BOUNDS ON OSCILLATORY INTEGRAL OPERATORS BASED ON MULTILINEAR ESTIMATES

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§1. Summary

Let $S \subset \mathbb{R}^n$ be a smooth, compact hyper-surface with positive definite second fundamental form. Let σ be its surface measure.

We prove the following result with respect to the Fourier restriction/extension problem.

Theorem 1. *Assume the exponent p satisfies*

$$\begin{cases} p > 2\dfrac{4n+3}{4n-3} & \text{if } n \equiv 0 \ (mod\ 3) \\ p > \dfrac{2n+1}{n-1} & \text{if } n \equiv 1 \ (mod\ 3) \\ p > \dfrac{4(n+1)}{2n-1} & \text{if } n \equiv 2 \ (mod\ 3). \end{cases} \tag{1.1}$$

Then the inequality

$$\|\hat{\mu}\|_p \lesssim C_p \Big\| \frac{d\mu}{d\sigma} \Big\|_\infty \tag{1.2}$$

holds for measures $\mu \ll \sigma$ such that $\frac{d\mu}{d\sigma} \in L^\infty(S, \sigma)$.

See §3. For $n = 3$ (resp. $n = 4$), the exponent in (1.2) is $\frac{10}{3}$ (resp. 3) and coincides with the condition $p \geq \frac{2(n+2)}{n}$ resulting from the bilinear L^2-approach in [T1]. For $n \geq 5$, the result is new.

Recall that, according to the restriction conjecture, due to E. Stein, cf. [St1], (1.1) should remain valid for all $p > \frac{2n}{n-1}$.

We also point out that if S is the $(n-1)$-sphere or paraboloid, then (1.2) may be strengthen to

$$\|\hat{\mu}\|_p \leq C_p \Big\| \frac{d\mu}{d\sigma} \Big\|_p \tag{1.2'}$$

for p satisfying (1.1) (the argument combines Theorem 1, the Maurey-Nikishin factorization theorem and invariance considerations, the usual way; cf. [B1]).

The main ingredient in our approach is the multilinear theory developed in [BCT] that we will recall in §5. In §2 we treat the case $n = 3$ to explain the method in its simplest form. In §4, the analysis is refined further and combined with T. Wolff's Kakeya maximal function estimate [Wo1] to establish (1.1) for $n = 3$ under the condition

$$p > 3\frac{3}{10}. \tag{1.3}$$

Thus we have the following small improvement of the $p > \frac{10}{3}$ result in 3D.

Theorem 2. *For $n = 3$ and S as above, we have*

$$\|\hat{\mu}\|_p \leq C_p \Big\| \frac{d\mu}{d\sigma} \Big\|_\infty \text{ for } p > 3\frac{3}{10} \tag{1.4}$$

assuming $\mu \ll \sigma$ and $\frac{d\mu}{d\sigma} \in L^\infty(S, d\sigma)$.

By using 'ε-removal lemmas', Theorems 1 and 2 may be derived from a weaker 'local' version, more precisely

Theorem 1′. *Let $n \geq 3$ and S as above.*

Denote

$$Q_R^{(p)} = \max \|\hat{\mu}\|_{L^p(B_R)}$$

where the maximum is taken over all measures $\mu \ll \sigma$ on S such that $\|\frac{d\mu}{d\sigma}\|_\infty \leq 1$. Then, for all $\varepsilon > 0$

$$Q_R^{(p)} \ll R^\varepsilon \tag{1.5}$$

provided p satisfies (1.1).

and

Theorem 2′. *Same statement for $n = 3$ and $p \geq 3\frac{3}{10}$.*

The use of such ε-removal lemmas is by now standard (cf. [T2]), but we will include an argument for completeness sake in the Appendix, since we process here $L^\infty - L^p$ inequalities rather than $L^p - L^p$ inequalities, as in [T2].

The technique used applies also in the variable coefficient (Hörmander) setting. Thus we consider oscillatory integral operators

$$(T_\lambda f)(x) = \int e^{i\lambda\psi(x,y)} f(y)dy \qquad (\|f\|_\infty \leq 1) \tag{1.6}$$

with real analytic phase function

$$\psi(x, y) = x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n \langle Ay, y\rangle + O(|x|\,|y|^3) + O(|x|^2|y|^2) \tag{1.7}$$

and A non-degenerate.

($x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ are restricted to a neighborhood of 0.)

Our concern is then in which range of p, a bound

$$\|T_\lambda f\|_p \leq c\lambda^{-\frac{n}{p}} \tag{1.8}$$

holds. Recall Stein's result [St2]

$$\|T_\lambda f\|_p \leq c\|f\|_2 \text{ for } p \geq \frac{2(n+1)}{n-1}. \tag{1.9}$$

Also, for n *odd*, there are examples showing that, replacing $\|f\|_2$ by $\|f\|_\infty$, an inequality (1.8) may only hold for $p \geq \frac{2(n+1)}{n-1}$ (see [B2]).

Lee observed in [L] that Stein's estimate may be improved if we make the additional hypothesis that A in (1.7) is positive (or negative) definite. He extended the bilinear approach from [T1] to the variable coefficient setting. In particular, he proved that (1.8) holds (up to a factor λ^ϵ) if $p \geq \frac{2(n+2)}{n}$. We will prove in §5 that (1.8) holds under the condition (1.1). Thus we have

Theorem 3. *Let T_λ be as above with A positive or negative definite in (1.7). Then*

$$\|T_\lambda f\|_p \leq C_p\lambda^{-\frac{n}{p}}\|f\|_\infty \tag{1.10}$$

holds for p satisfying (1.1).

If $n = 3$ or 4, Theorem 3 agrees with the results of [L], and for $n \geq 5$, it is new.

For n <u>even</u>, there is the following statement (with only the non-degeneracy assumption on A).

Theorem 4. *Let n be even and T_λ as above, assuming in (1.7) that A is non-degenerate. Then*

$$\|T_\lambda f\|_p \leq C_p \lambda^{-\frac{n}{p}} \|f\|_\infty \text{ for } p > \frac{2(n+2)}{n} \tag{1.11}$$

(apart from the endpoint, the condition on p in Theorem 4 was already previously observed to be best possible, cf. [B2].)

It turns out, rather surprisingly, that for $n = 3$ the exponent $\frac{10}{3}$ in Theorem 3 is also optimal. In §6, we describe a specific example (with A elliptic), making the comparison with the hyperbolic case, and explaining the role of the Kakeya compression phenomenon. For $n = 3$, in both elliptic and hyperbolic cases, there may be a curved Kakeya compression in a 2-dimensional set at the coarse scale $\frac{1}{\sqrt{\lambda}}$, but the local behaviour of the oscillatory integrals is different.

The proof of Theorems 3 and 4 is based on an application of Theorem 6.2 from [BCT], but we need a version without the extra λ^ε-factors. Hence, we proceed to 'ε-removal' at the multilinear stage (see Appendix), which also provides an alternative strategy to derive Theorem 1 directly, without passing through Theorem 1′ (let us point out that this ε-removal argument applies only to our particular application of [BCT], Theorem 6.2, see §5.)

Returning to curved Kakeya compression, it is shown that a curved Kakeya set in even dimension n has Minkowski dimension at least $\frac{n}{2} + 1$ (see §6). This statement was known to be optimal (see [B2]).

Details are given in §7 for $n = 4$, where it is shown how to derive this property from multi-linear Kakeya-type results. This strategy may be seen as the essence of our paper and is basically repeated to obtain the oscillatory integral bounds cited above.

Returning to Theorem 3, we should point out the application to the Bochner-Riesz multilinear problem. Recall that the Bochner-Riesz multiplier S_δ is defined by $(S_\delta f)^\wedge(\xi) = (1 - |\xi|^2)^\delta_+ \hat{f}(\xi)$. Equivalently $S_\delta f = f * K_\delta$, where K_δ has the asymptotic

$$K_\delta(x) \sim e^{e^{\pm 2\pi i|x|}} / |x|^{\frac{n+1}{2} + \delta}. \tag{1.12}$$

The problem is then to obtain the optimal condition on $\delta \geq 0$ to satisfy

$$\|S_\delta f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.13}$$

C. Fefferman's proof of the ball-multiplier conjecture implies that certainly $\delta > 0$ for $p \neq 2$ (note that the problem is self-dual). In view of (1.12), the condition

$$\delta > \max\Big(0, \Big|\frac{1}{2} - \frac{1}{p}\Big| n - \frac{1}{2}\Big) \tag{1.14}$$

is clearly necessary. It is conjectured that (1.14) also suffices for (1.13) to hold and this was proven for $n = 2$ in [C-S] and, independently, in [Hor].

