}^d(K_j) &\geqslant \liminf_j \mathcal{H}^d(J_j) + \mu(Q_{r(1-\sqrt{\varepsilon})}) - C\sqrt{\varepsilon} r^d \\
&\geqslant \liminf_j \mathcal{H}^d(J_j) + (\theta_0(1 - \sqrt{\varepsilon})^d - C\sqrt{\varepsilon})r^d.
\end{aligned}$$

Since, for ε small, this is in contradiction with K_j be a minimizing sequence, we finally conclude that $\theta(0) \geqslant 1$.

Step five: We now show that $\theta(x) \leqslant 1$ for every $x \in K$ such that the approximate tangent space to K exists. Again, for further purposes, we will actually show that $\theta(x) \leqslant 1$ for every $x \in K \setminus H$ such that (3.16) holds. Arguing by contradiction, we assume that $\theta(x) = 1 + \sigma > 1$. As usually, we assume that $x = 0$ and $\pi = \{y : y_{d+1}, ..., y_n = 0\}$. By the monotonicity of the density established in Step 2, for every $\varepsilon > 0$ we can find $r > 0$ such that

$$K \cap Q_r \subset R_{r,\varepsilon r}, \qquad 1 + \sigma \leqslant \frac{\mu(Q_r)}{r^d} \leqslant 1 + \sigma + \varepsilon\,\sigma. \tag{3.28}$$

Since $\mathcal{H}^d \llcorner K_j$ converges to μ we have

$$\mathcal{H}^d(K_j \cap Q_r) > \Big(1 + \frac{\sigma}{2}\Big)\, r^d, \qquad \mathcal{H}^d((K_j \cap Q_r) \setminus R_{r,\varepsilon r}) < \frac{\sigma}{4}\, r^d. \qquad \forall j \geqslant j_0(r), \tag{3.29}$$

Consider the map $P : \mathbb{R}^n \to \mathbb{R}^n \in D(0, r)$ with $\mathrm{Lip}\, P \leqslant 1 + C\sqrt{\varepsilon}$ defined in (3.22), which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane. By exploiting the fact that $\mathscr{P}(H)$ is a deformation class, we find that

$$\begin{aligned}\mathcal{H}^d(K_j \cap Q_r) - o_j(1) \leqslant{}& \underbrace{\mathcal{H}^d(P(K_j \cap R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1} + \underbrace{\mathcal{H}^d(P(K_j \cap (R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})))}_{I_2} \\ &+ \underbrace{\mathcal{H}^d(P(K_j \cap (Q_r \setminus R_{r,\varepsilon r})))}_{I_3}\,.\end{aligned}$$

By construction, $I_1 \leqslant r^d$, while, by (3.29),

$$I_3 \leqslant (\mathrm{Lip}\, P)^d\, \mathcal{H}^d(K_j \cap (Q_r \setminus R_{r,\varepsilon r})) < (1 + C\sqrt{\varepsilon})^d\, \frac{\sigma}{4}\, r^d\,.$$

Hence, as $j \to \infty$,

$$\Big(1 + \frac{\sigma}{2}\Big) r^d \leqslant r^d + \liminf_{j\to\infty} I_2 + (1 + C\sqrt{\varepsilon})^d\, \frac{\sigma}{4}\, r^d\,,$$

that is,

$$\Big(\frac{1}{2} - \frac{(1 + C\sqrt{\varepsilon})^d}{4}\Big)\,\sigma \leqslant \liminf_{j\to\infty} \frac{I_2}{r^d}\,. \tag{3.30}$$

By (3.28), we finally estimate that

$$\begin{aligned}\limsup_{j\to\infty}\ I_2 &\leqslant (1 + C\sqrt{\varepsilon})^d\, \mu(Q_r \setminus Q_{(1-\sqrt{\varepsilon})r}) \\ &\leqslant (1 + C\sqrt{\varepsilon})^d \Big((1 + \sigma + \varepsilon\sigma) - (1 + \sigma)(1 - \sqrt{\varepsilon})^d\Big)\, r^d.\end{aligned} \tag{3.31}$$

By choosing ε sufficiently small, (3.30) and (3.31) provide the desired contradiction. In particular, by combining this with the previous step we deduce that $\theta = 1$ for every x such that K admits an approximate tangent space at x, that is for $\mathcal{H}^d$ almost every x. Classical argument in measure theory then implies that $\mu = \mathcal{H}^d \llcorner K$.

Step six: We now show that the canonical density one rectifiable varifold associated to K is stationary in $\mathbb{R}^n \setminus H$. In particular, applying Allard's regularity theorem, see [79, Chapter 5], we will deduce that there exists an $\mathcal{H}^d$-negligible closed set $\Sigma \subset K$ such that $\Gamma = K \setminus \Sigma$ is a real analytic manifold. Since being a stationary varifold is a local property, to prove our claim it is enough to show that for every ball $B \subset\subset \mathbb{R}^n \setminus H$ we have

$$\mathcal{H}^d(K) \leqslant \mathcal{H}^d(\phi(K)) \tag{3.32}$$

whenever ϕ is a diffeomorphism such that $\mathrm{spt}\{\phi - \mathrm{Id}\} \subset B$. Indeed, by exploiting (3.32) with $\phi_t = \mathrm{Id} + tX$, $X \in C^1_c(B)$ we deduce the desired stationarity property.

To prove (3.32) we argue as in [37, Theorem 7]. Given $\varepsilon > 0$ we can find $\delta > 0$ and a compact set $\hat{K} \subset K \cap B$ with $\mathcal{H}^d((K \setminus \hat{K}) \cap B) < \varepsilon$ such that K admits an approximate tangent plane T_xK at every $x \in \hat{K}$,

$$\sup_{x \in \hat{K}} \sup_{y \in B_{x,\delta}} |\nabla\phi(x) - \nabla\phi(y)| \leqslant \varepsilon\,, \qquad \sup_{x \in \hat{K}} \sup_{y \in \hat{K} \cap B_{x,\delta}} d(T_xK, T_yK) < \varepsilon\,, \tag{3.33}$$

where d is a distance on $G(d)$, the d-dimensional Grassmanian. Moreover, denoting by $S_{x,r}$ the set of points in $B_{x,r}$ at distance at most $\varepsilon\, r$ from $x + T_xK$, then $K \cap B_{x,r} \subset S_{x,r}$ for every $r < \delta$ and $x \in \hat{K}$. By Besicovitch covering theorem we can find a finite disjoint family of closed balls $\{\overline{B}_i\}$ with $B_i = B_{x_i,r_i} \subset B \subset\subset \mathbb{R}^n \setminus H$, $x_i \in \hat{K}$, and $r_i < \delta$, such that $\mathcal{H}^d(\hat{K} \setminus \bigcup_i B_i) < \varepsilon$. By exploiting the construction of Step four, we can find $j(\varepsilon) \in \mathbb{N}$ and maps $P_i : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Lip}\,(P_i) \leqslant 1 + C\sqrt{\varepsilon}$ and $P_i = \mathrm{Id}$ on B_i^c, such that, for a certain $X_i \subset S_i = S_{x_i, \varepsilon\, r_i}$,

$$\begin{aligned} &P_i(X_i) \subset B_i \cap (x_i + T_{x_i}K)\,, \\ &\mathcal{H}^d\Big(P_i\big((K_j \cap B_i) \setminus X_i\big)\Big) \leqslant C\sqrt{\varepsilon}\, \omega_d\, r_i^d\,, \qquad \forall j \geqslant j(\varepsilon)\,. \end{aligned} \tag{3.34}$$

Denoting with J_d^T the d-dimensional tangential jacobian with respect to the plane T and by J_d^K the one with respect to K and exploiting (3.33), (3.34), the area formula and that $\omega_d\, r_i^d \leqslant \mathcal{H}^d(K \cap B_i)$ (by the monotonicity formula), and setting $\alpha_i = \mathcal{H}^d((K \setminus \hat{K}) \cap B_i)$, we get

$$\begin{aligned} \mathcal{H}^d(\phi(P_i(K_j \cap X_i))) &= \int_{P_i(K_j \cap X_i)} J_d^{T_{x_i}K}\phi(x)\, d\mathcal{H}^d(x) \leqslant (J_d^{T_{x_i}K}\phi(x_i) + \varepsilon)\, \omega_d\, r_i^d \\ &\leqslant (J_d^{T_{x_i}K}\phi(x_i) + \varepsilon)\, \mathcal{H}^d(K \cap B_i) \leqslant (J_d^{T_{x_i}K}\phi(x_i) + \varepsilon)\, (\mathcal{H}^d(\hat{K} \cap B_i) + \alpha_i) \\ &\leqslant \int_{\hat{K} \cap B_i} (J_d^K\phi(x) + 2\varepsilon)\; d\mathcal{H}^d(x) + ((\mathrm{Lip}\,\phi)^d + \varepsilon)\, \alpha_i \\ &= \mathcal{H}^d(\phi(\hat{K} \cap B_i)) + 2\varepsilon\, \mathcal{H}^d(\hat{K} \cap B_i) + ((\mathrm{Lip}\,\phi)^d + \varepsilon)\, \alpha_i\,, \end{aligned} \tag{3.35}$$

where in the last identity we have used the area formula and the injectivity of ϕ. Since $P_i = \mathrm{Id}$ on B_i^c, $\phi = \mathrm{Id}$ on B^c, $B_i \subset B$ and the balls B_i are disjoint, the map $\tilde{\phi}$ which is equal to ϕ on $B \setminus \cup_i B_i$, equal to the identity on B^c and equal to $\phi \circ P_i$ on B_i is well defined. Moreover, by (3.35), we get

$$\mathcal{H}^d(\tilde{\phi}(K_j)) \leqslant \mathcal{H}^d(\phi(K)) + C\varepsilon$$

where C depends only on K. By exploiting the definition of deformation class, we get that

$$\mathcal{H}^d(K) \leqslant \mathcal{H}^d(\tilde{\phi}(K_j)) + o_j(1) \leqslant \mathcal{H}^d(\phi(K)) + C\varepsilon + o_j(1).$$

Letting $j \to \infty$ and $\varepsilon \to 0$ we obtain (3.32).

Step seven: We finally address the dimension of the singular set. Recall that, by monotonicity, the density function

$$\Theta^d(K, x) = \lim_{r \to 0} \frac{\mathcal{H}^d(K \cap B_{x,r})}{\omega_d r^d}$$

is everywhere defined in $\mathbb{R}^n \setminus H$ and equals 1 $\mathcal{H}^d$-almost everywhere in K. Fixing $x \in K$ and a sequence $r_k \downarrow 0$, the monotonicity formula, the stationarity of $\mathcal{H}^d \llcorner K$ and the compactness theorem for integral varifolds [2, Theorem 6.4] imply that (up to subsequences)

$$\mathcal{H}^d \llcorner \left(\frac{K - x}{r_k}\right) \rightharpoonup V \quad \text{locally in the sense of varifolds,} \tag{3.36}$$

where

(a) V is a stationary integral varifold: in particular $\Theta^d(\|V\|, y) \geqslant 1$ for $y \in \operatorname{spt}(V)$;

(b) V is a cone, namely $(\delta_\lambda)_\# V = V$, where $\delta_\lambda(x) = \lambda x$, $\lambda > 0$;

(c) $\Theta^d(\|V\|, 0) = \Theta^d(K, x) \geqslant \Theta^d(\|V\|, y)$ for every $y \in \mathbb{R}^n$.

Recall that the tangent varifold V depends (in principle) on the sequence (r_k). We denote by $\operatorname{TanVar}(K, x)$ the (nonempty) set of all possible limits V as in (3.36), varying among all sequences along which (3.36) holds. Given a cone W we set

$$\operatorname{Spine}(W) := \{y \in \mathbb{R}^n : \Theta^d(\|W\|, y) = \Theta^d(\|W\|, 0)\}. \tag{3.37}$$

By [9, 2.26], $\operatorname{Spine}(W)$ is a vector subspace of $\mathbb{R}^n$, see also [84, Theorem 3.1]. We can stratify K in the following way: for every $k = 0, \dots, n$ we let

$$A_k := \{x \in K : \text{ for all } V \in \operatorname{TanVar}(K, x),\ \dim \operatorname{Spine}(V) \leqslant k\}.$$

Clearly $A_0 \subset \cdots \subset A_d = \cdots = A_n$; moreover the following holds: $\dim_{\mathcal{H}} A_k \leqslant k$, see [9, 2.28] and [84, Theorem 2.2]. In order to prove our claim, we need to show that $A_d \setminus A_{d-1} \subset K \setminus \Sigma$, where Σ, as in Step six. To this end we note that the monotonicity formula for stationary varifolds implies that if W is a d-dimensional stationary cone with $\dim \operatorname{Spine}(W) = d$, then $\|W\| = \Theta^d(\|W\|, 0)\mathcal{H}^d \llcorner T$ for some d-dimensional plane [1] T. In particular since every $x \in A_d \setminus A_{d-1}$ admits at least one flat tangent varifold, for every such x there exists a sequence r_k satisfying

$$\mathcal{H}^d \llcorner \frac{K - x}{r_k} \to m\mathcal{H}^d \llcorner T;$$

moreover $m = \Theta^d(K, x)$ by (c). But then, the very same proof of Step five above implies that $\Theta^d(K, x) = 1$. Thus every $x \in A_d \setminus A_{d-1}$ satisfies the hypotheses of Allard's regularity Theorem [2, Regularity Theorem, Section 8], implying that $K \cap Q_{x,\frac{r}{2}}$ is a real analytic submanifold. Equivalently $x \notin \Sigma$ and this concludes the proof. $\square$

1 Indeed up to a rotation $\operatorname{spt}(W) = \operatorname{Spine}(W) \times \Gamma$, where Γ is a cone in $\mathbb{R}^{n-d}$. If $\Gamma \neq \{0\}$ then $\Theta^d(\|W\|, 0) > \Theta^d(\|W\|, y)$ for any $y \in \operatorname{Spine}(W) \setminus \{0\}$, which contradicts (3.37).

3.4 ANISOTROPIC PLATEAU PROBLEM IN CODIMENSION ONE

The anisotropic Plateau problem aims at finding an energy minimizing surface spanning a given boundary when the energy functional is more general than the usual surface area (as in the standard Plateau problem) and is obtained integrating a general Lagrangian F over the surface. In particular, the integrand depends on the position and the tangent space to the surface.

As in the case of the area integrand, [33, 51, 77, 7, 32, 56, 37, 44], many definitions of boundary conditions (both homological and homotopical), as well as the type of competitors (currents, varifolds, sets) have been considered in the literature. An important existence, regularity and almost uniqueness result in arbitrary dimension and codimension was achieved by Almgren in [7], using refined techniques from geometric measure theory. In more recent times, Harrison and Pugh in [57, 58] investigated the anisotropic Plateau problem under a suitable cohomological definition of boundary.

In this section we present the result of our work [38]. We adopt the same strategy as in [37, 44], namely we prove a general compactness theorem for minimizing sequences in general classes of $(n-1)$-rectifiable sets. More precisely, we consider the measures naturally associated to any such sequence and we show that, if a sufficiently large class of deformations are admitted, any weak limit is induced by a rectifiable set, thus providing compactness and semicontinuity under very little assumptions.

One main difficulty in our approach is to prove the rectifiability of the support of the limiting measure. As already observed, the key ingredient in [37, 44] to obtain such rectifiability is the classical monotonicity formula for the mass ratio of the limiting measure, which allows to apply Preiss' rectifiability theorem for Radon measures [76, 35]. Such a strategy does not seem feasible for general anisotropic integrands, where the monotonicity of the mass ratio is unlikely to be true, as pointed out in [3]. The main goal of this section is to show how, in codimension one, the rectifiability of the limiting measure follows from the theory of Caccioppoli sets, bypassing the monotonicity formula and the deep result of Preiss. In particular, we are able to prove the results analogous to those of [37] with a strategy which has some similarities with the one used in [7].

The most general case of any codimension and anisotropic energies is addressed in the next Section, which describes our paper [42]. It uses however different and more sophisticated PDE techniques.

3.4.1 *Preliminary assumtpions*

We next outline a set of flexible requirements for $\mathscr{P}(\mathbf{F}, H)$, which is more specific for the codimension one case. These conditions will replace the ones of Definition 3.1 and Definition 3.2.

Definition 3.9 (Cup competitors)**.** Let $K \subset \mathbb{R}^n \setminus H$ and $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$. We introduce the following equivalence relation among points of $\overline{B_{x,r}} \setminus K$:

$$y_0 \sim_{K,x,r} y_1 \quad \Longleftrightarrow \quad \exists\, \gamma \in C^0([0,1], \overline{B_{x,r}} \setminus K) : \gamma(0) = y_0,\ \gamma(1) = y_1,\ \gamma(]0,1[) \subset B_{x,r}$$

(where x and r are clear from the context we will omit them and simply write $\sim_K$). We enumerate as $\{\Gamma_i(K,x,r)\}$ the equivalence classes in $\partial B_{x,r}/\sim_{K,x,r}$ (where the index i varies either among all natural numbers or belongs to a finite subset of them). The cup competitor associated to $\Gamma_i(x,r)$ for K in $B_{x,r}$ is

$$\big(K \setminus B_{x,r}\big) \cup \big((\partial B_{x,r}) \setminus \Gamma_i(K,x,r)\big)\,. \tag{3.38}$$

For further reference we also introduce the sets

$$\Omega_i(K,x,r) = \{z \in B_{x,r} \setminus K : \exists\, y \in \Gamma_i(K,x,r) \text{ such that } z \sim_{K,x,r} y\}\,. \tag{3.39}$$

The dependence on K, x and r will be sometimes suppressed if clear from the context. It is easy to see that the associated sets $\Omega_i(K,x,r)$ are connected components of $B_{x,r} \setminus K$ (possibly not all of them).

Definition 3.10 (Good class)**.** A family $\mathscr{P}(\mathbf{F},H)$ of relatively closed subsets $K \subset \mathbb{R}^n \setminus H$ is called a *good class* if for any $K \in \mathscr{P}(\mathbf{F},H)$, for every $x \in K$ and for a.e. $r \in (0, \mathrm{dist}(x,H))$ the following holds:

$$\inf\big\{\mathbf{F}(J) : J \in \mathscr{P}(\mathbf{F},H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\big\} \leqslant \mathbf{F}(L) \tag{3.40}$$

whenever L is any cup competitor for K in $B_{x,r}$.

Remark 3.11. Observe that the definition of cup competitors is a slight modification of that of [37], where $\Gamma_i(K,x,r)$ were taken to be connected components of $\partial B_{x,r} \setminus K$: observe however that, for every cup competitor in [37], we can find a cup competitor as above which has at most the same area, since each $\Gamma_i(K,x,r)$ is a union of connected components of $\partial B_{x,r} \setminus K$ and each connected component of $\partial B_{x,r} \setminus K$ is contained in at least one $\Gamma_i(K,x,r)$. Finally good classes in our paper [38] do not assume any kind of comparisons with cones, as it is the case of [37].

The point of our work [38] is that the notion of good class is enough to ensure that any weak* limit of a minimizing sequence is a rectifiable measure and that a suitable lower semicontinuity statement holds for energies $\mathbf{F}$ which satisfy the usual ellipticity condition of [50, 5.1.2], cf. Theorem 3.13 below. In particular, as shown in [50, 5.1.3-5.1.5], the convexity of the integrand F is a sufficient condition and we take it therefore as definition in this section (notice that in the next section we will replace this assumption with more useful requirements to address the higher codimesion case).

Definition 3.12 (Elliptic anisotropy, [50, 5.1.2-5.1.5])**.** F is elliptic if its even and positively 1-homogeneous extension to $\mathbb{R}^n \times (\Lambda_{n-1}(\mathbb{R}^n) \setminus \{0\})$ is C^2 and it is uniformly convex in the T variable on compact sets.

Actually the only points required in the proof of Theorem 3.13 are the lower semicontinuity of the functional $\mathbf{F}$ under the usual weak convergence of reduced boundaries of Caccioppoli sets and the following estimate on the oscillation of F over compact sets $W \subset\subset \mathbb{R}^n$:

$$\sup_{x,y \in W\, S,T \in G(n,n-1)} |F(x,T) - F(y,S)| \leqslant \omega_W(|x-y| + \|T-S\|), \tag{3.41}$$

where ω_W is a modulus of continuity which depends upon $G(W)$ and $\|\cdot\|$ is the standard metric on $G(n, n-1)$ defined as in [79, Chapter 8, Section 38]. In particular the C^2 regularity of the definition above can be considerably relaxed.

A minimizing sequence $\{K_j\} \subset \mathscr{P}(\mathbf{F}, H)$ in Problem (3.2) satisfies the property $\mathbf{F}(K_j) \to m_0$, and throughout the section we will assume m_0 to be finite.

3.4.2 *Main theorem*

We have all the definitions to state the main theorem of Section 3.4.

Theorem 3.13. *Let $H \subset \mathbb{R}^n$ be closed and $\mathscr{P}(\mathbf{F}, H)$ be a good class in the sense of Definition 3.10. Let $\{K_j\} \subset \mathscr{P}(\mathbf{F}, H)$ be a minimizing sequence and assume $m_0 < \infty$. Then, up to subsequences, the measures $\mu_j := F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^{n-1} \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to a measure $\mu = \theta \mathcal{H}^{n-1} \llcorner K$, where $K = \operatorname{spt} \mu$ is an $(n-1)$-rectifiable set and $\theta \geqslant c_0$ for some constant $c_0(F, n)$.*

Moreover, if F is elliptic as in Definition 3.12, then $\liminf_j \mathbf{F}(K_j) \geqslant \mathbf{F}(K)$ (that is $\theta(x) \geqslant F(x, T_x K)$) and in particular, if $K \in \mathscr{P}(\mathbf{F}, H)$, then K is a minimum for Problem (3.2) and thus $\theta(x) = F(x, T_x K)$.

Indeed the measure μ above is an $(n-1)$-dimensional rectifiable varifold. Since the proof of Theorem 3.13 does not exploit Preiss' rectifiability Theorem, when the Lagrangian is constant (i.e. up to a factor it is the area functional $F \equiv 1$) and we require the stronger energetic inequality in (3.40) to hold for any cup competitors as in (3.38) [37, Equation 1.2], then the same strategy gives a simpler proof of the conclusions of [37, Theorem 2], except for the monotonicity formula in [37, Equation (1.5)].

Finally, we remark that it is possible to obtain the useful additional information $\theta(x) = F(x, T_x K)$ in Theorem 3.13 even when we cannot directly infer that $K = \operatorname{spt} \mu$ belongs to the class $\mathscr{P}(\mathbf{F}, H)$, provided we allow the class of competitors to be also a deformation class as in Definition 3.2.

Proposition 3.14. *Assume that F is elliptic as in Definition 3.12 and that H, $\mathscr{P}(\mathbf{F}, H)$, $\{K_j\}$, μ and K are as in Theorem 3.13. If in addition $\mathscr{P}(\mathbf{F}, H)$ is a deformation class in the sense of Definition 3.2, then $\theta(x) = F(x, T_x K)$ for $\mathcal{H}^{n-1}$-a.e. $x \in K$.*

3.4.3 *Proof of Theorem 3.13*

Parts of the proofs follow the isotropic case treated in [37]: we will be brief on these arguments, hoping to convey the main ideas and in order to leave space to the original content.

The proof of Theorem 3.13 goes as follows: we consider the natural measures (μ_j) associated to a minimizing sequence (K_j) and extract a weak limit μ. We first recall that, as a consequence of minimality, μ enjoys density upper and lower bounds on $\operatorname{spt}(\mu)$, leading to the representation $\mu = \theta \mathcal{H}^{n-1} \llcorner \operatorname{spt}(\mu)$: this part follows almost verbatim the proof of [37]. Then, via an energy comparison argument, we exclude the presence of purely unrectifiable subsets of $\operatorname{spt}(\mu)$, which is the core novelty of the note. We then show that, if the Lagrangian is elliptic, then the energy is lower semicontinuous along (K_j). Finally, if we assume also that $\mathscr{P}(\mathbf{F}, H)$ is a deformation class, we show that $\theta(x) = F(x, T_x K)$.

Proof of Theorem 3.13: density bound

We prove lower ad upper density bounds for the limiting measure μ:

Lemma 3.15 (Density bounds)**.** *Suppose that $\mathscr{P}(\mathbf{F}, H)$ is a good class, that $\{K_j\} \subset \mathscr{P}(\mathbf{F}, H)$ is a minimizing sequence for problem* (3.2) *and that*

$$\mu_j = F(\cdot, T_{(\cdot)}K)\mathcal{H}^{n-1} \llcorner K_j \overset{*}{\rightharpoonup} \mu$$

in $\mathbb{R}^n \setminus H$. Then the limit measure μ enjoys density upper and lower bounds:

$$\theta_0\, \omega_{n-1} r^{n-1} \leqslant \mu(B_{x,r}) \leqslant \theta_0^{-1}\, \omega_{n-1} r^{n-1}\,, \qquad \forall x \in \operatorname{spt}\mu\,,\ \forall r < d_x := \operatorname{dist}(x, H) \tag{3.42}$$

for some positive constant $\theta_0 = \theta_0(n, F) > 0$.

Proof. The density lower bound can be proved as in [37, Theorem 2, Step 1] with the use of cup competitors only, since the energy $\mathbf{F}$ is comparable to the Hausdorff measure by (2.10). The notion of cup competitors in Definition 3.10 slightly differs from the notion in [37, Definition 1], however the key fact is that the latter have larger energy, cf. Remark 3.11. The existence of a density upper bound is trivially true, since we can use a generic sequence $\{\Gamma_j\}$ of cup competitors associated to $\{K_j\}$ in $B_{x,r}$. Observe that at least one Γ_j exists as long as $\partial B_{x,r} \setminus K_j \neq \emptyset$: on the other hand for a.e. radius r we have $\liminf_j \mathcal{H}^{n-1}(K_j) < \infty$ and we can assume the existence of a subsequence for which the Γ_j exist. Hence, by almost minimality

$$\begin{aligned}\mu(B_{x,r}) &\leqslant \limsup_j \mu_j(\overline{B_{x,r}}) \leqslant \limsup_j \mathbf{F}(\partial B_{x,r} \setminus \Gamma_j) \\ &\leqslant \limsup_j \Lambda\, \mathcal{H}^{n-1}(\partial B_{x,r} \setminus \Gamma_j) \leqslant \Lambda\, \sigma_{n-1}\, r^{n-1}\,.\end{aligned} \tag{3.43}$$

□

We remark that, if the requirement of being a good class were substituted by that of being a deformation class, the density lower bound could be proven as in Theorem 3.3, Step 1: note that although the bound in (3.8) is claimed for the area functional, the argument requires only the two-sided comparison of (2.10). Moreover, the upper bound could be obtained as in (3.43), but using the slightly different cup competitors defined in [37, Definition 1], which are proven to be deformed competitors in [37, Theorem 7, Step 1].

Proof Theorem 3.13: rectifiability

Up to extracting subsequences, we can assume the existence of a Radon measure μ on $\mathbb{R}^n \setminus H$ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu\,, \qquad \text{as Radon measures on } \mathbb{R}^n \setminus H\,. \tag{3.44}$$

We set $K = \operatorname{spt}\mu$ and from the differentiation theorem for Radon measures, see for instance [68, Theorem 6.9], and Lemma 3.15 we deduce that

$$\mu = \theta \mathcal{H}^{n-1} \llcorner K, \tag{3.45}$$

where K is a relatively closed set in $\mathbb{R}^n \setminus H$ and $\theta : K \to \mathbb{R}^+$ is a Borel function with $c_0 \leqslant \theta \leqslant C_0$.

We decompose $K = \mathcal{R} \cup \mathcal{N}$ into a rectifiable $\mathcal{R}$ and a purely unrectifiable $\mathcal{N}$ (see [79, Chapter 3, Section 13.1]) and assume by contradiction that $\mathcal{H}^{n-1}(\mathcal{N}) > 0$. Then, there is $x \in K$ such that

$$\Theta^{n-1}(\mathcal{R}, x) = \lim_{r\to 0} \frac{\mathcal{H}^{n-1}(\mathcal{R} \cap B_{x,r})}{\omega_{n-1} r^{n-1}} = 0, \qquad \Theta^{*n-1}(\mathcal{N}, x) = a > 0. \tag{3.46}$$

Without loss of generality, we assume that $x = 0$. The overall aim is to show that at 0 the density lower bound of Lemma 3.15 would be false, reaching therefore a contradiction.

For every $\rho > 0$, we let $\Omega_i(\rho)$, with $i \in \mathbb{N}$, be sets of (3.39) (where we omit the dependence on K and x). Observe that the $\Omega_i(\rho)$ are sets of finite perimeter (see for instance [50, 4.5.11]). If we denote, as usual, by $\partial^*\Omega_i(\rho)$ their reduced boundaries (in $B_{x,\rho}$), we know that $\partial^*\Omega_i(\rho) \subset K$. Moreover:

(a) by the rectifiability of the reduced boundary (cf. [50, 4.5.6]), $\partial^*\Omega_i(\rho) \subset \mathcal{R}$;

(b) each point $x \in \partial^*\Omega_i(\rho)$ belongs to at most another distinct $\partial^*\Omega_j(\rho)$, because at any point $y \in \partial^*\Omega$ of a Caccioppoli set Ω its blow-up is a half-space, cf. [50, 4.5.5].

Since in what follows we will often deal with subsets of the sphere ∂B_ρ, we will use the following notation:

- $\partial_{\partial B_\rho} A$ is the topological boundary of A as subset of ∂B_ρ;
- $\partial^*_{\partial B_\rho} A$ is the reduced boundary of A relative to ∂B_ρ.

Using the slicing theory for sets of finite perimeter we can infer that

$$\mathcal{H}^{n-2}(\partial^*_{\partial B_t}(\Omega_i(\rho) \cap \partial B_t) \setminus ((\partial^*\Omega_i(\rho)) \cap \partial B_t)) = 0 \qquad \text{for a.e. } t < \rho. \tag{3.47}$$

This can be for instance proved identifying $\Omega_i(\rho)$ and $\partial^*\Omega_i(\rho)$ with the corresponding integer rectifiable currents (see [79, Remark 27.7]) and then using the slicing theory for integer rectifiable currents (cf. [79, Chapter 6, Section 28]). Combining (a), (b) and (3.47) above we eventually achieve

$$\sum_i \mathcal{H}^{n-2} \llcorner \partial^*_{\partial B_t}(\Omega_i(\rho) \cap \partial B_t) \leqslant 2\mathcal{H}^{n-2} \llcorner \mathcal{R} \cap \partial B_t \quad \text{for a.e. } t < \rho. \tag{3.48}$$

Step 1. In this first step we show that, for every $\varepsilon_0 > 0$ and every $r_0 > 0$ small enough, there exists $\rho \in]r_0, 2r_0[$ satisfying

$$\max_i \left\{\mathcal{H}^{n-1}(\Gamma_i(\rho))\right\} \geqslant (\sigma_{n-1} - \varepsilon_0)\rho^{n-1}. \tag{3.49}$$

Indeed, by (3.46), we consider r_0 so small that $\mathcal{H}^{n-1}(\mathcal{R} \cap B_s(x)) \leqslant \varepsilon_0 s^{n-1}$ for every $s \leqslant 2r_0$. We first claim the existence of a closed set $R \subset]r_0, 2r_0[$ of positive measure such that the following holds $\forall \rho \in R$:

(i)
$$\lim_{\sigma\in R,\sigma\to\rho} \mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_\sigma) = \mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_\rho);$$

(ii) $\mathcal{H}^{n-1}(K\cap\partial B_\rho) = 0$;

(iii) $\mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_\rho) \leqslant C\varepsilon_0\rho^{n-2}$.

The existence of a set of positive measure R' such that (iii) holds at any $\rho \in R'$ is an obvious consequence of the coarea formula and of Chebycheff's inequality, provided the universal constant C is larger than 2^{n-1}. Moreover, condition (ii) holds at all but countable many radii. Next, since the map $t \mapsto \mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_t)$ is measurable, by Lusin's theorem we can select a closed subset R of R' with positive measure for which (i) holds at every radius.

Fix now a point $\rho \in R$ of density 1 for R: it turns out that ρ satisfies indeed condition (3.49). In order to show that estimate, we first choose $(\rho_k) \subset R$, $\rho_k \uparrow \rho$ such that (3.48) holds for $t = \rho_k$. Observe that, for every sequence of points $x_k \in \partial B_{\rho_k} \cap \Omega_i(\rho)$ converging to some x_∞, we have that $x_\infty \in \Gamma_i(\rho) \cup (K\cap\partial B_\rho)$, otherwise there would exist $\tau > 0$ such that $B_\tau(x_\infty)\cap\Omega_i(\rho) = \emptyset$, against the convergence of x_k to x_∞. In particular, rescaling everything at radius ρ, for every $\eta > 0$ there exists $k(\eta)$ such that, for all $k \geqslant k(\eta)$

$$E_{k,i} := \frac{\rho}{\rho_k}\left(\Omega_i(\rho)\cap\partial B_{\rho_k}\right) \subset U_\eta\left(\Gamma_i(\rho)\cup(K\cap\partial B_\rho)\right)\cap\partial B_\rho =: \Gamma_{i,\eta},$$

where U_η denotes the η-tubular neighborhood.

Observe that $\mathcal{H}^{n-1}(\Gamma_{i,\eta}) \downarrow \mathcal{H}^{n-1}(\Gamma_i(\rho))$ as $\eta \downarrow 0$, because $\partial_{\partial B_\rho}\Gamma_i(\rho) \subset K\cap\partial B_\rho$ and (ii) holds. On the other hand, for every $\eta > 0$, we can take Λ_η compact subset of $\Gamma_i(\rho)$ with $\mathcal{H}^{n-1}(\Gamma_i(\rho)\setminus\Lambda_\eta) < \eta$ and $U_\alpha(\Lambda_\eta)\cap B_\rho \subset \Omega_i(\rho)$ for some small $\alpha(\eta) > 0$. Therefore, for $\rho - \rho_k < \alpha(\eta)$, the following holds

$$E_{k,i} \supset \Lambda_\eta.$$

Since $\Lambda_\eta \subset E_{k,i} \subset \Gamma_{i,\eta}$ for every $k > k(\eta)$, $\Lambda_\eta \subset \Gamma_i(\rho) \subset \Gamma_{i,\eta}$ and $\mathcal{H}^{n-1}(\Gamma_{i,\eta}\setminus\Lambda_\eta) \downarrow 0$ as $\eta \downarrow 0$, we easily deduce that $E_{k,i} \to \Gamma_i(\rho)$ in $L^1(\partial B_\rho)$. Moreover, by (3.48)

$$\sum_i \mathcal{H}^{n-2}(\partial^*_{\partial B_{\rho_k}}(\Omega_i(\rho)\cap\partial B_{\rho_k})) \leqslant 2\mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_{\rho_k}). \tag{3.50}$$

The L^1 convergence shown above, the lower semicontinuity of the perimeter and the definition of $E_{k,i}$ imply that

$$\begin{aligned}\sum_i \mathcal{H}^{n-2}(\partial^*_{\partial B_\rho}\Gamma_i(\rho)) &\leqslant \liminf_k \sum_i \mathcal{H}^{n-2}(\partial^*_{\partial B_\rho}E_{k,i})\\ &= \liminf_k \left(\frac{\rho}{\rho_k}\right)^{n-2}\sum_i \mathcal{H}^{n-2}(\partial^*_{\partial B_{\rho_k}}(\Omega_i(\rho)\cap\partial B_{\rho_k})).\end{aligned}$$

Plugging (3.50), conditions (i) and (iii) in the previous equation, we get

$$\sum_i \mathcal{H}^{n-2}(\partial^*_{\partial B_\rho}\Gamma_i(\rho)) \leqslant 2\liminf_k \mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_{\rho_k}) = 2\mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_\rho) \leqslant C\varepsilon_0\rho^{n-2}.$$

Let us denote by $\Gamma_0(\rho)$ the element of largest $\mathcal{H}^{n-1}$ measure among the $\Gamma_i(\rho)$: applying the isoperimetric inequality [37, Lemma 9] in ∂B_ρ we get

$$\sigma_{n-1}\rho^{n-1} - \mathcal{H}^{n-1}(\Gamma_0(\rho)) = \sum_{i\geqslant 1} \mathcal{H}^{n-1}(\Gamma_i(\rho)) \leqslant C\left(\mathcal{H}^{n-2}(\mathcal{R}\cap\partial B_\rho)\right)^{\frac{n-1}{n-2}} \leqslant C\varepsilon_0^{\frac{n-1}{n-2}}\rho^{n-1},$$

namely $\mathcal{H}^{n-1}(\Gamma_0(\rho)) \geqslant (\sigma_{n-1} - C\varepsilon_0^{\frac{n-1}{n-2}})\rho^{n-1}$, which proves (3.49).

Step 2. In this second step we let $\Gamma_0^j(\rho)$ be a $\sim_{K_j}$-equivalence class of largest $\mathcal{H}^{n-1}$-measure in $\partial B_\rho \setminus K_j$ and we claim that:

$$\liminf_j \mathcal{H}^{n-1}(\Gamma_0^j(\rho)) \geqslant \mathcal{H}^{n-1}(\Gamma_0(\rho)), \tag{3.51}$$

(where, consistently, $\Gamma_0(\rho)$ is a $\sim_K$-equivalence class of largest measure in $\partial B_\rho \setminus K$; note that the latter estimate, combined with Step 1, implies, for ε_0 sufficiently small and j sufficiently large, that such equivalence classes of largest $\mathcal{H}^{n-1}$ measure are indeed uniquely determined).

Recall that $\Omega_0(\rho)$ is associated to $\Gamma_0(\rho)$ according to (3.39). Let us consider $\delta > 0$ sufficiently small and $\bar\Gamma \subset\subset \Gamma_0(\rho)$ verifying

$$\mathcal{H}^{n-1}(\bar\Gamma) \geqslant \mathcal{H}^{n-1}(\Gamma_0(\rho)) - \delta\,. \tag{3.52}$$

Next, by compactness, we can uniformly separate $\bar\Gamma$ and K, that is we can pick $\eta > 0$ sufficiently small so that

$$V := \bigcup_{s\in[\rho-\eta,\rho]} \frac{s}{\rho}\bar\Gamma = \left\{x \in \overline{B_\rho} \setminus B_{\rho-\eta} : \rho\frac{x}{|x|} \in \bar\Gamma\right\} \subset\subset \overline{B}_\rho \setminus K\,. \tag{3.53}$$

Next we choose an open connected subset of $\Omega_0(\rho)$ with smooth boundary, denoted by $\Omega(\rho)$, such that

$$|\Omega_0(\rho) \setminus \Omega(\rho)| < \delta\eta. \tag{3.54}$$

The set $\Omega(\rho)$ can be constructed as follows:

- first one considers $\Lambda \subset\subset \Omega_0(\rho)$ compact with $|\Omega_0(\rho) \setminus \Lambda| < \delta\eta$: this can be achieved for instance looking at a Whitney subdivision of $\Omega_0(\rho)$, taking the union of the cubes with side length bounded from below by a small number;
- Λ can be enlarged to become connected by adding, if needed, a finite number of arcs at positive distance from $\partial\Omega_0(\rho)$;
- we can finally take a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{\mathbb{R}^n\setminus\Omega_0(\rho)} = 0$, $f|_\Lambda = 1$ and $0 \leqslant f \leqslant 1$: by the Morse-Sard Theorem, one can choose $t \in]0,1[$ such that $\{f = t\}$ is a C^∞ submanifold.
- the connected component of $\{f > t\}$ containing Λ satisfies the required assumptions.

Since $\Omega(\rho) \subset\subset \Omega_0(\rho)$, by weak convergence $\mathcal{H}^{n-1}(K_j \cap \Omega(\rho)) \to 0$; moreover since $\Omega(\rho)$ is smooth and connected, it satisfies the isoperimetric inequality

$$|\Omega(\rho) \setminus \Omega^j(\rho)| \leqslant \mathrm{Iso}(\Omega(\rho))(\mathcal{H}^{n-1}(\Omega(\rho) \cap K_j))^{\frac{n}{n-1}} = o_j(1)\,. \tag{3.55}$$

where $\Omega^j(\rho)$ is the connected component of $\Omega(\rho) \setminus K_j$ of largest volume and $\mathrm{Iso}(\Omega(\rho))$ is the isoperimetric constant of the smooth connected domain $\Omega(\rho)$ (for the isoperimetric inequality see [50, Theorem 4.5.2(2)] and use the fact that $\partial^*\Omega^j(\rho) \subset K_j$, which has been observed above).

Obviously (3.53) implies $\mathcal{H}^{n-1}(K_j \cap V) \to 0$ and, by projecting $K_j \cap V$ on ∂B_ρ via the radial map $\Pi : B_\rho \ni x \mapsto \frac{\rho}{|x|}x \in \partial B_\rho$, we easily get that the set

$$\bar{\Gamma}^j := \{y \in \bar{\Gamma} : \Pi^{-1}(y) \cap V \cap K_j = \emptyset\}$$

verifies

$$\begin{aligned}\mathcal{H}^{n-1}(\bar{\Gamma}^j) &= \mathcal{H}^{n-1}(\bar{\Gamma}) - \mathcal{H}^{n-1}(\Pi(K_j \cap V)) \\ &\geqslant \mathcal{H}^{n-1}(\bar{\Gamma}) - (\mathrm{Lip}\,\Pi|_{B_\rho \setminus B_{\rho-\eta}})^{n-1}\mathcal{H}^{n-1}(K_j \cap V) = \mathcal{H}^{n-1}(\bar{\Gamma}) - o_j(1).\end{aligned} \tag{3.56}$$

We deduce from (3.56) that

$$|V \cap \Pi^{-1}(\bar{\Gamma}^j)| = |V| - |V \setminus \Pi^{-1}(\bar{\Gamma}^j)| \geqslant |V| - \eta\mathcal{H}^{n-1}(\bar{\Gamma} \setminus \bar{\Gamma}^j) \geqslant |V| - o_j(1). \tag{3.57}$$

The previous inequality, (3.53), (3.54) and (3.55) in turn imply that

$$\begin{aligned}|V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar{\Gamma}^j))| \quad &\leqslant \quad |V \setminus \Omega^j(\rho)| + |V \setminus \Pi^{-1}(\bar{\Gamma}^j)| \\ &\overset{(3.53),(3.57)}{\leqslant} |\Omega_0(\rho) \setminus \Omega(\rho)| + |\Omega(\rho) \setminus \Omega^j(\rho)| + o_j(1) \\ &\overset{(3.54),(3.55)}{\leqslant} \eta\delta + o_j(1).\end{aligned} \tag{3.58}$$

If $x \in V \cap \Omega^j(\rho) \cap \Pi^{-1}(\bar{\Gamma}^j)$, then $x \sim_{K_j} \Pi(x)$ using as a path simply the radial segment $[x, \Pi(x)[$; moreover we can always connect two points belonging to $V \cap \Omega^j(\rho) \cap \Pi^{-1}(\bar{\Gamma}^j)$ with a path inside $\Omega^j(\rho)$. But, by (3.58), the endpoints $\Pi(x)$ of these segments must cover all but a small fraction G_j of $\bar{\Gamma}^j$ of measure $o_j(1)$. Indeed we can estimate the complement set $G_j := \Pi(V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar{\Gamma}^j)))$ using the coarea formula and the self similarity of the shells:

$$\begin{aligned}\frac{\rho}{n}\left(1 - \left(1 - \frac{\eta}{\rho}\right)^n\right)\mathcal{H}^{n-1}(G_j) &= \int_{\rho-\eta}^{\rho} \mathcal{H}^{n-1}\left(\frac{t}{\rho}G_j\right)dt \\ &\leqslant |V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar{\Gamma}^j))| \leqslant \delta\eta + o_j(1),\end{aligned}$$

which yields, for η small enough (namely smaller than a positive constant $\eta_0(n, \rho)$),

$$\mathcal{H}^{n-1}(G_j) \leqslant 2\delta + o_j(1)\,. \tag{3.59}$$

By concatenating the paths we conclude that $\bar{\Gamma}^j \setminus G_j$ must be contained in a unique equivalence class $\Gamma_i^j(\rho)$. We remark that for the moment we do not know whether $\Gamma_i^j(\rho)$ is an equivalence class of $\partial B_\rho \setminus K_j$ with largest measure. Summarizing the inequalities achieved so far we conclude

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma_0^j(\rho)) \geqslant \mathcal{H}^{n-1}(\Gamma_i^j(\rho)) \;&\geqslant\; \mathcal{H}^{n-1}(\bar{\Gamma}^j \setminus G_j) \overset{(3.59)}{\geqslant} \mathcal{H}^{n-1}(\bar{\Gamma}^j) - 2\delta - o_j(1) \\ &\overset{(3.56)}{\geqslant} \mathcal{H}^{n-1}(\bar{\Gamma}) - o_j(1) - 2\delta \overset{(3.52)}{\geqslant} \mathcal{H}^{n-1}(\Gamma_0(\rho)) - o_j(1) - 3\delta\end{aligned}$$

In particular, letting first $j \uparrow \infty$ and then $\delta \downarrow 0$ we achieve (3.51).

Step 3. We recover a straightforward contradiction since, by the density lower bound (d.l.b.) proven in Lemma 3.15, the good class property (g.c.p.) of $\mathscr{P}(\mathbf{F}, H)$, the lower semicontinuity (l.s.) in the weak convergence (3.44) and the bound (2.10), we get

$$\begin{aligned}c_0\rho^{n-1} &\overset{\text{d.l.b.}}{\leqslant} \mu(B_{x,\rho}) \overset{\text{l.s.}}{\leqslant} \liminf_j \mathbf{F}(K_j, B_{x,\rho}) \\ &\overset{\text{g.c.p.}}{\leqslant} \liminf_j \mathbf{F}(\partial B_{x,\rho} \setminus \Gamma_0^j(\rho)) \overset{(2.10)}{\leqslant} \Lambda \liminf_j \mathcal{H}^{n-1}(\partial B_{x,\rho} \setminus \Gamma_0^j(\rho)).\end{aligned}$$

Plugging in (3.51) and (3.49) (both relative to the complementary sets), we get

$$c_0\rho^{n-1} \overset{(3.51)}{\leqslant} \Lambda\mathcal{H}^{n-1}(\partial B_{x,\rho} \setminus \Gamma_0(\rho)) \overset{(3.49)}{\leqslant} \Lambda\varepsilon_0\rho^{n-1},$$

which is false for ε_0 small enough. We conclude $\mathcal{H}^{n-1}(\mathcal{N}) = 0$, hence the rectifiability of the set K.

3.4.4 *Proof of Theorem 3.13: semicontinuity*

We are now ready to complete the proof of Theorem 3.13, namely to show that $\liminf_j \mathbf{F}(K_j) \geqslant \mathbf{F}(K)$ and $\mu = F(x, T_xK)\mathcal{H}^{n-1} \llcorner K$, when the integrand F is elliptic.
We claim indeed that $\theta(x) \geqslant F(x, T_xK)$ for every x where the rectifiable set K has an approximate tangent plane $\pi = T_xK$ and θ is approximately continuous. Let x be such point and assume, without loss of generality, that $x = 0$. We therefore have the following limit in the weak* topology:

$$\theta(r\cdot)\mathcal{H}^{n-1} \llcorner \frac{K}{r} \overset{*}{\rightharpoonup} \theta(0)\mathcal{H}^{n-1} \llcorner \pi \qquad \text{for} \qquad r \downarrow 0. \tag{3.60}$$

For a suitably chosen sequence $r_j \downarrow 0$, consider the corresponding rescaled sets $\tilde{K}_j := \frac{1}{r_j}K_j$ and rescaled measures $\tilde{\mu}_j := \tilde{F}_j\mathcal{H}^{n-1} \llcorner \tilde{K}_j$, where $\tilde{F}_j(y) := F(r_jy, T_y\tilde{K}_j)$. With a diagonal argument, if $r_j \downarrow 0$ sufficiently slow (since the blow-up to π in (3.60) happens on the full continuous limit $r \downarrow 0$), then we can assume that the $\tilde{\mu}_j$ are converging weakly* in $\mathbb{R}^n$ to $\tilde{\mu} = \theta(0)\mathcal{H}^{n-1} \llcorner \pi$. Note moreover that $\tilde{\mu}_j(B_1) \to \omega_{n-1}\theta(0)$ because $\tilde{\mu}(\partial B_1) = 0$.

Let $\tilde{\Omega}_j$ be the largest connected component of $B_1 \setminus \tilde{K}_j$. As already observed, $\tilde{\Omega}_j$ is a Caccioppoli set and $\partial^*\tilde{\Omega}_j \subset \tilde{K}_j$. Up to subsequences, we can assume that $\tilde{\Omega}_j$ converges as a Caccioppoli set to some $\tilde{\Omega}$ whose reduced boundary in B_1 must be contained in π. We thus have three alternatives:

(i) $\tilde{\Omega}$ is the lower or the upper half ball of $B_1 \setminus \pi$. In this case, the lower semicontinuity of the energy **F** on Caccioppoli sets (which follows from [50, 5.1.2 & 5.1.5]) implies

$$\begin{aligned}\omega_{n-1} r^{n-1} F(0,\pi) &\leqslant \liminf_{j\to\infty} \int_{\partial^*\Omega_j \cap B_1} F(0, T_y \partial^*\Omega_j) \, d\mathcal{H}^{n-1}(y) \\ &= \liminf_{j\to\infty} \int_{\partial^*\Omega_j \cap B_1} F(r_j y, T_y \partial^*\Omega_j) \, d\mathcal{H}^{n-1}(y) \\ &\leqslant \lim_{j\to\infty} \int_{\tilde{K}_j \cap B_1} \tilde{F}(y) \, d\mathcal{H}^{n-1}(y) = \lim_{j\to\infty} \tilde{\mu}_j(B_1) = \omega_{n-1}\theta(0),\end{aligned}$$

which is the desired inequality.

(ii) $\tilde{\Omega}$ is the whole B_1;

(iii) $\tilde{\Omega}$ is the empty set.

The third alternative is easy to exclude. Indeed in such case $|\tilde{\Omega}_j|$ converges to 0. On the other hand, if we consider one of the two connected components of $B_1 \setminus U_{1/100}(\pi)$, say A, we know that $\mathcal{H}^{n-1}(\tilde{K}_j \cap A)$ converges to 0 (since $\tilde{\mu}_j \rightharpoonup^* \theta(0)\mathcal{H}^{n-1} \llcorner \pi$). The relative isoperimetric inequality implies that the volume of the largest connected component of $A \setminus \tilde{K}_j$ converges to the volume of A (cf. the argument for (3.55)).

Consider next alternative (ii). We argue similarly to step 2 of the previous subsection. Consider a fixed $\varepsilon > 0$ and set $\bar{\Gamma}^+ = (\partial B_1)^+ \setminus U_{3\varepsilon}(\pi)$, where $(\partial B_1)^+ = \partial B_1 \cap \{x_n > 0\}$, having set x_n a coodinate direction orthogonal to π: in particular $\mathcal{H}^{n-1}(\bar{\Gamma}^+) \geqslant \sigma_{n-1}/2 - C\varepsilon$.

Similary to step 2 consider

$$V = \bigcup_{1-\varepsilon \leqslant s \leqslant 1} s\bar{\Gamma}^+$$

consisting of the segments $S_x := [(1-\varepsilon)x, x]$ for every $x \in \bar{\Gamma}^+$. In particular for ε sufficiently small we have $V \subset B_1 \setminus U_{2\varepsilon}(\pi)$ and thus we know that

(a) $\mathcal{H}^{n-1}(\tilde{K}_j \cap V) \to 0$;

(b) $|V \setminus \tilde{\Omega}_j| \to 0$.

In particular, if we consider, as in step 3 above, the set $\tilde{G}_j^+ \subset \bar{\Gamma}^+$ of points for which either $S_x \cap \tilde{\Omega}_j = \emptyset$ or $S_x \cap \tilde{K}_j \neq \emptyset$, we conclude that $\mathcal{H}^{n-1}(\tilde{G}_j^+) = o_j(1)$.

If we define symmetrically the sets $\bar{\Gamma}^-$ and $\tilde{G}_j^-$, the same argument gives us $\mathcal{H}^{n-1}(\bar{\Gamma}^-) \geqslant \sigma_{n-1}/2 - C\varepsilon$ as well as $\mathcal{H}^{n-1}(\tilde{G}_j^-) = o_j(1)$. Choosing now ε small and an appropriate diagonal sequence, we conclude the existence of a sequence of sets $\tilde{\Gamma}_j = \bar{\Gamma}^+ \cup \bar{\Gamma}^- \setminus (\tilde{G}_j^+ \cup \tilde{G}_j^-) \subset \partial B_1 \setminus \tilde{K}_j$ with the property that

- $\mathcal{H}^{n-1}(\tilde{\Gamma}_j)$ converges to σ_{n-1};
- any two points $x, y \in \tilde{\Gamma}_j$ can be connected in $\bar{B}_1(0)$ with an arc which does not intersect $\tilde{K}_j$.

Therefore each $\tilde{\Gamma}_j$ must be contained in a unique equivalence class $\Gamma_{i(j)}(\tilde{K}_j, 0, 1)$. Coming to the sets K_j by scaling backwards, we find a sequence of sets $\Gamma_{i(j)}(K_j, 0, r_j)$ in the equivalence classes of Definition 3.9 such that

$$\lim_{j\to\infty} \frac{\mathcal{H}^{n-1}(\partial B_{0,r_j} \setminus \Gamma_{i(j)}(K_j, 0, r_j))}{r_j^{n-1}} = 0\,.$$

Considering the bound $\mathbf{F}(K_j \cap B_{r_j}) \leqslant \Lambda\mathcal{H}^{n-1}(\partial B_{0,r_j} \setminus \Gamma_{i(j)}(K_j, 0, r_j))$, we can pass to the rescaled measures again to conclude $\tilde{\mu}_j(B_1(0)) = o_j(1)$, clearly contradicting the assumption that $\mu_j(B_1(0))$ converges to the positive number $\theta(0)$.

3.4.5 *Proof of Proposition 3.14*

We first need the following estimates.

Lemma 3.16. *Let* K *be the* $(n-1)$*-rectifiable set obtained in the previous section. For every* x *where* K *has an approximate tangent plane* T_xK*, let* O_x *be the special orthogonal transformation of* $\mathbb{R}^n$ *mapping* $\{x_1 = 0\}$ *onto* T_xK *and set* $\bar{Q}_{x,r} = O_x(Q_{x,r})$ *and* $\bar{R}_{x,r,\varepsilon r} = O_x(R_{x,r\varepsilon r})$*.*

Then at $\mathcal{H}^{n-1}$ *almost every* $x \in K$ *the following holds: for every* $\varepsilon > 0$ *there exist* $r_0 = r_0(x) \leqslant \frac{1}{\sqrt{n}}\,\mathrm{dist}(x, H)$ *such that, for* $r \leqslant r_0/2$*,*

$$\begin{aligned} &(\lambda\omega_{n-1} - \varepsilon)r^{n-1} \leqslant \mu(B_{x,r}) \leqslant (\theta(x)\omega_{n-1} + \varepsilon)r^{n-1}, \\ &(\theta(x) - \varepsilon)r^{n-1} < \mu(\bar{Q}_{x,r}) < (\theta(x) + \varepsilon)r^{n-1}, \end{aligned} \tag{3.61}$$

$$\sup_{y\in B_{x,r_0},\, S\in G(n,n-1)} |F(y, S) - F(x, S)| \leqslant \varepsilon. \tag{3.62}$$

Moreover, for almost every such r*, there exists* $j_0(r) \in \mathbb{N}$ *such that for every* $j \geqslant j_0$*:*

$$(\theta(x)\omega_{n-1} - \varepsilon)r^{n-1} \leqslant \mathbf{F}(K_j, B_{x,r}) \leqslant (\theta(x)\omega_{n-1} + \varepsilon)r^{n-1}, \tag{3.63}$$

$$(\theta(x) - \varepsilon)r^{n-1} \leqslant \mathbf{F}(K_j, \bar{Q}_{x,r}) \leqslant (\theta(x) + \varepsilon)r^{n-1}, \qquad \mathbf{F}(K_j, Q_{x,r} \setminus \bar{R}_{x,r,\varepsilon r}) < \varepsilon r^{n-1}. \tag{3.64}$$

Proof. Fix a point x where T_xK exists and θ is approximately continuous: for the sake of simplicity, we can assume $x = 0$ and that, after a rotation, the approximate tangent space at 0 coincides with $T_K = \{x^n = 0\}$. For almost every $r \leqslant r_0/2$ we can suppose that $\mu(\partial B_r) = \mu(\partial Q_r) = \mu(\partial R_{r,\varepsilon r}) = 0$; moreover by rectifiability and the density lower bound (3.42), we also know that $B_r \cap K \subset U_{\varepsilon r}(T_K)$ (see the proof above). The second equation in (3.64) follows than by weak convergence. We also know that, up to further reducing r_0, for $r \leqslant r_0/2$, (3.45) and [12, Theorem 2.83] imply that

$$(\theta(0)\omega_{n-1} - \varepsilon)r^{n-1} < \mu(B_r) < (\theta(0)\omega_{n-1} + \varepsilon)r^{n-1}, \tag{3.65}$$

$$(\theta(0) - \varepsilon)r^{n-1} < \mu(Q_r) < (\theta(0) + \varepsilon)r^{n-1}. \tag{3.66}$$

Again by weak convergence, we recover (3.63) and the first equation in (3.64). Moreover (3.62) is a consequence of (3.41). Finally (3.61) follows from (3.65), (3.66) and $\theta \geqslant \lambda$, whereas the latter bound is a consequence of the previous subsection where we have shown $\theta(x) \geqslant F(x, T_xK)$. $\square$

Assume w.l.o.g. $x = 0$. Arguing by contradiction, we assume that $\theta(0) = F(0, T_0 K) + \sigma$ for some $\sigma > 0$ and let $\varepsilon < \min\{\frac{\sigma}{2}, \frac{\lambda\sigma}{4\Lambda}\}$. As a consequence of (3.64), there exist r and $j_0 = j_0(r)$ such that

$$\mathbf{F}(K_j, Q_r) > \left(F(0,T) + \frac{\sigma}{2}\right) r^{n-1}, \qquad \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < \frac{\lambda\sigma}{4\Lambda} r^{n-1}, \qquad \forall j \geqslant j_0 . \quad (3.67)$$

Consider the map $P \in D(0,r)$ defined in [44, Equation 3.14] which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane T_K and satisfies $\|P - \mathrm{Id}\|_\infty + \mathrm{Lip}\,(P - \mathrm{Id}) \leqslant C\sqrt{\varepsilon}$. Exploiting the fact that $\mathscr{P}(\mathbf{F}, H)$ is a deformation class and by almost minimality of K_j, we find that

$$\begin{aligned}\mathbf{F}(K_j, Q_r) - o_j(1) \leqslant\ & \underbrace{\mathbf{F}(P(K_j), P(R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1} + \underbrace{\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_2} \\ & + \underbrace{\mathbf{F}(P(K_j), P(Q_r \setminus R_{r,\varepsilon r}))}_{I_3} .\end{aligned}$$

By the properties of P and (3.62), we get $I_1 \leqslant (F(0,T_K) + \varepsilon)r^{n-1}$, while, by (3.67) and equation (2.10)

$$I_3 \leqslant \frac{\Lambda}{\lambda} (\mathrm{Lip}\,P)^{n-1}\, \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < (1 + C\sqrt{\varepsilon})^{n-1} \frac{\sigma}{4} r^{n-1} .$$

Since $\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) \leqslant \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^{n-1}\mathbf{F}(K_j, R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})$ and $R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r} \subset Q_{(1-\sqrt{\varepsilon})r} \setminus Q_r$, by (3.64) we can also bound

$$\begin{aligned} I_2 = \mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) &\leqslant \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^{n-1}\mathbf{F}(K_j, Q_r \setminus Q_{(1-\sqrt{\varepsilon})r}) \\ &\leqslant C(1+C\sqrt{\varepsilon})^{n-1}\Big((F(0,T_K)+\sigma+\varepsilon) - (F(0,T_K)+\sigma-\varepsilon)(1-\sqrt{\varepsilon})^{n-1}\Big) r^{n-1} \leqslant C\sqrt{\varepsilon} r^{n-1}. \end{aligned}$$

Hence, as $j \to \infty$, by (3.61)

$$\left(F(0,T_K) + \frac{\sigma}{2}\right) r^{n-1} \leqslant (F(0,T_K) + \varepsilon) r^{n-1} + C\sqrt{\varepsilon} r^{n-1} + (1 + C\sqrt{\varepsilon})^{n-1} \frac{\sigma}{4} r^{n-1} \ :$$

dividing by r^{n-1} and letting $\varepsilon \downarrow 0$ provides the desired contradiction.

