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Line energies for gradient vector fields in the plane

Luigi Ambrosio, Camillo De Lellis, Carlo Mantegazza

Scuola Normale Superiore, Piazza Cavalieri 7, I-56100 Pisa, Italy
(e-mail: ambrosio@bibsns.sns.it / delellis@cibs.sns.it / mantegaz@cibs.sns.it)

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Abstract. In this paper we study the singular perturbation of $\int(1-|\nabla u|^2)^2$ by $\varepsilon^2|\nabla^2 u|^2$. This problem, which could be thought as the natural second order version of the classical singular perturbation of the potential energy $\int(1-u^2)^2$ by $\varepsilon^2|\nabla u|^2$, leads, as in the first order case, to energy concentration effects on hypersurfaces. In the two dimensional case we study the natural domain for the limiting energy and prove a compactness theorem in this class.

1 Introduction

This paper is devoted to the asymptotic behaviour of the functionals

$$F_\varepsilon(u) := \frac{1}{2}\int_\Omega \left(\varepsilon|\nabla^2 u|^2 + \frac{(1-|\nabla u|^2)^2}{\varepsilon}\right) dx \qquad \Omega \subset \mathbf{R}^n$$

as $\varepsilon \downarrow 0$. In the two dimensional case $n = 2$ we define a space of functions which seems to be the natural domain for the limiting energy and prove the equicoercivity of F_ε in this space.

The problem of studying the behaviour of F_ε as $\varepsilon \downarrow 0$ was raised more than 10 years ago by P. Aviles and Y. Giga in [7], in connection with the theory of smectic liquid crystals; more recently, G. Gioia and M. Ortiz considered in [19] the same functionals in the two dimensional case to model the energy deformation of thin film blisters undergoing a biaxial compression. In their model ε is proportional to the thickness of the blister, u denotes the vertical displacement and, neglecting the horizontal displacement, using the classical von Kármán theory of plates they proved that $\varepsilon^2|\nabla^2 u|^2$ represents

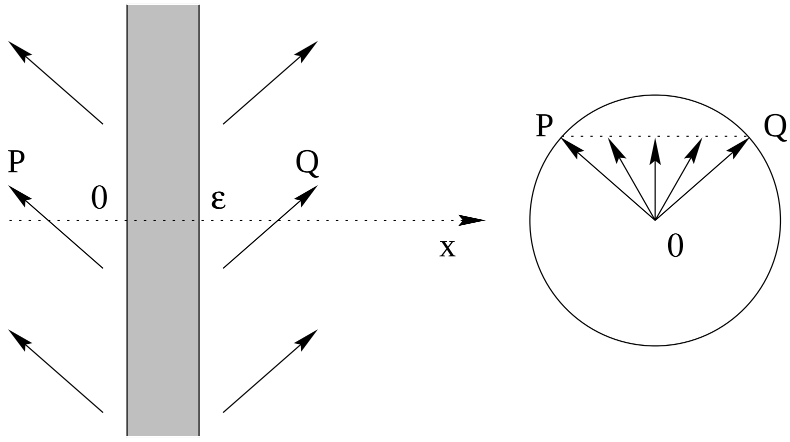

Fig. 1. The one dimensional ansatz

(under suitable isotropy assumptions) the bending energy of the film, while the elastic energy is represented by $\left(1-|\nabla u|^2\right)^2$. Hence, $\varepsilon F_\varepsilon$ is a singular perturbation of the elastic energy of the film.

It is clear that any admissible function for the limit problem must satisfy the eikonal equation

$$|\nabla u| = 1 \quad \mathcal{L}^n\text{-a.e. in } \Omega\,. \tag{1.1}$$

There are several heuristic arguments suggesting that the limit energy can concentrate on hypersurfaces: the strongest one is perhaps the analogy with the first order Modica–Mortola functionals (see [17], [16])

$$M_\varepsilon(u) := \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{(1-u^2)^2}{\varepsilon}\right)\,dx\ \ ,$$

whose Γ-limit is a constant multiple of the area functional. Notice that in M_ε there appears a "two well" potential, while the potential in F_ε has a single well, the unit circle. Notice also that $F_\varepsilon(u) = G_\varepsilon(\nabla u)$, where

$$G_\varepsilon(v) := \frac{1}{2}\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-|v|^2)^2}{\varepsilon}\right)\,dx\ \ ,$$

and, if we don't take into account the constraint that $\operatorname{curl} v = 0$, it is easy to prove that the Γ-limit of G_ε is identically 0 (see for instance [20] or the general formula given in [4]). Taking into account the zero curl constraint, instead, leads to the following ansatz, illustrated in Fig. 1: near to a jump discontinuity of the gradient the optimal transition layer is obtained keeping constant the tangential component of the gradients and making a sharp transition between the two normal ones.

This ansatz, first stated in [7], formally leads to the limiting energy

$$\frac{1}{6}\int_{J_{\nabla u}} |\nabla^+ u - \nabla^- u|^3 \, d\mathcal{H}^{n-1} \quad , \tag{1.2}$$

where $J_{\nabla u}$ is the jump set of ∇u and $\nabla^{\pm} u$ are the traces on both sides of the jump set.

The rigorous study of the asymptotic behaviour of F_ε is a very challenging mathematical problem, because many standard methods available for the analysis of first order problems are of difficult use in the second order ones (for instance truncation arguments). If $n > 2$ there is presently no idea on what the function space for the limit problem should be.

In the two dimensional case, the first significant progress was made by R.W. Kohn and W. Jin (see [13], [14]), who realized that the divergence of the vector field

$$\Sigma u := \left(u_1 \left(1 - u_2^2 - \frac{1}{3}u_1^2\right), -u_2 \left(1 - u_1^2 - \frac{1}{3}u_2^2\right)\right) \quad ,$$

namely $(1-|\nabla u|^2)(u_{11}-u_{22})$, can be used to estimate from below F_ε; they used this estimate to compute the limit as $\varepsilon \downarrow 0$ of the minimum problems

$$\min\left\{ F_\varepsilon(u) : \; u \in W^{2,2}(\Omega), \;\; u|_{\partial\Omega} = 0, \;\; \frac{\partial u}{\partial \nu} = -1 \right\} \tag{1.3}$$

(here ν is the outer normal to $\partial\Omega$) in some particular cases; their results are in agreement with the conjecture that the limit energy is given by (1.2). The Kohn–Jin argument implies that, for the limit problem, any admissible function must have the property that $\operatorname{div} \Sigma u$ (in the sense of distributions) is representable by a measure. By the eikonal constraint, Σu reduces to

$$\Xi u := \frac{2}{3}\left(u_1^3, -u_2^3\right) \; .$$

Aviles and Giga went further in [9], noticing that, by rotation invariance, the same property holds for the fields

$$\Xi_{\xi\eta} u := \frac{2}{3}\left(\left(\frac{\partial u}{\partial \xi}\right)^3 \xi - \left(\frac{\partial u}{\partial \eta}\right)^3 \eta \right)$$

where (ξ, η) is any orthonormal basis of $\mathbf{R}^2$. They proved that the supremum of the divergences of all these vector fields provides a functional $J : W^{1,3}(\Omega) \to [0, \infty]$ which is lower semicontinuous with respect to the strong $W^{1,3}(\Omega)$ convergence and which coincides with (1.2) if u solves

(1.1) and ∇u has bounded variation in Ω. Moreover, refining the Kohn–Jin argument, they proved that

$$J(u) \leq \liminf_{h\to\infty} F_{\varepsilon_h}(u_h)$$

for any sequence (u_h) converging to u in $W^{1,3}(\Omega)$. Hence, $\Gamma - \liminf F_\varepsilon$ (in the $W^{1,3}(\Omega)$ topology) is finite only if $J(u) < \infty$, and this strongly suggests that

$$AG_e(\Omega) := \left\{ u \in W^{1,3}(\Omega) :\ \text{(1.1) holds and } J(u) < \infty \right\}$$

is the natural function space for the limiting problem.

In our paper we answer to several questions raised in [9], the first one being whether $AG_e(\Omega)$ coincides with

$$BV^2_e(\Omega) := \left\{ u \in W^{1,3}(\Omega) :\ \text{(1.1) holds and } \nabla u \in BV(\Omega, \mathbf{R}^2) \right\} .$$

We show by a counterexample that there exist functions in $AG_e(\Omega)$ such that ∇u has not locally bounded variation in Ω. This negative result shows that a separate study is required for the space $AG_e(\Omega)$, for which the existing theory of BV functions can be used only as a useful analogy.

The second question raised in [9] concerns the compactness properties of $AG_e(\Omega)$ with respect to the strong $W^{1,3}(\Omega)$ convergence. We prove that for any constant $M > 0$ the sublevel sets

$$\{u \in AG_e(\Omega) :\ J(u) \leq M\}$$

are compact. Since J is lower semicontinuous with respect to the strong $W^{1,3}(\Omega)$ convergence, this provides existence of minimizers for the problem

$$(\mathcal{P}) \qquad \min \{J(u) :\ u \in AG_e(\Omega),\ u \geq 0,\ u|_{\partial\Omega} = 0\} .$$

A still open conjecture actually states that the distance function from $\partial\Omega$ is the minimizer in $(\mathcal{P})$. Our compactness theorem also takes into account the case when the eikonal equation is fulfilled only in the limit, and shows that the functionals F_ε are equicoercive, i.e., any sequence (u_h) with $F_{\varepsilon_h}(u_h)$ bounded and $\varepsilon_h \downarrow 0$ has a subsequence $(u_{h(k)})$ converging to some $u \in AG_e(\Omega)$. This result is of interest in view of the fact that Γ-convergence implies converge of minimizers to minimizers (and of the minimum values as well) only if the functionals are equicoercive.

We also study in $BV^2_e(\Omega)$ the functionals

$$J_\rho(u) := c_\rho \int_{J_{\nabla u}} \left|\nabla^+ u - \nabla^- u\right|^{\rho+1} \, d\mathcal{H}^1 ,$$

where c_ρ is a suitable normalization constant ($c_2 = 1/6$). As shown in [9], these functionals formally arise from a variant of F_ε, in which the ρ-th power of $|\nabla u|^2 - 1$ is taken into account. Positively answering to a conjecture in [9], we prove that for any $\rho > 2$ the functional J_ρ is not lower semicontinuous, thus showing that in this case a microstructure arises and the one-dimensional ansatz is violated. The lower semicontinuity for $\rho < 2$ is still an open problem.

We conclude noticing that we still don't know whether (1.2) is the limiting energy or not mainly because of two reasons: the first one is that this representation of $J(u)$ is known to be true only if ∇u has locally bounded variation in Ω. The second one is that we still don't know whether any function $u \in AG_e(\Omega)$ can be approximated by functions $u_\varepsilon \in W^{2,2}(\Omega)$ in such a way that $F_\varepsilon(u_\varepsilon)$ converge to $J(u)$ as $\varepsilon \downarrow 0$ (the so-called $\Gamma - \limsup$ inequality). In some sense, both problems are related to the conjecture that the limiting energy concentrates *only* on lines, a conjecture supported by the computations done in specific examples by Kohn and Jin. We hope to be able to attack these problems in a forthcoming paper.