In fact, Hörmander's approach consists in reducing the study of convolution by K_δ to some specific oscillatory integral operator T_λ, of the type considered above (note that regarding dimension, the $\mathbb{R}^d - \mathbb{R}^d$ problem is replaced by an $\mathbb{R}^{d-1} - \mathbb{R}^d$ problem in this reduction). As a corollary of our Theorem 3 together with the standard factorization and rotational invariance considerations (already mentioned above), we obtain (cf. [B2] for details).

Theorem 5. *Let $n \geq 3$. Then the Bochner-Riesz conjecture holds providing* $\max(p, p')$ *satisfies (1.1).*

On the geometric side, the Kakeya-type maximal function underlying the Bochner-Riesz operators (sometimes called 'Nikodym maximal function) involves also averaging over straight line segments and, for $n = 3$, T. Wolff's $\frac{5}{2}$-inequality is again known to hold (see [Wo1]). Thus in principle, one could expect the proof of Theorem 2 to carry over and lead to the validity of the Bochner-Riesz conjecture for $\max(p, p') \geq 3\frac{3}{10}$, if $n = 3$. We do not pursue the details of this matter here. In fact, it is well-possible that the exponent $3\frac{3}{10}$ from Theorem 2 may be improved further, by reorganizing and refining the method. No serious attempt was given to do so, as our primary goal is to show how to obtain some progress over the present results, keeping the arguments as simple as possible.

Finally, let us cite [T3] as a survey work on the problems discussed in this paper and where the reader will find many background material and references.

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§2. An Approach to the Restriction Problem in 3D

(alternative proof of the $L^{10/3}$-bound)

1. Consider the oscillatory integral operator

$$Tf(x) = \int e^{i\phi(x,y)} f(y) dy \qquad (|f| \leq 1)$$

where $y \in \Omega$ is a neighborhood of $0 \in \mathbb{R}^2$ and $x \in \mathbb{R}^3 \cap [|x| < R]$,

$$\phi(x, y) = x_1 y_1 + x_2 y_2 + x_3 \phi_1(y) \tag{1.1}$$

with $\phi_1(y) = y_1^2 + y_2^2$ (paraboloid), or more generally

$$\phi_1(y) = \langle Ay, y\rangle + O(|y|^3) \quad (A = \text{positive definite}) \tag{1.2}$$

(we will comment on the indefinite case at the end of this section).

The purpose of this section is to explain in a simple case how the multi-linear theory from [BCT] can be exploited to produce results in the usual restriction problem.

Given a phase function ϕ as above, we introduce at a given point $y \in \Omega$ the vector

$$Z = Z(y) = \partial_{y_1}(\nabla_x \phi) \wedge \partial_{y_2}(\nabla_x \phi) = (-\partial_1 \phi_1(y), -\partial_2 \phi_1(y), 1). \tag{1.3}$$

For simplicity, we carry the discussion for the case of the paraboloid, thus

$$\phi_1(y) = y_1^2 + y_2^2.$$

In this case, the transversality condition of $\{Z(y^{(i)}), i = 1, 2, 3\}$, where $y^{(i)}$ is restricted to some small disc $\Omega_i \subset \Omega$ (as needed for the trilinear L^3-bound from [BCT]) amounts to non-collinearity of $\Omega_1, \Omega_2, \Omega_3$.

Discussion of the general situation (1.2) would require to introduce the Gauss map associated to the surface

$$(y_1, y_2) \mapsto \big(y_1, y_2, \phi_1(y)\big).$$

(see §3.)

2. Fix K (a large parameter).

Partition $\Omega = \bigcup \Omega_\alpha, \Omega_\alpha$ balls of size $\frac{1}{K}$; $y_\alpha \in \Omega_\alpha$. There are $\sim K^2$ values of α. Write

$$\begin{aligned} Tf(x) &= \sum_\alpha e^{i\phi(x, y_\alpha)} \Big[\int_{\Omega_\alpha} e^{i[\phi(x,y) - \phi(x, y_\alpha)]} f(y) dy\Big] \\ &= \sum_\alpha e^{i\phi(x, y_\alpha)} (T_\alpha f)(x). \end{aligned} \tag{2.1}$$

Note that

$$|\nabla_x [\phi(x, y) - \phi(x, y_\alpha)]| \leq \frac{1}{K} \text{ for } y \in \Omega_\alpha.$$

Take a smooth rapidly decaying bumpfunction η s.t. $\hat{\eta}(\omega) = 1$ on $[\omega \in \mathbb{R}^3; |\omega| \leq 1]$. Let $\eta_K(x) = \frac{1}{K^3}\eta\big(\frac{x}{K}\big)$ satisfying $\hat{\eta}_K(\omega) = 1$ for $|\omega| < 1/K$.

Thus

$$T_\alpha f = T_\alpha f * \eta_K$$

and

$$|T_\alpha f(x)| \leq \int |T_\alpha f(z)|\ |\eta_K(x-z)| dz.$$

Restrict x to a ball $B(a, K) \subset \mathbb{R}^3$. Set $a = 0$.

For $x \in B(0, K)$

$$|T_\alpha f(x)| \leq \int |T_\alpha f(z)| \zeta_K(z) dz = c_\alpha \tag{2.2}$$

where

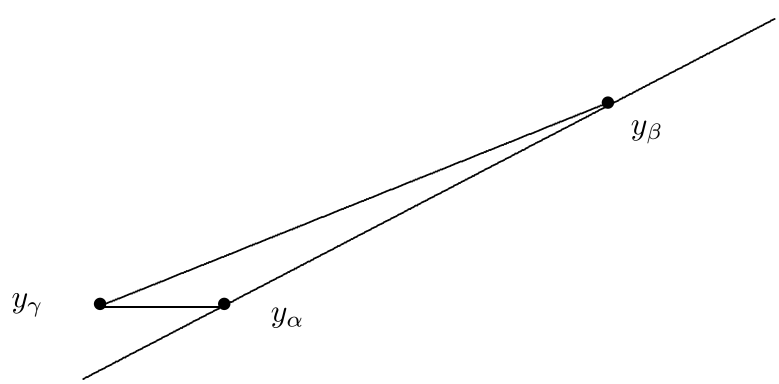
$$\zeta(x) = \max_{|x-x'|\leq 1} |\eta(x')|.$$

3. Denote $c_* = \max c_\alpha = c_{\alpha_*}$. Let $K_1 \ll K$ be a second large parameter. We distinguish several possibilities.

(3.1) **Non-coplanar interaction**.

There are α, β, γ such that $c_\alpha, c_\beta, c_\gamma > K^{-4} c_*$ and

$$|y_\alpha - y_\beta| \geq |y_\alpha - y_\gamma| \geq \operatorname{dist}\big(y_\gamma, \underbrace{y_\alpha + \mathbb{R}(y_\beta - y_\alpha)}_{\equiv \ell(y_\alpha, y_\beta)}\big) > 10^3 \frac{1}{K} \tag{3.1'}$$



In this situation we use the trilinear theory from [BCT].

(3.2) **Non-transverse interaction**.

If $|y_\alpha - y_{\alpha_*}| > \frac{1}{K_1}$, then $c_\alpha \leq K^{-4} c_*$. Here we use rescaling (cf. [T-V-V]).

(3.3) **Transverse coplanar interaction**.

There is α_{**} with $c_{\alpha_{**}} > K^{-4} c_*, |y_{\alpha_*} - y_{\alpha_{**}}| > \frac{1}{K_1}$.

Assuming (3.1) fails, it follows that moreover

$$c_\alpha \leq K^{-4} c_* \text{ if } \operatorname{dist}\big(y_\alpha, \ell(y_{\alpha_*}, y_{\alpha_{**}})\big) > 10^3 \frac{K_1}{K}.$$

In this case we rely on the by now standard square function estimates going back to A. Cordoba's work [C].