We obtain that $\theta(x) \leqslant F(x, T_x K)$ almost everywhere and, together with the previous step, $\mu = F(x, T_x K)\mathcal{H}^{n-1} \llcorner K$.

3.5 ANISOTROPIC PLATEAU PROBLEM IN HIGHER CODIMENSION

In this section we present our paper [43], which concludes the series of works on the Plateau problem presented in Sections 3.3 and 3.4: we here provide a general and flexible existence result for sets that minimize an anisotropic energy, which can be applied to several notions of boundary conditions. In the spirit of the previous sections, we use the direct methods of the calculus of variations to find a minimizing measure via standard compactness arguments, and then we aim at proving that it is actually a fairly regular surface.

We have employed in Sections 3.3 and 3.4 several techniques to first establish the rectifiability of the limit measure: in the case of the area integrand this property has been deduced from the powerful result due to Preiss (see Theorem 3.4), while for the anisotropic case in codimension one, it has been obtained as a consequence of the theory of sets of finite perimeter and of the isoperimetric inequality on the sphere.

These two techniques are no longer available in the case of anisotropic problems in higher codimension (in particular due to the lack of a monotonicity formula). Nevertheless, a new rectifiability criterion has been obtained in [42, Theorem 1.2], see Theorem 4.2, for varifolds having positive lower density and a bounded anisotropic first variation, extending the celebrated rectifiability theorem of Allard [2].

3.5.1 *Preliminary assumptions*

We will assume throughout this section the following ellipticity condition on the Lagrangian F, which has been introduced in [7], and that is a geometric version of quasiconvexity, cf. [71]:

Definition 3.17 (Elliptic anisotropy, [7, 1.2])**.** The anisotropic Lagrangian F is said to be *elliptic* if there exists $\Gamma \geqslant 0$ such that, whenever $x \in \mathbb{R}^n$ and D is a d-disk centered in x and with radius r, then the inequality

$$\mathbf{F}^x(K, B_{x,r}) - \mathbf{F}^x(D, B_{x,r}) \geqslant \Gamma(\mathcal{H}^d(K \cap B_{x,r}) - \mathcal{H}^d(D))$$

holds for every d-rectifiable set K such that $K \cap \overline{B_{x,r}}$ is closed, $K \cap \partial B_{x,r} = \partial D \times \{0\}$ and K cannot be deformed into $\partial D \times \{0\}$ via a map $\varphi \in D(x,r)$.

Remark 3.18. Given a d-rectifiable set K and a deformation $\varphi \in D(x,r)$, using property (2.10), we deduce the quasiminimality property

$$\mathbf{F}(\varphi(K)) \leqslant \Lambda \mathcal{H}^d(\varphi(K)) \leqslant \Lambda(\mathrm{Lip}\,(\varphi))^d \mathcal{H}^d(K) \leqslant \frac{\Lambda}{\lambda}(\mathrm{Lip}\,(\varphi))^d \mathbf{F}(K). \tag{3.68}$$

Moreover, whenever $U \subset\subset \mathbb{R}^n$, the following holds

$$\sup_{x,y \in U\ S,T \in G} |F(x,T) - F(y,S)| \leqslant \omega_U(|x-y| + \|T-S\|), \tag{3.69}$$

for some modulus of continuity ω_U for F in $G(U)$.

We want to employ a generalization of Allard's rectifiability theorem for anisotropic energies recently obtained in [42]: in order to do so, we also assume another ellipticity property on the Lagrangian F, called atomic condition, see Definition 4.1, which guarantees the validity of Theorem 4.2, proven in the next chapter. This ellipticity condition is equivalent, in codimension one, to the strict convexity of F (hence to Definition 3.17), as shown in Theorem 4.3. Unfortunately, in the general codimension case we are not yet able to relate it to Definition 3.17.

3.5.2 *Main theorem*

The following theorem is the main result of Section 3.5 and establishes the behavior of minimizing sequences.

Theorem 3.19. *Let* $H \subset \mathbb{R}^n$ *be closed and* $\mathscr{P}(\mathbf{F}, H)$ *be a deformation class in the sense of Definition 3.2. Assume the infimum in Plateau problem* (3.2) *is finite and let* $(K_j) \subset \mathscr{P}(\mathbf{F}, H)$ *be a minimizing sequence. Then, up to subsequences, the measures* $\mu_j := F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^d \llcorner K_j$ *converge weakly** *in* $\mathbb{R}^n \setminus H$ *to the measure* $\mu = F(\cdot, T_{(\cdot)}K)\mathcal{H}^d \llcorner K$*, where* $K = \operatorname{spt}\mu \setminus H$ *is a* d*-rectifiable set. Furthermore the integral varifold naturally associated to* μ *is* F*-stationary in* $\mathbb{R}^n \setminus H$*. In particular,* $\liminf_j \mathbf{F}(K_j) \geqslant \mathbf{F}(K)$ *and if* $K \in \mathscr{P}(\mathbf{F}, H)$*, then* K *is a minimum for* (3.2).

Remark 3.20. We observe that in case the set K provided by the Theorem 3.19 belongs to $\mathscr{P}(\mathbf{F}, H)$, it has minimal $\mathbf{F}$ energy with respect to deformations in the classes $D(x, r)$ of Definition 3.1, with $x \in K$ and $H \cap B_r(x) = \emptyset$.

While the union of these classes is strictly contained in the class of all Lipschitz deformations, however such union is rich enough to generate the comparison sets in [8] which are needed to prove the almost everywhere regularity of K, under the assumption of the strict ellipticity in Definition 3.17, see [8, III.1 and III.3].

3.5.3 *Proof of Theorem 3.19*

Since the infimum in Plateau problem (3.2) is finite, there exists a minimizing sequence $(K_j) \subset \mathscr{P}(\mathbf{F}, H)$ and a Radon measure μ on $\mathbb{R}^n \setminus H$ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu, \qquad \text{as Radon measures on } \mathbb{R}^n \setminus H, \tag{3.70}$$

where $\mu_j = F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^d \llcorner K_j$. We set $K = \operatorname{spt}\mu$ and consider also the canonical density one rectifiable varifolds V^j associated to K_j:

$$V^j := \mathcal{H}^d \llcorner K_j \otimes \delta_{T_x K_j}.$$

Since K_j is a minimizing sequence in (3.2) and $F \geqslant \lambda$, we have the bound (for j large) $\|V^j\|(\mathbb{R}^n) \leqslant \frac{2m_0}{\lambda}$, and therefore we can assume that V^j converges to V in the sense of varifolds.

We now prove that V is F-stationary in $\mathbb{R}^n \setminus H$: arguing by contradiction, if V were not F-stationary, we would be able to exhibit a competing sequence of sets with strictly lower energy.

Assume indeed the existence of $g \in C^1_c(\mathbb{R}^n \setminus H, \mathbb{R}^n)$ such that $\delta_F V(g) < 0$. By standard partition of unity argument for the compact set $\operatorname{supp}(g)$ in the open set $\mathbb{R}^n \setminus H$, we get the existence of a ball $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$ and a vector field (not relabeled) $g \in C^1_c(B_{x,r}, \mathbb{R}^n)$ such that $\delta_F V(g) = -2c < 0$. For an arbitrarily small time $T > 0$, we gat that $(\mathrm{Id} + Tg) \in D(x, r)$. Moreover, there exists an open set $B_{x,r} \subset A \subset \mathbb{R}^n$, satisfying $\|(\mathrm{Id} + Tg)^\# V\|(\partial A) = 0$. We consequently have

$$\mathbf{F}((\mathrm{Id} + Tg)^\# V, A) \leqslant -cT + \mathbf{F}(V, A).$$

By lower semicontinuity and by the hypothesis on ∂A, for j large enough it holds true:

$$\mathbf{F}((\mathrm{Id} + Tg)^\# V^j, A) - \frac{1}{j} \leqslant -cT + \mathbf{F}(V_j, A) + \frac{1}{j}.$$

Note that $\mathbf{F}((\mathrm{Id}+Tg)^{\#}V^j, A) = \mathbf{F}((\mathrm{Id}+Tg)(K_j), A)$ as well as $\mathbf{F}(V^j, A) = \mathbf{F}(K_j, A)$: adding to both members $\mathbf{F}(K_j, \mathbb{R}^n \setminus A)$ and noting that $(\mathrm{Id}+Tg)(K_j) \setminus A = K_j \setminus A$, we obtain

$$\mathbf{F}((\mathrm{Id}+Tg)(K_j), \mathbb{R}^n) \leqslant \frac{2}{j} - cT + \mathbf{F}(K_j, \mathbb{R}^n).$$

Since $(\mathrm{Id}+Tg) \in D(x,r)$ and $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$, this is a contradiction with the minimizing property of the sequence (K_j) in the deformation class $\mathscr{P}(\mathbf{F}, H)$.

The limit varifold V satisfies moreover a density lower bound: there exists $\theta_0 > 0$ such that

$$\|V\|(B_{x,r}) \geqslant \theta_0\, \omega_d r^d\,, \qquad x \in \operatorname{spt}\|V\| \text{ and } r < d_x := \operatorname{dist}(x,H). \tag{3.71}$$

The proof is similar to Thorem 3.3, Step 1, and we show it in the Appendix 3.5.4. Combining this property with the F-stationarity in $\mathbb{R}^n \setminus H$ and applying Theorem 4.2, we conclude that V is a d-rectifiable varifold and in turn, that $\mu = \mathbf{F}(V, \cdot) = \theta \mathcal{H}^d \llcorner \tilde{K}$ for some countably $\mathcal{H}^d$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since K is the support of μ, then $\mathcal{H}^d(\tilde{K} \setminus K) = 0$. On the other hand, by differentiation of Hausdorff measures, (3.71) yields $\mathcal{H}^d(K \setminus \tilde{K}) = 0$. Hence K is d-rectifiable and

$$\mu = \theta \mathcal{H}^d \llcorner K. \tag{3.72}$$

We now proceed to compute the exact value of the density θ: to this end, we need the following Lemma (see also [38, Lemma 3.2]), whose proof is postponed to the Appendix 3.5.4.

Lemma 3.21. *Let K be the d-rectifiable set obtained in the previous section. For every x where K has an approximate tangent plane T_xK, let O_x be the special orthogonal transformation of $\mathbb{R}^n$ mapping $\{x_{d+1} = \cdots = x_n = 0\}$ onto T_xK and set $\bar{Q}_{x,r} = O_x(Q_{x,r})$ and $\bar{R}_{x,r,\varepsilon r} = O_x(R_{x,r\varepsilon r})$. At almost every $x \in K$ the following holds: for every $\varepsilon > 0$ there exist $r_0 = r_0(x) \leqslant \frac{1}{\sqrt{n+1}} \operatorname{dist}(x,H)$ such that, for $r \leqslant r_0/2$,*

$$(\theta_0 \omega_d - \varepsilon) r^d \leqslant \mu(B_{x,r}) \leqslant (\theta(x)\omega_d + \varepsilon) r^d, \qquad (\theta(x) - \varepsilon) r^d < \mu(\bar{Q}_{x,r}) < (\theta(x) + \varepsilon) r^d, \tag{3.73}$$

$$\sup_{y \in B_{x,r_0},\, S \in G} |F(y,S) - F(x,S)| \leqslant \varepsilon, \tag{3.74}$$

where $\theta_0 = \theta_0(n,d)$ is the universal lower bound obtained in (3.71). Moreover, for almost every such r, there exists $j_0(r) \in \mathbb{N}$ such that for every $j \geqslant j_0$:

$$(\theta(x)\omega_d - \varepsilon) r^d \leqslant \mathbf{F}(K_j, B_{x,r}) \leqslant (\theta(x)\omega_d + \varepsilon) r^d, \tag{3.75}$$

$$(\theta(x) - \varepsilon) r^d \leqslant \mathbf{F}(K_j, \bar{Q}_{x,r}) \leqslant (\theta(x) + \varepsilon) r^d, \qquad \mathbf{F}(K_j, \bar{Q}_{x,r} \setminus \bar{R}_{x,r,\varepsilon r}) < \varepsilon r^d. \tag{3.76}$$

We are now ready to complete the proof of Theorem 5.2, namely to show that $\liminf_j \mathbf{F}(K_j) \geqslant \mathbf{F}(K)$ and moreover $\mu = F(x, T_xK)\mathcal{H}^d \llcorner K$.
For the lower semicontinuity, it will be enough to show that $\theta(x) \geqslant F(x, T_xK)$, where $x \in K$ satisfies the properties of Lemma 3.21. Assume w.l.o.g. $x = 0$. Let us fix $\varepsilon < r_0/2$ and choose a radius r such that both r and $(1-\sqrt{\varepsilon})r$ satisfy properties (3.73)-(3.76): in order to apply the ellipticity assumption in Definition 3.17 of F, we need to compare our set with $T_0K \cap \partial B_r$.

We reach this comparison with the help of a map $P \in D(0, r)$ that squeezes a large portion of B_r onto T_0K. Before doing this, we need to preliminary deform our competing sequence into another one, of approximately the same energy, whose associated measures are concentrated near T_0K. In turn this, with the help of the density lower bound 3.71, can be achieved by applying a polyhedral deformation in B_{2r} outside the slab $R_{0,2r,\varepsilon r}$: this construction is obtained as in Theorem 3.3, Step 4. Once we have ensured that, up to a deformation $\phi \in D(0, 2r)$,

$$\mathcal{H}^d(\phi(K_j) \cap B_{2r} \setminus R_{0,2r,\varepsilon r}) = 0 \tag{3.77}$$

(see equation (3.20)) we can proceed to the squeezing deformation. With abuse of notation, we will rename this new sequence $(\phi(K_j))$ with (K_j). Consider now a map S satisfying

- $S = \mathrm{Id}$ in $\overline{B}_{1-\sqrt{\varepsilon}} \cup (\mathbb{R}^n \setminus \overline{B}_{1+\sqrt{\varepsilon}})$,
- $S(\partial B \cap U_\varepsilon(T_0K)) = \partial B \cap T_0K$,
- S stretches $\partial B \setminus U_\varepsilon(T_0K)$ onto $\partial B \setminus T_0K$.

It is not hard to construct an extension $S \in D(0, 1)$ fulfilling the previous requirements and such that $S|_{B_{1+\sqrt{\varepsilon}}\setminus B}$ and $S|_{B\setminus B_{1-\sqrt{\varepsilon}}}$ are interpolations between the values of S on the three spheres $S|_{\partial B_{1+\sqrt{\varepsilon}}}$, $S|_{\partial B}$ and $S|_{B\setminus B_{1-\sqrt{\varepsilon}}}$. One can also assume that $\|S - \mathrm{Id}\|_\infty + \mathrm{Lip}\,(S - \mathrm{Id}) \leqslant C\sqrt{\varepsilon}$ and then obtain the desired map by rescaling $P(\cdot) = rS(\frac{\cdot}{r})$.

We set $K_j' := P(K_j)$ and since $K_j \cap B_{(1-\sqrt{\varepsilon})r} = K_j' \cap B_{(1-\sqrt{\varepsilon})r}$, using Remark 3.18 and property (3.75), we estimate:

$$\begin{aligned}
\mathbf{F}\left(K_j, B_r\right) &\geqslant \mathbf{F}\left(K_j', B_{(1-\sqrt{\varepsilon})r}\right) \geqslant \mathbf{F}\left(K_j', B_r\right) - \mathbf{F}\left(K_j', B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&\geqslant \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^d \mathbf{F}\left(K_j, B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&= \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^d \left(\mathbf{F}(K_j, B_r) - \mathbf{F}\left(K_j, B_{(1-\sqrt{\varepsilon})r}\right)\right) \\
&\geqslant \mathbf{F}(K_j', B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^d(\theta(0)\omega_d + \varepsilon)r^d + \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^d(\theta(0)\omega_d - \varepsilon)(1 - \sqrt{\varepsilon})^d r^d \\
&\geqslant \mathbf{F}(K_j', B_r) - C\sqrt{\varepsilon} r^d.
\end{aligned} \tag{3.78}$$

We furthermore observe that $K_j' \cap B_r$ cannot be deformed via any map $Q \in D(0, r)$ onto $\partial B_r \cap T_0K$. Otherwise, being $\mathscr{P}(\mathbf{F}, H)$ a deformation class, there would exist a competitor $J_j \in \mathscr{P}(\mathbf{F}, H)$, εr^d-close in energy to $K_j'' := Q(P(K_j))$, with $K_j'' \cap B_r = \emptyset$. Since $K_j'' \cap (\mathbb{R}^n \setminus B_{(1+\sqrt{\varepsilon})r}) = K_j \cap (\mathbb{R}^n \setminus B_{(1+\sqrt{\varepsilon})r})$, using (3.75) and equation (3.68), we would get:

$$\begin{aligned}
\mathbf{F}(K_j) - \mathbf{F}(J_j) &\geqslant \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r}\right) - \mathbf{F}\left(K_j'', B_{(1+\sqrt{\varepsilon})r}\right) - \varepsilon r^d \\
&\geqslant \mathbf{F}(K_j, B_r) + \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \mathbf{F}\left(K_j', B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \varepsilon r^d \\
&\geqslant (\theta_0\omega_d - \varepsilon)r^d + (\theta(0)\omega_d + \varepsilon)((1 + \sqrt{\varepsilon})^d - 1)\left(1 - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^d\right) r^d - \varepsilon r^d \\
&\geqslant (\theta_0\omega_d - C\sqrt{\varepsilon})r^d > 0,
\end{aligned}$$

which contradicts the minimizing property of the sequence $\{K_j\}$ if ε is small enough.

In order to apply the ellipticity condition in Definition 3.17, we want to costruct another closed set $\tilde{K}_j$ such that

(i) $\tilde{K}_j \cap \partial B_r = (\partial B_r \cap T_0 K)$,

(ii) $\tilde{K}_j \subset \overline{B_r}$ cannot be deformed via any map $Q \in D(0, r)$ onto $\partial B_r \cap T_0 K$,

(iii) $\mathbf{F}^0(\tilde{K}_j, B_r) = \mathbf{F}^0(K_j', B_r)$.

We can achieve this set in the following way

$$\tilde{K}_j := (\partial B_r \cap T_0 K) \cup (K_j' \cap \overline{B_r} \cap R_{0,r,\varepsilon r} \setminus \{\|x_{||}\| > r - \|x_\perp\|\}),$$

where $x_{||}$ and $x_\perp$ denote respectively the projections of x on $T_0 K$ and its orthogonal linear subspace. Indeed condition (i) is straightforward by construction, condition (ii) is a direct consequence of the fact that $K_j' \cap B_r$ cannot be deformed via any map $Q \in D(0, r)$ onto $\partial B_r \cap T_0 K$. Condition (iii) follows by (3.77) and the properties of the map S, for ε small enough.

Therefore, the ellipticity of F, (3.74), (3.75), (3.68) and (3.71) imply that

$$\mathbf{F}^0(T_0 K, B_r) \leqslant \mathbf{F}^0(\tilde{K}_j, B_r) = \mathbf{F}^0(K_j', B_r) \leqslant \mathbf{F}(K_j', B_r) + C\varepsilon r^d. \tag{3.79}$$

We can now sum up as follows

$$\begin{aligned}\theta(0)\omega_d r^d &\overset{(3.75)}{\geqslant} \mathbf{F}(K_j, B_r) - \varepsilon r^d \overset{(3.78)}{\geqslant} \mathbf{F}(K_j', B_r) - C\sqrt{\varepsilon} r^d \\ &\overset{(3.79)}{\geqslant} \mathbf{F}^0(T_0 K, B_r) - C\sqrt{\varepsilon} r^d = F(0, T_0 K)\omega_d r^d - C\sqrt{\varepsilon} r^d\end{aligned}$$

which easily implies the desired inequality $\theta(0) \geqslant F(0, T_0 K)$.

To get the last claim, we prove that $\theta(x) \leqslant F(x, T_x K)$ for almost every $x \in K$ (again satisfying the setting of Lemma 3.21). We remark that this concluding argument is in the spirit of Theorem 3.3, Step 5. Take as usual w.l.o.g. $x = 0$. Arguing by contradiction, we assume that $\theta(0) = F(0, T_0 K) + \sigma$ for some $\sigma > 0$ and let $\varepsilon < \min\{\frac{\sigma}{2}, \frac{\lambda\sigma}{4\Lambda}\}$. As a consequence of (3.76), there exist r and $j_0 = j_0(r)$ such that

$$\mathbf{F}(K_j, Q_r) > \left(F(0, T) + \frac{\sigma}{2}\right) r^d, \qquad \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < \frac{\lambda\sigma}{4\Lambda} r^d, \qquad \forall j \geqslant j_0. \tag{3.80}$$

Consider the map $P \in D(0, r)$ defined in [44, Equation 3.14] which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane $T_0 K$ and satisfies $\|P - \mathrm{Id}\|_\infty + \mathrm{Lip}\,(P - \mathrm{Id}) \leqslant C\sqrt{\varepsilon}$. Exploiting the fact that $\mathscr{P}(\mathbf{F}, H)$ is a deformation class and by almost minimality of K_j, we find that

$$\begin{aligned}\mathbf{F}(K_j, Q_r) - o_j(1) \leqslant{}& \underbrace{\mathbf{F}(P(K_j), P(R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1} + \underbrace{\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_2} \\ &+ \underbrace{\mathbf{F}(P(K_j), P(Q_r \setminus R_{r,\varepsilon r}))}_{I_3}.\end{aligned}$$

By the properties of P and (3.74), we get $I_1 \leqslant (F(0, T_0K) + \varepsilon)r^d$, while, by (3.80) and equation (3.68)

$$I_3 \leqslant \frac{\Lambda}{\lambda} (\operatorname{Lip} P)^d \, \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < (1 + C\sqrt{\varepsilon})^d \, \frac{\sigma}{4} \, r^d .$$

Since $\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) \leqslant \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^d \mathbf{F}(K_j, R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})$ and $R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r} \subset Q_{(1-\sqrt{\varepsilon})r} \setminus Q_r$, by (3.76) we can also bound

$$\begin{aligned} I_2 = \mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) &\leqslant \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^d \mathbf{F}(K_j, Q_r \setminus Q_{(1-\sqrt{\varepsilon})r}) \\ &\leqslant C(1 + C\sqrt{\varepsilon})^d \Big((F(0, T_0K) + \sigma + \varepsilon) - (F(0, T_0K) + \sigma - \varepsilon)(1 - \sqrt{\varepsilon})^d \Big) \, r^d \leqslant C\sqrt{\varepsilon} r^d. \end{aligned}$$

Hence, as $j \to \infty$, by (3.73)

$$\Big(F(0, T_0K) + \frac{\sigma}{2} \Big) r^d \leqslant (F(0, T_0K) + \varepsilon) r^d + C\sqrt{\varepsilon} r^d + (1 + C\sqrt{\varepsilon})^d \, \frac{\sigma}{4} \, r^d \ :$$

dividing by r^d and letting $\varepsilon \downarrow 0$ provides the desired contradiction.

We obtain that $\theta(x) \leqslant F(x, T_xK)$ almost everywhere and, together with the previous step, $\mu = F(x, T_xK)\mathcal{H}^d \llcorner K$.

3.5.4 *Appendix*

In this appendix, we prove for the reader's convenience some results concerning the density bounds that are used in the proof of the Theorem 3.19. These results are similar to the ones proven for Theorems 3.3 and 3.13.

Proof of the lower density estimates

Consider the limiting measure μ obtained in (3.70). We show in this section the existence of $\theta_0 = \theta_0(n, d) > 0$ such that

$$\mu(B_{x,r}) \geqslant \theta_0 \, \omega_d r^d , \qquad x \in \operatorname{spt} \mu \text{ and } r < d_x := \operatorname{dist}(x, H). \tag{3.81}$$

To this end, it is sufficient to prove the existence of $\beta = \beta(n, d) > 0$ such that

$$\mu(Q_{x,l}) \geqslant \beta \, l^d , \qquad x \in \operatorname{spt} \mu \text{ and } l < 2d_x/\sqrt{n} .$$

Let us assume by contradiction that there exist $x \in \operatorname{spt} \mu$ and $l < 2d_x/\sqrt{n}$ such that

$$\frac{\mu(Q_{x,l})^{\frac{1}{d}}}{l} < \beta.$$

We claim that this assumption, for β chosen sufficiently small depending only on d and n, implies that for some $l_\infty \in (0, l)$

$$\mu(Q_{x,l_\infty}) = 0, \tag{3.82}$$

which is a contradiction with the property of x to be a point of $\operatorname{spt} \mu$. In order to prove (3.82), we assume that $\mu(\partial Q_{x,l}) = 0$, which is true for a.e. l.

To prove (3.82), we construct a sequence of nested cubes $Q_i = Q_{x,l_i}$ such that, if β is sufficiently small, the following holds:

(i) $Q_0 = Q_{x,l}$;

(ii) $\mu(\partial Q_{x,l_i}) = 0$;

(iii) setting $m_i := \mu(Q_i)$ then:
$$\frac{m_i^{\frac{1}{d}}}{l_i} < \beta;$$

(iv) $m_{i+1} \leqslant (1 - \frac{1}{k_2})m_i$, where $k_2 := \frac{\Lambda k_1}{\lambda}$ and k_1 is the constant in Theorem 5.4 (6);

(v) $(1-4\varepsilon_i)l_i \geqslant l_{i+1} \geqslant (1-6\varepsilon_i)l_i$, where
$$\varepsilon_i := \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} \tag{3.83}$$
and $k = \max\{6, 6/(1-(\frac{k_2-1}{k_2})^{\frac{1}{d}})\}$ is a universal constant;

(vi) $\lim_i m_i = 0$ and $\lim_i l_i > 0$.

Following [32], we are going to construct the sequence of cubes by induction: the cube Q_0 satisfies by construction hypotheses (i)-(iii). Suppose that cubes until step i are already defined.

Setting $m_i^j := \mathbf{F}(K_j, Q_i)$, we cover Q_i with the family $\Lambda_{\varepsilon_i l_i}(Q_i)$ of closed cubes with edge length $\varepsilon_i l_i$ and we set C_1^i and C_2^i for the corresponding sets defined in (3.6). We define Q_{i+1} to be the internal cube given by the construction, and we note that C_2^i and Q_{i+1} are non-empty if, for instance,
$$\varepsilon_i = \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} < \frac{1}{k} \leqslant \frac{1}{6},$$
which is guaranteed by our choice of k. Observe moreover that $C_1^i \cup C_2^i$ is a strip of width at most $2\varepsilon_i l_i$ around ∂Q_i, hence the side l_{i+1} of Q_{i+1} satisfies $(1-4\varepsilon_i)l_i \leqslant l_{i+1} < (1-2\varepsilon_i)l_i$.

Now we apply Theorem 3.7 to Q_i with $E = K_j$ and $\varepsilon = \varepsilon_i l_i$, obtaining the map $\Phi_{i,j} = \Phi_{\varepsilon_i l_i, K_j}$. We claim that, for every j sufficiently large,
$$m_i^j \leqslant k_2(m_i^j - m_{i+1}^j) + o_j(1). \tag{3.84}$$
Indeed, since (K_j) is a minimizing sequence, by the definition of deformation class and by (2.10), we have that
$$\begin{aligned} m_i^j &\leqslant m_i + o_j(1) \leqslant \Lambda\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_i\right)\right) + o_j(1) \\ &= \Lambda\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) + \Lambda\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap (C_1^i \cup C_2^i)\right)\right) + o_j(1) \\ &\leqslant \Lambda k_1\mathcal{H}^d\left(K_j \cap (C_1^i \cup C_2^i)\right) + o_j(1) = k_2(m_i^j - m_{i+1}^j) + o_j(1). \end{aligned}$$
The last inequality holds because $\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) = 0$ for j large enough: otherwise, by property (5) of Theorem 5.4, there would exist $T \in \Lambda^*_{\varepsilon_i l_i, d}(Q_{i+1})$ such that $\mathcal{H}^d(\Phi_{i,j}(K_j \cap T)) = \mathcal{H}^d(T)$. Together with property (ii) and by (2.10), this would imply
$$l_i^d\varepsilon_i^d = \mathcal{H}^d(T) \leqslant \mathcal{H}^d\left(\Phi_{i,j}\left(K_j\right) \cap Q_i\right) \leqslant k_1\mathcal{H}^d\left(K_j \cap Q_i\right) \leqslant \frac{k_1}{\lambda}m_i^j \to \frac{k_1}{\lambda}m_i$$

and therefore, substituting (3.83),

$$\frac{m_i}{k^d \beta^d} \leqslant \frac{k_1}{\lambda} m_i,$$

which is false if β is sufficiently small ($m_i > 0$ because $x \in \mathrm{spt}(\mu)$). Passing to the limit in j in (3.84) we obtain (iv):

$$m_{i+1} \leqslant \frac{k_2 - 1}{k_2} m_i. \tag{3.85}$$

Since $l_{i+1} \geqslant (1 - 4\varepsilon_i) l_i$, we can slightly shrink the cube Q_{i+1} to a concentric cube Q'_{i+1} with $l'_{i+1} \geqslant (1 - 6\varepsilon_i) l_i > 0$, $\mu(\partial Q'_{i+1}) = 0$ and for which (iv) still holds, just getting a lower value for m_{i+1}. With a slight abuse of notation, we rename this last cube Q'_{i+1} as Q_{i+1}.

We now show (iii). Using (3.85) and condition (iii) for Q_i, we obtain

$$\frac{m_{i+1}^{\frac{1}{d}}}{l_{i+1}} \leqslant \left(\frac{k_2 - 1}{k_2}\right)^{\frac{1}{d}} \frac{m_i^{\frac{1}{d}}}{(1 - 6\varepsilon_i) l_i} < \left(\frac{k_2 - 1}{k_2}\right)^{\frac{1}{d}} \frac{\beta}{1 - 6\varepsilon_i}.$$

The last quantity will be less than β if

$$\left(\frac{k_2 - 1}{k_2}\right)^{\frac{1}{d}} \leqslant 1 - 6\varepsilon_i = 1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}. \tag{3.86}$$

In turn, inequality (3.86) is true because (iii) holds for Q_i, provided we choose $k \geqslant 6/\big(1 - (1 - 1/k_2)^{\frac{1}{d}}\big)$. Furthermore, estimating $\varepsilon_0 < 1/k$ by (iii) and (v), we also have $\varepsilon_{i+1} \leqslant \varepsilon_i$.

We are left to prove (vi): $\lim_i m_i = 0$ follows directly from (iv); regarding the non degeneracy of the cubes, note that

$$\begin{aligned}
\frac{l_\infty}{l_0} := \liminf_i \frac{l_i}{l_0} &\geqslant \prod_{i=0}^{\infty} (1 - 6\varepsilon_i) = \prod_{i=0}^{\infty} \left(1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}\right) \\
&\geqslant \prod_{i=0}^{\infty} \left(1 - \frac{6 m_0^{\frac{1}{d}}}{k\beta l_0 \prod_{h=0}^{i-1} (1 - 6\varepsilon_h)} \left(\frac{k_2 - 1}{k_2}\right)^{\frac{i}{d}}\right) \\
&\geqslant \prod_{i=0}^{\infty} \left(1 - \frac{6}{k(1 - 6\varepsilon_0)^i} \left(\frac{k_2 - 1}{k_2}\right)^{\frac{i}{d}}\right),
\end{aligned}$$

where we used $\varepsilon_h \leqslant \varepsilon_0$ in the last inequality. Since $\varepsilon_0 < 1/k$, the last product is strictly positive, provided

$$k > \frac{6}{1 - \left(\frac{k_2 - 1}{k_2}\right)^{\frac{1}{d}}},$$

which is guaranteed by our choice of k. We conclude that $l_\infty > 0$, which ensures claim (3.82).

Proof of Lemma 3.21

Fix a point x where $T_x K$ exists and θ is approximately continuous: for the sake of simplicity, we can assume $x = 0$ and that, after a rotation, the approximate tangent space at 0 coincides with $T_K = \{x^{d+1} = ... = x^n = 0\}$. For almost every $r \leqslant r_0/2$ we can suppose that $\mu(\partial B_r) = \mu(\partial Q_r) = \mu(\partial R_{r,\varepsilon r}) = 0$; moreover by rectifiability and the density lower bound (3.71), we also know that $B_r \cap K \subset U_{\varepsilon r}(T_K)$ (for the proof see [37, eq 2.11]). The second equation in (3.76) follows than by weak convergence. We also know that, up to further reducing r_0, for $r \leqslant r_0/2$, (3.72) and [12, Theorem 2.83] imply that

$$(\theta(0)\omega_d - \varepsilon)r^d < \mu(B_r) < (\theta(0)\omega_d + \varepsilon)r^d, \tag{3.87}$$

$$(\theta(0) - \varepsilon)r^d < \mu(Q_r) < (\theta(0) + \varepsilon)r^d. \tag{3.88}$$

Again by weak convergence, we recover (3.75) and the first equation in (3.76). Moreover (3.74) is a consequence of (3.69). Finally (3.73) follows from (3.87), (3.88) and (3.71).

3.6 APPLICATION 1: SOLUTION TO HARRISON-PUGH FORMULATION

We wish to apply Theorems 3.3, 3.13 and 3.19 to three definitions of boundary conditions. The first class of competitors is the natural generalization of the one considered by Harrison and Pugh in [56]. The main theorem of this section is Theorem 3.23, which has been first obtained by De Lellis, Ghiraldin and Maggi in [37] for the isotropic case in codimension one and then extended by us in joint work in [44],[38] and [43] respectively to the higher codimension isotropic case, codimension one anisotropic setting and higher codimension anisotropic case.

3.6.1 *Defintion of the class of competitors*

Definition 3.22. Let H be a closed set in $\mathbb{R}^n$.
Let us consider the family

$$\mathcal{C}_H = \left\{\gamma : S^{n-d} \to \mathbb{R}^n \setminus H : \gamma \text{ is a smooth embedding of } S^{n-d} \text{ into } \mathbb{R}^n\right\}.$$

We say that $\mathcal{C} \subset \mathcal{C}_H$ is closed by isotopy (with respect to H) if $\mathcal{C}$ contains all elements $\gamma' \in \mathcal{C}_H$ belonging to the same smooth isotopy class $[\gamma]$ in $\pi_{n-d}(\mathbb{R}^n \setminus H)$ of any $\gamma \in \mathcal{C}$, see [59, Ch. 8]. Given $\mathcal{C} \subset \mathcal{C}_H$ closed by isotopy, we say that a relatively closed subset K of $\mathbb{R}^n \setminus H$ is a $\mathcal{C}$-spanning set of H if

$$K \cap \gamma \neq \emptyset \text{ for every } \gamma \in \mathcal{C}.$$

We denote by $\mathcal{F}(H, \mathcal{C})$ the family of countably $\mathcal{H}^d$-rectifiable sets which are $\mathcal{C}$-spanning sets of H.

We can prove the following closure property for the class $\mathcal{F}(H, \mathcal{C})$:

Theorem 3.23. *Let H be a closed subset of $\mathbb{R}^n$ and $\mathcal{C}$ be closed by isotopy with respect to H. Then:*

(a) For any Lagrangian F, $\mathcal{F}(H, \mathcal{C})$ is a deformation class in the sense of Definition 3.2.

(b) If $\{K_j\} \subset \mathcal{F}(H, \mathcal{C})$ *is a minimizing sequence and* K *is any set associated to* $\{K_j\}$ *by Theorems 3.3, 3.13, 3.19, then* $K \in \mathcal{F}(H, \mathcal{C})$ *and thus* K *is a minimizer.*

(c) If $d = n - 1$, $\mathcal{F}(H, \mathcal{C})$ *is also a good class in the sense of Definition 3.10. Moreover, the set* K *in (b) is an* $(\mathbf{F}, 0, \infty)$*-minimal set in* $\mathbb{R}^n \setminus H$ *in the sense of Almgren* [8].

3.6.2 *A preliminary Lemma*

The proof of Theorem 3.23(c) relies on the following elementary geometric remark.

Lemma 3.24. *Assumne* $d = n - 1$. *If* $K \in \mathcal{F}(H, \mathcal{C})$, $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$, *and* $\gamma \in \mathcal{C}$, *then either* $\gamma \cap (K \setminus B_{x,r}) \neq \emptyset$, *or there exists a connected component* σ *of* $\gamma \cap \overline{B_{x,r}}$ *which is homeomorphic to an interval and whose end-points belong to two distinct equivalence classes* $\Gamma_i(x, r)$*'s of* $\partial B_{x,r} \setminus K$ *in the sense of Definition 3.10.*

Proof of Lemma 3.24. Step one: We first prove the lemma under the assumption that γ and $\partial B_{x,r}$ intersect transversally. Indeed, if this is the case, then we can find finitely many mutually disjoint closed arcs $I_i \subset S^1$, $I_i = [a_i, b_i]$, such that $\gamma \cap B_{x,r} = \bigcup_i \gamma((a_i, b_i))$ and $\gamma \cap \partial B_{x,r} = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}$. Arguing by contradiction we may assume that for every i there exists an equivalence class $\Gamma_i(x, r)$ of $\partial B_{x,r} \setminus K$ such that $\gamma(a_i), \gamma(b_i) \in \Gamma_i(x, r)$. By connectedness of the associated $\Omega_i(x, r)$ (see the discussion after Definition 3.10) and the definition of $\Gamma_i(x, r)$, for each i we can find a smooth embedding $\tau_i : I_i \to \Omega_i(x, r) \cup \Gamma_i(x, r)$ such that $\tau_i(a_i) = \gamma(a_i)$ and $\tau_i(b_i) = \gamma(b_i)$; moreover since $n \geqslant 1$, one can easily achieve this by enforcing $\tau_i(I_i) \cap \tau_j(I_j) = \emptyset$. Finally, we define $\bar{\gamma}$ by setting $\bar{\gamma} = \gamma$ on $S^1 \setminus \bigcup_i I_i$, and $\bar{\gamma} = \tau_i$ on I_i. In this way, $[\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^n \setminus H)$, with $\bar{\gamma} \cap K \setminus \overline{B_{x,r}} = \gamma \cap K \setminus \overline{B_{x,r}} = \emptyset$ and $\bar{\gamma} \cap K \cap \overline{B_{x,r}} = \emptyset$ by construction; that is, $\bar{\gamma} \cap K = \emptyset$. Since there exists $\widetilde{\gamma} \in \mathcal{C}_H$ with $[\widetilde{\gamma}] = [\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^n \setminus H)$ which is uniformly close to $\bar{\gamma}$, we infer $\widetilde{\gamma} \cap K = \emptyset$, and thus find a contradiction to $K \in \mathcal{F}(H, \mathcal{C})$.

Step two: We prove the lemma for any ball $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$. Since γ is a smooth embedding, by Sard's theorem we find that γ and $\partial B_{x,s}$ intersect transversally for a.e. $s > 0$. In particular, given ε small enough, for any such $s \in (r - \varepsilon, r)$ we can construct a smooth diffeomorphism $f_s : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_s = \mathrm{Id}$ on $\mathbb{R}^n \setminus B_{x,r+2\varepsilon}$ and $f_s(y) = x + (r/s)(y - x)$ for $y \in B_{x,r+\varepsilon}$, in such a way that

$$f_s \to \mathrm{Id} \text{ uniformly on } \mathbb{R}^n \text{ as } s \to r^- . \tag{3.89}$$

We claim that one can apply step one to $f_s \circ \gamma$. Indeed, the facts that $f_s \circ \gamma \in \mathcal{C}$ and $f_s \circ \gamma$ and $\partial B_{x,r}$ intersect transversally are straightforward; moreover, since $\mathrm{dist}(\gamma, K \cap \partial B_{x,r}) > 0$ and by (3.89) one easily entails that $(f_s \circ \gamma) \cap K \setminus B_{x,r} = \emptyset$. Hence, by step one, there exists a proper circular arc $I = [a_s, b_s] \subset S^1$ such that $f_s(\gamma(a_s)) \in \Gamma_i(x, r)$ and $f_s(\gamma(b_s)) \in \Gamma_j(x, r)$ where $\Gamma_i(x, r) \neq \Gamma_j(x, r)$ are two equivalence classes of $\partial B_{x,r} \setminus K$ and $(f_s \circ \gamma)(a_s, b_s) \subset B_{x,r}$. Up to subsequences, we can assume that $a_s \to \bar{a}$, $b_s \to \bar{b}$ and the arc $[a_s, b_s]$ converges to $[\bar{a}, \bar{b}]$. It follows that $\gamma(\bar{a})$ and $\gamma(\bar{b})$ must be belong to distinct equivalence classes of $\partial B_{x,r} \setminus K$, otherwise, by (3.89), $f_s(\gamma(a_s))$ and $f_s(\gamma(b_s))$ would belong to the same equivalence class for some s close enough to r. By (3.89), we also have $\gamma([\bar{a}, \bar{b}]) \subset \overline{B_{x,r}}$. □

3.6.3 *Proof of Theorem 3.23*

Step one: We start by proving that $\mathcal{F}(H, \mathcal{C})$ is a deformation class in the sense of Definition 3.2: let $\widetilde{K} \in \mathcal{F}(H, \mathcal{C})$, $x \in \widetilde{K}$, $r \in (0, \operatorname{dist}(x, H))$ and $\varphi \in D(x, r)$. We show that $\varphi(\widetilde{K}) \in \mathcal{F}(H, \mathcal{C})$ arguing by contradiction: assume that $\gamma(S^{n-d}) \cap \varphi(\widetilde{K}) = \emptyset$ for some $\gamma \in \mathcal{C}$ and, without loss of generality, suppose also that $\gamma(S^{n-d}) \cap (\widetilde{K} \setminus B_{x,r}) = \emptyset$. By Definition 3.1 there exists a sequence

$$(\varphi_j) \subset \mathfrak{D}(x, r) \qquad \text{such that} \qquad \lim_j \|\varphi_j - \varphi\|_{C^0} = 0.$$

Since $\gamma(S^{n-d})$ is compact and $\varphi_j = \mathrm{Id}$ outside $B_{x,r}$, for j sufficiently large $\gamma(S^{n-d}) \cap \varphi_j(\widetilde{K}) = \emptyset$; moreover φ_j is invertible, hence $\varphi_j^{-1}\left(\gamma(S^{n-d})\right) \cap \widetilde{K} = \emptyset$. But the property for φ_j of being isotopic to the identity implies $\varphi_j^{-1} \circ \gamma \in \mathcal{C}$, which contradicts $\widetilde{K} \in \mathcal{F}(H, \mathcal{C})$. This proves (a).

Step two: By step one, given a minimizing sequence $\{K_j\} \subset \mathcal{F}(H, \mathcal{C})$, we can find a set K with the properties stated in Theorems 3.3, 3.13, 3.19. In order to prove the statement in (b), we just need to show that $K \in \mathcal{F}(H, \mathcal{C})$. Suppose by contradiction that some $\gamma \in \mathcal{C}$ does not intersect K. Since both γ and K are compact, there exists a positive ε such that the tubular neighborhood $U_{2\varepsilon}(\gamma)$ does not intersect K and is contained in $\mathbb{R}^n \setminus H$. Hence $\mu(U_{2\varepsilon}(\gamma)) = 0$, and thus

$$\lim_{j \to \infty} \mathcal{H}^d(K_j \cap U_\varepsilon(\gamma)) = 0\,. \tag{3.90}$$

Observe that there is a diffeomorphism $\Phi : S^{n-d} \times D_\varepsilon \to U_\varepsilon(\gamma)$ such that $\Phi|_{S^{n-d} \times \{0\}} = \gamma$, where $D_\rho := \{y \in \mathbb{R}^d : |y| < \rho\}$. Denote by γ_y the parallel curve $\Phi|_{S^{n-d} \times \{y\}}$. Then $\gamma_y \in [\gamma] \in \pi_{n-d}(\mathbb{R}^n \setminus H)$ for every $y \in D_\varepsilon$. Thus we must have $K_j \cap (\gamma \times \{y\}) \neq \emptyset$ for every $y \in D_\varepsilon$ and every $j \in \mathbb{N}$. If we set $\hat{\pi} : S^{n-d} \times D_\varepsilon \to D_\varepsilon$ to be the projection on the second factor and define $\pi : U_\varepsilon(\gamma) \to D_\varepsilon$ as $\hat{\pi} \circ \Phi^{-1}$, then π is a Lipschitz map. The coarea formula then implies

$$\mathcal{H}^d(K_j \cap U_\varepsilon(\gamma)) \geqslant \frac{\omega_d\, \varepsilon^d}{(\mathrm{Lip}\,(\pi))^d} > 0\,,$$

which contradicts (3.90). This shows that $K \in \mathcal{F}(H, \mathcal{C})$, as claimed in (b).

Step three: From now on assume that $d = n - 1$. We show that $\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 3.10. To this end, we fix $V \in \mathcal{F}(H, \mathcal{C})$ and $x \in V$, and prove that for a.e. $r \in (0, \operatorname{dist}(x, H))$ one has $V' \in \mathcal{F}(H, \mathcal{C})$, where V' is a cup competitor of V in $B_{x,r}$. We thus fix $\gamma \in \mathcal{C}$ and, without loss of generality, we assume that $\gamma \cap (V \setminus B_{x,r}) = \emptyset$. By Lemma 3.24, γ has an arc contained in $\overline{B_{x,r}}$ homeomorphic to $[0, 1]$ and whose end-points belong to distinct equivalence classes of $\partial B_{x,r} \setminus V$; we denote by $\sigma : [0, 1] \to \overline{B_{x,r}}$ a parametrization of this arc. Since V' must contain all but one $\Gamma_i(x, r)$, either $\sigma(0)$ or $\sigma(1)$ belongs to $\gamma \cap V' \cap \partial B_{x,r}$.

Step four: We show that K is an $(\mathbf{F}, 0, \infty)$-minimal set, i.e.

$$\mathbf{F}(K) \leqslant \mathbf{F}(\varphi(K))$$

whenever $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map such that $\varphi = \mathrm{Id}$ on $\mathbb{R}^n \setminus B_{x,r}$ and $\varphi(B_{x,r}) \subset B_{x,r}$ for some $x \in \mathbb{R}^n \setminus H$ and $r < \operatorname{dist}(x, H)$. To this end, it suffices to show that given such a function φ, then $\varphi(K) \in \mathcal{F}(H, \mathcal{C})$. We fix $\gamma \in \mathcal{C}$ and directly assume that $\gamma \cap (K \setminus B_{x,\rho}) = \emptyset$ for some $\rho \in (r, \mathrm{dist}(x, H))$. By Lemma 3.24, there exist two distinct connected components A and A' of $B_{x,\rho} \setminus K$ and a connected component of $\gamma \cap \overline{B_{x,\rho}}$ having end-points $p \in \overline{A} \cap \partial B_{x,\rho}$ and $q \in \overline{A'} \cap \partial B_{x,\rho}$.

We complete the proof by showing that $p = \varphi(p)$ and $q = \varphi(q)$ are adherent to distinct connected components of $B_{x,\rho} \setminus \varphi(K)$. We argue by contradiction, and denote by Ω the connected component of $B_{x,\rho} \setminus \varphi(K)$ with $p, q \in \overline{\Omega}$. If h denotes the restriction of φ to $\overline{A}$, then the topological degree of h is defined on $\mathbb{R}^n \setminus h(\partial A)$, thus in Ω.

Since $\varphi = \mathrm{Id}$ in a neighborhood of $\partial B_{x,\rho}$, one has $\deg(h, p') = 1$ for every p' sufficiently close to p; since the degree is locally constant and Ω is connected, $\deg(h, \cdot) = 1$ on Ω. In particular, for every $y \in \Omega$, $\varphi^{-1}(y) \cap A \neq \emptyset$. We apply this with $y = q'$ for some $q' \in \Omega$ sufficiently close to q. Let $w \in \varphi^{-1}(q')$: since $\varphi = \mathrm{Id}$ on $\mathbb{R}^n \setminus B_{x,r}$, if $|q'| > r$ then $w = q'$, and thus $q' \in A$. In other words, every $q' \in B_{x,\rho}$ sufficiently close to q is contained in A. We may thus connect in A any pair of points $p', q' \in B_{x,\rho}$ which are sufficiently close to p and q respectively, that is to say, p and q can be connected in A. This contradicts $A \neq A'$, and completes the proof of the fact that K is a $(\mathbf{F}, 0, \infty)$-minimal set. This concludes the proof of (c).

3.7 APPLICATION 2: SOLUTION TO DAVID FORMULATION

The second type of boundary condition we want to consider is the one related to the notion of "sliding minimizers" introduced by David in [31, 30]. The main theorem of this section is Theorem 3.27, which has been first obtained by De Lellis, Ghiraldin and Maggi in [37] for the isotropic case in codimension one and then extended by us in joint work in [44] to the isotropic case in higher codimension.

3.7.1 *Defintion of the class of competitors*

Just through this section we will assum $F \equiv 1$ and we will denote $\mathscr{P}(H) := \mathscr{P}(\mathbf{F}, H)$.

Definition 3.25 (Sliding minimizers)**.** Let $H \subset \mathbb{R}^n$ be closed and $K_0 \subset \mathbb{R}^n \setminus H$ be relatively closed. We denote by $\Sigma(H)$ the family of Lipschitz maps $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a continuous map $\Phi : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(1, \cdot) = \varphi$, $\Phi(0, \cdot) = \mathrm{Id}$ and $\Phi(t, H) \subset H$ for every $t \in [0, 1]$. We then define

$$\mathcal{A}(H, K_0) = \{K : K = \varphi(K_0) \text{ for some } \varphi \in \Sigma(H)\}$$

and say that K_0 is a sliding minimizer if $\mathcal{H}^d(K_0) = \inf\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0)\}$.

Remark 3.26. For every $K_0 \subset \mathbb{R}^n \setminus H$ relatively closed and d-rectifiable, $\mathcal{A}(H, K_0)$ is a deformation class in the sense of Definition 3.2, since $D(x, r) \subset \Sigma(H)$ for every $B_{x,r} \subset \mathbb{R}^n \setminus H$.

Applying Theorem 3.3 to the framework of sliding minimizers we obtain the following result. Here and in the following $U_\delta(E)$ denotes the δ-neighborhood of a set $E \subset \mathbb{R}^n$.

Theorem 3.27. *Assume that*

(i) K_0 *is bounded* d*-rectifiable with* $\mathcal{H}^d(K_0) < \infty$*;*

(ii) $\mathcal{H}^d(H) = 0$ *and for every* $\eta > 0$ *there exist* $\delta > 0$ *and* $\Pi \in \Sigma(H)$ *such that*

$$\operatorname{Lip}\Pi \leqslant 1+\eta\,, \qquad \Pi(U_\delta(H)) \subset H\,. \tag{3.91}$$

Then, given any minimizing sequence (K_j) *in the Plateau problem corresponding to* $\mathscr{P}(H) = \mathcal{A}(H, K_0)$ *and any set* K *as in Theorem 3.3, we have*

$$\inf\left\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0)\right\} = \mathcal{H}^d(K) = \inf\left\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K)\right\}. \tag{3.92}$$

In particular K *is a sliding minimizer.*

Remark 3.28. It is far from obvious to prove the existence of a minimizer in the class $\mathcal{A}(H, K_0)$. It is indeed false in general that the sliding minimizer K in Theorem 3.27 belongs to $\mathcal{A}(H, K_0)$ (see the discussion in [37, Remark 8]).

3.7.2 *Proof of Theorem 3.27*

As already observed in Remark 3.26, $\mathcal{A}(H, K_0)$ is a deformation class and we can therefore apply Theorem 3.3.

Step one: We show that $\mathcal{H}^d(K_j) \to \mathcal{H}^d(K)$ and thus the first equality in (3.92). We first let $R_0 > 0$ be such that $H \subset B_{R_0}$ and consider the Lipschitz map $\varphi(x) := \min\{|x|, R_0\}x/|x|$. Obviously $\varphi \in \Sigma(H)$ and we easily compute

$$\mathcal{H}^d(K_j) - \varepsilon \leqslant \mathcal{H}^d(\varphi(K_j)) \leqslant \mathcal{H}^d(K_j \cap B_{2R_0}) + \frac{1}{2^d}\mathcal{H}^d(K_j \setminus B_{2R_0})\,.$$

This implies that $\mathcal{H}^d(K_j \setminus B_{2R_0}) \to 0$. In order to prove $\mathcal{H}^d(K_j) \to \mathcal{H}^d(K)$, we are left to show that there is no loss of mass at H. To this end, let us fix $\eta > 0$, and consider $\delta > 0$ and the map Π as in (3.91). Then, by $\Pi \in \Sigma(H)$ and by $\mathcal{H}^d(\Pi(U_\delta(H))) \leqslant \mathcal{H}^d(H) = 0$,

$$\begin{aligned}
\mathcal{H}^d(K) &\leqslant \limsup_{j\to\infty} \mathcal{H}^d(K_j) \leqslant \limsup_{j\to\infty} \mathcal{H}^d(\Pi(K_j)) \leqslant (1+\eta)^d \limsup_{j\to\infty} \mathcal{H}^d(K_j \setminus U_\delta(H)) \\
&= (1+\eta)^d \limsup_{j\to\infty} \mathcal{H}^d((K_j \cap \overline{B_{2R_0}}) \setminus U_\delta(H)) \\
&\leqslant (1+\eta)^d\, \mathcal{H}^d(K \cap \overline{B_{2R_0}}) \leqslant (1+\eta)^d\, \mathcal{H}^d(K)\,.
\end{aligned}$$

The arbitrariness of η implies that $\limsup_j \mathcal{H}^d(K_j) = \mathcal{H}^d(K)$.

Step two: To complete the proof, we show the second equality in (3.92), i.e. that K is a sliding minimizer. By Theorem 3.3, there exists an $\mathcal{H}^d$-negligible closed set $S \subset K$ such that $\Gamma = K \setminus S$ is a real analytic hypersurface. We may now exploit this fact to show that $\mathcal{H}^d(K) \leqslant \mathcal{H}^d(\phi(K))$ for every $\phi \in \Sigma(H)$, showing that K is a sliding minimizer (and hence an $(\mathbb{M}^\alpha, 0, \infty)$-minimal set). The idea is that, by regularity of Γ, at a fixed distance from the singular set one can project K_j directly onto K, rather than onto its affine tangent planes localized in balls. More precisely, since $\mathcal{H}^d(H \cup S) = 0$ and $\mathcal{H}^d(K) < \infty$ one has

$$\limsup_{j\to\infty} \mathcal{H}^d(K_j \cap U_\delta(H \cup S)) \leqslant \mathcal{H}^d(K \cap U_\delta(H \cup S)) =: \rho(\delta)\,, \tag{3.93}$$

where $\rho(\delta) \to 0$ as $\delta \to 0^+$. If $N_\varepsilon(A)$ denotes the normal ε-neighborhood upon $A \subset \Gamma$, then, by compactness of $\Gamma_\delta = \Gamma \setminus U_\delta(H \cup S)$, there exists $\varepsilon < \delta$ such that the projection onto Γ defines a smooth map $p : N_{2\varepsilon}(\Gamma_\delta) \to \Gamma_\delta$. We now define a Lipschitz map

$$f_{\varepsilon,\delta} : N_\varepsilon(\Gamma_\delta) \cup U_{\delta/2}(H \cup S) \cup (\mathbb{R}^n \setminus U_\delta(\Gamma)) \to \mathbb{R}^n$$

by setting $f_{\varepsilon,\delta} = p$ on $N_\varepsilon(\Gamma_\delta)$, and $f_{\varepsilon,\delta} = \mathrm{Id}$ on the remainder. Observe that

$$\lim_{\varepsilon \downarrow 0} \mathrm{Lip}\,(f_{\varepsilon,\delta}) = 1 < \infty\,.$$

For every δ we then choose $\varepsilon < \delta$ so that $f = f_{\varepsilon,\delta}$ has Lipschitz constant at most 2 and extend it to a Lipschitz map $\hat{f}$ on $\mathbb{R}^n$ with the same Lipschitz constant. Obviously $\hat{f}$ belongs to $\Sigma(H)$. We can then estimate

$$\mathcal{H}^d(\hat{f}(K_j) \setminus \Gamma_\delta) \leqslant (\mathrm{Lip}\,\hat{f})^d\, \mathcal{H}^d\big(K_j \setminus N_\varepsilon(\Gamma_\delta)\big)\,. \tag{3.94}$$

Observe that $\mathbb{R}^n \setminus N_\varepsilon(\Gamma_\delta) \subset\subset \mathbb{R}^n \setminus U_{\varepsilon/2}(K) \cup U_{2\delta}(H \cup S)$ and thus

$$\limsup_{j\to\infty} \mathcal{H}^d\big(K_j \setminus N_\varepsilon(\Gamma_\delta)\big) \leqslant \mathcal{H}^d(K \cap U_{2\delta}(H \cap S)) \overset{(3.93)}{\leqslant} \rho(2\delta)\,. \tag{3.95}$$

Combining (3.94) and (3.95)

$$\limsup_{j} \mathcal{H}^d(\hat{f}(K_j) \setminus \Gamma_\delta) \leqslant 2^d \rho(2\delta)\,.$$

On the other hand $\Gamma_\delta \subset K$. Thus, combining (3.94) and (3.95) with a standard diagonal argument, we achieve a sequence of maps $f_j \in \Sigma(H)$ such that $\mathcal{H}^d(f_j(K_j)) \setminus K) \to 0$. Since each K_j equals $\psi_j(K_0)$ for some $\psi_j \in \Sigma(H)$, we therefore conclude the existence of a sequence of maps $\{\varphi_j\} \subset \Sigma(H)$ such that $\mathcal{H}^d(\varphi_j(K_0) \setminus K) \to 0$.

