

Ramanujan Coverings of Graphs

Chris Hall*, Doron Puder† and William F. Sawin‡

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Abstract

Let G be a finite connected graph, and let ρ be the spectral radius of its universal cover. For example, if G is k -regular then $\rho = 2\sqrt{k-1}$. We show that for every r , there is an r -covering (a.k.a. an r -lift) of G where all the new eigenvalues are bounded from above by ρ . It follows that a bipartite Ramanujan graph has a Ramanujan r -covering for every r . This generalizes the $r = 2$ case due to Marcus, Spielman and Srivastava [MSS15a].

Every r -covering of G corresponds to a labeling of the edges of G by elements of the symmetric group S_r . We generalize this notion to labeling the edges by elements of various groups and present a broader scenario where Ramanujan coverings are guaranteed to exist.

In particular, this shows the existence of richer families of bipartite Ramanujan graphs than was known before. Inspired by [MSS15a], a crucial component of our proof is the existence of interlacing families of polynomials for complex reflection groups. The core argument of this component is taken from [MSS15b].

Another important ingredient of our proof is a new generalization of the matching polynomial of a graph. We define the r -th matching polynomial of G to be the average matching polynomial of all r -coverings of G . We show this polynomial shares many properties with the original matching polynomial. For example, it is real rooted with all its roots inside $[-\rho, \rho]$.

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1 Introduction

1.1 Ramanujan coverings

Let G be a finite, connected, undirected graph on n vertices and let A_G be its adjacency matrix. A_G The eigenvalues of A_G are real and we denote them by

$$\lambda_n \leq \dots \leq \lambda_2 \leq \lambda_1 = \text{pf}(G),$$

where $\lambda_1 = \text{pf}(G)$ is the Perron-Frobenius eigenvalue of A_G , referred to as the trivial eigenvalue. For $\text{pf}(G)$ example, $\text{pf}(G) = k$ for G k -regular. The smallest eigenvalue, λ_n , is at least $-\text{pf}(G)$, with equality if and only if G is bipartite. Denote by $\lambda(G)$ the largest absolute value of a non-trivial eigenvalue, $\lambda(G)$ namely $\lambda(G) = \max(\lambda_2, -\lambda_n)$. It is well know that $\lambda(G)$ provides a good estimate to different expansion properties of G : the smaller $\lambda(G)$ is, the better expanding G is (see [HLW06, Pud15]).

However, $\lambda(G)$ cannot be arbitrarily small. Let $\rho(G)$ be the spectral radius of the universal $\rho(G)$ covering tree of G . For instance, if G is k -regular then $\rho(G) = 2\sqrt{k-1}$. It is known that $\lambda(G)$ cannot be much smaller than $\rho(G)$, so graphs with $\lambda(G) \leq \rho(G)$ are considered optimal expanders (we elaborate in Section 2.1 below). Following [LPS88, HLW06] they are called **Ramanujan graphs**, and the interval $[-\rho(G), \rho(G)]$ called **the Ramanujan interval**. In the bipartite case, $\lambda(G) = |\lambda_n| = \text{pf}(G)$ is large, but G can still expand well in many senses (see Section 2.1), and the optimal scenario is when all other eigenvalues are within the Ramanujan interval, namely, when $\lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_2 \in [-\rho(G), \rho(G)]$. We call a bipartite graph with this property a **bipartite-Ramanujan graph**.

Let H be a topological r -sheeted covering of G (r -covering in short) with covering map $p: H \rightarrow G$. If $f: V(G) \rightarrow \mathbb{R}$ is an eigenfunction of G , then the composition $f \circ p$ is an eigenfunction of H with the same eigenvalue. Thus, out of the rn eigenvalues of H (considered as a multiset), n are induced from G and are referred to as **old eigenvalues**. The other $(r-1)n$ are called the **new eigenvalues** of H .

Definition 1.1. Let H be a covering of G . We say that H is a **Ramanujan Covering** of G if all the new eigenvalues of H are in $[-\rho(G), \rho(G)]$. We say H is a **one-sided Ramanujan covering** if all the new eigenvalues are bounded from above¹ by $\rho(G)$.

The existence of infinitely many k -regular Ramanujan graphs for every $k \geq 3$ is a long-standing open question. Bilu and Linial [BL06] suggest the following approach to solving this conjecture: start with your favorite k -regular Ramanujan graph (e.g. the complete graph on $k + 1$ vertices) and construct an infinite tower of Ramanujan 2-coverings. They conjecture that every (regular) graph has a Ramanujan 2-covering. This approach turned out to be very useful in the groundbreaking result of Marcus, Spielman and Srivastava [MSS15a], who proved that every graph has a one-sided Ramanujan 2-covering. This translates, as explained below, to that there are infinitely many k -regular *bipartite* Ramanujan graphs of every degree k .

In this paper, we generalize the result of [MSS15a] to coverings of every degree:

Theorem 1.2. *Every connected, loopless graph has a one-sided Ramanujan r -covering for every r .*

In fact, this result holds also for graphs with loops, as long as they are regular (Proposition 2.3), so the only obstruction is irregular graphs with loops. We stress that throughout this paper, all statements involving graphs hold not only for simple graphs, but also for graphs with multiple edges. Unless otherwise stated, the results also hold for graphs with loops.

A finite graph is bipartite if and only if its spectrum is symmetric around zero. In addition, every covering of a bipartite graph is bipartite. Thus, every one-sided Ramanujan covering of a bipartite graph is, in fact, a (full) Ramanujan covering. Therefore,

Corollary 1.3. *Every connected bipartite graph has a Ramanujan r -covering for every r .*

In the special case where the base graph is $\bullet \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \bullet$ (two vertices with k edges connecting them), Theorem 1.2 (and Corollary 1.3) were shown in [MSS15b], using a very different argument. In this regard, our result generalizes the 2-coverings result from [MSS15a] as well as the more recent result from [MSS15b].

In our view, the main contributions of this paper are the following. (i) It sheds new light on the work of Marcus-Spielman-Srivastava [MSS15a]: we show there is nothing special about $r = 2$ (2-covering of graphs), and that with the right framework, the ideas can be generalized to any $r \geq 2$. (ii) Our main result shows the existence of richer families of bipartite-Ramanujan graphs than was known before (see Corollary 2.2). (iii) We introduce a more general framework of group-based coverings of graphs, extend Theorem 1.2 to a more general setting and point to the heart of the matter – Properties (P1) and (P2) defined below. (iv) We introduce the d -matching-polynomial which has nice properties and seems to be an interesting object for its own right.

1.2 Group labeling of graphs and Ramanujan coverings

As mentioned above, we suppose that G is undirected, yet we regard it as an oriented graph. More precisely, we choose an orientation for each edge in G , and we write $E^+(G)$ for the resulting set of oriented edges and $E^-(G)$ for the edges with the opposite orientation. Finally, if e is an edge in $E^\pm(G)$, we write $-e$ for the corresponding edge in $E^\mp(G)$ with the opposite orientation, and we identify $E(G)$ with the disjoint union $E^+(G) \sqcup E^-(G)$. We let $h(e)$ and $t(e)$ denote the head

$E^+(G)$
 $E^-(G)$
 $-e$
 $h(e), t(e)$

¹We could also define a one-sided Ramanujan covering as having all its eigenvalues bounded from below by $-\rho(G)$. Every result stated in the paper about these coverings would still hold for the lower-bound case, unless stated otherwise.

vertex and tail vertex of $e \in E(G)$, respectively. We say that G is an *oriented undirected graph*.

Throughout this paper, the family of r -coverings of the graph G is defined via the following natural model, introduced in [AL02] and [Fri03]. The vertex set of every r -covering H is $\{v_i \mid v \in V(G), 1 \leq i \leq r\}$. Its edges are defined via a function $\sigma: E(G) \rightarrow S_r$ satisfying $\sigma(-e) = \sigma(e)^{-1}$ (occasionally, we denote $\sigma(e)$ by σ_e): for every $e \in E^+(G)$ we introduce in H the r edges connecting $h(e)_i$ to $t(e)_{\sigma_e(i)}$ for $1 \leq i \leq r$. See Figure 1.1.

Definition 1.4. Denote by $\mathcal{C}_{r,G}$ the probability space consisting of all r -coverings $\{\sigma: E(G) \rightarrow S_r \mid \sigma(-e) = \sigma(e)^{-1}\}$, endowed with uniform distribution.

Let $H \in \mathcal{C}_{r,G}$ correspond to $\sigma: E(G) \rightarrow S_r$ and let $f: V(H) \rightarrow \mathbb{C}$ be an eigenfunction of H with eigenvalue μ . Define $\bar{f}: V(G) \rightarrow \mathbb{C}^r$ in terms of f : for every $v \in V(G)$, let $\bar{f}(v)$ be the transpose of the vector $(f(v_1), f(v_2), \dots, f(v_r))$. Considering the permutations σ_e as permutation matrices, the collection of vectors $\{\bar{f}(v)\}_{v \in V(G)}$ satisfies the following equation for every $v \in V(G)$:

$$\sum_{e: t(e)=v} \sigma_e \bar{f}(h(e)) = \mu \cdot \bar{f}(v) \quad (1.1)$$

(note that every loop at v appears twice in the summation, once in each orientation). Conversely, every function $f: V(G) \rightarrow \mathbb{C}^r$ satisfying (1.1) for some fixed μ and every $v \in V(G)$, is an eigenfunction of H with eigenvalue μ .

This way of presenting coverings of G and their spectra suggests the following natural generalization: instead of picking the matrices σ_e from the group of permutation matrices, one can label the edges of G by matrices from any fixed subgroup of² $\text{GL}_d(\mathbb{C})$. Since the same group Γ may be embedded in different ways in $\text{GL}_d(\mathbb{C})$, even for varying values of d , the right notion here is that of group representations. Namely, a group Γ together with a finite dimensional representation π , which is simply a homomorphism $\pi: \Gamma \rightarrow \text{GL}_d(\mathbb{C})$ (in this case we say that π is d -dimensional).

Given a pair (Γ, π) , where π is d -dimensional, we define a **Γ -labeling** to be a function $\gamma: E(G) \rightarrow \Gamma$ satisfying $\gamma(-e) = \gamma(e)^{-1}$. The **π -spectrum** of the Γ -labeling γ is defined accordingly as the values μ satisfying

$$\sum_{e: t(e)=v} \pi(\gamma(e)) f_{h(e)} = \mu \cdot f_v \quad \forall v \in V(G)$$

for some $0 \neq f: V(G) \rightarrow \mathbb{C}^d$. More precisely, it is the spectrum of the $nd \times nd$ matrix $\mathbf{A}_{\gamma, \pi}$ obtained from A_G , the adjacency matrix of G , as follows: for every $u, v \in V(G)$, replace the (u, v) entry in A_G by the $d \times d$ block $\sum_{e: u \rightarrow v} \pi(\gamma(e))$ (the sum is over all edges from u to v , and is a zero $d \times d$ block if there are no such edges). Whenever Γ is finite, or more generally whenever π is a unitary representation, the spectrum of $\mathbf{A}_{\gamma, \pi}$ is real – see Claim 2.11.

²We change the notation of dimension from r to d deliberately, to avoid confusion: as explained below, the right point of view to consider r -coverings is via a subgroup of $\text{GL}_{r-1}(\mathbb{C})$ rather than of $\text{GL}_r(\mathbb{C})$.

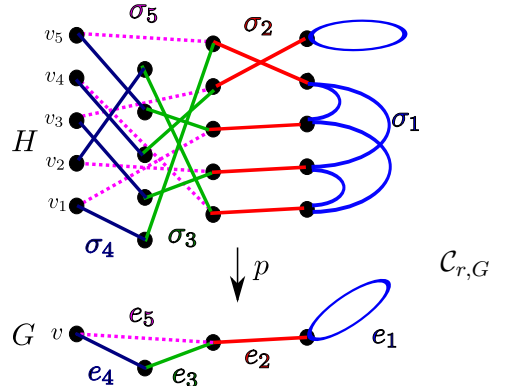


Figure 1.1: A 5-covering of a graph defined by permutations

Definition 1.5. Let Γ be a finite group. A Γ -labeling of the graph G is a function $\gamma: E(G) \rightarrow \Gamma$ satisfying $\gamma(-e) = \gamma(e)^{-1}$. Denote by $\mathcal{C}_{\Gamma, G}$ the probability space of all Γ -labelings of G endowed with uniform distribution. $\mathcal{C}_{\Gamma, G}$

Let $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ be a representation of Γ . For any Γ -labeling γ we say that $A_{\gamma, \pi}$ is a (Γ, π) -covering of the graph G . The spectrum of this (Γ, π) -covering is a multiset denoted $\mathrm{Spec}(A_{\gamma, \pi})$. $\mathrm{Spec}(A_{\gamma, \pi})$
The (Γ, π) -covering $A_{\gamma, \pi}$ is said to be **Ramanujan** if $\mathrm{Spec}(A_{\gamma, \pi}) \subseteq [-\rho(G), \rho(G)]$, and **one-sided Ramanujan** if all the eigenvalues of $A_{\gamma, \pi}$ are at most $\rho(G)$.

For example, if G consists of a single vertex with several loops, and if \mathcal{R} marks the regular representation³ of Γ , then a (Γ, \mathcal{R}) -covering $A_{\gamma, \mathcal{R}}$ of G is equivalent to the Cayley graph of Γ with respect to the set $\gamma(E(G))$. The non-trivial spectrum of this Cayley graph is given by the component $\mathcal{R} - \mathrm{triv}$ of \mathcal{R} (see Section 2.4 for some background). Hence, the Cayley graph is Ramanujan if and only if the corresponding $(\Gamma, \mathcal{R} - \mathrm{triv})$ -covering is Ramanujan.

As another example, the symmetric group S_r has an $(r - 1)$ -dimensional representation, called the *standard* representation and denoted std – see Section 2.4 for details. Every r -covering H of G corresponds to a unique (S_r, std) -covering, and, moreover, the new spectrum of H is precisely the spectrum of the corresponding (S_r, std) -covering – see Claim 2.12. In particular, a Ramanujan r -covering corresponds to a Ramanujan (S_r, std) -covering. When $r = 2$, the standard representation of $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ coincides with the sign representation, and this correspondence between 2-coverings and (S_2, std) -coverings appears already in [BL06] and is used in [MSS15a].

The following is, then, a natural generalization of the question concerning ordinary Ramanujan coverings of graphs:

Question 1.6. For which pairs (Γ, π) of a (finite) group Γ with a representation $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ is it guaranteed that every connected graph G has a (one-sided/full) Ramanujan (Γ, π) -covering?

In this language Theorem 1.2 states that every connected graph has a one-sided Ramanujan (S_r, std) -covering for every $r \geq 2$. There are limitations to the possible positive results one can hope for regarding Question 1.6 – see Remark 1.14.

Our proof of Theorem 1.2 exploits two group-theoretic properties of the pair (S_r, std) , and this theorem can be generalized to any pair (Γ, π) satisfying these two properties. The first property deals with the exterior powers⁴ of π :

Definition 1.7. Let Γ be a finite group and $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ a representation. We say that (Γ, π) satisfies $(\mathcal{P}1)$ if all exterior powers $\bigwedge^m \pi$, $0 \leq m \leq d$, are irreducible and non-isomorphic. $(\mathcal{P}1)$

The exterior power $\bigwedge^0 \pi$ is always the trivial representation mapping every $g \in \Gamma$ to $1 \in \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$. The next power, $\bigwedge^1 \pi$, is simply π itself. The last power, $\bigwedge^d \pi$, is the one-dimensional representation given by $\det \circ \pi: \Gamma \rightarrow \mathbb{C}^*$. Hence, if π is one dimensional, (Γ, π) satisfies $(\mathcal{P}1)$ if and only if π is non-trivial. For example, the sign representation of $\mathbb{Z}/2\mathbb{Z}$ used in [MSS15a] satisfies $(\mathcal{P}1)$. If π is 2-dimensional, (Γ, π) satisfies $(\mathcal{P}1)$ if and only if π is irreducible and $\pi(\Gamma) \not\subseteq \mathrm{SL}_2(\mathbb{C})$. We explain more in Section 2.4.

Denote by $\phi_{\gamma, \pi}$ the characteristic polynomial of the (Γ, π) -covering $A_{\gamma, \pi}$, namely $\phi_{\gamma, \pi}$

³Namely, \mathcal{R} is a $|\Gamma|$ -dimensional representation, and for every $g \in \Gamma$, the matrix $\mathcal{R}(g)$ is the permutation matrix describing the action of g on the elements of Γ by right multiplication.