4. Assume (3.1)

For $x \in B(0, K)$, by (2.2), (1.1)

$$|Tf(x)| \leq \sum_\alpha c_\alpha < K^2 c_* < K^6 (c_\alpha c_\beta c_\gamma)^{\frac{1}{3}}.$$

Hence, for $q \geq 3$

$$\begin{aligned} |Tf(x)|^q \leq |Tf(x)|^3 &\leq K^{18} \int |T_\alpha f|(z_1) |T_\beta f|(z_2) |T_\gamma f|(z_3)\, \zeta_K(z_1)\zeta_K(z_2)\zeta_K(z_3) dz_1 dz_2 dz_3 \\ &\leq K^{18} \sum_{\alpha,\beta,\gamma\,(3.1')} \int |T_\alpha f|(x - z_1) |T_\beta f|(x - z_2) |T_\gamma f|(x - z_3) \zeta_K(z_1)\zeta_K(z_2)\zeta_K(z_3) \end{aligned}$$

The corresponding contribution is estimated using the trilinear bound from [BCT]

$$\int_{B_R} |T_\alpha f|(x - z_1) |T_\beta f|(x - z_2) |T_\gamma f|(x - z_3) dx < R^\varepsilon . C(K) < R^{2\varepsilon} \tag{4.1}$$

5. Assume (3.2). For $x \in B(0, K)$, estimate

$$\begin{aligned} |Tf(x)| &\leq 10 \max_\tau \Big| \int_{\tilde\Omega_\tau} e^{i\phi(x,y)} f(y) dy \Big| + \sum_{|y_\alpha - y_{\alpha_*}| > \frac{1}{K_1}} c_\alpha \\ &\leq 10. \max_\tau |\tilde T_\tau f(x)| + K^{-2} c_* \end{aligned} \tag{5.1}$$

where $\Omega = \bigcup \tilde\Omega_\tau$ is a partition of Ω in balls of size $\frac{1}{K_1}$.

Thus (5.1) implies for $x \in B(0, K)$

$$|Tf(x)|^q \leq C \sum_{\tau=1}^{\sim K_1^2} |\tilde{T}_\tau f|^q(x) + CK^{-2q} \sum_{\alpha=1}^{\sim K^2} \int |T_\alpha f|^q(x-z)\zeta_K(z)dz. \tag{5.2}$$

The corresponding contribution is at most

$$C \sum_\tau \int_{B_R} |\tilde{T}_\tau f|^q + CK^{-2q} \sum_\alpha \int_{B_R} |T_\alpha f|^q. \tag{5.3}$$

At this point, we use the (parabolic) rescaling

$$\Big| \int_{|y-\bar{y}|<\rho} e^{i\phi(x,y)} f(y)dy \Big| =$$

$$y = \bar{y} + y'$$

$$\Big| \int_{|y'|<\rho} e^{i[(x_1+2\bar{y}_1x_3)y_1'+(x_2+2\bar{y}_2x_3)y_2'+x_3|y'|^2]} f(\bar{y}+y')dy' \Big| = (5.4)$$

and

$$\|(5.4)\|_{L^q(B_R)} \leq C\rho^2\rho^{-\frac{4}{q}} Q_{\rho R} \tag{5.5}$$

where we define

$$Q_R = \max_{|f|\leq 1} \|Tf\|_{L^q(B_R)}. \tag{5.6}$$

Substituting (5.5) in (5.3) gives the contribution ($\rho = \frac{1}{K_1}$ and $\rho = \frac{1}{K}$)

$$CK_1^2.K_1^{-2q+4}Q^q_{R/K_1} + CK^{-2q}.K^2.K^{-2q+4}Q^q_{R/K}$$

and hence for the L^q-norm

$$< CK_1^{-2(1-\frac{3}{q})} Q_{R/K_1} + CK^{-4+\frac{6}{\varepsilon}} Q_{R/K}. \tag{5.7}$$

6. Assume (3.3). Thus, denoting $\ell = \ell(y_{\alpha_*}, y_{\alpha_{**}})$, for $x \in B(a, R)$

$$\Big| \int_{\text{dist}\,(y,\ell)>10^4\frac{K_1}{K}} e^{i\phi(x,y)} f(y)dy \Big| \leq \sum_{\text{dist}\,(y_\alpha,\ell)>10^3\frac{K_1}{K}} |T_\alpha f(x)| < K^2K^{-4}c_*$$

$$< K^{-2} \int |T_{\alpha_*} f(a-z)|\zeta_K(z)dz. \tag{6.1}$$

Hence

$$\Big|\int_{\text{dist }(y,\ell)>10^4\frac{K_1}{K}} e^{i\phi(x,y)}f(y)dy\Big|^q < K^{-2q}\sum_{\alpha=1}^{\sim K^2}\int |T_\alpha f(x-z)|^q\zeta_K(z)dz \tag{6.2}$$

and by (5.5), the corresponding contribution is at most

$$K^{-2}.K^{\frac{2}{q}}.K^{\frac{4}{q}-2}\ Q_{R/K} < K^{-2}Q_{R/K}. \tag{6.3}$$

Considering the partition $\Omega = \bigcup \tilde\Omega_\tau$ in balls of size $\frac{1}{K_1}$ and fixing $x\in B(a,K)$, there are clearly the following alternatives

(6.4) $|Tf(x)| < C\max_\tau \Big|\int_{\tilde\Omega_\tau} e^{i\phi(x,y)}f(y)dy\Big|$.

(6.5) There are τ,τ' such that dist $(\tilde\Omega_\tau,\tilde\Omega_{\tau'}) > \frac{10^6}{K_1}$ and

$$\Big|\int_{\tilde\Omega_\tau} e^{i\phi(x,y)}f(y)dy\Big|, \Big|\int_{\tilde\Omega_{\tau'}} e^{i\phi(x,y)}f(y)dy\Big| > \frac{1}{10K_1^2}|Tf(x)|.$$

If (6.4), write

$$|Tf(x)| \leq C\Big[\sum_{\tau=1}^{\sim K_1^2}\Big|\int_{\tilde\Omega_\tau} e^{i\phi(x,y)}f(y)dy\Big|^q\Big]^{\frac{1}{q}} = (6.6)$$

and by (5.4), (5.5)

$$\|(6.6)\|_{L^q(B_R)} \leq K_1^{\frac{2}{q}}.K_1^{\frac{4}{q}-2}Q_{R/K_1} < K_1^{-2(1-\frac{3}{q})}Q_{R/K_1}. \tag{6.7}$$

Assume (6.5). Estimate further

$$\Big|\int_{\tilde\Omega_\tau} e^{i\phi(x,y)}f(y)dy\Big| \leq \Big|\sum_{\substack{\Omega_\alpha\subset\tilde\Omega_\tau\\ \text{dist }(y_\alpha,\ell)\leq 10^3\frac{K_1}{K}}} e^{i\phi(x,y_2)}(T_\alpha f)(x)\Big| + \sum_{\substack{\Omega_\alpha\subset\tilde\Omega_\tau\\ \text{dist }(y_\alpha,\ell)>10^3\frac{K_1}{K}}} |T_\alpha f|$$

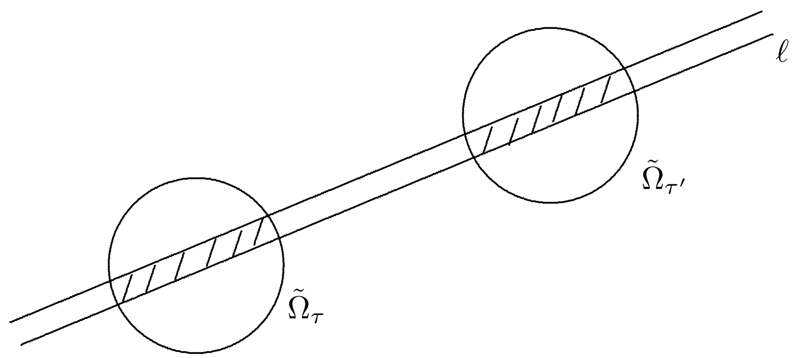
$$= (6.8) + (6.9)$$

and similarly for $|\int_{\tilde\Omega_{\tau'}} e^{i\phi(x,y)}f(y)dy|$.

The contribution of (6.9) was evaluated in (6.1), (6.3).