We are now ready to show the right identity in (3.92). Fix $\phi \in \Sigma(H)$. Then

$$\begin{aligned}\mathcal{H}^d(\phi(K)) &\geqslant \liminf_{j\to\infty} \mathcal{H}^d(\phi \circ \varphi_j(K_0)) \\ &\geqslant \inf\big\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0)\big\} = \mathcal{H}^d(K)\,.\end{aligned}$$

This shows that K is a sliding minimizer.

3.8 APPLICATION 3: SOLUTION TO REIFENBERG FORMULATION

The third type of boundary constraints we take into account is due to Reifenberg in [77], and it is given in terms of homology groups. The main theorem of this section is Theorem 3.32, which we got in joint work in [43].

3.8.1 *Defintion of the class of competitors and main result*

The original formulation of Plateau problem given by Reifenberg in [77] involves an algebraic notion of boundary described in terms of Čech homology groups. The particular choice of an homology theory defined on compact spaces and with coefficient groups that are abelian and compact has two motivations: with these assumptions the homology is well behaved under Hausdorff convergence of compact sets, and furthermore it satisfies the classical axioms of Eilenberg and Steenrod, enabling the use of the Mayer-Vietoris exact sequence [49, Chapter X].

Let G be a compact Abelian group and let K be a closed set in $\mathbb{R}^n$. For every $m \geqslant 0$ we denote with $\check{H}_m(K;G)$ (often omitting the explicit mention of the group G) the m^{th}-Čech homology groups of K with coefficients in G, [49, Chapter IX].

Recall that, if $H \subset K$ is a compact set, the inclusion map $i_{H,K} : H \to K$ induces a graded homomorphism between the homology groups (of every grade m, again often omitted)

$$i_{*\,H,K} : \check{H}_m(H,G) \to \check{H}_m(K,G).$$

(For any given continuous maps of compact spaces $f : X \to Y$, the induced homomorphisms f_* between homology groups are functorial). Note, in the next definition, the role of the dimension d, inherent of our variational problem.

Definition 3.29 (Boundary in the sense of Reifenberg)**.** Let G, H, K be as above and let $L \subset \check{H}_{d-1}(H,G)$ be a subgroup. We say that K has boundary L if

$$\mathrm{Ker}(i_{*\,H,K}) \supset L. \tag{3.96}$$

Definition 3.30 (Reifenberg class)**.** Given G a compact Abelian group and $H \subset \mathbb{R}^n$ a compact set, we let $\mathcal{R}(H)$ be the class of closed d-rectifiable subsets K of $\mathbb{R}^n \setminus H$ uniformly contained in a ball $B \supset\supset H$ and such that $K \cup H$ has boundary L in the sense of Definition 3.29.

Remark 3.31. We remark that $\mathcal{R}(H)$ is a deformation class in the sense of Definition 3.2, for every Lagrangian F. Indeed for every $K \in \mathcal{R}(H)$, every $x \in K$, $r \in (0, \mathrm{dist}(x,H))$ and $\varphi \in D(x,r)$ (which is in particular continuous),

$$\varphi(K \cup H) = \varphi(K) \cup H,$$

and moreover by functoriality $\mathrm{Ker}((\varphi \circ i_{H,K\cup H})_*) \supset \mathrm{Ker}(i_{*\,H,K\cup H}) \supset L$, which implies that $\varphi(K) \in \mathcal{R}(H)$. We can therefore apply Theorem 3.3 and Theorem 3.19 to the class $\mathcal{R}(H)$: we immediately obtain the existence of a relatively closed subset K of $\mathbb{R}^n \setminus H$ satisfying

$$\mathbf{F}(K) = \inf_{S \in \mathcal{R}(H)} \{\mathbf{F}(S)\}.$$

We address now the question whether K belongs to the Reifenberg class $\mathcal{R}(H)$. Recall the definition of Hausdorff distance between two compact sets C_1, C_2 of a metric space X:

$$d_{\mathcal{H}}(C_1, C_2) := \inf\{r > 0 : C_1 \subset U_r(C_2) \text{ and } C_2 \subset U_r(C_1)\}.$$

Theorem 3.32. *For every minimizing sequence* $(K_j) \subset \mathcal{R}(H)$ *the associated limit set given by Theorem 3.3 or Theorem 3.19 satisfies* $K \in \mathcal{R}(H)$.

Proof. We wish to construct another minimizing sequence, $(K_j^1) \subset \mathcal{R}(H)$, yielding the same set K but with the further property that

$$d_{\mathcal{H}}(K_j^1 \cup H, K \cup H) \to 0. \tag{3.97}$$

Step 1: Construction of the new sequence.

From Theorem 3.3 and Theorem 3.19 we know that $\mu_j := F(\cdot, T_{(\cdot)}K_j)\mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to the measure $\mu = F(\cdot, T_{(\cdot)}K)\mathcal{H}^d \llcorner K$. Then, for every $\varepsilon > 0$, there exists $j(\varepsilon)$ big enough so that for every $j \geqslant j(\varepsilon)$ we get

$$\mu_j(\mathbb{R}^n \setminus U_\varepsilon(K \cup H)) < \frac{\varepsilon^d}{\Lambda k_1 (4n)^d}, \tag{3.98}$$

where we denoted with $U_\varepsilon(K \cup H)$ the ε-tubular neighborhood of $K \cup H$, with Λ the costant in equation (2.10) and k_1 is the costant of Theorem 3.7.

We cover $U_{5\varepsilon}(K \cup H) \setminus U_{2\varepsilon}(K \cup H)$ with a complex Δ of closed cubes with side length equal to $\frac{\varepsilon}{4n}$ contained in $U_{6\varepsilon}(K \cup H) \setminus U_\varepsilon(K \cup H)$. We can apply an adaptation of the Deformation Theorem 3.7 relative to the set K_j and obtain a Lipschitz deformation $\varphi_j := \varphi_{\frac{\varepsilon}{4n}, K_j}$. Observe that $\varphi(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H)) \subset \Delta_d$ (the d-skeleton of the complex): we claim that

$$\varphi_j(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H)) \subset \Delta_{d-1}. \tag{3.99}$$

Otherwise by Theorem 3.7 $\varphi_j(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H))$ should contain an entire d-face of edge length $\frac{\varepsilon}{4n}$, leading together with (3.98) to a contradiction:

$$\begin{aligned}\frac{\varepsilon^d}{(4n)^d} &\leqslant \mathcal{H}^d(\varphi(K_j) \cap (U_{4\varepsilon}(K \cup H) \setminus U_{3\varepsilon}(K \cup H))) \leqslant \Lambda k_1 \mathcal{H}^d(K_j \setminus U_\varepsilon(K \cup H)) \\ &\leqslant \Lambda k_1 \mathbf{F}(K_j \setminus U_\varepsilon(K \cup H)) \leqslant \Lambda k_1 \mu_j(\mathbb{R}^n \setminus U_\varepsilon(K \cup H)) < \frac{\varepsilon^d}{(4n)^d}.\end{aligned}$$

Set $\widetilde{K_j} := \varphi_j(K_j)$: by (3.99) and the coarea formula [50, 3.2.22(3)], there exists $\alpha \in (3, 4)$ such that

$$\mathcal{H}^{d-1}(\widetilde{K_j} \cap \partial U_{\alpha\varepsilon}(K \cup H)) = 0. \tag{3.100}$$

We let

$$K_j^1 := \widetilde{K_j} \cap \overline{U_{\alpha\varepsilon}(K \cup H)} \quad \text{and} \quad K_j^2 := \widetilde{K_j} \setminus U_{\alpha\varepsilon}(K \cup H). \tag{3.101}$$

Step 2: proof of the property (3.97).

Recall that by construction,

$$\forall \varepsilon > 0 \quad K_j^1 \cup H \subset U_{4\varepsilon}(K \cup H). \tag{3.102}$$

If on the other hand there were $x \in K \cup H \setminus U_\varepsilon(K_{j(h)}^1 \cup H)$ for some subsequence $j(h)$, then necessarily $d(x, H) \geqslant \varepsilon$ as well as $d(x, K_{j(h)}^1) \geqslant \varepsilon$: the weak convergence $\mu_{j(h)} \xrightharpoonup{*} \mu$ would then fail the uniform density lower bounds (3.8),(3.71) on $B(x, \varepsilon/2)$. This implies (3.97).

Step 3: boundary constraint of the new sequence. To conclude the proof of Theorem 3.32, we need to check that $(K_j^1) \subset \mathcal{R}(H)$. By (3.101) we get

$$\widetilde{K_j} = K_j^1 \cup K_j^2, \qquad \text{and} \qquad K_j^1 \cap K_j^2 = \widetilde{K_j} \cap \partial U_{\alpha\varepsilon}(K)$$

and (3.100),(3.101) yield

$$\mathcal{H}^{d-1}((K_j^1 \cup H) \cap K_j^2) = \mathcal{H}^{d-1}(K_j^1 \cap K_j^2) = 0.$$

Therefore (since there cannot be $d-1$ cycles in $(K_j^1 \cup H) \cap K_j^2$, [60, Theorem VIII 3']):

$$\check{H}_{d-1}((K_j^1 \cup H) \cap K_j^2) = (0). \tag{3.103}$$

We furthermore observe that the sets $\widetilde{K_j}$ are obtained as deformations via Lipschitz maps strongly approximable via isotopies, and therefore belong to $\mathcal{R}(H)$. Since the map φ_j coincides with the identity on H, we have

$$i_{H,\widetilde{K_j} \cup H} = \varphi_j \circ i_{H,K_j \cup H};$$

moreover, trivially $i_{H,\widetilde{K_j}\cup H} = i_{K_j^1 \cup H,\widetilde{K_j}\cup H} \circ i_{H,K_j^1 \cup H}$. Hence by functoriality

$$\mathrm{Ker}(i_{*\,K_j^1 \cup H,\widetilde{K_j}\cup H} \circ i_{*\,H,K_j^1\cup H}) = \mathrm{Ker}(i_{*\,H,\widetilde{K_j}\cup H}) = \mathrm{Ker}((\varphi_j)_* \circ i_{*\,H,K_j \cup H}) \supset L.$$

We claim that $i_{*\,K_j^1 \cup H,\widetilde{K_j}\cup H}$ is injective: this implies that

$$\mathrm{Ker}(i_{*\,H,K_j^1 \cup H}) \supset L, \tag{3.104}$$

namely $(K_j^1) \subset \mathcal{R}(H)$.

Step 4: the map $i_{*\,K_j^1 \cup H,\widetilde{K_j}\cup H}$ is injective. We can write the Mayer-Vietoris sequence (which for the Čech homology holds true for compact spaces and with coefficients in a compact group, due to the necessity of having the excision axiom, [49, Theorem 7.6 p.248]) and use (3.103):

$$(0) \overset{(3.103)}{=} \check{H}_{d-1}((K_j^1 \cup H) \cap K_j^2) \xrightarrow{f} \check{H}_{d-1}(K_j^1 \cup H) \oplus \check{H}_{d-1}(K_j^2) \xrightarrow{g} \check{H}_{d-1}(\widetilde{K_j} \cup H) \longrightarrow \ldots$$

where $f = (i_{*\,(K_j^1 \cup H)\cap K_j^2, K_j^1 \cup H}, i_{*\,(K_j^1\cup H)\cap K_j^2, K_j^2})$ and $g(\sigma,\tau) = \sigma - \tau$. The exactness of the sequence implies that g is injective: in particular the map g is injective when restricted to the subgroup $\check{H}_{d-1}(K_j^1 \cup H) \oplus (0)$, where it coincides with $i_{*\,K_j^1 \cup H,\widetilde{K_j}\cup H}$. This concludes the proof of Step 4.

Step 5: boundary constraint for the limit set. Setting

$$Y_n := \overline{\bigcup_{j \geqslant n} K_j^1 \cup H},$$

by (3.97) we get

$$d_{\mathcal{H}}(Y_n, K \cup H) \to 0. \tag{3.105}$$

Therefore $K \cup H$ is the inverse limit of the sequence Y_n. Since the sets $(K^1_j \cup H)$ are in the Reifenberg class $\mathcal{R}(H)$, namely the inclusion (3.104) holds, by composing the two injections $i_{*\, K^1_n \cup H, Y_n}$ and $i_{*\, H, K^1_n \cup H}$ we obtain that

$$L \subset \mathrm{Ker}(i_{*\, H, Y_n}).$$

Since the Čech homology with coefficients in compact groups is continuous [49, Definition 2.3], the latter inclusion is stable under Hausdorff convergence, see [49, Theorem 3.1] (see also [77, Lemma 21A]): therefore, by (3.105), we conclude

$$L \subset \mathrm{Ker}(i_{*\, H, K \cup H}),$$

and eventually $K \in \mathcal{R}(H)$. □

Remark 3.33. Using the contravariance of cohomology theory, the same results can be obtained when considering a cohomological definition of boundary, again in the Čech theory, as introduced in [58]. In particular a new proof of their theorem can be obtained with our assumption on the Lagrangian as in Definition 4.1.

Note that in the cohomological definition of boundary all the Eilenberg-Steenrod axioms are satisfied even with a non-compact group G. This allows us to consider as coefficients set the natural group $\mathbb{Z}$.

Remark 3.34. We observe that any minimizer K as in Theorem 3.32 is also an $(\mathbf{F}, 0, \infty)$ minimal set in the sense of [8, Definition III.1]. Indeed the boundary condition introduced in Definition 3.29 is preserved under Lipschitz maps (not necessarily in $D(x, r)$). In particular, by [8, Theorem III.3(7)], if F is smooth and strictly elliptic (Γ in Definition 3.17 is strictly positive), then K is smooth away from the boundary, outside of a relative closed set of $\mathcal{H}^d$-measure zero (the Theorem gives actually $C^{1,\alpha}$ almost everywhere regularity for all $\alpha < 1/2$ if $F \in C^3$ and elliptic).

4 THE ANISOTROPIC COUNTERPART OF ALLARD'S RECTIFIABILITY THEOREM

4.1 INTRODUCTION

Allard's rectifiability Theorem, [2], asserts that every d-varifold in $\mathbb{R}^n$ with locally bounded (isotropic) first variation is d-rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive lower d-dimensional density. It is a natural question whether this result holds for varifolds whose first variation with respect to an anisotropic integrand is locally bounded.

More specifically, for an open set $\Omega \subset \mathbb{R}^n$ and a positive C^1 integrand

$$F : \Omega \times G(n, d) \to \mathbb{R}_{>0} := (0, +\infty),$$

it has been observed in (2.13) that the *anisotropic first variation* of a d-varifold $V \in \mathbf{V}_d(\Omega)$ acts on $g \in C^1_c(\Omega, \mathbb{R}^n)$ as

$$\delta_F V(g) \int_{\Omega \times G(n,d)} \Big[\langle d_x F(x, T), g(x)\rangle + B_F(x, T) : Dg(x)\Big] dV(x, T).$$

In the paper [42], in joint work with De Philippis and Ghiraldin, we answer to the following question:

Question. Is it true that for every $V \in \mathbf{V}_d(\Omega)$ such that $\delta_F V$ is a Radon measure in Ω, the associated varifold V_*, defined as in (2.5),

$$V_* := V \llcorner \{x \in \Omega : \Theta^d_*(x, V) > 0\} \times G(n, d) \tag{4.1}$$

is d-rectifiable?

We will show that this is true if (and only if in the case of autonomous integrands) F satisfies the following *atomic condition* at every point $x \in \Omega$.

Definition 4.1. For a given integrand $F \in C^1(\Omega \times G(n, d))$, $x \in \Omega$ and a Borel probability measure $\mu \in \mathcal{P}(G(n, d))$, let us define

$$A_x(\mu) := \int_{G(n,d)} B_F(x, T) d\mu(T) \in \mathbb{R}^n \otimes \mathbb{R}^n. \tag{4.2}$$

We say that F verifies the *atomic condition* (AC) at x if the following two conditions are satisfied:

(i) $\dim \mathrm{Ker} A_x(\mu) \leqslant n - d$ for all $\mu \in \mathcal{P}(G(n, d))$,

(ii) if $\dim \mathrm{Ker} A_x(\mu) = n - d$, then $\mu = \delta_{T_0}$ for some $T_0 \in G(n, d)$.

The following Theorem is the main result of this chapter:

Theorem 4.2. *Let* $F \in C^1(\Omega \times G(n,d), \mathbb{R}_{>0})$ *be a positive integrand and let us define*

$$\mathscr{V}_F(\Omega) = \Big\{ V \in \mathbf{V}_d(\Omega) : \delta_F V \text{ is a Radon measure} \Big\}. \tag{4.3}$$

Then we have the following:

(i) If F *satisfies the atomic condition at every* $x \in \Omega$*, then for every* $V \in \mathscr{V}_F(\Omega)$ *the associated varifold* V_* *defined in* (4.1) *is* d*-rectifiable.*

(ii) Assume that F *is autonomous, i.e. that* $F(x,T) \equiv F(T)$*: then every* V_* *associated to a varifold* $V \in \mathscr{V}_F(\Omega)$ *is* d*-rectifiable if and only if* F *satisfies the atomic condition.*

For the area integrand, $F(x,T) \equiv 1$, it is easy to verify that $B_F(x,T) = T$ where we are identifying $T \in G(n,d)$ with the matrix $T \in (\mathbb{R}^n \otimes \mathbb{R}^n)_{\mathrm{sym}}$ representing the orthogonal projection onto T. Since T is positive semidefinite (i.e. $T \geqslant 0$), it is easy to check that the (AC) condition is satisfied. In particular Theorem 4.2 provides a new independent proof of Allard's rectifiability theorem.

Since the atomic condition (AC) is essentially necessary to the validity of the rectifiability Theorem 4.2, it is relevant to relate it to the previous known notions of *ellipticity* (or *convexity*) of F with respect to the "plane" variable T. This task seems to be quite hard in the general case. For $d = (n-1)$ we can however completely characterize the integrands satisfying (AC). Referring to Section 4.5 for a more detailed discussion, we recall here that in this case the integrand F can be equivalently thought as a positive one-homogeneous even function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ via the identification

$$G(x, \lambda\nu) := |\lambda| F(x, \nu^\perp) \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \nu \in \mathbb{S}^{n-1}. \tag{4.4}$$

The atomic condition then turns out to be equivalent to the *strict convexity* of G, more precisely:

Theorem 4.3. *An integrand* $F : C^1(\Omega \times G(n, n-1), \mathbb{R}_{>0})$ *satisfies the atomic condition at* x *if and only if the function* $G(x,\cdot)$ *defined in* (4.4) *is strictly convex.*

As we said, we have not been able to obtain a simple characterization in the general situation when $2 \leqslant d \leqslant (n-2)$ (while for $d = 1$ the reader can easily verify that an analogous version of Theorem 4.3 holds true). In this respect, let us recall that the study of convexity notions for integrands defined on the d-dimensional Grassmannian is an active field of research, where several basic questions are still open, see [24, 23] and the survey [10].

Beside its theoretical interest, the above Theorem has some relevant applications in the study of existence of minimizers of geometric variational problems defined on class of rectifiable sets. It can be indeed shown that given an $\mathbf{F}$-minimizing sequence of sets, the limit of the varifolds naturally associated to them is F-stationary (i.e. it satisfies $\delta_F V = 0$) and has density bounded away from zero. Hence, if F satisfies (AC), this varifold is rectifiable and it can be shown that its support minimizes $\mathbf{F}$, see [43, 45].

We conclude this introduction with a brief sketch of the proof of Theorem 4.2. The original proof of Allard in [2] (see also [63] for a quantitative improvement under slightly more general assumptions) for varifolds with locally bounded variations with respect to the area integrand heavily relies on the monotonicity formula, which is strongly linked to the isotropy of the area integrand, [3]. A completely different strategy must hence be used to prove Theorem 4.2.

The idea is to use the notion of *tangent measure* introduced by Preiss, [76], in order to understand the local behavior of a varifold V with locally bounded first variation. Indeed at $\|V_*\|$ almost every point, condition (AC) is used to show that every tangent measure is translation invariant along *at least* d (fixed) directions, while the positivity of the lower d-dimensional density ensures that there exists at least one tangent measure that is invariant along *at most* d directions. The combination of these facts allows to show that the "Grassmannian part" of the varifold V_* at x is a Dirac delta δ_{T_x} on a fixed plane T_x, see Lemma 4.8. A key step is then to show that $\|V_*\| \ll \mathcal{H}^d$: this is achieved by using ideas borrowed from [5] and [41]. Once this is obtained, a simple rectifiability criterion, based on the results in [76] and stated in Lemma 4.5, allows to show that V_* is d-rectifiable.

4.2 PRELIMINARY RESULTS

We introduce some preliminary results. The next Lemma shows that $\mathrm{Tan}(x, \mu)$ is not trivial at μ-almost every point where μ has positive lower d-dimensional density and that furthermore there is always a tangent measure which looks at most d-dimensional on a prescribed ball (a similar argument can be used to show that $\mathrm{Tan}(x, \mu)$ is always not trivial at μ almost every point without any assumption on the d-dimensional density, see [12, Corollary 2.43]).

Lemma 4.4. *Let $\mu \in \mathcal{M}_+(\Omega)$ be a Radon measure. Then for every $x \in \Omega$ such that $\Theta^d_*(x, \mu) > 0$ and for every $t \in (0, 1)$, there exists a tangent measure $\sigma_t \in \mathrm{Tan}(x, \mu)$ satisfying*

$$\sigma_t(\overline{B_t}) \geqslant t^d. \tag{4.5}$$

Proof. *Step 1*: We claim that for every $x \in \Omega$ such that $\Theta^d_*(x, \mu) > 0$, it holds

$$\limsup_{r \to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} \geqslant t^d, \qquad \forall t \in (0, 1). \tag{4.6}$$

More precisely, we are going to show that

$$\{x \in \mathrm{spt}\mu : (4.6) \text{ fails}\} \subset \{x \in \mathrm{spt}\mu : \Theta^d_*(x, \mu) = 0\},$$

which clearly implies that (4.6) holds for every $x \in \Omega$ with positive lower d-dimensional density. Let indeed $x \in \mathrm{spt}\mu$ be such that (4.6) fails, then there exist $t_0 \in (0, 1)$, $\bar{\varepsilon} > 0$, and $\bar{r} > 0$ such that

$$\mu(B_{t_0 r}(x)) \leqslant (1 - \bar{\varepsilon}) t_0^d \, \mu(B_r(x)) \qquad \text{for all } r \leqslant \bar{r}.$$

Iterating this inequality, we deduce that

$$\mu(B_{t_0^k \bar{r}}(x)) \leqslant (1 - \bar{\varepsilon})^k t_0^{kd} \mu(B_{\bar{r}}(x)) \qquad \text{for all } k \in \mathbb{N}$$

and consequently

$$\Theta^d_*(x,\mu) \leqslant \lim_{k\to\infty} \frac{\mu(B_{t_0^k \bar{r}}(x))}{\omega_d (t_0^k \bar{r})^d} = 0.$$

Step 2: Let now x be a point satisfying (4.6) and let $t \in (0,1)$: there exists a sequence $r_j \downarrow 0$ (possibly depending on t and on x), such that

$$t^d \leqslant \limsup_{r\to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} = \lim_{j\to\infty} \frac{\mu(B_{tr_j}(x))}{\mu(B_{r_j}(x))} = \lim_{j\to\infty} \mu_{x,r_j}(B_t)$$

where μ_{x,r_j} is defined as in (2.2). Up to extracting a (not relabelled) subsequence,

$$\mu_{x,r_j} \overset{*}{\rightharpoonup} \sigma_t \in \mathrm{Tan}(x,\mu).$$

By upper semicontinuity:

$$\sigma_t(\overline{B_t}) \geqslant \limsup_j \mu_{x,r_j}(\overline{B_t}) \geqslant t^d$$

which is (4.5). $\square$

In order to prove Theorem 4.2 we need the following rectifiability criterion which is essentially [52, Theorem 4.5], see also [68, Theorem 16.7]. For the sake of readability, we postpone its proof to Appendix 4.7.

Lemma 4.5. *Let $\mu \in \mathcal{M}_+(\Omega)$ be a Radon measure such that the following two properties hold:*

(i) For μ-a.e. $x \in \Omega$, $0 < \Theta^d_*(x,\mu) \leqslant \Theta^{d*}(x,\mu) < +\infty$.

(ii) For μ-a.e. $x \in \Omega$ there exists $T_x \in G(n,d)$ such that every $\sigma \in \mathrm{Tan}(x,\mu)$ is translation invariant along T_x, i.e.

$$\int \partial_e \varphi \, d\sigma = 0 \qquad \text{for every } \varphi \in C^1_c(B) \text{ and every } e \in T_x.$$

Then μ is d-rectifiable, i.e. $\mu = \theta \mathcal{H}^d \llcorner K$ for some d-rectifiable set K and Borel function $\theta : \mathbb{R}^n \to \mathbb{R}_{>0}$. Furthermore $T_x K = T_x$ for μ-a.e. x.

The following Lemma is based on a simple Lebesgue point argument, combined with the separability of $C^0_c(G(\Omega))$, see for instance [36, Proposition 9].

Lemma 4.6. *For $\|V\|$-almost every point $x \in \Omega$ and every sequence $r_j \to 0$ there is a subsequence r_{j_i} such that*

$$V_{x,r_{j_i}}(dy,dT) = \|V\|_{x,r_{j_i}}(dy) \otimes \mu_{x+yr_{j_i}}(dT) \overset{*}{\rightharpoonup} \sigma(dy) \otimes \mu_x(dT) =: V^\infty(dy,dT) \in \mathrm{Tan}(x,V),$$

with $\sigma \in \mathrm{Tan}(x,\|V\|)$.

The key point of the Lemma above is that the "Grassmannian" part μ_y^∞ of a tangent varifold V^∞ equals μ_x for every $y \in \Omega$: it therefore neither depends on the space variable y, nor on the chosen blow-up sequence (r_j). Informally, we could write:

$$\text{" } \mathrm{Tan}(x, V) = \mathrm{Tan}(x, \|V\|) \otimes \mu_x(dT) \text{ "}.$$

We furthermore note that, as a consequence of (2.6),

$$\mathrm{Tan}(x, \|V_*\|) = \mathrm{Tan}(x, \|V\|) \quad \text{and} \quad \mathrm{Tan}(x, V_*) = \mathrm{Tan}(x, V) \qquad \|V_*\|\text{-a.e.} \tag{4.7}$$

We conclude this section with the following simple result which shows that every tangent varifold to a varifold having locally bounded anisotropic first variation is F_x-stationary (where F_x is defined in (2.11)).

Lemma 4.7. *Let $V \in \mathbf{V}_d(\Omega)$ be a d-dimensional varifold with locally bounded anisotropic first variation. Then, for $\|V\|$-almost every point, every $W \in \mathrm{Tan}(x, V)$ is F_x-stationary, i.e. $\delta_{F_x} W = 0$. Moreover, if $W(dy, dT) = \sigma(dy) \otimes \mu_x(dT)$ for some $\sigma \in \mathrm{Tan}(x, \|V\|)$ (which by Lemma 4.6 happens $\|V\|$-a.e. x), then*

$$\partial_e \sigma = 0 \qquad \textit{for all } e \in T_x := \mathrm{Im} A_x(\mu_x)^* \tag{4.8}$$

in the sense of distributions, where $A_x(\mu_x)$ is defined in (4.2).

Proof. Let x be a point such that the conclusion of Lemma 4.6 holds true and such that

$$\limsup_{r \to 0} \frac{|\delta_F V|(B_r(x))}{\|V\|(B_r(x))} = C_x < +\infty. \tag{4.9}$$

Note that, by Lemma 4.6 and Lebesgue-Besicovitch differentiation Theorem, [79, Theorem 4.7], this is the case for $\|V\|$-almost every point. We are going to prove the Lemma at every such a point.

Let r_i be a sequence such that $V_{x,r_i}(dy, dT) \overset{*}{\rightharpoonup} W(dy, dT) = \sigma(dy) \otimes \mu_x(dT)$, $\sigma \in \mathrm{Tan}(x, \|V\|)$. For $g \in C_c^1(B, \mathbb{R}^n)$, we define $g_i := g \circ \eta^{x,r_i} \in C_c^1(B_{r_i}(x), \mathbb{R}^n)$ and we compute

$$\begin{aligned}
\delta_{F_x} V_{x,r_i}(g) &= \int_{G(B)} B_F(x, T) : Dg(y)\, dV_{x,r_i}(y, T) \\
&= \frac{r_i^d}{\|V\|(B_{r_i}(x))} \int_{G(B_{r_i}(x))} B_F(x, T) : Dg\left(\eta^{x,r_i}(z)\right) J\eta^{x,r_i}(z, T)\, dV(z, T) \\
&= \frac{r_i}{\|V\|(B_{r_i}(x))} \int_{G(B_{r_i}(x))} B_F(x, T) : Dg_i(z)\, dV(z, T) \\
&= r_i \frac{\delta_{F_x} V(g_i)}{\|V\|(B_{r_i}(x))} = r_i \frac{\delta_F V(g_i) + \delta_{(F_x - F)} V(g_i)}{\|V\|(B_{r_i}(x))}.
\end{aligned}$$

Combining (2.17), (4.9) and since $r_i \|Dg_i\|_{C^0} = \|Dg\|_{C^0}$, we get

$$\begin{aligned}
|\delta_{F_x} V_{x,r_i}(g)| &\leqslant r_i \frac{|\delta_F V|(B_{r_i}(x)) \|g\|_\infty}{\|V\|(B_{r_i}(x))} \\
&\quad + r_i \frac{\|F - F_x\|_{C^1(B_{r_i}(x))} \|g_i\|_{C^1} \|V\|(B_{r_i}(x))}{\|V\|(B_{r_i}(x))} \\
&\leqslant r_i C_x \|g\|_\infty + o_{r_i}(1) \|g\|_{C^1} \to 0,
\end{aligned}$$

which implies $\delta_{F_x} W = 0$. Hence, recalling the Definition (4.2) of $A_x(\mu)$, for every $g \in C^1_c(B, \mathbb{R}^n)$:

$$0 = \delta_{F_x} W(g) = \int_B A_x(\mu_x) : Dg(y)\, d\sigma(y).$$

Therefore $A_x(\mu_x) D\sigma = 0$ in the sense of distributions, which is equivalent to (4.8), since $\mathrm{Ker} A_x(\mu_x) = (\mathrm{Im} A_x(\mu_x)^*)^\perp$. □

4.3 INTERMEDIATE LEMMATA

To prove the sufficiency part of Theorem 4.2, there are two key steps:

(i) Show that the "Grassmannian" part of the varifold V_* is concentrated on a single plane;

(ii) Show that $\|V_*\| \ll \mathcal{H}^d$.

In this section we prove these steps, in Lemma 4.8 and Lemma 4.11 respectively.

Lemma 4.8. *Let* F *be an integrand satisfying condition* (AC) *at every* x *in* Ω *and let* $V \in \mathscr{V}_F(\Omega)$, *see* (4.3). *Then, for* $\|V_*\|$*-a.e.* $x \in \Omega$, $\mu_x = \delta_{T_0}$ *for some* $T_0 \in G(n, d)$.

Proof. Let $t \leqslant t(d) \ll 1$ to be fixed later. By Lemmata 4.6, 4.7 and 4.4 and by (4.7), for $\|V_*\|$-a.e. x there exist a sequence $r_i \to 0$ and a tangent measure σ such that

$$\|V_*\|_{x,r_i} \overset{*}{\rightharpoonup} \sigma, \qquad (V_*)_{x,r_i} \overset{*}{\rightharpoonup} \sigma \otimes \mu_x, \qquad \sigma(\overline{B_t}) \geqslant t^d$$

and

$$\partial_e \sigma = 0 \quad \text{for all } e \in T_x = \mathrm{Im} A_x(\mu_x)^*.$$

Let us now show that if $t(d)$ is sufficiently small, then $\mu_x = \delta_{T_0}$. Assume by contradiction that μ_x is not a Dirac delta: from the (AC) condition of F, this implies that $\dim \mathrm{Ker} A_x(\mu_x)^* < n - d$ and consequently that $\dim(T_x) > d$. This means that σ is invariant by translation along at least $d+1$ directions and therefore there exists $Z \in G(n, d+1)$, a probability measure $\gamma \in \mathcal{P}(Z^\perp)$ defined in the linear space $Z^\perp$ and supported in $B^{n-d-1}_{1/\sqrt{2}}$, and a constant $c \in \mathbb{R}$, such that we can decompose σ in the cylinder $B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}} \subset Z \times Z^\perp$ as

$$\sigma \llcorner B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}} = c\mathcal{H}^{d+1} \llcorner (Z \cap B^{d+1}_{1/\sqrt{2}}) \otimes \gamma,$$

where $c \leqslant 2^{(d+1)/2} \omega_{d+1}^{-1}$ since $\sigma(B_1) \leqslant 1$. Taking $t(d) < \frac{1}{2\sqrt{2}}$, the ball $\overline{B_t}$ is contained in the cylinder $B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}}$ and hence

$$t^d \leqslant \sigma(\overline{B_t}) \leqslant \sigma(B^{d+1}_t \times B^{n-d-1}_{1/\sqrt{2}}) \leqslant C(d) t^{d+1},$$

which is a contradiction if $t(d) \ll 1$. □

The next Lemma is inspired by the "Strong Constancy Lemma" of Allard [5, Theorem 4], see also [41].

Lemma 4.9. *Let* $F_j : G(B) \to \mathbb{R}_{>0}$ *be a sequence of* C^1 *integrands and let* $V_j \in \mathbf{V}_d(G(B))$ *be a sequence of* d*-varifolds equi-compactly supported in* B *(i.e. such that* $\mathrm{spt}\|V_j\| \subset K \subset\subset B$*) with* $\|V_j\|(B) \leqslant 1$*. If there exist* $N > 0$ *and* $S \in G(n, d)$ *such that*

(1) $|\delta_{F_j} V_j|(B) + \|F_j\|_{C^1(G(B))} \leqslant N$,

(2) $|B_{F_j}(x, T) - B_{F_j}(x, S)| \leqslant \omega(|S - T|)$ *for some modulus of continuity independent on* j,

(3) $\delta_j := \int_{G(B)} |T - S| dV_j(z, T) \to 0$ *as* $j \to \infty$,

then, up to subsequences, there exists $\gamma \in L^1(B^d, \mathcal{H}^d \llcorner B^d)$ *such that for every* $0 < t < 1$

$$\left|(\Pi_S)_\#\big(F_j(z, S)\|V_j\|\big) - \gamma\mathcal{H}^d \llcorner B^d\right|(B^d_t) \longrightarrow 0, \tag{4.10}$$

where $\Pi_S : \mathbb{R}^n \to S$ *denotes the orthogonal projection onto* S *(which in this Lemma we do not identify with* S*).*

Proof. To simplify the notation let us simply set $\Pi = \Pi_S$; we will also denote with a prime the variables in the d-plane S so that $x' = \Pi(x)$. Let $u_j = \Pi_\#\big(F_j(z, S)\|V_j\|\big) \in \mathcal{M}_+(B^d)$: then

$$\langle u_j, \varphi \rangle = \int_{G(B)} \varphi(\Pi(z)) F_j(z, S) dV_j(z, T) \qquad \text{for all } \varphi \in C^0_c(B^d).$$

Let $e \in S$ and, for $\varphi \in C^1_c(B^d)$, let us denote by D' the gradient of φ with respect to the variables in S, so that $\Pi^*(D'\varphi)(\Pi(z)) = D(\varphi(\Pi(z)))$. We then have in the sense of distributions

$$\begin{aligned}
-\langle \partial'_e u_j, \varphi \rangle = \langle u_j, \partial'_e \varphi \rangle &= \int_{G(B)} \langle D'\varphi(\Pi(z)), e \rangle F_j(z, S)\, dV_j(z, T) \\
&= \int_{G(B)} \langle D'\varphi(\Pi(z)), e \rangle (F_j(z, S) - F_j(z, T))\, dV_j(z, T) \\
&\quad + \int_{G(B)} F_j(z, T)(S - T) : e \otimes \Pi^*(D'\varphi)(\Pi(z))\, dV_j(z, T) \\
&\quad + \int_{G(B)} \big(C_{F_j}(z, S) - C_{F_j}(z, T)\big) : e \otimes \Pi^*(D'\varphi)(\Pi(z))\, dV_j(z, T) \qquad (4.11) \\
&\quad - \int_{G(B)} \big\langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \big\rangle\, dV_j(z, T) \\
&\quad + \int_{G(B)} \big\langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \big\rangle\, dV_j(z, T) \\
&\quad + \int_{G(B)} \big(F_j(z, T)T + C_{F_j}(z, T)\big) : e \otimes D(\varphi(\Pi(z)))\, dV_j(z, T),
\end{aligned}$$

where we have used that

$$\mathrm{Id} : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = S : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = \langle D'\varphi(\Pi(z)), e \rangle$$

and $C_{F_j}(z,S) : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = 0$, since $D'\varphi$ and e belong to S, see (2.16). Let us define the distributions

$$\langle X_j^e, \psi\rangle := \int_{G(B)} \Big((F_j(z,S) - F_j(z,T))\mathrm{Id} + F_j(z,T)(S-T) \\ + (C_{F_j}(z,S) - C_{F_j}(z,T)) \Big) : e \otimes \Pi^*\psi(\Pi(z))\, dV_j(z,T)$$

and

$$\begin{aligned} \langle f_j^e, \varphi\rangle &:= \int_{G(B)} \big\langle d_z F_j(z,T), e\,\varphi(\Pi(z))\big\rangle\, dV_j(z,T), \\ \langle g_j^e, \varphi\rangle &:= -\int_{G(B)} \Big(\big\langle d_z F_j(z,T), e\,\varphi(\Pi(z))\big\rangle \\ &\qquad + \big(F_j(z,T)T + C_{F_j}(z,T)\big) : e \otimes \Pi^* D'\varphi(\Pi(z)) \Big)\, dV_j(z,T) \\ &= -\delta_{F_j} V_j(e\,\varphi \circ \Pi). \end{aligned}$$

By their very definition, X_j^e are vector valued Radon measures in $\mathcal{M}(B_1^d, \mathbb{R}^d)$ and, by the uniform bound on the C^1 norm of the F_j, (2.15) and assumptions (2) and (3):

$$\sup_{|e|=1} |X_j^e|(B_1^d) \to 0 \qquad \text{as } j \to \infty. \tag{4.12}$$

Moreover, by the mass bound $\|V_j\|(B) \leqslant 1$ and assumption (1), f_j^e and g_j^e are also Radon measures satisfying

$$\sup_j \sup_{|e|=1} |f_j^e|(B_1^d) + |g_j^e|(B_1^d) < +\infty. \tag{4.13}$$

Letting e vary in an orthonormal base $\{e_1, \dots, e_d\}$ of S, we can re-write (4.11) as

$$D'u_j = \operatorname{div}' X_j + f_j + g_j, \tag{4.14}$$

where $X_j = (X_j^{e_1}, \dots, X_j^{e_d}) \in \mathbb{R}^d \otimes \mathbb{R}^d$, $f_j = (f_j^{e_1}, \dots, f_j^{e_d})$ and $g_j = (g_j^{e_1}, \dots, g_j^{e_d})$.

Let us now choose an arbitrary sequence $\varepsilon_j \downarrow 0$ and a family of smooth approximation of the identity $\psi_{\varepsilon_j}(x') = \varepsilon_j^{-d}\psi(x'/\varepsilon_j)$, with $\psi \in C_c^\infty(B_1)$, $\psi \geqslant 0$. To prove (4.10) it is enough to show that $\{v_j := u_j \star \psi_{\varepsilon_j}\}$ is precompact in $L^1_{loc}(B_1^d)$. Note that by convolving (4.14) we get that v_j solves

$$Dv_j = \operatorname{div} Y_j + h_j, \tag{4.15}$$

where, to simplify the notation, we have set $D = D'$, $\operatorname{div} = \operatorname{div}'$ and

$$Y_j := X_j \star \psi_{\varepsilon_j} \in C_c^\infty(B_1^d, \mathbb{R}^n \otimes \mathbb{R}^n), \qquad h_j = (f_j + g_j) \star \psi_{\varepsilon_j} \in C_c^\infty(B_1^d, \mathbb{R}^n)$$

are smooth functions compactly supported in B_1^d. Note that, by (4.12), (4.13) and the positivity of u_j

$$v_j \geqslant 0, \qquad \int |Y_j| \to 0 \qquad \text{and} \qquad \sup_j \int |h_j| < +\infty.$$

We can solve the system (4.15) by taking another divergence and inverting the Laplacian using the potential theoretic solution (note that all the functions involved are compactly supported):

$$\nu_j = \Delta^{-1}\mathrm{div}\,(\mathrm{div}\, Y_j) + \Delta^{-1}\mathrm{div}\, h_j. \tag{4.16}$$

Recall that

$$\Delta^{-1} w = E \star w, \tag{4.17}$$

with $E(x) = -c_d|x|^{2-d}$ if $d \geqslant 3$ and $E(x) = c_2 \log|x|$ if $d = 2$, for some positive constants c_d, depending just on the dimension. Hence, denoting by P.V. the principal value,

$$\begin{aligned}\Delta^{-1}\mathrm{div}\,(\mathrm{div}\, Y_j)(x) &= K \star Y_j(x)\\ &:= \text{P.V. } c_d \int_{\mathbb{R}^d} \frac{(x-y)\otimes(x-y) - |x-y|^2 \mathrm{Id}}{|x-y|^{d+2}} : Y_j(y)\,dy,\end{aligned}$$

and

$$\Delta^{-1}\mathrm{div}\, h_j(x) = G \star h_j(x) := c_d \int_{\mathbb{R}^d} \left\langle \frac{x-y}{|x-y|^d}, h_j(y) \right\rangle dy.$$

By the Frechet-Kolomogorov compactness theorem, the operator $h \mapsto G \star h : L^1_c(B^d_1) \to L^1_{loc}(\mathbb{R}^d)$ is compact (where $L^1_c(B^d_1)$ are the L^1 functions with compact support in B^d_1). Indeed, for $M \geqslant 1$, by direct computation one verifies that

$$\int_{B^d_M} |G \star h(x+v) - G \star h(x)|dx \leqslant C|v| \log\left(\frac{eM}{|v|}\right) \int_{B^d_1} |h|dx, \qquad \forall v \in B^d_1. \tag{4.18}$$

In particular, $\{b_j := G \star h_j\}$ is precompact in $L^1_{loc}(\mathbb{R}^d)$. The first term is more subtle: the kernel K defines a Calderon-Zygmund operator $Y \mapsto K \star Y$ on Schwarz functions that can be extended to a bounded operator from L^1 to $L^{1,\infty}$, [54, Chapter 4]. In particular we can bound the quasi-norm of $a_j := K \star Y_j$ as

$$[a_j]_{L^{1,\infty}(\mathbb{R}^d)} := \sup_{\lambda>0} \lambda|\{|a_j| > \lambda\}| \leqslant C \int_{B^d_1} |Y_j| \to 0. \tag{4.19}$$

Moreover, $K \star Y_j \overset{*}{\rightharpoonup} 0$ in the sense of distributions, since $\langle K \star Y_j, \varphi\rangle = \langle Y_j, K \star \varphi\rangle \to 0$ for $\varphi \in C^1_c(\mathbb{R}^d)$. We can therefore write

$$0 \leqslant \nu_j = a_j + b_j,$$

with $a_j \to 0$ in $L^{1,\infty}$ by (4.19), $a_j \overset{*}{\rightharpoonup} 0$ in the sense of distributions and $\{b_j\}$ pre-compact in L^1_{loc} by (4.18). Lemma 4.10 below implies that ν_j is strongly precompact in L^1_{loc}, which is the desired conclusion. □

Lemma 4.10. *Let* $\{\nu_j\}, \{a_j\}, \{b_j\} \subset L^1(\mathbb{R}^d)$ *such that*

(i) $0 \leqslant \nu_j = a_j + b_j$,

(ii) $\{b_j\}$ *strongly precompact in* L^1_{loc},

(iii) $a_j \to 0$ *in* $L^{1,\infty}$ *and* $a_j \overset{*}{\rightharpoonup} 0$ *in the sense of distributions.*

Then $\{v_j\}$ *is strongly precompact in* L^1_{loc}.

Proof. It is enough to show that $\chi|a_j| \to 0$ in L^1 for $\chi \in C^\infty_c(\mathbb{R}^d)$, $\chi \geqslant 0$. The first condition implies that $a_j^- \leqslant |b_j|$, hence the sequence $\{\chi a_j^-\}$ is equi-integrable and thus, by (iii) and Vitali convergence Theorem, it converges to zero in L^1_{loc}, hence

$$\int \chi|a_j| = \int \chi a_j + 2\int \chi a_j^- \to 0,$$

where the first integral goes to zero by (iii). □

The following Lemma is a key step in the proof of Theorem 4.2:

Lemma 4.11. *Let* F *be an integrand satisfying condition* (AC) *at every* x *in* Ω *and let* $V \in \mathscr{V}_F(\Omega)$, *see* (4.3). *Then* $\|V_*\| \ll \mathcal{H}^d$.

Proof. Since by (2.7), $\Theta^d_*(\cdot, V) > 0$ $\|V_*\|$-a.e., classical differentiation theorems for measures imply that

$$\mathcal{H}^d \llcorner \{\Theta^d_*(\cdot, V) > \lambda\} \leqslant \frac{1}{\lambda}\|V_*\| \qquad \forall \lambda > 0,$$

see [68, Theorem 6.9]. Hence $\mathcal{H}^d \llcorner \{\Theta^d_*(\cdot, V) > 0\}$ is a σ-finite measure and by the Radon-Nikodym Theorem

$$\|V_*\| = f\mathcal{H}^d \llcorner \{\Theta^d_*(\cdot, V) > 0\} + \|V_*\|^s \tag{4.20}$$

for some psitive Borel function f and $\|V_*\|^s$ is concentrated on a set $E \subset \{\Theta^d_*(\cdot, V) > 0\}$ such that $\mathcal{H}^d(E) = 0$: in particular $\mathcal{H}^d(\Pi(E)) = 0$ whenever Π is an orthogonal projection onto a d-dimensional subspace of $\mathbb{R}^n$. Hence $\|V_*\|^s$ and $f\mathcal{H}^d \llcorner \{\Theta^d_*(\cdot, V) > 0\}$ are mutually singular Radon measure (the fact that they are Radon measures follows trivially from (4.20)).

We are going to show that $\|V_*\|^s = 0$, which clearly concludes the proof. To this aim, let us assume by contradiction that $\|V_*\|^s > 0$ and let us choose a point $\bar{x} \in \Omega$ and a sequence of radii $r_j \to 0$ such that:

(i)

$$\lim_{j\to\infty} \frac{\|V_*\|^s(B_{r_j}(\bar{x}))}{\|V_*\|(B_{r_j}(\bar{x}))} = \lim_{j\to\infty} \frac{\|V_*\|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} = 1. \tag{4.21}$$

(ii) There exists $\sigma \in \mathrm{Tan}(\bar{x}, \|V\|) = \mathrm{Tan}(\bar{x}, \|V_*\|) = \mathrm{Tan}(\bar{x}, \|V_*\|^s)$, with $\sigma \llcorner B_{1/2} \neq 0$.

(iii)

$$\limsup_{j\to\infty} \frac{|\delta_F V|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} \leqslant C_{\bar{x}} < +\infty. \tag{4.22}$$

(iv)

$$V_j := V_{\bar{x},r_j} \overset{*}{\rightharpoonup} \sigma \otimes \delta_S, \tag{4.23}$$

where $S \in G(n,d)$ and $\partial_e \sigma = 0$ for every $e \in S$.

Here the first, second and third conditions hold $\|V_*\|^s$-a.e. by simple measure theoretic arguments and by (2.6) and (4.7), and the fourth one holds $\|V_*\|^s$-a.e. as well by combining Lemma 4.7, Lemma 4.8 and (4.7).

Fix a smooth cutoff function χ with $0 \leqslant \chi \leqslant 1$, $\mathrm{spt}(\chi) \subset B_1$ and $\chi = 1$ in $B_{1/2}$ and define $W_j := \chi V_j$ so that

$$\|W_j\| = \chi f \mathcal{H}^d \llcorner \{\Theta^d_*(\cdot, V) > 0\} + \|W_j\|^s$$

where $\|W_j\|^s = \chi \|V_*\|^s$. In particular

$$(\Pi_S)_\# \|W_j\|^s \qquad \text{is concentrated on} \qquad E_j := \Pi_S\left(\frac{E-\bar{x}}{r_j}\right) \cap B_1^d, \tag{4.24}$$

and thus

$$\mathcal{H}^d(E_j) = 0. \tag{4.25}$$

Note furthermore that

$$\sup_j |\delta_{F_j} W_j|(\mathbb{R}^d) < +\infty, \tag{4.26}$$

where $F_j(z,T) = F(\bar{x} + r_j z, T)$. Indeed for $\varphi \in C^\infty_c(B_1, \mathbb{R}^n)$

$$\begin{aligned}
|\delta_{F_j} W_j(\varphi)| &= |\delta_{F_j}(\chi V_j)(\varphi)| \\
&= \Big| \int r_j \big\langle d_x F(\bar{x} + r_j z, T), \chi(z)\varphi(z) \big\rangle dV_j(z,T) \\
&\quad + \int B_F(\bar{x} + r_j z, T) : D\varphi(z)\, \chi(z) dV_j(z,T) \Big| \\
&= \Big| \int r_j \big\langle d_x F(\bar{x} + r_j z, T), \chi\varphi \big\rangle dV_j(z,T) \\
&\quad + \int B_F(\bar{x} + r_j z, T) : D(\chi\varphi)(z) dV_j(z,T) \\
&\quad - \int B_F(\bar{x} + r_j z, T) : D\chi(z) \otimes \varphi(z) dV_j(z,T) \Big| \\
&\leqslant |\delta_{F_j} V_j(\chi\varphi)| + \|F\|_{C^1} \|V_j\|(B_1) \|D\chi\|_\infty \|\varphi\|_\infty \\
&\leqslant r_j \frac{|\delta_F V|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} \|\varphi\|_\infty + \|F\|_{C^1} \|V_j\|(B_1) \|D\chi\|_\infty \|\varphi\|_\infty,
\end{aligned}$$

so that (4.26) follows from (4.22) and the fact that $\|V_j\|(B_1) \leqslant 1$. Finally, by (4.23),

$$\begin{aligned}
\lim_j \int_{G(B_1)} |T - S| dW_j(z,T) &= \lim_j \int_{G(B_1)} |T - S| \chi(z) dV_j(z,T) \\
&= \int_{G(B_1)} |T - S| \chi(z) d\delta_S(T) d\sigma(z) = 0.
\end{aligned}$$

Hence the sequences of integrands $\{F_j\}$ and of varifolds $\{W_j\}$ satisfy the assumptions of Lemma 4.9 (note indeed that $B_{F_j}(z,T) = B_F(\bar{x}+r_j z, T)$ so that assumption (2) in Lemma 4.9 is satisfied). Thus we deduce the existence of $\gamma \in L^1(\mathcal{H}^d \llcorner B_1^d)$ such that, along a (not relabelled) subsequence, for every $0 < t < 1$

$$\Big|(\Pi_S)_\#(F(\bar{x}+r_j(\cdot), S)\|W_j\|) - \gamma\mathcal{H}^d \llcorner B^d\Big|(B_t^d) \longrightarrow 0. \tag{4.27}$$

By (4.21) we can substitute $\|W_j\|^s$ for $\|W_j\|$ in (4.27) to get

$$\Big|(\Pi_S)_\#(F(\bar{x}+r_j(\cdot), S)\|W_j\|^s) - \gamma\mathcal{H}^d \llcorner B_1^d\Big|(B_t^d) \longrightarrow 0.$$

By point (ii) above, $F(\bar{x}+r_j(\cdot), S)\|W_j\|^s \overset{*}{\rightharpoonup} F(\bar{x}, S)\chi\sigma$ with $\sigma \llcorner B_{1/2} \neq 0$. Recalling that $F(\bar{x}, S) > 0$ we then have

$$\begin{aligned}
0 &< \big|(\Pi_S)_\#(F(\bar{x}, S)\chi\sigma)\big|(B_{1/2}^d) \\
&\leqslant \liminf_{j\to\infty} \big|(\Pi_S)_\#(F(\bar{x}+r_j(\cdot), S)\|W_j\|^s)\big|(B_{1/2}^d) \\
&= \liminf_{j\to\infty} \big|(\Pi_S)_\#(F(\bar{x}+r_j(\cdot), S)\|W_j\|^s)\big|(E_j \cap B_{1/2}^d) \\
&\leqslant \limsup_{j\to\infty} \Big|(\Pi_S)_\#(F(\bar{x}+r_j(\cdot), S)\|W_j\|^s) - \gamma\mathcal{H}^d \llcorner B^d\Big|(E_j \cap B_{1/2}^d) = 0,
\end{aligned}$$

since $(\Pi_S)_\#\|W_j\|^s$ is concentrated on E_j and $\mathcal{H}^d(E_j) = 0$, see (4.24) and (4.25). This contradiction concludes the proof.

□

4.4 PROOF OF THE MAIN THEOREM

Proof of Theorem 4.2. Step 1: Sufficiency. Let F be a C^1 integrand satisfying the (AC) condition at every $x \in \Omega$ and let $V \in \mathcal{V}_F(\Omega)$, we want to apply Lemma 4.5 to $\|V_*\|$. Note that, according to Lemma 4.11 and (2.7),

$$\mathcal{H}^d \llcorner \{x \in \Omega : \Theta_*^d(x, \|V_*\|) > 0\} \ll \|V_*\| \ll \mathcal{H}^d \llcorner \{x \in \Omega : \Theta_*^d(x, \|V_*\|) > 0\}.$$

Since, by [68, Theorem 6.9], $\mathcal{H}^d\big(\{x \in \Omega : \Theta^{d*}(x, \|V_*\|) = +\infty\}\big) = 0$, we deduce that

$$0 < \Theta_*^d(x, \|V_*\|) \leqslant \Theta^{d*}(x, \|V_*\|) < +\infty \qquad \text{for } \|V_*\|\text{-a.e. } x \in \Omega,$$

hence assumption (i) of Lemma 4.5 is satisfied. By Lemma 4.8, $V_* = \|V_*\| \otimes \delta_{T_x}$ for some $T_x \in G(n, d)$, and, combining this with Lemma 4.7 and (4.7), for $\|V_*\|$-almost every $x \in \Omega$ every $\sigma \in \operatorname{Tan}(x, \|V_*\|)$ is invariant along the directions of T_x, so that also assumption (ii) of Lemma 4.5 is satisfied. Hence

$$\|V_*\| = \theta\mathcal{H}^d \llcorner (K \cap \Omega),$$

for some rectifiable set K and Borel function θ. Moreover, again by Lemma 4.5, $T_x K = T_x$ for $\|V_*\|$-almost every x. This proves that V_* is d-rectifiable.