Added in proof. After the completion of this paper we learned of an independent work by A.Desimone, R.W.Kohn, S.Müller and F.Otto closely related to ours (see [10]); they obtain the compactness theorem with a different argument.

2 Notation and preliminary results

In this paper Ω denotes an open set in $\mathbf{R}^2$ and $\mathcal{L}^2$ and $\mathcal{H}^1$ denote respectively the Lebesgue measure in $\mathbf{R}^2$ and the Hausdorff 1-dimensional measure in $\mathbf{R}^2$. Given a Radon measure μ in Ω and a Borel set $B \subset \Omega$, the restriction $\mu \llcorner B$ is the Radon measure $\chi_B \mu$, i.e.

$$\mu \llcorner B(A) := \mu(A \cap B) \qquad \text{for any Borel set } A \subset \Omega\,.$$

For any $v \in L^1_{\mathrm{loc}}(\Omega, \mathbf{R}^p)$ and any $x \in \Omega$ the *approximate limit* of v at x, denoted by $\tilde{v}(x)$, is the unique $z \in \mathbf{R}^p$ satisfying

$$\lim_{\varrho \downarrow 0} \varrho^{-2} \int_{B_\varrho(x)} |u(y) - z|\, dy = 0\,.$$

We denote by S_v the set of *approximate discontinuity* points, i.e., the set of points where the approximate limit does not exist. Analogously, the *one sided approximate limits* $v^+(x),\ v^-(x)$ at x are vectors $a,\ b \in \mathbf{R}^p$ satisfying

$$\lim_{\varrho \downarrow 0} \varrho^{-2} \int_{B^+_\varrho(x)} |u(y) - a|\, dy = 0$$
$$\text{and} \quad \lim_{\varrho \downarrow 0} \varrho^{-2} \int_{B^-_\varrho(x)} |u(y) - b|\, dy = 0 \ ,$$

where $B^{\pm}_{\varrho}(x) = \{y \in B_{\varrho}(x) : \pm\langle y - x, \nu\rangle \geq 0\}$ are the two half balls corresponding to some unit vector ν. We denote by $J_v \subset S_v$ the set of *approximate jump* points, i.e., all points $x \in S_v$ such that the approximate limits $v^{\pm}(x)$ exists for some unit vector ν, denoted by $\nu_v(x)$. For any $x \in J_v$ the triplet

$$\big(v^+(x), v^-(x), \nu_v(x)\big)$$

is uniquely determined, up to a permutation of $(v^+(x), v^-(x))$ and a change of sign of $\nu_v(x)$. We use sometimes the abbreviation $[v(x)]^+_-$ for the *jump* $v^+(x) - v^-(x)$ of v at $x \in J_v$.

We now recall some facts about BV functions which will be used mainly in Sect. 3 (see [11] as a general reference on BV and [2], [3] for the decomposition of derivative). We denote by $BV(\Omega, \mathbf{R}^m)$ the space of $\mathbf{R}^m$-valued functions with bounded variation in Ω, i.e. the space of all functions $v \in L^1(\Omega, \mathbf{R}^m)$ whose distributional derivative is representable by a finite Radon measure in Ω. This measure, with values in $2 \times m$ matrices (2 columns, m rows) will be denoted by Dv. The measure Dv can be split in three mutually singular parts: the first one is the absolutely continuous part with respect to $\mathcal{L}^2$, whose density, denoted by ∇v, can also be interpreted as a differential of v, in an approximate sense. Another part of Dv is the jump part, denoted by $D_j v$ and defined as $Dv \llcorner J_v$; this part of the derivative is absolutely continuous with respect to $\mathcal{H}^1 \llcorner J_v$ and its density is $(v^+ - v^-) \otimes \nu_v$ (here $a \otimes b$ denotes the $2 \times m$ matrix tensor product of $a \in \mathbf{R}^m$ and $b \in \mathbf{R}^2$). Finally, the remaining part of the derivative, denoted by $D_c v$, is called Cantor part; the measure $D_c v$ is singular with respect to $\mathcal{L}^2$ and vanishes on any Borel set σ-finite with respect to $\mathcal{H}^1$. Summarizing, we have

$$Dv = D_a v + D_c v + D_j v = \nabla v \mathcal{L}^2 + D_c v + (v^+ - v^-) \otimes \nu_v \mathcal{H}^1 \llcorner J_v \,.$$

It can also be proved that $\mathcal{H}^1(S_v \setminus J_v) = 0$, hence $\tilde{v}$ is defined $\mathcal{H}^1$-a.e. out of J_v. Since $|Dv|$ vanishes on $\mathcal{H}^1$-negligible sets, $\tilde{v}$ is defined $|D_a v| + |D_c v|$-a.e. in Ω.

3 The Aviles–Giga space

Let (ξ, η) be an orthonormal basis of $\mathbf{R}^2$ and $u \in W^{1,3}_{\mathrm{loc}}(\Omega)$; we define

$$\Sigma_{\xi\eta} u := u_\xi \left(1 - u_\eta^2 - \frac{1}{3} u_\xi^2\right) \xi - u_\eta \left(1 - u_\xi^2 - \frac{1}{3} u_\eta^2\right) \eta$$

where u_ξ, u_η are abbreviations for the partial derivatives along ξ and η respectively. Hence, $\Sigma_{\xi\eta} u$ is a locally integrable vector field in Ω which coincides with

$$\Xi_{\xi\eta} u := \frac{2}{3}\left(u_\xi^3 \xi - u_\eta^3 \eta\right)$$

if u satisfies the eikonal equation (1.1).

Using the divergences of these fields we can define a space of functions already considered implicitely in [9]; for this reason we call it the Aviles–Giga space.

Definition 3.1 (Aviles–Giga space) *We say that $u \in W^{1,3}_{\rm loc}(\Omega)$ belongs to $AG(\Omega)$ if $\operatorname{div} \Sigma_{\xi\eta}u$ is (representable by) a measure in Ω for any orthonormal basis (ξ, η) of $\mathbf{R}^2$.*
We denote by $AG_e(\Omega)$ the class of all functions $u \in AG(\Omega)$ such that $|\nabla u| = 1$ $\mathcal{L}^2$-a.e. in Ω.

In the following we denote by (e_1, e_2) the canonical basis of $\mathbf{R}^2$ and by

$$\varepsilon_1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad \varepsilon_2 := \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

the basis obtained from (e_1, e_2) under a anti clockwise rotation of $\pi/4$. As for the first order derivative in the sense of distributions (with directions instead of orthonormal bases), only these two bases are sufficient to recover the divergences of all vector fields $\Sigma_{\xi\eta}u$.

Theorem 3.2 *For any $u \in W^{1,3}_{\rm loc}(\Omega)$ we have*

$$\Sigma_{\xi\eta}u = (\cos 2\theta)\Sigma_{e_1e_2}u + (\sin 2\theta)\Sigma_{\varepsilon_1\varepsilon_2}u$$

with $\xi = (\cos\theta, \sin\theta)$, $\eta = (-\sin\theta, \cos\theta)$. In particular $u \in AG(\Omega)$ if and only if $\operatorname{div} \Sigma_{e_1e_2}u$ and $\operatorname{div} \Sigma_{\varepsilon_1\varepsilon_2}u$ are both representable by finite measures in Ω.

The proof of this theorem follows by long but straightforward computations, so we omit it. Motivated by this theorem, for any $u \in AG(\Omega)$ we define the $\mathbf{R}^2$-valued measure

$$Iu = (I^1u, I^2u) := (\operatorname{div} \Sigma_{e_1e_2}u, \operatorname{div} \Sigma_{\varepsilon_1\varepsilon_2}u) .$$

Notice that, according to Theorem 3.2, $|Iu|$ is the supremum of $|\operatorname{div} \Sigma_{\xi\eta}u|$ among all orthonormal bases (ξ, η) of $\mathbf{R}^2$; the latter is the functional J considered by Aviles and Giga in [9].

The following closure and lower semicontinuity theorem is a direct consequence of Theorem 3.2.

Theorem 3.3 (Lower semicontinuity) *Let $(u_h) \subset AG(\Omega)$ be strongly converging in $W^{1,3}_{\rm loc}(\Omega)$ to u. If $|Iu_h|(\Omega)$ are equibounded, then $u \in AG(\Omega)$, the measures Iu_h weakly* converge to Iu in Ω and hence*

$$|Iu|(\Omega) \le \liminf_{h\to\infty} |Iu_h|(\Omega) .$$

Proof. Since the fields $\Sigma_{e_1e_2}u_h$ converge to $\Sigma_{e_1e_2}u$ in $L^1_{\rm loc}(\Omega,\mathbf{R}^2)$, their divergences converge in the sense of distributions. Since $|I^1u_h|(\Omega)$ are equibounded, it follows that $\mathrm{div}\,\Sigma_{e_1e_2}u$ is representable by a finite measure in Ω and I^1u_h weakly* converge to I^1u as $h\to\infty$. The proof for I^2u_h is analogous. □

We now compute Iu in several cases of interest. Since

$$\Sigma_{e_1e_2}u=\left(u_1\left(1-u_2^2-\frac{1}{3}u_1^2\right),-u_2\left(1-u_1^2-\frac{1}{3}u_2^2\right)\right)\quad, \tag{3.1}$$

if ∇u belongs to a Sobolev space the pointwise divergence of the field is given by $(1-|\nabla u|^2)(u_{11}-u_{22})$; analogously, representing with a long but straightforward computation $\Sigma_{\varepsilon_1\varepsilon_2}u$ in the canonical basis of $\mathbf{R}^2$, we get

$$\Sigma_{\varepsilon_1\varepsilon_2}u=\left(u_2-\frac{2}{3}u_2^3,u_1-\frac{2}{3}u_1^3\right) \tag{3.2}$$

whose pointwise divergence is $2(1-|\nabla u|^2)u_{12}$. Assuming enough integrability of the second derivatives these expressions for the divergences are integrable and give also the divergence in the sense of distributions.

Proposition 3.4 *If $u\in W^{2,3/2}_{\rm loc}(\Omega)$ then*

$$I^1u=(1-|\nabla u|^2)(u_{11}-u_{22})\mathcal{L}^2\,,\quad I^2u=2(1-|\nabla u|^2)u_{12}\mathcal{L}^2$$

In particular $|Iu|=\left|1-|\nabla u|^2\right||\lambda_1-\lambda_2|\mathcal{L}^2$, where $\lambda_1,\ \lambda_2$ are the eigenvalues of ∇^2u.