Thus it remains to obtain a bound on

$$\int_{B(a,K)} \Big| \sum_{\substack{\Omega_\alpha \subset \tilde\Omega_\tau \\ \text{dist}\,(y_\alpha,\ell)\le 10^3 \frac{K_1}{K}}} e^{i\phi(x,y_\alpha)}(T_\alpha f)(x)\Big|^{\frac q2} \Big| \sum_{\substack{\Omega_\alpha \subset \tilde\Omega_{\tau'} \\ \text{dist}\,(y_\alpha,\ell)\le 10^3 \frac{K_1}{K}}} e^{i\phi(x,y_\alpha)}(T_\alpha f)(x)\Big|^{\frac q2} dx \tag{6.10}$$



By Hölder's inequality, assuming $q<4$

$$(6.10) \lesssim K^{3(1-\frac q4)} \Big[\int_{B(a,K)} |\cdots|^2 \; |\cdots|^2 dx\Big]^{q/4} \tag{6.11}$$

Consider

$$\int_{B(a,K)} |\cdots|^2 \; |\cdots|^2 \le$$
$$\sum_{\substack{\Omega_{\alpha_1},\Omega_{\alpha_2}\subset\tilde\Omega_\tau\cap\Delta \\ \Omega_{\alpha_1'},\Omega_{\alpha_2'}\subset\tilde\Omega_{\tau'}\cap\Delta}} \Big|\int_{B(a,K)} T_{\alpha_1}f\; \overline{T_{\alpha_2}f}\; \overline{T_{\alpha_1'}f}\; T_{\alpha_2'}f \quad e^{i[\phi(x,y_{\alpha_1})-\phi(x,y_{\alpha_2})\cdots]}dx\Big| \tag{6.12}$$

where $\Delta = \Big\{y\in B(0,1); \text{dist}\,(y,\ell) < 10^3 \frac{K_1}{K}\Big\}$.

Rewriting

$$\phi(x,y_{\alpha_1})-\phi(x,y_{\alpha_2})-\phi(x,y_{\alpha_1'})+\phi(x,y_{\alpha_2'}) =$$
$$< (x_1,x_2), y_{\alpha_1}-y_{\alpha_2}-y_{\alpha_1'}+y_{\alpha_2'} > + x_3\big(\phi_1(y_{\alpha_1})-\phi_1(y_{\alpha_2})-\phi_1(y_{\alpha_1'})+\phi_1(y_{\alpha_2'})\big)$$

we see that in (6.12) we may restrict the summation to those quadruples $(\alpha_1, \alpha_2, \alpha_1', \alpha_2')$ for which

$$\begin{cases} |y_{\alpha_1} - y_{\alpha_2} - y_{\alpha_1'} + y_{\alpha_2'}| \lesssim \dfrac{1}{K} & (6.13) \\ |\phi_1(y_{\alpha_1}) - \phi_1(y_{\alpha_2}) - \phi_1(y_{\alpha_1'}) + \phi_1(y_{\alpha_2'})| \lesssim \dfrac{1}{K} & (6.13') \end{cases}$$

Let $\ell = b + \mathbb{R}v$ $(|v| = 1)$ and $|y_{\alpha_i} - (b + t_i v)| < 10^3 \frac{K_1}{K}, |y_{\alpha_i'} - (b + t_i' v)| < 10^3 \frac{K_1}{K}$.

Recall from (6.5) that

$$|t_1 - t_2|, |t_1' - t_2'| \leq \frac{2}{K_1}, |t_1 - t_1'| > \frac{10^6}{K_1}.$$

Hence (6.13), (6.13') imply by the preceding

$$\begin{cases} |t_1 - t_2 - t_1' + t_2'| \lesssim C\dfrac{K_1}{K} & (6.14) \\ |t_1^2 - t_2^2 - (t_1')^2 + (t_2')^2| \lesssim C\dfrac{K_1}{K} & (6.14') \end{cases}$$

and we obtain from the separation property that

$$|(t_1 + t_2) - (t_1' + t_2')| \lesssim C\frac{K_1^2}{K}. \tag{6.14''}$$

Hence $|t_1 - t_2|, |t_1' - t_2'| < C\frac{K_1^2}{K}$, thus $|y_{\alpha_1} - y_{\alpha_2}|, |y_{\alpha_1'} - y_{\alpha_2'}| < C\frac{K_1^2}{K}$.

Consequently

$$(6.12) \lesssim K_1^8 \sum_{\substack{\Omega_\alpha \subset \tilde{\Omega}_\tau \cap \Delta \\ \Omega_{\alpha'} \subset \tilde{\Omega}_{\tau'} \cap \Delta}} \int_{B(a,K)} |(T_\alpha f)(x)|^2 |(T_{\alpha'} f)(x)|^2 dx \tag{6.15}$$

and

$$\begin{aligned} (6.10), (6.11) &\lesssim K^{3(1-\frac{q}{4})} K_1^{2q} K^{\frac{3q}{4}} \Big[\sum_{\Omega_\alpha \subset \tilde{\Omega}_\tau \cap \Delta} c_\alpha^2\Big]^{\frac{q}{4}} \Big[\sum_{\Omega_{\alpha'} \subset \tilde{\Omega}_{\tau'} \cap \Delta} c_{\alpha'}^2\Big]^{\frac{q}{4}} \\ &\lesssim K^3 K_1^{2q} \Big(\frac{K}{K_1}\Big)^{(\frac{q}{2}-1)} \Big[\sum c_\alpha^q\Big] \qquad (6.16) \\ &< K_1^{\frac{3q}{2}+1} K^{\frac{q}{2}-1} \sum_\alpha \int \Big[\int_{B(a,K)} |T_\alpha f(x-z)|^q dx\Big] \zeta_K(z) dz. \qquad (6.16') \end{aligned}$$

Summing over the balls $B(a, K)$ implies an estimate

$$K_1^{\frac{3}{2}+\frac{1}{q}} K^{\frac{1}{2}-\frac{1}{q}} \Big(\sum_\alpha \|T_\alpha f\|^q_{L^q(B_R)}\Big)^{\frac{1}{q}} < K_1^{\frac{3}{2}+\frac{1}{q}} K^{5/q-3/2}\, Q_{R/K}. \tag{6.17}$$

Collecting contributions (4.1), (5.7), (6.7), (6.3), (6.17) implies that

$$Q_R \lesssim C(K)R^\varepsilon + K_1^{-2(1-\frac{3}{q})} Q_{R/K_1} + K^{-2}\, Q_{R/K} + K_1^{\frac{3}{2}+\frac{1}{q}} K^{\frac{5}{q}-\frac{3}{2}}\, Q_{R/K} \tag{6.18}$$

and hence an appropriate choice of K_1, K shows that

$$Q_R \ll R^\varepsilon \text{ for } q > \frac{10}{3}. \tag{6.19}$$

Remark. The use of different scales in previous analysis (and even more so in §3) is reminiscent of the 'induction on scales' approach form [Wo2] and [T1], although the present argument is considerably simpler. In particular, it suffices to take K, K_1 to be large constants, rather than R-dependent (i.e. R^ε-factors), though this point is inessential.

(7). One may also consider the hyperbolic case, for instance

$$\phi(x, y) = x_1y_1 + x_2y_2 + x_3y_1y_2. \tag{7.1}$$

The hyperbolic case was studied by Vargas in [V], adapting the bilinear method. She proved the same estimates in the hyperbolic case that Tao proved in the elliptic case - in particular that the restriction operator is bounded from L^∞ into L^p for $p > 10/3$. Our method gives nearly the same estimate, losing a factor of R^ϵ.

The preceding may be repeated verbatim, except for the analysis of (6.13'). The condition becomes $(v_1^2 + v_2^2 = 1)$

$$|v_1|\ |v_2|\ |t_1^2 - t_2^2 - (t_1')^2 + (t_1')^2| \lesssim C\frac{K_1}{K} \tag{7.2}$$

and the case where v_1 or v_2 is small has to be treated separately.

Suppose $|v_2| < \frac{1}{K_1}$. Let $\Omega = \bigcup_{1\le s\lesssim K_1} \omega_s$ be a partition in horizontal stripes of width $\frac{1}{K_1}$. Recalling (6.1)-(6.3), for $x \in B(a, R)$, the only significant contribution to $Tf(x)$ is given by

$$2\max_s \Big|\int_{\omega_s} e^{i\phi(x,y)} f(y)dy\Big| \lesssim \Big[\sum_s \Big|\int_{\omega_s} e^{i\phi(x,y)} f(y)dy\Big|^q\Big]^{\frac{1}{q}} \tag{7.3}$$

since $\ell = b + tv = b + te_1 + 0(\frac{1}{K_1})$ by assumption on v.

The contribution of (7.3) is at most

$$K_1^{\frac{1}{q}} . \Big\| \int_\omega e^{i\phi(x,y)} f(y) dy \Big\|_{L^q(B_R)} \tag{7.4}$$

where $\omega = [0,1] \times [0, \frac{1}{K_1}]$.

A rescaling $(x, y) \mapsto (x_1, K_1 x_2, K_1 x_3; K_1 x_3; y_1, \frac{1}{K_1} y_2)$ shows that

$$\Big\| \int_\omega e^{i\phi(x,y)} f(y) dy \Big\|_{L^q(B_R)} \leq K_1^{-1+\frac{2}{q}} Q_R$$

which in (6.18) gives an extra term $K_1^{-1+\frac{3}{q}} Q_R$.