Step 2: Necessity. Let us now assume that $F(x,T) \equiv F(T)$ does not depend on the point, but just on the tangent plane and let us suppose that F does not verify the atomic condition (AC). We will show the existence of a varifold $V \in \mathscr{V}_F(\mathbb{R}^n)$, with positive lower d-dimensional density (namely $V = V_*$), which is not d-rectifiable. Indeed the negation of (AC) means that there exists a probability measure μ on $G(n,d)$, such that one of the following cases happens:

1) $\dim \operatorname{Ker} A(\mu) = \dim \operatorname{Ker} A(\mu)^* > n - d$

2) $\dim \operatorname{Ker} A(\mu) = \dim \operatorname{Ker} A(\mu)^* = n - d$ and $\mu \neq \delta_{T_0}$,

where $A(\mu) := \int_{G(n,d)} B_F(T)\, d\mu(T)$ and $B_F(T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is constant in x. Let $W := \operatorname{Im} A(\mu)^*$, $k = \dim W \leqslant d$ and let us define the varifold

$$V(dx, dT) := \mathcal{H}^k \llcorner W(dx) \otimes \mu(dT) \in \mathbf{V}_d(\mathbb{R}^n).$$

Clearly V is not d-rectifiable since either $k < d$ or $\mu \neq \delta_W$. We start by noticing that $V = V_*$, indeed for $x \in W$

$$\Theta^d(x,V) = \lim_{r \to 0} \frac{\mathcal{H}^k(B_r(x) \cap W)}{\omega_d r^d} = \begin{cases} 1 & \text{if } k = d \\ +\infty & \text{if } k < d. \end{cases} \tag{4.28}$$

Let us now prove that $V \in \mathscr{V}_F(\mathbb{R}^n)$. For every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\delta_F V(g) = \int_W A(\mu) : Dg \, d\mathcal{H}^k = -\langle g, A(\mu) D(\mathcal{H}^k \llcorner W) \rangle = 0$$

since $D(\mathcal{H}^k \llcorner W) \in W^\perp = [\operatorname{Im} A(\mu)^*]^\perp = \operatorname{Ker} A(\mu)$. Hence V is F-stationary and in particular $V \in \mathscr{V}_F(\mathbb{R}^n)$ which, together with (4.28) concludes the proof. $\square$

4.5 PROOF OF THE EQUIVALENCE OF THE ELLIPTICITY DEFINITIONS IN CODIMENSION ONE

In this section we prove Theorem 4.3. As explained in the introduction, it is convenient to identify the Grassmannian $G(n, n-1)$ with the projective space $\mathbb{RP}^{n-1} = \mathbb{S}^{n-1}/\pm$ via the map

$$\mathbb{S}^{n-1} \ni \pm\nu \mapsto \nu^\perp.$$

Hence an $(n-1)$-varifold V can be thought as a positive Radon measure $V \in \mathcal{M}_+(\Omega \times \mathbb{S}^{n-1})$ even in the $\mathbb{S}^{n-1}$ variable, i.e. such that

$$V(A \times S) = V(A \times (-S)) \qquad \text{for all } A \subset \Omega,\ S \subset \mathbb{S}^{n-1}.$$

In the same way, we identify the integrand $F : \Omega \times G(n, n-1) \to \mathbb{R}_{>0}$ with a positively one homogeneous even function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ via the equality

$$G(x, \lambda\nu) := |\lambda| F(x, \nu^\perp) \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \nu \in \mathbb{S}^{n-1}. \tag{4.29}$$

Note that $G \in C^1(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ and that by one-homogeneity:

$$\langle d_e G(x,e), e\rangle = G(x,e) \qquad \text{for all } e \in \mathbb{R}^n \setminus \{0\}. \tag{4.30}$$

With these identifications, it is a simple calculation to check that:

$$\begin{aligned}\delta_F V(g) = &\int_{\Omega\times\mathbb{S}^{n-1}} \langle d_x G(x,\nu), g(x)\rangle \, dV(x,\nu)\\ &+ \int_{\Omega\times\mathbb{S}^{n-1}} \Big(G(x,\nu)\mathrm{Id} - \nu\otimes d_\nu G(x,\nu)\Big) : Dg(x)\, dV(x,\nu),\end{aligned}$$

see for instance [5, Section 3] or [40, Lemma A.4]. In particular, under the correspondence (4.29)

$$B_F(x,T) = G(x,\nu)\mathrm{Id} - \nu\otimes d_\nu G(x,\nu) =: B_G(x,\nu), \qquad T = \nu^\perp.$$

Note that $B_G(x,\nu) = B_G(x,-\nu)$ since $G(x,\nu)$ is even. Hence the atomic condition at x can be re-phrased as:

(i) $\dim \mathrm{Ker} A_x(\mu) \leqslant 1$ for all *even* probability measures $\mu \in \mathcal{P}_{\mathrm{even}}(\mathbb{S}^{n-1})$,

(ii) if $\dim \mathrm{Ker} A_x(\mu) = 1$ then $\mu = (\delta_{\nu_0} + \delta_{-\nu_0})/2$ for some $\nu_0 \in \mathbb{S}^{n-1}$,

where

$$A_x(\mu) = \int_{\mathbb{S}^{n-1}} B_G(x,\nu)d\mu(\nu).$$

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Since the (AC) condition deals only with the behavior of the frozen integrand $G_x(\nu) = G(x,\nu)$, for the whole proof x is fixed and for the sake of readability we drop the dependence on x.

Step 1: Sufficiency. Let us assume that $G : \mathbb{R}^n \to \mathbb{R}$ is even, one-homogeneous and strictly convex. We will show that the requirements (i) and (ii) in the (AC) condition are satisfied. First note that, by one-homogeneity, the strict convexity of G is equivalent to:

$$G(\nu) > \langle d_\nu G(\bar\nu), \nu\rangle \qquad \text{for all } \bar\nu, \nu \in \mathbb{S}^{n-1} \text{ and } \nu \neq \pm\bar\nu. \tag{4.31}$$

Plugging $-\nu$ in (4.31) and exploiting the fact that G is even we obtain

$$G(\nu) > |\langle d_\nu G(\bar\nu), \nu\rangle| \qquad \text{for all } \bar\nu, \nu \in \mathbb{S}^{n-1} \text{ and } \nu \neq \pm\bar\nu. \tag{4.32}$$

Let now $\mu \in \mathcal{P}_{\mathrm{even}}(\mathbb{S}^{n-1})$ be an even probability measure,

$$A(\mu) = \int_{\mathbb{S}^{n-1}} \Big(G(\nu)\mathrm{Id} - \nu\otimes d_\nu G(\nu)\Big)d\mu(\nu)$$

and assume there exists $\bar\nu \in \mathrm{Ker}A(\mu) \cap \mathbb{S}^{n-1}$. We then have

$$\begin{aligned}0 &= \langle d_\nu G(\bar\nu), A(\mu)\bar\nu\rangle\\ &= \int_{\mathbb{S}^{n-1}} \Big\{(G(\bar\nu)G(\nu) - \langle d_\nu G(\bar\nu),\nu\rangle\langle d_\nu G(\nu),\bar\nu\rangle\Big\}d\mu(\nu)\\ &\geqslant \int_{\mathbb{S}^{n-1}} \Big\{G(\bar\nu)G(\nu) - \big|\langle d_\nu G(\bar\nu),\nu\rangle\big|\big|\langle d_\nu G(\nu),\bar\nu\rangle\big|\Big\}d\mu(\nu)\end{aligned}$$

where we have used (4.30). Inequality (4.32) implies however that the integrand in the last line of the above equation is strictly positive, unless $\nu = \pm\bar{\nu}$ for all $\nu \in \operatorname{spt}\mu$, which immediately implies that the (AC) condition is satisfied.

Step 2: Necessity. Let us assume that G (or equivalently F) satisfies the (AC) condition, let $\nu, \bar{\nu} \in \mathbb{S}^{n-1}$, $\nu \neq \pm\bar{\nu}$ and define

$$\mu = \frac{1}{4}\big(\delta_\nu + \delta_{-\nu} + \delta_{\bar{\nu}} + \delta_{-\bar{\nu}}\big).$$

Then the matrix

$$A(\mu) = \frac{1}{2}B_G(\nu) + \frac{1}{2}B_G(\bar{\nu})$$

has full rank. In particular the vectors $A(\mu)\nu, A(\mu)\bar{\nu}$ are linearly independent. On the other hand

$$2A(\mu)\nu = B_G(\bar{\nu})\nu = G(\bar{\nu})\nu - \langle d_\nu G(\bar{\nu}), \nu\rangle\bar{\nu}$$
$$2A(\mu)\bar{\nu} = B_G(\nu)\bar{\nu} = G(\nu)\bar{\nu} - \langle d_\nu G(\nu), \bar{\nu}\rangle\nu$$

and thus, these two vectors are linearly independent if and only if

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu\rangle\langle d_\nu G(\nu), \bar{\nu}\rangle \neq 0.$$

Since G is positive and $\mathbb{S}^{n-1} \setminus \{\pm\bar{\nu}\}$ is connected for $n \geqslant 3$, the above equation implies that

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu\rangle\langle d_\nu G(\nu), \bar{\nu}\rangle > 0 \qquad \text{for all } \nu \neq \pm\bar{\nu}. \tag{4.33}$$

Exploiting that G is even, the same can be deduced also if $n = 2$. We now show that (4.33) implies (4.31) and thus the strict convexity of G (actually Step 1 of the proof shows that they are equivalent). Let $\bar{\nu}$ be fixed and let us define the linear projection

$$P_{\bar{\nu}}\nu = \frac{\langle d_\nu G(\bar{\nu}), \nu\rangle}{G(\bar{\nu})}\,\bar{\nu}.$$

We note that by (4.30) $P_{\bar{\nu}}$ is actually a projection, i.e. $P_{\bar{\nu}} \circ P_{\bar{\nu}} = P_{\bar{\nu}}$. Hence, setting $\nu_t = t\nu + (1-t)P_{\bar{\nu}}\nu$ for $t \in [0,1]$, we have $P_{\bar{\nu}}\nu_t = P_{\bar{\nu}}\nu$. Thus

$$\nu_t - P_{\bar{\nu}}\nu_t = t(\nu - P_{\bar{\nu}}\nu). \tag{4.34}$$

Hence, if we define $g(t) = G(\nu_t)$, we have, for $t \in (0,1)$,

$$tg'(t) = t\langle d_\nu G(\nu_t), \nu - P_{\bar{\nu}}\nu\rangle = \langle d_\nu G(\nu_t), \nu_t - P_{\bar{\nu}}\nu_t\rangle > 0,$$

where in the second equality we have used equation (4.34) and the last inequality follows from (4.33) with $\nu = \nu_t$, and $t > 0$. Hence, exploiting also the one-homogeneity of G,

$$G(\nu) = g(1) > g(0) = G(P_{\bar{\nu}}\nu) = \frac{\langle d_\nu G(\bar{\nu}), \nu\rangle}{G(\bar{\nu})}G(\bar{\nu}) = \langle d_\nu G(\bar{\nu}), \nu\rangle$$

which proves (4.31) and concludes the proof. □

4.6 APPENDIX: FIRST VARIATION WITH RESPECT TO ANISOTROPIC INTEGRANDS

In this section we compute the **F**-first variation of a varifold V. To this end we recall that, by identifying a d-plane T with the orthogonal projection onto T, we can embed $G(n,d)$ into $\mathbb{R}^n \otimes \mathbb{R}^n$. Indeed we have

$$G(n,d) \approx \Big\{ T \in \mathbb{R}^n \otimes \mathbb{R}^n : T \circ T = T, \quad T^* = T, \quad \operatorname{tr} T = d \Big\}. \tag{4.35}$$

With this identification, let $T(t) \in G(n,d)$ be a smooth curve such that $T(0) = T$. Differentiating the above equalities we get

$$T'(0) = T'(0) \circ T + T \circ T'(0), \qquad (T'(0))^* = T'(0), \qquad \operatorname{tr} T'(0) = 0. \tag{4.36}$$

In particular from the first equality above we obtain

$$T \circ T'(0) \circ T = 0, \qquad T^\perp \circ T'(0) \circ T^\perp = 0.$$

Hence

$$\operatorname{Tan}_T G(n,d) \subset \big\{ S \in \mathbb{R}^n \otimes \mathbb{R}^n : S^* = S, \quad T \circ S \circ T = 0, \quad T^\perp \circ S \circ T^\perp = 0 \big\}.$$

Since $\dim \operatorname{Tan}_T G(n,d) = \dim G(n,d) = d(n-d)$ the above inclusion is actually an equality. To compute the anisotropic first variation of a varifold we need the following simple Lemma:

Lemma 4.12. *Let* $T \in G(n,d)$ *and* $L \in \mathbb{R}^n \otimes \mathbb{R}^n$, *and let us define* $T(t) \in G(n,d)$ *as the orthogonal projection onto* $(\mathrm{Id} + tL)(T)$ *(recall the identification* (4.35)*). Then*

$$T'(0) = T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \in \operatorname{Tan}_T G(n,d).$$

Proof. One easily checks that $T(t)$ is a smooth function of T for t small. Since

$$T(t) \circ (\mathrm{Id} + tL) \circ T = (\mathrm{Id} + tL) \circ T,$$

differentiating we get

$$T'(0) \circ T = (\mathrm{Id} - T) \circ L \circ T = T^\perp \circ L \circ T. \tag{4.37}$$

Using that $(T'(0))^* = T'(0)$, $T^* = T$, the first equation in (4.36) and (4.37), one obtains

$$\begin{aligned} T'(0) &= T'(0) \circ T + T \circ T'(0) \\ &= T'(0) \circ T + (T'(0) \circ T)^* = T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^*, \end{aligned}$$

and this concludes the proof. □

We are now ready to compute the first variation of an anisotropic energy:

Lemma 4.13. *Let* $F \in C^1(\Omega \times G(n,d))$ *and* $V \in \mathbf{V}_d(\Omega)$, *then for* $g \in C^1_c(\Omega, \mathbb{R}^n)$ *we have*

$$\delta_F V(g) = \int_{G(\Omega)} \Big[\langle d_x F(x,T), g(x) \rangle + B_F(x,T) : Dg(x) \Big] \, dV(x,T), \tag{4.38}$$

where the matrix $B_F(x,T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ *is uniquely defined by*

$$B_F(x,T) : L := F(x,T)(T : L) + \big\langle d_T F(x,T),\, T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \big\rangle \tag{4.39}$$

for all $L \in \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. For $g \in C^1_c(\Omega, \mathbb{R}^n)$ let $\varphi_t(x) = x + tg(x)$ which is a diffeomorphism of Ω into itself for $t \ll 1$. We have

$$\begin{aligned}
\delta_F V(g) &= \frac{d}{dt}\mathbf{F}(\varphi_t^{\#}V)\Big|_{t=0}\\
&= \frac{d}{dt}\int_{G(\Omega)} F(\varphi_t(x), d\varphi_t(T))J\varphi_t(x,T)dV(x,T)\Big|_{t=0}\\
&= \int_{G(\Omega)} \frac{d}{dt}F(\varphi_t(x),T)dV(x,T)\Big|_{t=0} + \int_{G(\Omega)} \frac{d}{dt}F(x, d\varphi_t(T))dV(x,T)\Big|_{t=0}\\
&\quad + \int_{G(\Omega)} F(x,T)\frac{d}{dt}J\varphi_t(x,T)\Big|_{t=0} dV(x,T).
\end{aligned}$$

Equation (4.38) then follows by the definition of $B_F(x,T)$, (4.39), and the equalities

$$\frac{d}{dt}F(\varphi_t(x),T)\Big|_{t=0} = \langle d_x F(x,T), g(x)\rangle, \tag{4.40}$$

$$\frac{d}{dt}J\varphi_t(x,T)\Big|_{t=0} = T : Dg(x), \tag{4.41}$$

$$\frac{d}{dt}F(x, d\varphi_t(T))\Big|_{t=0} = \Big\langle d_T F(x,T),\, T^{\perp}\circ Dg(x)\circ T + (T^{\perp}\circ Dg(x)\circ T)^*\Big\rangle. \tag{4.42}$$

Here (4.40) is trivial, (4.41) is a classical computation, see for instance [79, Section 2.5], and (4.42) follows from Lemma 4.12. $\square$

4.7 APPENDIX: PROOF OF THE RECTIFIABILITY LEMMA

In this Section we prove Lemma 4.5. Let us start recalling the following rectifiability criterion due to Preiss, see [76, Theorem 5.3].

Theorem 4.14. *Let μ be a measure on $\mathbb{R}^n$ and assume that at μ-a.e. x the following two conditions are satisfied:*

(I) If we set $\alpha = \alpha_d = 1 - 2^{-d-6}$ and

$$E_r(x) := \left\{ z \in B_r(x) :\ \exists s \in (0,r) \textit{ satisfying } \frac{\mu(B_s(z))}{\omega_d s^d} \leqslant \alpha \frac{\mu(B_r(x))}{\omega_d r^d} \right\},$$

then

$$\liminf_{r\to 0} \frac{\mu(E_r(x))}{\mu(B_r(x))} = 0;$$

(II) If we set $\beta = \beta_d = 2^{-d-9}d^{-4}$ and

$$F_r(x) := \sup_{T\in G(n,d)} \left\{ \inf_{z\in(x+T)\cap B_r(x)} \frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))} \right\},$$

then

$$\liminf_{r\to 0} F_r(x) > 0.$$

Then μ is a d-rectifiable measure.

Proof of Lemma 4.5. By replacing μ with $\mu \llcorner \Omega'$, where $\Omega' \subset\subset \Omega$, we can assume that μ is defined on the whole $\mathbb{R}^n$. We are going to prove that μ verifies conditions (I) and (II) in Theorem 4.14.

Let us start by verifying condition (I). Given $\varepsilon, m > 0$, let

$$E(\varepsilon, m) := \left\{ z \in \mathbb{R}^n : \quad \frac{\mu(B_r(z))}{\omega_d r^d} > m \text{ for all } r \in (0, \varepsilon) \right\},$$

and, for $\alpha = \alpha_d$ as in Theorem 4.14 and $\gamma \in (1, 1/\alpha)$, set

$$\widehat{E}(\varepsilon, m) := E(\varepsilon, \alpha\gamma m) \setminus \bigcup_{k=1}^{\infty} E\left(\frac{\varepsilon}{k}, m\right).$$

If x is such that $0 < \Theta^d_*(x, \mu) < +\infty$, then $x \in \widehat{E}(\bar{\varepsilon}, \bar{m})$ for some positive $\bar{\varepsilon}$ and $\bar{m}$ such that $\alpha\gamma\bar{m} < \Theta^d_*(x, \mu) < \bar{m}$, hence

$$\{0 < \Theta^d_*(x, \mu) < +\infty\} \subset \bigcup_{m>0} \bigcup_{\varepsilon>0} \widehat{E}(\varepsilon, m).$$

Let now $x \in \widehat{E}(\varepsilon, m)$ be a density point for $\widehat{E}(\varepsilon, m)$:

$$\lim_{r \to 0} \frac{\mu(B_r(x) \setminus \widehat{E}(\varepsilon, m))}{\mu(B_r(x))} = 0. \tag{4.43}$$

Note that $x \in \widehat{E}(\varepsilon, m)$ implies that $\alpha\gamma m \leqslant \Theta^d_*(x, \mu) \leqslant m < \gamma m$. Hence, if $(r_k)_k$ is a sequence verifying $r_k \to 0$, $r_k < \varepsilon$ and such that $\Theta^d_*(x, \mu) = \lim_k \mu(B_{r_k}(x))/\omega_d r_k^d$, then, for k large enough,

$$E_{r_k}(x) \subset B_{r_k}(x) \setminus E(\varepsilon, \alpha\gamma m) \subset B_{r_k}(x) \setminus \widehat{E}(\varepsilon, m),$$

which, together with (4.43), proves that μ verifies condition (I).

We now verify condition (II). Let x be a point such that all the tangent measures at x are translation invariant in the directions of T_x and such that $0 < \Theta^d_*(x, \mu) \leqslant \Theta^{d*}(x, \mu) < +\infty$. Note that the latter condition implies that for every $\sigma \in \mathrm{Tan}(x, \mu)$

$$\frac{\Theta^d_*(x, \mu)}{\Theta^{d*}(x, \mu)} t^d \leqslant \sigma(B_t) \leqslant \frac{\Theta^{d*}(x, \mu)}{\Theta^d_*(x, \mu)} t^d \qquad \text{for all } t \in (0, 1).$$

In particular, $0 \in \operatorname{spt} \sigma$ for all $\sigma \in \mathrm{Tan}(x, \mu)$. Let us choose a sequence $r_i \to 0$ and $z_{r_i} \in (x + T_x) \cap B_{r_i}(x)$, such that

$$\liminf_{r \to 0} \left\{ \inf_{z \in (x+T_x) \cap B_r(x)} \frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))} \right\} = \lim_{i \to \infty} \frac{\mu(B_{\beta r_i}(z_{r_i}))}{\mu(B_{r_i}(x))}$$
$$\geqslant \lim_{i \to \infty} \mu_{x, r_i}\left(B_\beta \left(\frac{z_{r_i} - x}{r_i} \right) \right),$$

where μ_{x,r_i} is defined in (2.2) and $\beta = \beta_d$ is as in Theorem 4.14. Up to subsequences we have that

$$\lim_{i \to \infty} \mu_{x, r_i} \overset{*}{\rightharpoonup} \sigma \in \mathrm{Tan}(x, \mu) \qquad \text{and} \qquad \lim_{i \to \infty} \frac{z_{r_i} - x}{r_i} = z \in \bar{B} \cap T_x.$$

Hence

$$\liminf_{r\to 0}\left\{\inf_{z_r\in(x+T_x)\cap B_r(x)}\frac{\mu(B_{\beta r}(z_r))}{\mu(B_r(x))}\right\}\geqslant \sigma(B_\beta(z)).$$

Let $z'\in B_{\beta/2}(z)\cap T_x$ such that $B_{\beta/2}(z')\subset B_\beta(z)\cap B$. Since σ is translation invariant in the directions of T_x

$$\sigma(B_\beta(z))\geqslant \sigma(B_{\frac{\beta}{2}}(z'))=\sigma(B_{\frac{\beta}{2}}(0))>0,$$

where in the last inequality we have used that $0\in \operatorname{spt}\sigma$. Thus

$$\liminf_{r\to 0}F_r(x)\geqslant \liminf_{r\to 0}\left\{\inf_{z\in(x+T_x)\cap B_r(x)}\frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))}\right\}>0,$$

implying that also condition (II) in Theorem 4.14 is satisfied. Hence μ is d-rectifiable. In particular for μ-a.e. x, $\operatorname{Tan}(x,\mu)=\{\omega_d^{-1}\mathcal{H}^d\llcorner(T_xK\cap B)\}$. Since, by assumption, μ is invariant along the directions of T_x, this implies that $T_x=T_xK$ and concludes the proof. $\square$

5 COMPACTNESS FOR INTEGRAL VARIFOLDS: THE ANISOTROPIC SETTING

5.1 INTRODUCTION

The aim of this chapter is to present our paper [45], where we extend the results [37, 44, 38, 43], presented in Chapter 3, to the minimization of an anisotropic energy on classes of rectifiable varifolds in any dimension and codimension, see Theorem 5.2. The limit of a minimizing sequence of varifolds with density uniformly bounded from below is proven to be rectifiable. Moreover, with the further assumption that the minimizing sequence is made of integral varifolds with uniformly locally bounded anisotropic first variation, the limiting varifold turns out to be also integral.

We remark that every sequence of rectifiable (resp. integral) varifolds enjoying a uniform bound on the mass and on the isotropic first variation is precompact in the space of rectifiable (resp. integral) varifolds. This has been proved by Allard in [2, Section 6.4], see also [79, Theorem 42.7 and Remark 42.8].

One of the main results of this work is indeed an anisotropic counterpart of the aforementioned compactness for integral varifolds, in the assumption that the limiting varifold has positive lower density, see Theorem 5.7.

The additional tool available in the isotropic setting is the monotonicity formula for the mass ratio of stationary varifolds, which ensures that the density function is upper semicontinuous with respect to the convergence of varifolds. This property allows the limiting varifold to inherit the lower density bound of the sequence.

The monotonicity formula is deeply linked to the isotropic case, see [3]. Nonetheless, given a minimizing sequence of varifolds for an elliptic integrand, we are able to get a density lower bound for the limiting varifold via a deformation theorem for rectifiable varifolds, see Theorem 5.4. We can obtain it modifying [32, Proposition 3.1], proved by David and Semmes for closed sets. Thanks to the density lower bound and the anisotropic stationarity of the limiting varifold, we can conclude directly its rectifiability applying the main theorem of [42], see Theorem 4.2.

The integrality result requires additional work, see Lemma 5.7: the idea is to blow-up every varifold of the minimizing sequence in a point in which the limiting varifold has Grassmannian part supported on a single d-plane S (note that this property holds $\|V\|$-a.e. by the previously proved rectifiability). Applying a result proved in [42], see Lemma 4.9, on a diagonal sequence of blown-up varifolds, we get that roughly speaking their projections on S converge in total variation to an L^1 function on S. This function is integer valued thanks to the integrality assumption on the minimizing sequence and coincides with the density of the limiting varifold in the blow-up point, which is consequently an integer. Since the argument holds true for $\|V\|$-a.e. point, the limiting varifold turns out to be integral.

5.2 SETTING

In the entire Chapter 5, we will asume that the anisotropy introduced in (2.9) is defined on the whole $G(\mathbb{R}^n)$, i.e. $\Omega = \mathbb{R}^n$. Assume to have a class of varifolds $\mathscr{P}(\mathbf{F}, H) \subseteq \mathbf{R}_d(\mathbb{R}^n)$ encoding a notion of boundary: one can then formulate the anisotropic Plateau problem with multiplicity by asking whether the infimum

$$m_0 := \inf \left\{ \mathbf{F}(V) : V \in \mathscr{P}(\mathbf{F}, H) \right\} \tag{5.1}$$

is achieved by some varifold (which is the limit of a minimizing sequence), if it belongs to the chosen class $\mathscr{P}(\mathbf{F}, H)$ and which additional regularity properties it satisfies. We will say that a sequence $(V_j)_{j \in \mathbb{N}} \subseteq \mathscr{P}(\mathbf{F}, H)$ is a *minimizing sequence* if $\mathbf{F}(V_j) \downarrow m_0$.

We need to introduce some minimal requirements for the class $\mathscr{P}(\mathbf{F}, H)$. This is an adaptation of Definition 3.2 to the setting of varifolds.

Definition 5.1 (Deformed competitors and deformation class)**.** Let $H \subseteq \mathbb{R}^n$ be a closed set and $V \in \mathbf{R}_d(\mathbb{R}^n)$. A *deformed competitor* for V in $B_{x,r}$ is any varifold

$$\varphi^\# V \in \mathbf{R}_d(\mathbb{R}^n) \quad \text{where} \quad \varphi \in D(x, r) \qquad (D(x,r) \text{ is as in Definition 3.1}).$$

We say that $\mathscr{P}(\mathbf{F}, H)$ is a *d*eformation class with respect to H and F if $\mathscr{P}(\mathbf{F}, H) \subseteq \mathbf{R}_d(\mathbb{R}^n)$ and for every $V \in \mathscr{P}(\mathbf{F}, H)$ it holds:

- $\mathrm{conc}(V)$ is a relatively closed subset of $\mathbb{R}^n \setminus H$;
- for every $x \in \mathbb{R}^n \setminus H$ and for a.e. $r \in (0, \mathrm{dist}(x, H))$

  $$\inf \left\{ \mathbf{F}(W) : W \in \mathscr{P}(\mathbf{F}, H), W \llcorner G\big((\overline{B_{x,r}})^c\big) = V \llcorner G\big((\overline{B_{x,r}})^c\big) \right\} \leqslant \mathbf{F}(L),$$

  whenever L is any deformed competitor for V in $B_{x,r}$.

5.2.1 *The main result*

We can now state our main result:

Theorem 5.2. *Let $F \in C^1(G(\mathbb{R}^n))$ be a Lagrangian satisfying the atomic condition as in Definition 4.1 at every point $x \in \mathbb{R}^n$ and enjoying the bounds* (2.10)*. Let $H \subseteq \mathbb{R}^n$ be a closed set and $\mathscr{P}(\mathbf{F}, H)$ be a deformation class with respect to H and F. Assume the infimum in Plateau problem* (5.1) *is finite and let $(V_j)_{j \in \mathbb{N}} \subseteq \mathscr{P}(\mathbf{F}, H)$ be a minimizing sequence. Then, up to subsequences, V_j converges to a d-varifold $V \in \mathbf{V}_d(\mathbb{R}^n)$ with the following properties:*

(a) $\liminf_j \mathbf{F}(V_j) \geqslant \mathbf{F}(V)$*;*

(b) if $V \in \mathscr{P}(\mathbf{F}, H)$, then V is a minimum for (5.1)*;*

(c) V is F-stationary in $\mathbb{R}^n \setminus H$.

Furthermore:

(d) *if the minimizing sequence* $(V_j)_{j\in\mathbb{N}}$ *enjoys a uniform density lower bound in* $\mathbb{R}^n \setminus H$, *i.e. there exists* $\delta > 0$ *such that:*

$$\Theta^d(x, V_j) \geqslant \delta, \qquad \text{for } \|V_j\|\text{-a.e. } x \in \mathbb{R}^n \setminus H,\ \forall j \in \mathbb{N},$$

then $V \llcorner G(\mathbb{R}^n \setminus H) \in \mathbf{R}_d(\mathbb{R}^n)$ *and* $conc(V)$ *is relatively closed in* $\mathbb{R}^n \setminus H$;

(e) *if the minimizing sequence* $(V_j)_{j\in\mathbb{N}}$ *satisfies* $(V_j \llcorner G(\mathbb{R}^n \setminus H))_{j\in\mathbb{N}} \subseteq \mathbf{I}_d(\mathbb{R}^n)$ *and*

$$\sup_j |\delta_F V_j|(W) < \infty, \qquad \forall W \subset\subset \mathbb{R}^n \setminus H, \tag{5.2}$$

then $V \llcorner G(\mathbb{R}^n \setminus H) \in \mathbf{I}_d(\mathbb{R}^n)$.

Remark 5.3. If the assumption $(V_j \llcorner G(\mathbb{R}^n \setminus H))_{j\in\mathbb{N}} \subseteq \mathbf{I}_d(\mathbb{R}^n)$ required in the condition (e) of Theorem 5.2 is satisfied, also condition (d) applies, with the trivial density lower bound $\delta = 1$.

5.3 PRELIMINARY RESULTS

A key result we are going to use is a deformation theorem for rectifiable varifolds with density bigger or equal than one, that we prove in this section. It is the analogous of the deformation theorem for closed sets, due to David and Semmes [32, Proposition 3.1], see Theorem 3.7, and of the one for rectifiable currents [79, 50].

The proof relies on the one of [32, Proposition 3.1].

We will use cimilar notation to the one introduced in Section 3.3.1. We recall it for completeness. Given a closed cube $Q = Q_{x,l}$ and $\varepsilon > 0$, we cover Q with a grid of closed smaller cubes with edge length $\varepsilon \ll l$, with non empty intersection with $\mathrm{Int}(Q)$ and such that the decomposition is centered in x (i.e. one of the subcubes is centered in x). The family of these smaller cubes is denoted $\Lambda_\varepsilon(Q)$. We set

$$\begin{aligned} C_1 &:= \bigcup \{T \cap Q : T \in \Lambda_\varepsilon(Q), T \cap \partial Q \neq \emptyset\}, \\ C_2 &:= \bigcup \{T \in \Lambda_\varepsilon(Q) : (T \cap Q) \not\subseteq C_1, T \cap \partial C_1 \neq \emptyset\}, \\ Q^1 &:= \overline{Q \setminus (C_1 \cup C_2)} \end{aligned} \tag{5.3}$$

and consequently

$$\Lambda_\varepsilon(Q^1 \cup C_2) := \left\{T \in \Lambda_\varepsilon(Q) : T \subseteq (Q^1 \cup C_2)\right\}.$$

For each nonnegative integer $m \leqslant n$, let $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ denote the collection of all m-dimensional faces of cubes in $\Lambda_\varepsilon(Q^1 \cup C_2)$ and $\Lambda^*_{\varepsilon,m}(Q^1 \cup C_2)$ will be the set of the elements of $\Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ which are not contained in $\partial(Q^1 \cup C_2)$. We also let $S_{\varepsilon,m}(Q^1 \cup C_2) := \bigcup \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$ be the m-skeleton of order ε in $Q^1 \cup C_2$.

Theorem 5.4. *Given* $x_0 \in \mathbb{R}^n$, $r > 0$, *a closed cube* $Q \subseteq B_{x_0,r}$ *and* $V \in \mathbf{R}_d(\mathbb{R}^n)$ *such that:*

$$V = \theta \mathcal{H}^d \llcorner K \otimes \delta_{T_x K}, \qquad \text{where} \qquad \theta(x) \geqslant 1 \quad \text{for } \mathcal{H}^d \llcorner K - \text{a.e. } x \in Q,$$

$$K \cap Q \text{ is a closed set} \quad \text{and} \quad \|V\|(Q) < +\infty.$$

Then there exists a map $\Phi_{\varepsilon,V} \in D(x_0, r)$ *satisfying the following properties:*

(1) $\Phi_{\varepsilon,V}(x) = x$ *for* $x \in \mathbb{R}^n \setminus (Q^1 \cup C_2)$*;*

(2) $\Phi_{\varepsilon,V}(x) = x$ *for* $x \in S_{\varepsilon,d-1}(Q^1 \cup C_2)$*;*

(3) $\Phi_{\varepsilon,V}(K \cap (Q^1 \cup C_2)) \subseteq S_{\varepsilon,d}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2)$*;*

(4) $\Phi_{\varepsilon,V}(T) \subseteq T$ *for every* $T \in \Lambda_{\varepsilon,m}(Q^1 \cup C_2)$*, with* $m = d, \dots, n$*;*

(5) $\|(\Phi_{\varepsilon,V})^\# V\|(T) \leqslant k_1 \|V\|(T)$ *for every* $T \in \Lambda_\varepsilon(Q^1 \cup C_2)$*;*

(6) either $\|(\Phi_{\varepsilon,V})^\# V\|(T) = 0$ *or* $\|(\Phi_{\varepsilon,V})^\# V\|(T) \geqslant \mathcal{H}^d(T)$*, for every* $T \in \Lambda^*_{\varepsilon,d}(Q^1)$*;*

where k_1 *depends only on* n *and* d *(but neither on* ε *nor on* V*).*

Proof. Our map $\Phi_{\varepsilon,V}$ can be obtained as the last element of a finite sequence $\Phi_n, \Phi_{n-1}, \dots, \Phi_d, \Phi_{d-1}$ of Lipschitz maps on $\mathbb{R}^n$. The maps Φ_m with $m = d, \dots, n$ will satisfy the analogous of $(1) - (5)$, with (2) and (3) replaced by

$$\Phi_m(x) = x \text{ for } x \in S_{\varepsilon,m}(Q^1 \cup C_2),$$

$$\Phi_m(K \cap (Q^1 \cup C_2)) \subseteq S_{\varepsilon,m}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2).$$

The last map Φ_{d-1} will be constructed in order to satisfy also property (6).

We start with $\Phi_n(x) := x$, which verifies all the required conditions. Suppose that, for a given $m > d$, we have already built $\Phi_n, \Phi_{n-1}, \dots, \Phi_m$. We want to define Φ_{m-1} as

$$\Phi_{m-1} := \psi_{m-1} \circ \Phi_m, \tag{5.4}$$

where ψ_{m-1} is a Lipschitz map in $\mathbb{R}^n$ given by the following Lemma:

Lemma 5.5. *The exists a Lipschitz map* $\psi_{m-1} : Q^1 \cup C_2 \to Q^1 \cup C_2$ *such that:*

$$\psi_{m-1}(x) = x \text{ for } x \in S_{\varepsilon,m-1}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2),$$

$$\psi_{m-1}(\Phi_m(K \cap (Q^1 \cup C_2))) \subseteq S_{\varepsilon,m-1}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2),$$

$$\psi_{m-1}(T) \subseteq T \text{ for every } T \in \Lambda_{\varepsilon,m}(Q^1 \cup C_2), \text{ with } m = d, \dots, n,$$

and

$$\|(\psi_{m-1} \circ \Phi_m)^\# V\|(T) \leqslant C \|(\Phi_m)^\# V\|(T) \text{ for every } T \in \Lambda_\varepsilon(Q^1 \cup C_2),$$

where C *depends only on* m *and* d*.*

Assuming Lemma 5.5, we can easily extend ψ_{m-1} to be the identity outside $Q^1 \cup C_2$ and the map Φ_{m-1} defined in (5.4) satisfies the desired properties.

To conclude, we need to construct Φ_{d-1} in order to satisfy also condition (6). We proceed in a way analogous to the one used in [44, Theorem 2.4], see Theorem 3.7.

We want to set

$$\Phi_{d-1} := \Psi \circ \Phi_d,$$

where Ψ will be defined below. We first define Ψ on every $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ distinguishing two cases

(a) if either $\|(\Phi_d)^\# V\|(T) = 0$ or $\|(\Phi_d)^\# V\|(T) \geqslant \mathcal{H}^d(T)$ or $T \notin \Lambda^*_{\varepsilon,d}(Q^1)$, then we set $\Psi_{|T} := \mathrm{Id}$;

(b) otherwise, since the varifold density $\Theta(x, V)$ is bigger or equal than one for $\|V\|$-a.e. $x \in Q$, the same holds for $(\Phi_d)^\# V$, because Φ_d is a Lipschitz map. We infer that

$$\mathcal{H}^d(T) > \|(\Phi_d)^\# V\|(T) \geqslant \mathcal{H}^d(\Phi_d(K \cap Q) \cap T).$$

Since $\Phi_d(K \cap Q)$ is compact ($K \cap Q$ is compact by assumption), there exists $y_T \in T$ and $\delta_T > 0$ such that $B_{\delta_T, y_T} \cap \Phi_d(K \cap Q) = \emptyset$; we define

$$\Psi_{|T}(x) := x + \alpha(x - y_T) \min\left\{1, \frac{|x - y_T|}{\delta_T}\right\},$$

where $\alpha > 0$ is such that the point $x + \alpha(x - y_T) \in (\partial T) \times \{0\}^{n-d}$.

The second step is to define Ψ on every $T' \in \Lambda_{\varepsilon,d+1}(Q^1 \cup C_2)$. Without loss of generality, we can assume T' centered in 0. We divide T' in pyramids $P_{T,T'}$ with base $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ and vertex 0. Assuming $T \subseteq \{x_{d+1} = -\frac{\varepsilon}{2}, x_{d+2}, ..., x_n = 0\}$ and $T' \subseteq \{x_{d+2}, ..., x_n = 0\}$, we set

$$\Psi_{|P_{T,T'}}(x) := -\frac{2x_{d+1}}{\varepsilon} \Psi_{|T}\left(-\frac{x}{x_{d+1}} \frac{\varepsilon}{2}\right).$$

We iterate this procedure on all the dimensions till to n, defining it well in $Q^1 \cup C_2$. Since $\Psi_{|\partial(Q^1 \cup C_2)} = \mathrm{Id}$, we can extend the map as the identity outside $Q^1 \cup C_2$.

By construction of Ψ, if we denote

$$(\Psi \circ \Phi_d)^\# V = \tilde{\theta} \mathcal{H}^d \llcorner (\Psi \circ \Phi_d)(K) \otimes \delta_{T_x (\Psi \circ \Phi_d)(K)},$$

we get that $\tilde{\theta} = 0$ in the interior of T, and we can assume this is true also at the boundary since $\mathcal{H}^d((\partial T) \times \{0\}^{n-d}) = 0$ and $\tilde{\theta}$ is defined for $\mathcal{H}^d$-a.e. $x \in (\Psi \circ \Phi_d)(K)$.

We consequently obtain:

$$\|(\Psi \circ \Phi_d)^\# V\|(T) = \int_{(\Psi \circ \Phi_d)(K) \cap T} \tilde{\theta} \, d\mathcal{H}^d = 0,$$

and so property (6) is now satisfied.

In addition, one can easily check that $\Psi \in D(x_0, r)$ and thus, since $\Phi_d \in D(x_0, r)$ and the class $D(x_0, r)$ is closed by composition, then also $\Phi_{d-1} \in D(x_0, r)$.

This concludes the proof of Theorem 5.4 provided we prove Lemma 5.5.

The proof of Lemma 5.5 can be repeated verbatim as the one of [32, Lemma 3.10], if we replace [32, Lemma 3.22] with the following:

Lemma 5.6. *Let T be an m-dimensional closed cube with $m > d$ and $\tilde{V} \in \mathbf{R}_d(\mathbb{R}^n)$ such that:*

$$\tilde{V} = \tilde{\theta} \mathcal{H}^d \llcorner F \otimes \delta_{T_x F}, \qquad \text{where} \qquad F \subset T \text{ is a closed } d\text{-rectifiable set} \quad \text{and} \quad \|\tilde{V}\|(T) < +\infty.$$

For every $z \in T \setminus F$, we define $\varepsilon_z := d(z, F) > 0$. We consider a Lipschitz map $\eta_{z,T} : T \to T$, which satisfies in $T \setminus B_{z,\varepsilon_z}$ the conditions:

$$\eta_{z,T}(x) \in \partial T, \qquad \eta_{z,T}(x) - x = c(x - z), \qquad c = c(x, z, T) > 0, \quad \forall x \in T \setminus B_{z,\varepsilon_z}.$$

Then

$$\int_{z\in\left(\frac{1}{2}T\right)\setminus F} \|(\eta_{z,T})^{\#}\tilde{V}\|(T)\, d\mathcal{H}^m(z) \leqslant C(\mathrm{diam}(T))^m \|\tilde{V}\|(T), \tag{5.5}$$

where C *depends just on* m *and* d.

Proof of Lemma 5.6. For a given point z, if we denote

$$(\eta_{z,T})^{\#}\tilde{V} = \bar{\theta}\mathcal{H}^d \llcorner \eta_{z,T}(F) \otimes \delta_{T_x \eta_{z,T}(F)},$$

by (2.4) we compute

$$\|(\eta_{z,T})^{\#}\tilde{V}\|(T) = \int_F \tilde{\theta} J_F \eta_{z,T}\, d\mathcal{H}^d. \tag{5.6}$$

Moreover, for every $x \in T \setminus \overline{B_{z,\varepsilon_z}}$, we have

$$\begin{aligned} J_F\eta_{z,T}(x,\pi) &\leqslant C|D\eta_{z,T}|^d \leqslant C\left(\lim_{y\to x} \frac{|\eta_{z,T}(x)-\eta_{z,T}(y)|}{|x-y|}\right)^d \\ &\leqslant C\left(\lim_{y\to x}\frac{|x-y|\mathrm{diam}(T)}{|x-y|\cdot|x-z|}\right)^d \leqslant C\frac{(\mathrm{diam}(T))^d}{|x-z|^d}, \end{aligned} \tag{5.7}$$

where C depends just on m and d. Plugging (5.7) in (5.6), we infer that

$$\|(\eta_{z,T})^{\#}\tilde{V}\|(T) \leqslant C(\mathrm{diam}(T))^d \left\|\frac{1}{|\cdot - z|^d}\tilde{V}\right\|(T).$$

Integrating this estimate over $\left(\frac{1}{2}T\right)\setminus F$ and applying Fubini's theorem, we get

$$\int_{z\in\left(\frac{1}{2}T\right)\setminus F} \|(\eta_{z,T})^{\#}\tilde{V}\|(T)\, d\mathcal{H}^m(z) \leqslant C(\mathrm{diam}(T))^d \int_T \left(\int_T \frac{1}{|x-z|^d}\, d\mathcal{H}^m(z)\right) d\|\tilde{V}\|(x).$$

Since the integral in z on the right hand side is finite because $m > d$ and its value is less or equal than $C(\mathrm{diam}(T))^{m-d}$, we conclude the estimate (5.5) as we wanted to prove. □

Lemma 5.6 allows us to prove Lemma 5.5 as for [32, Lemma 3.10]. Our proof is now concluded. □

5.4 AN INTEGRALITY THEOREM

In this section, we prove an integrality theorem of independent interest, which is going to be applied in the proof of Theorem 5.2.

Theorem 5.7. *Let* $F \in C^1(G(\mathbb{R}^n), \mathbb{R}_{>0})$ *be a positive integrand satisfying the* atomic condition *as in Definition 4.1 at every* $x \in \mathbb{R}^n$. *Let* $U \subseteq \mathbb{R}^n$ *be an open set and* $(V_j)_{j\in\mathbb{N}} \subseteq \mathbf{I}_d(\mathbb{R}^n)$ *be a sequence of integral varifolds converging to a varifold* V. *Assume that* V *enjoys the density lower bound*

$$\Theta^d_*(x,V) > 0 \quad \textit{for } \|V\|\text{-}a.e.\ x \in U \tag{5.8}$$

and that the sequence $(V_j)_{j\in\mathbb{N}}$ *satisfies*

$$\sup_{j\in\mathbb{N}} |\delta_F V_j|(W) < \infty, \qquad \forall\, W \subset\subset U; \tag{5.9}$$

then $V \llcorner G(U) \in \mathbf{I}_d(\mathbb{R}^n)$.

Proof. By the assumption (5.9) and by the lower semicontinuity of the total variation of the anisotropic first variation with respect to varifolds convergence, we get that $(\delta_F V) \llcorner U$ is a Radon measure. Moreover, V enjoys the density lower bound (5.8). Since F satisfies the atomic condition as in Definition 4.1 at every $x \in \mathbb{R}^n$, we are in the hypotheses to apply Theorem 4.2 and to conclude that V is a d-rectifiable varifold.

We now prove that the limiting varifold V is integral.

Since $V \llcorner G(U) \in \mathbf{R}_d(\mathbb{R}^n)$, it can be represented as

$$V \llcorner G(U) := \Theta(\cdot, V)\mathcal{H}^d \llcorner K \otimes \delta_{T_x K},$$

where K is a d-rectifiable set, $\Theta(\cdot, V) \in L^1(\mathbb{R}^n; \mathcal{H}^d)$ and $T_x K$ denotes the tangent space of K at x.

By assumption (5.9), we know that there exists $\nu \in \mathcal{M}_+(U)$ such that $|\delta_F V_j|$ converges weakly in the sense of measures to ν in U. By Besicovitch differentiation theorem (see [12, Theorem 2.22]) we get that for $\|V\|$- a.e. point x in U

$$\limsup_{r \to 0} \frac{\nu(B_{x,r})}{\|V\|(B_{x,r})} = C_{\bar{x}} < +\infty. \tag{5.10}$$

We fix a point $\bar{x} \in U$ such that $\Theta(\bar{x}, V)$ and $T_{\bar{x}} K$ exist, $\Theta(\bar{x}, V) \in (0, +\infty)$ (this is true at $\|V\|$-a.e. point in U by the rectifiability of $V \llcorner G(U)$) and such that (5.10) holds. Assume w.l.o.g. that $T_{\bar{x}} K = \mathbb{R}^d \times \{0\}^{n-d}$; we denote $S := T_{\bar{x}} K$ and with $\Pi_S : \mathbb{R}^n \to S$ and $\Pi_{S^\perp} : \mathbb{R}^n \to S^\perp$ the orthogonal projections respectively onto S and $S^\perp$.

There exists a sequence of radii $(r_k)_{k \in \mathbb{N}} \downarrow 0$ such that $\nu(\partial B_{\bar{x}, r_k}) = 0$ and consequently there exists $j_k := j(r_k)$ big enough so that

$$|\delta_F V_{j_k}|(B_{\bar{x}, r_k}) = (1 + o_{r_k}(1))\nu(B_{\bar{x}, r_k}). \tag{5.11}$$

Combining (5.10) and (5.11), we obtain

$$\limsup_{k \to \infty} \frac{|\delta_F V_{j_k}|(B_{\bar{x}, r_k})}{\|V\|(B_{\bar{x}, r_k})} = \limsup_{k \to \infty} \frac{\nu(B_{\bar{x}, r_k})}{\|V\|(B_{\bar{x}, r_k})} = C_{\bar{x}} < +\infty,$$

and for k big enough we conclude

$$|\delta_F V_{j_k}|(B_{\bar{x}, r_k}) \leqslant 2C_{\bar{x}} \|V\|(B_{\bar{x}, r_k}). \tag{5.12}$$

For every $k \in \mathbb{N}$, we consider the rescaling transformation $\eta^{\bar{x}, r_k} : \mathbb{R}^n \to \mathbb{R}^n$, $\eta^{\bar{x}, r_k}(y) = \frac{y - \bar{x}}{r_k}$. We define

$$V^k := \left((\eta^{\bar{x}, r_k})^\# V\right) \qquad \text{and} \qquad V_j^k := \left((\eta^{\bar{x}, r_k})^\# V_j\right).$$

Since $V_j \rightharpoonup V$, for every $k \in \mathbb{N}$

$$V_j^k \rightharpoonup V^k \qquad \text{as} \qquad j \to \infty.$$

But, since $\Theta(\bar{x}, V) < +\infty$, we get that V^k are locally bounded uniformly with respect to k and we infer

$$V^k \rightharpoonup \Theta(\bar{x}, V)\mathcal{H}^d \llcorner S \otimes \delta_S, \qquad \text{as } k \to \infty.$$

Via a diagonal argument, up to extract another (not relabeled) subsequence j_k, if we define $\tilde{V}^k := V^k_{j_k}$, we get

$$\|V_{j_k}\|(B_{\bar{x},r_k}) \leqslant 2\|V\|(B_{\bar{x},r_k}) \leqslant 4\Theta(\bar{x},V)r_k^d, \tag{5.13}$$

$$\|\tilde{V}^k\|(B^d_1 \times B^{n-d}_1 \setminus B^d_1 \times B^{n-d}_{\frac{1}{2}}) = o_{r_k}(1), \tag{5.14}$$

$$\|\tilde{V}^k\|(B^d_1 \times B^{n-d}_1) \leqslant 2\Theta(\bar{x},V), \tag{5.15}$$

and the convergence

$$\tilde{V}^k \rightharpoonup \Theta(\bar{x},V)\mathcal{H}^d \llcorner S \otimes \delta_S, \qquad \text{as } k \to \infty. \tag{5.16}$$

We consider $\chi_1 \in C^\infty_c(B^d_{\sqrt{2}/2})$ with $\chi_1 \equiv 1$ in $B^d_{\frac{1}{2}}$, $\chi_2 \in C^\infty_c(B^{n-d}_{\sqrt{2}/2})$ with $\chi_2 \equiv 1$ in $B^{n-d}_{1/2}$ and we define $\chi \in C^\infty_c(B_1)$ as $\chi(x) := \chi_1(\Pi_S(x))\chi_2(\Pi_{S^\perp}(x))$.

We denote $F_k(z,T) = F(\bar{x}+r_k z,T)$ and define the family of varifolds $W_k := \chi\tilde{V}^k$ equicompactly supported in B_1. We claim that

$$\sup_{k\in\mathbb{N}} |\delta_{F_k} W_k|(B_1) < +\infty. \tag{5.17}$$

Indeed, we define $\chi_k := \chi \circ \eta^{\bar{x},r_k} \in C^\infty_c(B_{\bar{x},r_k})$ and for every $\varphi \in C^\infty_c(B_1,\mathbb{R}^n)$ we consider the map $\varphi_k := \varphi \circ \eta^{\bar{x},r_k} \in C^\infty_c(B_{\bar{x},r_k},\mathbb{R}^n)$, so that

$$\|\chi_k\|_\infty \leqslant \|\chi\|_\infty \leqslant 1, \qquad r_k\|\nabla\chi_k\|_\infty \leqslant \|\nabla\chi\|_\infty \qquad \text{and} \qquad \|\varphi_k\|_\infty \leqslant \|\varphi\|_\infty. \tag{5.18}$$

Thanks to (2.17) and (2.18), we compute

$$\begin{aligned}
|\delta_{F_k} W_k(\varphi)| &= |\delta_{F_k}(\chi\tilde{V}^k)(\varphi)| \\
&= \Big| \int \big\langle d_z F_k(z,T), \chi(z)\varphi(z)\big\rangle d\tilde{V}^k(z,T) \\
&\quad + \int B_{F_k}(z,T) : D\varphi(z)\,\chi(z) d\tilde{V}^k(z,T) \Big| \\
&= \Big| \int \big\langle d_z F_k(\eta^{\bar{x},r_k}(y),T), \chi(\eta^{\bar{x},r_k}(y))\varphi(\eta^{\bar{x},r_k}(y))\big\rangle J\eta^{\bar{x},r_k}(y,T) dV_{j_k}(y,T) \\
&\quad + \int B_{F_k}(\eta^{\bar{x},r_k}(y),T) : D\varphi(\eta^{\bar{x},r_k}(y))\,\chi(\eta^{\bar{x},r_k}(y)) J\eta^{\bar{x},r_k}(y,T) dV_{j_k}(y,T) \Big| \\
&\overset{(2.18)}{=} \Big| r_k^{1-d} \int \big\langle d_y F(y,T), \chi_k(y)\varphi_k(y)\big\rangle dV_{j_k}(y,T) \\
&\quad + r_k^{1-d} \int B_F(y,T) : D\varphi_k(y)\,\chi_k(y) dV_{j_k}(y,T) \Big| \\
&= \Big| r_k^{1-d} \int \big\langle d_y F(y,T), \chi_k(y)\varphi_k(y)\big\rangle dV_{j_k}(y,T) \\
&\quad + r_k^{1-d} \int B_F(y,T) : D(\varphi_k\chi_k)(y)\, dV_{j_k}(y,T) \\
&\quad - r_k^{1-d} \int B_F(y,T) : \nabla\chi_k(y) \otimes \varphi_k(y)\, dV_{j_k}(y,T) \Big| \\
&\overset{(2.17)}{\leqslant} r_k^{1-d} |\delta_F V_{j_k}(\chi_k\varphi_k)| + r_k^{1-d} \|F\|_{C^1(B_{\bar{x},r_k})} \|V_{j_k}\|(B_{\bar{x},r_k}) \|\nabla\chi_k\|_\infty \|\varphi_k\|_\infty,
\end{aligned}$$

which, combined with (5.12), (5.13) and (5.18), gives

$$
\begin{aligned}
|\delta_{F_k} W_k(\varphi)| &\overset{(5.13),(5.18)}{\leqslant} r_k^{1-d} |\delta_F V_{j_k}|(B_{\bar{x},r_k}) \|\varphi\|_\infty + 4\|F\|_{C^1(B_{\bar{x},r_k})} \Theta(\bar{x}, V) \|\nabla\chi\|_\infty \|\varphi\|_\infty \\
&\overset{(5.12)}{\leqslant} 2r_k^{1-d} C_{\bar{x}} \|V\|(B_{\bar{x},r_k}) \|\varphi\|_\infty + 4\|F\|_{C^1(B_{\bar{x},r_k})} \Theta(\bar{x}, V) \|\nabla\chi\|_\infty \|\varphi\|_\infty \\
&\overset{(5.13)}{\leqslant} \left[4r_k C_{\bar{x}} \Theta(\bar{x}, V) + 4\|F\|_{C^1(B_{\bar{x},r_k})} \Theta(\bar{x}, V) \|\nabla\chi\|_\infty\right] \|\varphi\|_\infty.
\end{aligned}
$$

This inequality implies (5.17). Finally, by (5.16),

$$
\begin{aligned}
\lim_{k\to\infty} \int_{G(B_1)} |T - S| dW_k(z, T) &= \lim_{k\to\infty} \int_{G(B_1)} |T - S| \chi(z) d\tilde{V}^k(z, T) \\
&= \int_{G(B_1)} |T - S| \Theta(\bar{x}, V) \chi(z) d\delta_S(T) d\mathcal{H}^d \llcorner S(z) = 0.
\end{aligned}
$$

Hence the sequence $(W_k)_{k\in\mathbb{N}}$ satisfies the assumptions of Lemma 4.9, indeed we observe that $B_{F_k}(z, T) = B_F(\bar{x} + r_k z, T)$, so that assumption (2) in Lemma 4.9 is satisfied. Thus we deduce that there exists $\gamma \in L^1(\mathcal{H}^d \llcorner B_1^d)$ such that, along a (not relabeled) subsequence, for every $0 < t < 1$

$$
\left| (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|W_k\|) - \gamma\mathcal{H}^d \llcorner B_1^d \right| (B_t^d) \longrightarrow 0. \tag{5.19}
$$

Since $\chi_1 \equiv 1$ in $B_{1/2}^d$, thanks to (5.14) we get

$$
\begin{aligned}
&(\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|W_k\|)(B_{1/2}^d) \\
&\qquad = (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|(\chi_2 \circ \Pi_{S^\perp})\tilde{V}^k\|)(B_{1/2}^d) \\
&\qquad = (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|\tilde{V}^k \llcorner (B_1^d \times B_1^{n-d})\|)(B_{1/2}^d) - o_{r_k}(1),
\end{aligned}
$$

which we plug in (5.19) to obtain

$$
\left| (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|) - \gamma\mathcal{H}^d \llcorner B_1^d \right| (B_{1/2}^d) \longrightarrow 0. \tag{5.20}
$$

But, thanks to (5.15)

$$
\begin{aligned}
&\left| (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|) - (\Pi_S)_\#(F(\bar{x}, S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|) \right| (B_{1/2}^d) \\
&\quad = \left| (\Pi_S)_\#(F(\bar{x} + r_k(\cdot), S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\| - F(\bar{x}, S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|) \right| (B_{1/2}^d) \\
&\quad \leqslant \left| (F(\bar{x} + r_k(\cdot), S) - F(\bar{x}, S))\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\| \right| (B_{1/2}^d \times B_1^{n-d}) \\
&\quad \leqslant \sup_{z\in B_{\bar{x},2}} \left| (F(\bar{x} + r_k z, S) - F(\bar{x}, S)) \right| \|\tilde{V}^k\|(B_1^d \times B_1^{n-d}) \\
&\quad \overset{(5.15)}{\leqslant} 2\Theta(\bar{x}, V)\|F\|_{C^1(B_{\bar{x},2})} r_k \longrightarrow 0.
\end{aligned} \tag{5.21}
$$

Plugging (5.21) in (5.20), we conclude by triangular inequality that

$$
\left| (\Pi_S)_\#(F(\bar{x}, S)\|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|) - \gamma\mathcal{H}^d \llcorner B_1^d \right| (B_{1/2}^d) \longrightarrow 0. \tag{5.22}
$$

Since $\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}$ is an integral varifold, then $(\Pi_S)_\# \|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\|$ is a d-rectifiable measure in $\mathbb{R}^d \approx S$ with integer d-density $\theta_k(\cdot) \in L^1(B_1^d, \mathbb{N}, \mathcal{L}^d)$. By (5.22), we deduce that

$$F(\bar{x}, S)\theta_k(\cdot) \longrightarrow \gamma(\cdot) \qquad \text{in } L^1(B_{1/2}^d, \mathcal{L}^d)$$

and consequently, up to subsequences, $(\theta_k(x))_k \subset \mathbb{N}$ converges for $\mathcal{H}^d$-a.e. $x \in B_{1/2}^d$ to $\frac{\gamma(x)}{F(\bar{x},S)} \in \mathbb{N}$. By (5.16), we also know that

$$(\Pi_S)_\# \|\tilde{V}^k \llcorner B_1^d \times B_1^{n-d}\| \rightharpoonup \Theta(\bar{x}, V)\mathcal{L}^d \llcorner B_1^d$$

and by uniqueness of the limit, we infer that $\frac{\gamma(\cdot)}{F(\bar{x},S)} \equiv \Theta(\bar{x}, V)$ in $B_{1/2}^d$. But $\frac{\gamma(\cdot)}{F(\bar{x},S)}$ is integer valued in $B_{1/2}^d$, so we conclude that $\Theta(\bar{x}, V) \in \mathbb{N}$ and that V is an integral varifold. $\square$

Remark 5.8. We recall that the isotropic version of Theorem 5.7 above has been proved in [2, Section 6.4], without the density assumption (5.8), which is a consequence of the monotonicity formula in the isotropic setting. If one were able to preserve in the limit varifold V the lower density bound of the sequence V_j of Theorem 5.7, one would get further applications. For instance, it would positively answer to a question raised by Tonegawa in the setting of anisotropic mean curvature flows, see [81, Section 4.1, p. 116].

5.5 PROOF OF THE MAIN THEOREM

Up to extracting subsequences, we can assume the existence of $V \in \mathbf{V}_d(\mathbb{R}^n)$ such that

$$V_j \overset{*}{\rightharpoonup} V.$$

We remark that condition (a) of Theorem 5.2 is automatically satisfied by the lower semicontinuity of the functional $\mathbf{F}(\cdot)$ with respect to varifolds convergence. This implies straightforwardly also condition (b). For the remaining properties, we divide the argument in several steps.

5.5.1 *Proof of Theorem 5.2: stationarity of the limiting varifold*

In this section we prove property (c).