⁴See Section 2.4 for a definition of exterior powers, irreducible representations and isomorphism of representations.

$$\phi_{\gamma,\pi}(x) \stackrel{\text{def}}{=} \det(xI - A_{\gamma,\pi}) = \prod_{\mu \in \text{Spec}(A_{\gamma,\pi})} (x - \mu). \quad (1.2)$$

Along this paper, the default distribution on Γ -labelings of a graph G is the one defined by $\mathcal{C}_{\Gamma,G}$ (see Definition 1.5). Hence, when Γ and G are understood from the context, we use the notation $\mathbb{E}_{\gamma}[\phi_{\gamma,\pi}(x)]$ to denote the expected characteristic polynomial of a random (Γ, π) -covering, the expectation being over the space $\mathcal{C}_{\Gamma,G}$ of Γ -labelings.

The following theorem describes the role of Property $(\mathcal{P}1)$ in our proof:

Theorem 1.8. *Let the graph G be connected. For every pair (Γ, π) satisfying $(\mathcal{P}1)$ with $\dim(\pi) = d$, the following holds:*

$$\mathbb{E}_{\gamma}[\phi_{\gamma,\pi}(x)] = \mathbb{E}_{H \in \mathcal{C}_{d,G}} \mathcal{M}_H(x), \quad (1.3)$$

where $\mathcal{M}_H(x)$ is the matching polynomial⁵ of H . In particular, as long as $(\mathcal{P}1)$ holds, $\mathbb{E}_{\gamma}[\phi_{\gamma,\pi}(x)]$ depends only on d and not on (Γ, π) .

Namely, if (Γ, π) satisfies $(\mathcal{P}1)$, then the expected characteristic polynomial of a random (Γ, π) -covering is equal to the expected matching polynomial of a d -covering of G . In particular, as we show below, std is a d -dimensional representation of S_{d+1} satisfying $(\mathcal{P}1)$, and so

$$\frac{\mathbb{E}_{H \in \mathcal{C}_{d+1,G}} [\det[xI - A_H]]}{\det[xI - A_G]} = \mathbb{E}_{H \in \mathcal{C}_{d,G}} \mathcal{M}_H(x).$$

This generalizes an old result from [GG81] for the case $d = 1$, which is essential in [MSS15a]: the expected characteristic polynomial of a 2-covering of G is equal to the characteristic polynomial of G times the matching polynomial of G . Together with Theorem 2.7 below, we get that whenever (Γ, π) satisfies $(\mathcal{P}1)$ and G has no loops, the expected characteristic polynomial $\mathbb{E}_{\gamma}[\phi_{\gamma,\pi}(x)]$ has only real-roots, all of which lie inside the Ramanujan interval $[-\rho, \rho]$. We call the right hand side of (1.3) the **d -matching polynomial** of G – see Definition 2.5.

To define the second property we need the notion of *pseudo-reflections*: a matrix $A \in \text{GL}_d(\mathbb{C})$ is called a **pseudo-reflection** if A has finite order and $\text{rank}(A - I) = 1$. Equivalently, A is a pseudo-reflection if it is conjugate to a diagonal matrix of the form

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

with $\lambda \neq 1$ some root of unity.

Definition 1.9. Let Γ be a finite group and $\pi: \Gamma \rightarrow \text{GL}_d(\mathbb{C})$ a representation. We say that (Γ, π) satisfies $(\mathcal{P}2)$ if $\pi(\Gamma)$ is a complex reflection group, namely, if it is generated by pseudo-reflections. ($\mathcal{P}2$)

Complex reflection groups are a generalization of Coxeter groups. The most well known example is the group of permutation matrices in $\text{GL}_d(\mathbb{C})$: this group is generated by genuine reflections – transpositions. In the related case of (S_r, std) , the image of every transposition is a pseudo-reflection as well, hence (S_r, std) satisfies $(\mathcal{P}2)$ – see Section 2.4. The complete classification of pairs (Γ, π) satisfying $(\mathcal{P}2)$ is well known – see Section 5.

The following theorem manifests the role of $(\mathcal{P}2)$ in our proof:

⁵See (2.1).

Theorem 1.10. *Let G be a finite, loopless graph. For every pair (Γ, π) satisfying $(\mathcal{P}2)$, the following holds:*

- $\mathbb{E}_\gamma [\phi_{\gamma, \pi}(x)]$ is real rooted.
- There exists a (Γ, π) -covering $A_{\gamma, \pi}$ with largest eigenvalue at most the largest root of $\mathbb{E}_\gamma [\phi_{\gamma, \pi}(x)]$.

The proof of Theorem 1.10 is based on the method of interlacing polynomials. The core of the argument is inspired by [MSS15b, Theorem 3.3].

We can now state our generalization of Theorem 1.2:

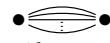
Theorem 1.11. *Let Γ be a finite group and $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ a representation such that (Γ, π) satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$. Then every connected, loopless graph G has a one-sided Ramanujan (Γ, π) -covering.*

In Section 5 we elaborate the combinatorial consequences of Theorem 1.11.

Some Remarks

Remark 1.12. Some of the above definitions and results apply also to compact groups Γ . For example, if Γ is compact, we can let $\mathcal{C}_{\Gamma, G}$ be the probability space of all Γ -labelings of G endowed with Haar measure on $\Gamma^{E^+(G)}$. If $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ is unitary (see Section 2.4), Property $(\mathcal{P}1)$ makes perfect sense, and Theorem 1.8 (and its proof) apply. Two interesting instances are specified in Corollary 5.6. There are also ways to generalize $(\mathcal{P}2)$ for compact groups in a way that the proof of Theorem 1.10 will apply.

Remark 1.13. Suppose that π is an r -dimensional permutation representation of Γ , meaning that $\pi(\Gamma)$ is a group of permutation matrices in $\mathrm{GL}_r(\mathbb{C})$ (see Section 5.4). Suppose further that $|\Gamma|$ is much smaller than $r!$. This means that (Γ, π) -coverings of a graph G correspond to a small subset of all r -coverings of G . Every permutation representation is composed of⁶ the trivial representation and an $(r - 1)$ -dimensional representation π' . A positive answer to Question 1.6 for π' can yield a relatively fast algorithm to construct (bipartite-) Ramanujan graphs.

A concrete compelling example is given by $\Gamma = \mathrm{PSL}_2(\mathbb{F}_q)$ and π the action of Γ by permutations on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. In this case, π is $(q + 1)$ -dimensional, and π' is irreducible of dimension q . Here $|\Gamma| \approx q^3/2$. The pair $(\mathrm{PSL}_2(\mathbb{F}_q), \pi')$ satisfies neither $(\mathcal{P}1)$ nor $(\mathcal{P}2)$, so the results in this paper do not apply. However, there are only, roughly, q^{3m} disjoint $(\mathrm{PSL}_2(\mathbb{F}_q), \pi')$ -coverings of a graph G with m edges. So a positive answer to Question 1.6 in this case means, for example, one can construct a Ramanujan $(q + 1)$ -covering of the graph with k edges connecting 2 vertices , namely, a bipartite k -regular Ramanujan graph on $2(q + 1)$ vertices, in time, roughly, q^{3k} . This example is especially compelling because this subgroup of permutations is sparse and well-understood, and also because the group $\mathrm{PSL}_2(q)$ has proven useful before in constructing Ramanujan graphs: the explicit construction of Ramanujan graphs in [LPS88, Mar88] uses Cayley graphs of such groups.

Remark 1.14. Returning to Question 1.6, we stress that not every pair (Γ, π) guarantees Ramanujan coverings. For example, let \mathcal{R} denote the regular representation of Γ (see Footnote 3) and assume

⁶See Section 2.4 for the meaning of this.

that $\text{rank}(\Gamma) > \text{rank}(\pi_1(G))$, where $\text{rank}(\Gamma)$ marks the minimal size of a generating set of Γ . Then there is no surjective homomorphism $\pi_1(G) \rightarrow \Gamma$ (see Claim 2.10), so every (Γ, \mathcal{R}) -covering is necessarily disconnected, and the new spectrum contains the trivial, Perron-Frobenius eigenvalue of G .

Another counterexample to Question 1.6 is that of regular representations of abelian groups. In particular, several authors asked about the existence of Ramanujan “**shift lifts**” of graphs: whether every graph has an r -covering where all edges are labeled by cyclic shift permutations, namely, by powers of the permutation $(1\ 2\ \dots\ r) \in S_r$. This is equivalent to (C_r, \mathcal{R}) -coverings where C_r is the cyclic group of size r . We claim there cannot be Ramanujan r -coverings of this kind when r is large. The reason is the same argument showing that large abelian groups do not admit Ramanujan Cayley graphs with a small number of generators: for example, the balls in such a covering grow only polynomially, where in expander graphs they grow exponentially fast. Concretely, simulations we conducted show that the bouquet with one vertex and two loops has no Ramanujan 36-covering where the loops are labeled by cyclic permutations. It also does not have any one-sided Ramanujan 73-covering with cyclic permutations.

1.3 Overview of the proof

The proof of Theorem 1.2 and its generalization, Theorem 1.11, follows the general proof strategy from [MSS15a]. A key point in this strategy, is the following elementary yet very useful fact:

Fact 1.15 (E.g. proof of [MSS15a, Lemma 4.2]). *Assume that $f, g \in \mathbb{R}[x]$ are two polynomials of degree n so that $(1 - \lambda)f + \lambda g$ is real rooted for every $\lambda \in [0, 1]$. Then, for every $1 \leq i \leq n$, the i -th root of $(1 - \lambda)f + \lambda g$ moves monotonically when λ moves from 0 to 1.*

Namely, if the roots of a polynomial h are all real and denoted $r_n(h) \leq \dots \leq r_2(h) \leq r_1(h)$, then Fact 1.15 means that the function $\lambda \mapsto r_i((1 - \lambda)f + \lambda g)$ is monotone (non-decreasing or non-increasing) for every i . We elaborate in Section 2.5.

The starting point of the strategy of [MSS15a] is that instead of considering “discrete” coverings of the graph G , one can consider convex combinations of coverings, or more precisely, convex combinations of *characteristic polynomials* of coverings. Concretely, let $\Delta_r(G)$ be the simplex of all probability distributions on r -coverings of the graph G . The simplex $\Delta_r(G)$ has $|\mathcal{C}_{r,G}| = (r!)^{|E^+(G)|}$ vertices. To the vertex corresponding to the r -covering $H \in \mathcal{C}_{r,G}$ we associate the characteristic polynomial of the *new* spectrum of H , namely,

$$\phi_H = \frac{\det(xI - A_H)}{\det(xI - A_G)} = \prod_{\mu \in \text{New Spectrum of } H} (x - \mu) = \det(xI - A_{\sigma, \text{std}})$$

where std is the standard $(r - 1)$ -dimensional representation of S_r , and σ is the S_r -labeling of G corresponding to H .

Every point $p \in \Delta_r(G)$ is associated with a polynomial ϕ_p , the corresponding weighted average of the $\{\phi_H\}_{H \in \mathcal{C}_{r,G}}$: if $p = \sum_{H \in \mathcal{C}_{r,G}} a_H \cdot H$, then $\phi_p = \sum_{H \in \mathcal{C}_{r,G}} a_H \cdot \phi_H$. The proof now consists of two main parts:

- (i) Find a Ramanujan point $p_{Ram} \in \Delta_r(G)$, namely a point whose corresponding polynomial $\phi_{p_{Ram}}$ is real rooted with all its roots inside the Ramanujan interval $[-\rho, \rho]$.
- (ii) find a real-rooted region inside $\Delta_r(G)$ which contains p_{Ram} and which allows one to use Fact 1.15 and move along straight lines from p_{Ram} to one of the vertices.

We now explain each part in greater detail and explain how it is obtained in the current paper:

Part (i): a Ramanujan point inside $\Delta_r(G)$

In the case $r = 2$ studied in [MSS15a], the center of $\Delta_r(G)$ is a Ramanujan point. This is the point corresponding to the uniform distribution over all 2-coverings of G . As mentioned above, the polynomial at this point is the matching polynomial of G [GG81]. The fact that this polynomial is Ramanujan, i.e. is real-rooted with all its roots in $[-\rho, \rho]$, is a classical fact due to Heilmann and Lieb [HL72] – see Theorem 2.4 below.

It turns out that for larger values of r , the center point of $\Delta_r(G)$ is still Ramanujan, yet this result is new, more involved, and relies on Property (P1). Theorem 1.8 states that the polynomial associated with the center point is the $(r - 1)$ -matching polynomial of G , defined as the average of the matching polynomials of all $(r - 1)$ -coverings of G . In Section 2.2 we explain why every real root of this polynomial lies in $[-\rho, \rho]$. That all its roots are real follows from the fact that the center point of $\Delta_r(G)$ is contained in the real-rooted region we find in Part (ii) below.

Part (ii): a real-rooted region inside $\Delta_r(G)$

The main technical part of [MSS15a] consists of showing that if $p \in \Delta_2(G)$ is a probability distribution on 2-coverings of G in which the S_2 -label of every $e \in E^+(G)$ is chosen independently, then the associated polynomial ϕ_p is real rooted. This gives a real-rooted region inside $\Delta_2(G)$. For example, if the S_2 -label of every $e \in E^+(G)$ is uniform among the two possibilities, we get the center point, which provides yet another proof for that the matching polynomial of G is real rooted.

The crux of the matter is that among this family of distributions, if one perturbs the distribution only on one particular edge $e \in E^+(G)$, the corresponding point in $\Delta_2(G)$ moves along a straight interval, which allows us to use Fact 1.15.

More concretely, order the edges in $E^+(G)$ by e_1, e_2, \dots in an arbitrary fashion. Start at the Ramanujan point p_{Ram} – the center point of $\Delta_2(G)$. Let q_1 denote the point in $\Delta_2(G)$ where e_1 is labeled by $\text{id} \in S_2$ and all remaining edges are labeled uniformly at random and independently. Let q_{-1} be another point with the same definition except that e_1 is labeled by $(12) \in S_2$. Now p_{Ram} lies on the straight interval connecting q_1 and q_{-1} . Note that every point on this interval corresponds to a random 2-covering in which every edge is labeled independently from the others, and the associated polynomial is, therefore, real rooted. By Fact 1.15, the largest root of $\phi_{p_{Ram}}$ lies between the largest root of ϕ_{q_1} and the largest root of $\phi_{q_{-1}}$. Hence, one of the two points q_1 or q_{-1} has largest root at most the one of p_{Ram} , and in particular at most ρ . Assume without loss of generality that q_{-1} has largest root at most ρ . Now repeat this process, this time by perturbing the distribution on e_2 . Let $q_{-1,1} \in \Delta_2(G)$ denote the point where $\sigma(e_1) = (12)$, $\sigma(e_2) = \text{id}$ and all remaining edges are distributed uniformly and independently, and let $q_{-1,-1}$ be defined similarly, only with $\sigma(e_2) = (12)$. Since q_{-1} lies on the straight line between $q_{-1,1}$ and $q_{-1,-1}$, the largest root of one of these two latter points is at most that of q_{-1} . If we go on and

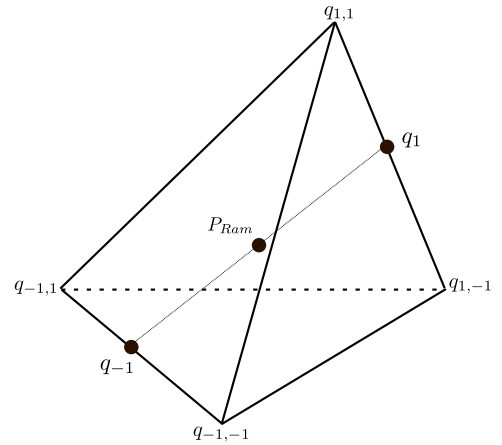


Figure 1.2: The simplex $\Delta_2(G)$ when $|E^+(G)| = 2$

gradually choose a deterministic value for every $e \in E^+(G)$ while not increasing the largest root, we end up with a vertex of $\Delta_2(G)$ representing a one-side Ramanujan 2-covering of G . This is illustrated in Figure 1.2.

For larger values of r , the definition of the real-rooted region is more subtle. Simple independence of the edges does not suffice⁷. Instead, we define the following real-rooted region in $\Delta_r(G)$. This follows the ideas in Section 3 of [MSS15b], and generalizes the real-rooted region in $\Delta_2(G)$ from [MSS15a].

Proposition. *Let $p \in \Delta_r(G)$ be a probability distribution of r -coverings of G satisfying that for every $e \in E^+(G)$, the random labeling of e*

1. *is independent of the labelings on other edges, and*
2. *is equal to a product of independent random variables $X_{e,1}, X_{e,2}, \dots, X_{e,\ell(e)}$, where each $X_{e,i}$ has (at most) two values in its support: the identity permutation and some transposition.*

Then ϕ_p is real rooted.