Proof. The Sobolev embedding theorem implies that $|\nabla u|\in L^6_{\rm loc}(\Omega)$, hence $(1-|\nabla u|^2)|\nabla^2u|$ is locally integrable in Ω, by Hölder inequality. A smoothing argument also proves that

$$\begin{aligned}&\mathrm{div}\Sigma_{e_1e_2}u=(1-|\nabla u|^2)(u_{11}-u_{22})\mathcal{L}^2\quad,\\&\mathrm{div}\Sigma_{\varepsilon_1\varepsilon_2}u=2(1-|\nabla u|^2)u_{12}\mathcal{L}^2\end{aligned}$$

in the sense of distributions. The last part of the statement can be obtained noticing that the modulus of the difference between the eigenvalues of a symmetric matrix A is

$$\sqrt{|A_{11}-A_{22}|^2+4|A_{12}|^2}\,.$$

□

Now we analyze Iu in the case when ∇u is bounded and is a function with bounded variation. To this aim, in the following proposition we recall some facts about measure distributional derivatives of second order.

Proposition 3.5 (BV gradients) *Let $u \in W^{1,1}(\Omega)$ and assume that $\nabla u \in BV(\Omega, \mathbf{R}^2)$. Then $D^2u = D\nabla u$ is a symmetric matrix valued measure with finite total variation in Ω representable as*

$$\begin{aligned} D^2u &= D^2_a u + D^2_c u + D^2_j u \\ &= \nabla^2 u \mathcal{L}^2 + H|D^2_c u| + (\nabla^+ u - \nabla^- u) \otimes \nu_{\nabla u} \mathcal{H}^1 \llcorner J_{\nabla u}\,, \end{aligned}$$

where

(i) $\nabla^2 u(x)$ is a symmetric matrix for $\mathcal{L}^2$-a.e. $x \in \Omega$;
(ii) $H(x)$ is a unit symmetric matrix with rank 1 for $|D^2_c u|$-a.e. $x \in \Omega$;
(iii) $\nu_{\nabla u}(x)$ is parallel to $\nabla^+ u(x) - \nabla^- u(x)$ for $\mathcal{H}^1$-a.e. $x \in J_{\nabla u}$.

Proof. The decomposition of D^2u in three parts is a general property of BV functions, and since D^2u is symmetric all these parts (being mutually singular) are symmetric. Statement (i) follows by the identity $D^2_a u = \nabla^2 u \mathcal{L}^2$. Statement (ii) follows by Alberti rank one theorem (see [1]) and finally statement (iii) can be proved noticing that a tensor product $a \otimes b$ is symmetric if and only if a is parallel to b. □

The chain rule for the computation of the derivative of $f \circ v$ with $u \in BV$ and f Lipschitz and continuously differentiable has been first proved in [21] (see also [5] for the case when f is not C^1); it turns out that the "diffuse" part of derivative, made by absolutely continuous part and Cantor part, obeys to the classical chain rule, while the jump part obeys to the natural transformation rule for jumps.

Proposition 3.6 (Vol'pert chain rule) *Let $v \in BV_{\mathrm{loc}}(\Omega, \mathbf{R}^p)$ and let $f \in C^1(\mathbf{R}^p)$ with bounded gradient. Then $w = f \circ v \in BV_{\mathrm{loc}}(\Omega)$ and*

$$\begin{cases} \nabla w = \langle \nabla f(v), \nabla v \rangle, \quad D_c w = \langle \nabla f(\tilde{v}), D_c v \rangle \\ \\ D_j w = (f(v^+) - f(v^-))\, \nu_v \mathcal{H}^1 \llcorner J_v\,. \end{cases}$$

If ∇u is bounded in Ω and belongs to $BV(\Omega, \mathbf{R}^2)$ the measure Iu can computed using Vol'pert chain rule in BV; using (3.1) and (3.2) we obtain

$$I^1 u \llcorner (\Omega \setminus J_{\nabla u}) = \left(1 - |\widetilde{\nabla u}|^2\right) (D_{11} u - D_{22} u) \llcorner (\Omega \setminus J_{\nabla u}) \tag{3.3}$$

$$I^2 u \llcorner (\Omega \setminus J_{\nabla u}) = 2 \left(1 - |\widetilde{\nabla u}|^2\right) D_{12} u \llcorner (\Omega \setminus J_{\nabla u})\,. \tag{3.4}$$

Moreover, for any orthonormal basis (ξ, η) of $\mathbf{R}^2$ we have

$$\begin{aligned} &\mathrm{div}\, \Sigma_{\xi\eta} u \llcorner J_{\nabla u} = \\ &\quad \left\{ [\langle \Sigma_{\xi\eta} u, \xi \rangle]^+_- \langle \xi, \nu_{\nabla u} \rangle - [\langle \Sigma_{\xi\eta} u, \eta \rangle]^+_- \langle \eta, \nu_{\nabla u} \rangle \right\} \mathcal{H}^1 \llcorner J_{\nabla u}\,. \end{aligned} \tag{3.5}$$

In particular, (3.3), (3.4) and (3.5) imply the following result. For $a,\ b \in \mathbf{R}^2$ and $\nu \in \mathbf{S}^1$ we define $\Psi(a,b,\nu)$ as the supremum of

$$\begin{aligned}&\left[a_\xi\left(1-a_\eta^2-\frac{1}{3}a_\xi^2\right)-b_\xi\left(1-b_\eta^2-\frac{1}{3}b_\xi^2\right)\right]\langle\xi,\nu\rangle\\ &-\left[a_\eta\left(1-a_\xi^2-\frac{1}{3}a_\eta^2\right)-b_\eta\left(1-b_\xi^2-\frac{1}{3}b_\eta^2\right)\right]\langle\eta,\nu\rangle\end{aligned} \tag{3.6}$$

among all orthonormal bases (ξ,η) of $\mathbf{R}^2$ (here a_ξ is the component of a along ξ and a_η, b_ξ, b_η are defined analogously). We set

$$\Phi(a,b) := \Psi\left(a,b,\frac{b-a}{|b-a|}\right) \qquad \forall a,\ b\in\mathbf{R}^2,\ a\neq b\,. \tag{3.7}$$

Lemma 3.7 *Let Φ be defined as in (3.7). Then*

$$\Phi(a,b)=\frac{1}{6}|a-b|\left|6-3|a|^2-3|b|^2+|a-b|^2\right|\,.$$

Proof. First of all we compute the expression in (3.6) with $\nu=(a-b)/|a-b|$ in the following Cartesian system: the unit vector e_1, which gives the x_1 axis, is perpendicular to $a-b$ and such that $\langle a,e_1\rangle\geq 0$; the unit vector e_2 is, obviously, perpendicular to e_1 and such that $\langle a,e_2\rangle\geq 0$. In this system of coordinates we have

$$a=(r,s)\ ,\quad b=(r,-t)\ ,\quad \xi=(x,y)\ ,\quad \eta=(-y,x)\ ,$$

where $r,\ s,\ t,\ x,\ y$ are real numbers such that $r\geq 0, s\geq 0, t\geq 0$ and $x^2+y^2=1$. Now we substitute $a_\xi=rx+sy$, $b_\xi=rx-ty$, $a_\eta=-ry+sx$, $b_\eta=-ry-tx$, $\langle\xi,\nu\rangle=y$ and $\langle\eta,\nu\rangle=x$ in (3.6) to obtain:

$$\begin{aligned}&y\Bigg[(rx+sy)\left(1-(sx-ry)^2-\frac{(rx+sy)^2}{3}\right)\\ &-(rx-ty)\left(1-(-tx-ry)^2-\frac{(rx-ty)^2}{3}\right)\Bigg]\\ &-\Bigg[x(sx-ry)\left(1-(rx+sy)^2-\frac{(sx-ry)^2}{3}\right)\\ &+(tx+ry)\left(1-(rx-ty)^2-\frac{(tx+ry)^2}{3}\right)\Bigg]\end{aligned}$$

and a straightforward computation gives

$$(y^2-x^2)\frac{s+t}{3}\left(3-\frac{3}{2}(r^2+s^2)-\frac{3}{2}(r^2+t^2)+\frac{(s+t)^2}{2}\right)\,.$$

It is now obvious that the maximum of this expression is

$$\frac{1}{6}|s+t|\left|6-3(r^2+s^2)-3(r^2+t^2)+(s+t)^2\right| .$$

Finally we observe that $s+t=|a-b|$, $r^2+s^2=|a|^2$ and $r^2+t^2=|b|^2$; this completes the proof. $\square$

Theorem 3.8 (Representation of $|Iu|$) *Let $u \in W^{1,\infty}(\Omega)$ be such that $\nabla u \in BV(\Omega, \mathbf{R}^2)$. Then $u \in AG(\Omega)$ and*

$$\begin{aligned}|Iu| = &\left|1-|\nabla u|^2\right| |\lambda_1-\lambda_2| \mathcal{L}^2 + \left|1-|\widetilde{\nabla u}|^2\right| |D_c^2 u| \\ &+\frac{1}{6}|\nabla^+ u - \nabla^- u| \left|6-3|\nabla^+ u|^2 - 3|\nabla^- u|^2 \right.\\ &\left. +|\nabla^+ u - \nabla^- u|^2\right| \mathcal{H}^1 \llcorner J_{\nabla u} ,\end{aligned}$$

where λ_i are the eigenvalues of $\nabla^2 u$.

Proof. Let $D^2u = H|D^2u|$ be a polar representation of D^2u, set $A = \Omega \setminus J_{\nabla u}$ and notice that $H(x) = \nabla^2 u(x)/|\nabla^2 u(x)|$ for $\mathcal{L}^2$-a.e. $x \in \Omega$ while, by Theorem 3.5(ii), the rank of $H(x)$ is 1 for $|D_c^2 u|$-a.e. x; by (3.3) and (3.4), we get

$$\begin{aligned} I^1 u \llcorner A &= \left(1-|\widetilde{\nabla u}|^2\right)(H_{11}-H_{22})|D^2u| \llcorner A \\ I^2 u \llcorner A &= 2\left(1-|\widetilde{\nabla u}|^2\right) H_{12}|D^2 u| \llcorner A \end{aligned}$$

so that, arguing as in Proposition 3.4, we obtain

$$|Iu| \llcorner A = \left|1-|\widetilde{\nabla u}|^2\right| |\mu_1-\mu_2| |D^2 u| \llcorner A$$

where $\mu_1,\ \mu_2$ are the eigenvalues of H; since the norm of H is 1, the difference $|\mu_1-\mu_2|$ is 1 $|D_c^2 u|$-a.e. in Ω and is $|\lambda_1-\lambda_2|/|\nabla^2 u|$ $\mathcal{L}^2$-a.e. in Ω; since

$$|D^2 u| \llcorner A = |\nabla^2 u| \mathcal{L}^2 + |D_c^2 u|$$

the representation of $|Iu| \llcorner A$ follows.