§3. Higher Dimensional Restriction Estimates

The method presented in §2 easily generalizes to arbitrary dimension, considering the Fourier restriction/extension problem for a smooth, compact hyper-surface S in $\mathbb{R}^n$ with positive definite second fundamental form. For $x \in S$, denote $x' \in S^{(n-1)}$ the normal vector at the point x and let $\sim : S^{(n-1)} \to S$ be the Gauss map. Thus $\widetilde{x'} = x$.

In this section, we establish Theorem 1′, implying in turn Theorem 1 by the 'ε-removal lemma' presented in the Appendix.

1. Let $U_1, \dots, U_n \subset S$ be small caps such that $|x_1' \wedge \cdots \wedge x_n'| > c$ for $x_i \in U_i$.

Let M be large and $\mathcal{D}_i \subset U_i$ $(1 \leq i \leq n)$ discrete sets of $\frac{1}{M}$-separated points.

Let $B_M \subset \mathbb{R}^n$ be a ball of radius M. Then, for $q = \frac{2n}{n-1}$

$$\fint_{B_M} \prod_{i=1}^n \Big| \sum_{\xi \in \mathcal{D}_i} a(\xi) e^{ix.\xi} \Big|^{q/n} \ll M^\varepsilon \prod_{i=1}^n \Big[\sum_{\xi \in \mathcal{D}_i} |a(\xi)|^2 \Big]^{\frac{q}{2n}}. \tag{1.1}$$

Proof.

This is just a discretized version of Theorem 1.16 in [BCT] as our assumption on $U_1, \dots, U_n$ ensures the required transversality condition (see the discussion in the beginning of §5).

We can assume B_M centered at 0. Introduce functions g_i on U_i defined by

$$\begin{cases} g_i(\zeta) = a(\xi) \text{ if } |\zeta - \xi| < \dfrac{c}{M}, \xi \in \mathcal{D}_i \\ g_i(\zeta) = 0 \ \text{ otherwise} \end{cases} \tag{1.2}$$

($c>0$ a small constant). One may then replace $\sum_{\xi\in\mathcal{D}_i} a(\xi)e^{ix.\xi}$ by $c' M^{n-1}\int_S g_i(\zeta)e^{ix.\zeta}\sigma(d\zeta)$ if $x\in B_M$. Hence

$$\begin{aligned} &\int_{B_M}\prod_{i=1}^n\Big|\sum_{\zeta\in\mathcal{D}_i} a(\xi)e^{ix.\xi}\Big|^{q/n}dx \lesssim \\ &M^{(n-1)q}\int_{B_M}\prod_{i=1}^n\Big|\int_S g_i(\zeta)e^{ix\zeta}\sigma(d\zeta)\Big|^{q/n}dx \overset{\text{[BCT]}}{\ll} \\ &M^{(n-1)q+\varepsilon}\prod_{i=1}^n\|g_i\|^{q/n}_{L^2(U_i)}\sim M^{\frac{n-1}{2}q+\varepsilon}\prod_{i=1}^n\Big[\sum_{\xi\in\mathcal{D}_i}|a(\xi)|^2\Big]^{\frac{q}{2n}}. \end{aligned} \tag{1.3}$$

Since $⨍_{B_M}$ refers to the average, (1.1) follows, since $q=\frac{2n}{n-1}$.

2. Let $S\subset\mathbb{R}^n$ be as above and $2\le m\le n$. Let V be an m-dimensional subspace of $\mathbb{R}^n$, $P_1,\dots,P_m\in S$ such that

$$P'_1,\dots,P'_m\in V \text{ and } |P'_1\wedge\cdots\wedge P'_m|>c \tag{2.1}$$

and $U_1,\dots,U_m\subset S$ sufficiently small neighborhoods of $P_1,\dots,P_m$.

Let M be large and $\mathcal{D}_i\subset U_i$ ($1\le i\le m$) discrete sets of $\frac{1}{M}$-separated points $\xi\in S$ such that $\operatorname{dist}(\xi',V)<\frac{c}{M}$. Let $g_i\in L^\infty(U_i)(1\le i\le m)$. Then letting $q=\frac{2m}{m-1}$

$$\begin{aligned} &⨍_{B_M}\prod_{i=1}^m\Big|\sum_{\xi\in\mathcal{D}_i}\Big(\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta)e^{ix.\zeta}\sigma(d\zeta)\Big)\Big|^{q/m}dx \ll \\ &M^\varepsilon\Big\{⨍_{B_M}\prod_{i=1}^m\Big[\sum_{\xi\in\mathcal{D}_i}\Big|\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta)e^{ix.\zeta}\sigma(d\zeta)\Big|^2\Big]^{1/2m}\Big\}^q. \end{aligned} \tag{2.2}$$

Proof.

Performing a rotation, we may assume $V=[e_1,\dots,e_m]$ and denote $\tilde V$ the image of $V\cap S^{(n-1)}$ under the Gauss map. Let again B_M be centered at 0. For each $\xi\in\bigcup_{i=1}^m\mathcal{D}_i$ there is by assumption some $\hat\xi\in S\cap\tilde V$, $|\xi-\hat\xi|<\frac{c}{M}$. Write

$$\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta)e^{ix.\zeta}\sigma(d\zeta)=e^{ix.\hat\xi}\int_{|\zeta-\xi|<\frac{c}{M}} g_i(\zeta)e^{ix.(\zeta-\hat\xi)}\sigma(d\zeta). \tag{2.3}$$

Since in the second factor of (2.3), $|\zeta - \hat{\xi}| = o(\frac{1}{M})$, we may view it as a constant $a(\xi)$ on $B_M \subset \mathbb{R}^n$.

Thus we need to estimate

$$\fint_{B_M} \Big\{ \prod_{i=1}^m \Big| \sum_{\xi \in \mathcal{D}_i} e^{ix.\hat{\xi}} a(\xi) \Big|^{q/m} \Big\} dx. \tag{2.4}$$

Writing $x = (u, v) \in B_M^{(m)} \times B_M^{(n-m)}$, (2.4) may be bounded by

$$\max_{v \in B_M^{(n-m)}} \fint_{B_M^{(m)}} \Big\{ \prod_{i=1}^m \Big| \sum_{\xi \in \mathcal{D}_i} e^{iu.\pi_m(\hat{\xi})} a_v(\xi) \Big|^{q/m} \Big\} du \tag{2.5}$$

with $a_v(\xi) = e^{iv.\hat{\xi}} a(\xi)$.

Since S has positive definite second fundamental form, $\pi_m(S \cap \tilde{V}) \subset V = [e_1, \dots, e_m]$ is a hypersurface in V with same property and the normal vector at $\pi_m(\hat{\xi}) = (\hat{\xi})' \in V$. Since (2.1), application of (1.1) with n replaced by m and $\mathcal{D}_i$ by $\{\pi_m \hat{\xi}; \xi \in \mathcal{D}_i\}$ gives the estimate on (2.5)

$$\ll M^\varepsilon \prod_{i=1}^m \Big[\sum_{\xi \in \mathcal{D}_i} |a(\xi)|^2 \Big]^{q/2m}$$

and (2.2) follows.

3. Essential use is made of scaling.

Denote $Q_R^{(p)}$ a bound on

$$\Big\| \int_S g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_R)}$$

with $g \in L^\infty(S), |g| \le 1$ and with S as specified in the beginning of §3.

Parametrize S (locally) as

$$\begin{cases} \xi_i = y_i \quad (1 \le i \le n-1) \\ \xi_n = y_1^2 + \cdots + y_{n-1}^2 + O(|y|^3) \end{cases} \tag{3.1}$$

with y taken in a small neighborhood of 0.