The statement of Proposition 4.4 is slightly more general and applies to any pseudo-reflections and not only transpositions. To illustrate this proposition, consider the case $r = 3$. Following [MSS15b, Lemma 3.5], define for every $e \in E^+(G)$ three independent random variables X_e, Y_e and Z_e taking values in S_3 :

$$X_e = \begin{cases} \text{id} & \text{with prob } \frac{1}{2} \\ (1\ 2) & \text{with prob } \frac{1}{2} \end{cases} \quad Y_e = \begin{cases} \text{id} & \text{with prob } \frac{1}{3} \\ (1\ 3) & \text{with prob } \frac{2}{3} \end{cases} \quad Z_e = \begin{cases} \text{id} & \text{with prob } \frac{1}{2} \\ (1\ 2) & \text{with prob } \frac{1}{2} \end{cases}. \quad (1.4)$$

The random permutation $X_e \cdot Y_e \cdot Z_e$ has uniform distribution in S_3 . This shows that the center point $p \in \Delta_3(G)$ satisfies the assumptions in the Proposition and therefore ϕ_p is real rooted. Together with Theorem 1.8 that implies that all real roots of ϕ_p are in the Ramanujan interval $[-\rho, \rho]$, we obtain that the center point is Ramanujan.

In Remark 4.7 we explain how the explicit construction of random variables in (1.4) can be generalized to every r , showing that the center point of $\Delta_r(G)$ satisfies the assumptions in Proposition 4.4 and is, therefore, real rooted. However, the crucial feature of the family of r -coverings of G is that $\text{std}(S_r) \leq \text{GL}_{r-1}(\mathbb{C})$ is a complex reflection group, i.e. that (S_r, std) satisfies (P2). In Section 4.3 we give an alternative, non-constructive argument which shows that for every pair (Γ, π) satisfying (P2), the center point of the corresponding simplex satisfies the assumptions of Proposition 4.4.

Finally, to find a one-sided Ramanujan r -covering of G , we imitate the process illustrated in Figure 1.2, only at each stage we perturb and then fix the value of one of the independent random variables $X_{e,i}$. For example, in the case $r = 3$ and the variables constructed in (1.4), this translates to the $3|E^+(G)|$ -step process illustrated in Figure 1.3. At each step, we determine the value of one of the $3|E^+(G)|$ variables so that the maximal root of the associated polynomial never increases.

⁷Consider, for instance, the cycle of length two $\bullet \text{---} \text{---} \bullet$. Define a distribution P on its 3-coverings by labeling one edge deterministically with the identity permutation and the other edge with either the identity or a 3-cycle $(1\ 2\ 3)$, each with probability $\frac{1}{2}$. The average characteristic polynomial of $A_{\sigma, \text{std}}$ is then $\frac{(x^2-4)^2 + (x^2-1)^2}{2}$ which is *not* real rooted.

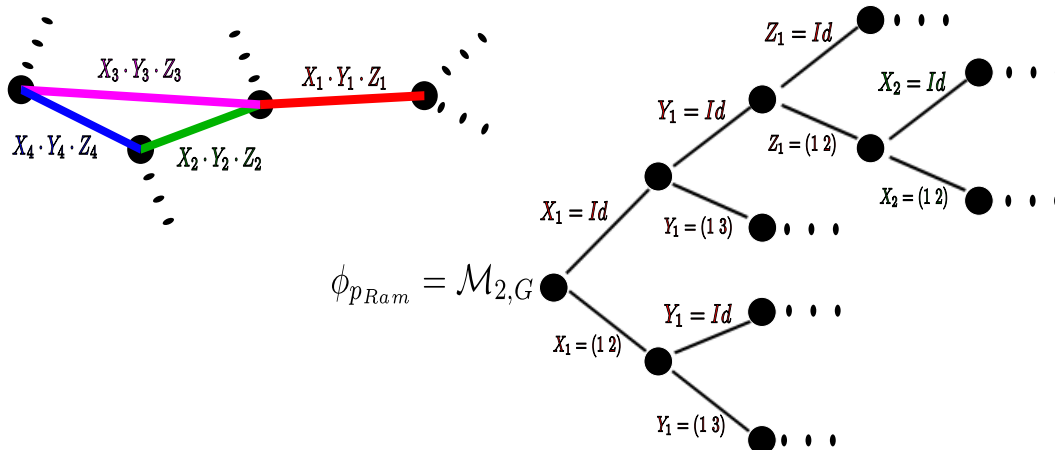


Figure 1.3: On the left is a piece of the graph G with three random variables associated to every edge. On the right is a piece of the corresponding binary tree of polynomials.

The rest of the paper is organized as follows. In Section 2 we give more background details, prove some preliminary results and reduce all results to proving Theorems 1.8 and 1.10. Section 3 is dedicated to property ($\mathcal{P}1$) and the proof of Theorem 1.8, while in Section 4 we study Property ($\mathcal{P}2$) and prove Theorem 1.10. In Section 5, we study groups satisfying the two properties and present further combinatorial applications of Theorem 1.11. We end in Section 6 with a list of open questions arising from the discussion in this paper.

2 Background and Preliminary Claims

In this section we give more background material, prove some preliminary claims, and reduce all the results from Section 1 to the proofs of Theorems 1.8 and 1.10.

2.1 Expander and Ramanujan Graphs

As in Section 1, let G be a finite connected graph on n vertices and A_G its adjacency matrix. Recall that $\mathfrak{pf}(G)$ is the Perron-Frobenius eigenvalue of A_G , that $\lambda_n \leq \dots \leq \lambda_2 \leq \lambda_1 = \mathfrak{pf}(G)$ are its entire spectrum, and that $\lambda(G) = \max(\lambda_2, -\lambda_n)$. The graph G is considered to be well-expanding if it is “highly” connected. This can be measured by different combinatorial properties of G , most commonly by its Cheeger constant, by the rate of convergence of a random walk on G , or by how well the number of edges between any two sets of vertices approximates the corresponding number in a random graph (via the so-called Expander Mixing Lemma)⁸. All these properties can be measured, at least approximately, by the spectrum of G , and especially by $\lambda(G)$ and the spectral gap $\mathfrak{pf}(G) - \lambda(G)$: the smaller $\lambda(G)$ and the bigger the spectral gap is, the better expanding G is⁹. (See [HLW06] and [Pud15, Appendix B] and the references therein.)

⁸In this sense, Ramanujan graphs resemble random graphs. The converse is also true in certain regimes of random graphs: see [Pud15] and the references therein.

⁹More precisely, the Cheeger inequality relates the Cheeger constant of a graph with the value of $\lambda_2(G)$.

Yet, $\lambda(G)$ cannot be arbitrarily small. Let T be the universal covering tree of G . We think of all the finite graphs covered by T as one family. For example, for any $k \geq 2$, all finite k -regular graphs constitute a single such family of graphs: they are all covered by the k -regular tree. Let $\rho(T)$ be the spectral radius of T . This number is the spectral radius of the adjacency operator A_T acting on $\ell^2(V(T))$ by

$$(A_T f)(v) = \sum_{u \sim v} f(u),$$

and for the k -regular tree, this number is $2\sqrt{k-1}$. The spectral radius of T plays an important role in the theory of expansion of the corresponding family of graphs:

Theorem 2.1 (This version appears in [Cio06, Thm 6], a slightly weaker version appeared already in [Gre95, Thm 2.11]). *Let T be a tree with finite quotients and ρ its spectral radius. For every $\varepsilon > 0$, there exists $c = c(T, \varepsilon)$, $0 < c < 1$, such that if G is a finite graph with n vertices which is covered by T , then at least cn of its eigenvalues satisfy $\lambda_i \geq \rho - \varepsilon$. In particular, $\lambda(G) \geq \rho - o_n(1)$ (with the $o_n(1)$ term depending only on T).*

The last statement of the theorem, restricted to regular graphs, is due to Alon-Boppana [Nil91]. Thus, graphs G satisfying $\lambda(G) \leq \rho(G)$ are considered to be optimal expanders. Following the terminology of [LPS88], they are called Ramanujan graphs.

The seminal works [LPS88, Mar88, Mor94] provide an infinite family of k -regular Ramanujan graphs whenever $k-1$ is a prime power. Lubotzky [Lub94, Problem 10.7.3] asked whether for every $k \geq 3$ there are infinitely many k -regular Ramanujan graphs¹⁰. We stress that this only hints at a much stronger phenomena. For example, it is known [Fri08] that as $n \rightarrow \infty$, almost all k -regular graphs are nearly Ramanujan, in the following sense: for every $\varepsilon > 0$, the non-trivial spectrum of a random k -regular graph falls in $[-\rho - \varepsilon, \rho + \varepsilon]$ with probability tending to 1. Moreover, it is conjectured that among all k -regular graphs on n vertices the proportion of Ramanujan graphs tends to a constant in $(0, 1)$ as $n \rightarrow \infty$ (e.g. [MNS08]).

Recall that we consider families of finite graphs defined by a common universal covering tree. In the regular case, every family has at least one Ramanujan graph (e.g. the complete graph on $k+1$ vertices). Other families may contain no Ramanujan graphs at all. For example, the family of (k, ℓ) -biregular graphs, all covered by the (k, ℓ) -biregular tree, consists entirely of bipartite graphs, so none of them is Ramanujan in the strict sense. There also exist families with no Ramanujan graphs, not even bipartite-Ramanujan ones [LN98]. In these cases there are certain “bad” eigenvalues outside the Ramanujan interval appearing in every finite graph in the family. Still, it makes sense to look for optimal expanders under these constraints, namely, for graphs in the family where all other eigenvalues lie in the Ramanujan interval. For example, bipartite-Ramanujan graphs are optimal expanders in many combinatorial senses within the family of bipartite graphs (e.g. [GP14, Lemma 10]). The strategy of constructing Ramanujan coverings fits this general goal: find any graph in the family which is optimal (has all its values in the Ramanujan interval except for the bad ones) and construct Ramanujan coverings to obtain more optimal graphs in the same family. Of course, connected coverings of a graph G are covered by the same tree as G .

Marcus, Spielman and Srivastava have already shown that every graph has a one-sided Ramanujan 2-covering [MSS15a]. Thus, if a family of graphs contains at least one Ramanujan graph

¹⁰In fact, Lubotzky’s original definition of Ramanujan graphs included also *bipartite*-Ramanujan graphs. Thus, [MSS15a] answers this question positively.

(bipartite or not), then it has infinitely many bipartite-Ramanujan graphs¹¹. More recently, they have showed that for any $k \geq 3$, the graph $\bullet \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \bullet$ (two vertices with k edges connecting them) has a Ramanujan r -covering for every r [MSS15b]. It follows there are k -regular bipartite-Ramanujan graphs, not necessarily simple, on $2r$ vertices for every r . Our proof to the more general result, Theorem 1.2, is very different. It also yields the existence of a richer family of bipartite-Ramanujan graphs than was known before.

Corollary 2.2. *Consider the family of all finite graphs which are covered by some given common universal covering tree. If the family contains a (simple) bipartite-Ramanujan graph on n vertices, then it also contains (simple, respectively) bipartite-Ramanujan graphs on nr vertices for every $r \in \mathbb{Z}_{\geq 1}$.*

In particular, there is a simple k -regular, bipartite-Ramanujan graph on $2kr$ vertices for every r . There is also a simple, (k, ℓ) -biregular, bipartite-Ramanujan graph on $(k + \ell)r$ vertices for every r .

The last statement follows by constructing Ramanujan r -coverings of the full k -regular bipartite graph on $2k$ vertices, or of the full (k, ℓ) -biregular bipartite graph on $k + \ell$ vertices, both of which are bipartite-Ramanujan.

As of now, we cannot extend all the results in this paper to graphs with loops (and see Question 6.6). However, we can extend Theorem 1.2 to regular graphs with loops. We now give the short proof of this extension, assuming Theorem 1.2:

Proposition 2.3. *Let G be a regular finite graph, possibly with loops. Then G has a one-sided Ramanujan r -covering for every r .*

We remark that in this proposition the proof does not yield the analogous result for coverings with new spectrum bounded from below by $-\rho(G)$.

Proof. Let G be any finite connected graph with n vertices and m edges. Subdivide each of its edges by introducing a new vertex in its middle, to obtain a new, bipartite graph H , with n vertices on one side and m on the other. Clearly, there is a one-to-one correspondence between (isomorphism types of) r -coverings of G and (isomorphism types of) r -coverings of H . The rank of A_H is at most $2n$, so H has eigenvalue 0 with multiplicity (at least) $m - n$. As H is bipartite, the remaining $2n$ eigenvalues are symmetric around zero, and their squares are the eigenvalues of $A_G + D_G$, where A_G is the adjacency matrix of G and D_G is diagonal with the degrees of the vertices.

Now assume G is k -regular, and let μ be an eigenvalue of G . Then $\pm\sqrt{\mu + k}$ are eigenvalues of H (and these are precisely all the eigenvalues of H , aside to the $m - n$ zeros). By Corollary 1.3, H has a Ramanujan r -covering \hat{H}_r for every r . Since the spectral radius of the $(k, 2)$ -biregular tree is¹² $\sqrt{k - 1} + 1$, every eigenvalue μ of the corresponding r -covering \hat{G}_r satisfies $\sqrt{\mu + k} \leq \sqrt{k - 1} + 1$, i.e., $\mu \leq 2\sqrt{k - 1}$. \square

The exact same argument can be used to extend also the statement of Theorem 1.11 to regular graphs with loops: if G is regular, possibly with loops, and (Γ, π) satisfies (P1) and (P2), then G has a one-sided Ramanujan (Γ, π) -covering.

¹¹Given a Ramanujan graph, its “double cover” — the 2-covering with all permutations being non-identity — is bipartite-Ramanujan.

¹²In general, the spectral radius of the (k, ℓ) -biregular tree is $\sqrt{k - 1} + \sqrt{\ell - 1}$.

2.2 The d -Matching Polynomial

An important ingredient in our proof of Theorem 1.2 is a new family of polynomials associated to a given graph. These polynomials generalize the well-known matching polynomial of a graph defined by Heilmann and Lieb [HL72]: let m_i be the number of matchings in G with i edges, and set $m_0 = 1$. The matching polynomial of G is

$$\mathcal{M}_G(x) \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i m_i x^{n-2i} \in \mathbb{Z}[x]. \quad (2.1)$$

The following is a crucial ingredient in the proof of the main result of [MSS15a]:

Theorem 2.4. [HL72]¹³ *The matching polynomial \mathcal{M}_G of every finite connected graph G is real rooted with all its roots lying in the Ramanujan interval $[-\rho(G), \rho(G)]$.*

Definition 2.5. Let $d \in \mathbb{Z}_{\geq 1}$. The **d -matching polynomial** of a finite graph G , denoted $\mathcal{M}_{d,G}$, $\mathcal{M}_{d,G}$ is the average of the matching polynomials of all d -coverings of G (in $\mathcal{C}_{d,G}$ – see Definition 1.4).

For example, if G is K_4 minus an edge, then

$$\mathcal{M}_{3,G}(x) = x^{12} - 15x^{10} + 81x^8 - 189x^6 + 180x^4 - \frac{178}{3}x^2 + 4.$$

Of course, $\mathcal{M}_{1,G} = \mathcal{M}_G$ is the usual matching polynomial of G (a graph is the only 1-covering of itself). Note that these generalized matching polynomials of G are monic, but their other coefficients need not be integer valued. However, they seem to share many of the nice properties of the usual matching polynomial. For instance,

Corollary 2.6. *Every real root of $\mathcal{M}_{d,G}$ lies inside the Ramanujan interval $[-\rho(G), \rho(G)]$.*

Proof. Every covering of G belongs to the same family as G (even when the covering is not connected, each component is covered by the same tree as G). Recall that n denotes the number of vertices of G . The ordinary matching polynomial of every $H \in \mathcal{C}_{d,G}$ is a degree- nd monic polynomial. By Theorem 2.4, it is strictly positive in the interval $(\rho(G), \infty)$, and is either strictly positive or strictly negative in $(-\infty, -\rho(G))$ depending only on the parity of nd . The corollary now follows by the definition of $\mathcal{M}_{d,G}$ as the average of such polynomials. \square

In fact, all roots of $\mathcal{M}_{d,G}$ are real¹⁴. For this, we use the full strength of Theorems 1.8 and 1.10. We show (Fact 2.13 below) that for every d , there is a pair (Γ, π) of a group Γ and a d -dimensional representation π which satisfies both (P1) and (P2). From theorem 1.8 we obtain that $\mathcal{M}_{d,G}(x) = \mathbb{E}_\gamma[\phi_{\gamma,\pi}(x)]$ (the expectation over $\mathcal{C}_{\Gamma,G}$), and from Theorem 1.10 we obtain that $\mathbb{E}_\gamma[\phi_{\gamma,\pi}(x)]$ is real-rooted. We wonder if there is a more direct proof of the real-rootedness of $\mathcal{M}_{d,G}$ (see Question 6.5). This yields:

¹³Actually, [HL72] shows that \mathcal{M}_G satisfies this statement only when G is regular. Apparently, the case of irregular graphs was first noticed in [MSS15a], even though some of the original proofs of [HL72] work in the irregular case as well.

¹⁴Except when G has loops and then we do not know if this necessarily holds.