Assume by contradiction that there exists a smooth vector field ψ compactly supported in $\mathbb{R}^n \setminus H$ such that $\delta_F V(\psi) < 0$. By standard partition of unity argument for the compact set supp(ψ) in the open set $\mathbb{R}^n \setminus H$, using the linearity of $\delta_F V(\cdot)$, we get the existence of a ball $B_{x,r} \subset\subset \mathbb{R}^n \setminus H$ and of a vector field (not relabeled) $\psi \in C_c^1(B_{x,r}, \mathbb{R}^n)$ such that

$$\delta_F V(\psi) \leqslant -2C < 0.$$

There exists a map $\varphi : t \in \mathbb{R} \mapsto \varphi_t \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ solving the following ODE:

$$\begin{cases} \frac{\partial \varphi_t(x)}{\partial t} = \psi(\varphi_t(x)) & \forall x \in \mathbb{R}^n, \\ \varphi_0(x) = x & \forall x \in \mathbb{R}^n. \end{cases}$$

Notice that one can choose an $\varepsilon > 0$ small enough to have that $\varphi_t \in D(x, r)$ for every $t \in [0, \varepsilon]$. We set

$$V_t := (\varphi_t)^{\#}V, \qquad \text{and } V_t^j = (\varphi_t)^{\#}V_j.$$

By continuity of the functional $\delta_F Z(\psi)$ with respect to Z, up to take a smaller $\varepsilon > 0$, we get that

$$\delta_F V_t(\psi) \leqslant -C < 0, \quad \forall t \in [0, \varepsilon].$$

Integrating the last inequality, we conclude that

$$\mathbf{F}(V_\varepsilon) \leqslant \mathbf{F}(V) - C\varepsilon. \tag{5.23}$$

We fix an $\alpha \in [1, 2]$ such that

$$\|V_\varepsilon\|(\partial B_{x,\alpha r}) = 0, \quad \text{and consequently} \quad \mathbf{F}(V_\varepsilon, \partial B_{x,\alpha r}) = 0. \tag{5.24}$$

We notice that equation (5.24) can be read as

$$\mathbf{F}(V, \partial B_{x,\alpha r}) = 0, \quad \text{because} \quad \varphi_\varepsilon = \mathrm{Id} \text{ in } B^c_{x,r}. \tag{5.25}$$

Since $V_\varepsilon^j \rightharpoonup V_\varepsilon$ and $V^j \rightharpoonup V$, thanks to the equalities (5.24) and (5.25), one can infer

$$\mathbf{F}(V_\varepsilon, \overline{B_{x,\alpha r}}) = \lim_j \mathbf{F}(V_\varepsilon^j, \overline{B_{x,\alpha r}}), \quad \text{and} \quad \mathbf{F}(V, \overline{B_{x,\alpha r}}) = \lim_j \mathbf{F}(V^j, \overline{B_{x,\alpha r}}). \tag{5.26}$$

Moreover, from (5.23) and the fact that $\varphi_\varepsilon = \mathrm{Id}$ in $B^c_{x,r}$, we also get

$$\mathbf{F}(V_\varepsilon, \overline{B_{x,\alpha r}}) \leqslant \mathbf{F}(V, \overline{B_{x,\alpha r}}) - C\varepsilon. \tag{5.27}$$

Using (5.26) and (5.27), we infer

$$\begin{aligned}
\liminf_j \mathbf{F}(V_\varepsilon^j) &= \liminf_j \left(\mathbf{F}(V_\varepsilon^j, \overline{B_{x,\alpha r}}) + \mathbf{F}\big(V_\varepsilon^j, \big(\overline{B_{x,\alpha r}}\big)^c\big)\right) \\
&\leqslant \limsup_j \mathbf{F}(V_\varepsilon^j, \overline{B_{x,\alpha r}}) + \liminf_j \mathbf{F}\big(V_\varepsilon^j, \big(\overline{B_{x,\alpha r}}\big)^c\big) \\
&\overset{(5.26)}{=} \mathbf{F}(V_\varepsilon, \overline{B_{x,\alpha r}}) + \liminf_j \mathbf{F}\big(V_\varepsilon^j, \big(\overline{B_{x,\alpha r}}\big)^c\big) \\
&\overset{(5.27)}{\leqslant} \mathbf{F}(V, \overline{B_{x,\alpha r}}) - C\varepsilon + \liminf_j \mathbf{F}\big(V_\varepsilon^j, \big(\overline{B_{x,\alpha r}}\big)^c\big) \\
&\overset{(5.26)}{=} \lim_j \mathbf{F}(V^j, \overline{B_{x,\alpha r}}) - C\varepsilon + \liminf_j \mathbf{F}\big(V_\varepsilon^j, \big(\overline{B_{x,\alpha r}}\big)^c\big) \\
&\leqslant \liminf_j \mathbf{F}(V^j) - C\varepsilon.
\end{aligned} \tag{5.28}$$

Since $\varphi_\varepsilon \in D(x, r)$, by definition of deformation class, see Definition 5.1, there exists a new sequence $(\tilde{V}_j)_{j\in\mathbb{N}} \subseteq \mathscr{P}(\mathbf{F}, H)$, such that

$$\mathbf{F}(\tilde{V}_j) \leqslant \mathbf{F}(V_\varepsilon^j) + \frac{C\varepsilon}{4},$$

and passing to the lower limit on j, we get

$$\liminf_j \mathbf{F}(\tilde{V}_j) \leqslant \liminf_j \mathbf{F}(V^j) - \frac{3C\varepsilon}{4},$$

which contradicts the minimality of the sequence $(V_j)_{j\in\mathbb{N}}$.

5.5.2 *Proof of Theorem 5.2: lower density estimates*

In this section we show that if there exists $\delta > 0$ such that

$$\Theta^d(x, V_j) \geqslant \delta, \qquad \text{for } \|V_j\|\text{-a.e. } x \in \mathbb{R}^n \setminus H, \quad \forall j \in \mathbb{N},$$

then there exist $\theta_0 = \theta_0(n, d, \delta, \lambda, \Lambda) > 0$ such that

$$\|V\|(B_{x,r}) \geqslant \theta_0\, \omega_d r^d\,, \qquad x \in \operatorname{spt}\|V\| \text{ and } r < d_x := \operatorname{dist}(x, H). \tag{5.29}$$

To this end, by (2.10), it is sufficient to prove the existence of $\beta = \beta(n, d, \delta, \lambda, \Lambda) > 0$ such that

$$\mathbf{F}(V, Q_{x,l}) \geqslant \beta\, l^d\,, \qquad x \in \operatorname{spt}\|V\| \text{ and } l < 2d_x/\sqrt{n}\,.$$

We adapt the argument of [44, Theorem 1.3, Step one] to the anisotropic energy and taking into account the varifolds multiplicity. Let us assume by contradiction that there exist $x \in \operatorname{spt}\|V\|$ and $l < 2d_x/\sqrt{n}$ such that

$$\frac{\mathbf{F}(V, Q_{x,l})^{\frac{1}{d}}}{l} < \beta.$$

We claim that this assumption, for β chosen sufficiently small depending only on n, d, δ, λ and Λ, implies that for some $l_\infty \in (0, l)$

$$\mathbf{F}(V, Q_{x,l_\infty}) = 0, \tag{5.30}$$

which is a contradiction with $x \in \operatorname{spt}\|V\|$. In order to prove (5.30), we assume that $\mathbf{F}(V, \partial Q_{x,l}) = 0$, which is true for a.e. $l \in \mathbb{R}_{>0}$.

To prove (5.30), we construct a sequence of nested cubes $Q_i := Q_{x,l_i}$ such that, if β is sufficiently small, the following holds:

(i) $Q_0 = Q_{x,l}$;

(ii) $\mathbf{F}(V, \partial Q_i) = 0$;

(iii) setting $m_i := \mathbf{F}(V, Q_i)$ then:
$$\frac{m_i^{\frac{1}{d}}}{l_i} < \beta;$$

(iv) $m_{i+1} \leqslant (1 - \frac{1}{k_2})m_i$, where $k_2 := \frac{\Lambda k_1}{\lambda}$ and k_1 is the constant in Theorem 5.4;

(v) $(1 - 4\varepsilon_i)l_i \geqslant l_{i+1} \geqslant (1 - 6\varepsilon_i)l_i$, where
$$\varepsilon_i := \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} \tag{5.31}$$
and $k = \max\{6, 6/(1 - (\frac{k_2-1}{k_2})^{\frac{1}{d}})\}$ is a universal constant.

(vi) $\lim_i m_i = 0$ and $\lim_i l_i > 0$.

Following [32], we are going to construct the sequence of cubes by induction: the cube Q_0 satisfies by construction hypotheses (i)-(iii). Suppose that cubes until step i are already defined.

Setting $m_i^j := \mathbf{F}(V_j, Q_i)$, we cover Q_i with the family $\Lambda_{\varepsilon_i l_i}(Q_i)$ of closed cubes with edge length $\varepsilon_i l_i$ as described in Section 5.3 and we set C_1^i and C_2^i for the corresponding sets defined in (5.3). We define Q_{i+1} to be the internal cube given by the construction, and we note that C_2^i and Q_{i+1} are non-empty if, for instance,

$$\varepsilon_i = \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} < \frac{1}{k} \leqslant \frac{1}{6},$$

which is guaranteed by our choice of k. Observe moreover that $C_1^i \cup C_2^i$ is a strip of width at most $2\varepsilon_i l_i$ around ∂Q_i, hence the side l_{i+1} of Q_{i+1} satisfies $(1-4\varepsilon_i)l_i \leqslant l_{i+1} < (1-2\varepsilon_i)l_i$.

We denote with K_j the concentration set of V_j (that is $V_j := \theta_j \mathcal{H}^d \llcorner K_j \otimes \delta_{T_x K_j}$), where $\theta_j \in L^1(K_j; [\delta, +\infty); \mathcal{H}^d))$ and apply Theorem 5.4 to Q_i, $V_j^\delta := \frac{1}{\delta} V_j$ and $\varepsilon = \varepsilon_i l_i$, obtaining the map $\Phi_{i,j} = \Phi_{\varepsilon_i l_i, V_j^\delta}$. Notice that we are in the hypotheses to apply Theorem 5.4, since the rescaled varifolds V_j^δ have density bigger or equal than one in Q_i, K_j is a relatively closed subset of $\mathbb{R}^n \setminus H$ and $Q_0 \cap H = \emptyset$.

We claim that, for every j sufficiently large,

$$m_i^j \leqslant k_2(m_i^j - m_{i+1}^j) + o_j(1). \tag{5.32}$$

Indeed, since $(V_j)_{j \in \mathbb{N}}$ is a minimizing sequence in the class $\mathscr{P}(\mathbf{F}, H)$, then (V_j^δ) is a minimizing sequence in the class

$$\mathscr{P}_\delta(\mathbf{F}, H) := \left\{ \frac{1}{\delta} W \,:\, W \in \mathscr{P}(\mathbf{F}, H) \right\}.$$

Since we are just rescaling the density of the varifolds and $\mathscr{P}(\mathbf{F}, H)$ is a deformation class, also $\mathscr{P}_\delta(\mathbf{F}, H)$ is a deformation class and by (2.10), we have that

$$\begin{aligned}
\frac{1}{\delta} m_i^j = \mathbf{F}(V_j^\delta, Q_i) \leqslant \frac{1}{\delta} m_i + o_j(1) &\leqslant \Lambda \|(\Phi_{i,j})^\# V_j^\delta\|(Q_i) + o_j(1) \\
&= \Lambda \|(\Phi_{i,j})^\# V_j^\delta\|(Q_{i+1}) + \Lambda \|(\Phi_{i,j})^\# V_j^\delta\| \left(C_1^i \cup C_2^i\right) + o_j(1) \\
&\leqslant \Lambda \|(\Phi_{i,j})^\# V_j^\delta\| \left(C_1^i \cup C_2^i\right) + o_j(1) \leqslant \frac{k_2}{\delta}(m_i^j - m_{i+1}^j) + o_j(1).
\end{aligned}$$

The last inequality holds because $\|(\Phi_{i,j})^\# V_j^\delta\|(Q_{i+1}) = 0$ for j large enough: otherwise, by property (6) of Theorem 5.4, there would exist $T \in \Lambda^*_{\varepsilon_i l_i, d}(Q_{i+1})$ such that $\|(\Phi_{i,j})^\# V_j^\delta\|(T) \geqslant \mathcal{H}^d(T)$. Together with property (ii) and by (2.10), this would imply

$$l_i^d \varepsilon_i^d = \mathcal{H}^d(T) \leqslant \|(\Phi_{i,j})^\# V_j^\delta\|(T) \leqslant \frac{k_1}{\delta}\|V_j\|(Q_i) \leqslant \frac{k_1}{\delta\lambda} m_i^j \to \frac{k_1}{\delta\lambda} m_i$$

and therefore, substituting (5.31),

$$\frac{m_i}{k^d \beta^d} \leqslant \frac{k_1}{\delta\lambda} m_i,$$

which is false if β is sufficiently small ($m_i > 0$ because $x \in \operatorname{spt}(\|V\|)$). Passing to the limit in j in (5.32) we obtain (iv):

$$m_{i+1} \leqslant \frac{k_2 - 1}{k_2} m_i. \tag{5.33}$$

Since $l_{i+1} \geqslant (1-4\varepsilon_i)l_i$, we can slightly shrink the cube Q_{i+1} to a concentric cube Q'_{i+1} with $l'_{i+1} \geqslant (1-6\varepsilon_i)l_i > 0$, $\mathbf{F}(V, \partial Q'_{i+1}) = 0$ and for which (iv) still holds, just getting a lower value for m_{i+1}. With a slight abuse of notation, we rename this last cube Q'_{i+1} as Q_{i+1}.

We now show (iii). Using (5.33) and condition (iii) for Q_i, we obtain

$$\frac{m_{i+1}^{\frac{1}{d}}}{l_{i+1}} \leqslant \left(\frac{k_2-1}{k_2}\right)^{\frac{1}{d}} \frac{m_i^{\frac{1}{d}}}{(1-6\varepsilon_i)l_i} < \left(\frac{k_2-1}{k_2}\right)^{\frac{1}{d}} \frac{\beta}{1-6\varepsilon_i}.$$

The last quantity will be less than β if

$$\left(\frac{k_2-1}{k_2}\right)^{\frac{1}{d}} \leqslant 1 - 6\varepsilon_i = 1 - \frac{6}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i}. \tag{5.34}$$

In turn, inequality (5.34) is true because (iii) holds for Q_i, provided we choose $k \geqslant 6/\big(1 - (1-1/k_2)^{\frac{1}{d}}\big)$. Furthermore, estimating $\varepsilon_0 < 1/k$ by (iii) and (v), we also have $\varepsilon_{i+1} \leqslant \varepsilon_i$.

We are left to prove (vi): $\lim_i m_i = 0$ follows directly from (iv); regarding the non degeneracy of the cubes, note that

$$\begin{aligned}
\frac{l_\infty}{l_0} := \liminf_i \frac{l_i}{l_0} &\geqslant \prod_{i=0}^{\infty}(1-6\varepsilon_i) = \prod_{i=0}^{\infty}\left(1 - \frac{6}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i}\right) \\
&\geqslant \prod_{i=0}^{\infty}\left(1 - \frac{6m_0^{\frac{1}{d}}}{k\beta l_0 \prod_{h=0}^{i-1}(1-6\varepsilon_h)}\left(\frac{k_2-1}{k_2}\right)^{\frac{i}{d}}\right) \\
&\geqslant \prod_{i=0}^{\infty}\left(1 - \frac{6}{k(1-6\varepsilon_0)^i}\left(\frac{k_2-1}{k_2}\right)^{\frac{i}{d}}\right),
\end{aligned}$$

where we used $\varepsilon_h \leqslant \varepsilon_0$ in the last inequality. Since $\varepsilon_0 < 1/k$, the last product is strictly positive, provided

$$k > \frac{6}{1 - \left(\frac{k_2-1}{k_2}\right)^{\frac{1}{d}}},$$

which is guaranteed by our choice of k. We conclude that $l_\infty > 0$, which ensures claim (5.30).

5.5.3 *Proof of Theorem 5.2: rectifiability of the limiting varifold*

In this section, we prove condition (d). Indeed, with the assumption on the uniform density lower bound of the minimizing sequence, by the previous step we know that V enjoys the denisty lower bound (5.29). Moreover, by condition (c), it is F-stationarity in $\mathbb{R}^n \setminus H$. Since F is as in Definition 4.1, we are in the hypotheses to apply Theorem 4.2 and to conclude that $V \llcorner G(\mathbb{R}^n \setminus H)$ is a d-rectifiable varifold.

Moreover, by the previous step, for every $x \in \operatorname{spt}\|V\| \setminus H$ we have (5.29). It follows that

$$\operatorname{spt}\|V\| \setminus H \subseteq \operatorname{conc}(V) \subseteq \operatorname{spt}\|V\|.$$

We conclude that $\operatorname{conc}(V) \setminus H = \operatorname{spt}\|V\| \setminus H$ and consequently that the concentration set is relatively closed in $\mathbb{R}^n \setminus H$.

5.5.4 *Proof of Theorem 5.2: integrality of the limiting varifold*

In this section we prove that, under the further assumption that the minimizing sequence is made of integral varifolds satisfying

$$\sup_j |\delta_F V_j|(W) < \infty, \qquad \forall\, W \subset\subset (\mathbb{R}^n \setminus H),$$

then $V \llcorner G(\mathbb{R}^n \setminus H)$ is integral. Indeed, we already know that V enjoys the density lower bound (5.29).

We are in the hypotheses to apply Theorem 5.7 with $U := \mathbb{R}^n \setminus H$ and conclude that condition (e) holds.

Part II

STABILITY IN BRANCHED TRANSPORT

6

NOTATION OF PART II

We introduce in this chapter the Notation of Part II. Since Part I and Part II address different problems, the notation for Part II will be self-contained. The main difference with the notation previously adopted will be the ambient space $\mathbb{R}^d$ instead of $\mathbb{R}^n$.

6.0.1 *Measures and rectifiable sets*

Given a locally compact separable metric space Y, we denote by $\mathcal{M}(Y)$ the set of Radon measures in Y, namely the set of (possibly signed) measures on the σ-algebra of Borel sets of Y that are locally finite and inner regular. We denote also by $\mathcal{M}_+(Y)$ the subset of positive measures and by $\mathcal{P}(Y)$ the subset of probability measures, i.e. those poitive measures μ such that $\mu(Y) = 1$.

We denote by $|\mu|$ the total variation measure associated to μ. The negative and positive parts of μ are the positive measures defined respectively by

$$\frac{|\mu| - \mu}{2} \quad \text{and} \quad \frac{|\mu| + \mu}{2}.$$

For $\mu, \nu \in \mathcal{M}(Y)$, we write $\mu \leqslant \nu$ in case $\mu(A) \leqslant \nu(A)$ for every Borel set A. Given a measure μ we denote by

$$\operatorname{spt}(\mu) := \bigcap \{C \subset Y : C \text{ is closed and } |\mu|(Y \setminus C) = 0\}$$

its *support*. We say that μ is *supported* on a Borel set E if $|\mu|(Y \setminus E) = 0$. For a Borel set E, $\mu \llcorner E$ is the restriction of μ to E, i.e. the measure defined by

$$[\mu \llcorner E](A) = \mu(E \cap A) \qquad \text{for every Borel set } A.$$

We say that two measures μ and ν are mutually singular if there exists a Borel set E such that $\mu = \mu \llcorner E$ and $\nu = \nu \llcorner E^c$.

For a measure $\mu \in \mathcal{M}(Y)$ and a Borel map $\eta : Y \to Z$ between two metric spaces we let $\eta_\sharp \mu \in \mathcal{M}(Z)$ be the push-forward measure, namely

$$\eta_\sharp \mu(A) := \mu(\eta^{-1}(A)), \qquad \text{for every Borel set } A \subset Z.$$

We use $\mathcal{L}^d$ and $\mathcal{H}^k$ to denote respectively the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and the k-dimensional Hausdorff measure, see [79].

A set $K \subset \mathbb{R}^d$ is said to be *countably* k*-rectifiable* (or simply k*-rectifiable*) if it can be covered, up to an $\mathcal{H}^k$-negligible set, by countably many k-dimensional submanifolds of class C^1. At $\mathcal{H}^k$-a.e. point x of a k-rectifiable set E, a notion of (unoriented) tangent k-plane is well-defined: we denote it by $\operatorname{Tan}(E, x)$.

6.0.2 *Currents*

We recall here the basic terminology related to k-dimensional currents. We refer to the introductory presentation given in the standard textbooks [79], [61] for further details. The most complete reference remains the treatise [50].

Let us denote by $\Lambda^k(\mathbb{R}^d)$ the vector space of k-covectors in $\mathbb{R}^d$. A k-*dimensional current* T in $\mathbb{R}^d$ is a continuous linear functional on the space $\mathscr{D}^k(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d; \Lambda^k(\mathbb{R}^d))$ of smooth and compactly supported differential k-forms on $\mathbb{R}^d$. Hence the space $\mathscr{D}_k(\mathbb{R}^d)$ of k-dimensional currents in $\mathbb{R}^d$ is endowed with the natural notion of weak* convergence. For a sequence $(T_n)_{n\in\mathbb{N}}$ of k-dimensional currents converging to a current T, we use the standard notation $T_n \rightharpoonup T$. With ∂T we denote the *boundary* of T, that is the $(k-1)$-dimensional current defined via

$$\langle \partial T, \omega \rangle := \langle T, d\omega \rangle \quad \text{for every } \omega \in \mathscr{D}^{k-1}(\mathbb{R}^d).$$

The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T, \omega\rangle$ over all k-forms ω such that $|\omega| \leqslant 1$ everywhere (here with $|\omega|$ we denoted the comass norm of ω).

By the Radon–Nikodým Theorem, a k-dimensional current T with finite mass can be identified with the vector-valued measure $T = \vec{T}\|T\|$ where $\|T\|$ is a finite positive measure and $\vec{T}$ is a unit k-vector field. Hence, the action of T on a k-form ω is given by

$$\langle T, \omega \rangle = \int_{\mathbb{R}^d} \langle \omega(x), \vec{T}(x) \rangle d\|T\|(x)\,.$$

In particular a 0-current with finite mass can be identified with a real-valued Radon measure and the mass of the current coincides with the total variation (or mass) of the corresponding measure. We will tacitly use such identification several times through the next chapters.

For a current T with finite mass, we will denote by spt(T) its *support*, defined as the support of the associated measure $\|T\|$. A current T is called *normal* if both T and ∂T have finite mass; we denote the set of normal k-currents in $\mathbb{R}^d$ by $\mathbf{N}_k(\mathbb{R}^d)$. Given a normal 1-current T, we denote by $\partial_+ T$ and $\partial_- T$ respectively the positive and the negative part of the (finite) measure ∂T. It is well-known that, if T is a normal current with compact support and $\partial T = \mu^+ - \mu^-$, (where not necessarily μ^+ and μ^- are mutually singular) it holds

$$\mathbb{M}(\mu^+) = \mathbb{M}(\mu^-). \tag{6.1}$$

In particular:

$$\mathbb{M}(\partial T) = 2\mathbb{M}(\partial_- T) = 2\mathbb{M}(\partial_+ T). \tag{6.2}$$

Given a Borel set $A \subseteq \mathbb{R}^d$, we define the restriction of a current T with finite mass to A as

$$\langle T \llcorner A, \omega \rangle := \int_A \langle \omega(x), \vec{T}(x) \rangle d\|T\|(x)\,.$$

Notice that the restriction of a normal current to a Borel set is a current with finite mass, but it might fail to be normal.

A k-dimensional *rectifiable current* is a current $T = T[E, \tau, \theta]$, which can be represented as

$$\langle T, \omega \rangle = \int_E \langle \omega(x), \tau(x) \rangle \, \theta(x) d\mathcal{H}^k(x) , \tag{6.3}$$

where E is a k-rectifiable set, $\tau(x)$ is a unit simple k-vector field defined on E which at $\mathcal{H}^k$-a.e $x \in E$ spans the approximate tangent space $\mathrm{Tan}(E, x)$ and $\theta : E \to \mathbb{R}$ is a function such that $\int_E |\theta| d\mathcal{H}^k < \infty$. We denote by $\mathbf{R}_k(\mathbb{R}^d)$ the space of k-dimensional rectifiable currents in $\mathbb{R}^d$. Modulo changing sign to the orientation τ, we can always assume that θ takes non-negative values. We will tacitly make such assumption through the next chapters, unless we specify elsewhere. It is easy to see that for $T \in \mathbf{R}_k(\mathbb{R}^d)$ it holds

$$\mathbb{M}(T) = \int_E \theta(x) d\mathcal{H}^k(x); \tag{6.4}$$

in particular, any rectifiable current has finite mass. We remark that the rectifiable currents we are considering all have finite mass and compact support.

A k-dimensional *polyhedral chain* $P \in \mathbf{P}_k(\mathbb{R}^d)$ is a rectifiable current which can be written as a linear combination

$$P = \sum_{i=1}^{N} \theta_i [\sigma_i], \tag{6.5}$$

where $\theta_i \in (0, \infty)$, the σ_i's are non-overlapping, oriented, k-dimensional, convex polytopes (finite unions of k-simplexes) in $\mathbb{R}^n$ and $[\sigma_i] = [\sigma_i, \tau_i, 1]$, τ_i being a constant k-vector orienting σ_i. If $P \in \mathbf{P}_k(\mathbb{R}^d)$, then its *flat norm* is defined by

$$\mathbb{F}(P) := \inf\{\mathbb{M}(S) + \mathbb{M}(P - \partial S) : \ S \in \mathbf{P}_{k+1}(\mathbb{R}^n)\}.$$

Flat k-chains can be therefore defined to be the $\mathbb{F}$-*completion* of $\mathbf{P}_k(\mathbb{R}^d)$ in $\mathscr{D}_k(\mathbb{R}^d)$.

We remark that for the spaces of currents considered above the following chain of inclusions holds:

$$\mathbf{P}_k(\mathbb{R}^d) \subset \mathbf{R}_k(\mathbb{R}^d) \subset \mathbf{F}_k(\mathbb{R}^d) \cap \{T \in \mathscr{D}_k(\mathbb{R}^d) : \ \mathbb{M}(T) < \infty\}. \tag{6.6}$$

The flat norm $\mathbb{F}$ extends to a functional (still denoted $\mathbb{F}$) on $\mathscr{D}_k(\mathbb{R}^d)$, which coincides on $\mathbf{F}_k(\mathbb{R}^d)$ with the completion of the flat norm on $\mathbf{P}_k(\mathbb{R}^d)$, by setting:

$$\mathbb{F}(T) := \inf\{\mathbb{M}(S) + \mathbb{M}(T - \partial S) : \ S \in \mathscr{D}_{k+1}(\mathbb{R}^d)\}. \tag{6.7}$$

The main reason for our interest on this notion is the fact that the flat norm metrizes the weak* convergence of normal currents in a compact set with equi-bounded masses and masses of the boundaries. This fact can be easily deduced from [50, Theorem 4.2.17(1)].

In the sequel, we will also use the following equivalent characterization of the flat norm of a flat chain (cf. [50, 4.1.12] and [70, 4.5]). If $T \in \mathbf{F}_k(\mathbb{R}^d)$ and $B \subset \mathbb{R}^d$ is a ball such that $\mathrm{spt}(T) \subset B$, then

$$\mathbb{F}(T) = \sup\{\langle T, \omega \rangle : \ \omega \in \mathscr{D}^k(\mathbb{R}^d) \text{ with } \|\omega\|_{C^0(B;\Lambda^k(\mathbb{R}^d))} \leqslant 1, \ \|d\omega\|_{C^0(B;\Lambda^{k+1}(\mathbb{R}^d))} \leqslant 1\}. \tag{6.8}$$

7 IMPROVED STABILITY OF OPTIMAL TRAFFIC PATHS

7.1 INTRODUCTION

The branched transport problem is a variant of the classical Monge-Kantorovich problem, where the cost of the transportation does not depend only on the initial and the final spatial distribution of the mass that one wants to transfer, but also on the paths along which the mass particles move. It was introduced to model systems which naturally show ramifications, such as roots systems of trees and leaf ribs, the nervous, the bronchial and the cardiovascular systems, but also to describe other supply-demand distribution networks, like irrigation networks, electric power supply, water distribution, etc. In all of the many different formulations of the problem, the main feature is the fact that the cost functional is designed in order to privilege large flows and to prevent diffusion; indeed the transport actually happens on a 1-dimensional network.

To translate this principle in mathematical terms, one can consider costs which are proportional to a power $\alpha \in (0, 1)$ of the flow. Roughly speaking, it is preferable to transport two positive masses m_1 and m_2 together, rather than separately, because $(m_1 + m_2)^\alpha < m_1^\alpha + m_2^\alpha$. Obviously the smaller is α and the stronger is the grouping effect.

Different costs and descriptions have been introduced in order to model such problem: one of the first proposals came by Gilbert in [53], who considered finite directed weighted graphs G with straight edges $e \in E(G)$ "connecting" two discrete measures, and a weight function $w : E(G) \to (0, \infty)$. The cost of G is defined to be:

$$\sum_{e \in E(G)} w(e)^\alpha \mathcal{H}^1(e). \tag{7.1}$$

Later Xia has extended this model to a continuous framework using Radon vector-valued measures, or, equivalantly, 1-dimensional currents, called in this context "traffic paths" (see [85]).

In [64, 13], new objects called "traffic plans" have been introduced and studied. Roughly speaking, a traffic plan is a measure on the set of Lipschitz paths, where each path represents the trajectory of a single particle. All these formulations were proved to be equivalent (see [15] and references therein) and in particular the link between the last two of them is encoded in a deep result, due to Smirnov, on the structure of acyclic, normal 1-dimensional currents (see Theorem 7.9).

A rich variety of branched transportation problems can be described through these objects: in all of them existence [85, 64, 13, 14, 22, 75] and (partially) regularity theory [86, 21, 46, 47, 69, 87, 18] are well-established. It is, instead, a challenging problem to perform numerical simulations.

The main reference on the topic is the book [15], which is an almost up-to-date overview on the results in the field. To witness the current research activity on this topic we refer also to the recent works [65], where currents with coefficients in a normed group are used to propose a rephrasing of the discrete problem which could be considered as a convex problem, to [19], which proves the equivalence of several formulations of the urban planning model, including two different regimes of transportation and to [20], which provides a new convexification of the 2-dimensional problem, used to perform numerical simulations.

Other techniques have been recently introduced, with the aim to tackle this and similar problems numerically. For instance [72] provides a Modica-Mortola-type approximation of the branched transportation problem and in [25] the authors introduce a family of approximating energies, modeled on the Ambrosio-Tortorelli functional (see also [17]). Numerical simulations with a different aim are implemented in the recent works [67] and [16]. Here the novel formulations of the Steiner-tree problem and the Gilbert-Steiner problem, introduced in [66] and [65], are exploited to find numerical calibrations: functional-analytic tools which can be used to prove the minimality of a given configuration.

A natural question of special relevance in view of numerical simulations, is whether the optima are stable with respect to variations of the initial and final distribution of mass. In order to introduce this question more precisely and to state our main result, let us give some informal definitions. More technical definitions will be introduced in Section 7.2 and used along this chapter. Nevertheless, the simplified notation introduced here suffices to formulate the question and our main result.

Given two finite positive measures μ^-, μ^+ on the set $X := \overline{B_R(0)} \subset \mathbb{R}^d$ with $\mu^-(X) = \mu^+(X)$, a traffic path connecting μ^- to μ^+ is a vector-valued measure $T = \vec{T}(\mathcal{H}^1 \llcorner E)$, supported on a set $E \subset X$, which is contained in a countable union of curves of class C^1, having distributional divergence

$$\operatorname{div} T = \mu^+ - \mu^-.$$

The α-mass of T is defined as the quantity

$$\mathbb{M}^\alpha(T) := \int_E |\vec{T}(x)|^\alpha d\mathcal{H}^1(x).$$

We say that T is an optimal traffic path, and we write $T \in \mathbf{OTP}(\mu^-, \mu^+)$ if

$$\mathbb{M}^\alpha(T) \leqslant \mathbb{M}^\alpha(S), \text{ for every traffic path } S \text{ with } \operatorname{div} S = \mu^+ - \mu^-.$$

We address the following question about the stability of optimal traffic paths, raised in [15, Problem 15.1].

Question. Let $\alpha \leqslant 1 - \frac{1}{d}$. Let $(\mu_n^-)_{n\in\mathbb{N}}, (\mu_n^+)_{n\in\mathbb{N}}$ be finite measures on X and for every n let $T_n \in \mathbf{OTP}(\mu_n^-, \mu_n^+)$, with $\mathbb{M}^\alpha(T_n)$ uniformly bounded. Assume that T_n converges to a vector-valued measure T where $\operatorname{div} T = \mu^+ - \mu^-$ and $\mu^\pm$ are finite measures. Is it true that $T \in \mathbf{OTP}(\mu^-, \mu^+)$?

The threshold

$$\alpha = 1 - \frac{1}{d} \tag{7.2}$$

appears in several contexts in the literature. Firstly, when α is above this value any two probability measures with compact support in $\mathbb{R}^d$ can be connected with finite cost (see

Proposition 7.6). Secondly, above this value the answer to the previous question is positive and the minimum cost between two given measures is continuous with respect to the weak* convergence of measures (see [15, Lemma 6.11 and Proposition 6.12]). Finally, above the threshold interior regularity holds (see [15, Theorem 8.14]) and actually the stability property plays an important role in the proof of such result. The finiteness of the cost, as well as the continuity of the minimum cost, fails for values of α smaller or equal to the value (7.2) (see [26] for an example of failure of continuity). Surprisingly enough, the stability of optimal plans still holds, at least under mild additional assumptions. The main result of this chapter provides a positive answer to the stability question for α below the critical threshold (7.2), when the supports of the limit measures $\mu^{\pm}$ are disjoint and "not too big"; nothing is instead assumed on the approximating sequence $(\mu_n^{\pm})_{n\in\mathbb{N}}$.

Theorem 7.1. *Let $\alpha > 1 - \frac{1}{d-1}$. Let $A^-, A^+ \subset X$ be measurable sets and μ^-, μ^+ be finite measures on X with $\mu^-(X) = \mu^+(X)$, $\operatorname{spt}(\mu^+) \cap \operatorname{spt}(\mu^-) = \emptyset$,*

$$\mathcal{H}^1(A^- \cup A^+) = 0 \qquad \text{and} \qquad \mu^-(X \setminus A^-) = \mu^+(X \setminus A^+) = 0. \tag{7.3}$$

Let $(\mu_n^-)_{n\in\mathbb{N}}, (\mu_n^+)_{n\in\mathbb{N}}$ be finite measures on X such that $\mu_n^-(X) = \mu_n^+(X)$ and

$$\mu_n^{\pm} \rightharpoonup \mu^{\pm}. \tag{7.4}$$

For every $n \in \mathbb{N}$ let $T_n \in \boldsymbol{OTP}(\mu_n^-, \mu_n^+)$ be an optimal traffic path and assume that there exists a traffic path T and a constant $C > 0$ such that

$$T_n \rightharpoonup T \qquad \text{and} \qquad \mathbb{M}^{\alpha}(T_n) \leqslant C.$$

Then T is optimal, namely

$$T \in \boldsymbol{OTP}(\mu^-, \mu^+).$$

Remark 7.2.

1. Notice that in the plane (namely, for $d = 2$) our result cover all possible exponents $\alpha \in (0, 1)$.
2. The actual notion of traffic path as well as the notion of convergence mentioned in Question 7.1 and denoted in Theorem 7.1 by $T_n \rightharpoonup T$, are slightly different from those used in this introduction (see Subsection 6.0.2). For our purposes, it is important to observe that the convergence of traffic paths T_n to T implies the convergence of $\operatorname{div} T_n$ to $\operatorname{div} T$, weakly in the sense of measures.
3. The assumptions that the supports of μ^- and μ^+ are disjoint is recurrent in the literature. For example it is assumed in the proof of interior regularity properties of optimal traffic plans (see [15, Chapter 8]). Moreover such hypothesis could be dropped if we assume that either μ^- or μ^+ are finite atomic measures. However we will not pursue this in the present chapter.
4. The restriction that $\mu^{\pm}$ are supported on $\mathcal{H}^1$-null sets is essential for our proof (even though we can relax such assumption in some special case, see [26]). On the other hand, restrictions on the "size" of sets supporting the measures $\mu^{\pm}$ are recurrent assumptions in previous works (see [15, Chapter 10] and [46]). Requiring (7.3) for supporting Borel

sets A^+ and A^- rather than for the (closed) supports of $\mu^\pm$, allows one to apply the theorem to more cases; for instance, as soon as the limit measures are supported on any countable set (possibly dense in an open subset of X).

5. There is a subtle reason for our choice to use traffic paths, rather than traffic plans, which is related to a known issue about the definition of the cost for traffic plans (see the discussion at the beginning of [15, Chapter 4]). Nevertheless we are able to prove a weaker version of our main result also for traffic plans: roughly speaking one should assume additionally the Hausdorff convergence of the supports of $\mu_n^\pm$ to the supports of $\mu^\pm$. This problem and other versions of the stability results with weaker assumptions on $\mu^\pm$ in some special settings are addressed in [26].

On the structure of the chapter

A few words are worthwhile concerning the organization of this chapter. In Section 7.2 we introduce the setting and preliminaries and in Section 7.3 we collect some properties of optimal traffic paths which we use extensively through this chapter. In particular, in Proposition 7.10 we prove a result about the representation of optimal traffic paths as weighted collections of curves, which paves the way for several new operations on traffic paths introduced in this chapter. We conclude Section 7.3 raising the main question on the stability of optimal traffic paths and recalling the results which are already available in the literature. Section 7.4 requires some explanation: there we prove a result on the lower semi-continuity of the transportation cost. Clearly such property is already used by many other authors. The reason for our attention on that issue is twofold: firstly we want to throw light on a point that is partially overlooked in some previous works (see Remark 7.5), secondly we need a stronger (localized) version of the usual semi-continuity. Section 7.5 deserves particular attention at a first reading, since it gives a heuristic presentation of the proof of Theorem 7.1 and sheds light on several lemmas used therein. We kept the presentation as informal as possible, so that it is possible to follow the fundamental ideas of the chapter even without being used to the notions and definitions of Section 7.2. Section 7.6 contains several preliminary lemmas, covering results and new techniques which are the ingredients of the proof of the main theorem. Eventually, in Section 7.7, we prove Theorem 7.1.

7.2 SETTING AND PRELIMINARIES

7.2.1 α-*mass*

For fixed $\alpha \in [0,1)$, we define also the α-*mass* of a current $T \in \mathbf{R}_k(\mathbb{R}^d) \cup \mathbf{N}_k(\mathbb{R}^d)$ by

$$\mathbb{M}^\alpha(T) := \begin{cases} \int_E \theta^\alpha(x)\, d\mathcal{H}^k(x) & \text{if } T \in \mathbf{R}_k(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{7.5}$$

One elementary property of this functional is its *sub-additivity*, namely

$$\mathbb{M}^\alpha(T_1+T_2) \leqslant \mathbb{M}^\alpha(T_1)+\mathbb{M}^\alpha(T_2) \qquad \text{for every } T_1, T_2 \in \mathbf{R}_k(\mathbb{R}^d)\cup\mathbf{N}_k(\mathbb{R}^d). \tag{7.6}$$

Indeed, the inequality is trivial if T_1 or T_2 is not rectifiable. In turn, if $T_i = T[E_i, \tau_i, \theta_i]$, $i = 1, 2$, the multiplicity θ of $T_1 + T_2$ is obtained as the sum of the multiplicities of T_1 and T_2 with possible signs, so that $\theta \leqslant \theta_1 + \theta_2$. Since moreover the inequality $(\theta_1+\theta_2)^\alpha \leqslant \theta_1^\alpha + \theta_2^\alpha$ holds for every $\theta_1, \theta_2 \in [0, \infty)$, we deduce that

$$\mathbb{M}^\alpha(T_1+T_2) \leqslant \int_{E_1\cup E_2} (\theta_1+\theta_2)^\alpha \, d\mathcal{H}^k \leqslant \int_{E_1\cup E_2} \theta_1^\alpha + \theta_2^\alpha \, d\mathcal{H}^k = \mathbb{M}^\alpha(T_1)+\mathbb{M}^\alpha(T_2).$$

7.2.2 *Traffic paths*

Fix $R > 0$. Along this chapter, by X we denote the closed ball of radius R in $\mathbb{R}^d$ centered at the origin. Following [85] and [15], given two positive measures $\mu^-, \mu^+ \in \mathcal{M}_+(X)$ with the same total variation, we define the set $\mathbf{TP}(\mu^-, \mu^+)$ of the *traffic paths* connecting μ^- to μ^+ as

$$\mathbf{TP}(\mu^-,\mu^+) := \{T \in \mathbf{N}_1(\mathbb{R}^d) : \mathrm{spt}(T) \subset X, \partial T = \mu^+ - \mu^-\},$$

and the *minimal transport energy* associated to μ^-, μ^+ as

$$\mathbb{M}^\alpha(\mu^-,\mu^+) := \inf\{\mathbb{M}^\alpha(T) : T \in \mathbf{TP}(\mu^-,\mu^+)\}.$$

Moreover we define the set of *optimal traffic paths* connecting μ^- to μ^+ by

$$\mathbf{OTP}(\mu^-,\mu^+) := \{T \in \mathbf{TP}(\mu^-,\mu^+) : \mathbb{M}^\alpha(T) = \mathbb{M}^\alpha(\mu^-,\mu^+)\}. \tag{7.7}$$

Given a rectifiable current T with compact support in $\mathbb{R}^d$ and a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^m$, we denote by $f_\sharp T$ the push-forward of T according to f, i.e the rectifiable current in $\mathbb{R}^m$ defined by

$$\langle f_\sharp T, \omega\rangle := \langle T, f^\sharp \omega\rangle, \qquad \text{for every } \omega \in \mathscr{D}^k(\mathbb{R}^m)$$

where $f^\sharp\omega$ is the pull-back of the form ω.

A consequence of the following proposition is that, in order to minimize the α-mass among currents with boundary in X, it is not restrictive to consider only currents supported in X. Indeed the projection onto X reduces the α-mass. See also [39, Lemma 3.2.4 (2)].

Proposition 7.3. *Let* $T \in \mathbf{R}_1(\mathbb{R}^d)$ *and let* $f : \mathbb{R}^d \to \mathbb{R}^m$ *be an* L*-Lipschitz map. Then* $\mathbb{M}^\alpha(f_\sharp T) \leqslant L\mathbb{M}^\alpha(T)$.

Proof. If $T = T[E, \tau, \theta]$, combining the Area Formula (see [79, (8.5)]) and the fact that $(a + b)^\alpha \leqslant a^\alpha + b^\alpha$ for every $a, b > 0$, we get

$$\begin{aligned} \mathbb{M}^\alpha(f_\sharp T) &\leqslant \int_{f(E)} \left(\int_{f^{-1}(y)} \theta(x) d\mathcal{H}^0(x)\right)^\alpha d\mathcal{H}^1(y) \\ &\leqslant \int_{f(E)} \int_{f^{-1}(y)} \theta^\alpha(x) d\mathcal{H}^0(x) d\mathcal{H}^1(y) \\ &= \int_E J_f(x)\theta^\alpha(x) d\mathcal{H}^1(x) \leqslant L \int_E \theta^\alpha(x) d\mathcal{H}^1(x) = L\mathbb{M}^\alpha(T). \end{aligned} \tag{7.8}$$

$\square$

Remark 7.4. We notice that, given two measures $\mu^-, \mu^+ \in \mathcal{M}_+(X)$ with the same total variation and a rectifiable current $R \in \mathbf{R}_1(\mathbb{R}^d)$ with $\mathbb{M}^\alpha(R) < \infty$ and $\partial R = \mu^+ - \mu^-$, there exists $R' \in \mathbf{R}_1(X)$ with $\partial R' = \mu^+ - \mu^-$ and

$$\mathbb{M}^\alpha(R') \leqslant \mathbb{M}^\alpha(R).$$

More precisely, if R is not supported on X, then one can find R' such that

$$\mathbb{M}^\alpha(R') < \mathbb{M}^\alpha(R).$$

The proof of this fact is easily obtained by choosing R' as the push-forward of the current R according to the closest-point projection π onto X and applying Proposition 7.3, observing that π has local Lipschitz constant strictly smaller than 1 at all points of $\mathbb{R}^d \setminus X$.

Remark 7.5 (Comparison with costs studied in the literature). The original definition of "cost" of a traffic path slightly differs from the α-mass defined above. Indeed in [85, Definition 3.1] the author defines the cost of a traffic path as the lower semi-continuous relaxation on the space of normal currents of the functional (7.1) defined on *polyhedral chains*. In [86, Section 3], the author notices that, in the class of rectifiable currents, his definition of cost coincides with the α-mass defined in (7.5). The proof of this fact is only sketched in [82, Section 6] and is proved in detail in [28], see next chapter. To keep the present chapter self-contained, in our exposition we prefer not to rely on this fact, but we stick to the notion of cost given by our definition of α-mass. We will prove independently in Section 7.4 that the α-mass is lower semi-continuous, together with a localized version of this result that does not appear in the literature. Since several results in previous works (see for instance Theorem 7.6) are first proven for polyhedral chains and then extended by lower semi-continuity, their validity in our setting does not rely on the equivalence between the two costs.

7.3 KNOWN RESULTS ON OPTIMAL TRAFFIC PATHS

In this section we collect some of the known properties of optimal traffic paths. The presentation does not aim to be exhaustive, but we only recall the facts used in the proof of Theorem 7.1.

7.3.1 *Existence of traffic paths with finite cost*

We begin with the observation that the existence of elements with finite α-mass in $\mathbf{TP}(\mu^-, \mu^+)$ is not guaranteed in general. For example in [46, Theorem 1.2] it is proved that there exists no traffic path with finite α-mass connecting a Dirac delta to the Lebesgue measure on a ball if $\alpha \leqslant 1 - \frac{1}{d}$. On the other hand, if the exponent α is larger than such critical threshold, then not only the existence of traffic paths with finite α-mass is guaranteed, but one also has a quantitative upper bound on the minimal transport energy.

Theorem 7.6 ([85, Proposition 3.1]). *Let $\alpha > 1 - \frac{1}{d}$ and $\mu^-, \mu^+ \in \mathcal{M}_+(\mathbb{R}^d)$ be two measures with equal mass M supported on a set of diameter L. Then*

$$\mathbb{M}^\alpha(\mu^-, \mu^+) \leqslant C_{\alpha,d} M^\alpha L,$$

where $C_{\alpha,d}$ is a constant depending only on α and d.

7.3.2 *Structure of optimal traffic paths*

An important information about the structure of optimal traffic paths (more in general, about traffic paths of finite α-mass) is their rectifiability, which follows immediately from the definition of α-mass. Some further piece of information comes from the fact that optimal traffic paths do not "contain cycles". A current T with finite mass is called *acyclic* if there exists no non-trivial current S such that

$$\partial S = 0 \qquad \text{and} \qquad \mathbb{M}(T) = \mathbb{M}(T - S) + \mathbb{M}(S).$$

The following theorem states that optimal traffic paths with finite cost are acyclic. Even though in [73] several definitions of cost are considered, the proof of such theorem is given exactly for our cost (7.5).

Theorem 7.7 ([73, Theorem 10.1])**.** *Let* $\mu^-, \mu^+ \in \mathcal{M}_+(\mathbb{R}^d)$ *and* $T \in \boldsymbol{OTP}(\mu^-, \mu^+)$ *with finite* α*-mass. Then* T *is acyclic.*

The power of this result relies in the possibility to represent acyclic normal 1-currents as weighted collections of Lipschitz paths. Before stating this result, we introduce some notation.

We denote by Lip the space of 1-Lipschitz curves $\gamma : [0, \infty) \to \mathbb{R}^d$. For $\gamma \in \text{Lip}$ we denote by $T_0(\gamma)$, the value

$$T_0(\gamma) := \sup\{t : \gamma \text{ is constant on } [0, t]\}$$

and by $T_\infty(\gamma)$ the (possibly infinite) value

$$T_\infty(\gamma) := \inf\{t : \gamma \text{ is constant on } [t, \infty)\}.$$

Given a Lipschitz curve with finite length $\gamma : [0, \infty) \to \mathbb{R}^d$, we call $\gamma(\infty) := \lim_{t\to\infty} \gamma(t)$. We say that a curve $\gamma \in \text{Lip}$ of finite length is *simple* if $\gamma(s) \neq \gamma(t)$ for every $T_0(\gamma) \leqslant s < t \leqslant T_\infty(\gamma)$ such that γ is non-constant in the interval $[s, t]$.

To a Lipschitz simple curve with finite length $\gamma : [0, \infty) \to \mathbb{R}^d$, we associate canonically the rectifiable 1-dimensional current

$$R_\gamma := R_\gamma[\text{Im}(\gamma), \frac{\gamma'}{|\gamma'|}, 1].$$

It follows immediately from (6.4) that

$$\mathbb{M}(R_\gamma) = \mathcal{H}^1(\text{Im}(\gamma)) \tag{7.9}$$

and it is easy to verify that

$$\partial R_\gamma = \delta_{\gamma(\infty)} - \delta_{\gamma(0)}. \tag{7.10}$$

Since γ is simple, if it is also non-constant, then $\gamma(\infty) \neq \gamma(0)$ and $\mathbb{M}(\partial R_\gamma) = 2$.

In the following definition, we consider a class of normal currents that can be written as a weighted superposition of Lipschitz simple curves with finite length.

Definition 7.8 (Good decomposition). Let $T \in \mathbf{N}_1(\mathbb{R}^d)$ and let $\pi \in \mathcal{M}_+(\mathrm{Lip})$ be a finite nonnegative measure, supported on the set of curves with finite length, such that

$$T = \int_{\mathrm{Lip}} R_\gamma \, d\pi(\gamma), \tag{7.11}$$

in the sense of [1, Section 2.3].

We say that π is a good decomposition of T if π is supported on non-constant, simple curves and satisfies the equalities

$$\mathbb{M}(T) = \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) \, d\pi(\gamma) = \int_{\mathrm{Lip}} \mathcal{H}^1(\mathrm{Im}(\gamma)) \, d\pi(\gamma)\,; \tag{7.12}$$

$$\mathbb{M}(\partial T) = \int_{\mathrm{Lip}} \mathbb{M}(\partial R_\gamma) \, d\pi(\gamma) = 2\pi(\mathrm{Lip})\,. \tag{7.13}$$

Concretely, (7.11) means that, representing T as a vector-valued measure $\vec{T}\|T\|$, for every smooth compactly supported vector field $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ it holds

$$\int_{\mathbb{R}^d} \varphi \cdot \vec{T} \, d\|T\| = \int_{\mathrm{Lip}} \int_0^\infty \varphi(\gamma(t)) \cdot \gamma'(t) \, dt \, d\pi(\gamma) \tag{7.14}$$

The following theorem, due to Smirnov ([80]), shows that any acyclic, normal, 1-dimensional current has a good decomposition.

Theorem 7.9 ([74, Theorem 5.1]). *Let $T = \vec{T}\|T\| \in \mathbf{N}_1(\mathbb{R}^d)$ be an acyclic normal 1-current. Then there is a Borel finite measure π on Lip such that T can be decomposed as*

$$T = \int_{\mathrm{Lip}} R_\gamma \, d\pi(\gamma)$$

and π is a good decomposition of T.

In the following proposition we collect some useful properties of good decompositions. Further properties will be given in Proposition 7.18.

Proposition 7.10 (Properties of good decompositions). *If $T \in \mathbf{N}_1(\mathbb{R}^d)$ has a good decomposition π as in (7.11), the following statements hold:*

1. *The positive and the negative parts of the signed measure ∂T are*

$$\partial_- T = \int_{\mathrm{Lip}} \delta_{\gamma(0)} \, d\pi(\gamma) \qquad \textit{and} \qquad \partial_+ T = \int_{\mathrm{Lip}} \delta_{\gamma(\infty)} \, d\pi(\gamma). \tag{7.15}$$

2. *If $T = T[E, \tau, \theta]$ is rectifiable, then*

$$\theta(x) = \pi(\{\gamma : x \in \mathit{Im}(\gamma)\}) \qquad \textit{for } \mathcal{H}^1\textit{-a.e. } x \in E. \tag{7.16}$$

3. *For every* $\pi' \leqslant \pi$ *the representation*

$$T' := \int_{\mathrm{Lip}} R_\gamma d\pi'(\gamma) \tag{7.17}$$

is a good decomposition of T'*; moreover, if* $T = T[E, \tau, \theta]$ *is rectifiable, then* T' *can be written as* $T' = T[E, \theta', \tau]$ *with* $\theta' \leqslant \min\{\theta, \pi'(\mathrm{Lip}\,)\}$.

4. *If* $\mathbb{M}^\alpha(T) < \infty$*, for every* $\varepsilon > 0$ *there exists* $\delta := \delta(T, \varepsilon) > 0$ *such that for every* $\pi' \leqslant \pi$ *with* $\pi'(\mathrm{Lip}\,) \leqslant \delta$ *we have*

$$\mathbb{M}^\alpha(T') \leqslant \varepsilon, \tag{7.18}$$

where T' *is defined by* (7.17).

Proof. Proof of (1). It follows from the expression in (7.11), from the linearity of the boundary operator and from (7.10) that

$$\partial T = \int_{\mathrm{Lip}} \partial R_\gamma d\pi(\gamma) = \int_{\mathrm{Lip}} \delta_{\gamma(\infty)}\, d\pi(\gamma) - \int_{\mathrm{Lip}} \delta_{\gamma(0)}\, d\pi(\gamma) =: S_\infty - S_0.$$

By the subadditivity of the mass and by (7.13)

$$\begin{aligned}\mathbb{M}(S_\infty) + \mathbb{M}(S_0) &\leqslant \int_{\mathrm{Lip}} \mathbb{M}(\delta_{\gamma(\infty)}) d\pi(\gamma) + \int_{\mathrm{Lip}} \mathbb{M}(\delta_{\gamma(0)}) d\pi(\gamma) \\ &= \int_{\mathrm{Lip}} \mathbb{M}(\partial R_\gamma) d\pi(\gamma) = \mathbb{M}(\partial T) = \mathbb{M}(S_\infty - S_0)\end{aligned}$$

From this, we deduce that equality holds in the previous chain of inequalities and that there is no cancellation between S_∞ and S_0, namely, they are mutually singular measures. This, in turn, implies that they represent the positive and negative part of the measure $\partial T = S_\infty - S_0$.

Proof of (2). We compute, for every smooth compactly supported test function $\phi : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} \phi\theta d\mathcal{H}^1 \llcorner E = \int_{\mathrm{Lip}} \left(\int_{\mathbb{R}^d} \phi \mathbb{1}_{\mathrm{Im}\gamma} d\mathcal{H}^1 \llcorner E \right) d\pi = \int_{\mathbb{R}^d} \phi \left(\int_{\mathrm{Lip}} \mathbb{1}_{\mathrm{Im}\gamma} d\pi \right) d\mathcal{H}^1 \llcorner E,$$

where in the first equality we used [1, Theorem 5.5 (iii)], which states that (7.11) induces an analogous equality between the associated positive measures, and the fact that π-a.e. γ is simple.

Proof of (3). We write $T = T' + (T - T')$ and, since $T - T'$ is "parametrized" by $\pi - \pi'$, we have that

$$\mathbb{M}(T') \leqslant \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d\pi'(\gamma), \quad \text{and} \quad \mathbb{M}(T - T') \leqslant \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d(\pi - \pi')(\gamma). \tag{7.19}$$

We conclude that

$$\begin{aligned}\mathbb{M}(T) &\leqslant \mathbb{M}(T') + \mathbb{M}(T - T') \\ &\leqslant \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d\pi'(\gamma) + \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d(\pi - \pi')(\gamma) = \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d\pi(\gamma).\end{aligned} \tag{7.20}$$

Since π represents a good decomposition of T, by (7.12) it follows that equality must hold at each step in the previous inequality. In particular, from (7.19), we deduce that

$$\mathbb{M}(T') = \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) d\pi'(\gamma).$$

The same argument applied to the current $\partial T'$ leads to the proof that the property (7.13) holds for the good decomposition of T'.

Since the decomposition (7.17) is good, then, by the formula (7.16), we get that for $\mathcal{H}^1$-a.e. $x \in E$

$$\begin{aligned} \theta'(x) &= \pi'(\{\gamma : x \in \mathrm{Im}(\gamma)\}) \\ &\leqslant \min\{\pi(\{\gamma : x \in \mathrm{Im}(\gamma)\}), \pi'(\mathrm{Lip}\,)\} = \min\{\theta(x), \pi'(\mathrm{Lip}\,)\}. \end{aligned}$$

This concludes the proof of (3).

Proof of (4). By the previous point, applied to the good decomposition of T' given in (7.17), it follows that

$$\theta'(x) \leqslant \min\{\theta, \delta\}.$$

Therefore

$$\mathbb{M}^\alpha(T') \leqslant \int_E \min\{\theta(x), \delta\}^\alpha \, d\mathcal{H}^1(x)$$

and the right-hand side converges to 0 as $\delta \to 0$ by the Lebesgue dominated convergence Theorem. □

7.3.3 *Stability of optimal traffic paths*

Our work [27] addresses Question 7.1, which we can now rephrase in rigorous terms as follows.

For every $n \in \mathbb{N}$, let $\mu_n^-, \mu_n^+ \in \mathcal{M}_+(X)$ with the same mass and let $T_n \in \mathbf{OTP}(\mu_n^-, \mu_n^+)$, with $\mathbb{M}^\alpha(T_n)$ uniformly bounded. Assume

$$T_n \rightharpoonup T, \quad \text{and} \quad \mu_n^\pm \rightharpoonup \mu^\pm$$

where $\partial T = \mu^+ - \mu^-$ and $\mu^\pm \in \mathcal{M}_+(X)$. Is it true that $T \in \mathbf{OTP}(\mu^-, \mu^+)$?

The answer is relatively simple for $\alpha \in (1 - 1/d, 1]$, relying on the fact that the minimal transport energy $\mathbb{M}^\alpha(\nu_n, \nu)$ metrizes the weak*-convergence of probability measures $\nu_n \rightharpoonup \nu$, as stated in the following lemma.

Lemma 7.11 ([15, Lemma 6.11])**.** *Let $\alpha > 1 - \frac{1}{d}$ and $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ be a sequence of probability measures weakly converging to $\nu \in \mathcal{P}(X)$. Then we have that*

$$\lim_{n\to\infty} \mathbb{M}^\alpha(\nu_n, \nu) = 0.$$

From Lemma 7.11 one can easily deduce the following stability result for optimal traffic paths.