In order to represent $|Iu|$ on A we notice that (3.5) gives

$$|Iu| \llcorner J_{\nabla u} = \Psi(\nabla^+ u, \nabla^- u, \nu_{\nabla u}) \mathcal{H}^1 \llcorner J_{\nabla u}$$

because $|Iu| = \sup |\mathrm{div} \Sigma_{\xi\eta} u|$ among all orthonormal bases (ξ, η) of $\mathbf{R}^2$. Since Ψ is odd with respect to ν and $\nu_{\nabla u}$ is parallel to $[\nabla u]^+_-$ the representation of $|Iu|$ on $J_{\nabla u}$ is achieved. $\square$

If u is a solution of (1.1) the absolutely continuous part and the Cantor part of $D^2 u$ give no contribution and we obtain the following result, first proved in [9].

Theorem 3.9 *If u satisfies (1.1) and $\nabla u \in BV(\Omega, \mathbf{R}^2)$ then $u \in AG_e(\Omega)$ and*

$$|Iu| = \frac{1}{6}|\nabla^+ u - \nabla^- u|^3 \mathcal{H}^1 \llcorner J_{\nabla u}\,. \tag{3.8}$$

Unfortunately the converse in Theorem 3.8 is not true, not even if $u \in AG_e(\Omega)$: indeed, we will show that there exist functions $u \in AG_e(\Omega)$ such that ∇u has not locally bounded variation in Ω. More precisely, we show how a function $u : \mathbf{R}^2 \to \mathbf{R}$ which belongs to $AG_e(U)$ for every bounded open set $U \subset \mathbf{R}^2$ and such that $\nabla u \notin BV_{\rm loc}(\mathbf{R}^2, \mathbf{R}^2)$ can be constructed.

First of all, for any integer $n \geq 1$ we define

$$\theta_n = \frac{\frac{1}{n} + \frac{1}{n+1}}{2} \cdot \frac{\pi}{4} \;\; ;$$

then we construct a lozenge R_0^n whose vertices, denoted by A_n, B_n, C_n, D_n, have the following coordinates:

$$A_n = \left(-1 - \frac{1}{n}, 0\right) \quad , \quad B_n = \left(0, \left(1 + \frac{1}{n}\right)\tan\theta_n\right)$$

$$C_n = \left(1 + \frac{1}{n}, 0\right) \quad , \quad D_n = \left(0, -\left(1 + \frac{1}{n}\right)\tan\theta_n\right) \; .$$

Between R_0^n and R_0^{n+1} we construct lozenges R_i^n ($1 \leq i \leq 2[(n+1)/2]$) homothetics to R_0^n as follows (see Fig. 2): setting $a_n = (1/n - 1/(n+1))/(2[(n+1)/2])$, where [] denotes the integer part, the coordinates of the vertices of R_i^n are

$$\begin{aligned} A_i^n &= \left(-1 - \frac{1}{n} + ia_n, 0\right) \\ C_i^n &= \left(1 + \frac{1}{n} - ia_n, 0\right) \\ B_i^n &= \left(0, \left(1 + \frac{1}{n} - ia_n\right)\tan\theta_n\right) \\ D_i^n &= \left(0, -\left(1 + \frac{1}{n} - ia_n\right)\tan\theta_n\right) . \end{aligned}$$

Now let us define the vector field x_n in the following way.
We draw R_i^k ($1 \leq k \leq n$, $0 \leq i \leq 2[(n+1)/2]$) and the axes $\{x_i = 0\}$. These lines divide $\mathbf{R}^2$ in a finite number of connected components. We put $x_n = \left(-\sqrt{2}, \sqrt{2}\right)/2$ in the unbounded connected component of the first quadrant and in the other regions we define x_n by reflection along the lines drawn (with the additional condition $|x_n| = 1$).

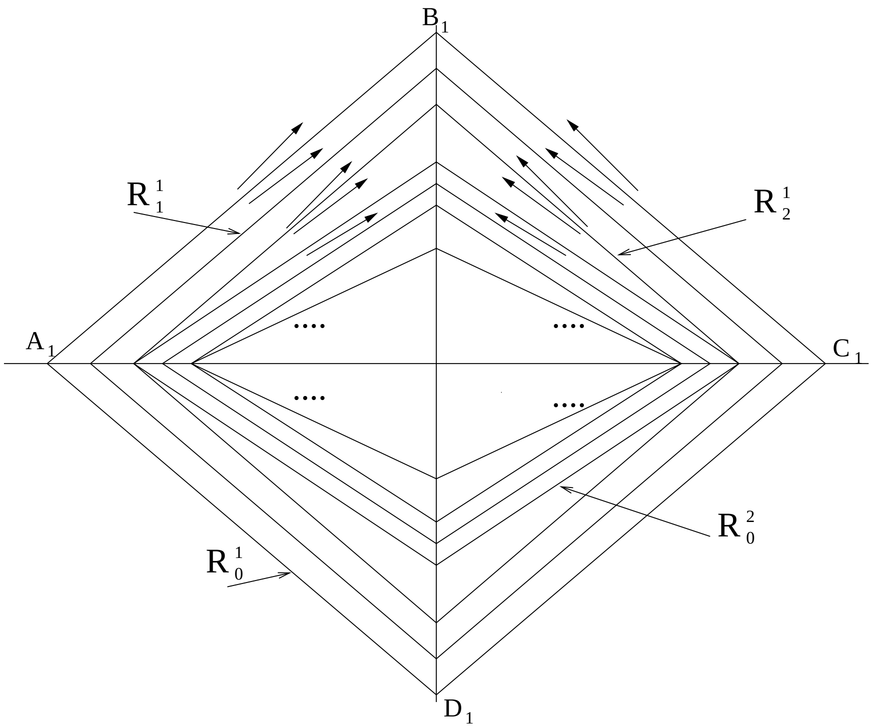

Fig. 2. The vector field $x_n/3$

The function $x_n : \mathbf{R}^2 \to \mathbf{R}^2$ is piecewise constant and mirror symmetric with respect to both the axes. Let us see what are the values of x_n in the first quadrant; if we denote by φ_n the angle between x_n and the x_1 axis we have:
– $\varphi_n = \pi - \pi/4(k+1)$ in the region which lies between R^k_{2i} and R^k_{2i+1};
– $\varphi_n = \pi - \pi/4k$ in the region between R^k_{2i-1} and R^k_{2i};
– $\varphi_n = \pi - \pi/4(k+1)$ in the region between $R^k_{2[(n+1)/2]}$ and R^{k+1}_0.

It is easy to verify that the function x_n so defined is a gradient: let u_n be the function such that $\nabla u_n = x_n$ and $u_n(0,0) = 0$. Then $u_n \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^2)$ and $|\nabla u_n| = 1$ $\mathcal{L}^2$-a.e.; this implies that there exists a function $u \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^2)$ such that $u_n \to u$ in $W^{1,p}_{\text{loc}}(\mathbf{R}^2)$ for any $p < \infty$. For every neighborhood U of the segment $L = \{|x_1| \le 1,\, x_2 = 0\}$ we have $\nabla u|_{\mathbf{R}^2 \setminus U} \equiv \nabla u_n|_{\mathbf{R}^2 \setminus U}$ for n large enough. This means that $\nabla u \in BV_{\text{loc}}(\mathbf{R}^2 \setminus U, \mathbf{R}^2)$ and hence that $u \in AG_{\text{loc}}(\mathbf{R}^2 \setminus U)$.

Now we want to show that $u \in AG(U)$ for every bounded open set U. Since $\nabla u_n \to \nabla u$ $\mathcal{L}^2$-a.e. in $\mathbf{R}^2$ it follows that $(\partial_\xi u_n)^3 \to (\partial_\xi u)^3$ $\mathcal{L}^2$-a.e. for every $\xi \in \mathbf{R}^2$. Then $\operatorname{div} \Sigma_{\xi\eta} u_n$ converge to $\operatorname{div} \Sigma_{\xi\eta} u$ in the sense of distributions for every orthonormal basis (ξ, η) of $\mathbf{R}^2$.

We notice that for any pair of unit vectors a, b we have $|a-b| = 2|\sin(\varphi/2)|$, where φ is the angle between them. Hence Theorem 3.8 gives

$$\begin{aligned}
6\,|\mathrm{div}\,\Sigma_{\xi\eta}u_n|\,(U) \leq 6|Iu_n|(U) &= \int_{J_{\nabla u_n}} \left|[\nabla u_n]^{+}_{-}\right|^3 \, d\mathcal{H}^1 \\
&\leq 8\mathcal{H}^1\left(\{x_1x_2=0\}\cap U\}\right) \\
&\quad +8\sum_{k=1}^{n}\sum_{i=0}^{2[(k+1)/2]} \mathcal{H}^1(U\cap\partial R_i^k)\sin^3\left(\frac{\pi}{8k(k+1)}\right) \\
&\leq K_1 + K_2\sum_{k=1}^{\infty}\frac{k+2}{(2k(k+1))^3}
\end{aligned}$$

where K_1 and K_2 are two constants. This proves that $\mathrm{div}\,\Sigma_{\xi\eta}u$ is representable by a measure with finite total variation in U.

It is easy to prove that $\nabla u \notin BV(U,\mathbf{R}^2)$ whenever $U\cap L\neq\emptyset$. If we call U_n the open set bounded by R_0^n we have that $\nabla u_n|_{U\setminus\overline{U}_n} \equiv \nabla u|_{U\setminus\overline{U}_n}$. So $\nabla u \in BV(U\setminus\overline{U}_n,\mathbf{R}^2)$ and

$$\begin{aligned}
|D\nabla u|(U\setminus\overline{U}_n) = |D\nabla u_n|(U\setminus\overline{U}_n) &\geq \int_{(U\setminus\overline{U}_n)\cap J_{\nabla u_n}} \left|[\nabla u_n]^{+}_{-}\right| \, d\mathcal{H}^1 \\
\geq K_3\sum_{k=1}^{n}\sum_{i=0}^{2[(k+1)/2]}\frac{\mathcal{H}^1(U\cap\partial R_i^k)}{k(k+1)}\,.
\end{aligned}$$

But since $U\cap L$ is not empty there exists a strictly positive constant a such that $\mathcal{H}^1(U\cap\partial R_i^k)\geq a$ if k is large enough. This implies that $\nabla u\notin BV(U,\mathbf{R}^2)$.

4 Compactness

The main result of this paper is the following compactness theorem.

Theorem 4.1 (Compactness) *Let $\Omega\subset\mathbf{R}^2$ be open and bounded and let $(u_h)\subset AG(\Omega)$ be such that*

(i) the total variations $|Iu_h|(\Omega)$ and $\|u_h\|_1$ are equibounded;
(ii) $|\nabla u_h|\to 1$ in $L^3(\Omega)$.

Then (u_h) has a subsequence strongly converging in $W^{1,3}(\Omega)$ to $u\in AG_e(\Omega)$.

This result, together with Theorem 3.3, implies that the minimum problem

$$\min\left\{|Iu|(\Omega):\ u\in AG_e(\Omega),\ u\geq 0,\ u|_{\partial\Omega}=0\right\}$$

has at least one solution for any bounded open set Ω; notice that we cannot write the Neumann boundary condition in (1.3) because a theory of traces for the functions in $AG_e(\Omega)$ is still to be developed; however, due to the eikonal constraint, the condition $u \geq 0$ in Ω could heuristically be considered as a substitute of the Neumann boundary condition.