Let U_ρ be a ρ-cap on S and evaluate

$$\Big\| \int_{U_\rho} g(\xi) e^{ix.\xi} \sigma(d\xi) \Big\|_{L^p(B_R)}.$$

Thus in (3.1) we restrict y to a ball $B(a,\rho)\subset\mathbb{R}^{n-1}$ and evaluate

$$\Big\|\int_{B(a,\rho)} g(y)e^{i[x_1y_1+\cdots+x_{n-1}y_{n-1}+x_n(|y|^2+O(|y|^3))]}dy\Big\|_{L^p(B_R)}. \tag{3.2}$$

A shift $y\mapsto y-a$ and a change of variables $x_i'=x_i+x_n(2a_i+\cdots)(1\le i<n)$ permits to set $a=0$. Rescale $y=\rho y'$ to obtain

$$\rho^{n-1}\Big\|\int_{B(0,1)} g(\rho y')e^{i[\rho x_1y_1'+\cdots+\rho x_{n-1}y_{n-1}'+\rho^2x_n(|y'|^2+\rho O(|y'|^3))]}dy'\Big\|_{L^p(B_R)}$$

and a further rescaling in x, $x_i'=\rho x_i(1\le i\le n-1), x_n'=\rho^2x_n$, gives

$$\begin{aligned}&\rho^{n-1-(n+1)/p}\Big\|\int_{B(0,1)} g(\rho y')e^{i[x_1'y_1'+\cdots+x_{n-1}'y_{n-1}'+x_n'(|y'|^2+\rho O(|y'|^3))]}dy'\Big\|_{L^p(B_{\rho R})}\\ &\le \rho^{n-1-(n+1)/p}\ Q^{(p)}_{\rho R}\end{aligned} \tag{3.3}$$

4. Let $g\in L^\infty(S), |g|\le 1$ and consider for $x\in B_R$

$$\int_S g(\xi)e^{ix.\xi}\sigma(d\xi). \tag{4.1}$$

Let

$$R^\varepsilon\gg K_n\gg K_{n-1}\gg\cdots\gg K_1$$

be suitably chosen.

Start decomposing $S=\bigcup_\alpha U_\alpha(\frac{1}{K_n})$ in caps of size $\frac{1}{K_n}$ and write

$$(4.1)=\sum_\alpha\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)=\sum_\alpha c_\alpha(x).$$

Fixing x, there are 2 possibilities

(4.2) There are $\alpha_1,\alpha_2,\dots,\alpha_n$ such that

$$|c_{\alpha_1}(x)|,\dots,|c_{\alpha_n}(x)|>K_n^{-n}\max_\alpha|c_\alpha(x)| \tag{4.3}$$

and

$$|\xi_1'\wedge\cdots\wedge\xi_n'|>c(K_n)\text{ for }\xi_i\in U_{\alpha_i}. \tag{4.4}$$

(4.5) The negation of (4.2), which implies that there is an $(n-1)$-dim subspace V_{n-1} such that

$$|c_\alpha(x)| \le K_n^{-n} \max_\alpha |c_\alpha(x)| \ \text{ if } \ \operatorname{dist}(U_\alpha, \tilde V_{n-1}) \gtrsim \frac{1}{K_n}.$$

If (4.2), clearly by (4.3)

$$\Big|\int_S g(\xi)e^{ix.\xi}\sigma(d\xi)\Big| \le K_n^{n-1}\max|c_\alpha(x)| \le K_n^{2n-1}\Big[\prod_{i=1}^n |c_{\alpha_i}(x)|\Big]^{\frac{1}{n}}$$

and

$$\int_{x(4.2)}\Big|\int_S g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p \lesssim K_n^{p(2n-1)} \sum_{\substack{\alpha_1,\dots,\alpha_n\\(4.4)}} \int_{B_R}\prod_{i=1}^n\Big|\int_{U_{\alpha_i}(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{\frac{p}{n}}. \tag{4.6}$$

In view of (4.4), the [BCT]-estimate applies to each (4.6) term. Thus

$$\int_{B_R}\prod_{i=1}^n\Big|\int_{U_{\alpha_i}(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{\frac{2}{n-1}}dx \ll C(K_n)R^\varepsilon. \tag{4.7}$$

Assuming

$$p \ge \frac{2n}{n-1} \tag{4.8}$$

we see that

$$(4.6) < C(K_n)R^\varepsilon \tag{4.9}$$

(here and in the sequel, $C(K)$ refers to some power of K.)

Next consider the case (4.5). Thus

$$\begin{aligned}|(4.1)| &\le \Big|\int_{\operatorname{dist}(\xi,\tilde V_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big| + \frac{1}{K_n}\max_\alpha\Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|\\ &= (4.10)+(4.11)\end{aligned}$$

where V_{n-1} depends on x.

Note that, using the argument explained earlier in §1, we may view $|c_\alpha(x)|$ as essentially constant on balls of size K_n (literally speaking, this is of course incorrect and what was done is a replacement of $|c_\alpha(x)|$ by a majorant $|c_\alpha| * \eta_{K_n}, \eta_K(x) = \frac{1}{K^d}\eta(\frac{x}{K})$ and η a suitable bump-function – we do not repeat these technicalities here.)

Thus the bound (4.10) + (4.11) may be considered valid on $B(x, K_n)$, with a same linear space V_{n-1}.

The contribution of (4.11) to $\| \int g(\xi)e^{ix.\xi}\sigma(d\xi)\|_p$ is bounded by

$$\frac{1}{K_n}\Big(\sum_\alpha \Big\| \int_{U_\alpha} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big\|_p^p\Big)^{\frac{1}{p}} \lesssim \frac{1}{K_n}.K_n^{\frac{n-1}{p}}.\Big(\frac{1}{K_n}\Big)^{n-1-\frac{n+1}{p}} Q^{(p)}_{R/K_n}$$
$$= \Big(\frac{1}{K_n}\Big)^{n(1-\frac{2}{p})} Q^{(p)}_{R/K_n} < \frac{1}{K_n} Q^{(p)}_R. \tag{4.12}$$

Consider the term (4.10). Proceeding similarly, write for $x \in B(\bar{x}, K_n)$

$$\int_{\text{dist }(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi) =$$
$$\sum_\alpha \int_{U_\alpha(\frac{1}{K_{n-1}})\cap[\text{dist }|\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi) = \sum_\alpha c_\alpha^{(n-1)}(x). \tag{4.13}$$

We distinguish the cases

(4.14) There are $\alpha_1, \dots, \alpha_{n-1}$ such that

$$|c^{(n-1)}_{\alpha_1}(x)|, \dots, |c^{(n-1)}_{\alpha_{n-1}}(x)| > K_{n-1}^{-(n-1)} \max_\alpha |c_\alpha^{(n-1)}(x)| \tag{4.15}$$

and

$$|\xi_1' \wedge \dots \wedge \xi'_{n-1}| > c(K_{n-1}) \ \text{ for } \ \xi_i \in U_{\alpha_i}\Big(\frac{1}{K_{n-1}}\Big). \tag{4.16}$$

(4.17) Negation of (4.14), implying that there is an $(n-2)$-dim subspace $V_{n-2} \subset V_{n-1}$ (depending on x) such that

$$|c_\alpha^{(n-1)}(x)| < K_{n-1}^{-(n-1)} \max_\alpha |c_\alpha^{(n-1)}(x)| \text{ for dist }(U_\alpha, \tilde{V}_{n-2}) \gtrsim \frac{1}{K_{n-1}}.$$

This space V_{n-2} can then again be taken the same on a K_{n-1}-neighborhood of x.

We analyze the contribution of (4.14). By (4.15)

$$|(4.13)| < K_{n-1}^{2n-3}\Big[\prod_{i=1}^{n-1} |c^{(n-1)}_{\alpha_i}(x)|\Big]^{\frac{1}{n-1}} \tag{4.18}$$

and hence

$$\int\limits_{\substack{B(\bar{x},K_n)\\ x \text{ satisfies } (4.14)}} \Big| \int\limits_{\text{dist}\,(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p \leq$$

$$K_{n-1}^{p(2n-3)} \sum_{\substack{\alpha_1,\dots,\alpha_{n-1}\\ (4.16)}} \int_{B(\bar{x},K_n)} \Big\{ \prod_{i=1}^{n-1} \Big| \int_{U_{\alpha_i}(\frac{1}{K_{n-1}})\cap[\text{dist}\,(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^{p/n-1}\Big\} \tag{4.19}$$

We use the bound (2.2) to estimate the individual integrals

$$\int_{B(\bar{x},K_n)} \Big\{ \prod_{i=1}^{n-1} \Big| \int_{U_{\alpha_i}(\frac{1}{K_{n-1}})\cap[\text{dist}\,(\xi,\tilde{V}_{n-1})\lesssim\frac{1}{K_n}]} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|\Big\}^{\frac{q}{n-1}} \text{ with } q=\frac{2(n-1)}{n-2}. \tag{4.20}$$

Thus $m=n-1, V=V_{n-1}$ and P_i is the center of $U_{\alpha_i}(\frac{1}{K_{n-1}})$. Let $M=K_n$ and $\mathcal{D}_i$ the centers of a cover of $U_{\alpha_i}(\frac{1}{K_{n-1}})$ by caps $U_\alpha(\frac{1}{K_n})$.