Theorem 2.7. *Let G be a finite, connected¹⁵, loopless graph. For every $d \in \mathbb{Z}_{\geq 1}$, the polynomial $\mathcal{M}_{d,G}$ is real rooted with all its roots contained in the Ramanujan interval $[-\rho(G), \rho(G)]$.*

In the proof of Theorem 1.8, which gives an alternative definition for $\mathcal{M}_{d,G}$, we will use a precise formula for this polynomial which we now develop. Every $H \in \mathcal{C}_{d,G}$, a d -covering of G , has exactly d edges covering any specific edge in G , and, likewise, d vertices covering every vertex of G . Thus, one can think of $\mathcal{M}_{d,G}$ as a generating function of multi-matchings in G : each edge in G can be picked multiple times so that each vertex is covered by at most d edges. We think of such a multi-matching as a function $m: E^+(G) \rightarrow \mathbb{Z}_{\geq 0}$. The weight associated to every multi-matching m is equal to the average number of ordinary matchings projecting to m in a random d -covering of G . Namely, the weight is the average number of matchings in $H \in \mathcal{C}_{d,G}$ with exactly $m(e)$ edges projecting to e , for every $e \in E^+(G)$.

To write an explicit formula, we extend m to all $E(G)$ by $m(-e) = m(e)$. We also denote by $e_{v,1}, \dots, e_{v,\deg(v)}$ the edges in $E(G)$ emanating from a vertex $v \in V(G)$ (in an arbitrary order, loops at v appearing twice, of course), and by $m(v)$ the number of edges incident with v in the multi-matching. Namely,

$$m(v) = \sum_{i=1}^{\deg(v)} m(e_{v,i}).$$

Finally, we denote by $|m|$ the total number of edges in m (with multiplicity), so $|m| = |m| \sum_{e \in E^+(G)} m(e)$.

Definition 2.8. A **d -multi-matching** of a graph G is a function $m: E(G) \rightarrow \mathbb{Z}_{\geq 0}$ with $m(-e) = m(e)$ for every $e \in E(G)$ and $m(v) \leq d$ for every $v \in V(G)$. We denote the set of d -multi-matchings of G by $\text{MultiMatch}_d(G)$.

$\text{MultiMatch}_d(G)$

Proposition 2.9. *Let m be a d -multi-matching of G . Denote¹⁶*

$$W_d(m) = \frac{\prod_{v \in V(G)} \binom{d}{m(e_{v,1}), \dots, m(e_{v,\deg(v)})}}{\prod_{e \in E^+(G)} \binom{d}{m(e)}}. \quad (2.2)$$

Then,

$$\mathcal{M}_{d,G}(x) = \sum_{m \in \text{MultiMatch}_d(G)} (-1)^{|m|} \cdot W_d(m) \cdot x^{nd-2|m|}. \quad (2.3)$$

Proof. Every matching of a d -covering $H \in \mathcal{C}_{d,G}$ projects to a unique multi-matching m of G covering every vertex of G at most d times. Thus, it is enough to show that $W_d(m)$ is exactly the average number of ordinary matchings projecting to m in a random $H \in \mathcal{C}_{d,G}$. Every such matching in H contains exactly $m(e)$ edges in the fiber above every $e \in E(G)$. Assume we know, for each $e \in E(G)$, which vertices in H are covered by the $m(e)$ edges above it. So there are $m(e)$ specific vertices in the fiber above $h(e)$, and $m(e)$ specific vertices in the fiber above $t(e)$. The probability

¹⁵Connectivity here is required only because of the way $\rho(G)$ was defined. The real-rootedness holds for any finite graph. In the general case, the d -matching polynomial is the product of the d -matching polynomials of the different connected components, so the statement remains true for non connected graphs if $\rho(G)$ is defined as the maximum of $\rho(G_i)$ over the different components G_i of G .

¹⁶We use the notation $\binom{b}{a_1, a_2, \dots, a_k}$ to denote the multinomial coefficient $\frac{b!}{a_1! \dots a_k! (b - \sum a_i)!}$.

that a random permutation in S_d matches specific $m(e)$ elements in $\{1, \dots, d\}$ to specific $m(e)$ elements in $\{1, \dots, d\}$ is

$$\frac{m(e)!(d - m(e))!}{d!} = \binom{d}{m(e)}^{-1}.$$

Thus, the denominator of $W_d(m)$ is equal to the probability that a random d -covering has a matching which projects to m and agrees with the particular choice of vertices. We are done as the numerator is exactly the number of possible choices of vertices. (Recall that since we deal with ordinary matchings in H , every vertex is covered by at most one edge, so the set of vertices in the fiber above $v \in V(G)$ which are matched by the pre-image of $e_{v,i}$ is disjoint from those covered by the pre-image of $e_{v,j}$ whenever $i \neq j$.) Finally, we remark that the formula and proof remain valid also for graphs with multiple edges or loops. \square

The proof of Theorem 1.8 in Section 3 will consist of showing that $\mathbb{E}_\gamma[\phi_\gamma(x)]$ is equal to the expression in (2.3).

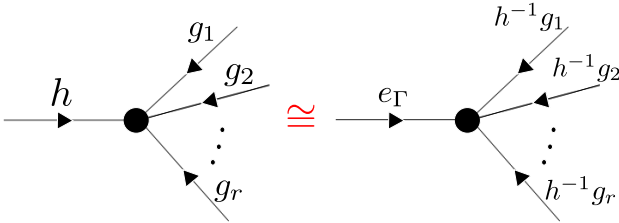
To summarize, here is what this paper shows about the d -matching polynomial $\mathcal{M}_{d,G}$ of the graph G :

- It can be defined by any of the following:
 1. $\mathbb{E}_{H \in \mathcal{C}_{d,G}}[\mathcal{M}_H]$ — the average matching polynomial of a random d -covering of G
 2. $\mathbb{E}_{H \in \mathcal{C}_{d+1,G}}\left[\frac{\det(xI - A_H)}{\det(xI - A_G)}\right] = \mathbb{E}_{H \in \mathcal{C}_{d+1,G}}\left[\prod_{\mu \in \text{newSpec}(H)}(x - \mu)\right]$ - the average “new part” of the characteristic polynomial of a random $(d + 1)$ -covering H of G
 3. $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma,G}}[\phi_{\gamma,\pi}]$ - the average characteristic polynomial of a random (Γ, π) -covering of G whenever (Γ, π) satisfies (P1) and π is d -dimensional
 4. $\sum_{m \in \text{MultiMatch}_d(G)} (-1)^{|m|} \cdot W_d(m) \cdot x^{nd - 2|m|}$, with $W_d(m)$ defined as in (2.2).
- If G has no loops, then $\mathcal{M}_{d,G}$ is real-rooted with all its roots in the Ramanujan interval.

2.3 Group Labelings of Graphs

The model $\mathcal{C}_{r,G}$ we use for a random r -covering of a graph G is based on a uniformly random labeling $\gamma: E(G) \rightarrow S_r$. This is generalized in Definition 1.5 to $\mathcal{C}_{\Gamma,G}$, a probability-space of random Γ -labelings of the graph G . There are natural equivalent ways to obtain the same distribution on (isomorphism) types of r -coverings or Γ -labelings. Although the following will not be used in the rest of the paper, we choose to state it here, albeit loosely, for the sake of completeness.

Two r -covering H_1 and H_2 of G are isomorphic if there is a graph isomorphism between them which respects the covering maps. A similar equivalence relation can be given for Γ -labelings. This is the equivalence relation generated, for example, by the equivalence of the following two labelings of the edges incident to some vertex:



(here e_Γ is the identity element of Γ). For example, if the Γ -labelings γ_1 and γ_2 of G are isomorphic, then $\text{Spec}(A_{\gamma_1, \pi}) = \text{Spec}(A_{\gamma_2, \pi})$ for any finite dimensional representation π of Γ .

Claim 2.10. *Let G be a finite connected graph and Γ a finite group. Let T be a spanning tree of G . The following three probability models yield the same distribution on isomorphism types of Γ -labelings of G :*

1. $\mathcal{C}_{\Gamma, G}$ — uniform distribution on labelings $\gamma: E^+(G) \rightarrow \Gamma$
2. uniform distribution on homomorphisms $\pi_1(G) \rightarrow \Gamma$
3. an arbitrary fixed Γ -labeling of $E^+(T)$ (e.g., with the constant identity labeling) and a uniform distribution on labelings of the remaining edges $E^+(G) \setminus E(T)$.

2.4 Group Representations

Let Γ be a group. A (complex, finite-dimensional) **representation**¹⁷ of Γ is any group homomorphism $\pi: \Gamma \rightarrow \text{GL}_d(\mathbb{C})$ for some $d \in \mathbb{Z}_{\geq 1}$; if Γ is a topological group, we also demand π to be continuous. We then say π is a d -dimensional representation. The representation is called **faithful** if π is injective. Two d -dimensional representations π_1 and π_2 are **isomorphic** if they are conjugate to each other in the following sense: there is some $B \in \text{GL}_d(\mathbb{C})$ such that $\pi_2(g) = B^{-1}\pi_1(g)B$ for every $g \in \Gamma$. The **trivial** representation is the constant function $\text{triv}: \Gamma \rightarrow \text{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ mapping all elements to 1. The direct sum of two representations π_1 and π_2 of dimensions d_1 and d_2 , respectively, is a $(d_1 + d_2)$ -dimensional representation $\pi_1 \oplus \pi_2: \Gamma \rightarrow \text{GL}_{d_1+d_2}(\mathbb{C})$ where $(\pi_1 \oplus \pi_2)(g)$ is a block-diagonal matrix, with a $d_1 \times d_1$ block of $\pi_1(g)$ and a $d_2 \times d_2$ block of $\pi_2(g)$. A representation π is called **irreducible** if is not isomorphic to the direct sum of two representations¹⁸. Otherwise, it is called **reducible**.

Let $U(d)$ be the unitary group, that is, the subgroups of matrices $A \in \text{GL}_d(\mathbb{C})$ whose inverse is A^* , the conjugate-transpose of A . The representation π is called **unitary** if its image in $\text{GL}_d(\mathbb{C})$ is conjugate to a subgroup of $U(d)$. In other words, it is isomorphic to a representation $\Gamma \rightarrow U(d)$. All representations of finite groups are unitary: e.g., conjugate π by $B = \left(\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \pi(g)^* \pi(g)\right)^{1/2}$ to obtain a unitary image.

Claim 2.11. *Let π be a unitary representation of Γ and $A_{\gamma, \pi}$ a (Γ, π) -covering of some graph G . Then the spectrum of $A_{\gamma, \pi}$ is real.*

Proof. It is easy to see that $\text{Spec}(A_{\gamma, \pi}) = \text{Spec}(A_{\gamma, \pi'})$ whenever π and π' are isomorphic. Thus, assume without loss of generality that $\pi(\Gamma) \subseteq U(d)$. Then, by definition, $A_{\gamma, \pi}$ is Hermitian, and the statement follows. \square

The r -dimensional representation π of S_r mapping every $\sigma \in S_r$ to the corresponding permutation matrix is reducible: the 1-dimensional subspace of constant vectors $\langle \mathbf{1} \rangle \leq \mathbb{C}^r$ is invariant under this representation. The action of this representation on the orthogonal complement $\langle \mathbf{1} \rangle^\perp$ is an $(r - 1)$ -dimensional irreducible representation of S_r called the **standard** representation and denoted std . The action on $\langle \mathbf{1} \rangle$ is isomorphic to the trivial representation. Thus, $\pi \cong \text{std} \oplus \text{triv}$.

¹⁷A standard reference for the subject of group representations is [FH91].

¹⁸Equivalently, π is irreducible if it has no non-trivial invariant subspace, namely, no $\{0\} \neq W \leq \mathbb{C}^d$ with $\pi(g)(W) \leq W$ for every $g \in \Gamma$.

Claim 2.12. *If $\gamma: E(G) \rightarrow S_r$ is an S_r -labeling of G , then the new spectrum of the r -covering of G associated with γ is equal to the spectrum of $A_{\gamma, \text{std}}$.*

In particular, every (one-sided) Ramanujan r -covering of G corresponds to a unique (one-sided, respectively) Ramanujan (S_r, std) -covering of G .

Proof. For any Γ -labeling γ of the graph G and any two representations π_1 and π_2 , it is clear that $\text{Spec}(A_{\gamma, \pi_1 \oplus \pi_2})$ is the disjoint union (as multisets) of $\text{Spec}(A_{\gamma, \pi_1})$ and $\text{Spec}(A_{\gamma, \pi_2})$. The claim follows as $A_{\gamma, \text{triv}} = A_G$ for any Γ -labeling γ . \square

In this language, Theorem 1.2 says that every graph G has a one-sided Ramanujan (S_r, std) -covering. This theorem will follow from Theorem 1.11 if we show that the pair (S_r, std) satisfies both (P1) and (P2). Before showing this, let us recall what exterior powers of representations are.

Let $V = \mathbb{C}^d$. The m -th exterior power of V , $\bigwedge^m V$, is the quotient of the tensor power $\bigotimes^m V$ by the subspace spanned by $\{v_1 \otimes v_2 \otimes \dots \otimes v_m \mid v_i = v_j \text{ for some } i \neq j\}$. It is a $\binom{d}{m}$ -dimensional vector space. The representative of $v_1 \otimes \dots \otimes v_m$ is denoted $v_1 \wedge \dots \wedge v_m$ and we have $v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \dots \wedge v_{\sigma(m)} = \text{sgn}(\sigma) \cdot v_1 \wedge v_2 \wedge \dots \wedge v_m$ for any permutation $\sigma \in S_m$.

Now let $\pi: \Gamma \rightarrow \text{GL}_d(\mathbb{C})$ be a d -dimensional representation. Its m -th exterior power, denoted $\bigwedge^m \pi$, is a $\binom{d}{m}$ -dimensional representation depicting an action of Γ on $\bigwedge^m V$. This action is given by

$$g \cdot (v_1 \wedge \dots \wedge v_m) \stackrel{\text{def}}{=} (g \cdot v_1) \wedge \dots \wedge (g \cdot v_m).$$

Fact 2.13. *For every $r \in \mathbb{Z}_{\geq 2}$, the pair (S_r, std) of the symmetric group S_r with its standard, $(r-1)$ -dimensional representation std satisfies both (P1) and (P2).*

Proof. That the exterior powers

$$\bigwedge^0 \text{std} = \text{triv}, \bigwedge^1 \text{std} = \text{std}, \bigwedge^2 \text{std}, \dots, \bigwedge^{r-1} \text{std} = \text{sign}$$

of std are all irreducible and non-isomorphic to each other is a classical fact. More concretely, $\bigwedge^m \text{std}$ is the irreducible representation corresponding to the hook-shaped Young diagram with $m+1$ rows $(r-m, 1, 1, \dots, 1)$, and distinct Young diagrams correspond to distinct irreducible representations (see Chapter 4 and, in particular, Exercise 4.6 in [FH91]). Hence (S_r, std) satisfies (P1).

The symmetric group S_r is generated by transpositions (permutations with $r-2$ fixed points and a single 2-cycle). The image of a transposition under $\pi \cong \text{triv} \oplus \text{std}$ is a pseudo-reflection (with spectrum $\{-1, 1, 1, \dots, 1\}$). Because the spectrum of $\text{triv}(\sigma)$ is $\{1\}$ for any $\sigma \in S_r$, we get that $\text{Spec}(\text{std}(\sigma)) = \{-1, 1, \dots, 1\}$ (with $r-2$ ones) whenever σ is a transposition, namely, $\text{std}(\sigma)$ is a pseudo-reflection. Thus (S_r, std) satisfies (P2). \square

Claim 2.12 and Fact 2.13 show, then, why Theorem 1.2 is a special case of Theorem 1.11. Fact 2.13 also shows that for every d there is a pair (Γ, π) satisfying (P1) and (P2) with $\dim(\pi) = d$. This, together with Theorems 1.8 and 1.10, yields that $\mathcal{M}_{d,G}$ is real-rooted for every loopless G and every d . Adding Corollary 2.6 we obtain Theorem 2.7. Since Theorem 1.11 follows from Theorems 1.8, 1.10 and 2.7, it remains to prove Theorems 1.8 and 1.10.

In Section 3 below, we prove Theorem 1.8 and show that whenever the pair (Γ, π) satisfies (P1), the polynomial $\mathbb{E}_\gamma[\phi_{\gamma, \pi}]$ is equal to $\mathcal{M}_{d,G}$. The crux of this proof is a calculation of $\mathbb{E}_\gamma[\phi_{\gamma, \pi}] = \mathbb{E}_\gamma[\det(xI - A_{\gamma, \pi})]$ by minors of the $d \times d$ blocks, noticing that the determinant of an m -minor of $\pi(g)$ corresponds to an entry (matrix coefficient) of $(\bigwedge^m \pi)(g)$, and using the Peter-Weyl Theorem (Theorem 3.3 below) for matrix coefficients.