The proof of Theorem 4.1 is based on some preliminary results, some of which have an independent interest. The first one is a Jensen-type inequality.

Lemma 4.2 (Nonconvex Jensen inequality) *For any probability measure μ in $[-1,1] = I$ we have*

$$\int_I t^3 \, d\mu(t) \geq \left(\int_I t \, d\mu(t) \right)^3$$

provided $\lambda = \int_I t \, d\mu(t)$ satisfies $2\lambda^3 + 3\lambda^2 \geq 1$.

Proof. Let λ be such that $2\lambda^3 + 3\lambda^2 \geq 1$, let $\mathcal{M}$ be the set of probability measures in I and let

$$A := \left\{ \mu \in \mathcal{M} : \int_I t \, d\sigma(t) = \lambda \right\} .$$

We will prove that the minimum of $\mu \mapsto \int_I t^3 \, d\mu(t)$ on A is λ^3. Since A is a singleton if $\lambda = -1$, in the following we assume that $\lambda > -1$. Let A' be the collection of all $\mu \in A$ whose support is $\{-1\} \cup [0,1]$. We first prove that the infimum of $\mu \mapsto \int_I t^3 \, d\mu(t)$ in A' equals the infimum in A. Indeed, if $\mu \in A$ we set

$$\tilde{\mu} := -\int_{[-1,0)} t \, d\mu(t) \delta_{-1} + a\delta_0 + \mu \llcorner (0,1]$$

with

$$a := 1 - \mu\left((0,1]\right) + \int_{[-1,0)} t \, d\mu(t) = \mu\left([-1,0]\right) + \int_{[-1,0)} t \, d\mu(t) \geq 0 \, .$$

By construction $\tilde{\mu} \in A'$ and

$$\int_I t^3 \, d(\mu - \tilde{\mu})(t) = \int_{[-1,0)} t^3 \, d(\mu - \tilde{\mu})(t) = \int_{[-1,0)} (t^3 - t) \, d\mu(t) \geq 0 \, .$$

Denoting by A'' the subset of A' made by all measures whose support is $\{-1, a\}$ for some $a \geq 0$, the infimum of $\mu \mapsto \int_I t^3 \, d\mu(t)$ in A' is equal to the infimum on A''; in fact, for any $\mu \in A'$ we can set

$$\tilde{\mu} := \mu \llcorner \{-1\} + \mu\left((-1,1]\right) \delta_a \quad \text{with} \quad a := \frac{1}{\mu\left((-1,1]\right)} \int_{(-1,1]} t \, d\mu(t)$$

to obtain a measure $\tilde{\mu} \in A''$ such that, by Jensen inequality, $\int_I t^3 \, d\tilde{\mu}(t) \leq \int_I t^3 \, d\mu(t)$.

Let $\mu \in A''$ and set $k = \mu(\{-1\})$; then

$$a = \frac{\lambda + k}{1 - k}$$

and since $|a| \leq 1$ we obtain that $k \leq (1 - \lambda)/2 < 1$. Hence, we need only to prove that the function

$$F(k) := (1 - k) \left(\frac{\lambda + k}{1 - k} \right)^3 - k \qquad k \in \left[0, \frac{1 - \lambda}{2} \right]$$

achieves its minimum at $k = 0$. Indeed, a straightforward computation shows that F is convex, and since

$$F'(0) = 2\lambda^3 + 3\lambda^2 - 1 \geq 0$$

the statement follows. □

The second ingredient in the proof of Theorem 4.1 is Hodge decomposition, together with truncation of gradients. Recent related results on this topic are given in [15] and [12] (see also [22]). We first show that any vector field v can be written as $\nabla\varphi + \varepsilon$, where $\varphi \in W^{1,1}$ and the L^1 norm of ε is controlled by $|\mathrm{curl}\, v|$; a refined version of the decomposition, given in Corollary 4.4, provides a function $\varphi \in W^{1,\infty}$.

Theorem 4.3 (Hodge decomposition) *For any $v \in L^1(B_1, \mathbf{R}^2)$ such that $\mathrm{curl}\, v$ is (representable by) a finite measure in B_1 and any $C > 1$ there exists $\varphi \in W^{1,1}(B_1)$ such that*

$$\int_{B_1} |v - \nabla\varphi| \, dx \leq C |\mathrm{curl}\, v|(B_1) \, . \tag{4.1}$$

Proof. Assume first that, in addition, $v \in C^\infty(B_1, \mathbf{R}^2)$, set $u(x) = v(x) - v(0)$ and define $\phi \in C^\infty(B_1)$ by $\phi(x) = \int_0^1 \langle u(tx), x \rangle \, dt$. We compute

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1}(x) &= \int_0^1 \frac{\partial u_1}{\partial x_1}(tx) t x_1 + \frac{\partial u_2}{\partial x_1}(tx) t x_2 + u_1(tx) \, dt \\
&= \int_0^1 \frac{\partial u_1}{\partial x_1}(tx) t x_1 + \frac{\partial u_1}{\partial x_2}(tx) t x_2 + u_1(tx) \, dt - \int_0^1 \mathrm{curl}\, u(tx) t x_2 \, dx \\
&= \int_0^1 \frac{d}{dt}[t u_1(tx)] \, dt - \int_0^1 \mathrm{curl}\, u(tx) t x_2 \, dx \\
&= u_1(x) - \int_0^1 \mathrm{curl}\, u(tx) t x_2 \, dx \, .
\end{aligned}$$

An analogous computation for $\partial\phi/\partial x_2$ proves that

$$
\begin{aligned}
\|u-\nabla\phi\|_1 &\le \int_0^1\int_{B_1} t|x||\mathrm{curl}\,u|(tx)\,dxdt \\
&= \int_0^1\int_{B_t} \frac{|y|}{t^2}|\mathrm{curl}\,u|(y)\,dy\,dt \\
&= \int_{B_1}\int_0^1 \chi_{B_t}(y)\frac{|y|}{t^2}|\mathrm{curl}\,u|(y)\,dtdy \\
&= \int_B\left(\int_{|y|}^1 \frac{|y|}{t^2}\,dt\right)|\mathrm{curl}\,u|(y)\,dy \\
&= \int_{B_1}(1-|y|)|\mathrm{curl}\,u|(y)\,dy \le \int_{B_1}|\mathrm{curl}\,v|(y)\,dy\,.
\end{aligned}
$$

Then, $\varphi(x)=\phi(x)+\langle v(0),x\rangle$ satisfies (4.1) with $C=1$.

In the general case we fix $\delta>0$ and $\tau>1$ such that $\tau+2\delta\le C$ and find $t\in(1,\tau]$ such that $v_t(x)=v(x/t)$ satisfies

$$\int_{B_1}|v_t-v|\,dx<\delta|\mathrm{curl}\,v|(B_1)$$

(notice that the case $|\mathrm{curl}\,v|(B_1)=0$ is trivial). The function $v_t*\rho_\varepsilon$ is smooth in B_1 as soon as $\varepsilon<t-1$; we choose $\varepsilon<t-1$ so small that

$$\int_{B_1}|v_t*\rho_\varepsilon-v_t|\,dx<\delta|\mathrm{curl}\,v|(B_1).$$

By the previous construction we can find $\varphi\in C^\infty(B_1)$ such that

$$
\begin{aligned}
\int_{B_1}|v_t*\rho_\varepsilon-\nabla\varphi|\,dx &\le |\mathrm{curl}\,(v_t*\rho_\varepsilon)|(B_1)=|(\mathrm{curl}\,v_t)*\rho_\varepsilon|(B_1) \\
&\le |\mathrm{curl}\,v_t|(B_t)=t|\mathrm{curl}\,v|(B_1)\le\tau|\mathrm{curl}\,v|(B_1)\,.
\end{aligned}
$$

Taking into account our choices of t and ε the proof is achieved. □

Corollary 4.4 *There exists a constant c with the following property: for $v\in L^1(B_1,\mathbf{R}^2)$ such that $\mathrm{curl}\,v$ is (representable by) a finite measure in B_1 and any $\lambda>0$ there exist $\phi\in W^{1,\infty}(B_{1/3})$ and a Borel set $E\subset B_{1/3}$ such that $\|\nabla\phi\|_\infty\le c\lambda$,*

$$\mathcal{L}^2(E)\le\frac{c}{\lambda}\left[\int_{B_1}(|v|-\lambda)^+\,dx+|\mathrm{curl}\,v|(B_1)\right]$$

and

$$\int_{B_{1/3}}|v-\nabla\phi|\,dx\le c\left[|\mathrm{curl}\,v|(B_1)+\int_E|v|\,dx+\int_{B_1}(|v|-\lambda)^+dx\right]. \tag{4.2}$$

Proof. In this proof we denote by c_i some constants depending only on the dimension (2 in this case). For any nonnegative $f \in L^1(B_1)$ we denote by Mf its (local) maximal function, defined by

$$Mf(x) := \sup\left\{\frac{\int_{B_\varrho(x)} |f|\,dy}{\pi\varrho^2} :\ B_\varrho(x) \subset B_1\right\} \qquad x \in B_1\,.$$

We will use the weak L^1 estimate

$$\mathcal{L}^2\left(\{Mf > \lambda\}\right) \leq \frac{c_1}{\lambda}\int_{B_1} |f|\,dx \qquad \forall \lambda > 0 \tag{4.3}$$

which is a straightforward consequence of the definition of Mf and Besicovitch covering theorem.

Let φ be given by Theorem 4.3 with $C = 2$ and let

$$F := \left\{x \in B_{1/3} :\ M|\nabla\varphi|(x) \leq 3\lambda\right\}\,.$$

We recall the estimate

$$\frac{1}{\pi\varrho^2}\int_{B_\varrho(x)} \frac{|\varphi(z) - \tilde{\varphi}(x)|}{|z - x|}\,dz \leq \int_0^1 \frac{\int_{B_{t\varrho}(x)} |\nabla\varphi|\,dy}{\pi(t\varrho)^2}\,dt \leq M|\nabla\varphi|(x)$$

for any ball $B_\varrho(x) \subset B_1$ centered at some point $x \in B_1 \setminus S_\varphi$ (see for instance (2.5) and Theorem 2.3 of [6]). This estimate easily implies that the restriction of $\tilde{\varphi}$ to $F \setminus S_\varphi$ is a Lipschitz function, with Lipschitz constant less than $c_2\lambda$; we denote by ϕ any Lipschitz extension of this function to the whole of $B_{1/3}$. By construction, ϕ coincides with φ $\mathcal{L}^2$-a.e. in F.