By (2.2) we get an estimate

$$(4.20) \ll K_n^\varepsilon C(K_{n-1}) \Big\{ \int_{B(\bar{x},K_n)} \prod_{i=1}^{n-1} \Big[\sum_\alpha^{(i)} \Big| \int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2(n-1)}}\Big\}^q \tag{4.21}$$

where in $\sum^{(i)}$ the sum is over those α such that $U_\alpha(\frac{1}{K_n})\subset U_{\alpha_i}(\frac{1}{K_{n-1}})$ and $U_\alpha(\frac{1}{K_n})\cap\tilde{V}_{n-1}\neq\phi$. Clearly

$$(4.21) \ll K_n^\varepsilon\ C(K_{n-1}) \Big\{ \int_{B(\bar{x},K_n)} \Big[\sum_{U_\alpha(\frac{1}{K_n})\cap\tilde{V}_{n-1}\neq\phi} \Big| \int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2}}\Big\}^q. \tag{4.22}$$

If

$$p \geq \frac{2(n-1)}{n-2} = q \tag{4.23}$$

the contribution of (4.15) may be estimated replacing p by $q=\frac{2(n-1)}{n-2}$, and using the [BCT] bound (4.7) with n replaced by $n-1$ and K_n by K_{n-1}. This gives a bound R^ε.

Thus we assume

$$p < \frac{2(n-1)}{n-2}. \tag{4.24}$$

Then

$$(4.19)^{1/p} \ll C(K_{n-1})K_n^\varepsilon \int_{B(\bar x, K_n)} \Big[\sum_{U_a(\frac{1}{K_n})\cap \tilde V_{n-1}\neq\phi} \Big| \int_{U_\alpha(\frac{1}{K_n})} g(\xi) e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac 12}. \tag{4.25}$$

Note that $U_\alpha(\frac{1}{K_n})\cap \tilde V_{n-1}\neq \phi$ for $\sim K_n^{n-2}$ values of α.

Hence, by Hölder's inequality, the integrand in (4.25) is at most

$$K_n^{(n-2)(\frac 12-\frac 1p)}\Big[\sum_\alpha \Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p\Big]^{\frac 1p} \tag{4.26}$$

where α is unrestricted in the α-summation. Substituting (4.26) in (4.25) gives

$$(4.19) \ll C(K_{n-1})K_n^{(n-2)(\frac p2-1)+\varepsilon} \int_{B(\bar x,K_n)} \Big[\sum_\alpha \Big|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^p\Big]$$

and integrating over B_R permits to bound the (4.14)-contribution by

$$C(K_{n-1})K_n^{(n-2)(\frac 12-\frac 1p)+\varepsilon}\Big[\sum_\alpha \Big\|\int_{U_\alpha(\frac{1}{K_n})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big\|^p_{L^p(B_R)}\Big]^{1/p}. \tag{4.27}$$

Invoking again the rescaling inequality (3.3), this gives

$$C(K_{n-1})K_n^{(n-2)(\frac 12-\frac 1p)+\frac{n-1}{p}-(n-1)+\frac{n+1}{p}+\varepsilon}Q_{R/K_n} = C(K_{n-1})K_n^{\frac{n+2}{p}-\frac n2+\varepsilon}. \tag{4.28}$$

Taking K_n sufficiently large compared with K_{n-1}, we see that the (4.14)-contribution is taken care of if either $p\geq \frac{2(n-1)}{n-2}$ or

$$p>2+\frac 4n. \tag{4.29}$$

Thus we impose

$$p>\min\Big(\frac{2(n-1)}{n-2}, \frac{2(n+2)}{n}\Big). \tag{4.30}$$

Next we need to consider the contribution of (4.17).

The analysis is analogous to the preceding, replacing $n-1$ by $n-2$ and K_n byK_{n-1}. More precisely, if

$$p<\frac{2(n-2)}{n-3} \tag{4.31}$$

the local estimate (4.25) becomes

$$c(K_{n-2})K_{n-1}^{\varepsilon} \int_{B(\bar{x},K_{n-1})} \Big[\sum_{U_\alpha(\frac{1}{K_{n-1}})\cap \tilde{V}_{n-2}\neq\phi} \Big| \int_{U_\alpha(\frac{1}{K_{n-1}})} g(\xi)e^{ix.\xi}\sigma(d\xi)\Big|^2\Big]^{\frac{1}{2}} \tag{4.32}$$

and $U_\alpha(\frac{1}{K_{n-1}}) \cap \tilde{V}_{n-2} \neq \phi$ for $\sim K_{n-1}^{n-3}$ values of α.

This leads to the condition on p

$$p > \min\Big(\frac{2(n-2)}{n-3}, \frac{2(n+3)}{n+1}\Big). \tag{4.33}$$

The continuation of the process is clear.

Eventually we see that the exponent p needs to satisfy

$$p > 2\min\Big\{\frac{k}{k-1}, \frac{2n-k+1}{2n-k-1}\Big\} \text{ for all } 2 \leq k \leq n. \tag{4.34}$$

Hence we obtain.

Theorem 1'.

$Q_R^{(p)} \ll R^\varepsilon$ provided

$$p \geq 2\frac{4n+3}{4n-3} \qquad \text{if } n \equiv 0 \pmod 3$$

$$p \geq \frac{2n+1}{n-1} \qquad \text{if } n \equiv 1 \pmod 3$$

$$p \geq \frac{4(n+1)}{2n-1} \qquad \text{if } n \equiv 2 \pmod 3.$$

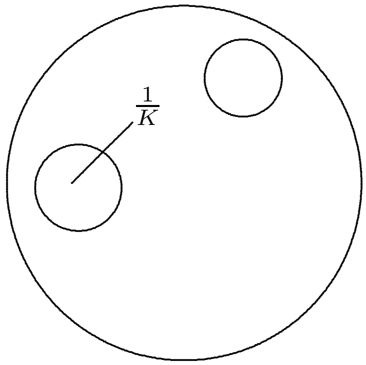
§4 Improving Upon the Exponent in the 3D Restriction Problem

We consider the case of the paraboloid (though the argument generalizes).

Going back to the analysis in §2, the main idea is to collect the contributions obtained at different scales, rather than performing an induction on scale argument. This will allow us to bring into play also T. Wolff's $\frac{5}{2}$-bound for the Kakeya maximal function. (see [Wo1]).

1. Representation at scale 1

Fix large parameters $K \gg K_1 \gg 1$



$$f_i = f|_{B(a_i, \frac{1}{K})}$$

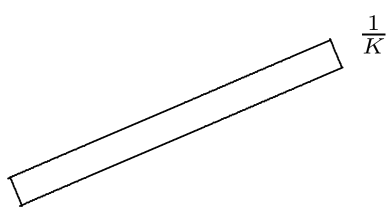
Recalling the analysis in §2, we have

$$|Tf| \leq C(K) \max_{\substack{i_1, i_2, i_3 \\ non-collinear}} (|Tf_{i_1}|.|Tf_{i_2}|\ |Tf_{i_3}|)^{\frac{1}{3}} + \max_{\substack{\mathcal{L} \\ \text{dist}\,(\mathcal{L}', \mathcal{L}'') > \frac{1}{K_1}}} \Big|\sum_{i \in \mathcal{L}'} Tf_i\Big|^{\frac{1}{2}} \Big|\sum_{i \in \mathcal{L}''} Tf_i\Big|^{\frac{1}{2}}$$

$$+ \max_a \big|T(f|_{B(a, \frac{1}{K_1})})\big|$$

$$= (1.1) + (1.2) + (1.3).$$

Here $\mathcal{L}', \mathcal{L}'' \subset \mathcal{L}$ are separated segments of a 'line' $\mathcal{L}$.



Since

$$\Big[\fint_{B(a,K)} (1.2)^4 \Big]^{\frac{1}{4}} \leq C(K_1) \Big(\sum_{i \in \mathcal{L}} |Tf_i|^2 \Big)^{\frac{1}{2}}$$

we may write

$$(1.2) = \phi. \Big(\sum_{i \in \mathcal{L}} |Tf_i|^2 \Big)^{\frac{1}{2}}$$

with

$$\left(\fint_{B(a,K)}|\phi|^4\right)^{\frac{1}{4}} < c(K_1)$$

and ϕ constant on balls of radius 1.

In what follows, we identify small discs $\subset \Omega$ and the corresponding caps $\subset S$ obtained as image under the map $y \mapsto (y_1, y_2, y_1^2 + y_2^2)$, which are both denoted by τ.

2. Representation of Tf_τ (by rescaling).

Let τ be a δ-cap and rescale.