Theorem 7.12 ([15, Proposition 6.12])**.** *Let $\alpha > 1 - \frac{1}{d}$. Assume that $(\mu_n^-)_{n\in\mathbb{N}}, (\mu_n^+)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ converge (weakly in the sense of measures) respectively to $\mu^-, \mu^+ \in \mathcal{P}(X)$. Let $T_n \in \boldsymbol{OTP}(\mu_n^-, \mu_n^+)$ satisfying*

$$\sup_{n\in\mathbb{N}} \mathbb{M}^\alpha(T_n) < \infty.$$

If $T_n \rightharpoonup T$ for some current T, then $T \in \boldsymbol{OTP}(\mu^-, \mu^+)$.

Indeed, assuming by contradiction that Theorem 7.12 does not hold for a sequence $T_n \rightharpoonup T$, we find a contradiction by considering an energy competitor for T_n (n large enough) as follows. We take the optimal transport T_{opt} for the limit problem and we add two traffic paths of arbitrarily small energy that connect respectively μ_n^- to μ^-, and μ^+ to μ_n^+. This strategy fails for $\alpha \leqslant 1 - \frac{1}{d}$, since Lemma 7.11 does not hold below the critical threshold (an example of such phenomenon is provided in [26]). For this reason, we develop in the following sections a more involved strategy to prove the stability of optimal traffic paths.

7.4 LOWER SEMI-CONTINUITY OF THE α-MASS

This section is devoted to the proof of a lower semi-continuity result for the α-mass. The statement will be split in two parts. On one side, we prove the lower semi-continuity for normal currents, which for example allows one to prove the classical existence of optimal traffic paths in (7.7) (see [15, Proposition 3.41]). On the other side, our strategy of proof of Theorem 7.1 requires to work with rectifiable currents with boundary of possibly infinite mass, obtained as restriction of normal rectifiable currents to Borel sets. Therefore for rectifiable currents we prove a localized version of the usual lower semi-continuity.

Theorem 7.13. *Let $k \geqslant 0$, $\alpha \in (0, 1]$, $(T_n)_{n\in\mathbb{N}}$ be a sequence of k-dimensional currents in X, and T be a k-dimensional current with*

$$\lim_{n\to\infty} \mathbb{F}(T_n - T) = 0.$$

1. *If the T_n's and T are rectifiable and A is an open subset of X, then*

$$\mathbb{M}^\alpha(T \llcorner A) \leqslant \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner A). \tag{7.21}$$

2. *If T_n and T are normal, then*

$$\mathbb{M}^\alpha(T) \leqslant \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n). \tag{7.22}$$

Using Theorem 7.13(2) and the compactness of normal currents (see [50, 4.2.17(1)]), the existence of optimal transport paths in (7.7) follows via the direct method of the Calculus of Variations.

Corollary 7.14. *Let $\alpha \in (0, 1]$. Given two measures $\mu^-, \mu^+ \in \mathcal{M}_+(X)$ such that $\mathbb{M}^\alpha(\mu^-, \mu^+) < +\infty$, there exists a current $T \in \boldsymbol{OTP}(\mu^-, \mu^+)$.*

The proof of the first part of Theorem 7.13 employs a characterization of rectifiability by slicing. The proof of the second point is carried out by slicing our rectifiable currents and reducing the theorem to the lower semi-continuity of 0-dimensional currents, following some ideas in [39, Lemma 3.2.14]. For this reason, we need to recall some further preliminaries on the slicing of currents. Let $k \leqslant d$, let $I(d,k)$ be the set of multi-indices of order k in $\mathbb{R}^d$, i.e. the set of k-tuples $(i_1,\dots,i_k)$ with

$$1 \leqslant i_1 < \dots < i_k \leqslant d,$$

let $\{e_1,\dots,e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$, and let V_I be the k-plane spanned by $\{e_{i_1},\dots,e_{i_k}\}$ for every $I = (i_1,\dots,i_k) \in I(d,k)$. Given a k-plane V, we denote p_V the orthogonal projection on V. If $V = V_I$ for some I, we simply write p_I instead of p_{V_I}. Given a current $T \in \mathbf{N}_k(\mathbb{R}^d)$ with compact support, a Lipschitz function $p : \mathbb{R}^d \to \mathbb{R}^k$ and $y \in \mathbb{R}^k$, we denote by $\langle T, p, y\rangle$ the 0-dimensional *slice* of T in $p^{-1}(y)$ (see [50, Section 4.3] or [79, Section 28] for the case $k = 1$). In this chapter, we will employ the notion of slicing only to apply two deep known results (contained in Theorem 7.15 and Lemma 7.16). The following theorem shows that the rectifiability of a current is equivalent to the rectifiability of a suitable family of slices.

Theorem 7.15 ([83]). *Let $T \in \mathbf{N}_k(\mathbb{R}^d)$. Then $T \in \mathbf{R}_k(\mathbb{R}^d)$ if and only if*

$$\langle T, p_I, y\rangle \text{ is rectifiable for every } I \in I(d,k) \text{ and for } \mathcal{H}^k\text{-a.e. } y \in V_I.$$

By $Gr(d,k)$ we denote the Grassmannian of k-dimensional planes in $\mathbb{R}^d$ and by $\gamma_{d,k}$ we denote the Haar measure on $Gr(d,k)$, i.e. the unique probability measure on $Gr(d,k)$ which is invariant under the action of orthogonal transformations (see [61, Section 2.1.4]).

In the following lemma, we collect some known properties of slices and their behaviour with respect to the α-mass and the flat norm. The bounds (7.23) and (7.24) below are proved in [39, Corollary 3.2.5(5) and Remark 3.2.11] respectively. The integral-geometric equality is a consequence of [50, 3.2.26; 2.10.15; 4.3.8] (see also [39, (21)]).

Lemma 7.16. *Let $R \in \mathbf{R}_k(\mathbb{R}^d)$ and $N \in \mathbf{N}_k(\mathbb{R}^d)$. Then for every $V \in Gr(d,k)$ we have*

$$\int_{\mathbb{R}^k} \mathbb{M}^\alpha(\langle R, p_V, y\rangle)\, dy \leqslant \mathbb{M}^\alpha(R), \tag{7.23}$$

$$\int_{\mathbb{R}^k} \mathbb{F}(\langle N, p_V, y\rangle)\, dy \leqslant \mathbb{F}(N). \tag{7.24}$$

Moreover, there exists $c = c(d,k)$ such that the following integral-geometric equality holds:

$$\mathbb{M}^\alpha(R) = c \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{M}^\alpha\big(\langle R, p_V, y\rangle\big)\, d(\gamma_{d,k} \otimes \mathcal{H}^k)(V,y). \tag{7.25}$$

Proof of Theorem 7.13(1). Step 1: the case $k = 0$. Since a 0-dimensional rectifiable current $T = T[E, 1, \theta]$ is a signed, atomic measure, we write

$$T \llcorner A = \sum_{i\in\mathbb{N}} \theta_i \delta_{x_i}$$

for $(x_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}^d$ distinct and for $(\theta_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$ (with possible signs). Fix $\varepsilon > 0$ and let $I \subseteq \mathbb{N}$ be a finite set such that

$$\mathbb{M}^\alpha(T \llcorner A) - \sum_{i\in I} |\theta_i|^\alpha \leqslant \varepsilon \qquad \text{if } \mathbb{M}^\alpha(T \llcorner A) < \infty \tag{7.26}$$

and

$$\sum_{i\in I} |\theta_i|^\alpha \geqslant \frac{1}{\varepsilon} \qquad \text{otherwise.} \tag{7.27}$$

Up to reordering the sequences $(x_i)_{i\in\mathbb{N}}$ and $(\theta_i)_{i\in\mathbb{N}}$, we may assume that $I = \{1, ..., N\}$ for some $N := N(\varepsilon)$. Set

$$r := \frac{1}{4} \min \Big\{ \min\{d(x_i, x_j) : 1 \leqslant i < j \leqslant N\}, \min\{d(x_i, A^c) : 1 \leqslant i \leqslant N\} \Big\}.$$

Since $\lim_{n\to\infty} \mathbb{F}(T_n - T) = 0$, then $T_n \rightharpoonup T$ weakly in the sense of measures. Hence for every $i \in \{1, ..., N\}$

$$\mathbb{M}(T \llcorner B(x_i, r)) \leqslant \liminf_{n\to\infty} \mathbb{M}(T_n \llcorner B(x_i, r)), \qquad \text{for every } i \in \{1, ..., N\}. \tag{7.28}$$

By (7.28) and the elementary inequality $\big(\sum_{i\in\mathbb{N}} |a_i|\big)^\alpha \leqslant \sum_{i\in\mathbb{N}} |a_i|^\alpha$ for any $(a_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$, we deduce that for every $i \in \{1, ..., N\}$

$$\begin{aligned} |\theta_i|^\alpha &\leqslant \big(\mathbb{M}(T \llcorner B(x_i, r))\big)^\alpha \leqslant \liminf_{n\to\infty} \big(\mathbb{M}(T_n \llcorner B(x_i, r))\big)^\alpha \\ &\leqslant \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner B(x_i, r)). \end{aligned} \tag{7.29}$$

Adding over i and observing that the balls $B(x_i, r)$ are disjoint by the choice of r, we find that

$$\sum_{i\in I} |\theta_i|^\alpha \leqslant \liminf_{n\to\infty} \sum_{i=1}^N \mathbb{M}^\alpha(T_n \llcorner B(x_i, r)) \leqslant \liminf_{n\to\infty} \sum_{i=1}^N \mathbb{M}^\alpha(T_n \llcorner A).$$

By (7.26) (or (7.27) in the case that $\mathbb{M}^\alpha(T \llcorner A) = \infty$) and since ε is arbitrary, we find (7.21).

Step 2 (Reduction to $k = 0$ through integral-geometric equality). We prove now Theorem 7.13(1) for $k > 0$. Up to subsequences, we can assume

$$\lim_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner A) = \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner A).$$

Integrating in $V \in Gr(d, k)$ the second inequality in Lemma 7.16 we get

$$\lim_{n\to\infty} \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{F}(\langle T_n - T, p_V, y\rangle) d(\gamma_{d,k} \otimes \mathcal{H}^k)(V, y) \leqslant \lim_{n\to\infty} \mathbb{F}(T_n - T) = 0.$$

Since the integrand $\mathbb{F}(\langle T_n - T, p_V, y\rangle)$ is converging to 0 in L^1, up to subsequences, we get

$$\lim_{n\to\infty} \mathbb{F}(\langle T_n - T, p_V, y\rangle) = 0 \qquad \text{for } \gamma_{d,k} \otimes \mathcal{H}^k\text{-a.e. } (V, y) \in Gr(d, k) \times \mathbb{R}^k.$$

We conclude from Step 1 that

$$\mathbb{M}^\alpha(\langle T, p_V, y\rangle \llcorner A) \leqslant \liminf_{n\to\infty} \mathbb{M}^\alpha(\langle T_n, p_V, y\rangle \llcorner A).$$

By [11, (5.15)], for $\mathcal{H}^k$-a.e. y

$$\langle T \llcorner A, p_V, y\rangle = \langle T, p_V, y\rangle \llcorner A. \tag{7.30}$$

By (7.30), we get the inequality

$$\mathbb{M}^\alpha(\langle T \llcorner A, p_V, y\rangle) \leqslant \liminf_{n\to\infty} \mathbb{M}^\alpha(\langle T_n \llcorner A, p_V, y\rangle). \tag{7.31}$$

The conclusion follows applying twice the integral-geometric equality (7.25). Indeed, using the semi-continuity proved for $k = 0$ and Fatou's lemma, we get

$$\begin{aligned}
\mathbb{M}^\alpha(T \llcorner A) &= c \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{M}^\alpha\big(\langle T \llcorner A, p_V, y\rangle\big)\, d(\gamma_{d,k} \otimes \mathcal{H}^k)(V, y) \\
&\overset{(7.31)}{\leqslant} c \int_{Gr(d,k)\times\mathbb{R}^k} \liminf_{n\to\infty} \mathbb{M}^\alpha\big(\langle T_n \llcorner A, p_V, y\rangle\big)\, d(\gamma_{d,k} \otimes \mathcal{H}^k)(V, y) \\
&\leqslant c \liminf_{n\to\infty} \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{M}^\alpha\big(\langle T_n \llcorner A, p_V, y\rangle\big)\, d(\gamma_{d,k} \otimes \mathcal{H}^k)(V, y) \\
&= \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n \llcorner A).
\end{aligned} \tag{7.32}$$

This concludes the proof of Step 2, so the proof of Theorem 7.13(1) is complete. □

In order to prove Theorem 7.13(2), the only property which is missing at this stage is the fact that a normal, non-rectifiable k-current cannot be approximated with rectifiable currents with uniformly bounded mass, α-mass, and mass of the boundary. This is proved in the following lemma.

Lemma 7.17. *Let* $(T_n) \subset \mathbf{R}_k(\mathbb{R}^d)$ *and let us assume that*

$$\sup_{n\in\mathbb{N}}\{\mathbb{M}^\alpha(T_n)\} \leqslant C < +\infty.$$

If $\lim_{n\to\infty} \mathbb{F}(T_n - T) = 0$ *for some* $T \in \mathbf{N}_k(\mathbb{R}^d)$*, then* T *is in fact rectifiable.*

Proof. Step 1: the case $k = 0$. We prove the lemma for $k = 0$, recalling that a 0-dimensional rectifiable current $T = T[E, \tau, \theta]$, with $\tau(x) = \pm 1$, is an atomic signed measure (i.e. a measure supported on a countable set). More precisely, we prove the following claim: let $(T_n)_{n\in\mathbb{N}}$ be a sequence of 0-rectifiable currents $T_n = T[E_n, \tau_n, \theta_n]$ such that $\lim_{n\to\infty} \mathbb{F}(T_n - T) = 0$ for some $T \in \mathbf{N}_0(\mathbb{R}^d)$ and $\mathbb{M}^\alpha(T_n) \leqslant C$ for some $C > 0$; then T is 0-rectifiable.

Indeed, fix $\delta > 0$. For any $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{M}(T_n \llcorner \{x : \theta_n(x) < \delta\}) &= \int_{E_n\cap\{\theta_n<\delta\}} \theta_n(x)\, d\mathcal{H}^k(x) \\
&\leqslant \delta^{1-\alpha} \int_{E_n\cap\{\theta_n<\delta\}} \theta_n(x)^\alpha\, d\mathcal{H}^k(x) \leqslant \mathbb{M}^\alpha(T_n)\delta^{1-\alpha} \leqslant C\delta^{1-\alpha}.
\end{aligned}$$

Therefore, up to subsequences the measure $T_n \llcorner \{x : \theta_n(x) \geqslant \delta\}$ converges to a discrete measure T_1 (indeed the support of the measures $T_n \llcorner \{x : \theta_n(x) \geqslant \delta\}$ consists of a finite number of points, which is uniformly bounded with respect to n, due to the bound on

$\mathbb{M}^\alpha(T_n)$), and the sequence $(T_n \llcorner \{x : \theta_n(x) < \delta\})_{n\in\mathbb{N}}$ converges to a signed measure T_2 of mass less or equal than $C\delta^{1-\alpha}$.

By the arbitrariness of δ, we conclude that the measure T_2 has arbitrarily small mass and that the measure T_1 is purely atomic. Since $T = T_1 + T_2$, the statement follows.

Step 2. We prove the claim for $k > 0$.

We apply the inequalities in Lemma 7.16 to our sequence $(T_n)_{n\in\mathbb{N}}$ to deduce that

$$\int_{\mathbb{R}^k} \mathbb{M}^\alpha(\langle T_n, p_I, y\rangle)\, dy \leqslant \mathbb{M}^\alpha(T_n) \leqslant C. \tag{7.33}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} \mathbb{F}(\langle T_n - T, p_I, y\rangle)\, dy \leqslant C \lim_{n\to\infty} \mathbb{F}(T_n - T) = 0.$$

Since the sequence of non-negative functions $(\mathbb{F}(\langle T_n - T, p_I, \cdot\rangle))_{n\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^k)$ to 0, up to a (not relabelled) subsequence, we get the pointwise convergence

$$\lim_{n\to\infty} \mathbb{F}(\langle T_n - T, p_I, y\rangle) = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^k.$$

Moreover, by Fatou lemma and (7.33) we know that for every $I \in I(d,k)$

$$\int_{\mathbb{R}^k} \liminf_{n\to\infty} \mathbb{M}^\alpha(\langle T_n, p_I, y\rangle)\, dy \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^k} \mathbb{M}^\alpha(\langle T_n, p_I, y\rangle)\, dy < \infty.$$

Therefore, we have that

$$\liminf_{n\to\infty} \mathbb{M}^\alpha(\langle T_n, p_I, y\rangle) < \infty \qquad \text{for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^k.$$

Hence we are in the position to apply Step 1 to a.e. slice $\langle T_n, p_I, y\rangle$ to a y-dependent subsequence and deduce that

$$\langle T, p_I, y\rangle \text{ is } 0\text{-rectifiable for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^k,\ I \in I(d,k).$$

Finally, we employ Theorem 7.15 to infer that this property of the slices implies that T is rectifiable. □

Proof of Theorem 7.13(2). Let $(T_n) \subset \mathbf{N}_k(\mathbb{R}^d)$ and $T \in \mathbf{N}_k(\mathbb{R}^d)$ be such that $\lim_{n\to\infty} \mathbb{F}(T_n - T) = 0$. If T is rectifiable, then (7.22) follows by Theorem 7.13(1) and the fact that non-rectifiable currents have infinite α-mass. Otherwise if T is non-rectifiable, then (7.22) follows from Lemma 7.17. □

7.5 IDEAS FOR THE PROOF OF THE MAIN THEOREM

Since the proof of Theorem 7.1 develops some new geometric ideas in order to construct a suitable competitor for a minimization problem, we introduce informally the strategy in this section, assuming some significant simplifications, before entering the technical details of the actual argument. At the end of this section of heuristics we give some hints on how to remove the further assumptions.

We can easily reduce to the case that $\mu^{\pm}, \mu_n^{\pm} \in \mathcal{P}(X)$. By contradiction, we assume that there exists a sequence $T_n \rightharpoonup T$ of optimizers such that T is not an optimizer, namely there exists T_{opt} and $\Delta > 0$ with

$$\mathbb{M}^{\alpha}(T_{opt}) \leqslant \mathbb{M}^{\alpha}(T) - \Delta, \qquad \partial T_{opt} = \partial T = \mu^+ - \mu^-.$$

We aim to find a contradiction by defining a suitable competitor $\tilde{T}_n$ for T_n for some n large enough, that "almost follows" T_{opt} instead of T, and satisfies the estimates

$$\mathbb{M}^{\alpha}(\tilde{T}_n) \leqslant \mathbb{M}^{\alpha}(T_n) - \frac{\Delta}{8}, \qquad \partial \tilde{T}_n = \partial T_n = \mu_n^+ - \mu_n^-.$$

(1) Covering of $A^{\pm}$. First, we choose a countable covering of the sets $A^{\pm}$ supporting $\mu^{\pm}$, denoted by $\{B_i^{\pm} = B^{\pm}(x_i, r_i)\}_{i\in\mathbb{N}}$ (see Figure (1a)) such that

$$\sum_{i=1}^{\infty} r_i, \quad \mathbb{M}^{\alpha}\Big(T \llcorner \bigcup_{i=1}^{\infty} B_i^{\pm}\Big), \quad \text{and} \quad \mathbb{M}^{\alpha}\Big(T_{opt} \llcorner \bigcup_{i=1}^{\infty} B_i^{\pm}\Big) \text{ are arbitrarily small.} \tag{7.34}$$

This choice is made possible by the assumption that the measures $\mu^{\pm}$ are supported on sets of $\mathcal{H}^1$-measure 0 and by the fact that $\mathbb{M}^{\alpha}$ is absolutely continuous with respect to $\mathcal{H}^1$. We also select a finite number $N^{\pm}$ such that

$$\mu^{\pm}\Big(\Big(\bigcup_{i=1}^{N^{\pm}} B_i^{\pm}\Big)^c\Big) \text{ is small.} \tag{7.35}$$

For simplicity, in this section we make the assumption that the balls $B_i^{\pm}$ are pairwise disjoint and that the coverings are finite, namely the quantity in (7.35) is 0.

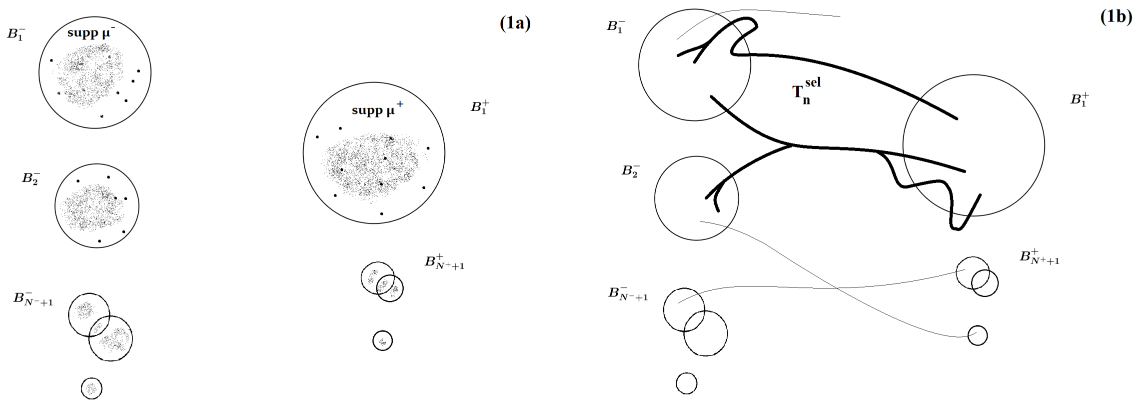

Figure 1: Figure (1a) shows the supports of μ^+ and μ^- and the covering introduced in (1). In Figure (1b) we represented the traffic path T_n and the selection of its curves that begin (respectively end) in the first N^- (respectively N^+) balls.

(2) Representation of T_n. Using Theorem 7.9, we represent each T_n and T_{opt} by a collection of curves weighted by the probability measures π_n and π_{opt} in $\mathcal{P}(\mathrm{Lip}\,)$, namely

$$T_n = \int_{\mathrm{Lip}} R_{\gamma}\, d\pi_n(\gamma), \qquad T_{opt} = \int_{\mathrm{Lip}} R_{\gamma}\, d\pi_{opt}(\gamma).$$

This representation is essential in order to build an energy competitor for the traffic path T_n.

Intuitively, in the competitor that we want to construct, the mass particles, whose original spatial distribution is represented by μ_n^-, will move for an initial stretch along the curves in the support of π_n, as long as these curves remain in the balls where they begin. Then, they will be connected to the curves in the support of π_{opt} via a "cheap" transport supported on the spheres ∂B_i^-. Subsequently the particles will move along the curves in the support of π_{opt}, until they reach the spheres ∂B_i^+. From there, another cheap transport supported on the spheres will connect them back to the curves in the support of π_n and finally they will be transported to their final destination along the curves of π_n. Observe that in the process we may have changed the final destination of each single particle, but we preserved the global final particle distribution.

Let us describe now the strategy more in detail. First, we define π_n^{sel} as the restriction of π_n to curves that start in $\cup_{i=1}^{N^-} B_i^-$ and end in $\cup_{i=1}^{N^+} B_i^+$. We associate to this π_n^{sel} a new current T_n^{sel}, as represented in Figure (1b), and we notice that the remaining $\pi_n - \pi_n^{sel}$ carries little mass, by (7.35) and by the fact that $\partial T_n \rightharpoonup \partial T$. We make the further simplifying assumption that

$$T_n - T_n^{sel} = 0, \tag{7.36}$$

even though this is a big simplification since this term cannot be seen as an error in energy.

(3) Construction of a competitor $\tilde{T}_n^{sel}$ *for* T_n^{sel}. We follow the curves representing T_n^{sel} from their starting point, which, by (7.36) is assumed to be in some B_i^- with $i \in \{1, ..., N^-\}$, up to the first time when they touch ∂B_i^-. In this way, we define $T_n^{sel,-}$ as in Figure (2a). Similarly, we define $T_n^{sel,+}$ as the restriction of the curves in T_n^{sel} from the last time when they touch ∂B_i^+ up to their final point in B_i^+ (see again Figure (2a)).

In a similar way, we define T_{opt}^{restr} restricting the curves representing T_{opt} from the first time they exit $\cup_{i=1}^{N^-} B_i^-$ up to the last time they enter $\cup_{i=1}^{N^+} \overline{B}_i^+$ (see Figure (2b)).

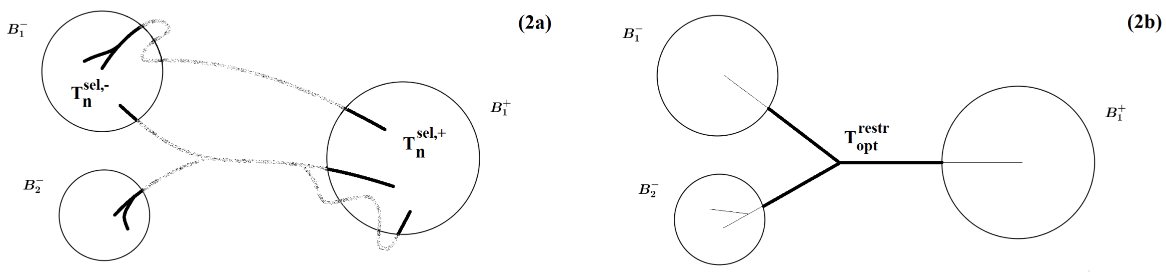

Figure 2: In Figure (2a) we mark $T_n^{sel,\pm}$ and in Figure (2b) we mark T_{opt}^{restr}.

We make the further simplifying assumption that $\mu_n^\pm$ and $\mu^\pm$ have the same quantity of mass in each of the balls $B_i^\pm$, $i = 1, ..., N^\pm$, namely

$$\mu_n^\pm(B_i^\pm) = \mu^\pm(B_i^\pm) \qquad \text{for every } i = 1, ..., N^\pm, \tag{7.37}$$

or, in other words, that

$$\partial_\pm T_n^{sel}(B_i^\pm) = \partial_\pm T_n(B_i^\pm) = \partial_\pm T_{opt}(B_i^\pm) \qquad \text{for every } i = 1, ..., N^\pm. \tag{7.38}$$

We notice that this also implies that

$$\partial_+ T_n^{sel,-}(\partial B_i^-) = \partial_- T_n^{sel,-}(B_i^-) = \partial_- T_n^{sel}(B_i^-) = \partial_- T_{opt}(B_i^-) = \partial_- T_{opt}^{restr}(\partial B_i^-) \quad (7.39)$$

(and a similar equality holds for $\partial_- T_n^{sel,+}(\partial B_i^+)$). Indeed, the first equality holds because the traffic path $T_n^{sel,-}$ transports all the mass inside B_i^- on the boundary of B_i^-; the last inequality holds because π_{opt}-a.e. curve exit from $\cup_{i=1}^{N^-} B_i^-$, since it has to end in $\cup_{i=1}^{N^+} \overline{B}_i^+$.

Next, we consider a traffic path $T_n^{conn,-}$ that connects $\partial_+ T_n^{sel,-}$ to $\partial_- T_{opt}^{restr}$ on $\cup_i \partial B_i^-$. By (7.39), these two measures can be connected since they have the same mass. Moreover, by a modification of Theorem 7.6 (see Lemma 7.20), the two measures can be connected with finite (and actually small) cost, since they are supported on the union of the $(d-1)$-dimensional spheres ∂B_i^-, and since by assumption in our theorem we required that $\alpha > 1 - \frac{1}{d-1}$. The cost of this transport is estimated through Lemma 7.20 by

$$\mathbb{M}^\alpha(T_n^{conn,-}) \leqslant \sum_{i=1}^{N^-} C_{\alpha,d} r_i^-, \quad (7.40)$$

which is small by (7.34).

In a similar way we define a traffic path $T_n^{conn,+}$ that connects $\partial_- T_n^{sel,+}$ to $\partial_+ T_{opt}^{restr}$ on $\cup_i \partial B_i^+$ and enjoys the estimate

$$\mathbb{M}^\alpha(T_n^{conn,+}) \leqslant \sum_{i=1}^{N^+} C_{\alpha,d} r_i^+. \quad (7.41)$$

Finally, we define (see Figure (3))

$$\tilde{T}_n^{sel} := T_n^{sel,-} + T_n^{conn,-} + T_{opt}^{restr} + T_n^{conn,+} + T_n^{sel,+}.$$

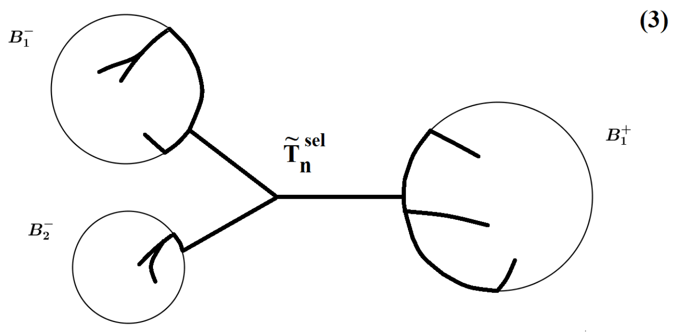

Figure 3: The energy competitor $\tilde{T}_n^{sel}$.

(4) Energy estimate for $\tilde{T}_n^{sel}$ and contradiction. We show finally that the competitor $\tilde{T}_n^{sel}$ has strictly less energy than T_n. Since by construction it has the same marginals, then we reach a contradiction. Indeed, by the subadditivity of the α-mass, we have

$$\mathbb{M}^\alpha(\tilde{T}_n^{sel}) \leqslant \mathbb{M}^\alpha(T_n^{sel,-}) + \mathbb{M}^\alpha(T_n^{conn,-}) + \mathbb{M}^\alpha(T_{opt}^{restr}) + \mathbb{M}^\alpha(T_n^{conn,+}) + \mathbb{M}^\alpha(T_n^{sel,+})$$

(7.42)

By the estimates on the energy of the connections in (7.40) and (7.41) and by the smallness assumptions on the rays, we estimate two terms in the right-hand side of (7.42)

$$\mathbb{M}^\alpha(T_n^{conn,-}) + \mathbb{M}^\alpha(T_n^{conn,+}) \leqslant \frac{\Delta}{4}. \tag{7.43}$$

Regarding the first and last terms in the right-hand side of (7.42), we estimate them with the full energy of T_n inside the balls of the coverings

$$\mathbb{M}^\alpha(T_n^{sel,\pm}) \leqslant \mathbb{M}^\alpha\Big(T_n \llcorner \Big(\cup_{i=1}^{N^\pm} \overline{B}_i^\pm \Big)\Big). \tag{7.44}$$

To bound the energy of T_{opt}^{restr}, we first estimate it with the energy of the whole T_{opt}. Thanks to the energy gap between T_{opt} and T and (7.34), the latter can be estimated choosing the energy of T inside the coverings below $\Delta/4$:

$$\mathbb{M}^\alpha(T_{opt}^{restr}) \leqslant \mathbb{M}^\alpha(T_{opt}) \leqslant \mathbb{M}^\alpha(T) - \Delta \leqslant \mathbb{M}^\alpha\Big(T \llcorner \Big(\big(\cup_{i=1}^{N^-} \overline{B}_i^- \big) \cup \big(\cup_{i=1}^{N^+} \overline{B}_i^+ \big) \Big)^c \Big) - \frac{3\Delta}{4}$$

By the lower semi-continuity of the α-mass on open sets (see Theorem 7.13(1)) we deduce that for n large enough

$$\mathbb{M}^\alpha(T_{opt}^{restr}) \leqslant \mathbb{M}^\alpha\Big(T_n^{sel} \llcorner \Big(\big(\cup_{i=1}^{N^-} \overline{B}_i^- \big) \cup \big(\cup_{i=1}^{N^+} \overline{B}_i^+ \big) \Big)^c \Big) - \frac{\Delta}{2} \tag{7.45}$$

Using (7.43), (7.44), (7.45) to estimate each term in the right-hand side of (7.42) and noticing that the α-mass is additive on traffic paths supported on disjoint sets, we find that

$$\begin{aligned} \mathbb{M}^\alpha(\tilde{T}_n^{sel}) \leqslant & \mathbb{M}^\alpha\Big(T_n^{sel,-} \llcorner \Big(\cup_{i=1}^{N^-} \overline{B}_i^- \Big)\Big) + \mathbb{M}^\alpha\Big(T_n^{sel} \llcorner \Big(\big(\cup_{i=1}^{N^-} \overline{B}_i^- \big) \cup \big(\cup_{i=1}^{N^+} \overline{B}_i^+ \big) \Big)^c \Big) \\ & + \mathbb{M}^\alpha\Big(T_n^{sel,+} \llcorner \Big(\cup_{i=1}^{N^+} \overline{B}_i^+ \Big)\Big) - \frac{\Delta}{4} = \mathbb{M}^\alpha(T_n^{sel}) - \frac{\Delta}{4}. \end{aligned}$$

This gives a contradiction to the optimality of the energy of T_n.

Removing some of the simplifying assumptions that we made in the sketch above is a delicate task and requires new ideas. We briefly describe our strategy.

In (1), we assumed that the balls $B_i^\pm$ are mutually disjoint. If this is not the case, we consider the sets

$$C_i^\pm := B_i^\pm \setminus \Big(\cup_{j=1}^{i-1} B_j^\pm \Big)$$

as a disjoint cover of the sets $A^\pm$. Then we modify the definition of $T_n^{sel,-}$: for every $= 1, ..., N^-$, we stop every curve starting in C_i^- as soon as it touches ∂B_i^-. The choice to let these curves arrive up to ∂B_i^- (and not only up to ∂C_i^-) is related to the fact that ∂B_i^- has a nicer geometry than ∂C_i^- and in particular ensures that the estimate (7.40) holds. Similarly, we modify $T_n^{sel,+}$.

To remove the assumption $T_n - T_n^{sel} = 0$ in (7.36), we consider $\tilde{T}_n^{sel} + T_n - T_n^{sel}$ as an energy competitor for T_n. To make an energy estimate on this object, we notice first that $T_n - T_n^{sel}$ has small pointwise multiplicity (intensity of flow), since its boundary has small mass and it is made by simple paths (see Proposition 7.10(2)). Secondly, we prove that the α-mass, which in general is sub-additive, is "almost additive" between currents which have multiplicities of very different magnitude at every point (Lemma 7.24) and that a suitable lower semi-continuity result holds, involving the restriction of the energy to points with sufficiently large multiplicity (Lemma 7.22).

Finally, we need to remove the assumption (7.37) : this is another delicate point. Given any $\varepsilon > 0$, by choosing n large enough, we may assume that

$$\partial_\pm T_n^{sel,\pm}(B_i^\pm) \leqslant (1+\varepsilon)\partial_\pm T_{opt}(B_i^\pm). \tag{7.46}$$

Then we use the whole $(1+\varepsilon)T_{opt}$ as a transport outside the balls $\cup_{i=1}^{N^\pm} B_i^\pm$. In view of (7.46), this transport might move too much mass from $\cup_{i=1}^{N^-}\partial B_i^-$ to $\cup_{i=1}^{N^+}\partial B_i^+$; however, the amount of mass in excess is small. Hence, we build another transport with small energy which brings back the mass in excess thanks to Proposition 7.25.

7.6 PRELIMINARIES FOR THE PROOF OF THE MAIN THEOREM

7.6.1 *Restriction of curves to open sets*

Let $A \subseteq \mathbb{R}^d$ be a measurable set. For every $\gamma \in \mathrm{Lip}$, we define the first time O_A in which a curve leaves a set A

$$O_A(\gamma) := \inf\{t : \gamma(t) \notin A\},$$

and the last time E_A in which a curve enters in a set A

$$E_A(\gamma) := \sup\{t : \gamma(t) \in A^c\}.$$

We define the restriction of curves on an interval as a map $res : \mathrm{Lip} \times \{(s,t) \in [0,\infty]^2 : s \leqslant t\} \to \mathrm{Lip}$

$$[res(a,b)(\gamma)](t) = \begin{cases} \gamma(a) & \text{for } t \leqslant a \\ \gamma(t) & \text{for } a < t < b \\ \gamma(b) & \text{for } t \geqslant b. \end{cases} \tag{7.47}$$

In the following, we will often consider the restriction of a curve gamma on a certain set, or more in general, the restriction of γ from an initial time depending on γ itself $I(\gamma)$ up to a final time $F(\gamma)$. In this case, we will shorten $res(I,F)(\gamma) := res(I(\gamma),F(\gamma))(\gamma)$.

The previous definition allows us to state an additional property of good decompositions.

Proposition 7.18. *Let $T \in \mathbf{N}_1(\mathbb{R}^d)$ have a good decomposition π as in (7.11), and consider two measurable functions $I, F : \mathrm{Lip} \to \mathbb{R}$ with $I \leqslant F$. Let us assume that $\int_{\mathrm{Lip}} \delta_{\gamma(I(\gamma))} d\pi(\gamma)$ and $\int_{\mathrm{Lip}} \delta_{\gamma(F(\gamma))} d\pi(\gamma)$ are mutually singular. Then the current*

$$\tilde{T} := \int_{\mathrm{Lip}} R_{res(I,F)(\gamma)} d\pi(\gamma) \tag{7.48}$$

has the good decomposition

$$\tilde{T} := \int_{\mathrm{Lip}} R_\gamma \, d\tilde{\pi}(\gamma), \qquad \textit{with } \tilde{\pi} = (\mathrm{res}(I,F))_\sharp \pi.$$

Moreover, if $T = T[E,\tau,\theta]$ *is rectifiable, then* $\tilde{T}$ *can be written as* $\tilde{T} = T[E,\tau,\tilde{\theta}]$*, with* $\tilde{\theta} \leqslant \theta$*.*

Remark 7.19. With the notation of the previous proposition, we notice that the assumptions that $\int_{\mathrm{Lip}} \delta_{\gamma(I(\gamma))} d\pi(\gamma)$ and $\int_{\mathrm{Lip}} \delta_{\gamma(F(\gamma))} d\pi(\gamma)$ are mutually singular in Proposition 7.18 is equivalent to the existence of two disjoint sets $E^-, E^+ \subseteq \mathbb{R}^d$ such that $\gamma(I(\gamma)) \in E^-$ and $\gamma(F(\gamma)) \in E^+$ for π-a.e. γ.

Proof of Proposition 7.18. Proof of the good decomposition property. By Remark 7.19, it is easy to see that

$$\gamma(I(\gamma)) \neq \gamma(F(\gamma)) \quad \text{for } \pi-\text{a.e. } \gamma, \tag{7.49}$$

and so $R_{\mathrm{res}(I,F)(\gamma)}$ is a non-constant simple curve, for π-a.e γ. Moreover, setting $T = \tilde{T} + T^{\mathrm{resid}}$ with

$$T^{\mathrm{resid}} := \int_{\mathrm{Lip}} R_{\mathrm{res}(0,I)(\gamma)} + R_{\mathrm{res}(F,\infty)(\gamma)} \, d\pi(\gamma),$$

we have, by the sub-additivity of the mass

$$\begin{aligned} \mathbb{M}(T) &\leqslant \mathbb{M}(\tilde{T}) + \mathbb{M}(T^{\mathrm{resid}}) \\ &\leqslant \int_{\mathrm{Lip}} \big(\mathbb{M}(R_{\mathrm{res}(0,I)(\gamma)}) + \mathbb{M}(R_{\mathrm{res}(I,F)(\gamma)}) + \mathbb{M}(R_{\mathrm{res}(F,\infty)(\gamma)})\big) \, d\pi(\gamma) \\ &= \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) \, d\pi(\gamma), \end{aligned} \tag{7.50}$$

where in the last line we use that π-a.e. curve γ is simple. Since, by (7.12), equality holds between the first and the last term, every inequality should be an equality and in particular

$$\mathbb{M}(\tilde{T}) = \int_{\mathrm{Lip}} \mathbb{M}(R_{\mathrm{res}(I,F)(\gamma)}) \, d\pi(\gamma) = \int_{\mathrm{Lip}} \mathbb{M}(R_\gamma) \, d\tilde{\pi}(\gamma).$$

In order to obtain the same equality for $\partial\tilde{T}$, we first notice that, by (7.15), it holds

$$\partial\tilde{T} = \int_{\mathrm{Lip}} \partial R_{\mathrm{res}(I,F)(\gamma)} \, d\pi(\gamma) = \int_{\mathrm{Lip}} \big(\delta_{\gamma(F(\gamma))} - \delta_{\gamma(I(\gamma))}\big) \, d\pi(\gamma).$$

By assumption, the measures $\int_{\mathrm{Lip}} \delta_{\gamma(I(\gamma))} d\pi(\gamma)$ and $\int_{\mathrm{Lip}} \delta_{\gamma(F(\gamma))} d\pi(\gamma)$ are mutually singular. Hence,

$$\partial_-\tilde{T} = \int_{\mathrm{Lip}} \delta_{\gamma(I(\gamma))} \, d\pi(\gamma) \quad \text{and} \quad \partial_+\tilde{T} = \int_{\mathrm{Lip}} \delta_{\gamma(F(\gamma))} \, d\pi(\gamma),$$

which yields, by (6.2),

$$\mathbb{M}(\partial\tilde{T}) = 2\mathbb{M}(\partial_-\tilde{T}) = 2\pi(\mathrm{Lip}\,) = \int_{\mathrm{Lip}} \mathbb{M}(\partial R_\gamma) \, d\tilde{\pi}(\gamma).$$

This concludes the proof that (7.48) is a good decomposition.

Proof of the estimate on the multiplicity. By the good decomposition property proved above and the formula (7.16), we get that for $\mathcal{H}^1$-a.e. $x \in E$

$$\begin{aligned} 0 \leqslant \tilde{\theta}(x) = \tilde{\pi}(\{\gamma : x \in \mathrm{Im}(\gamma)\}) &= \pi(\{\gamma : x \in \mathrm{Im}(\mathrm{res}(I, F)(\gamma))\}) \\ &\leqslant \pi(\{\gamma : x \in \mathrm{Im}(\gamma)\}) = \theta(x), \end{aligned} \tag{7.51}$$

where in the inequality we used that if $x \in \mathrm{Im}(\mathrm{res}(I, F)(\gamma))$ then $x \in \mathrm{Im}(\gamma)$. This concludes the proof of the claim. □

7.6.2 *Dimension reduction*

The next lemma is a fundamental tool for the proof of our main result. Indeed it allows us to transport measures which are supported on $(d-1)$-dimensional spheres, decreasing the critical threshold for which we have quantitative upper bounds on the minimal transport energy (see Theorem 7.6). Its proof is a simple combination of Theorem 7.6 and Proposition 7.3.

Lemma 7.20. *Let $\alpha > 1 - \frac{1}{d-1}$. Given two measures μ^- and μ^+ with mass M in $\mathbb{R}^d$ supported on $\partial B(x, r)$, there exists a current $T \in \textbf{TP}(\mu^-, \mu^+)$ such that*

$$\mathbb{M}^\alpha(T) \leqslant C_{\alpha,d} M^\alpha r,$$

where $C_{\alpha,d}$ is a constant depending only on α and d.

Proof. In this proof we denote by $B^{d-1}(0, r)$ the open ball in $\mathbb{R}^{d-1}$ centred at 0 with radius r. Let $p \in \partial B(x, r)$ such that $\mu^\pm(\{p\}) = 0$. It is easy to see that there exists a constant $C := C(d)$ and a 1-Lipschitz function $f : \overline{B^{d-1}(0, Cr)} \to \partial B(x, r) \subset \mathbb{R}^d$ which "wraps" $B^{d-1}(0, Cr)$ onto $\partial B(x, r) \setminus \{p\}$. More precisely, we can require that

$$f^{-1}(\{p\}) = \partial B^{d-1}(0, Cr) \qquad \text{and} \qquad f \text{ is injective on } B^{d-1}(0, Cr).$$

Let $\nu^\pm := [(f|_{\partial B(x,r)\setminus\{p\}})^{-1}]_\sharp \mu^\pm$ and observe that $f_\sharp \nu^\pm = \mu^\pm$. By Theorem 7.6, there exists $S \in \mathbf{TP}(\nu^-, \nu^+)$ with $\mathbb{M}^\alpha(S) \leqslant C_{\alpha,d} M^\alpha 2r$.

We observe that $T := f_\sharp S$ belongs to $\mathbf{TP}(\mu^-, \mu^+)$, indeed

$$\partial(f_\sharp S) = f_\sharp \partial S = f_\sharp((f|_{\partial B(x,r)\setminus\{p\}})_\sharp^{-1} \mu^+ - (f|_{\partial B(x,r)\setminus\{p\}})_\sharp^{-1} \mu^-) = \mu^+ - \mu^- = \partial S,$$

and trivially T is supported on $\partial B(x, r)$. The estimate on the α-mass of T follows immediately from Proposition 7.3. □

7.6.3 *Covering results*

In this subsection we prove two elementary covering results. Referring to the notation introduced in Section 7.5, Lemma 7.21 allows us to cover the sets $A^\pm$ with balls satisfying (7.34) such that for every $n \in \mathbb{N}$ almost no curve in the representation of T_n begins or ends on the corresponding spheres. With Lemma 7.22 we want to guarantee that it is possible to cover the sets $\mathrm{spt}(\mu^-)$ and $\mathrm{spt}(\mu^+)$, which by assumption are disjoint, with two disjoint families of small balls. This time we do not require any smallness assumption on the sum of the radii, but we want to control the number of balls in each family.

Lemma 7.21. *Consider a family of* 1*-currents* $T, T', (T_n)_{n\in\mathbb{N}} \in \mathbf{N}_1(\mathbb{R}^d) \cap \mathbf{R}_1(\mathbb{R}^d)$*, such that* $\mathbb{M}^\alpha(T), \mathbb{M}^\alpha(T') < +\infty$ *and* $\partial_\pm T = \partial_\pm T'$*. Given a set* A *such that* $\mathcal{H}^1(A) = 0$*, and* $\varepsilon > 0$*, there exists a covering of* A *with open balls* $(B(x_i, r_i))_{i\in\mathbb{N}}$ *such that*

$$\partial_\pm T(\partial B(x_i, r_i)) = \partial_\pm T_n(\partial B(x_i, r_i)) = 0 \qquad \textit{for every } i, n \in \mathbb{N},$$

$$\mathbb{M}^\alpha\Big(T \llcorner \bigcup_{i\in\mathbb{N}} \overline{B(x_i, r_i)}\Big) < \varepsilon \qquad \textit{and} \qquad \mathbb{M}^\alpha\Big(T' \llcorner \bigcup_{i\in\mathbb{N}} \overline{B(x_i, r_i)}\Big) < \varepsilon, \tag{7.52}$$

and

$$\sum_{i=1}^{\infty} r_i < \varepsilon. \tag{7.53}$$

Proof. We define on $\mathbb{R}^d$ the finite measure ν by

$$\nu(E) = \mathbb{M}^\alpha(T \llcorner E) + \mathbb{M}^\alpha(T' \llcorner E) \quad \text{for every Borel set } E$$

and we observe that ν vanishes on $\mathcal{H}^1$-null sets.

Since $\mathcal{H}^1(A) = 0$, for every $j \in \mathbb{N}$ we can find a covering of A with balls $\{B(x_i^{(j)}, r_i^{(j)})\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} r_i^{(j)} < \frac{1}{2^{j+1}},$$

and moreover, since for every point x there are only countably many radii r such that $\partial_\pm T(\partial B(x, r)) \neq 0$ or $\partial_\pm T_n(\partial B(x, r)) \neq 0$ for some n, then we can also assume (possibly enlarging slightly the previous radii) that

$$\partial_\pm T(\partial B(x_i^{(j)}, r_i^{(j)})) = \partial_\pm T_n(\partial B(x_i^{(j)}, r_i^{(j)})) = 0 \qquad \text{for every } i, n \in \mathbb{N}.$$

We define

$$A^{(j)} = \bigcup_{i\in\mathbb{N}} \overline{B(x_i^{(j)}, 2r_i^{(j)})}.$$

We consider the decreasing sequence of sets and their intersection

$$(B^{(j)})_{j\in\mathbb{N}} := \bigcup_{j'\geqslant j} A^{(j')}, \qquad B = \bigcap_{j\in\mathbb{N}} B^{(j)}.$$

We notice that $A^{(j)} \subseteq B^{(j)}$ for every $j \in \mathbb{N}$ and that $\mathcal{H}^1(B) = 0$, because B can be covered with each $B^{(j)}$, which in turn is made by balls whose radii satisfy the estimate

$$\sum_{j'\geqslant j} \sum_{i\in\mathbb{N}} r_i^{(j')} < \sum_{j'\geqslant j} \frac{1}{2^{j'+1}} = \frac{1}{2^j}.$$

We consequently have on the decreasing sequence of sets $(B^{(j)})_{j\in\mathbb{N}}$:

$$\lim_{j\to\infty} \nu(B^{(j)}) = \nu(\cap_j B^{(j)}) = \nu(B)$$

and we conclude that $\nu(B) = 0$ and that

$$\lim_{j\to\infty} \nu(A^{(j)}) \leqslant \lim_{j\to\infty} \nu(B^{(j)}) = 0.$$

Therefore, choosing j large enough, the covering $(B(x_i^{(j)}, r_i^{(j)}))_{i\in\mathbb{N}}$ satisfies the conditions in (7.52), (7.53). □

Lemma 7.22. *Given $r > 0$ and $K \subset X$, with K compact. There exists a finite number $M := M(X, r)$ and a family of balls $\{B(x_i, r_i)\}_{i=1}^M$, covering K, such that*

$$r_i < \frac{r}{3}, \qquad x_i \in K.$$

Proof. We cover K with balls $B(x, r/4)$, $x \in K$ and, by Vitali's covering theorem, we can extract a finite sub-covering, indexed by $\{1, ..., M\}$ such that the balls $\{B(x_j, r/20)\}_{j=1,...,M}$ are disjoint and the balls $\{B(x_j, r/4)\}_{j=1,...,M}$ cover K. By the disjointness of $\{B(x_j, r/20)\}_{j=1,...,M}$ and since these balls are all contained in $U_r(X) := \{y \in \mathbb{R}^d : \mathrm{dist}(y, X) < r\}$, it follows that

$$M|B(0, r/20)| \leqslant |U_r(X)|,$$

which completes the proof of the lemma. □

7.6.4 *A semi-continuity and a quasi-additivity result*

In this subsection we collect two results which allow us to get rid of the simplifying assumption (7.36) in the sketch of Section 7.5. Lemma 7.23 improves Theorem 7.13(1), allowing us to consider in the right hand side of the inequality (7.21) only the portion of the currents T_n which have sufficiently high multiplicity. Lemma 7.24 states that the α-mass is "quasi-additive" if the two addenda have multiplicities of different orders of magnitude.

Lemma 7.23. *Let $C > 0$, $A \subseteq \mathbb{R}^d$ an open set, and let $T' = T[E', \tau', \theta'] \in \mathbf{R}_1(\mathbb{R}^d)$ and $T := T[E, \tau, \theta] \in \mathbf{R}_1(\mathbb{R}^d)$ be rectifiable 1-currents with*

$$\mathbb{M}^\alpha(T'), \mathbb{M}^\alpha(T) \leqslant C. \tag{7.54}$$

Then, for every $\varepsilon > 0$ there exists $\delta := \delta(d, \alpha, \varepsilon, C, A, T) > 0$ (independent of T') such that, if $\mathbb{F}(T - T') \leqslant \delta$,

$$\mathbb{M}^\alpha(T' \llcorner \{x \in A : \theta'(x) > \delta\}) \geqslant \mathbb{M}^\alpha(T \llcorner A) - \varepsilon. \tag{7.55}$$

Proof. For every $\delta > 0$, by (7.54) it holds

$$\mathbb{M}(T' \llcorner \{\theta' \leqslant \delta\}) < \delta^{1-\alpha} \mathbb{M}^\alpha(T' \llcorner \{\theta' \leqslant \delta\}) < \delta^{1-\alpha} C. \tag{7.56}$$

Hence,

$$\begin{aligned}\mathbb{F}(T - T'\llcorner\{\theta' > \delta\}) &\leqslant \mathbb{F}(T - T') + \mathbb{F}(T' - T'\llcorner\{\theta' > \delta\}) \\ &= \mathbb{F}(T - T') + \mathbb{F}(T'\llcorner\{\theta' \leqslant \delta\}) \\ &\leqslant \mathbb{F}(T - T') + \mathbb{M}(T'\llcorner\{\theta' \leqslant \delta\}) \\ &\leqslant \mathbb{F}(T - T') + C\delta^{1-\alpha} \leqslant \delta + C\delta^{1-\alpha}.\end{aligned} \tag{7.57}$$

By the lower semi-continuity of the α-mass with respect to the flat convergence (as stated in Theorem 7.13(1)), there exists $\delta_0 := \delta_0(d, \alpha, \varepsilon, A, T)$ such that for any rectifiable 1-current $\tilde{T}$ satisfying $\mathbb{F}(\tilde{T} - T) \leqslant \delta_0$ we have $\mathbb{M}^\alpha(\tilde{T}\llcorner A) \geqslant \mathbb{M}^\alpha(T\llcorner A) - \varepsilon$. We conclude the proof choosing δ sufficiently small so that $\delta + C\delta^{1-\alpha} \leqslant \delta_0$. □

Lemma 7.24. *Let $\varepsilon \in (0, 1/4)$, $T_1 = T[E_1, \tau_1, \theta_1], T_2 = T[E_2, \tau_2, \theta_2] \in \mathbf{R}_1(\mathbb{R}^d)$ be rectifiable 1-currents with $\theta_1 < \varepsilon\theta_2$, $\mathcal{H}^1$-a.e. on $E_1 \cap E_2$. Then*

$$(1 + 4\varepsilon^\alpha)\mathbb{M}^\alpha(T_1 + T_2) \geqslant \mathbb{M}^\alpha(T_1) + \mathbb{M}^\alpha(T_2). \tag{7.58}$$

Proof. Firstly we observe that on $E_1 \cap E_2$ we have

$$2\varepsilon(\theta_2 - \theta_1) \geqslant \theta_1; \quad (1 + 2\varepsilon)(\theta_2 - \theta_1) \geqslant \theta_2. \tag{7.59}$$

Now we compute

$$\begin{aligned}(1{+}4\varepsilon^\alpha)\mathbb{M}^\alpha(T_1 + T_2) &= (1 + 4\varepsilon^\alpha)\mathbb{M}^\alpha(T_1\llcorner(E_1 \setminus E_2)) \\ &\quad + (1 + 4\varepsilon^\alpha)\mathbb{M}^\alpha(T_2\llcorner(E_2 \setminus E_1)) + (1 + 4\varepsilon^\alpha)\mathbb{M}^\alpha((T_1 + T_2)\llcorner(E_1 \cap E_2)) \\ &\geqslant \mathbb{M}^\alpha(T_1\llcorner(E_1 \setminus E_2)) + \mathbb{M}^\alpha(T_2\llcorner(E_2 \setminus E_1)) \\ &\quad + \big((2\varepsilon)^\alpha + (1 + 2\varepsilon)^\alpha\big)\mathbb{M}^\alpha((T_1 + T_2)\llcorner(E_1 \cap E_2)).\end{aligned}$$

We estimate the last term thanks to (7.59) to get

$$\begin{aligned}&\big((2\varepsilon)^\alpha + (1 + 2\varepsilon)^\alpha\big)\mathbb{M}^\alpha((T_1 + T_2)\llcorner(E_1 \cap E_2)) \\ &\qquad \geqslant \mathbb{M}^\alpha(T_1\llcorner(E_1 \cap E_2)) + \mathbb{M}^\alpha(T_2\llcorner(E_1 \cap E_2)).\end{aligned}$$

Putting together the previous two inequalities, we get (7.58). □

7.6.5 *Absolute continuity of the transportation cost*

The next proposition is the fundamental tool to get rid of the simplifying assumption (7.38) in the sketch of Section 7.5. It ensures that if there exists a traffic path of finite cost transporting a measure μ^- onto a measure μ^+, then a transportation between two "small" sub-measures of μ^- and μ^+ of equal mass is cheap.

Proposition 7.25. *Let $\mu^-, \mu^+ \in \mathcal{M}_+(X)$, be non-trivial measures with $\mu^-(X) = \mu^+(X) < \infty$. Assume*

$$\operatorname{spt}(\mu^-) \cap \operatorname{spt}(\mu^+) = \emptyset,$$

with $\mathbb{M}^\alpha(\mu^-, \mu^+) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of measures $\nu^- \leqslant \mu^-$ and $\nu^+ \leqslant \mu^+$ verifying

$$\nu^-(X) = \nu^+(X) \leqslant \delta,$$

then $\mathbb{M}^\alpha(\nu^-, \nu^+) \leqslant \varepsilon$.

Proof. Without loss of generality, we may assume $\mu^-, \mu^+ \in \mathcal{P}(X)$. By assumption, there exists $T \in \mathbf{OTP}(\mu^-, \mu^+)$, such that $\mathbb{M}^\alpha(T) < +\infty$.