Let $E = B_{1/3} \setminus F$; since

$$M(|\nabla\varphi|) \leq \lambda + M((|v| - \lambda)^+) + M(|v - \nabla\varphi|)$$

by (4.3) we infer

$$\mathcal{L}^2(E) \leq \frac{c_1}{\lambda}\left[\int_{B_1} (|v| - \lambda)^+\ dx + 2|\mathrm{curl}\, v|(B_1)\right]\,.$$

Finally, we prove (4.2). To this aim, since $v = \nabla\varphi + \varepsilon$ with $\|\varepsilon\|_1 \leq 2|\mathrm{curl}\, v|(B_1)$, we need only to estimate the L^1 norm of $|\nabla\varphi - \nabla\phi|$ on $B_{1/3}$. We have

$$\begin{aligned}\int_{B_{1/3}} |\nabla\varphi - \nabla\phi|\,dx &= \int_E |\nabla\varphi - \nabla\phi|\,dx \leq \int_E |\varepsilon| + |v| + c_2\lambda\,dx\\ &\leq \int_E |v| + c_2\lambda\,dx + 2|\mathrm{curl}\, v|(B_1)\,.\end{aligned}$$

□

The final ingredient of our proof of the compactness theorem is the theory of (gradient) Young measures; although the proof could be written in a more elementary way without using Young measures, we believe that this tool reduces in a substantial way the computations, thus making our proof much more readable.

Let us recall some basic facts, referring for instance to [18] for a more systematic presentation. Let $r \in [1, \infty]$, let $(w_h) \subset L^r(\Omega, \mathbf{R}^m)$ be a sequence and let $x \mapsto \nu_x$ be a measurable map which assigns to any $x \in \Omega$ a probability measure in $\mathbf{R}^m$ (here measurable means that $x \mapsto \int f(x,y)\, d\nu_x$ is Lebesgue measurable for any bounded $\mathcal{L} \otimes \mathbf{B}(\mathbf{R}^m)$-measurable function f); we say that (w_h) generates the Young measure (ν_x) if

$$\lim_{h \to \infty} \int_\Omega f\left(x, w_h(x)\right)\, dx = \int_\Omega \left(\int_{\mathbf{R}^m} f(x,p)\, d\nu_x(p) \right) dx \tag{4.4}$$

for any bounded Carathéodory function $f(x,y)$. We will use the following well known result (see for instance [18], Theorem 3.1):

Theorem 4.5 *Let $r \in (1, \infty)$ and let $(w_h) \subset L^r(\Omega, \mathbf{R}^m)$ be a bounded sequence. Then*

(i) there exists a subsequence $(w_{h(k)})$ generating a Young measure (ν_x) such that

$$\int_{\mathbf{R}^m} |p|^r\, d\nu_x(p) < \infty \quad \textit{for } \mathcal{L}^2\textit{-a.e. } x \in \Omega\,;$$

(ii) if $|w_h|^r$ are equiintegrable and (w_h) generates (ν_x),then (4.4) holds for any Carathéodory function f such that

$$|f(x,p)| \le g(x) + c|p|^r \qquad \forall (x,p) \in \Omega \times \mathbf{R}^m$$

for some $c \ge 0$, $g \in L^1(\Omega)$.

We first prove the compactness theorem under the extra assumptions that Ω is the unit ball B_1, $|Iu_h|(B_1)$ tend to 0 and u_h uniformly converge to a linear function u ; the general case will then be recovered by a simple blow-up argument, taking into account that, under the rescaling preserving the eikonal equation, Iu scales as a length. The information that $|Iu_h|(B_1)$ is infinitesimal, and not only bounded, will be used in an essential way to approximate the fields $\Sigma_{e_1 e_2} u_h$ and $\Sigma_{\varepsilon_1 \varepsilon_2} u_h$ by rotations, by the angle $\pi/2$, of suitable gradients; this leads to an integration by parts formula showing that the eikonal equation is preserved in the limit.

Proposition 4.6 *Let $(v_h) \subset AG(B_1)$ be a sequence such that*

(a) $|\nabla v_h|$ converge in $L^3(B_1)$ to 1;
(b) $|Iv_h|(B_1) \to 0$ as $h \to \infty$;

(c) (v_h) *uniformly converge in* B_1 *to a linear function* $u(x) = \langle L, x\rangle$.
Then $|L| = 1$.

Proof. Up to a rotation we can assume that $L = (\alpha, 0)$ with $0 \le \alpha \le 1$. We also assume that $|\nabla v_h|$ converges to 1 $\mathcal{L}^2$-a.e. in B_1 and (∇u_h) generate a Young measure (ν_x). Setting

$$\beta(x) := \int_{\mathbf{R}^2} p_1^3 \, d\nu_x(p) \ , \qquad \gamma(x) := \int_{\mathbf{R}^2} p_2^3 \, d\nu_x(p)$$

the equiintegrability of $|\nabla v_h|^3$ implies that v_{h1}^3 and v_{h2}^3 weakly converge in $L^1(B_1)$ to β and γ respectively. Notice that since

$$\int_{B_1} \left(\int_{\mathbf{R}^2} \left|1 - |p|^2\right| \, d\nu_x(p) \right) dx = \lim_{h\to\infty} \int_{B_1} \left|1 - |\nabla v_h|^2\right| \, dx = 0$$

the measures ν_x are supported on the unit circle for $\mathcal{L}^2$-a.e. $x \in B_1$, hence β and γ are bounded functions.
Step 1. In this step we prove that

$$\beta(x_0) \left(\alpha - \frac{2}{3}\beta(x_0) \right) \ge \frac{1}{3} \tag{4.5}$$

for any Lebesgue point $x_0 = (x_{01}, x_{02}) \in B_{1/3}$ of β and γ. Let us define the vector fields

$$\Psi_h(x) := \left(v_{h2}(1 - v_{h1}^2 - \frac{1}{3}v_{h2}^2), v_{h1}(1 - v_{h2}^2 - \frac{1}{3}v_{h1}^2) \right)$$
$$\Phi_h(x) := \left(v_{h1} - \frac{2}{3}v_{h1}^3, -v_{h2} + \frac{2}{3}v_{h2}^3 \right) .$$

Notice that, by a straightforward computation, the Jacobian determinant of the matrix M_h having Ψ_h, Φ_h as rows is

$$-\frac{2}{9}(v_{h1}^2 + v_{h2}^2)^3 + (v_{h1}^2 + v_{h2}^2)^2 - (v_{h1}^2 + v_{h2}^2)$$

hence $\det M_h$ converges to $-2/9$ $\mathcal{L}^2$-a.e. in B_1.

Since the components of Ψ_h are cubic polynomials in the derivatives of v_h, there exists a constant λ_0 such that $|\Psi_h| \le \lambda_0(1 + |\nabla v_h|^3)$; since Ψ_h is the rotation of $\Sigma_{e_1e_2} v_h$ by the angle $\pi/2$ we get

$$|\mathrm{curl}\, \Psi_h| = |\mathrm{div}\, \Sigma_{e_1e_2} v_h| \le |I v_h|$$

and hence by Corollary 4.4 with $\lambda = 2\lambda_0$ we can represent Ψ_h in $B_{1/3}$ as $\nabla\psi_h + \varepsilon_h$ for suitable functions $\psi_h \in W^{1,\infty}(B_{1/3})$ with $\|\nabla\psi_h\|_\infty \le 2c\lambda_0$ and ε_h satisfying

$$\int_{B_{1/3}} |\varepsilon_h| \, dx \le c \left[|I v_h|(B_1) + \int_{E_h} |\Psi_h| \, dx + \int_{B_1} (|\Psi_h| - \lambda)^+ dx \right]$$

for suitable Borel sets $E_h \subset B_{1/3}$ whose Lebesgue measure can be estimated with

$$\frac{c}{\lambda}\left[\int_{B_1}(|\Psi_h|-\lambda)^+\,dx+|Iv_h|(B_1)\right].$$

Since $|\Psi_h|-\lambda \le \lambda_0(|\nabla v_h|^3-1)$, the strong L^1 convergence of $|\nabla v_h|^3$ to 1 implies that $\mathcal{L}^2(E_h)$ tends to 0 as $h\to\infty$. In turn, this fact together with the equiintegrability of $|\Psi_h|$ (which is implied by the equiintegrability of $|\nabla v_h|^3$) gives that $\int_{B_{1/3}}|\varepsilon_h|\,dx$ tends to 0 as $h\to\infty$. Since, taking into account the strong $L^3(B_1)$ convergence of $|\nabla v_h|$ to 1, Ψ_h weakly converge in $L^1(B_1,\mathbf{R}^2)$ to $(2\gamma/3,2\beta/3)$ as $h\to\infty$, if we normalize ψ_h assuming that $\psi_h(x_0)=0$ we obtain that ψ_h uniformly converge in $\overline{B}_{1/3}$ to the function ψ_∞ equal to 0 at x_0 and having $\frac{2}{3}(\gamma,\beta)$ as gradient. Our choice of x_0 gives

$$\psi_\infty(x)=\frac{2}{3}\gamma(x_0)(x_1-x_{01})+\frac{2}{3}\beta(x_0)(x_2-x_{02})+o(|x-x_0|)\,.$$

By a similar argument, since Φ_h is the rotation of $\Sigma_{\varepsilon_1\varepsilon_2}v_h$ by the angle $\pi/2$, we can also represent Φ_h in $B_{1/3}$ as $\nabla\phi_h+\eta_h$ with $\eta_h\to 0$ in $L^1(B_{1/3},\mathbf{R}^2)$ and (ϕ_h) bounded in $W^{1,\infty}(B_{1/3})$. Since Φ_h weakly converge in $L^1(B_1,\mathbf{R}^2)$ to $(\alpha-2\beta/3,2\gamma/3)$ as $h\to\infty$, the same is true in $B_{1/3}$ for $\nabla\phi_h$. We assume also, possibly extracting a subsequence, that ε_h and η_h converge to 0 $\mathcal{L}^2$-a.e. in $B_{1/3}$.