2.5 Interlacing Polynomials

A central theme of [MSS15a] as well as of the current paper is showing that certain polynomials are real rooted. The main tool used in the proof is that of polynomials with interlacing roots or polynomials with common interlacing. The two elementary facts below, similar in spirit, show that in certain situations interlacement is equivalent to real-rootedness. Proofs can be found in [Fis08]. Following [MSS15b], we use these two facts in the proof of Theorem 1.10 in Section 4.

Definition 2.14. Let $f, g \in \mathbb{R}[x]$ be real rooted, $n = \deg(f)$ and $\alpha_n \leq \dots \leq \alpha_1$ the roots of f .

1. We say that f and g **interlace** if $\deg(g) = n - 1$ and the roots $\beta_{n-1} \leq \dots \leq \beta_1$ of g satisfy

$$\alpha_n \leq \beta_{n-1} \leq \alpha_{n-1} \leq \dots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1.$$

2. We say that f and g **have common interlacing** if $\deg(g) = n$, its leading coefficient has the same sign as that of f , and its roots $\beta_n \leq \dots \leq \beta_1$ satisfying

$$\{\alpha_n, \beta_n\} \leq \{\alpha_{n-1}, \beta_{n-1}\} \leq \dots \leq \{\alpha_2, \beta_2\} \leq \{\alpha_1, \beta_1\}$$

(i.e., $\alpha_{i+1} \leq \beta_i$ and $\beta_{i+1} \leq \alpha_i$ for every i).

The second definition can be extended to any set of polynomials: the i -th root of any of them is bigger than (or equal to) the $(i + 1)$ -st root of any other.

Fact 2.15. Let $f, g \in \mathbb{R}[x]$ with $\deg(f) = n$ and $\deg(g) = n - 1$. The polynomials f and g interlace if and only if $f + \alpha g$ is real rooted for every $\alpha \in \mathbb{R}$.

Moreover, in this case, the roots change monotonically as α grows.

More precisely, give a real-rooted h , let $r_{\deg(h)}(h) \leq \dots \leq r_1(h)$ be the roots of h , and write $r_0(h) = +\infty$ and $r_{\deg(h)+1}(h) = -\infty$. With this notation, if the leading coefficients of f and g have the same sign, then $\alpha \mapsto r_i(f + \alpha g)$ is monotonically decreasing for every $1 \leq i \leq n$, with $r_i(f + \alpha g)$ decreasing from $r_{i-1}(g)$ to $r_i(g)$. If the leading coefficients have opposite signs, the roots are monotonically increasing.

Fact 2.16. Let $f_1, \dots, f_m \in \mathbb{R}[x]$ have the same degree n . These polynomials have a common interlacing if and only if¹⁹ the average $\lambda_1 f_1 + \dots + \lambda_m f_m$ is real rooted for every $\lambda_1, \dots, \lambda_m$ with $\lambda_i \geq 0$ and $\sum \lambda_i = 1$.

Moreover, in this case, $r_i(\lambda_1 f_1 + \dots + \lambda_m f_m)$ lies in the convex hull of $r_i(f_1), \dots, r_i(f_m)$ for every $1 \leq i \leq n$.

The last property is a generalization of Fact 1.15. The simple argument of the proof appears, for example, in [MSS15a, Lemma 4.2].

¹⁹A more general claim, just as easy, states that the set of polynomials $\{f_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}[x]$ is interlacing if and only if for any probability measure on A , the expected polynomial is real rooted.

3 Property ($\mathcal{P}1$) and the Proof of Theorem 1.8

Recall that G is an undirected oriented graph with n vertices. In this section we assume the pair (Γ, π) satisfies ($\mathcal{P}1$), namely that Γ is finite and π is d -dimensional such that its exterior powers $\bigwedge^0 \pi, \dots, \bigwedge^d \pi$ are irreducible and non-isomorphic. We need to show that $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}} [\phi_{\gamma, \pi}] = \mathcal{M}_{d, G}$. We stress the proof is valid also in the more general case of Γ being compact and π unitary – see Remark 1.12 and Corollary 5.6.

For every Γ -labeling γ of G , we represent the matrix $A_{\gamma, \pi} \in M_{nd}(\mathbb{C})$ as a sum of $|E(G)|$ matrices as follows. For every $e \in E(G)$, let $\mathbf{A}_{\gamma, \pi}(e) \in M_{nd}(\mathbb{C})$ be the $nd \times nd$ matrix composed of n^2 blocks of size $d \times d$ each. All blocks are zero blocks except for the one corresponding to e , the block $(h(e), t(e))$, in which we put $\pi(\gamma(e))$. Clearly,

$$A_{\gamma, \pi} = \sum_{e \in E(G)} A_{\gamma, \pi}(e). \quad (3.1)$$

In order to analyze the expected characteristic polynomial of this sum of matrices, we begin with a technical lemma, giving the determinant of a sum of matrices as a formula in terms of the determinants of their minors. This lemma is used in Section 3.3 where we complete the proof of Theorem 1.8.

3.1 Determinant of Sum of Matrices

Let $A_1, \dots, A_q \in M_d(\mathbb{C})$ be $d \times d$ matrices. The determinant $|A_1 + \dots + A_q|$ can be thought of as a double sum. First, sum $\text{sgn}(\sigma) \prod_{i=1}^d (A_1 + \dots + A_q)_{i, \sigma(i)}$ over all permutations $\sigma \in S_d$. Then, for each term and each $i \in [d] = \{1, \dots, d\}$, choose $s_\sigma(i) \in [q]$ which marks which of the q summands is taken in the entry $(i, \sigma(i))$. Namely,

$$|A_1 + \dots + A_q| = \sum_{\sigma \in S_d} \text{sgn}(\sigma) \sum_{s_\sigma: [d] \rightarrow [q]} \prod_{i=1}^d [A_{s_\sigma(i)}]_{i, \sigma(i)}. \quad (3.2)$$

The idea of Lemma 3.1 below is to group the terms in this double sum in a different fashion: first, for every $j \in [q]$, choose from which rows $R_j \subseteq [d]$ and from which columns $C_j \subseteq [d]$ the entry is taken from A_j . Then, vary over all permutations σ that respect these constraints, namely the permutations for which $\sigma(R_j) = C_j$. With this in mind, we define:

$$T(q, d) = \left\{ \left(\dot{R}, \dot{C} \right) \left| \begin{array}{l} \dot{R} = (R_1, \dots, R_q), \dot{C} = (C_1, \dots, C_q) \text{ are partitions of } [d] \text{ into } q \text{ parts} \\ \text{such that } |R_\ell| = |C_\ell| \text{ for all } 1 \leq \ell \leq q \end{array} \right. \right\},$$

and the corresponding permutations:

$$\text{Sym}(\dot{R}, \dot{C}) \stackrel{\text{def}}{=} \{ \sigma \in S_d \mid \sigma(R_\ell) = C_\ell \text{ for all } \ell \}.$$

Finally, for each such pair of partitions $(\dot{R}, \dot{C}) \in T(q, d)$, we need a “relative sign”, denoted $\text{Sgn}(\dot{R}, \dot{C})$, which will enable us to calculate the sign of every $\sigma \in \text{Sym}(\dot{R}, \dot{C})$ based solely on the signs of the permutation-matrix σ restricted to the minors (R_ℓ, C_ℓ) . This is the sign of the

permutation-matrix obtained by assigning, for each ℓ , the identity matrix $I_{|R_\ell|}$ to the (R_ℓ, C_ℓ) minor. For example, if $\dot{R} = (\{1, 3, 5\}, \{2, 4\})$ and $\dot{C} = (\{3, 4, 5\}, \{1, 2\})$, then

$$\operatorname{sgn}(\dot{R}, \dot{C}) = \operatorname{sgn} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.1. *If $A_1, \dots, A_q \in M_d(\mathbb{C})$ are $d \times d$ matrices, then*

$$|A_1 + \dots + A_q| = \sum_{(\dot{R}, \dot{C}) \in T(q, d)} \operatorname{sgn}(\dot{R}, \dot{C}) \prod_{\ell=1}^q |A_\ell|_{R_\ell, C_\ell},$$

where for $R, C \subseteq [d]$ with $|R| = |C|$,

$$|A|_{R, C} = \begin{cases} \det((a_{i,j})_{i \in R, j \in C}) & \text{if } |R| = |C| \geq 1 \\ 1 & \text{if } R = C = \emptyset \end{cases}$$

marks the determinant of the (R, C) -minor of A .

Proof. For every $\sigma \in S_d$ and $s_\sigma : [d] \rightarrow [q]$ as in (3.2), the pair of partitions (\dot{R}, \dot{C}) satisfying $R_\ell = s_\sigma^{-1}(\ell)$ and $C_\ell = \sigma(R_\ell)$, for each ℓ , is the unique pair in $T(q, d)$ which respects σ and s_σ , and then $\sigma \in \operatorname{Sym}(\dot{R}, \dot{C})$. Therefore,

$$|A_1 + \dots + A_q| = \sum_{(\dot{R}, \dot{C}) \in T(q, d)} \sum_{\sigma \in \operatorname{Sym}(\dot{R}, \dot{C})} \operatorname{sgn}(\sigma) \sum_{\substack{s_\sigma : [d] \rightarrow [q] \\ s_\sigma^{-1}(\ell) = R_\ell \forall \ell}} \prod_{i=1}^d [A_{s_\sigma(i)}]_{i, \sigma(i)}.$$

Now, specifying a permutation $\sigma \in \operatorname{Sym}(\dot{R}, \dot{C})$ is equivalent to specifying the permutation σ_ℓ induced by σ on each of the minors (R_ℓ, C_ℓ) . Thus, $\operatorname{Sym}(\dot{R}, \dot{C}) \cong S_{|R_1|} \times \dots \times S_{|R_q|}$ (as sets) via $\sigma \mapsto (\sigma_1, \dots, \sigma_q)$. It is easy to see that the signs of these permutations are related by

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\dot{R}, \dot{C}) \cdot \operatorname{sgn}(\sigma_1) \cdot \dots \cdot \operatorname{sgn}(\sigma_q).$$

Thus, if $R_\ell(i)$ is the i -th element in R_ℓ , we get

$$\begin{aligned} |A_1 + \dots + A_q| &= \sum_{(\dot{R}, \dot{C}) \in T(q, d)} \operatorname{sgn}(\dot{R}, \dot{C}) \prod_{\ell=1}^q \left[\sum_{\sigma_\ell \in S_{|R_\ell|}} \operatorname{sgn}(\sigma_\ell) \prod_{i=1}^{|R_\ell|} [A_\ell]_{R_\ell(i), C_\ell(\sigma_\ell(i))} \right] \\ &= \sum_{(\dot{R}, \dot{C}) \in T(q, d)} \operatorname{sgn}(\dot{R}, \dot{C}) \prod_{\ell=1}^q |A_\ell|_{R_\ell, C_\ell}. \end{aligned}$$

□

3.2 Matrix Coefficients

Recall that if π is a d -dimensional representation, then $\bigwedge^m \pi$ is $\binom{d}{m}$ -dimensional, and if $\{v_1, \dots, v_d\}$ is a basis for \mathbb{C}^d , then $\{v_{i_1} \wedge \dots \wedge v_{i_m} \mid 1 \leq i_1 < i_2 < \dots < i_m \leq d\}$ is a basis for $\bigwedge^m (\mathbb{C}^d)$ (see Section 2.4). The following standard claim and classical theorem explain the role in Theorem 1.8 of the conditions on $\bigwedge^m \pi$, for $0 \leq m \leq d$, as defined in property (P1):

Claim 3.2 (e.g. [KRY09, Theorem 6.6.3]). *If the matrices $\pi(g)$ are given in terms of the basis $V = \{v_1, \dots, v_d\}$ and $(\bigwedge^m \pi)(g)$ in terms of the basis $\{v_{i_1} \wedge \dots \wedge v_{i_m} \mid 1 \leq i_1 < i_2 < \dots < i_m \leq d\}$, then the entry (matrix coefficient) of $(\bigwedge^m \pi)(g)$ in row $v_{i_1} \wedge \dots \wedge v_{i_m}$ and column $v_{j_1} \wedge \dots \wedge v_{j_m}$ is given by the minor-determinant $|\pi(g)|_{\{i_1, \dots, i_m\}, \{j_1, \dots, j_m\}}$.*

Theorem 3.3 (Peter-Weyl, see e.g. [Bum04, Chapter 2]). *The matrix coefficients of the irreducible representations of a finite (compact) group Γ are an orthogonal basis of $L^2(\Gamma)$. More precisely, if $\pi_1: \Gamma \rightarrow U(d_1)$ and $\pi_2: \Gamma \rightarrow U(d_2)$ are irreducible non-isomorphic unitary representations of Γ , then*

$$\mathbb{E}_{g \in \Gamma} \left[\pi_1(g)_{i_1, j_1} \cdot \overline{\pi_2(g)_{i_2, j_2}} \right] = 0$$

for every $i_1, j_1 \in [d_1]$ and $i_2, j_2 \in [d_2]$, the expectation taken according to the uniform (Haar, respectively) measure of Γ . Moreover, if $\pi: \Gamma \rightarrow U(d)$ is an irreducible representation, then

$$\mathbb{E}_{g \in \Gamma} \left[\pi(g)_{i_1, j_1} \cdot \overline{\pi(g)_{i_2, j_2}} \right] = \begin{cases} \frac{1}{d} & (i_1, j_1) = (i_2, j_2) \\ 0 & \text{otherwise} \end{cases}.$$

We now have all the tools needed to prove Theorem 1.8.

3.3 Proof of Theorem 1.8

Assume without loss of generality that $\pi: \Gamma \rightarrow U(d)$ maps the elements of Γ to unitary matrices, so that for every $e \in E(G)$, $A_{\gamma, \pi}(-e)$ is the conjugate-transpose matrix $A_{\gamma, \pi}(e)^*$.

Recall (3.1). We analyze the expected characteristic polynomial

$$\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}} [\phi_{\gamma, \pi}] = \mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}} [\det(xI - A_{\gamma, \pi})] = \mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}} \left[\det \left(xI - \sum_{e \in E(G)} A_{\gamma, \pi}(e) \right) \right]. \quad (3.3)$$

Our goal is to show it is equal to the formula given for $\mathcal{M}_{d, G}$ in Proposition 2.9. We use Lemma 3.1 to rewrite the determinant in the right hand side of (3.3). We now let

$$T = T(1 + |E(G)|, nd) = \left\{ \left(\dot{R}, \dot{C} \right) \left| \begin{array}{l} \dot{R} \text{ and } \dot{C} \text{ are partitions of } [nd] \text{ to } 1 + |E(G)| \text{ parts} \\ \text{indexed by } \{x\} \cup E(G), \\ \text{with } R_x = C_x \text{ and } |R_e| = |C_e| \text{ for all } e \in E(G) \end{array} \right. \right\}.$$

By Lemma 3.1,

$$\phi_{\gamma, \pi} = \sum_{(\dot{R}, \dot{C}) \in T} \text{sgn}(\dot{R}, \dot{C}) \cdot x^{|\dot{R}_x|} \prod_{e \in E(G)} (-1)^{|R_e|} |A_{\gamma, \pi}(e)|_{R_e, C_e}.$$

Taking expected values gives

$$\begin{aligned} \mathbb{E}_\gamma [\phi_{\gamma,\pi}] &= \sum_{(\dot{R}, \dot{C}) \in T} \operatorname{sgn}(\dot{R}, \dot{C}) \cdot x^{|\dot{R}_x|} \mathbb{E}_\gamma \left[\prod_{e \in E(G)} (-1)^{|R_e|} |A_{\gamma,\pi}(e)|_{R_e, C_e} \right] \\ &= \sum_{(\dot{R}, \dot{C}) \in T} \operatorname{sgn}(\dot{R}, \dot{C}) \cdot x^{|\dot{R}_x|} (-1)^{nd - |\dot{R}_x|} \prod_{e \in E^+(G)} \mathbb{E}_\gamma [|A_{\gamma,\pi}(e)|_{R_e, C_e} \cdot |A_{\gamma,\pi}(-e)|_{R_{-e}, C_{-e}}], \end{aligned} \quad (3.4)$$

since the $A_{\gamma,\pi}(e)$ are independent except for the pairs $A_{\gamma,\pi}(e)$ and $A_{\gamma,\pi}(-e)$.