Let $T = \int_{\mathrm{Lip}} R_\gamma \, d\pi(\gamma)$ be a good decomposition of T and define the finite measures $\pi^\pm \in \mathcal{M}_+(\mathrm{Lip})$, prescribing their Radon–Nikodým densities w.r.t. π, as

$$d\pi^-(\gamma) := \frac{d\nu^-}{d\mu^-}(\gamma(0))\, d\pi(\gamma), \qquad d\pi^+(\gamma) := \frac{d\nu^+}{d\mu^+}(\gamma(\infty))\, d\pi(\gamma).$$

We denote

$$T^\pm = T[E^\pm, \tau^\pm, \theta^\pm] := \int_{\mathrm{Lip}} R_\gamma \, d\pi^\pm(\gamma). \tag{7.60}$$

Let us consider $\delta > 0$ as fixed. For the moment we only require that $\delta < \delta_0 := \delta(\varepsilon/4)$ of Proposition 7.10(4). Further restrictions will be given later. Since $\pi^\pm(\mathrm{Lip}) = \nu^\pm(X) \leqslant \delta$, from Proposition 7.10 (3) and (4) we deduce that the decompositions in (7.60) are good and that

$$\mathbb{M}^\alpha(T^\pm) \leqslant \frac{\varepsilon}{4}. \tag{7.61}$$

By (7.15) we can write the boundaries of $T^\pm$ in terms of the decomposition as

$$\partial_- T^\pm = \int_{\mathrm{Lip}} \delta_{\gamma(0)} \, d\pi^\pm(\gamma) \qquad \text{and} \qquad \partial_+ T^\pm = \int_{\mathrm{Lip}} \delta_{\gamma(\infty)} \, d\pi^\pm(\gamma). \tag{7.62}$$

We apply Lemma 7.22 twice to $K := \mathrm{spt}(\mu^\pm)$ and $r := \frac{1}{3}\mathrm{dist}(\mathrm{spt}(\mu^-), \mathrm{spt}(\mu^+))$ to find a finite covering of $\mathrm{spt}(\mu^\pm)$ made by at most $M(X, r)$ open balls

$$B_i^\pm := B(x_i^\pm, r_i^\pm) \qquad i = 1, \dots, M^\pm.$$

For every $i = 1, \dots, M^\pm$ let us define

$$C^\pm := \bigcup_i B_i^\pm.$$

By the choice of r, the sets C^+ and C^- are disjoint. Hence, since $\mathrm{spt}(\partial_\pm T) \subseteq C^\pm$ and since (7.62) is in force, then $\pi^\pm$-a.e. $\gamma \in \mathrm{Lip}$ verifies

$$\gamma(0) \in C^- \qquad \text{and} \qquad \gamma(\infty) \in C^+. \tag{7.63}$$

We define the rectifiable 1-currents

$$\begin{aligned} T^{\mathrm{cut},-} &= T[E^{\mathrm{cut},-}, \tau^{\mathrm{cut},-}, \theta^{\mathrm{cut},-}] := \int_{\mathrm{Lip}} R_{\mathrm{res}(0, O_{C^-})(\gamma)} \, d\pi^-(\gamma), \\ T^{\mathrm{cut},+} &= T[E^{\mathrm{cut},+}, \tau^{\mathrm{cut},+}, \theta^{\mathrm{cut},+}] := \int_{\mathrm{Lip}} R_{\mathrm{res}(E_{C^+}, \infty)(\gamma)} \, d\pi^+(\gamma). \end{aligned} \tag{7.64}$$

By Proposition 7.18, (7.64) are good decompositions. Here we use a little abuse of notation, since the good decomposition of $T^{\mathrm{cut},-}$ would be the push-forward measure

$$\big(\mathrm{res}(0, O_{C^-})(\cdot)\big)_\sharp \pi^-$$

and similarly for $T^{cut,+}$. In particular, by point (1) of Proposition 7.10 it holds

$$\partial_- T^{cut,-} = \int_{\mathrm{Lip}} \delta_{\gamma(0)} d\pi^-(\gamma), \qquad \partial_+ T^{cut,-} = \int_{\mathrm{Lip}} \delta_{\gamma(O_{C^-})} d\pi^-(\gamma) \tag{7.65}$$

Hence we deduce

$$\mathrm{spt}(\partial_+ T^{cut,-}) \subseteq \partial C^- \qquad \text{and} \qquad \mathrm{spt}(\partial_- T^{cut,+}) \subseteq \partial C^+.$$

By the good decomposition property of $T^{cut,-}$ and of T^- and by Proposition 7.18 for $\mathcal{H}^1$-a.e. $x \in E^- \cap E^{cut,-}$ we have that

$$\theta^{cut,-}(x) \leqslant \theta^-(x). \tag{7.66}$$

Thanks to (7.61), we deduce that $T^{cut,\pm}$ have small energy

$$\mathbb{M}^\alpha(T^{cut,-}) = \int_{E^{cut,-}} (\theta^{cut,-})^\alpha d\mathcal{H}^1 \leqslant \int_{E^-} (\theta^-)^\alpha d\mathcal{H}^1 = \mathbb{M}^\alpha(T^-) \leqslant \frac{\varepsilon}{4}. \tag{7.67}$$

With similar computations we can prove the same energy estimate for $T^{cut,+}$.

Let $\{y_1^-, ..., y_{M^-}^-\}_{i=1,...,M^-} \subseteq \mathbb{R}^d$ and $\{y_1^+, ..., y_{M^+}^+\}_{i=1,...,M^+} \subseteq \mathbb{R}^d$ be two sets of distinct points such that $y_i^\pm \in \partial B_i^\pm$ for every $i = 1, ..., M^\pm$. For every $i = 1, ..., M^-$ we define the weight $w_i^\pm \in (0, \infty)$ as

$$w_i^- := (\partial_+ T^{cut,-})\Big(\partial B_i^- \setminus \bigcup_{j=1}^{i-1} \partial B_j^-\Big)$$

and

$$w_i^+ := (\partial_- T^{cut,+})\Big(\partial B_i^+ \setminus \bigcup_{j=1}^{i-1} \partial B_j^+\Big).$$

We consider the measures $\sigma^\pm := \sum_{i=1}^{M^\pm} w_i^\pm \delta_{y_i^\pm}$, whose total mass is equal to $\nu^\pm(X) \leqslant \delta$. Indeed we proved in (7.65), that $\partial_- T^{cut,-} = \partial_- T_-$ and consequently

$$\sigma^-(X) = \partial_+ T^{cut,-}(X) = \partial_- T^{cut,-}(X) = \partial_- T_-(X) = \nu^-(X) \leqslant \delta$$

and analogously

$$\sigma^+(X) = \partial_- T^{cut,+}(X) = \partial_+ T^{cut,+}(X) = \partial_+ T(X) = \nu^+(X) \leqslant \delta.$$

We claim that there exists $T^{conn,-} \in \mathbf{TP}(\partial_+ T^{cut,-}, \sigma^-)$ with

$$\mathbb{M}^\alpha(T^{conn,-}) \leqslant C(d, \alpha, X, r)\delta.$$

Similarly, we claim that there exists $T^{conn,+} \in \mathbf{TP}(\partial_- T^{cut,+}, \sigma^+)$ with

$$\mathbb{M}^\alpha(T^{conn,+}) \leqslant C(d, \alpha, X, r)\delta.$$

Indeed let us consider for every $i = 1, ..., M^-$ an optimal traffic path

$$T_i^{conn,-} \in \mathbf{OTP}\big((\partial_+ T^{cut,-}) \llcorner \big(\partial B_i^- \setminus \cup_{j=1}^{i-1} \partial B_j^-\big), w_i \delta_{y_i^-}\big)$$

and observe that, by Lemma 7.20

$$\mathbb{M}^{\alpha}(T_i^{conn,-}) \leqslant C(d,\alpha)\delta r.$$

If we consider now

$$T^{conn,-} := \sum_{i=1}^{M^-} T_i^{conn,-},$$

we notice that $T^{conn,-} \in \mathbf{TP}(\partial_+ T^{cut,-}, \sigma^-)$ and by the sub-additivity of the α-mass (7.6) we obtain that

$$\mathbb{M}^{\alpha}(T^{conn,-}) \leqslant \sum_{i=1}^{M^-} \mathbb{M}^{\alpha}(T_i^{conn,-}) \leqslant M^-(X,r)C(d,\alpha)\delta r \leqslant C(d,\alpha,X,r)\delta$$

and this proves the claim.

Finally we observe that there exists $T^{graph} \in \mathbf{TP}\big(\sigma^-, \sigma^+\big)$ with

$$\mathbb{M}^{\alpha}(T^{graph}) \leqslant \delta^{\alpha} C(d,X).$$

The simplest way to find such traffic path is to connect all the points in the support of $\sigma^{\pm}$ to a fixed point in X. The estimate of its α-mass is trivial. Overall, we find that

$$T^{new} := T^{cut,-} + T^{conn,-} + T^{graph} + T^{conn,+} + T^{cut,+} \in \mathbf{TP}(\nu^-, \nu^+)$$

and its energy is estimated using the sub-additivity (7.6) and the previous estimates (observing that $\delta \leqslant \delta^{\alpha}$ for $\delta \leqslant 1$)

$$\mathbb{M}^{\alpha}(T^{new}) \leqslant C(d,\alpha,X,r)\delta^{\alpha} + \frac{\varepsilon}{2}.$$

By choosing δ sufficiently small, we obtain that the last quantity is less than or equal to ε. This concludes the proof of the lemma.

$\square$

Corollary 7.26. *Let* $\mu^-, \mu^+ \in \mathcal{P}(X)$. *Assume*

$$\operatorname{spt}(\mu^-) \cap \operatorname{spt}(\mu^+) = \emptyset,$$

with $\mathbb{M}^{\alpha}(\mu^-, \mu^+) < \infty$. *Then for every pair of sequences* $(\mu_n^-)_{n\in\mathbb{N}}$ *and* $(\mu_n^+)_{n\in\mathbb{N}}$ *with* $\mu_n^-(\mathbb{R}^d) = \mu_n^+(\mathbb{R}^d)$, $\mu_n^- \leqslant \mu^-$, $\mu_n^+ \leqslant \mu^+$ *for every* $n \in \mathbb{N}$ *and with*

$$\lim_{n\to\infty} \mu^-(X) - \mu_n^-(X) = 0,$$

we have that

$$\lim_{n\to\infty} \mathbb{M}^{\alpha}(\mu_n^-, \mu_n^+) = \mathbb{M}^{\alpha}(\mu^-, \mu^+).$$

Proof. By the lower semi-continuity of the α-mass (Theorem 7.13(1)), we only need to show that

$$\limsup_{n\to\infty} \mathbb{M}^\alpha(\mu_n^-, \mu_n^+) \leqslant \mathbb{M}^\alpha(\mu^-, \mu^+). \tag{7.68}$$

Indeed, if we assume (7.68), by Corollary 7.14, and by the compactness of normal currents (see [50, 4.2.17(1)]) we can consider a sequence of optimizers $(T_{n_k})_{k\in\mathbb{N}}$, where $T_{n_k} \in \mathbf{OTP}(\mu_{n_k}^-, \mu_{n_k}^+)$ converge to a traffic path $T \in \mathbf{TP}(\mu^-, \mu^+)$ with finite cost and

$$\lim_{k\to\infty} \mathbb{M}^\alpha(T_{n_k}) = \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n).$$

Hence we compute

$$\mathbb{M}^\alpha(\mu^-, \mu^+) \leqslant \mathbb{M}^\alpha(T) \overset{(7.21)}{\leqslant} \liminf_{k\to\infty} \mathbb{M}^\alpha(T_{n_k}) = \liminf_{n\to\infty} \mathbb{M}^\alpha(T_n).$$

In order to prove (7.68), we let $T \in \mathbf{OTP}(\mu^-, \mu^+)$. Since by assumption the measures $\mu^- - \mu_n^-$ and $\mu^+ - \mu_n^+$ are non-negative, are converging to 0 and, for each fixed n, they have the same mass, we deduce by point (4) of Proposition 7.10 that, denoting by T_n' any optimal path in $\mathbf{OTP}(\mu^- - \mu_n^-, \mu^+ - \mu_n^+)$,

$$\lim_{n\to\infty} \mathbb{M}^\alpha(T_n') = 0.$$

Let $T_n = T - T_n' \in \mathbf{TP}(\mu_n^-, \mu_n^+)$. By the sub-additivity of the α-mass (7.6)

$$\mathbb{M}^\alpha(T_n) \leqslant \mathbb{M}^\alpha(T) + \mathbb{M}^\alpha(T_n').$$

Letting $n \to \infty$ we obtain (7.68). □

Remark 7.27. From this observation the stability follows as in the case $\alpha > 1 - 1/d$ as soon as the approximating sequences are sub-measures of μ^- and μ^+ respectively. In particular, if μ^- is a Dirac delta and μ^+ is an atomic measure, then an optimal traffic path connecting μ^- to μ^+ can be obtained as the limit of the optimal traffic paths connecting the correct "rescaled" measure of μ^- to the discrete measure obtained restricting μ^+ to suitable sets of finitely many points.

7.7 PROOF OF THE MAIN THEOREM

Up to rescaling, we can assume that μ^- and μ^+ are probability measures. Moreover, without loss of generality we can assume that μ_n^- and μ_n^+ are also probability measures and they are mutually singular. Indeed, assuming the validity of Theorem 7.1 in this special case, it is easy to deduce its validity in general, using the following argument. Denoting ν_n^- and ν_n^+ respectively the negative and the positive part of the measure $\mu_n^+ - \mu_n^-$, since the supports of μ^- and μ^+ are disjoint, we have that $\nu_n^- \rightharpoonup \mu^-$ and $\nu_n^+ \rightharpoonup \mu^+$. Moreover, since the ambient is a compact set, $\nu_n^\pm(X) \to \mu^\pm(X) = 1$. Now, denoting $\eta_n := \nu_n^-(X) = \nu_n^+(X)$, we are in the poisiton to apply Theorem 7.1 in the special case above for the approximating measures $\eta_n^{-1}\nu_n^\pm$, the limiting measures $\mu^\pm$, the optimal traffic paths $\eta_n^{-1}T_n$ and the limit traffic path T. Since $\eta_n \to 1$ and $T_n \rightharpoonup T$ is in force, then $\eta_n^{-1}T_n \rightharpoonup T$ is satisfied.

By contradiction, we assume T is not optimal, i.e.

$$\mathbb{M}^\alpha(T) \geqslant \mathbb{M}^\alpha(T_{opt}) + \Delta, \tag{7.69}$$

for some $\Delta > 0$ and for some T_{opt} with $\partial_\pm T_{opt} = \partial_\pm T$.

Step 1: construction of the coverings of A^- *and* A^+. Let $C_{\alpha,d}$ be the constant in Lemma 7.20. We claim that there exists a (finite or countable) family of balls $\{B_i^\pm = B(x_i^\pm, r_i^\pm)\}_{i\in I^\pm}$ covering respectively $A^- \cap \operatorname{spt}(\mu^-)$ and $A^+ \cap \operatorname{spt}(\mu^+)$, such that

$$\overline{\Big(\bigcup_{i\in I^-} B_i^-\Big)} \cap \overline{\Big(\bigcup_{i\in I^+} B_i^+\Big)} = \emptyset, \tag{7.70}$$

$$\sum_{i\in I^\pm} r_i^\pm < \frac{\Delta}{128 C_{\alpha,d}}, \tag{7.71}$$

$$\mathbb{M}^\alpha\Big(T \llcorner \bigcup_{i\in I^\pm} \overline{B_i^\pm}\Big) \leqslant \frac{\Delta}{128}, \qquad \mathbb{M}^\alpha\Big(T_{opt} \llcorner \bigcup_{i\in I^\pm} \overline{B_i^\pm}\Big) \leqslant \frac{\Delta}{128}, \tag{7.72}$$

$$\mu^\pm(\partial B_i^\pm) = 0, \qquad \mu_n^\pm(\partial B_i^\pm) = 0 \qquad \forall i \in I^\pm,\ n \in \mathbb{N}. \tag{7.73}$$

For simplicity, we assume $I^\pm$ to be either $\mathbb{N}$ or a set of the form $\{1, \dots, M^\pm\}$. Finally, up to removing certain balls, we can assume the two coverings to be not redundant, namely, we can assume that

$$\mu^\pm\Big(B_i^\pm \setminus \bigcup_{1\leqslant j<i} B_j^\pm\Big) \neq 0, \qquad \forall i \in I^\pm. \tag{7.74}$$

Since we have removed only balls that do not carry measure, the new set of balls still covers $A^- \cap \operatorname{spt}(\mu^-)$ and $A^+ \cap \operatorname{spt}(\mu^+)$ up to a set of $\mu^\pm$-measure 0.

We now prove the claim of this Step 1. Let d_0 be the distance between $\operatorname{spt}(\mu^-)$ and $\operatorname{spt}(\mu^+)$, which is positive since the supports $\operatorname{spt}(\mu^-)$ and $\operatorname{spt}(\mu^+)$ are compact and disjoint. Applying Lemma 7.21 with $\varepsilon = \min\{\Delta/(128C_{\alpha,d}), \Delta/128, d_0/4\}$ and $T' = T_{opt}$, we can find two finite coverings satisfying (7.70), (7.71), (7.72), and (7.73).

Step 2: choice of $N^\pm$. Let $\varepsilon_1 > 0$ to be chosen later. We choose $N^\pm$ satisfying

$$\mu^\pm\Big(\bigcup_{j=1}^{N^\pm} B_j^\pm\Big) > 1 - \frac{\varepsilon_1}{4}.$$

Step 3: choice of n. Let $\varepsilon_2 > 0$ to be chosen later. For every $i \in I^\pm$ we define

$$C_i^\pm = B_i^\pm \setminus \Big(\cup_{j=1}^{i-1} B_j^\pm\Big).$$

By (7.74) the coverings are not redundant, that is, for every $i \in I^{\pm}$,

$$\mu^{\pm}(C_i^{\pm}) > 0. \tag{7.75}$$

We claim that we can fix n large enough so that the following properties hold:

$$\mathbb{F}(T_n - T) \leqslant \varepsilon_2, \tag{7.76}$$

$$\mu_n^{\pm}(C_i^{\pm}) \leqslant (1+\varepsilon_2)\mu^{\pm}(C_i^{\pm}), \qquad \forall i = 1, \cdots, N^{\pm}, \tag{7.77}$$

$$\mu_n^{\pm}\Big(\mathbb{R}^d \setminus \bigcup_{i=1}^{N^{\pm}} C_i^{\pm}\Big) \leqslant \frac{\varepsilon_1}{2}. \tag{7.78}$$

Indeed, since $T_n \in \mathbf{OTP}(\mu_n^-, \mu_n^+)$, by Theorem 7.7, Theorem 7.9 and Proposition 7.10(2), $T_n = T_n[E_n, \tau_n, \theta_n]$ admits a good decomposition $\pi_n \in \mathcal{P}(\mathrm{Lip}\,)$ and its multiplicity θ_n verifies $\theta_n \leqslant 1$. Consequently we get

$$\mathbb{M}(T_n) = \int_{E_n} \theta_n(x) d\mathcal{H}^1(x) \leqslant \int_{E_n} \theta_n^{\alpha}(x) d\mathcal{H}^1(x) \leqslant \mathbb{M}^{\alpha}(T_n) \leqslant C.$$

Moreover

$$\mathbb{M}(\partial T_n) = \mu_n^-(\mathbb{R}^d) + \mu_n^+(\mathbb{R}^d) = 2 < +\infty.$$

By the discussion after the definition of flat norm (6.7), the uniform bounds on the mass of the currents T_n and on the mass of their boundaries guarantees that the weak* convergence implies (7.76), for n sufficiently large. By (7.73) and since $\mu_n^{\pm} = \partial_{\pm} T_n$ weakly converges to $\mu^{\pm} = \partial_{\pm} T$, we observe that

$$\mu^{\pm}(\partial C_i) = 0, \qquad \forall i = 1, \cdots, N^{\pm}$$

and therefore

$$\lim_{n \to \infty} \mu_n^{\pm}(C_i^{\pm}) = \mu^{\pm}(C_i^{\pm}), \qquad \forall i = 1, \cdots, N^{\pm}.$$

Since the right-hand side in the previous equality is non-zero thanks to (7.75), we obtain (7.77) for n large enough.

We fix n large enough to satisfy the conditions in this step. Up to the end of the proof, we will always refer to this choice of n.

Step 4: good decomposition of T_n *and selection.* Let us define

$$\pi_n^{sel} := \pi_n \llcorner \Big\{\gamma : \gamma(0) \in \bigcup_{i=1}^{N^-} C_i^- \text{ and } \gamma(\infty) \in \bigcup_{i=1}^{N^+} C_i^+\Big\}. \tag{7.79}$$

Let us consider T_n^{sel} to be the 1-dimensional current obtained from T_n selecting only those curves that begin inside the first N^- balls and end inside the first N^+ balls, i.e.

$$T_n^{sel} := \int_{\mathrm{Lip}} R_\gamma \, d\pi_n^{sel}(\gamma).$$

Notice that, by Proposition 7.10(3), π_n^{sel} is a good decomposition of T_n^{sel}; in particular by Proposition 7.10(1)

$$\partial_- T_n^{sel} = \int_{\mathrm{Lip}} \delta_{\gamma(0)}\, d\pi_n^{sel}(\gamma)$$

is supported on $\bigcup_{i=1}^{N^-} C_i^-$ and it satisfies $\partial_- T_n^{sel} \leqslant \partial_- T_n = \mu_n^-$.

For the same reason, $\pi_n - \pi_n^{sel}$ is a good decomposition of $T_n - T_n^{sel}$ and, denoting by $\tilde\theta_n$ the multiplicity of $T_n - T_n^{sel}$, we have the bound

$$\tilde\theta_n \leqslant \min\{\theta_n, (\pi_n - \pi_n^{sel})(\mathrm{Lip}\,)\}. \tag{7.80}$$

Next we estimate

$$\begin{aligned}(\pi_n - \pi_n^{sel})(\mathrm{Lip}\,) &= \pi_n\Big(\Big\{\gamma : \gamma(0) \notin \bigcup_{i=1}^{N^-} C_i^- \text{ or } \gamma(\infty) \notin \bigcup_{i=1}^{N^+} C_i^+\Big\}\Big) \\ &\leqslant \pi_n\Big(\Big\{\gamma : \gamma(0) \notin \bigcup_{i=1}^{N^-} C_i^-\Big\}\Big) + \pi_n\Big(\Big\{\gamma : \gamma(\infty) \notin \bigcup_{i=1}^{N^+} C_i^+\Big\}\Big).\end{aligned} \tag{7.81}$$

By the good decomposition of T_n (and in particular by (7.15)) for every Borel set $A \subseteq \mathbb{R}^d$

$$\pi_n\big(\{\gamma : \gamma(0) \in A\}\big) = \partial_- T_n(A) = \mu_n^-(A);$$

hence, by (7.78)

$$\pi_n\Big(\Big\{\gamma : \gamma(0) \notin \bigcup_{i=1}^{N^-} C_i^-\Big\}\Big) = \mu_n^-\Big(\Big(\bigcup_{i=1}^{N^-} C_i^-\Big)^c\Big) \leqslant \varepsilon_1/2.$$

A similar inequality holds for the second term in the right-hand side of (7.81). Overall, it follows

$$(\pi_n - \pi_n^{sel})(\mathrm{Lip}\,) \leqslant \varepsilon_1. \tag{7.82}$$

We also notice that T_n and T_n^{sel} are close in flat norm by (7.80) and (7.82)

$$\begin{aligned}\mathbb{F}(T_n - T_n^{sel}) \leqslant \mathbb{M}(T_n - T_n^{sel}) &= \int_{E_n} \tilde\theta_n\, d\mathcal{H}^1 \\ &\leqslant \varepsilon_1^{1-\alpha} \int_{E_n} \tilde\theta_n^\alpha d\mathcal{H}^1 \leqslant \varepsilon_1^{1-\alpha} \int_{E_n} \theta_n^\alpha d\mathcal{H}^1 \leqslant C\varepsilon_1^{1-\alpha}.\end{aligned} \tag{7.83}$$

Step 5: restriction of T_n *inside the covering.* We decompose π_n^{sel} into the sum of finitely many, pairwise singular measures $\pi_{n,i}^{sel,-}$, according to the starting points of the associated curves, i.e. for every $i = 1, ..., N^-$ we denote

$$\pi_{n,i}^{sel,-} := \pi_n^{sel} \llcorner \Big\{\gamma : \gamma(0) \in C_i^-\Big\}, \tag{7.84}$$

and we notice that, using (7.79),

$$\sum_{i=1}^{N^-} \pi_{n,i}^{sel,-} = \pi_n^{sel}. \tag{7.85}$$

We "cut" the current T_n^{sel} considering the curves in its decomposition only up to the first time when they leave the ball where they begin, i.e. we define

$$T_{n,i}^{sel,-} := \int_{\mathrm{Lip}} R_{res(0,O_{B_i^-})(\gamma)}\, d\pi_{n,i}^{sel,-}(\gamma), \qquad T_n^{sel,-} := \sum_{i=1}^{N^-} T_{n,i}^{sel,-}. \tag{7.86}$$

The measure

$$\sum_{i=1}^{N^-} (res(0, O_{B_i^-})(\cdot))_\sharp \pi_{n,i}^{sel,-}$$

is a good decomposition of $T_n^{sel,-}$: this is a consequence of Remark 7.19 applied to $I(\gamma) := \gamma(0)$,

$$F(\gamma) := \begin{cases} O_{B_i^-}(\gamma), & \text{if } \gamma(0) \in C_i^-, \text{ for some } i = 1, \cdots, N^- \\ 0, & \text{otherwise} \end{cases},$$

$E^- := (\cup_{i=1}^{N^-} B_i^-) \setminus (\cup_{i=1}^{N^-} \partial B_i^-)$ and $E^+ := \cup_{i=1}^{N^-} \partial B_i^-$. Notice that the assumption of the Remark are satisfied in view of (7.73).

Using this fact, by (7.15), (7.85) and (7.86), we get

$$\partial_- T_n^{sel,-} = \partial_- T_n^{sel}. \tag{7.87}$$

Analogously we define

$$\pi_{n,j}^{sel,+} := \pi_n^{sel} \llcorner \left\{\gamma : \gamma(\infty) \in C_j^+\right\} \qquad \text{for every } j = 1, ..., N^+, \tag{7.88}$$

and we "cut" the current T_n^{sel} considering the curves in its decomposition only from the last time when they enter in the ball where they end, i.e. we define

$$T_{n,j}^{sel,+} := \int_{\mathrm{Lip}} R_{res(E_{B_j^+},\infty)(\gamma)}\, d\pi_{n,j}^{sel,+}(\gamma), \qquad T_n^{sel,+} = \sum_{j=1}^{N^+} T_{n,j}^{sel,+}. \tag{7.89}$$

Arguing as for (7.87), we get

$$\partial_+ T_n^{sel,+} = \partial_+ T_n^{sel}, \tag{7.90}$$

and combining (7.87) and (7.90), we derive

$$\partial T_n^{sel,-} + \partial T_n^{sel,+} = \partial T_n^{sel} + \partial_+ T_n^{sel,-} - \partial_- T_n^{sel,+}. \tag{7.91}$$

Step 6: good decomposition of T_{opt} *and restriction outside the covering.* Let π_{opt} be a good decomposition of T_{opt}. Let us decompose π_{opt} into the sum of countably many, mutually singular measures $\pi_{opt,i,j}$, according to the starting and the ending points of the associated curves, i.e., for every $i \in I^-$ and $j \in I^+$ we denote

$$\pi_{opt,i,j} := \pi \llcorner \left\{\gamma : \gamma(0) \in C_i^- \text{ and } \gamma(\infty) \in C_j^+\right\}.$$

We denote by $T_{opt,i,j}$ the traffic path associated to $\pi_{opt,i,j}$. Now we "cut" the current T_{opt} considering the curves in its decomposition only from the first time when they leave the ball where they begin, up to the last time when they enter in the ball where they end, i.e. we define

$$T^{restr}_{opt,i,j} := \int_{Lip} R_{res(O_{B^-_i}, E_{B^+_j})(\gamma)}\, d\pi_{opt,i,j}(\gamma), \qquad T^{restr}_{opt} := \sum_{i \in I^-, j \in I^+} T^{restr}_{opt,i,j}.$$

Notice that, by Remark 7.19 and (7.70), this formula gives a good decomposition of T^{restr}_{opt}. Here we use the same abuse of notation, as in (7.64). By Proposition 7.18, we have that the multiplicity of T^{restr}_{opt} is pointwise bounded by the multiplicity of T_{opt}, so that

$$\mathbb{M}^\alpha\Big(T^{restr}_{opt} \llcorner \Big(\big(\cup_{i=1}^{N^-} \overline{B}^-_i\big) \cup \big(\cup_{i=1}^{N^+} \overline{B}^+_i\big)\Big)^c\Big) \leqslant \mathbb{M}^\alpha(T_{opt}), \tag{7.92}$$

and by (7.72)

$$\mathbb{M}^\alpha\Big(T^{restr}_{opt} \llcorner \big(\cup_{i=1}^{N^\pm} \overline{B}^\pm_i\big)\Big) \leqslant \mathbb{M}^\alpha\Big(T_{opt} \llcorner \big(\cup_{i=1}^{N^\pm} \overline{B}^\pm_i\big)\Big) \leqslant \frac{\Delta}{128}. \tag{7.93}$$

We observe that:

$$\sum_{j \in I^+} \partial_- T^{restr}_{opt,i,j}(\partial B^-_i) = \sum_{j \in I^+} \partial_- T_{opt,i,j}(C^-_i) = \partial_- T_{opt}(C^-_i) = \mu^-(C^-_i), \tag{7.94}$$

where the first equality follows because the first (resp. second) term can be seen as the total mass of the positive (resp. negative) part of the boundary of

$$\sum_{j \in I^+} \int_{Lip} R_{(0, res(O_{B^-_i}))(\gamma)}\, d\pi_{opt,i,j}(\gamma).$$

This is true because, by Remark 7.19 and (7.73), this formula gives a good decomposition (with the usual abuse of notation).

Step 7: connection along the spheres. By Proposition 7.10(1) we have $\partial_\pm T^{sel}_n \leqslant \partial_\pm T_n = \mu^\pm_n$. We deduce that

$$\begin{aligned} \mu^-_n(C^-_i) &\geqslant \partial_- T^{sel}_n(C^-_i) \overset{(7.87)}{=} \partial_- T^{sel,-}_n(C^-_i) \\ &\overset{(7.84)}{=} \partial_- T^{sel,-}_{n,i}(\mathbb{R}^d) \overset{(6.2)}{=} \partial_+ T^{sel,-}_{n,i}(\mathbb{R}^d) = \partial_+ T^{sel,-}_{n,i}(\partial B^-_i) \end{aligned} \tag{7.95}$$

and similarly

$$\mu^+_n(C^+_j) \geqslant \partial_- T^{sel,+}_{n,j}(\partial B^+_j). \tag{7.96}$$

Combining this with (7.77), it follows that, for every $i \in I^-$,

$$\partial_+ T^{sel,-}_{n,i}(\partial B^-_i) \leqslant (1+\varepsilon_2)\mu^-(C^-_i) \overset{(7.94)}{=} (1+\varepsilon_2) \sum_{j \in I^+} \partial_- T^{restr}_{opt,i,j}(\partial B^-_i)$$

and analogously, for every $j \in I^+$,

$$\partial_- T^{sel,+}_{n,j}(\partial B^+_j) \leqslant (1+\varepsilon_2)\mu^+(C^+_j) = (1+\varepsilon_2) \sum_{i \in I^-} \partial_+ T^{restr}_{opt,i,j}(\partial B^+_j).$$

Hence, for every $i \in I^-$, we denote

$$\alpha_i^- := \frac{\partial_+ T_{n,i}^{sel,-}(\partial B_i^-)}{(1+\varepsilon_2)\sum_{j\in I^+}\partial_- T_{opt,i,j}^{restr}(\partial B_i^-)} \in [0,1] \tag{7.97}$$

and, for every $j \in I^+$,

$$\alpha_j^+ := \frac{\partial_- T_{n,j}^{sel,+}(\partial B_j^+)}{(1+\varepsilon_2)\sum_{i\in I^-}\partial_+ T_{opt,i,j}^{restr}(\partial B_j^+)} \in [0,1]. \tag{7.98}$$

We define

$$T_{n,i}^{conn,-} \in \mathbf{TP}\Big(\partial_+ T_{n,i}^{sel,-}, \alpha_i^-(1+\varepsilon_2)\sum_{j\in I^+}\partial_- T_{opt,i,j}^{restr}\Big) \tag{7.99}$$

to be the traffic path given by Lemma 7.20 (supported on ∂B_i^-). The lemma can be applied since the two marginals in (7.99) are supported on ∂B_i^- and they have same total mass, as a consequence of (7.97). Its cost is estimated by

$$\mathbb{M}^\alpha\big(T_{n,i}^{conn,-}\big) \leqslant C_{\alpha,d}\big(\partial_+ T_{n,i}^{sel,-}(\partial B_i^-)\big)^\alpha r_i^- \leqslant C_{\alpha,d} r_i^-. \tag{7.100}$$

Analogously, we define a traffic path

$$T_{n,j}^{conn,+} \in \mathbf{TP}\Big(\alpha_j^+(1+\varepsilon_2)\sum_{i\in I^-}\partial_+ T_{opt,i,j}^{restr}, \partial_- T_{n,j}^{sel,+}\Big), \tag{7.101}$$

supported on ∂B_j^+, whose cost is again estimated by

$$\mathbb{M}^\alpha\big(T_{n,j}^{conn,+}\big) \leqslant C_{\alpha,d}\big(\partial_- T_{n,j}^{sel,+}(\partial B_j^+)\big)^\alpha r_j^+ \leqslant C_{\alpha,d} r_j^+. \tag{7.102}$$

Finally, we define the traffic paths

$$T_n^{conn,-} := \sum_{i=1}^{N^-} T_{n,i}^{conn,-} \qquad \text{and} \qquad T_n^{conn,+} := \sum_{j=1}^{N^+} T_{n,j}^{conn,+}.$$

We denote

$$\sigma_n^+ := (1+\varepsilon_2)\sum_{i=1}^{N^-}\alpha_i^-\sum_{j\in I^+}\partial_- T_{opt,i,j}^{restr}, \tag{7.103}$$

$$\sigma_n^- := (1+\varepsilon_2)\sum_{j=1}^{N^+}\alpha_j^+\sum_{i\in I^-}\partial_+ T_{opt,i,j}^{restr}, \tag{7.104}$$

from (7.99), (7.101), (7.86) and (7.89), we infer

$$T_n^{conn,-} \in \mathbf{TP}(\partial_+ T_n^{sel,-}, \sigma_n^+), \qquad \text{and} \qquad T_n^{conn,+} \in \mathbf{TP}(\sigma_n^-, \partial_- T_n^{sel,+}). \tag{7.105}$$

Using the fact that $\pi_n \in \mathcal{P}(\mathrm{Lip})$, one gets

$$\begin{aligned}\sigma_n^+(\mathbb{R}^d) &\overset{(7.105),(6.1)}{=} \partial_+ T_n^{sel,-}(\mathbb{R}^d) \overset{(6.2)}{=} \partial_- T_n^{sel,-}(\mathbb{R}^d) \\ &\overset{(7.87)}{=} \partial_- T_n^{sel}(\mathbb{R}^d) \overset{(7.15)}{=} \pi_n^{sel}(\mathrm{Lip}) \overset{(7.82)}{\geqslant} 1-\varepsilon_1.\end{aligned} \tag{7.106}$$

Using the sub-additivity of the α-mass, we get the energy estimate

$$\mathbb{M}^\alpha\big(T_n^{conn,\pm}\big) \overset{(7.100),(7.102)}{\leqslant} \sum_{i=1}^{N^\pm} C_{\alpha,d} r_i^\pm \overset{(7.71)}{\leqslant} \frac{\Delta}{128}. \tag{7.107}$$

Step 8: bringing back the mass in excess. Denoting

$$\nu^- := (1+\varepsilon_2) \sum_{i\in I^-, j\in I^+} \partial_- T_{opt,i,j}^{restr}, \quad \text{and} \quad \nu^+ := (1+\varepsilon_2) \sum_{i\in I^-, j\in I^+} \partial_+ T_{opt,i,j}^{restr}, \tag{7.108}$$

we get that

$$(1+\varepsilon_2) T_{opt}^{restr} \in \mathbf{TP}(\nu^-, \nu^+). \tag{7.109}$$

We define the two non-negative measures

$$\nu_n^- := \nu^- - \sigma_n^+, \qquad \nu_n^+ := \nu^+ - \sigma_n^-. \tag{7.110}$$

Since by (7.97), (7.98) $\alpha_i^-, \alpha_j^+ \in [0,1]$, comparing (7.105) with (7.108), we get

$$\sigma_n^+ \leqslant \nu^-, \quad \sigma_n^- \leqslant \nu^+, \quad \nu_n^- \leqslant \nu^-, \quad \text{and} \quad \nu_n^+ \leqslant \nu^+. \tag{7.111}$$

We claim that

$$\nu_n^-(\mathbb{R}^d) = \nu_n^+(\mathbb{R}^d) \leqslant \varepsilon_1 + \varepsilon_2. \tag{7.112}$$

Indeed we can compute

$$\begin{aligned}\sigma_n^+(\mathbb{R}^d) &\overset{(7.105),(6.1)}{=} \partial_+ T_n^{sel,-}(\mathbb{R}^d) \overset{(6.2)}{=} \partial_- T_n^{sel,-}(\mathbb{R}^d) \overset{(7.87)}{=} \partial_- T_n^{sel}(\mathbb{R}^d) \\ &\overset{(6.2)}{=} \partial_+ T_n^{sel}(\mathbb{R}^d) \overset{(7.90)}{=} \partial_+ T_n^{sel,+}(\mathbb{R}^d) \overset{(6.2)}{=} \partial_- T_n^{sel,+}(\mathbb{R}^d) \overset{(7.105),(6.1)}{=} \sigma_n^-(\mathbb{R}^d),\end{aligned}$$

which, together with (7.110) and the fact that $\nu^-(\mathbb{R}^d) = \nu^+(\mathbb{R}^d)$, implies $\nu_n^-(\mathbb{R}^d) = \nu_n^+(\mathbb{R}^d)$. Since $\sigma_n^+ \leqslant \nu^-$, we can estimate

$$\begin{aligned}\nu_n^-(\mathbb{R}^d) = \nu^-(\mathbb{R}^d) - \sigma_n^+(\mathbb{R}^d) &\overset{(7.108),(7.94)}{\leqslant} (1+\varepsilon_2)\mu^-(\mathbb{R}^d) - \sigma_n^+(\mathbb{R}^d) \\ &\overset{(7.106)}{\leqslant} (1+\varepsilon_2) - (1-\varepsilon_1) = \varepsilon_1 + \varepsilon_2,\end{aligned}$$

getting the claim (7.112).

Therefore, by (7.111), (7.112) and (7.109), we can apply Proposition 7.25 to prove the existence of a path

$$T^{back} \in \mathbf{TP}(\nu_n^+, \nu_n^-) \tag{7.113}$$

with

$$\mathbb{M}^{\alpha}(T^{back}) \leqslant \frac{\Delta}{128}, \tag{7.114}$$

provided ε_1 and ε_2 are chosen small enough.

From (7.105), (7.109), (7.113), and (7.110) we compute

$$\begin{aligned}
&\partial T_n^{conn,-} + (1+\varepsilon_2)\partial T_{opt}^{restr} + \partial T^{back} + \partial T_n^{conn,+} \\
&= \sigma_n^+ - \partial_+ T_n^{sel,-} + \nu^+ - \nu^- + \nu_n^- - \nu_n^+ + \partial_- T_n^{sel,+} - \sigma_n^- \\
&= \partial_- T_n^{sel,+} - \partial_+ T_n^{sel,-}.
\end{aligned} \tag{7.115}$$

Step 9: definition of a competitor for T_n^{sel}. Eventually we define

$$\tilde{T}_n^{sel} := T_n^{sel,-} + T_n^{conn,-} + (1+\varepsilon_2)T_{opt}^{restr} + T^{back} + T_n^{conn,+} + T_n^{sel,+}.$$

We show that it has the same boundary of T_n^{sel}

$$\partial \tilde{T}_n^{sel} = \partial T_n^{sel}. \tag{7.116}$$

Indeed, using (7.91) and (7.115), we get

$$\begin{aligned}
\partial \tilde{T}_n^{sel} &= \partial T_n^{sel,-} + \partial T_n^{conn,-} + (1+\varepsilon_2)\partial T_{opt}^{restr} + \partial T^{back} + \partial T_n^{conn,+} + \partial T_n^{sel,+} \\
&\overset{(7.91),(7.115)}{=} \partial T_n^{sel} + \partial_+ T_n^{sel,-} - \partial_- T_n^{sel,+} + \partial_- T_n^{sel,+} - \partial_+ T_n^{sel,-} = \partial T_n^{sel}.
\end{aligned} \tag{7.117}$$

Step 10: estimates on the energy of the competitor. In the following we denote by U the union of our two closed coverings

$$U^{\pm} := \cup_{i=1}^{N^{\pm}} \overline{B}_i^{\pm} \qquad U := U^+ \cup U^-.$$

We claim that the competitor $\tilde{T}_n^{sel}$ for T_n^{sel} enjoys the following estimate

$$\mathbb{M}^{\alpha}\left(\tilde{T}_n^{sel} \llcorner U^c\right) \leqslant \mathbb{M}^{\alpha}(T_{opt}) + \frac{\Delta}{4} \tag{7.118}$$

and that

$$\mathbb{M}^{\alpha}\left((\tilde{T}_n^{sel} - T_n^{sel,\pm}) \llcorner U^{\pm}\right) \leqslant \frac{\Delta}{32}. \tag{7.119}$$

We first focus on (7.118). By their definition, the currents $T_n^{conn,\pm}$, $T_n^{sel,\pm}$ are supported on the sets $U^{\pm}$; hence, they are supported on sets disjoint from U^c. Using (7.92) and $C\varepsilon_2 \leqslant \frac{\Delta}{8}$, we can compute

$$\begin{aligned}
\mathbb{M}^{\alpha}\left(\tilde{T}_n^{sel} \llcorner U^c\right) &= \mathbb{M}^{\alpha}(((1+\varepsilon_2)T_{opt}^{restr} + T^{back}) \llcorner U^c) \\
&\leqslant (1+\varepsilon_2)^{\alpha}\mathbb{M}^{\alpha}(T_{opt}^{restr} \llcorner U^c) + \mathbb{M}^{\alpha}(T^{back}) \\
&\leqslant \mathbb{M}^{\alpha}(T_{opt}) + C\varepsilon_2 + \frac{\Delta}{128} \leqslant \mathbb{M}^{\alpha}(T_{opt}) + \frac{\Delta}{4}.
\end{aligned} \tag{7.120}$$

To prove (7.119) (we show it for the choice $\pm = -$), it is enough to show that

$$\mathbb{M}^\alpha\big((\tilde{T}_n^{sel} - T_n^{sel,-})\llcorner U^-\big) \leqslant \frac{\Delta}{32}. \tag{7.121}$$

Using again that the currents $T_n^{conn,+}$, $T_n^{sel,+}$ are supported on the set U^+, we estimate, by the subadditivity of the α-mass,

$$\begin{aligned}\mathbb{M}^\alpha\Big(\big(\tilde{T}_n^{sel} - T_n^{sel,-}\big)\llcorner U^-\Big) &\leqslant \mathbb{M}^\alpha\big(T_n^{conn,-}\llcorner U^- + (1+\varepsilon_2)T_{opt}^{restr}\llcorner U^- + T^{back}\llcorner U^-\big)\\ &\leqslant \mathbb{M}^\alpha\big(T_n^{conn,-}\big) + (1+\varepsilon_2)^\alpha\mathbb{M}^\alpha\big(T_{opt}^{restr}\llcorner U^-\big) + \mathbb{M}^\alpha\big(T^{back}\big) \leqslant \frac{\Delta}{32},\end{aligned}$$

where in the last inequality we used $\varepsilon_2 \leqslant 1/4$, (7.93), (7.107), (7.114). This concludes the proof of (7.119).

Step 11: definition of a competitor for T_n. We define $\overline{T}_n := \tilde{T}_n^{sel} + T_n - T_n^{sel}$ as a competitor for the α-mass optimizer T_n, with the aim to prove that the former has less α-mass than the latter. Indeed, by (7.117), $\partial\overline{T}_n = \partial T_n$ and consequently

$$\tilde{T}_n^{sel} + T_n - T_n^{sel} \in \mathbf{TP}(\mu_n^-, \mu_n^+).$$

We split its energy as

$$\mathbb{M}^\alpha(\overline{T}_n) = \mathbb{M}^\alpha\big(\overline{T}_n\llcorner U\big) + \mathbb{M}^\alpha\big(\overline{T}_n\llcorner U^c\big) \tag{7.122}$$

For the first term, the additivity of the α-mass on disjoint sets gives

$$\mathbb{M}^\alpha\big(\overline{T}_n\llcorner U\big) = \mathbb{M}^\alpha\big((\tilde{T}_n^{sel} + T_n - T_n^{sel})\llcorner U^+\big) + \mathbb{M}^\alpha\big((\tilde{T}_n^{sel} + T_n - T_n^{sel})\llcorner U^-\big). \tag{7.123}$$

We estimate each term by means of (7.119); since the proof is the same, we do it for the first term in the right-hand side

$$\begin{aligned}&\mathbb{M}^\alpha\big((\tilde{T}_n^{sel} + T_n - T_n^{sel})\llcorner U^-\big)\\ &\leqslant \mathbb{M}^\alpha\big((\tilde{T}_n^{sel} - T_n^{sel,-})\llcorner U^-\big) + \mathbb{M}^\alpha\big((T_n^{sel,-} + T_n - T_n^{sel})\llcorner U^-\big)\\ &\leqslant \frac{\Delta}{32} + \mathbb{M}^\alpha\big((T_n^{sel,-} + T_n - T_n^{sel})\llcorner U^-\big)\end{aligned} \tag{7.124}$$

The latter can be estimated by noticing that it is a “part of an optimum” with

$$\mathbb{M}^\alpha\big((T_n^{sel,-} + T_n - T_n^{sel})\llcorner U^-\big) \leqslant \mathbb{M}^\alpha\big(T_n\llcorner U^-\big). \tag{7.125}$$

Indeed we apply Proposition 7.18 with $T = T_n^{sel}$ and $\tilde{T} = T_n^{sel,-}$, to obtain that

$$T_n^{sel} - T_n^{sel,-} = \beta T_n^{sel}, \qquad \text{where } \beta : \mathbb{R}^d \to [0,1],$$

and Proposition 7.10(3) with $T = T_n$ and $T' = T_n^{sel}$, to obtain that $T_n^{sel} = \varphi T_n$, where $\varphi : \mathbb{R}^d \to [0,1]$, and therefore

$$T_n - (T_n^{sel} - T_n^{sel,-}) = T_n - \beta T_n^{sel} = (1-\varphi\beta)T_n, \qquad \text{where } [1-\varphi\beta] : \mathbb{R}^d \to [0,1].$$

We can conclude that

$$\mathbb{M}^\alpha\big((T_n^{sel,-}+T_n-T_n^{sel})\llcorner U^-\big)\leqslant \sup_{x\in\mathbb{R}^d}\{1-\beta(x)\varphi(x)\}^\alpha\mathbb{M}^\alpha\big(T_n\llcorner U^-\big)\leqslant \mathbb{M}^\alpha\big(T_n\llcorner U^-\big),$$

which is exactly (7.125).

Putting together (7.123), (7.124), (7.125), we get an estimate for the first term in the right-hand side of (7.122)

$$\mathbb{M}^\alpha\big(\overline{T}_n\llcorner U\big)\leqslant \frac{\Delta}{16}+\mathbb{M}^\alpha\big(T_n\llcorner U\big). \tag{7.126}$$

The second term in (7.122) can be instead estimated through the sub-additivity of the α-mass, the energy bound on the competitor $\tilde{T}_n^{sel}$ in (7.118), and the energy gap in (7.69)

$$\begin{aligned}\mathbb{M}^\alpha\big(\overline{T}_n\llcorner U^c\big)&\leqslant \mathbb{M}^\alpha\big(\tilde{T}_n^{sel}\llcorner U^c\big)+\mathbb{M}^\alpha\big((T_n-T_n^{sel})\llcorner U^c\big)\\&\leqslant \mathbb{M}^\alpha(T_{opt})+\frac{\Delta}{4}+\mathbb{M}^\alpha\big((T_n-T_n^{sel})\llcorner U^c\big)\\&\leqslant \mathbb{M}^\alpha(T)-\frac{3\Delta}{4}+\mathbb{M}^\alpha\big((T_n-T_n^{sel})\llcorner U^c\big).\end{aligned} \tag{7.127}$$

We fix δ obtained from Lemma 7.23 with the choices $A=U^c$ and $\varepsilon=\Delta/8$. The conclusion of the lemma holds for $T'=T_n^{sel}$, provided we add the following further constraints on ε_1 and ε_2:

$$\varepsilon_2\leqslant\frac{\delta}{2},\qquad C\varepsilon_1^{1-\alpha}\leqslant\frac{\delta}{2},\qquad \varepsilon_1\leqslant\frac{\delta}{4},\qquad 16\varepsilon_1^\alpha C\leqslant\delta^\alpha\Delta. \tag{7.128}$$

By sub-additivity of flat norm, (7.76) and (7.83), we find that

$$\mathbb{F}(T_n^{sel}-T)\leqslant\mathbb{F}(T_n-T)+\mathbb{F}(T_n^{sel}-T_n)\leqslant\varepsilon_2+C\varepsilon_1^{1-\alpha}\leqslant\delta.$$

Using the previous inequality and Lemma 7.23,

$$\begin{aligned}\mathbb{M}^\alpha(T)&=\mathbb{M}^\alpha(T\llcorner U^c)+\mathbb{M}^\alpha(T\llcorner U)\\&\leqslant \mathbb{M}^\alpha\big(T_n^{sel}\llcorner\big(U^c\cap\{\theta_n^{sel}>\delta\}\big)\big)+\mathbb{M}^\alpha(T\llcorner U)+\frac{\Delta}{8}\\&\overset{(7.72)}{\leqslant} \mathbb{M}^\alpha\big(T_n^{sel}\llcorner\big(U^c\cap\{\theta_n^{sel}>\delta\}\big)\big)+\frac{\Delta}{4}.\end{aligned}$$

Substituting the previous inequality in (7.127), we find

$$\mathbb{M}^\alpha\big(\overline{T}_n\llcorner U^c\big)\leqslant\mathbb{M}^\alpha\big(T_n^{sel}\llcorner\big(U^c\cap\{\theta_n^{sel}>\delta\}\big)\big)-\frac{\Delta}{2}+\mathbb{M}^\alpha\big((T_n-T_n^{sel})\llcorner U^c\big). \tag{7.129}$$

We claim that it is possible to apply Lemma 7.24 with $T_1=\big(T_n-T_n^{sel}\big)\llcorner U^c$, $T_2=T_n^{sel}\llcorner\big(U^c\cap\{\theta_n^{sel}>\delta\}\big)$, and $\varepsilon=\varepsilon_1/\delta$. Indeed, by (7.80) $T_n-T_n^{sel}$ has multiplicity less than or equal to ε_1 and by (7.128) we have $\delta\geqslant 4\varepsilon_1$. Consequently, by (7.129)

$$\begin{aligned}&\mathbb{M}^\alpha\big(\overline{T}_n\llcorner U^c\big)\\&\leqslant\Big(1+4\Big(\frac{\varepsilon_1}{\delta}\Big)^\alpha\Big)\mathbb{M}^\alpha\big(T_n^{sel}\llcorner\big(U^c\cap\{\theta_n^{sel}>\delta\}\big)+\big(T_n-T_n^{sel}\big)\llcorner U^c\big)-\frac{\Delta}{2}\\&=\Big(1+4\Big(\frac{\varepsilon_1}{\delta}\Big)^\alpha\Big)\mathbb{M}^\alpha\big(\beta T_n\llcorner U^c\big)-\frac{\Delta}{2},\end{aligned} \tag{7.130}$$

where $\beta : \mathbb{R}^d \to [0,1]$. Since by hypothesis $\mathbb{M}^\alpha\big(T_n\big) \leqslant C$, using (7.128), we find that

$$\mathbb{M}^\alpha\big(\overline{T}_n \llcorner U^c\big) \leqslant \mathbb{M}^\alpha\big(T_n \llcorner U^c\big) + 4\Big(\frac{\varepsilon_1}{\delta}\Big)^\alpha C - \frac{\Delta}{2} \overset{(7.128)}{\leqslant} \mathbb{M}^\alpha\big(T_n \llcorner U^c\big) - \frac{\Delta}{4}. \tag{7.131}$$

Putting together (7.126) and (7.131), we find that

$$\mathbb{M}^\alpha(\overline{T}_n) \leqslant \mathbb{M}^\alpha\big(T_n \llcorner U\big) + \mathbb{M}^\alpha\big(T_n \llcorner U^c\big) - \frac{\Delta}{8} < \mathbb{M}^\alpha(T_n),$$

which is a contradiction to the optimality of T_n.

8

RELAXATION OF FUNCTIONALS ON POLYHEDRAL CHAINS

8.1 INTRODUCTION

Let $H : \mathbb{R} \to [0, \infty)$ be an even, subadditive, and lower semicontinuous function, with $H(0) = 0$. The function H naturally induces a functional Φ_H on the set $\mathbf{P}_k(\mathbb{R}^d)$ of polyhedral k-chains in $\mathbb{R}^d$. For every polyhedral k-chain of the form $P = \sum_{i=1}^{N} \theta_i[\sigma_i]$ (with non-overlapping k-simplexes σ_i), we set

$$\Phi_H(P) := \sum_{i=1}^{N} H(\theta_i)\mathcal{H}^k(\sigma_i).$$

It is easy to see that the above assumptions on H are necessary for the functional Φ_H to be (well defined and) lower semicontinuous on polyhedral chains with respect to convergence in flat norm. In this chapter we present our paper [28], a joint work with Colombo, Marchese and Stuvard, where we prove that the assumptions on H are also sufficient, and moreover we show that the lower semicontinuous envelope of Φ_H coincides on rectifiable k-currents with the H-mass, namely the functional

$$\mathbb{M}_H(R) := \int_E H(\theta(x))\, d\mathcal{H}^k(x), \quad \text{for every rectifiable k-current } R = R[E, \tau, \theta].$$

The validity of such a representation has recently attracted some attention. For instance, it is clearly assumed in [85] for the choice $H(x) = |x|^\alpha$, with $\alpha \in (0, 1)$, in order to prove some regularity properties of minimizers of problems related to branched transportation (see also [73], [15], [75] and Remark 7.5 of Chapter 7) and in [25] in order to define suitable approximations of the Steiner problem, with the choice $H(x) = (1 + \beta|x|)\mathbf{1}_{\mathbb{R}\setminus\{0\}}$, where $\beta > 0$ and $\mathbf{1}_A$ denotes the indicator function of the Borel set A.

We finally remark that in the last section of [82] the author sketches a strategy to prove an analogous version of the main theorem of this chapter (Theorem 8.3 below) in the framework of flat chains with coefficients in a normed abelian group G. Motivated by the relevance of such result for real valued flat chains, the ultimate aim of our note [28] is to present a self-contained complete proof of it when $G = \mathbb{R}$.

8.2 SETTING AND MAIN RESULT

Assumptions 1. *In what follows, we will consider a Borel function* $H\colon \mathbb{R} \to [0, \infty)$ *satisfying the following hypotheses:*

(H1) $H(0) = 0$ *and* H *is even, namely* $H(-\theta) = H(\theta)$ *for every* $\theta \in \mathbb{R}$;

(H2) H *is subadditive, namely* $H(\theta_1 + \theta_2) \leqslant H(\theta_1) + H(\theta_2)$ *for every* $\theta_1, \theta_2 \in \mathbb{R}$;

(H3) H *is lower semicontinuous, namely* $H(\theta) \leqslant \liminf_{j\to\infty} H(\theta_j)$ *whenever* θ_j *is a sequence of real numbers such that* $|\theta - \theta_j| \searrow 0$ *when* $j \uparrow \infty$.

Remark 8.1. Observe that the hypotheses (H2) and (H3) imply that H is in fact countably subadditive, namely

$$H\left(\sum_{j=1}^{\infty} \theta_j\right) \leqslant \sum_{j=1}^{\infty} H(\theta_j),$$

for any sequence $\{\theta_j\}_{j=1}^{\infty} \subset \mathbb{R}$ such that $\sum_{j=1}^{\infty} \theta_j$ converges.

Remark 8.2. Let $\tilde{H}\colon [0,\infty) \to [0,\infty)$ be any Borel function satisfying:

($\tilde{H}1$) $\tilde{H}(0) = 0$;

($\tilde{H}2$) $\tilde{H}$ is subadditive and monotone non-decreasing, i.e. $\tilde{H}(\theta_1) \leqslant \tilde{H}(\theta_2)$ for any $0 \leqslant \theta_1 \leqslant \theta_2$;

($\tilde{H}3$) $\tilde{H}$ is lower semicontinuous,

and let $H\colon \mathbb{R} \to [0,\infty)$ be the even extension of $\tilde{H}$, that is set $H(\theta) := \tilde{H}(|\theta|)$ for every $\theta \in \mathbb{R}$. Then, the function H satisfies Assumption 1.

Let H be as in Assumptions 1. We define a functional $\Phi_H\colon \mathbf{P}_k(\mathbb{R}^d) \to [0,\infty)$ as follows. Assume $P \in \mathbf{P}_k(\mathbb{R}^d)$ is as in (6.5). Then, we set

$$\Phi_H(P) := \sum_{i=1}^{N} H(\theta_i)\mathcal{H}^k(\sigma_i). \tag{8.1}$$

The functional Φ_H naturally extends to a functional $\mathbb{M}_H$, called the H-mass, defined on $\mathbf{R}_k(\mathbb{R}^d)$ by

$$\mathbb{M}_H(R) := \int_E H(\theta(x))\, d\mathcal{H}^k(x), \quad \text{for every } R = R[E,\tau,\theta] \in \mathbf{R}_k(\mathbb{R}^d). \tag{8.2}$$

We also define the functional $F_H\colon \mathbf{F}_k(\mathbb{R}^d) \to [0,\infty]$ to be the lower semicontinuous envelope of Φ_H. More precisely, for every $T \in \mathbf{F}_k(\mathbb{R}^d)$ we set

$$F_H(T) := \inf\left\{ \liminf_{j\to\infty} \Phi_H(P_j) : \ P_j \in \mathbf{P}_k(\mathbb{R}^d) \text{ with } \mathbb{F}(T - P_j) \searrow 0 \right\}. \tag{8.3}$$

The main result of this chapter is the following theorem.

Theorem 8.3. *Let* H *satisfy Assumption 1. Then,* $F_H \equiv \mathbb{M}_H$ *on* $\mathbf{R}_k(\mathbb{R}^d)$.

In order to prove Theorem 8.3, we adopt the following strategy. First, we show that the functional $\mathbb{M}_H$ is lower semicontinuous on rectifiable currents, with respect to the flat convergence, as in the following proposition, with $A = \mathbb{R}^d$.

Proposition 8.4. *Let* H *satisfy Assumption 1, and let* $A \subset \mathbb{R}^d$ *be open. Let* $T_j, T \in \mathbf{R}_k(\mathbb{R}^d)$ *be rectifiable* k*-currents such that* $\mathbb{F}(T - T_j) \searrow 0$ *as* $j \to \infty$. *Then*

$$\mathbb{M}_H(T \llcorner A) \leqslant \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A). \tag{8.4}$$

Next, we observe that, as an immediate consequence of Proposition 8.4 and of the properties of the lower semicontinuous envelope, it holds

$$\mathbb{M}_H(R) \leqslant F_H(R) \quad \text{for every } R \in \mathbf{R}_k(\mathbb{R}^d). \tag{8.5}$$

The opposite inequality, which completes the proof of Theorem 8.3, is obtained as a consequence of the following proposition, which provides a polyhedral approximation in flat norm of any rectifiable k-current R with H-mass and mass close to those of the given R.