We consider now the $\mathbf{R}^2$-valued functions $\varphi_h=(\psi_h,\phi_h)$; this family is bounded in $W^{1,\infty}(B_{1/3},\mathbf{R}^2)$ and

$$\lim_{h\to\infty}\det\nabla\varphi_h(x)=-\frac{2}{9}\qquad\text{for }\mathcal{L}^2\text{-a.e. }x\in B_{1/3}\,. \tag{4.6}$$

Let $\varrho\in(0,1/3]$ and let $g(x)=(\varrho-r)^+$, with $r=|x-x_0|$; we can pass to the limit as $h\to\infty$ in the identity

$$\begin{aligned}
&-\int_{B_\varrho(x_0)} g\det\nabla\varphi_h\,dx\\
&\quad=-\int_{B_\varrho(x_0)} g\,\mathrm{div}\,(\psi_h\phi_{h2},-\psi_h\phi_{h1})\\
&\quad=\int_{B_\varrho(x_0)}\psi_h(g_1\phi_{h2}-g_2\phi_{h1})\,dx\\
&\quad=-\int_{B_\varrho(x_0)}\frac{\psi_h}{r}\left((x_1-x_{01})\phi_{h2}-(x_2-x_{02})\phi_{h1}\right)\,dx
\end{aligned}$$

and use (4.6) and our choice of x_0 to obtain

$$\begin{aligned}
&\frac{1}{3}\int_{B_\varrho(x_0)}(\varrho - r)\,dx \\
&= -\frac{3}{2}\int_{B_\varrho(x_0)}\frac{\psi_\infty}{r}\left((x_1-x_{01})\frac{2\gamma}{3}-(x_2-x_{02}\left(\alpha-\frac{2}{3}\beta\right)\right)dx \\
&= \beta(x_0)\left(\alpha(x_0)-\frac{2}{3}\beta(x_0)\right)\int_{B_\varrho(x_0)}\frac{(x_2-x_{02})^2}{r}\,dx \\
&+ \left(\alpha-\frac{4}{3}\beta(x_0)\right)\gamma(x_0)\int_{B_\varrho(x_0)}\frac{(x_1-x_{01})(x_2-x_{02})}{r}\,dx \\
&- \frac{2}{3}\gamma^2(x_0)\int_{B_\varrho(x_0)}\frac{(x_1-x_{01})^2}{r}\,dx + o(\varrho^3)\,.
\end{aligned}$$

Since

$$\int_{B_\varrho(x_0)}(\varrho - r)\,dx = \int_{B_\varrho(x_0)}\frac{(x_2-x_{02})^2}{r}\,dx = \frac{\pi\varrho^3}{3}$$

and

$$\int_{B_\varrho(x_0)}\frac{(x_1-x_{01})(x_2-x_{02})}{r}\,dx = 0$$

the inequality (4.5) follows.

Step 2. In this step we prove that $\alpha = 1$. We fix x_0 in (4.5) and set $\beta = \beta(x_0)$. First, we notice that $\alpha \geq 2\sqrt{2}/3$, otherwise the polynomial

$$P(\beta) := -\frac{2}{3}\beta^2 + \alpha\beta$$

would be always strictly less that $1/3$.

Since $|\nabla u_h|$ converge in $L^3(B_1)$ to 1, the measure ν is supported on ∂B_1, hence the measure $\mu(B) = \nu(B\times\mathbf{R})$ is supported on $[-1,1]$. Using the identities

$$\beta = \int_{\mathbf{R}} t^3\,d\mu(t)\ \ , \qquad \alpha = \int_{\mathbf{R}} t\,d\mu(t)$$

and the inequality

$$3\alpha^2 + 2\alpha^3 \geq 3\cdot\frac{8}{9} > 1$$

from Lemma 4.2 we conclude that $\beta \geq \alpha^3$. Since P achieves its maximum at $3\alpha/4$, and since $\alpha^3 \geq 3\alpha/4$ for $\alpha \geq 2\sqrt{2}/3$, we obtain

$$\frac{1}{3} \leq P(\beta) \leq P(\alpha^3) = \alpha^4 - \frac{2}{3}\alpha^6\,.$$

A simple computation proves that the function $f(\alpha) = \alpha^4 - 2\alpha^6/3$ is strictly increasing in $[0,1]$, hence $f(\alpha) \geq f(1) = 1/3$ implies $\alpha = 1$. $\square$

Proof of Theorem 4.1. Possibly extracting a subsequence we can assume that (u_h) weakly converge to some function u in $W^{1,3}(\Omega)$. The weak lower semicontinuity of $u \mapsto |\nabla u|^3$ implies that $|\nabla u| \le 1$ $\mathcal{L}^2$-a.e. in Ω. In order to prove *strong* convergence of (u_h) in $W^{1,3}(\Omega)$ we need only to show that u is a solution of (1.1). In fact, passing to the limit as $h \to \infty$ in the identity

$$\int_\Omega |\nabla u_h - \nabla u|^2 \, dx = 2 \int_\Omega (1 - \langle \nabla u_h, \nabla u \rangle) \, dx + \int_\Omega |\nabla u_h|^2 - 1 \, dx$$

we obtain that (∇u_h) strongly converges to ∇u in $L^2(\Omega, \mathbf{R}^2)$ and therefore, being $|\nabla u_h|^3$ equiintegrable, Vitali theorem implies the convergence also in $L^3(\Omega, \mathbf{R}^2)$.

Let us fix a blow-up point $x_0 \in \Omega$ satisfying

(i) $\mu\left(B_\varrho(x_0)\right)/\varrho^2$ is bounded as $\varrho \downarrow 0$;
(ii) x_0 is a Lebesgue point of ∇u

and let us prove that $|\nabla u(x_0)| = 1$. Define the rescaled functions

$$u_h^\varrho(y) = \frac{1}{\varrho} u_h(x_0 + \varrho y) \qquad y \in B_1, \ \varrho > 0$$

and notice that $\nabla u_h^\varrho(y) = \nabla u_h(x_0 + \varrho y)$. Since

$$\limsup_{h\to\infty} |Iu_h^\varrho|(B_1) = \limsup_{h\to\infty} \varrho^{-1} |Iu_h| \left(B_\varrho(x_0)\right) \le \varrho^{-1} \mu\left(\overline{B}_\varrho(x_0)\right) \ ,$$

for $\varrho_i = 1/i$ we can find, by condition (i), integers h_i so large that $|Iu_{h_i}^{\varrho_i}|(B_1)$ is infinitesimal as $i \to \infty$. Analogously, since

$$\begin{aligned} &\lim_{h\to\infty} \int_{B_1} \left||\nabla u_h^\varrho| - 1\right|^3 \, dy \\ &\quad = \lim_{h\to\infty} \varrho^{-2} \int_{B_\varrho(x_0)} ||\nabla u_h| - 1|^3 \, dx = 0 \qquad \forall \varrho > 0 \end{aligned}$$

we can also assume that $|\nabla u_{h_i}^{\varrho_i}|$ converge in $L^3(B_1)$ to 1. Finally, since for ϱ fixed ∇u_h^ϱ weakly converge in $L^3(B_1, \mathbf{R}^2)$ to $\nabla u(x_0 + \varrho y)$, by condition (ii) we can choose h_i so that $\nabla u_{h_i}^{\varrho_i}$ weakly converge in $L^3(B_1, \mathbf{R}^2)$ to $\nabla u(x_0)$.

Then by Proposition 4.6 with $v_i = u_{h_i}^{\varrho_i}$ we infer that $|\nabla u(x_0)| = 1$. $\square$

5 Lower semicontinuity

We have seen in Sect. 3 that $u \mapsto |Iu|(\Omega)$ is lower semicontinuous with respect to the $W^{1,3}_{\text{loc}}(\Omega)$ convergence, and that

$$|Iu|(\Omega) = \frac{1}{6} \int_{J_{\nabla u}} |\nabla^+ u - \nabla^- u|^3 \, d\mathcal{H}^1$$

if $\nabla u \in BV_{\mathrm{loc}}(\Omega)$. In this section we consider, more generally, the lower semicontinuity of the functionals

$$I_\beta(u) := \int_{J_{\nabla u}} |\nabla^+ u - \nabla^- u|^\beta \, d\mathcal{H}^1 \tag{5.1}$$

in the class

$$BV_e^2(\Omega) := \left\{ u \in W^{1,\infty}(\Omega) : \text{ (1.1) holds and } \nabla u \in BV(\Omega, \mathbf{R}^2) \right\}$$

contained in $AG_e(\Omega)$.

The interest of these functionals is again related to a singular perturbation problem: in fact, the formal analysis in [7] and [19], based on the ansatz of constancy of tangential gradient in the transition layer, suggests these functionals arise (up to a normalization constant c_β) from the Γ-limit of the functionals

$$\int_\Omega \left(\varepsilon |\nabla^2 u|^2 + \frac{\left(|\nabla u|^2 - 1\right)^{\beta-1}}{\varepsilon} \right) dx \, .$$

The main result of the section is that I_β is *not* lower semicontinuous if $\beta > 3$; this also shows that for $\beta > 3$ the one-dimensional ansatz underlying the formal analysis is no longer true, and indeed Fig. 2 below shows how this ansatz can be violated.

Throughout this section we assume for simplicity that Ω is bounded and consider only the $W^{1,3}(\Omega)$ convergence (equivalent, by the eikonal constraint, to any $W^{1,p}(\Omega)$ strong convergence with $p < \infty$). We first state in Lemma 5.1 a necessary condition for lower semicontinuity, and then prove that this condition is violated for any $\beta > 3$. In the statement of the lemma we denote by $\nabla^* v$ the trace on the boundary of ∇v.

Lemma 5.1 *Let $R = (-a, a) \times (-b, b)$ be a rectangle and assume the existence of $v \in BV_e^2(R)$ and $\varphi \in [0, 2\pi]$ such that*

$$\nabla^* v(x_1, b) = \nabla^* v(x_1, -b) \quad \textit{for } \mathcal{L}^1\textit{-a.e. } x_1 \in (-a, a)$$

$$\nabla^* v(\pm a, x_2) = (\pm \cos\varphi, \sin\varphi) \quad \textit{for } \mathcal{L}^1\textit{-a.e. } x_2 \in (-b, b)$$

and

$$\int_{J_{\nabla v}} |\nabla^+ v - \nabla^- v|^\beta \, d\mathcal{H}^1 < 2b(2\cos\varphi)^\beta \, .$$

Then the functional I_β is not lower semicontinuous in $BV_e^2(\Omega)$ for any bounded open set $\Omega \subset \mathbf{R}^2$.

Proof. Without loss of generality, by a scaling argument we can assume that Ω contains the $(-a,a) \times (-b,b)$; let $u \in BV_e^2(\Omega)$ whose gradient is $(\cos\varphi, \sin\varphi)$ in the half plane $\{x_1 > 0\}$, is $(-\cos\varphi, \sin\varphi)$ in the half plane $\{x_1 < 0\}$ and $u(0) = 0$. For $h \geq 1$ integer we divide the strip $(-a/h, a/h) \times (-b,b)$ in h rectangles R_i with centers $(0, y_i)$ and side length $2a/h$, $2b/h$. Then, we can define

$$u_h(x_1,x_2) := \begin{cases} u(x_1,x_2) & \text{if } |x_1| \geq a/h; \\ \\ v(hx_1, hx_2 + y_i) & \text{if } (x_1,x_2) \in R_i\,. \end{cases}$$

By the trace conditions on ∇v, the gradient of u_h does not jump across ∂R_i, hence $I_\beta(u_h) - I_\beta(u)$ is given by

$$\sum_{i=1}^{h} \left\{ \int_{R_i \cap J_{\nabla u}} |\nabla^+ u - \nabla^- u|^\beta \, d\mathcal{H}^1 - \int_{R_i \cap J_{\nabla u}} |\nabla^+ u - \nabla^- u|^\beta \, d\mathcal{H}^1 \right\}$$
$$= \int_R |\nabla^+ v - \nabla^- v|^\beta \, d\mathcal{H}^1 - 2b(2\cos\varphi)^\beta < 0\,.$$

Since (u_h) converges to u, letting $h \to \infty$ we find that the lower semicontinuity fails along this sequence. □

Proposition 5.2 *The functional I_β is not lower semicontinuous in $BV_e^2(\Omega)$ if $\beta > 3$.*

Proof. We consider the function $\widetilde{v}(x_1,x_2) = -|x_1|$ in the strip $\mathbf{R} \times (-b,b)$ for a positive value b. We fix an angle $\theta \in (0, \pi/4)$ and we construct the function $v_\theta : \mathbf{R} \times (-b,b)$ whose gradient is discontinuous along the edges in Fig. 3 below and which coincides with $\widetilde{v}(x_1,x_2)$ outside the quadrilateral $ABCD$.