Since $A_{\gamma,\pi}(-e) = A_{\gamma,\pi}(e)^*$, the term inside the expectation in the right hand side of (3.4) is equal to

$$\mathbb{E}_\gamma [|A_{\gamma,\pi}(e)|_{R_e, C_e} \cdot |A_{\gamma,\pi}(e)^*|_{R_{-e}, C_{-e}}] = \mathbb{E}_\gamma [|A_{\gamma,\pi}(e)|_{R_e, C_e} \cdot \overline{|A_{\gamma,\pi}(e)|_{C_{-e}, R_{-e}}}] .$$

Clearly, this term is zero, unless the minors we choose for e and $-e$ are inside the $d \times d$ blocks corresponding to e and $-e$, respectively. That is, if B_v denotes the set of d indices of rows and columns corresponding to the vertex $v \in V(G)$, then this term is zero unless $R_e, C_{-e} \subseteq B_{h(e)}$ and $C_e, R_{-e} \subseteq B_{t(e)}$. If this is the case, we can think of R_e, C_e, R_{-e}, C_{-e} as subsets of $[d]$, so Claim 3.2 yields this term is

$$\mathbb{E}_\gamma \left[\left(\left(\bigwedge^{|R_e|} \pi \right) (\gamma(e)) \right)_{R_e, C_e} \cdot \overline{\left(\left(\bigwedge^{|R_{-e}|} \pi \right) (\gamma(e)) \right)_{C_{-e}, R_{-e}}} \right],$$

where we identify an m -subset of $[d]$ with a basis element of $\bigwedge^m(\mathbb{C}^d)$ in the obvious way. Finally, by the Peter-Weyl Theorem (Theorem 3.3) and our assumptions on the exterior powers $\bigwedge^m \pi$ for $0 \leq m \leq d$, this expectation is zero unless $|R_e| = |R_{-e}|$, $R_e = C_{-e}$, and $C_e = R_{-e}$. If all these equalities hold, the expectation is $\binom{d}{|R_e|}^{-1}$.

Define $T^{\text{sym}} \subseteq T$ to be the subset of T containing the partitions for which the expectation in (3.4) is not zero. Namely,

$$T^{\text{sym}} = \left\{ \left(\dot{R}, \dot{C} \right) \left| \begin{array}{l} \dot{R} \text{ and } \dot{C} \text{ are partitions of } [nd] \text{ to } |E(G)| + 1 \text{ parts} \\ \text{indexed by } \{x\} \cup E(G), \\ \text{with } R_x = C_x, \text{ and for all } e \in E^+(G) \\ |R_e| = |C_e|, C_{-e} = R_e, R_{-e} = C_e, R_e \subseteq B_{h(e)} \text{ and } R_{-e} \subseteq B_{t(e)} \end{array} \right. \right\} .$$

Our discussion shows that

$$\mathbb{E}_\gamma [\phi_{\gamma,\pi}] = \sum_{(\dot{R}, \dot{C}) \in T^{\text{sym}}} \operatorname{sgn}(\dot{R}, \dot{C}) \cdot x^{|\dot{R}_x|} (-1)^{nd - |\dot{R}_x|} \prod_{e \in E^+(G)} \frac{1}{\binom{d}{|R_e|}} .$$

Now, notice that because $|R_{-e}| = |R_e|$, we get that $nd - |R_x| = \sum_{e \in E(G)} |R_e|$ is even, so $(-1)^{nd - |R_x|} = 1$ for every $(\dot{R}, \dot{C}) \in T^{\text{sym}}$. Because of the conditions $C_{-e} = R_e$ and $R_{-e} = C_e$ on the partitions in T^{sym} , the permutation matrix defining $\operatorname{sgn}(\dot{R}, \dot{C})$ is symmetric. Thus, the corresponding permutation is an involution, with exactly $|R_x|$ fixed points and $\frac{nd - |R_x|}{2}$ 2-cycles²⁰,

²⁰In particular, if $(\dot{R}, \dot{C}) \in T^{\text{sym}}$ then $R_e \cap C_e = \emptyset$, even for loops.

so $\text{sgn}(\dot{R}, \dot{C}) = (-1)^{(nd - |R_x|)/2}$. Hence,

$$\mathbb{E}_\gamma [\phi_{\gamma, \pi}] = \sum_{(\dot{R}, \dot{C}) \in T^{\text{sym}}} (-1)^{(nd - |R_x|)/2} \cdot x^{|R_x|} \prod_{e \in E^+(G)} \frac{1}{\binom{d}{|R_e|}}.$$

Recall the definition of a d -multi-matching given in Definition 2.8: it is a function $m: E(G) \rightarrow \mathbb{Z}_{\geq 0}$ such that $m(-e) = m(e)$ for every $e \in E(G)$ and $m(v) \leq d$ for every $v \in V(G)$, where $m(v)$ is the sum of m on all oriented edges emanating from v . For every $(\dot{R}, \dot{C}) \in T^{\text{sym}}$, consider the map $\eta(\dot{R}, \dot{C}): E(G) \rightarrow \mathbb{Z}_{\geq 0}$ given by $e \mapsto |R_e|$. We claim this is a d -multi-matching. Indeed, for every $v \in V(G)$,

$$\eta(\dot{R}, \dot{C})(v) = \sum_{e \in E(G): h(e)=v} |R_e| \leq d$$

as \dot{R} is a partition and $R_e \subseteq B_{h(e)}$ when $h(e) = v$.

Finally, for every $(\dot{R}, \dot{C}) \in T^{\text{sym}}$, \dot{C} is completely determined by \dot{R} . Denote by $e_{v,1}, \dots, e_{v, \deg(v)}$ the oriented edges emanating from v . Then, for every d -multi-matching m , the number of partitions $(\dot{R}, \dot{C}) \in T^{\text{sym}}$ associated to m is exactly

$$\prod_{v \in V(G)} \binom{d}{m(e_{v,1}), \dots, m(e_{v, \deg(v)})}.$$

We obtain

$$\begin{aligned} \mathbb{E}_\gamma [\phi_{\gamma, \pi}] &= \sum_{m \in \text{MultiMatch}_d(G)} \sum_{(\dot{R}, \dot{C}) \in T^{\text{sym}}: \eta(\dot{R}, \dot{C})=m} (-1)^{(nd - |R_x|)/2} \cdot x^{|R_x|} \prod_{e \in E^+(G)} \frac{1}{\binom{d}{|R_e|}} \\ &= \sum_{m \in \text{MultiMatch}_d(G)} (-1)^{|m|} x^{nd - 2|m|} \frac{\prod_{v \in V(G)} \binom{d}{m(e_{v,1}), \dots, m(e_{v, \deg(v)})}}{\prod_{e \in E^+(G)} \binom{d}{m(e)}}, \end{aligned}$$

where the summation is over all d -multi-matchings m of G , and $|m| = \sum_{e \in E^+(G)} m(e)$. This is precisely the formula for $\mathcal{M}_{d,G}$ from Proposition 2.9, so the proof of Theorem 1.8 is complete. \square

Remark 3.4. To give further intuition for Theorem 1.8, we remark that its statement for one particular family of pairs (Γ, π) satisfying $(\mathcal{P}1)$ follows readily from well known results. This is the family of signed permutation groups: for every d , let $\pi(\Gamma)$ be the subgroup of $\text{GL}_d(\mathbb{C})$ consisting of matrices with entries in $\{0, \pm 1\}$ and with exactly one non-zero entry in every row and in every column. That these pairs satisfy $(\mathcal{P}1)$ (and also $(\mathcal{P}2)$) is well known - see Section 5. But here, every (Γ, π) -covering of a graph G corresponds to a d -covering H plus a signing (a $(\mathbb{Z}/2\mathbb{Z}, \text{sign})$ -covering) of H . The special case of Theorem 1.8 for $(\mathbb{Z}/2\mathbb{Z}, \text{sign})$ -coverings, known at least since [GG81], shows that the expected characteristic polynomial over all signings of H equals $\mathcal{M}_{1,H}$, hence the average of all signings over all possible d -coverings H is $\mathcal{M}_{d,G}$, by definition. This gives a shorter route to proving Theorem 2.7.

4 Property ($\mathcal{P}2$) and the Proof of Theorem 1.10

The main goal of this Section is to show that the expected characteristic polynomial of a random (Γ, π) -covering is real rooted for certain distributions on coverings, and in particular that when (Γ, π) satisfies ($\mathcal{P}2$), this is true for the uniform distribution. The main component of the proof is Theorem 4.2, showing that for certain distributions of Hermitian ($A^* = A$) matrices, the expected characteristic polynomial is real rooted. This theorem imitates and generalizes the argument of [MSS15b, Thm 3.3]. We repeat the argument in Section 4.1, because we need a more general statement, but we refer the interested reader to [MSS15b, Section 3] for some more elaborated concepts and notions. Theorem 4.2 is a generalization of the fact that the characteristic polynomials²¹ $\phi(A)$ and $\phi(BAB^*)$ interlace whenever $A \in M_d(\mathbb{C})$ is Hermitian and $B \in U(d)$ satisfies $\text{rank}(B - I_d) = 1$.

4.1 Average Characteristic Polynomial of Sum of Random Matrices

Definition 4.1. We say that the random variable W taking values in $U(d)$ is a **rank-1 random variable** if every two different possible values B_1 and B_2 satisfy $\text{rank}(B_1 B_2^{-1} - I_d) = 1$.

It is not hard to see that W is a rank-1 random variable if and only if it takes values in some $P\Lambda Q$ where $P, Q \in U(d)$ and $\Lambda \leq U(d)$ is the subgroup of diagonal matrices Λ

$$\left\{ \left(\begin{array}{cccc} \lambda & & & \\ & 1 & & \\ & & \dots & \\ & & & 1 \end{array} \right) \in U(d) \mid |\lambda| = 1 \right\}.$$

Theorem 4.2. Let $m \in \mathbb{Z}_{\geq 1}$, let $\ell(1), \dots, \ell(m) \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{W} = \{W_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq \ell(i)}$ be a set of independent rank-1 random variables taking values in $U(d)$. If $A_1, \dots, A_m \in M_d(\mathbb{C})$ are Hermitian matrices, then the expected characteristic polynomial

$$P_{\mathcal{W}}(A_1, \dots, A_m) \stackrel{\text{def}}{=} \mathbb{E}_{\mathcal{W}} \left[\phi \left(\sum_{i=1}^m W_{i,1} \dots W_{i,\ell(i)} A_i W_{i,\ell(i)}^* \dots W_{i,1}^* \right) \right]$$

is real rooted.

Note that the characteristic polynomial of a Hermitian matrix is in $\mathbb{R}[x]$, and so $P_{\mathcal{W}}(A_1, \dots, A_m)$, which is an average of such polynomials, is also in $\mathbb{R}[x]$.

Lemma 4.3. In the notation of Theorem 4.2, assume that $P_{\mathcal{W}}(A_1, \dots, A_m)$ is real rooted whenever $A_1, \dots, A_m \in M_d(\mathbb{C})$ are Hermitian. Then, for every $v \in \mathbb{C}^d$, the roots $\alpha_d \leq \dots \leq \alpha_1$ of $P_{\mathcal{W}}(A_1, \dots, A_i + vv^*, \dots, A_m)$ and the roots $\beta_d \leq \dots \leq \beta_1$ of $P_{\mathcal{W}}(A_1, \dots, A_m)$ satisfy

$$\beta_d \leq \alpha_d \leq \beta_{d-1} \leq \alpha_{d-1} \leq \dots \leq \beta_1 \leq \alpha_1.$$

In other words, the polynomials $P_{\mathcal{W}}(A_1, \dots, A_i + vv^*, \dots, A_m)$ and $P_{\mathcal{W}}(A_1, \dots, A_m)$ interlace in a strong sense.

²¹Recall that for any matrix A , we denote its characteristic polynomial by $\phi(A) = \det(xI - A)$.

Proof. Denote $Q(A_1, \dots, A_m) = \left[\frac{\partial}{\partial \mu} P_{\mathcal{W}}(A_1, A_2, \dots, A_i + \mu vv^*, \dots, A_m) \right]_{\mu=0}$. We claim that²²

$$P_{\mathcal{W}}(A_1, \dots, A_i + \mu vv^*, \dots, A_m) = P_{\mathcal{W}}(A_1, \dots, A_m) + \mu \cdot Q(A_1, \dots, A_m) \quad (4.1)$$

for every $\mu \in \mathbb{C}$. To see this, it is enough to show (4.1) in the case \mathcal{W} is constant (that is, $W_{i,j}$ is constant for every i, j), the general statement will then follow by linearity of expectation and of taking the derivative. For a constant \mathcal{W} , we only need to show that for any Hermitian $A \in M_d(\mathbb{C})$, the characteristic polynomial $\phi(A + \mu vv^*)$ is linear in μ . By conjugating A and vv^* by some unitary matrix, we may assume $v^* = (\alpha, 0, 0, \dots, 0)$, and then the claim is clear by expanding the determinant of $xI - (A + \mu vv^*)$ along, say, the first row.

If A_i is Hermitian, then for every $\mu \in \mathbb{R}$, $A_i + \mu vv^*$ is Hermitian, so by our assumption the left hand side of (4.1) is real rooted. Therefore, the right hand side $P_{\mathcal{W}}(A_1, \dots, A_m) + \mu \cdot Q(A_1, \dots, A_m)$ is also real rooted for every $\mu \in \mathbb{R}$. Note that while $P_{\mathcal{W}}(A_1, \dots, A_m)$ is monic of degree d (in the variable x), $Q(A_1, \dots, A_m)$ is a polynomial of degree $d - 1$ with a negative leading coefficient (because $\left[\frac{\partial}{\partial \mu} \det(xI - (A + \mu vv^*)) \right]_{\mu=0}$ has this property when \mathcal{W} is constant). It follows from Fact 2.15 that $P_{\mathcal{W}}(A_1, \dots, A_m)$ and $Q(A_1, \dots, A_m)$ interlace, and the real roots $\vartheta_{d-1} \leq \dots \leq \vartheta_1$ of $Q(A_1, \dots, A_m)$ satisfy

$$\beta_d \leq \vartheta_{d-1} \leq \beta_{d-1} \leq \dots \leq \beta_2 \leq \vartheta_1 \leq \beta_1.$$

Moreover, the i -th root of $P_{\mathcal{W}}(A_1, \dots, A_m) + \mu \cdot Q(A_1, \dots, A_m)$ grows continuously from β_i to ϑ_{i-1} as μ grows (the largest root grows from β_1 to $+\infty$). In particular, $\mu = 1$ corresponds to $P_{\mathcal{W}}(A_1, \dots, A_i + vv^*, \dots, A_m)$, so the roots $\alpha_d \leq \dots \leq \alpha_1$ satisfy

$$\beta_d \leq \alpha_d \leq \vartheta_{d-1} \leq \beta_{d-1} \leq \alpha_{d-1} \leq \vartheta_{d-1} \leq \dots \leq \beta_2 \leq \alpha_2 \leq \vartheta_1 \leq \beta_1 \leq \alpha_1.$$

□

Proof of Theorem 4.2. We prove by induction on the number of $W_{i,j}$'s, namely, induction on $\ell(\mathcal{W}) = \ell(1) + \dots + \ell(m)$. The statement is clear for $\ell(\mathcal{W}) = 0$. Given \mathcal{W} with $\ell(\mathcal{W}) > 0$, assume without loss of generality that $\ell(1) > 0$, and denote by \mathcal{W}' the set of random variables $\mathcal{W} \setminus \{W_{1,\ell(1)}\}$. We assume, by the induction hypothesis, that $P_{\mathcal{W}'}(A_1, \dots, A_m)$ is real rooted for all Hermitian matrices A_1, \dots, A_m .

For clarity we assume $W_{1,\ell(1)}$ takes only finitely many values, but the same argument works in the general case (Fact 2.16 has a variant for a set of polynomials of any cardinality). Let $B_1, \dots, B_t \in U(d)$ be the possible values of $W_{1,\ell(1)}$, obtained with respective probabilities p_1, \dots, p_t . We need to show that

$$P_{\mathcal{W}}(A_1, \dots, A_m) = p_1 \cdot P_{\mathcal{W}'}(B_1 A_1 B_1^*, A_2, \dots, A_m) + \dots + p_t \cdot P_{\mathcal{W}'}(B_t A_1 B_t^*, A_2, \dots, A_m)$$

is real rooted for all Hermitian matrices $A_1, \dots, A_m \in M_d(\mathbb{C})$. By Fact 2.16, it is enough to show the t polynomials $P_{\mathcal{W}'}(B_j A_1 B_j^*, A_2, \dots, A_m)$ ($1 \leq j \leq t$) have a common interlacing. By definition, this is equivalent to showing that any two of them have a common interlacing. Thus, it is enough to show that if $B, C \in U(d)$ satisfy $\text{rank}(BC^{-1} - I_d) = 1$, then the polynomials $P_{\mathcal{W}'}(BA_1 B^*, A_2, \dots, A_m)$ and $P_{\mathcal{W}'}(CA_1 C^*, A_2, \dots, A_m)$ have common interlacing.

²²This property is called ‘‘Rank-1 linearity’’ in [MSS15b].