Proposition 8.5. *Let* H *be any Borel function satisfying* (H1) *in Assumption 1, and let* $R \in \mathbf{R}_k(\mathbb{R}^d)$ *be rectifiable. For every* $\varepsilon > 0$ *there exists a polyhedral* k*-chain* $P \in \mathbf{P}_k(\mathbb{R}^d)$ *such that*

$$\mathbb{F}(R - P) \leqslant \varepsilon, \qquad \Phi_H(P) \leqslant \mathbb{M}_H(R) + \varepsilon \qquad \text{and} \qquad \mathbb{M}(P) \leqslant \mathbb{M}(R) + \varepsilon. \tag{8.6}$$

Theorem 8.3 characterizes the lower semicontinuous envelope F_H on rectifiable currents to be the (possibly infinite) H-mass $\mathbb{M}_H$. Without further assumptions on H, the lower semicontinuous envelope F_H can have finite values on flat chains which are non-rectifiable (for instance, the choice $H(\theta) := |\theta|$ induces the mass functional $F_H = \mathbb{M}$). If instead we add the natural hypothesis that H is monotone non-decreasing on $[0, \infty)$, then there is a simple necessary and sufficient condition which prevents this to happen in the case of flat chains with finite mass, thus allowing us to obtain an explicit representation for F_H on all flat chains with finite mass.

Proposition 8.6. *Let* H *be as in Assumption 1 and monotone non-decreasing on* $[0, \infty)$*. The condition*

$$\lim_{\theta \searrow 0^+} \frac{H(\theta)}{\theta} = +\infty. \tag{8.7}$$

holds if and only if

$$F_H(T) = \begin{cases} \mathbb{M}_H(T) & \text{for } T \in \mathbf{R}_k(\mathbb{R}^d), \\ +\infty & \text{for } T \in (\mathbf{F}_k(\mathbb{R}^d) \cap \{T \in \mathscr{D}_k(\mathbb{R}^d) : \ \mathbb{M}(T) < \infty\}) \setminus \mathbf{R}_k(\mathbb{R}^d). \end{cases} \tag{8.8}$$

8.3 PROOF OF THE LOWER SEMICONTINUITY

This section is devoted to the proof of Proposition 8.4. It is carried out by slicing the rectifiable currents T_j and T and reducing the proposition to the lower semicontinuity of 0-dimensional currents. Some of the techniques here adopted are borrowed from [39, Lemma 3.2.14].

We recall some preliminaries on the slicing of currents. Given $k \leqslant d$, let $I(k, d)$ be the set of k-tuples $(i_1, \dots, i_k)$ with

$$1 \leqslant i_1 < \ldots < i_k \leqslant d.$$

Let $\{e_1, \dots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. For any $I = (i_1, \dots, i_k) \in I(d, k)$, let V_I be the k-plane spanned by $\{e_{i_1}, \dots, e_{i_k}\}$. Given a k-plane V, we will denote p_V the orthogonal projection onto V. If $V = V_I$ for some I, we write p_I in place of p_{V_I}. Given a current $T \in \mathbf{F}_k(\mathbb{R}^d)$, a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^m$ for some $m \leqslant k$ and $y \in \mathbb{R}^m$, we denote by

$\langle T, f, y\rangle$ the $(k-m)$-dimensional *slice* of T in $f^{-1}(y)$ (see [50, Section 4.3]). Intuitively, this can be thought as the "intersection" of the current T with the level set $f^{-1}(y)$.

Let us denote by $Gr(d,k)$ the Grassmannian of k-dimensional planes in $\mathbb{R}^d$, and by $\gamma_{d,k}$ the Haar measure on $Gr(d,k)$ (see [61, Section 2.1.4]).

In the following lemma, we prove a version of the integral-geometric equality for the H-mass, which is a consequence of [50, 3.2.26; 2.10.15] (see also [39, (21)]). We observe that the hypotheses (H2) and (H3) on the function H are not needed here, and indeed Lemma 8.7 below is valid for any Borel function H for which the H-mass $\mathbb{M}_H$ is well defined.

Lemma 8.7. *Let* $E \subseteq \mathbb{R}^d$ *be* k*-rectifiable. Then there exists* $c = c(d,k)$ *such that the following integral-geometric equality holds:*

$$\mathcal{H}^k(E) = c\int_{Gr(d,k)}\int_{\mathbb{R}^k}\mathcal{H}^0(p_V^{-1}(\{y\})\cap E)\,d\mathcal{H}^k(y)\,d\gamma_{d,k}(V). \tag{8.9}$$

In particular, if $R \in \mathbf{R}_k(\mathbb{R}^d)$,

$$\mathbb{M}_H(R) = c\int_{Gr(d,k)\times\mathbb{R}^k}\mathbb{M}_H\big(\langle R, p_V, y\rangle\big)\,d(\gamma_{d,k}\otimes\mathcal{H}^k)(V,y). \tag{8.10}$$

Proof. The equality (8.9) is proved in [50, 3.2.26; 2.10.15]. For any Borel set $A\subset\mathbb{R}^d$, denoting $f=\mathbb{1}_A$, (8.9) implies that

$$\int_E f(x)\,d\mathcal{H}^k(x) = c\int_{Gr(d,k)}\int_{\mathbb{R}^k}\int_E f(x)\,\mathbb{1}_{p_V^{-1}(\{y\})}(x)\,d\mathcal{H}^0(x)\,d\mathcal{H}^k(y)\,d\gamma_{d,k}(V). \tag{8.11}$$

Since the previous equality is linear in f, it holds also when f is piecewise constant. Since the measure $\mathcal{H}^k\llcorner E$ is σ-finite, the equality can be extended to any measurable function $f\in L^1(\mathcal{H}^k\llcorner E)$. The case $f\notin L^1(\mathcal{H}^k\llcorner E)$ follows from the Monotone Convergence Theorem via a simple truncation argument.

Taking $R = R[E,\theta,\tau]$, and applying (8.11) with $f(x) = H(\theta(x))$, we deduce that

$$\mathbb{M}_H(R) = c\int_{Gr(d,k)}\int_{\mathbb{R}^k}\int_{E\cap p_V^{-1}(\{y\})}H(\theta(x))\,d\mathcal{H}^0(x)\,d\mathcal{H}^k(y)\,d\gamma_{d,k}(V).$$

We observe that the right-hand side coincides with the right-hand side in (8.10) since for $\mathcal{H}^k$-a.e. $y\in\mathbb{R}^k$ the 0-dimensional current $\langle R, p_V, y\rangle$ is concentrated on the set $E\cap p_V^{-1}(y)$ and its density at any $x\in E\cap p_V^{-1}(y)$ is $\theta(x)$. □

We prove the lower semicontinuity in (8.4) by an explicit computation in the case $k=0$. Then, by slicing, we get the proof for $k>0$, too.

Proof of Proposition 8.4. Step 1: the case $k=0$. Let $T_j := T_j[E_j,\tau_j,\theta_j], T := T[E,\tau,\theta]\in\mathbf{R}_0(\mathbb{R}^d)$ be such that $\mathbb{F}(T-T_j)\searrow 0$ as $j\to\infty$. Since $T\llcorner A$ is a signed, atomic measure, we write

$$T\llcorner A = \sum_{i\in\mathbb{N}}\tau(x_i)\theta(x_i)\delta_{x_i}$$

for distinct points $\{x_i\}_{i\in\mathbb{N}} \subseteq E \cap A$, orientations $\tau(x_i) \in \{-1, 1\}$, and for $\theta(x_i) > 0$. Fix $\varepsilon > 0$ and let $I \subset \mathbb{N}$ be a finite set such that

$$\mathbb{M}_H(T \llcorner A) - \sum_{i\in I} H(\theta(x_i)) \leqslant \varepsilon \qquad \text{if } \mathbb{M}_H(T \llcorner A) < \infty \tag{8.12}$$

and

$$\sum_{i\in I} H(\theta(x_i)) \geqslant \frac{1}{\varepsilon} \qquad \text{otherwise.} \tag{8.13}$$

Up to reordering the indexes, we may assume that $I = \{1, ..., N\}$ for some $N = N(\varepsilon)$. Since H is positive, even, and lower semicontinuous, for every $i \in \{1, \dots, N\}$ it is possible to determine $\eta_i = \eta_i(\varepsilon, \theta(x_i)) > 0$ such that

$$H(\theta) \geqslant (1-\varepsilon)H(\theta(x_i)) \quad \text{ for every } |\theta - \tau(x_i)\theta(x_i)| < \eta_i. \tag{8.14}$$

Moreover, for every $i \in \{1, \dots, N\}$ there exists $0 < r_i < \min\{\mathrm{dist}(x_i, \partial A), 1\}$ such that the balls $B(x_i, 2r_i)$ are pairwise disjoint, and moreover such that for every $\rho \leqslant r_i$ it holds

$$\left| \tau(x_i)\theta(x_i) - \sum_{x\in E\cap B(x_i,\rho)} \tau(x)\theta(x) \right| \leqslant \frac{\eta_i}{2}. \tag{8.15}$$

Our next aim is to prove that in sufficiently small balls and for j large enough, the sum of the multiplicities of T_j (with sign) is close to the sum of the multiplicities of T. In order to do this, we would like to test the current $T - T_j$ with the indicator function of each ball. Since this test is not admissible, we have to consider a smooth and compactly supported extension of it outside the ball, provided we can prove that the flat convergence of T_j to T localizes to the ball. From this, our claimed convergence of the signed multiplicities follows by the characterization of the flat norm in (6.8).

To make this formal, we define $\eta_0 := \min_{1\leqslant i\leqslant N} \eta_i$ and $r_0 := \min_{1\leqslant i\leqslant N} r_i$. Let j_0 be such that

$$\mathbb{F}(T - T_j) \leqslant \frac{\eta_0 r_0}{16} \quad \text{ for every } j \geqslant j_0.$$

By the definition (6.7) of flat norm, there exist $R_j \in \mathscr{D}_0(\mathbb{R}^d)$, $S_j \in \mathscr{D}_1(\mathbb{R}^d)$ such that $T - T_j = R_j + \partial S_j$ with $\mathbb{M}(R_j) + \mathbb{M}(S_j) \leqslant \frac{\eta_0 r_0}{8}$ for every $j \geqslant j_0$. Observe that the mass and the mass of the boundary of both R_j and S_j are finite, and thus by [50, 4.1.12] it holds $R_j \in \mathbf{F}_0(\mathbb{R}^d)$ and $S_j \in \mathbf{F}_1(\mathbb{R}^d)$. We want to deduce that for every $i \in \{1, \dots, N\}$ there exists $\rho_i \in \left(\frac{r_0}{2}, r_0\right)$ such that

$$\mathbb{F}((T - T_j) \llcorner B(x_i, \rho_i)) \leqslant \frac{\eta_0}{2}.$$

Indeed, for any fixed $i \in \{1, \dots, N\}$ one has that for a.e. $\rho \in \left(\frac{r_0}{2}, r_0\right)$

$$\begin{aligned} (T - T_j) \llcorner B(x_i, \rho) &= R_j \llcorner B(x_i, \rho) + (\partial S_j) \llcorner B(x_i, \rho) \\ &= R_j \llcorner B(x_i, \rho) - \langle S_j, d(x_i, \cdot), \rho \rangle + \partial \left(S_j \llcorner B(x_i, \rho)\right), \end{aligned} \tag{8.16}$$

where $d(x_i, z) := |x_i - z|$ and where the last identity holds by the definition of slicing for normal currents (cf. [50, 4.2.1]). On the other hand, by [50, 4.2.1] we have

$$\int_{\frac{r_0}{2}}^{r_0} \mathbb{M}(\langle S_j, d(x_i, \cdot), \rho\rangle)\, d\rho \leqslant \mathbb{M}(S_j \llcorner (B(x_i, r_0) \setminus B(x_i, \frac{r_0}{2}))) \leqslant \frac{\eta_0 r_0}{8}.$$

Hence, there exists $\rho_i \in \left(\frac{r_0}{2}, r_0\right)$ such that

$$\mathbb{M}(\langle S_j, d(x_i, \cdot), \rho_i\rangle) \leqslant \frac{\eta_0}{4}. \tag{8.17}$$

We conclude from (8.16) that

$$\begin{aligned} \mathbb{F}((T - T_j) \llcorner B(x_i, \rho_i)) &\leqslant \mathbb{M}(R_j \llcorner B(x_i, \rho_i)) + \mathbb{M}(\langle S_j, d(x_i, \cdot), \rho_i\rangle) + \mathbb{M}(S_j \llcorner B(x_i, \rho_i)) \\ &\overset{(8.17)}{\leqslant} \frac{\eta_0 r_0}{4} + \frac{\eta_0}{4} \leqslant \frac{\eta_0}{2}. \end{aligned} \tag{8.18}$$

Using the characterization of the flat norm in (6.8), and testing the currents $(T - T_j) \llcorner B(x_i, \rho_i)$ with any smooth and compactly supported function $\phi_i \colon \mathbb{R}^d \to \mathbb{R}$ which is identically 1 on $B(x_i, \rho_i)$, we obtain

$$\left| \sum_{x \in E \cap B(x_i, \rho_i)} \tau(x)\theta(x) - \sum_{y \in E_j \cap B(x_i, \rho_i)} \tau_j(y)\theta_j(y) \right| \leqslant \frac{\eta_0}{2}. \tag{8.19}$$

Combining (8.19) with (8.15), we deduce by triangle inequality that

$$\left| \tau(x_i)\theta(x_i) - \sum_{y \in E_j \cap B(x_i, \rho_i)} \tau_j(y)\theta_j(y) \right| \leqslant \eta_i. \tag{8.20}$$

Finally, using (8.14) and the fact that H is countably subadditive (cf. Remark 8.1), we conclude that for every $j \geqslant j_0$

$$\begin{aligned} H(\theta(x_i)) &\leqslant \frac{1}{1-\varepsilon} H\left(\sum_{y \in E_j \cap B(x_i, \rho_i)} \tau_j(y)\theta_j(y) \right) \\ &\leqslant \frac{1}{1-\varepsilon} \sum_{y \in E_j \cap B(x_i, \rho_i)} H(\theta_j(y)) \\ &= \frac{1}{1-\varepsilon} \mathbb{M}_H(T_j \llcorner B(x_i, \rho_i)). \end{aligned}$$

Summing over i, since the balls $B(x_i, \rho_i)$ are pairwise disjoint, we get that

$$\sum_{i \in I} H(\theta_i) \leqslant \frac{1}{1-\varepsilon} \liminf_{j \to \infty} \sum_{i=1}^{N} \mathbb{M}_H(T_j \llcorner B(x_i, \rho_i)) \leqslant \frac{1}{1-\varepsilon} \liminf_{j \to \infty} \mathbb{M}_H(T_j \llcorner A).$$

By (8.12) (or (8.13) in the case that $\mathbb{M}_H(T \llcorner A) = \infty$) and since ε is arbitrary, we find (8.4).

Step 2 (Reduction to $k = 0$ *through integral-geometric equality).* We prove now Proposition 8.4 for $k > 0$. Up to subsequences, we can assume

$$\lim_{j\to\infty} \mathbb{M}_H(T_j \llcorner A) = \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A).$$

By [50, 4.3.1], for every $V \in Gr(d,k)$ it holds

$$\int_{\mathbb{R}^k} \mathbb{F}(\langle T_j - T, p_V, y\rangle)\, dy \leqslant \mathbb{F}(T_j - T), \tag{8.21}$$

Integrating the inequality (8.21) in $V \in Gr(d,k)$ and using that $\gamma_{d,k}$ is a probability measure on $Gr(d,k)$ we get

$$\lim_{j\to\infty} \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{F}(\langle T_j - T, p_V, y\rangle) d(\gamma_{d,k} \otimes \mathcal{H}^k)(V,y) \leqslant \lim_{j\to\infty} \mathbb{F}(T_j - T) = 0.$$

Since the integrand $\mathbb{F}(\langle T_j - T, p_V, y\rangle)$ is converging to 0 in L^1, up to subsequences, we get

$$\lim_{j\to\infty} \mathbb{F}(\langle T_j - T, p_V, y\rangle) = 0 \qquad \text{for } \gamma_{d,k} \otimes \mathcal{H}^k\text{-a.e. } (V,y) \in Gr(d,k) \times \mathbb{R}^k.$$

We conclude from Step 1 that

$$\mathbb{M}_H(\langle T, p_V, y\rangle \llcorner A) \leqslant \liminf_{j\to\infty} \mathbb{M}_H(\langle T_j, p_V, y\rangle \llcorner A) \qquad \text{for } \gamma_{d,k} \otimes \mathcal{H}^k\text{-a.e. } (V,y) \in Gr(d,k) \times \mathbb{R}^k. \tag{8.22}$$

By [11, (5.15)], for every $V \in Gr(d,k)$ one has $\langle T, p_V, y\rangle \llcorner A = \langle T \llcorner A, p_V, y\rangle$ for $\mathcal{H}^k$-a.e. $y \in \mathbb{R}^k$.

In order to conclude, we apply twice the integral-geometric equality (8.10). Indeed, using (8.22) and Fatou's lemma, we get

$$\begin{aligned}
\mathbb{M}_H(T \llcorner A) &= c \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{M}_H\big(\langle T \llcorner A, p_V, y\rangle\big) d(\gamma_{d,k} \otimes \mathcal{H}^k)(V,y) \\
&\leqslant c \int_{Gr(d,k)\times\mathbb{R}^k} \liminf_{j\to\infty} \mathbb{M}_H\big(\langle T_j \llcorner A, p_V, y\rangle\big) d(\gamma_{d,k} \otimes \mathcal{H}^k)(V,y) \\
&\leqslant c \liminf_{n\to\infty} \int_{Gr(d,k)\times\mathbb{R}^k} \mathbb{M}_H\big(\langle T_j \llcorner A, p_V, y\rangle\big) d(\gamma_{d,k} \otimes \mathcal{H}^k)(V,y) \\
&= \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A).
\end{aligned} \tag{8.23}$$

This concludes the proof of Step 2, so the proof of Proposition 8.4 is complete. □

8.4 PROOF OF THE POLYHEDRAL APPROXIMATION

In this section we prove Proposition 8.5. In order to do this, we will consider a family of pairwise disjoint balls which contain the entire mass of the current R, up to a small error. Then, we replace in any of these balls the current R with a k-dimensional disc with constant multiplicity. Afterwards, we further approximate each disc with polyhedral chains.

We begin with the following lemma, where we prove that, at many points x in the k-rectifiable set supporting the current R and at sufficiently small scales (depending on the point), R is close in the flat norm to the tangent k-plane at x weighted with the multiplicity of R at x.

In this section, given the k-current $R = R[E, \tau, \theta]$, for a.e. $x \in E$ we denote with π_x the affine k-plane through x spanned by the (simple) k-vector $\tau(x)$ and with $S_{x,\rho}$ the k-current

$$S_{x,\rho} := [B(x, \rho) \cap \pi_x, \tau(x), \theta(x)].$$

Lemma 8.8. *Let $\varepsilon > 0$, and let $R = R[E, \tau, \theta]$ be a rectifiable k-current in $\mathbb{R}^d$. There exists a subset $E' \subset E$ such that the following holds:*

(i) $\mathbb{M}(R \llcorner (E \setminus E')) \leqslant \varepsilon$;

(ii) *for every $x \in E'$ there exists $r = r(x) > 0$ such that for any $0 < \rho \leqslant r$*

$$\mathbb{F}(R \llcorner (E' \cap B(x, \rho)) - S_{x,\rho}) \leqslant \varepsilon \mathbb{M}(R \llcorner B(x, \rho)). \tag{8.24}$$

Proof. Since E is countably k-rectifiable, there exist countably many linear k-dimensional planes Π_i and C^1 and globally Lipschitz maps $f_i : \Pi_i \to \Pi_i^\perp$ such that

$$E \subset E_0 \cup \bigcup_{i=1}^{\infty} \mathrm{Graph}(f_i),$$

with $\mathcal{H}^k(E_0) = 0$. We will denote $\Sigma_i := \mathrm{Graph}(f_i) \subset \mathbb{R}^d$. For every $x \in \bigcup_{i=1}^{\infty} \Sigma_i$, we let $i(x)$ be the first index such that $x \in \Sigma_i$. Then, for every $i \geqslant 1$, we define $R_i := R_i[E \cap \Sigma_i, \tau, \theta_i]$, where

$$\theta_i(x) := \begin{cases} \theta(x) & \text{if } i = i(x) \\ 0 & \text{otherwise.} \end{cases} \tag{8.25}$$

Clearly, $R = \sum_{i=1}^{\infty} R_i$ and $\mathbb{M}(R) = \sum_{i=1}^{\infty} \mathbb{M}(R_i)$. Hence, there exists $N = N(\varepsilon)$ such that

$$\sum_{i \geqslant N+1} \mathbb{M}(R_i) \leqslant \varepsilon. \tag{8.26}$$

Now, recall that x is a Lebesgue point of the function θ_i with respect to the Radon measure $\mathcal{H}^k \llcorner \Sigma_i$ if

$$\lim_{r \to 0} \frac{1}{\mathcal{H}^k(\Sigma_i \cap B(x, r))} \int_{\Sigma_i \cap B(x,r)} |\theta_i(y) - \theta_i(x)| \, d\mathcal{H}^k(y) = 0.$$

We define the set $E' \subset E$ by

$$\begin{aligned} E' := \Big\{ x \in E \cap \bigcup_{i=1}^{N} \Sigma_i \quad & \text{such that } x \text{ is a Lebesgue point of } \theta_i \\ & \text{with respect to } \mathcal{H}^k \llcorner \Sigma_i \text{ for every } i \in \{1, \dots, N\} \Big\}, \end{aligned} \tag{8.27}$$

and we observe that (i) follows from (8.26) and [12, Corollary 2.23].

Let us set

$$L := \max\{\mathrm{Lip}\,(f_i) : \ i = 1, \dots, N\}. \tag{8.28}$$

Fix $i \in \{1, \dots, N\}$. For every $x \in \Sigma_i$ there exists $r > 0$ such that if $\Sigma_j \cap B(x, \sqrt{d}r) \neq \emptyset$, then $x \in \Sigma_j$ for every $j \in \{1, \dots, N\}$.

Now, fix any point $x \in E'$, and fix an index $j \in \{1, \dots, N\}$ such that $x \in \Sigma_j$. If $j = i(x)$, then $\theta_j(x) = \theta(x) > 0$. Since by the definition of E'

$$\lim_{r \to 0} \frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x, r)))}{\mathcal{H}^k(\Sigma_j \cap B(x, r))} = \theta_j(x), \tag{8.29}$$

then there exists $r > 0$ such that for any $0 < \rho \leqslant \sqrt{d}r$

$$\frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x, \rho)))}{\mathcal{H}^k(\Sigma_j \cap B(x, \rho))} \geqslant \frac{\theta_j(x)}{2}. \tag{8.30}$$

Again by [12, Corollary 2.23] applied with $\mu = \mathcal{H}^k \llcorner \Sigma_j$ and $f = \theta_j$, there exists a radius $r > 0$ (depending on x) such that

$$\begin{aligned} \int_{\Sigma_j \cap B(x,\rho)} |\theta_j(y) - \theta_j(x)| \, d\mathcal{H}^k(y) &\leqslant \varepsilon \frac{\theta_j(x)}{2} \mathcal{H}^k(\Sigma_j \cap B(x, \rho)) \\ &\leqslant \varepsilon \frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x, \rho)))}{\mathcal{H}^k(\Sigma_j \cap B(x, \rho))} \mathcal{H}^k(\Sigma_j \cap B(x, \rho)) \\ &\leqslant \varepsilon \mathbb{M}(R_j \llcorner B(x, \rho)), \end{aligned} \tag{8.31}$$

for every $0 < \rho \leqslant \sqrt{d}r$.

If, instead, $j \neq i(x)$, then $\theta_j(x) = 0$ and therefore there exists a radius $r > 0$ (depending on x) such that for every $0 < \rho \leqslant \sqrt{d}r$

$$\begin{aligned} \int_{\Sigma_j \cap B(x,\rho)} \theta_j(y) \, d\mathcal{H}^k(y) &\leqslant \frac{\varepsilon \theta_{i(x)}(x)}{N(1+L)^k} \mathcal{H}^k(\Sigma_j \cap B(x, \rho)) \\ &\leqslant \frac{\varepsilon}{N} \theta_{i(x)}(x) \omega_k \rho^k \\ &\overset{(8.30)}{\leqslant} 2 \frac{\varepsilon}{N} \mathbb{M}(R_{i(x)} \llcorner B(x, \rho)), \end{aligned} \tag{8.32}$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$.

Fix any point $x \in E'$ and let $i = i(x)$. By possibly reparametrizing $f_i|_{\Pi_i \cap B(x,r)}$ from the k-plane tangent to Σ_i at x, translating and tilting such a plane, we can assume that $x = 0$, $\Pi_i = \{x_{k+1} = \dots = x_d = 0\}$ and $\nabla f_i(x) = 0$. By possibly choosing a smaller radius $r = r(x) > 0$, we may also assume that

$$|\nabla f_i| \leqslant \varepsilon \quad \text{in } \Pi_i \cap B(x, r). \tag{8.33}$$

With these conventions, the current $S_{x,\rho}$ in the statement reads $S_{x,\rho} = S_{x,\rho}[B(0, \rho) \cap \Pi_i, \tau(0), \theta_i(0)]$. We let $F_i \colon \Pi_i \times \Pi_i^\perp \to \mathbb{R}^d$ be given by $F_i(z, w) := (z, f_i(z))$, and we set $\tilde{R}_i := (F_i)_\sharp S_{x,\rho} \in \mathbf{R}_k(\mathbb{R}^d)$.

By (8.33) and the homotopy formula (cf. [79, 26.23]) applied with $g = F_i$ and $f(z,w) := (z,0)$, we have, denoting $C(x,\rho) := (B(x,\rho) \cap \Pi_i) \times \Pi_i^\perp$,

$$\begin{aligned}\mathbb{F}(\tilde{R}_i - S_{x,\rho}) &\leqslant C\|g-f\|_{L^\infty(C(x,\rho))}\left(\mathbb{M}(S_{x,\rho}) + \mathbb{M}(\partial S_{x,\rho})\right)\\ &\leqslant C\varepsilon\rho\left(\mathbb{M}(S_{x,\rho}) + \mathbb{M}(\partial S_{x,\rho})\right)\\ &\leqslant C\varepsilon\theta(x)\omega_k\rho^k\\ &\leqslant C\varepsilon\theta(x)\mathcal{H}^k(\Sigma_j \cap B(x,\rho))\\ &\overset{(8.30)}{\leqslant} C\varepsilon\mathbb{M}(R_i \llcorner B(x,\rho)).\end{aligned} \tag{8.34}$$

Now, observe that, if we denote by ξ_i the orientation of Σ_i induced by the orientation of $\Pi_i \times \Pi_i^\perp$ via F_i, the rectifiable current $\tilde{R}_i$ reads $\tilde{R}_i = \tilde{R}_i[\Sigma_i \cap C(x,\rho), \xi_i, \theta_i(x)]$ (cf. [79, 27.2]). Therefore, we can compute

$$\begin{aligned}\mathbb{M}(R_i \llcorner B(x,\rho) - \tilde{R}_i) &\leqslant \mathbb{M}(R_i \llcorner B(x,\rho) - \tilde{R}_i \llcorner B(x,\rho)) + \mathbb{M}(\tilde{R}_i \llcorner (C(x,\rho) \setminus B(x,\rho)))\\ &\overset{(8.31)}{\leqslant} \varepsilon\mathbb{M}(R_i \llcorner B(x,\rho)) + \mathbb{M}(\tilde{R}_i \llcorner (C(x,\rho) \setminus B(x,\rho)))\\ &\overset{(8.33)}{\leqslant} \varepsilon\mathbb{M}(R_i \llcorner B(x,\rho)) + C\varepsilon\theta_i(x)\mathcal{H}^k(\Sigma_i \cap B(x,\rho))\\ &\overset{(8.30)}{\leqslant} C\varepsilon\mathbb{M}(R_i \llcorner B(x,\rho)).\end{aligned} \tag{8.35}$$

Hence, we conclude:

$$\begin{aligned}\mathbb{F}(R \llcorner E' \cap B(x,\rho) - S_{x,\rho}) &\leqslant \mathbb{F}(R_{i(x)} \llcorner B(x,\rho) - S_{x,\rho}) + \sum_{\substack{j=1\\ j\neq i(x)}}^{N} \mathbb{M}(R_j \llcorner B(x,\rho))\\ &\overset{(8.32)}{\leqslant} \mathbb{F}(R_{i(x)} \llcorner B(x,\rho) - \tilde{R}_i) + \mathbb{F}(\tilde{R}_i - S_{x,\rho}) + 2\varepsilon\mathbb{M}(R_{i(x)} \llcorner B(x,\rho))\\ &\overset{(8.34),(8.35)}{\leqslant} C\varepsilon\mathbb{M}(R \llcorner B(x,\rho)).\end{aligned} \tag{8.36}$$

This proves (8.24). $\square$

A straightforward iteration argument yields the following corollary.

Corollary 8.9. *Let $R = R[E,\tau,\theta]$ be a rectifiable k-current in $\mathbb{R}^d$. Then, for $\mathcal{H}^k$-a.e. $x \in E$*

$$\lim_{r\to 0} \frac{\mathbb{F}(R \llcorner B(x,r) - S_{x,\rho})}{\mathbb{M}(R \llcorner B(x,r))} = 0. \tag{8.37}$$

Proof. For every $i \in \mathbb{N}$ define the set E_i to be the set E' given by Lemma 8.8 applied to R with $\varepsilon = 2^{-i-1}$, and let $F_i \subset E_i$ be the set of Lebesgue points of $\mathbf{1}_{E_i}$ (inside E_i) with respect to $\theta\mathcal{H}^k \llcorner E$. By [12, Corollary 2.23], the set F_i equals the set E_i up to a set of $\mathcal{H}^k$-measure 0 and for every $x \in F_i$ and for ρ sufficiently small (possibly depending on x) it holds

$$\begin{aligned}\mathbb{M}(R \llcorner B(x,\rho) - R \llcorner (E_i \cap B(x,\rho))) &= \int_{(E\setminus E_i)\cap B(x,\rho)} \theta \, d\mathcal{H}^k\\ &\leqslant 2^{-i-1}\int_{E\cap B(x,\rho)} \theta \, d\mathcal{H}^k = 2^{-i-1}\mathbb{M}(R \llcorner B(x,\rho)).\end{aligned}$$

Hence by Lemma 8.8 for every $x \in F_i$ there exists $r_i(x) > 0$ such that for every $0 < \rho < r_i(x)$

$$\begin{aligned}\mathbb{F}(R\llcorner B(x,\rho) - S_{x,\rho}) &\leqslant \mathbb{M}(R\llcorner B(x,\rho) - R\llcorner(E_i \cap B(x,\rho))) + \mathbb{F}(R\llcorner(E_i \cap B(x,\rho)) - S_{x,\rho}) \\ &\leqslant 2^{-i}\mathbb{M}(R\llcorner B(x,\rho))\end{aligned}$$

and

$$\mathbb{M}(R\llcorner(E \setminus F_i)) \leqslant 2^{-i-1}.$$

Denoting $F := \bigcup_{i\in\mathbb{N}} \bigcap_{j\geqslant i} F_j$, and noticing that $E \setminus F = E \cap F^c = E \cap \bigcap_{i\in\mathbb{N}} \bigcup_{j\geqslant i} F_j^c$ is contained in $\bigcup_{j\geqslant i} F_j^c$ for every $i \in \mathbb{N}$, we have

$$\mathbb{M}(R\llcorner(E \setminus F)) \leqslant \lim_{i\to\infty} \sum_{j=i}^{\infty} \mathbb{M}(R\llcorner(E \setminus F_j)) \leqslant \lim_{i\to\infty} \sum_{j=i}^{\infty} \frac{1}{2^j} = 0$$

and this implies that $\mathcal{H}^k(E \setminus F) = 0$. Since every $x \in F$ belongs definitively to every F_j (namely, for every $x \in F$ there exists $i_0(x) \in \mathbb{N}$ such that $x \in F_i$ for every $i \geqslant i_0(x)$), we obtain (8.37). □

Proof of Proposition 8.5. Let R be represented by $R = R[E, \tau, \theta]$ with $\theta \in L^1(\mathcal{H}^k\llcorner E; (0, \infty))$. We denote

$$\mu := \theta\mathcal{H}^k\llcorner E.$$

Moreover, if $\mathbb{M}_H(R) < +\infty$, we define the positive finite measure

$$\nu := H(\theta)\mathcal{H}^k\llcorner E.$$

Fix $\varepsilon > 0$. We make the following

Claim: There exists a finite family of mutually disjoint balls $\{B_i\}_{i=1}^N$ with $B_i := B(x_i, r_i)$, such that the following properties are satisfied:

(i)
$$r_i \leqslant \varepsilon \qquad \forall\, i = 1, \dots, N \qquad \text{and} \qquad \mu(\mathbb{R}^d \setminus (\cup_{i=1}^N B_i)) \leqslant \varepsilon;$$

(ii) if we denote $R_i := R\llcorner B_i$ and $S_i := S_{x_i,r_i}$, then
$$\mathbb{F}(R_i - S_i) \leqslant \varepsilon\mu(B_i);$$

(iii)
$$|\mu(B_i) - \theta(x_i)\omega_k r_i^k| \leqslant \varepsilon\mu(B_i), \qquad \forall\, i = 1, \dots, N;$$

(iv) if $\mathbb{M}_H(R) < +\infty$, then it holds
$$H(\theta(x_i))\omega_k r_i^k \leqslant (1 + \varepsilon)\nu(B_i), \qquad \forall\, i = 1, \dots, N.$$

Let us for the moment assume the validity of the claim and see how to conclude the proof of the proposition.

By point (iii) in the claim we deduce

$$\mathbb{M}(S_i) \leqslant (1 + \varepsilon)\mathbb{M}(R_i). \tag{8.38}$$

and by point (iv) we get

$$\mathbb{M}_H(S_i) \leqslant (1+\varepsilon)\mathbb{M}_H(R_i). \tag{8.39}$$

On the other hand, we can find a polyhedral chain $P_i \in \mathbf{P}_k(\mathbb{R}^d)$ (supported on $\pi_i \cap B_i$, $\pi_i := \pi_{x_i}$), such that

$$\mathbb{F}(P_i - S_i) \leqslant \varepsilon\mu(B_i), \qquad \mathbb{M}_H(P_i) \leqslant \mathbb{M}_H(S_i) \qquad \text{and} \qquad \mathbb{M}(P_i) \leqslant \mathbb{M}(S_i). \tag{8.40}$$

Indeed, it is enough to approximate the k-dimensional current S_i with simplexes with constant multiplicity and supported in $B_i \cap \pi_i$.

To conclude, we denote $P := \sum_{i=1}^N P_i$ and we estimate

$$\begin{aligned}\mathbb{F}(R-P) &\leqslant \sum_{i=1}^N \mathbb{F}(R_i - P_i) + \mathbb{M}(R \llcorner (\mathbb{R}^d \setminus (\cup_{i=1}^N B_i)))\\ &\overset{(i)}{\leqslant} \varepsilon + \sum_{i=1}^N \mathbb{F}(R_i - S_i) + \sum_{i=1}^N \mathbb{F}(S_i - P_i) \overset{(ii),(8.40)}{\leqslant} \varepsilon + 2\sum_{i=1}^N \varepsilon\mu(B_i) \leqslant \varepsilon + 2\varepsilon\mathbb{M}(R).\end{aligned} \tag{8.41}$$

Moreover

$$\mathbb{M}_H(P) = \sum_{i=1}^N \mathbb{M}_H(P_i) \overset{(8.40)}{\leqslant} \sum_{i=1}^N \mathbb{M}_H(S_i) \overset{(8.39)}{\leqslant} (1+\varepsilon)\sum_{i=1}^N \mathbb{M}_H(R_i) \leqslant (1+\varepsilon)\mathbb{M}_H(R) \tag{8.42}$$

and

$$\mathbb{M}(P) = \sum_{i=1}^N \mathbb{M}(P_i) \overset{(8.40)}{\leqslant} \sum_{i=1}^N \mathbb{M}(S_i) \overset{(8.38)}{\leqslant} (1+\varepsilon)\sum_{i=1}^N \mathbb{M}(R_i) \leqslant (1+\varepsilon)\mathbb{M}(R). \tag{8.43}$$

Proof of the Claim: Consider the set F of points $x \in E$ such that the following properties hold:

1. x satisfies
$$\lim_{r\to 0} \frac{\mathbb{F}(R \llcorner B(x,r) - S_{x,r})}{\mathbb{M}(R \llcorner B(x,r))} = 0;$$
2. denoting $\eta_{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ the map $y \mapsto \frac{y-x}{r}$, we have the following convergences of measures for $r \to 0$:
$$\mu_{x,r} := r^{-k}(\eta_{x,r})_\#(\mu \llcorner B(x,r)) \rightharpoonup \theta(x)\mathcal{H}^k \llcorner((x + \mathrm{span}(\tau(x))) \cap B(0,1)), \tag{8.44}$$
and
$$\nu_{x,r} := r^{-k}(\eta_{x,r})_\#(\nu \llcorner B(x,r)) \rightharpoonup H(\theta(x))\mathcal{H}^k \llcorner((x + \mathrm{span}(\tau(x))) \cap B(0,1)). \tag{8.45}$$

We observe that properties (1) and (2) hold for μ-a.e. point. Indeed the fact that (1) holds for μ-a.e. x follows from Corollary 8.9, while the fact that (2) holds for μ-a.e. x is a consequence of [35, Theorem 4.8]. Moreover, by (8.44) and by (8.45), for every $x \in F$ there exists a radius $r(x) < \varepsilon$ such that

$$|\mu_{x,r}(B(0,1)) - \theta(x)\omega_k| \leqslant \frac{\varepsilon}{2}\theta(x)\omega_k, \qquad \text{for a.e. } r < r(x).$$

This inequality implies that

$$|\mu(B(x,r)) - \theta(x)\omega_k r^k| \leqslant \frac{\varepsilon}{2}\theta(x)\omega_k r^k, \qquad \text{for a.e. } r < r(x), \tag{8.46}$$

so that in particular

$$\theta(x)\left(1-\frac{\varepsilon}{2}\right)\omega_k r^k \leqslant \mu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$

Plugging the last inequality in the right-hand side of (8.46), we get

$$|\mu(B(x,r)) - \theta(x)\omega_k r^k| \leqslant \frac{\varepsilon}{2-\varepsilon}\mu(B(x,r)) \leqslant \varepsilon\mu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$

which gives condition (iii) of the Claim.

Analogously, we get that

$$|\nu(B(x,r)) - H(\theta(x))\omega_k r^k| \leqslant \varepsilon\nu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$

The validity of the claim is then obtained via the Vitali-Besicovitch covering theorem ([12, Theorem 2.19]).

□

8.5 PROOF OF THE REPRESENTATION ON ALL FLAT CHAINS WITH FINITE MASS

In this section we prove Proposition 8.6. We first observe that the condition (8.7) is necessary for the validity of (8.8). Indeed, consider a map H as in Assumption 1 for which (8.7) does not hold. It means that there exists a constant $C > 0$ and a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ converging to 0 such that $H(\theta_i) \leqslant C\theta_i$ for every $i \in \mathbb{N}$. We consider now the sequence of polyhedral k-chains $\{P_i\}_{i\in\mathbb{N}}$ supported in the unit cube $[0,1]^n$ and defined as

$$P_i := \sum_{j=1}^{N_i} [\pi_i^j \cap [0,1]^d, \tau, \theta_i],$$

where for i fixed, π_i^j are k-planes parallel to $\{x_{k+1} = \ldots = x_d = 0\}$ whose last $(d-k)$ coordinates are "uniformly distributed" in $[0,1]^{d-k}$, τ is a fixed orientation for all the k-planes π_i^j not depending on i or j and $N_i := \min\{N \in \mathbb{N} : N\theta_i \geqslant 1\}$. Since $\theta_i \to 0$, then $N_i \to \infty$. For i large enough, so that $\theta_i N_i \leqslant 2$, we can compute

$$\Phi_H(P_i) = \sum_{j=1}^{N_i} \Phi_H([\pi_i^j \cap [0,1]^d, \tau, \theta_i]) = N_i H(\theta_i) \leqslant C N_i \theta_i \leqslant 2C.$$

Nevertheless, since $\theta_i N_i \to 1$, then the sequence $\{P_i\}_{i\in\mathbb{N}}$ converges in flat norm to the k-current T, acting on k-forms as

$$\langle T, \omega\rangle = \int_{[0,1]^d} \langle \omega(x), \tau\rangle \, d\mathcal{L}^d(x),$$

which belongs to $(\mathbf{F}_k(\mathbb{R}^d) \cap \{T \in \mathscr{D}_k(\mathbb{R}^d) : \mathbb{M}(T) < \infty\}) \setminus \mathbf{R}_k(\mathbb{R}^d)$. Clearly, $F_H(T) \leqslant 2C$.

We show now that, if H is also monotone non-decreasing on $[0, \infty)$, then condition (8.7) is also sufficient to the validity of (8.8). The proof is a consequence of the definition of F_H in (8.3) and the following Lemma (see also [27, Lemma 4.5]):

Lemma 8.10. *Assume* H *is as in Assumption 1, is monotone non-decreasing on* $[0, \infty)$*, and satisfies* (8.7). *Let* $\{R_j\}_{j\in\mathbb{N}} \subset \mathbf{R}_k(\mathbb{R}^d)$ *and let us assume that*

$$\sup_{j\in\mathbb{N}} \mathbb{M}_H(R_j) \leqslant C < +\infty.$$

If $\lim_{j\to\infty} \mathbb{F}(R_j - T) = 0$ *for some* $T \in \mathbf{F}_k(\mathbb{R}^d)$ *with finite mass, then* T *is in fact rectifiable.*

Proof. Step 1. We prove the lemma for $k = 0$, recalling that a 0-dimensional rectifiable current $R = R[E, \tau, \theta]$, with $\tau(x) = \pm 1$, is an atomic signed measure (i.e. a measure supported on a countable set).

We observe that (8.7) implies that there exists $\delta_0 > 0$ such that $H(\theta) > 0$ for every $\theta \in (0, \delta_0)$. We define the monotone non-decreasing function $f : [0, \delta_0) \to [0, +\infty)$ given by

$$f(\theta) := \begin{cases} \sup_{t\in(0,\theta]} \dfrac{t}{H(t)} & \text{if } 0 < \theta < \delta_0, \\ 0 & \text{if } \theta = 0. \end{cases}$$

By assumption (8.7), f is continuous in 0 and $H(\theta)f(\theta) \geqslant \theta$. Fix any $\delta \in (0, \delta_0)$. For any $j \in \mathbb{N}$

$$\begin{aligned}\mathbb{M}(R_j \llcorner \{x : \theta_j(x) < \delta\}) &= \int_{E_j\cap\{\theta_j<\delta\}} \theta_j(x)\, d\mathcal{H}^k(x) \leqslant \int_{E_j\cap\{\theta_j<\delta\}} f(\theta_j(x))H(\theta_j(x))\, d\mathcal{H}^k(x) \\ &\leqslant f(\delta) \int_{E_j\cap\{\theta_j<\delta\}} H(\theta_j(x))\, d\mathcal{H}^k(x) \leqslant f(\delta)\mathbb{M}_H(R_j) \leqslant Cf(\delta).\end{aligned}$$

Therefore, up to subsequences the sequence $\{R_j \llcorner \{x : \theta_j(x) < \delta\}\}_{j\in\mathbb{N}}$ converges to a signed measure R_2 of mass less than or equal to $Cf(\delta)$. On the other hand, using the upper bound on $\mathbb{M}_H(R_j)$ and the monotonicity of H, we deduce that the measures $R_j \llcorner \{x : \theta_j(x) \geqslant \delta\}$ are supported on a uniformly (with respect to j) bounded number of points, and converge to a discrete measure R_1. Hence, for any $\varepsilon > 0$, the limit T can be written as the sum of a discrete measure R_1 and of an error R_2 with mass less than or equal to ε. Since ε is arbitrary, the statement follows.

Step 2. We prove the claim for $k > 0$.

We apply [50, 4.3.1] to the sequence $\{R_j\}_{j\in\mathbb{N}}$ to deduce that for any $I \in I(d, k)$

$$\lim_{j\to\infty} \int_{\mathbb{R}^k} \mathbb{F}(\langle R_j - T, p_I, y\rangle)\, dy \leqslant \lim_{j\to\infty} \mathbb{F}(R_j - T) = 0.$$

Since the sequence of non-negative functions $\{\mathbb{F}(\langle R_j - T, p_I, \cdot\rangle)\}_{j\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^k)$ to 0, up to a (not relabelled) subsequence, we get the pointwise convergence

$$\lim_{j\to\infty} \mathbb{F}(\langle R_j - T, p_I, y\rangle) = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^k, \text{ for every } I \in I(d,k).$$

We apply the Fatou lemma and [39, Corollary 3.2.5(5)] to the sequence $\{R_j\}_{j\in\mathbb{N}}$ to deduce

$$\int_{\mathbb{R}^k} \liminf_{j\to\infty} \mathbb{M}_H(\langle R_j, p_I, y\rangle)\, dy \leqslant \liminf_{j\to\infty} \int_{\mathbb{R}^k} \mathbb{M}_H(\langle R_j, p_I, y\rangle)\, dy \leqslant \liminf_{j\to\infty} \mathbb{M}_H(R_j) \leqslant C. \tag{8.47}$$

Hence the integrand in the left-hand side is finite a.e., namely $\liminf_{j\to\infty} \mathbb{M}_H(\langle R_j, p_I, y\rangle) < \infty$ for $\mathcal{H}^k$-a.e. $y \in \mathbb{R}^k$, for every $I \in I(d,k)$. Hence we are can apply Step 1 to a.e. slice $\langle R_j, p_I, y\rangle$ to a y-dependent subsequence and deduce that

$$\langle T, p_I, y\rangle \text{ is } 0\text{-rectifiable for } \mathcal{H}^k\text{-a.e. } y \in \mathbb{R}^k, \text{ for every } I \in I(d,k). \tag{8.48}$$

To conclude the proof we employ Theorem [83, Rectifiable slices theorem, pp. 166-167], see Theorem 7.15, which ensures that a finite mass flat chain T is rectifiable if and only if property (8.48) holds. □

BIBLIOGRAPHY

[1] G. Alberti and A. Marchese. On the differentiability of Lipschitz functions with respect to measures in the Euclidean space. *Geom. Funct. Anal.*, 26(1):1–66, 2016.

[2] W. K. Allard. On the first variation of a varifold. *Ann. Math.*, 95:417–491, 1972.

[3] W. K. Allard. A characterization of the area integrand. In *Symposia Mathematica, Vol. XIV (Convegno di Teoria Geometrica dell'Integrazione e Varietà Minimali, INDAM, Rome, 1973)*, pages 429–444. Academic Press, London, 1974.

[4] W. K. Allard. On the first variation of a varifold: boundary behaviour. *Ann. Math.*, 101: 418–446, 1975.

[5] W. K. Allard and F. J. Almgren, Jr., editors. *Geometric measure theory and the calculus of variations*, volume 44 of *Proceedings of Symposia in Pure Mathematics*, 1986. American Mathematical Society, Providence, RI.

[6] F. Almgren. Deformations and multiple-valued functions. Geometric measure theory and the calculus of variations. pages 29–130, 1986.

[7] F. J. Jr Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. Math.*, 87:321–391, 1968.

[8] F. J. Jr. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(165):viii+199 pp, 1976.

[9] F. J. Jr. Almgren. *Almgren's big regularity paper.* Q*-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension* 2, volume 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000. With a preface by Jean E. Taylor and Vladimir Scheffer.

[10] J. C. Alvarez-Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 1–48. Cambridge Univ. Press, Cambridge, 2004.

[11] L. Ambrosio and B. Kirchheim. Currents in metric spaces. *Acta Mathematica*, 185(1): 1–80, 2000.

[12] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[13] M. Bernot, V. Caselles, and J.-M. Morel. Traffic Plans. *Publicacions Matematiques*, 49(2): 417–451, 2005.

[14] M. Bernot, V. Caselles, and J.-M. Morel. The structure of branched transportation networks. *Calc. Var. Partial Differential Equations*, 32(3):279–317, 2008.

[15] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks. Models and theory.* Lecture Notes in Mathematics, 1955. Springer, Berlin, 2009.

[16] M. Bonafini, G. Orlandi, and E. Oudet. Variational approximation of functionals defined on 1-dimensional connected sets: the planar case. 2016. preprint https://arxiv.org/abs/1610.03839.

[17] M. Bonnivard, A. Lemenant, and F. Santambrogio. Approximation of length minimization problems among compact connected sets. *SIAM J. Math. Anal.*, 47(2):1489–1529, 2015.

[18] A. Brancolini and S. Solimini. Fractal regularity results on optimal irrigation patterns. *J. Math. Pures Appl. (9)*, 102(5):854–890, 2014.

[19] A. Brancolini and B. Wirth. Equivalent formulations for the branched transport and urban planning problems. *J. Math. Pures Appl.*, 106(4):695–724, 2016.

[20] A. Brancolini, C. Rossmanithand, and B. Wirth. Optimal micropatterns in 2d transport networks and their relation to image inpainting. Preprint, 2016.

[21] A. Brancolini, G. Buttazzo, and F. Santambrogio. Path functionals over Wasserstein spaces. *J. Eur. Math. Soc. (JEMS)*, 8(3):415–434, 2006.

[22] L. Brasco, G. Buttazzo, and F. Santambrogio. A Benamou-Brenier approach to branched transport. *SIAM J. Math. Anal.*, 43(2):1023–1040, 2011.

[23] H. Busemann, G. Ewald, and G. C. Shephard. Convex bodies and convexity on Grassmann cones. V. Totally convex functionals. *Arch. Math. (Basel)*, 13:512–526, 1962.

[24] H. Busemann, G. Ewald, and G. C. Shephard. Convex bodies and convexity on Grassmann cones. I–IV. *Math. Ann.*, 151:1–41, 1963.

[25] A. Chambolle, B. Merlet, and L. Ferrari. A simple phase-field approximation of the Steiner problem in dimension two. 2016. Available on ArXiv: https://arxiv.org/abs/1609.00519.

[26] M. Colombo, A. De Rosa, and A. Marchese. On the stability of branched transportation: Lagrange formulation. Paper in preparation.

[27] M. Colombo, A. De Rosa, and A. Marchese. Improved stability of optimal traffic paths. *Available on arxiv at https://arxiv.org/abs/1701.07300*, 2017.

[28] M. Colombo, A. De Rosa, A. Marchese, and S. Stuvard. On the lower semicontinuous envelope of functionals defined on polyhedral chains. 2017. Available on: https://arxiv.org/abs/1703.01938.

[29] T. D'Arcy. *On Growth and Form*. Cambridge University Press, 1942.

[30] G. David. Regularity of minimal and almost minimal sets and cones: J. Taylor's theorem for beginners. In *Analysis and geometry of metric measure spaces*, volume 56 of *CRM Proc. Lecture Notes*, pages 67–117. Amer. Math. Soc., Providence, RI, 2013.

[31] G. David. Should we solve Plateau's problem again? In *Advances in Analysis: the legacy of Elias M. Stein*. Princeton University Press, 2014.

[32] G. David and S. Semmes. Uniform rectifiability and quasiminimizing sets of arbitrary codimension. pages viii+132 pp., 2000.

[33] E. De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r-dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[34] E. De Giorgi. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r-dimensioni. *Ricerche Mat.*, 4:95–113, 1955.

[35] C. De Lellis. *Rectifiable sets, densities and tangent measures*. Zurich Lectures in Advanced Mathematics. European Mathematical Society, Zürich, 2008.

[36] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, 170(2):137–184, 2003.

[37] C. De Lellis, F. Ghiraldin, and F. Maggi. A direct approach to Plateau's problem. *JEMS*, 2014.

[38] C. De Lellis, A. De Rosa, and F. Ghiraldin. A direct approach to the anisotropic Plateau's problem. *Available on arxiv*, 2016.

[39] T. De Pauw and R. Hardt. Size minimization and approximating problems. *Calc. Var. Partial Differential Equations*, 17(4):405–442, 2003.

[40] G. De Philippis and F. Maggi. Dimensional estimates for singular sets in geometric variational problems with free boundaries. *J. Reine Angew. Math.* , 2014.

[41] G. De Philippis and F. Rindler. On the structure of -free measures and applications. *Ann. of Math.*, 2016.

[42] G. De Philippis, A. De Rosa, and F. Ghiraldin. Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies. *Accepted on Comm. Pure Appl. Math.*, 2016. Available on arxiv: https://arxiv.org/abs/1609.01908.

[43] G. De Philippis, A. De Rosa, and F. Ghiraldin. Existence results for minimizers of parametric elliptic functionals. 2016. Forthcoming.

[44] G. De Philippis, A. De Rosa, and F. Ghiraldin. A direct approach to Plateau's problem in any codimension. *Adv. in Math.*, 288:59–80, January 2015.

[45] A. De Rosa. Minimization of anisotropic energies in classes of rectifiable varifolds. 2016. Available on arxiv at https://arxiv.org/abs/1611.07929.

[46] G. Devillanova and S. Solimini. On the dimension of an irrigable measure. *Rend. Sem. Mat. Univ. Padova*, 117:1–49, 2007.

[47] G. Devillanova and S. Solimini. Elementary properties of optimal irrigation patterns. *Calc. Var. Partial Differential Equations*, 28(3):317–349, 2007.

[48] J. Douglas. Solution of the problem of Plateau. *Trans. Amer. Math. Soc.*, 33(1):263–321, 1931.

[49] S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*. Princeton University Press, 1952.

[50] H. Federer. *Geometric measure theory*, volume 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York Inc., New York, 1969.

[51] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. Math.*, 72:458–520, 1960.

[52] I. Fragalà and C. Mantegazza. On some notions of tangent space to a measure. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(2):331–342, 1999.

[53] E. N. Gilbert. Minimum cost communication networks. *Bell System Tech. J.*, 46:2209–2227, 1967.

[54] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[55] J. Harrison. Soap film solutions to Plateau's problem. *J. Geom. Anal.*, 24(1):271–297, 2014.

[56] J. Harrison and H. Pugh. Existence and soap film regularity of solutions to Plateau's problem. 2013. arXiv:1310.0508.

[57] J. Harrison and H. Pugh. Solutions To Lipschitz Variational Problems With Cohomological Spanning Conditions. 2015. arxiv:1506.01692.

[58] J. Harrison and H. Pugh. General Methods of Elliptic Minimization . 2016. https://arxiv.org/abs/1603.04492.

[59] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. ISBN 0-387-90148-5. Corrected reprint of the 1976 original.

[60] W. Hurewicz and H. Wallman. *Dimension Theory*. Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.

[61] S. G. Krantz and H. R. Parks. *Geometric integration theory*, volume 80 of *Cornerstones*. Birkhäuser Boston, Inc., Boston, MA, 2008.

[62] J. L. Lagrange. Essai d'une nouvelle methode pour determiner les maxima et les minima des formules integrales indefies. *Miscellania Taurinensia II*, 1:173–195, 1760–1761.

[63] S. Luckhaus. Uniform rectifiability from mean curvature bounds. *Z. Anal. Anwend.*, 27 (4):451–461, 2008.

[64] F. Maddalena, S. Solimini, and J.M. Morel. A variational model of irrigation patterns. *Interfaces and Free Boundaries*, 5:391–416, 2003.

[65] A. Marchese and A. Massaccesi. An optimal irrigation network with infinitely many branching points. *ESAIM: COCV*, 22(2):543–561, 2016.

[66] A. Marchese and A. Massaccesi. The steiner tree problem revisited through rectifiable G-currents. *Advances in Calculus of Variations*, 9(1):19–39, 2016.

[67] A. Massaccesi, E. Oudet, and B. Velichkov. Numerical calibration of steiner trees. 2016. Available online: http://cvgmt.sns.it/paper/2995/.

[68] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[69] J.-M. Morel and F. Santambrogio. The regularity of optimal irrigation patterns. *Arch. Ration. Mech. Anal.*, 195(2):499–531, 2010.

[70] F. Morgan. *Geometric measure theory. A beginner's guide. Fourth edition*. Elsevier/Academic Press, Amsterdam, 2009.

[71] C. B. Jr Morrey. *Multiple integrals in the calculus of variations*. Springer-Verlag, Berlin Heidelberg New York, 1966.

[72] E. Oudet and F. Santambrogio. A Modica-Mortola approximation for branched transport and applications. *Arch. Ration. Mech. Anal.*, 201(1):115–142, 2011.

[73] E. Paolini and E. Stepanov. Optimal transportation networks as flat chains. *Interfaces Free Bound.*, 8(4):393–436, 2006.

[74] E. Paolini and E. Stepanov. Decomposition of acyclic normal currents in a metric space. *J. Funct. Anal.*, 263(11):3358–3390, 2012.

[75] P. Pegon. On the lagrangian branched transport model and the equivalence with its eulerian formulation. Preprint, 2016.

[76] D. Preiss. Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability, and densities. *Ann. of Math. (2)*, 125(3):537–643, 1987.

[77] E. R. Reifenberg. Solution of the Plateau problem for m-dimensional surfaces of varying topological type. *Acta Math.*, 104:1–92, 1960.

[78] E. R. Reifenberg. On Plateau's problem. *Ann. of Math. (2)*, 80:15–21, 1964.

[79] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[80] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5:206–238, 1993.

[81] Y. Tonegawa. Some remarks on the level set flow by anisotropic curvature. *Calculus of Variations and Partial Differential Equations*, 10(2):101–118, 2000.

[82] B. White. The deformation theorem for flat chains. *Acta Mathematica*, 183(2):255–271, 1999.

[83] B. White. Rectifiability of flat chains. *Ann. of Math. (2)*, 150(1):165–184, 1999.

[84] Brian White. Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *J. Reine Angew. Math.*, 488:1–35, 1997.

[85] Q. Xia. Optimal paths related to transport problems. *Commun. Contemp. Math.*, 5: 251–279, 2003.

[86] Q. Xia. Interior regularity of optimal transport paths. *Calc. Var.*, 20:283–299, 2004.

[87] Q. Xia. Boundary regularity of optimal transport paths. *Adv. Calc. Var.*, 4(2):153–174, 2011.