Setting $F(\theta) = I_\beta(v_\theta)$, we are going to study the function F in $(0,\pi/4)$ in order to show the existence of $\theta_0 \in (0,\pi/4)$ such that

$$I_\beta(v_{\theta_0}) = F(\theta_0) < 2^{\beta+1} b = I_\beta(\widetilde{v}) \ \ .$$

By Lemma 5.1 with $\varphi = 0$, $v = v_{\theta_0}$ and R equal to the smallest rectangle containing the quadrilateral $ABCD$, this implies that I_β is not lower semicontinuous.

Taking into account the reflection property of the gradient of v_θ along the singularity lines, elementary trigonometry yields

$$F(\theta) = b\Big[\, 2^{\beta+2}(\sin\theta)^{\beta-1} + 2^{\beta+1}(\cos 2\theta)^\beta + 2^{2\beta+1}(\sin\theta)^{\beta-1}(\cos\theta)^{\beta+1}\Big]$$
$$= 2^{\beta+1} b \left[2(\sin\theta)^{\beta-1} + (\cos 2\theta)^\beta + 2^\beta (\sin\theta)^{\beta-1}(\cos\theta)^{\beta+1}\right] \ \ .$$

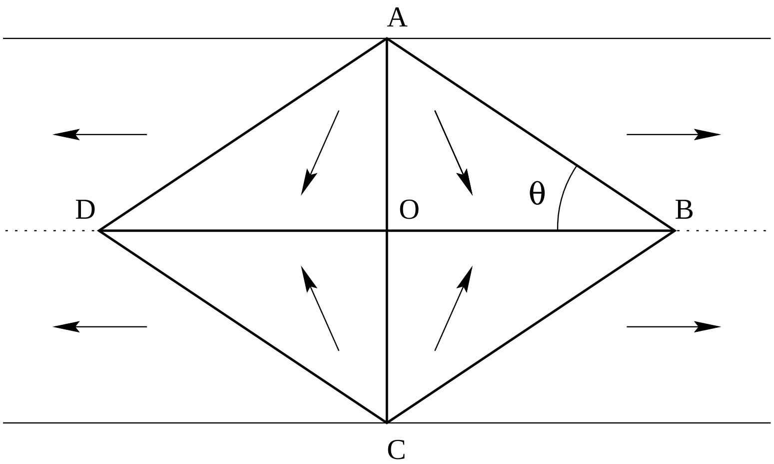

Fig. 3. The gradient of the basic function v

We have immediately that

$$\lim_{\theta \downarrow 0} F(\theta) = 2^{\beta+1} b = I_\beta(\widetilde{v}) \,.$$

Being interested in the values of the functions F near 0, we can operate the change of variables $t = \sin\theta$ and divide by the constant $2^{\beta+1}b$, thus reducing to study the function

$$\begin{aligned} f(t) := \frac{F(\arcsin t)}{2^{\beta+1}b} &= 2t^{\beta-1} + \left(1 - 2t^2\right)^\beta + 2^\beta t^{\beta-1}\left(1 - t^2\right)^{\frac{\beta+1}{2}} \\ &= 2t^{2+\varepsilon}\left(1 + 2^{3+\varepsilon}t^{2+\varepsilon}\left(1 - t^2\right)^{2+\frac{\varepsilon}{2}}\right) + \left(1 - 2t^2\right)^{3+\varepsilon} \quad , \end{aligned}$$

where we also replaced $3+\varepsilon$ for β. We want to show that there exists $t_0 > 0$ such that $f(t_0) < f(0) = 1$.

It is easy to see that the first and second derivatives at $t = 0$ of the first term of the last row vanish because of the factor $t^{2+\varepsilon}$, hence we only need to consider the term $g(t) = (1 - 2t^2)^{3+\varepsilon}$, whose derivatives are

$$\begin{aligned} g'(t) &= -4t(3+\varepsilon)\left(1 - 2t^2\right)^{2+\varepsilon} \\ g''(t) &= -4(3+\varepsilon)\left(1 - 2t^2\right)^{2+\varepsilon} + 16t^2(3+\varepsilon)(2+\varepsilon)\left(1 - 2t^2\right)^{1+\varepsilon} \,. \end{aligned}$$

hence, $f'(0) = g'(0) = 0$ and $f''(0) = g''(0) = -4(3+\varepsilon) < 0$. It follows that the function f is strictly decreasing in a right interval of $t = 0$, thus for $t_0 > 0$ sufficiently small we have $f(t_0) < f(0) = 1$. $\square$

6 Γ-convergence

In this section we consider the asymptotic behaviour of F_ε as $\varepsilon \downarrow 0$. We first prove an a priori estimate on $|\nabla u|^6$ and then, repeating essentially the argument of [9], the lower bound on $\Gamma - \liminf F_\varepsilon$ (the Γ-limits are computed with respect to the $W^{1,3}(\Omega)$ topology). Throughout this section we assume that Ω is a bounded open set with Lipschitz boundary.

Theorem 6.1 (Lower bound) *For any $u \in W^{2,2}(\Omega)$ and any $\varepsilon > 0$ we have*

$$\int_\Omega |\nabla u|^6 \, dx \leq \left(c + \frac{d\varepsilon^2}{\mathcal{L}^2(\Omega)} \right) F_\varepsilon^2(u) + e\mathcal{L}^2(\Omega) \tag{6.1}$$

for some absolute constants c, d, e and

$$|Iu|(A) \leq F_\varepsilon(u) + C(A)\varepsilon^{1/2} F_\varepsilon^{1/2}(u) \|\nabla u\|_2 \tag{6.2}$$

whenever A is open and $A \subset\subset \Omega$. In particular, $\Gamma - \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u) < \infty$ implies $u \in AG_e(\Omega)$ and

$$\Gamma - \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u) \geq |Iu|(\Omega) \, .$$

Proof. Let $v = |\nabla u|$: since $|\nabla v| \leq |\nabla^2 u|$, using Young inequality we get

$$F_\varepsilon(u) \geq \int_\Omega |1 - v^2||\nabla v| = \int_\Omega |\nabla(\Phi \circ v)| \, dx$$

with $\Phi(t) = \int_0^t |1 - \tau^2| \, d\tau$. By the Poincarè inequality we get

$$\int_\Omega |\Phi \circ v - m|^2 \leq C_1 F_\varepsilon^2(u) \tag{6.3}$$

where m is the mean value of $\Phi \circ v$ on Ω. On the other hand, using the estimate $|\Phi(t)| \leq a(1 - t^2)^2 + b$ for suitable constants a, b we get

$$m\mathcal{L}^2(\Omega) \leq 2a\varepsilon F_\varepsilon(u) + b\mathcal{L}^2(r) \, .$$

Using this inequality in (6.3) and taking into account that Φ has a cubic behaviour at infinity the inequality (6.1) follows.

The inequality (6.2) follows by Proposition 3.4 and the representation of the determinant in divergence form: in fact, for any $\phi \in C_c^1(\Omega)$ such that

$0 \le \phi \le 1$ and ϕ is identically equal to 1 on A, since $|\nabla^2 u|^2$ is the sum of the squares of the eigenvalues of $\nabla^2 u$ we can estimate $|Iu|(A)$ by

$$\begin{aligned}
&\int_\Omega \phi(1-|\nabla u|^2)|\lambda_1-\lambda_2|\,dx \\
&\quad \le \frac{1}{2}\int_\Omega \phi\left[\varepsilon(\lambda_1-\lambda_2)^2+\frac{\left(1-|\nabla u|^2\right)^2}{\varepsilon}\right]\,dx \\
&\quad = F_\varepsilon(u)-\varepsilon\int_\Omega \phi\lambda_1\lambda_2\,dx = F_\varepsilon(u)-\varepsilon\int_\Omega \phi\det(\nabla^2 u)\,dx \\
&\quad = F_\varepsilon(u)+\varepsilon\int_\Omega u_x(u_{yy}\phi_x-u_{xy}\phi_y)\,dx \\
&\quad \le F_\varepsilon(u)+2\varepsilon^{1/2}F_\varepsilon^{1/2}(u)\|\nabla\phi\|_\infty\|\nabla u\|_2\,.
\end{aligned}$$

In order to prove the final part of the statement we fix an open set $A \subset\subset \Omega$ and use (6.1) and (6.2) to obtain

$$|Iu|(A) \le \liminf_{h\to\infty}|Iu_h|(A) \le \liminf_{h\to\infty} F_{\varepsilon_h}(u_h)$$

for any sequence $(\varepsilon_h) \downarrow 0$ and any sequence $(u_h) \subset W^{2,2}(\Omega)$ converging to u such that $F_{\varepsilon_h}(u_h)$ is bounded. Eventually, letting $A \uparrow \Omega$ we obtain that $u \in AG(\Omega)$ and the inequality

$$\int_\Omega \left|1-|\nabla u_h|^2\right|\,dx \le 2\varepsilon_h F_{\varepsilon_h}(u_h)$$

implies that $u \in AG_e(\Omega)$. □

A simple consequence of Theorem 4.1 and the apriori estimate of $|\nabla u|^6$ is the equicoercivity of F_ε.

Theorem 6.2 (Equicoercivity) *Let $(\varepsilon_h) \downarrow 0$ and let $u_h \in W^{2,2}(\Omega)$ such that $F_{\varepsilon_h}(u_{\varepsilon_h})$ are equibounded. Then (u_h) has a subsequence strongly converging in $W^{1,3}(\Omega)$ to $u \in AG_e(\Omega)$.*

Proof. By (6.2) the sequence $(|Iu_h|(A))$ is bounded in any open set $A \subset\subset \Omega$ and (6.1) gives that $(|\nabla u_h|)$ is bounded in $L^6(\Omega)$. Since $(1-|\nabla u_h|^2)^2 \le c\varepsilon_h$, it follows that $|\nabla u_h|^2$ converge to 1 in $L^2(\Omega)$, hence $|\nabla u_h|$ converge to 1 in $L^2(\Omega)$. Since $(|\nabla u_h|)$ is bounded in $L^6(\Omega)$ we obtain strong convergence to 1 in $L^r(\Omega)$ for any $r < 6$, thus the hypotheses of Theorem 4.1 are fulfilled in any open set $A \subset\subset \Omega$. Then, the conclusion follows by a diagonal argument. □

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