By replacing A_1 with CA_1C^* and writing $D = BC^{-1}$ we need to prove that $P_{\mathcal{W}'}(DA_1D^*, A_2, \dots, A_m)$ and $P_{\mathcal{W}'}(A_1, A_2, \dots, A_m)$ have a common interlacing (now $\text{rank}(D - I_d) = 1$). If $DA_1D^* = A_1$ the statement is obvious. Otherwise, we claim that $DA_1D^* - A_1$ is a rank-2 trace-0 Hermitian matrix: by unitary conjugation we may assume

$$D = \begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \Lambda,$$

and direct calculation then shows that

$$DA_1D^* - A_1 = \begin{pmatrix} 0 & \lambda a_{1,2} & \cdots & \lambda a_{1,d} \\ \bar{\lambda} a_{2,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\lambda} a_{d,1} & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\pm\nu$ ($\nu \in \mathbb{R}_{>0}$) be the non-zero eigenvalues of $DA_1D^* - A_1$. By spectral decomposition, $DA_1D^* - A_1 = uu^* - vv^*$ for some vectors $u, v \in \mathbb{C}^d$ of length $\sqrt{\nu}$. Consider also $P_{\mathcal{W}'}(A_1 - vv^*, A_2, \dots, A_m)$ and denote

$$\begin{aligned} \alpha_d \leq \dots \leq \alpha_1 & \quad \text{the roots of } P_{\mathcal{W}'}(A_1, A_2, \dots, A_m) \\ \beta_d \leq \dots \leq \beta_1 & \quad \text{the roots of } P_{\mathcal{W}'}(A_1 - vv^*, A_2, \dots, A_m) \\ \gamma_d \leq \dots \leq \gamma_1 & \quad \text{the roots of } P_{\mathcal{W}'}(DA_1D^*, A_2, \dots, A_m). \end{aligned}$$

The assumptions of Lemma 4.3 are satisfied for \mathcal{W}' by the induction hypothesis. We can apply this lemma on $P_{\mathcal{W}'}(A_1 - vv^*, A_2, \dots, A_m)$ and $P_{\mathcal{W}'}(A_1, A_2, \dots, A_m)$ to obtain that

$$\beta_d \leq \alpha_d \leq \beta_{d-1} \leq \alpha_{d-1} \leq \dots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1.$$

Similarly, we can apply the lemma on

$$P_{\mathcal{W}'}(A_1 - vv^*, A_2, \dots, A_m) \text{ and } P_{\mathcal{W}'}(DA_1D^*, A_2, \dots, A_m) = P_{\mathcal{W}'}(A_1 - vv^* + uu^*, A_2, \dots, A_m)$$

to obtain that

$$\beta_d \leq \gamma_d \leq \beta_{d-1} \leq \gamma_{d-1} \leq \dots \leq \beta_2 \leq \gamma_2 \leq \beta_1 \leq \gamma_1.$$

It follows that $P_{\mathcal{W}'}(DA_1D^*, A_2, \dots, A_m)$ and $P_{\mathcal{W}'}(A_1, A_2, \dots, A_m)$ have a common interlacing. This completes the proof. \square

4.2 Average Characteristic Polynomial of Random Coverings

Let G be a finite graph *without loops*, Γ a group and $\pi: \Gamma \rightarrow \text{GL}_d(\mathbb{C})$ a unitary representation. We now deduce from Theorem 4.2 that for certain distributions of (Γ, π) -coverings of G , the average characteristic polynomial is real rooted. Recall that $\phi_{\gamma, \pi}$ denotes the characteristic polynomial of $A_{\gamma, \pi}$ – see (1.2).

Proposition 4.4. *Let X_1, \dots, X_r be independent random variables, each taking values in the space of Γ -labelings of G . Suppose that all possible values of X_i agree on all edges in $E^+(G)$ except (possibly) for one, and on that edge suppose that $\pi(X(e))$ is a rank-1 random variable of matrices in $U(d)$, as in Definition 4.1. Then $\mathbb{E}_{X_1 \dots X_r} [\phi_{X_1 \dots X_r, \pi}]$ is real rooted.*

Proof. As we noted in the proof of Claim 2.11, for any Γ -labeling γ , we have $\phi_{\gamma, \pi} = \phi_{\gamma, \pi'}$ whenever π and π' are isomorphic, so we assume without loss of generality that $\pi(\Gamma) \subseteq U(d)$. For every Γ -labeling γ , the matrix $A_{\gamma, \pi}$ is a $nd \times nd$ matrix composed of n^2 blocks of size $d \times d$. The blocks are indexed by ordered pairs of vertices of G . Similarly to a notation we used on Page 20, for any $e \in E^+(G)$, we let $A_{\gamma, \pi}^\pm(e) \in M_{nd}$ be the matrix with zero blocks except for the blocks corresponding to e and to $-e$. In the block $(h(e), t(e))$ we have $\pi(\gamma(e))$ and in the block $(t(e), h(e))$ we have $\pi(\gamma(-e)) = \pi(\gamma(e))^*$. It is clear that $A_{\gamma, \pi}^\pm(e)$ is Hermitian and that $A_{\gamma, \pi}^\pm(e)$

$$A_{\gamma, \pi} = \sum_{e \in E^+(G)} A_{\gamma, \pi}^\pm(e).$$

For every random Γ -labeling X of G and $e \in E^+(G)$, let $W_e(X)$ be the following random matrix in $U(nd)$:

$$W_e(X) = \begin{pmatrix} I_d & & & & \\ & \ddots & & & \\ & & \pi(X(e)) & & \\ & & & \ddots & \\ & & & & I_d \end{pmatrix}$$

where all non-diagonal $d \times d$ blocks are zeros, the $(h(e), h(e))$ block is $\pi(X(e))$, and the remaining diagonal blocks are I_d . Also let $\mathbf{1}: E(G) \rightarrow \Gamma$ be the trivial labeling which labels all edges by the identity element of Γ . With these notations, we have²³

$$A_{X_1 \dots X_r, \pi}^\pm(e) = W_e(X_1) \cdot \dots \cdot W_e(X_r) A_{\mathbf{1}, \pi}^\pm(e) W_e(X_r)^* \cdot \dots \cdot W_e(X_1)^*,$$

and

$$A_{X_1 \dots X_r, \pi} = \sum_{e \in E^+(G)} A_{X_1 \dots X_r, \pi}^\pm(e).$$

By assumption, the random Γ -labeling X_i is constant on all edges except for on one edge e . Thus, $\{X_i(e)\}_{e \in E^+(G)}$ is a set of independent variables. Moreover, the set $\{W_e(X_i)\}_{e \in E^+(G), 1 \leq i \leq r}$ is a set of independent variables taking values in $U(nd)$, and every $W_e(X_i)$ is a rank-1 random variable by the hypothesis on the values of X_i . The proposition now follows by applying Theorem 4.2. \square

Repeating the line of argument we explained in Section 1.3, we deduce:

Corollary 4.5. *In the notation of Proposition 4.4, there is a Γ -labeling $\gamma = \gamma_1 \cdots \gamma_r$ of G , with γ_i in the support of X_i , such that the largest root of $\phi_{\gamma, \pi}$ is at most the largest root of $\mathbb{E}_{X_1 \dots X_r} [\phi_{X_1 \dots X_r, \pi}]$.*

Proof. We prove by induction on r . Consider the support $\text{Supp}(X_1) \subseteq \{\Gamma\text{-labelings of } G\}$. Note that if X_1, \dots, X_r satisfy the assumptions of Proposition 4.4, then so would X'_1, X_2, \dots, X_r when

²³This formula is exactly the place this proof breaks for loops.

X'_1 is any random variable with the same support as X_1 . Therefore, by Fact 2.16, all polynomials in the family

$$\{\mathbb{E}_{X_2 \dots X_r} [\phi_{\gamma \cdot X_2 \dots X_r, \pi}]\}_{\gamma \in \text{Supp}(X_1)}$$

have a common interlacing. In particular, there is a polynomial in this family, say the one defined by $\gamma_1 \in \text{Supp}(X_1)$, with maximal root at most the maximal root of $\mathbb{E}_{X_1 \dots X_r} [\phi_{X_1 \dots X_r, \pi}]$. If $r = 1$ we are done. If $r \geq 2$, define $X'_2 = \gamma_1 \cdot X_2$. The random variables X'_2, X_3, \dots, X_r still satisfy the hypotheses of Proposition 4.4. Hence, by the induction hypothesis there is a Γ -labeling $\gamma = \gamma'_2 \cdot \gamma_3 \cdots \gamma_r$ of G with $\gamma'_2 \in \text{Supp}(X'_2)$ and $\gamma_i \in \text{Supp}(X_i)$ for $3 \leq i \leq r$, such that the largest root of $\phi_{\gamma, \pi}$ is at most the largest root of $\mathbb{E}_{X'_2, X_3 \dots X_r} [\phi_{X'_2, X_3 \dots X_r, \pi}]$, which in turn is at most the largest root of $\mathbb{E}_{X_1 \dots X_r} [\phi_{X_1 \dots X_r, \pi}]$. The statement of the corollary is now satisfied with $\gamma_1, \gamma_1^{-1} \gamma'_2, \gamma_3, \dots, \gamma_r$. \square

4.3 Proof of Theorem 1.10

We finally have all the tools needed to prove Theorem 1.10. Let G be a finite, loopless graph, and let (Γ, π) satisfy property (P2), that is, Γ is a finite group, $\pi: G \rightarrow \text{GL}_d(\mathbb{C})$ is a representation and $\pi(\Gamma)$ is a complex reflection group (i.e., generated by pseudo-reflections). Assume that Γ is generated by g_1, \dots, g_s , where $\pi(g_i)$ is a pseudo-reflection for all i , i.e., $\text{rank}(\pi(g_i) - I_d) = 1$. We first show that a certain lazy random walk on Γ , where in each step we use only one of the g_i 's, converges to the uniform distribution:

Claim 4.6. *Define a random walk $\{a_n\}_{n=0}^\infty$ on Γ as follows: $a_0 = \mathbf{1}_\Gamma$ (the identity element of Γ), and for $n \geq 1$*

$$a_n = \begin{cases} g_{n \bmod s} \cdot a_{n-1} & \text{with probability } \frac{1}{3} \\ (g_{n \bmod s})^{-1} \cdot a_{n-1} & \text{with probability } \frac{1}{3} \\ a_{n-1} & \text{with probability } \frac{1}{3} \end{cases}.$$

Then a_1, a_2, \dots converges to the uniform distribution on Γ .

Proof. Consider a_n as an element of the group-ring $\mathbb{C}[\Gamma]$ so that the coefficient of g is $\text{Prob}[a_n = g]$. Then for $n \geq 1$,

$$a_{s \cdot n} = \left(\frac{1}{3} \mathbf{1}_\Gamma + \frac{1}{3} g_s + \frac{1}{3} g_s^{-1} \right) \cdots \left(\frac{1}{3} \mathbf{1}_\Gamma + \frac{1}{3} g_1 + \frac{1}{3} g_1^{-1} \right) \cdot a_{s \cdot (n-1)}.$$

The s -steps random walk $\{a_{s \cdot n}\}_{n=0}^\infty$ is defined by the distribution

$$h = \left(\frac{1}{3} \mathbf{1}_\Gamma + \frac{1}{3} g_s + \frac{1}{3} g_s^{-1} \right) \cdots \left(\frac{1}{3} \mathbf{1}_\Gamma + \frac{1}{3} g_1 + \frac{1}{3} g_1^{-1} \right),$$

which satisfies $\text{supp}(h^n) = \Gamma$ for all large n . Thus, it converges to the only stationary distribution of this Markov chain: the uniform distribution. The same argument applies to $\{a_{s \cdot n + i}\}_{n=0}^\infty$ for any remainder $1 \leq i \leq s-1$. \square

Now define random Γ -labelings $\{Z_n\}_{n=1}^\infty$ of G as follows: let $\varepsilon = |E^+(G)|$ and enumerate the edges of G in an arbitrary order, so $E^+(G) = \{e_1, \dots, e_\varepsilon\}$. For $i \geq 1$ and $1 \leq j \leq \varepsilon$ define $X_{i,j}$

to be the random Γ -labeling of G which labels every edge besides e_j with the identity element $\mathbf{1}_\Gamma$, and

$$X_{i,j}(e_j) = \begin{cases} g_{i \bmod s} & \text{with probability } \frac{1}{3} \\ (g_{i \bmod s})^{-1} & \text{with probability } \frac{1}{3} \\ \mathbf{1}_\Gamma & \text{with probability } \frac{1}{3} \end{cases}.$$

Now define $Y_i = X_{i,1} \cdots X_{i,\varepsilon}$ and $Z_n = Y_1 Y_2 \cdots Y_n$. By definition, each random Γ -labeling $X_{i,j}$ is constant on every edge except one, and on the remaining edge the ratio of every two values is a pseudo-reflection. Proposition 4.4 yields, therefore, that $\mathbb{E}_{Z_n}[\phi_{Z_n, \pi}]$ is real rooted. By Claim 4.6, the random Γ -labelings Z_n converge, as $n \rightarrow \infty$, to the uniform distribution $\mathcal{C}_{\Gamma, G}$ of all Γ -labelings of G . Since the map $Z \rightarrow \mathbb{E}_Z[\phi_{Z, \pi}]$ is a continuous map from the space of distributions of Γ -labelings of G to $\mathbb{R}[x]$, we get that $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}]$ is real rooted, thus the first statement of Theorem 1.10 holds.

Finally, by Corollary 4.5, for every n , there is a Γ -labeling γ_n of G so that the largest root of $\phi_{\gamma_n, \pi}$ is at most the largest root of $\mathbb{E}_{Z_n}[\phi_{Z_n, \pi}]$. Because the set of Γ -labeling of G is finite, the γ_n have an accumulation point γ_0 . As the largest root of $\mathbb{E}_{Z_n}[\phi_{Z_n, \pi}]$ converges, as $n \rightarrow \infty$, to the largest root of $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}]$, the largest root of $\phi_{\gamma_0, \pi}$ is at most the largest root of $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}]$. This completes the proof of Theorem 1.10.

Remark 4.7. When $\Gamma = S_r$ is the symmetric group, [MSS15b, Lemma 3.5] gives a specific sequence of $2^{r-1} - 1$ rank-1 random permutations (“random swaps” in their terminology) the product of which is the uniform distribution on S_d . In fact, $\binom{r}{2}$ random swap are enough. This can be seen by the following inductive construction: let X be a uniformly random permutation in $S_{r-1} \leq S_r$, and define

$$Y_1 = \begin{pmatrix} (1 \ r) & \frac{1}{r} \\ \text{id} & \frac{r-1}{r} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} (2 \ r) & \frac{1}{r-1} \\ \text{id} & \frac{r-2}{r-1} \end{pmatrix}, \quad \dots, \quad Y_{r-1} = \begin{pmatrix} (r-1 \ r) & \frac{1}{2} \\ \text{id} & \frac{1}{2} \end{pmatrix}.$$

Then $X \cdot Y_1 \cdots Y_{r-1}$ gives a uniform distribution on S_r .

5 On Pairs Satisfying $(\mathcal{P}1)$ and $(\mathcal{P}2)$ and Further Applications

In this Section we say a few words about pairs (Γ, π) of a group and a representation satisfying properties $(\mathcal{P}1)$ and/or $(\mathcal{P}2)$, and elaborate on the combinatorial applications of Theorem 1.11, alongside the existence of one-sided Ramanujan r -coverings as stated in Theorem 1.2. We begin with $(\mathcal{P}2)$, where a complete classification is known.

5.1 Complex Reflection Groups

Recall that the pair (Γ, π) satisfies $(\mathcal{P}2)$ if Γ is finite and $\pi(\Gamma)$ is a complex reflection group, namely generated by pseudo-reflections: elements $A \in \text{GL}_d(\mathbb{C})$ of finite order with $\text{rank}(A - I_d) = 1$. If π is not faithful (not injective), it factors through the faithful $\bar{\pi}: \Gamma/\ker \pi \rightarrow \text{GL}_d(\mathbb{C})$, and (Γ, π) satisfies $(\mathcal{P}2)$ if and only if $(\Gamma/\ker \pi, \bar{\pi})$ does. In addition, if π is faithful but reducible and (Γ, π) satisfies $(\mathcal{P}2)$, then necessarily there are pairs (Γ_1, π_1) and (Γ_2, π_2) satisfying $(\mathcal{P}2)$ with $\Gamma \cong \Gamma_1 \times \Gamma_2$ and $\pi \cong (\pi_1, 1) \oplus (1, \pi_2)$.

Hence, the classification of pairs satisfying (P2) boils down to classifying finite, irreducible complex reflection groups: finite-order matrix groups inside $\mathrm{GL}_d(\mathbb{C})$ which are generated by pseudo-reflections and have no non-zero invariant proper subspaces of \mathbb{C}^d . This classification was established in 1954 by Shephard and Todd:

Theorem 5.1. [ST54] *Any finite irreducible complex reflection group W is one the following:*

1. $W \leq \mathrm{GL}_d(\mathbb{C})$ is isomorphic to S_{d+1} ($d \geq 2$), via the standard representation of S_{d+1} (see the paragraph preceding Claim 2.12).
2. $W = G(m, k, d)$ with $m, d \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{Z}_{\geq 1}$ and $k|m$. This is a generalization of signed permutations groups: the group $G(m, k, d) \leq \mathrm{GL}_d(\mathbb{C})$ consists of monomial matrices (matrices with exactly one non-zero entry in every row and every column), the non-zero entries are m -th roots of unity (not necessarily primitive), and their product is a $\frac{m}{k}$ -th root of unity. This a group of order $\frac{d! \cdot m^d}{k}$. For example,

$$\begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \\ \zeta^4 & 0 & 0 \end{pmatrix}$$

where $\zeta = e^{\frac{2\pi i}{6}}$, is an element of $G(6, 2, 3)$.

3. $W = \mathbb{Z}/m\mathbb{Z} \leq \mathrm{GL}_1(\mathbb{C})$ with $m \in \mathbb{Z}_{\geq 2}$, sometimes denoted $G(m, 1, 1)$, is the cyclic group of order m whose elements are m -th roots of unity.
4. W is one of 34 exceptional finite irreducible complex reflection groups of different dimensions d , $2 \leq d \leq 8$.

We remark that a finite complex reflection group which is conjugate to a subgroup of $\mathrm{GL}_d(\mathbb{R})$ (matrices with real entries) is, by definition, a finite Coxeter group. All groups listed in the theorem are finite complex reflection groups, and all irreducible except for $G(2, 2, 2)$.

Theorem 5.2 (Steinberg, [GM06, Thm 4.6]). *If (Γ, π) satisfies (P2) and π is irreducible, then (Γ, π) satisfies (P1) as well.*

Namely, if Γ is a finite group, $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ an irreducible representation and $\pi(\Gamma)$ is a complex reflection group, then the exterior powers $\bigwedge^m \pi$, $0 \leq m \leq d$, are irreducible and non-isomorphic.

It is evident that if π is reducible, the pair (Γ, π) does not satisfy (P1). Thus,

Corollary 5.3. *The pairs (Γ, π) satisfying both (P1) and (P2) are precisely the irreducible finite complex reflection groups²⁴.*

Finally, consider two pairs (Γ_1, π_1) and (Γ_2, π_2) , and a third pair (Γ, π) constructed as their direct product: $\Gamma \cong \Gamma_1 \times \Gamma_2$ and $\pi \cong (\pi_1, 1) \oplus (1, \pi_2)$. Constructing a (Γ, π) -covering of a graph is equivalent to constructing independent coverings, one for (Γ_1, π_1) and one for (Γ_2, π_2) . We conclude:

Corollary 5.4. *Let (Γ, π) satisfy (P2), and let G be a finite graph with no loops. Then,*

²⁴To be precise, this is true for faithful representations. If π factors through $\bar{\pi}: \Gamma/\ker \pi \rightarrow \mathrm{GL}_d(\mathbb{C})$, then (Γ, π) satisfies (P1) and (P2) if and only if so does $(\Gamma/\ker \pi, \bar{\pi})$.

- If d_1, \dots, d_r are the dimensions of the irreducible components of π , then

$$\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}] = \mathcal{M}_{d_1, G} \cdot \dots \cdot \mathcal{M}_{d_r, G}.$$

- $\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}]$ is real rooted and there is some labeling with smaller largest root.
- There is a one-sided Ramanujan (Γ, π) -covering of G .

5.2 Pairs Satisfying $(\mathcal{P}1)$

The list in Theorem 5.1 does not exhaust all pairs (Γ, π) (with π faithful) satisfying $(\mathcal{P}1)$. Even when restricting to finite groups, there are pairs satisfying $(\mathcal{P}1)$ but not $(\mathcal{P}2)$. A handful of such examples arises from the observation that $(\mathcal{P}1)$ is preserved by passing to bigger groups:

Claim 5.5. *Let Γ be a group, $\pi: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C})$ a representation and $H \leq \Gamma$ a subgroup. If $(H, \pi|_H)$ satisfies $(\mathcal{P}1)$ then so does (Γ, π) .*

Proof. It is clear that if $\bigwedge^m \pi$ cannot have an invariant proper subspace if $(\bigwedge^m \pi)|_H$ has none. An isomorphism of $\bigwedge^m \pi$ and $\bigwedge^{d-m} \pi$ induces an isomorphism on the same representation restricted to H . \square

For example, we can increase $\mathrm{std}(S_r)$ by adding some scalar matrix of finite order m as an extra generator, and obtain a d -dimensional faithful representation of $S_{d+1} \times \mathbb{Z}/m\mathbb{Z}$ which satisfies $(\mathcal{P}1)$.

There are also pairs with Γ finite which do not contain any complex reflection group. For instance, consider the index-2 subgroup Γ of $G(2, 1, 3)$ where we restrict to *even* permutation 3×3 matrices with ± 1 signing of every non-zero entry. The natural 3-dimensional representation of this group satisfies $(\mathcal{P}1)$, but does not contain any complex reflection group. We are not aware of a full classification of pairs (Γ, π) satisfying $(\mathcal{P}1)$, even when Γ is finite.

There are some interesting examples of pairs (Γ, π) satisfying $(\mathcal{P}1)$ where Γ is infinite and compact. For example, the standard representation π of the orthogonal group $O(d)$ or of the unitary group $U(d)$, is such (by, e.g., Claim 5.5 and the fact one can identify $\mathrm{std}(S_{d+1})$ as a subgroup of $O(d)$ or of $U(d)$).

Corollary 5.6. *Let $\Gamma = O(d)$ or $\Gamma = U(d)$, and let π be the standard d -dimensional representation. Then, for every finite graph G ,*

$$\mathbb{E}_{\gamma \in \mathcal{C}_{\Gamma, G}}[\phi_{\gamma, \pi}] = \mathcal{M}_{d, G}.$$

5.3 Applications of Theorem 1.11

In this section we elaborate the combinatorial consequences of Theorem 1.11 stating that if (Γ, π) satisfies both $(\mathcal{P}1)$ and $(\mathcal{P}2)$, then there is a one-sided Ramanujan (Γ, π) -covering of G whenever G is finite with no loops. Corollary 5.3 tells us exactly what pairs satisfy the conditions of the theorem. The most interesting consequence, based on the pair (S_r, std) , was already stated as Theorem 1.2: every G as above has a one-sided Ramanujan r -covering for every r .

Another interesting application stems from one-dimensional representations (item (3) in Theorem 5.1):

Corollary 5.7. *For every $m \in \mathbb{Z}_{\geq 2}$ and every loopless²⁵ finite graph G , there is a labeling of the oriented edges of G by m -th roots of unity (with $\gamma(-e) = \gamma(e)^{-1}$, as usual), such that the resulting spectrum is one-sided Ramanujan.*

Of course, the result for m follows from the result for m' whenever $1 \neq m'|m$. For $m = 2$ this is the main result of [MSS15a]. As this corollary deals only with one-dimensional representations, the original proof of [MSS15a] can be relatively easily adapted to show it. This was noticed also by [LPV14].

Recall that all irreducible representations of abelian groups are one-dimensional. Therefore, given an abelian group Γ and a finite graph G , there is a Γ -labeling of G which yields a one-sided Ramanujan (Γ, π) -covering for any irreducible representation π of Γ . However, this certainly does not guarantee the existence of a single Γ -labeling which is “Ramanujan” for all irreducible representations simultaneously. In fact, such a Γ -labeling does not exist in general – see Remark 1.14.

Still, in the special case where $\Gamma = \mathbb{Z}/3\mathbb{Z}$ is the cyclic group of order 3, there are only two non-trivial representations π_1 and π_2 , and one is the complex conjugate of the other. Hence, $\phi_{\gamma, \pi_2} = \phi_{\gamma, \pi_1}$ for any Γ -labeling γ , and so the spectra are identical, and we get, as noticed by [LPV14, CV15]²⁶:

Corollary 5.8. *Every finite graph G has a one-sided Ramanujan 3-covering, where the permutation above every edge is cyclic.*

From the third infinite family of complex reflection groups (item (2) in Theorem 5.1), we do not get any significant combinatorial implications. If $\Gamma = G(m, k, d)$, Theorem 1.11 guarantees that every graph has a one-sided Ramanujan “signed d -covering”: a d -covering of G where every oriented edge is then labeled by an m -th root of unity, and such that the product of roots in every fiber of edges is an $\frac{m}{k}$ -th root of unity. But Corollary 5.7 shows that every d -covering of G can be edge-labeled by m -th roots of unity so that the resulting spectrum is one-sided Ramanujan. If $k < m$, we can label by $\frac{m}{k}$ -th roots, so applying Theorem 1.11 on Γ yields nothing new. If $k = m$, the statement of the theorem cannot be (easily) derived from former results: we get that G has a d -covering with edges labeled by m -th roots of unity, so that the product of the labels on every fiber is 1, and the resulting spectrum is one-sided Ramanujan.

5.4 Permutation Representations

Every group action of Γ on a finite set X yields a representation π of dimension $|X|$. In this case, π can be taken to map Γ into permutation matrices, hence (Γ, π) -coverings of a graph G correspond to topological $|X|$ -coverings of G (with permutations restricted to the set $\pi(\Gamma)$). Such representations are called permutation representations. For instance, the natural action of S_r on $\{1, \dots, r\}$ yields the set of all r -coverings from Theorem 1.2. The action of $\mathbb{Z}/3\mathbb{Z}$ by cyclic shifts on a set of size 3 yields the regular representation of this group and the coverings in Corollary 5.8. In general, the regular representation of a group is always of this kind.

It is interesting to consider the set \mathcal{A} of all possible pairs (Γ, π) where Γ is a finite group and π a permutation representation, so that every graph has a (one-sided) Ramanujan (Γ, π) -covering.

²⁵In this special case it is actually possible to prove the result even for graphs with loops: the proof of Proposition 4.4 does not break.

²⁶Interestingly, it is also shown in [CV15] that every graph has a one-sided Ramanujan 4-covering with cyclic permutations. This does not seem to follow from the results in the current paper.

Of course, the action of Γ on X must be transitive: otherwise, the coverings are never connected. Observe this set is closed under two “operations”:

1. If $\Lambda \leq \Gamma$ and $(\Lambda, \pi|_\Lambda)$ is in \mathcal{A} , then so is (Γ, π) .
2. The set \mathcal{A} is closed under towers of coverings: a Ramanujan covering of a Ramanujan covering is a Ramanujan covering of the original graph. In algebraic terms this corresponds to wreath products of groups. Namely, if (Γ, π) and (Λ, ρ) are both in \mathcal{A} with respect to actions on the sets X and Y , respectively, then so is the pair $(\Gamma \text{wr}_X \Lambda, \psi)$, where

$$\Gamma \text{wr}_Y \Lambda = \left(\prod_{y \in Y} \Gamma_y \right) \rtimes \Lambda$$

is the restricted wreath product (Γ_y is a copy of Γ for every $y \in Y$, and Λ acts by permuting the copies according to its action on Y), and ψ is based on the action of $\Gamma \text{wr}_Y \Lambda$ on the set $X \times Y$ by

$$(\{g_y\}, \ell) \cdot (x, y) = (g_y \cdot x, \ell \cdot y).$$

In this language, for example, a tower of 2-coverings, as considered by [BL06] and [MSS15a], corresponds to a pair (Γ, π) with Γ a nested wreath product of $\mathbb{Z}/2\mathbb{Z}$. See also [Mak15, Chapter 5] and the references therein.

6 Open Questions

We finish with some open questions arising naturally from the discussion in this paper.

Question 6.1. Irreducible representations and one-sided Ramanujan coverings: Which pairs (Γ, π) of a finite group and an irreducible representation guarantee the existence of one-sided Ramanujan (Γ, π) -coverings for every finite graph? Can the statement of Theorem 1.11 be extended to more pairs? Does $(\mathcal{P}1)$ suffice? In fact, we are not aware of a single example of a pair (Γ, π) with π non-trivial and irreducible and a finite graph G so that there is no (one-sided) Ramanujan (Γ, π) -covering of G . See also Remark 1.13.

Question 6.2. Full Ramanujan coverings: The previous question can be asked for full (two-sided) Ramanujan coverings as well. The difference is that in this case nothing is known for general graphs. The case $(\mathbb{Z}/2\mathbb{Z}, \pi)$ with π the non-trivial one-dimensional representation is the Bilu-Linial Conjecture [BL06]. Proving it would yield the existence of infinitely many k -regular non-bipartite Ramanujan graphs for every $k \geq 3$.

Question 6.3. Even “Fuller” Ramanujan coverings: There is another, stronger sense, of Ramanujan graphs: graphs where the non-trivial spectrum is contained in the spectrum of the universal covering tree. The spectrum of the k -regular tree is precisely $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, so this coincides with the standard definition of Ramanujan. But in other families of graphs, the spectrum of the tree is not necessarily connected, and then the current definition is stronger. For example, if $k \geq \ell$, the spectrum of the (k, ℓ) -biregular tree is

$$\left[-\sqrt{k-1} - \sqrt{\ell-1}, -\sqrt{k-1} + \sqrt{\ell-1} \right] \cup \{0\} \cup \left[\sqrt{k-1} - \sqrt{\ell-1}, \sqrt{k-1} + \sqrt{\ell-1} \right].$$

Does every graph have a Ramanujan r -covering (or 2-covering) in this stricter sense?

Question 6.4. Regular representations and Cayley graphs: Let Γ be finite and \mathcal{R} its regular representation. Such pairs are especially interesting as (Γ, \mathcal{R}) -coverings of graphs generalize the notion of Cayley graphs (these are (Γ, \mathcal{R}) -coverings of bouquets). For example, for certain families of finite groups, mostly simple groups of Lie type, it is known that random Cayley graphs are expanding uniformly (e.g. [BG08, BGGT15]). Can this be extended to random (Γ, \mathcal{R}) -coverings of graphs, at least, say, when G is a good expander itself?

Question 6.5. The d -matching polynomial: This paper shows that $\mathcal{M}_{d,G}$, the d -matching polynomial of the graph G share quite a few properties with the classical matching polynomial, $\mathcal{M}_{1,G}$. But $\mathcal{M}_{1,G}$ has many more interesting properties (a good reference is [God93]). What parts of this theory can be generalized to $\mathcal{M}_{d,G}$? In particular, it would be desirable to find a more direct proof to the real-rootedness of $\mathcal{M}_{d,G}$. Such a proof may allow to extend this real-rootedness result to graph with loops.

Question 6.6. Loops: Some results in this paper hold for graphs with loops (e.g. Theorem 1.8). We conjecture that, in fact, all the results hold for graphs with loops. In particular, we conjecture that any finite graph G with loops should have a one-sided Ramanujan r -covering (Theorem 1.2), that $\mathcal{M}_{d,G}$ is real-rooted for every d (Theorem 2.7) and that if (Γ, π) satisfies (P1) and (P2), then G has a one-sided Ramanujan (Γ, π) -covering (Theorem 1.11). (And see Question 6.7.)

If true, this would yield, for example, that if A is a uniformly random permutation matrix, or Haar-random orthogonal or unitary matrix in $U(d)$, then $\mathbb{E}[\phi(A + A^*)]$ is real-rooted.

Question 6.7. Another interlacing family of characteristic polynomials: The one argument in this paper that breaks for loops is in the proof of Proposition 4.4. The problem is that if e is a loop, then $\pi(\gamma(e))$ and $\pi(\gamma(-e))$ lie in the same $d \times d$ block of $A_{\gamma, \pi}$. One way to extend the arguments for loops is to prove the following parallel of Theorem 4.2, which we believe should hold:

For a matrix A denote $A^{\mathcal{HERM}} \stackrel{\text{def}}{=} A + A^*$. If $\mathcal{W} = \{W_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq \ell(i)}$ is defined as in Theorem 4.2, then

$$\mathbb{E}_{\mathcal{W}} \left[\phi \left([W_{1,1} \dots W_{1,\ell(1)} A_1 + \dots + W_{m,1} \dots W_{m,\ell(m)} A_m]^{\mathcal{HERM}} \right) \right]$$

is real-rooted for every $A_1, \dots, A_m \in M_d(\mathbb{C})$.

If true, this statement generalizes the fact that the characteristic polynomials $\phi(A + A^*)$ and $\phi(BA + (BA)^*)$ interlace whenever $A, B \in GL_d(\mathbb{C})$ with B a pseudo-reflection.

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Chris Hall,
 Department of Mathematics,
 University of Wyoming
 Laramie, WY 82071 USA
 chall14@uwyo.edu

Doron Puder,
 School of Mathematics,
 Institute for Advanced Study,
 Einstein Drive, Princeton, NJ 08540 USA

doronpuder@gmail.com

William F. Sawin,
Department of Mathematics,
Princeton University
Fine Hall, Washington Road
Princeton NJ 08544-1000 USA
wsawin@math.princeton.edu