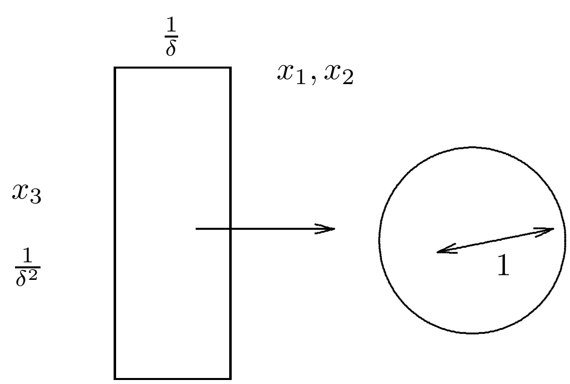
Up to linear transformation of the form

$$\begin{cases} x_1' = x_1 + a_1 x_3 \\ x_2' = x_2 + a_2 x_3 \\ x_3' = x_3 \end{cases}$$

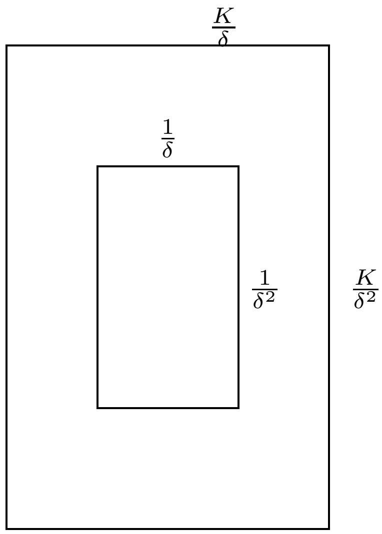
and reduction to scale 1 by transformation

$$\begin{cases} x_1' = \delta x_1 \\ x_2' = \delta x_2 \\ x_3' = \delta^2 x_3 \end{cases}$$

we obtain



Applying at unit scale the representation from (1) and scaling back, we obtain on the $(\frac{K}{\delta} \times \frac{K}{\delta} \times \frac{K}{\delta^2})$-box



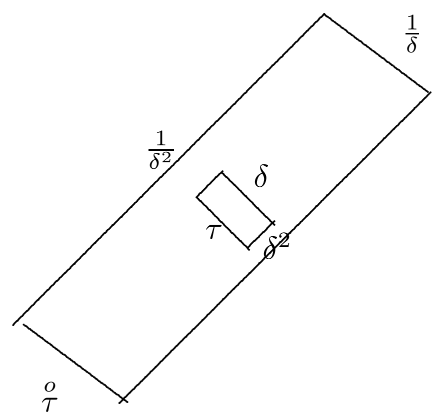
$$|Tf_\tau| \le C(K) \max_{\substack{\tau_1,\tau_2,\tau_3 \\ \text{non-collinear}}} |Tf\tau_1|^{\frac{1}{3}} |Tf\tau_2|^{\frac{1}{3}} . |Tf\tau_3|^{\frac{1}{3}} \tag{2.1}$$

$$+\phi_\tau \max_{\mathcal{L}} \Big(\sum_{\tau_i \in \mathcal{L}} |Tf\tau_i|^2 \Big)^{\frac{1}{2}} \text{ where } \tau_i \text{ is a } \tfrac{\delta}{K}\text{-cap} \tag{2.2}$$

$$+ \max_{\substack{\tau' \subset \tau \\ \delta/K_1\text{-cap}}} |Tf\tau'| \tag{2.3}$$

Given a δ-cap τ, denote $\overset{o}{\tau}$ the polar set

$$\tau \to \overset{o}{\tau} = (\tfrac{1}{\delta} \times \tfrac{1}{\delta} \times \tfrac{1}{\delta^2}) \text{ box}$$



On every $K\overset{o}{\tau}$-box B, ϕ_τ satisfies

$$\begin{aligned} \fint_B \phi_\tau^4 &= \frac{1}{|B|} \int_B \phi_\tau^4 \\ &= \frac{\delta^4}{K^3} . \delta^{-4} \int_{B(a,K)} \phi_\tau(\delta^{-1}x_1', \delta^{-1}x_2', \delta^{-2}x_3')^4 \, dx_1' dx_2' dx_3' \qquad (2.4) \\ &< C(K_1) \end{aligned}$$

and ϕ_τ is essentially constant on $\overset{o}{\tau}$-boxes.

3. Iteration

Apply the decomposition (2.1)-(2.3) to each Tf_{τ_i} in (2.2) and $Tf_{\tau'}$ in (2.3).

Considering Tf_{τ_i}, let ϕ_{τ_i} be the corresponding factor appearing in (2.2).

Thus ϕ_{τ_i} is constant on $\overset{o}{\tau}_i$-boxes and $⨍_{B'} \phi^4_{\tau_i} < C(K_1)$ if B' is a $K\overset{o}{\tau}_i$-box.

Let B' be a $K\overset{o}{\tau}_i$-box and subdivide B' as

$$B' = \bigcup B'_\alpha$$

with B'_α $\overset{o}{\tau}_i$-boxes. Then

$$\int_{B'} \phi^4_\tau \phi^4_{\tau_i} \sim \sum_\alpha \Big[\phi_{\tau_i}\Big|_{B'_\alpha}\Big]^4 \int_{B'_\alpha} \phi^4_\tau. \tag{3.1}$$

Note that $\overset{o}{\tau}_i$ is an $\big[\frac{K}{\delta} \times \frac{K}{\delta} \times \frac{K^2}{\delta^2}\big]$-box in direction ξ_i-normal at τ_i. Let ξ be any normal for τ. Thus $\measuredangle(\xi, \xi_i) < \delta$ and $K\overset{o}{\tau}$ is contained in $\big[2\frac{K}{\delta} \times 2\frac{K}{\delta} \times 2\frac{K}{\delta^2}\big]$-box in direction ξ_i. It follows that $\overset{o}{\tau}_i$ may be partitioned in $K\overset{o}{\tau}$-boxes B and hence by (2.4)

$$⨍_{B'_\alpha} \phi^4_\tau \le \max_B ⨍_B \phi^4_\tau < C(K_1). \tag{3.2}$$

Substituting (3.2) in (3.1) gives

$$C(K_1) \sum_\alpha \int_{B'_\alpha} \phi^4_{\tau_i} = C(K_1) \int_{B'} \phi^4_{\tau_i} < C(K_1)^2 |B'|. \tag{3.3}$$

Note also that in (2.2) $\mathcal{L}$ consists of at most K $\frac{\delta}{K}$-discs. Iteration of (2.1)-(2.3), where we terminate the process for (2.1) and continue for (2.2), gives a representation

$$|Tf| \le$$

$$R^\varepsilon \max_{1>\delta>\frac{1}{\sqrt{R}}} \max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} \Big(\phi_\tau |Tf_{\tau_1}|^{1/3} |Tf_{\tau_2}|^{1/3} |Tf_{\tau_3}|^{1/3}\Big)^2\Big]^{\frac{1}{2}} \tag{3.4}$$

$$+ \max_{\mathcal{E}_{\frac{1}{\sqrt{R}}}} \Big[\sum_{\tau\in\mathcal{E}} (\phi_\tau |Tf_\tau|)^2\Big]^{\frac{1}{2}} \tag{3.5}$$

where

(3.6) $\mathcal{E}_\delta$ consists of at most $\frac{1}{\delta}$ disjoint δ-caps τ

(3.7) $\tau_1, \tau_2, \tau_3 \subset \tau$ are $\frac{1}{K\delta}$-size and non-collinear

(3.8) $\fint_B \phi_\tau^4 < C(K_1)^{\frac{\log\frac{1}{\delta}}{\log K}} < R^{\frac{\log C(K_1)}{\log K}} \ll R^\varepsilon$ if B is a $\overset{o}{\tau}$-box.

Fix dyadic $1 > \delta > \frac{1}{\sqrt{R}}$ and consider

$$\max_{\mathcal{E}_\delta} \Big[\sum_{\tau \in \mathcal{E}_\delta} (\phi_\tau |Tf_{\tau_1}|^{1/3} \, |Tf_{\tau_2}|^{1/3} \, |Tf_{\tau_3}|^{1/3})^2 \Big]^{\frac{1}{2}} \tag{3.9}$$

with $\mathcal{E}_\delta$ and τ_1, τ_2, τ_3 as above.

In what follows, we will make several estimates on (3.9) considering various norms.

4. We assume $|f| \leq 1$. By rescaling, for $\tau_1, \tau_2, \tau_3 \subset \tau$ as in (3.7),

$$\int_{B_R} |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}| \leq \delta^2 \int_{B_{\delta R}} |Tg_{U_1}| \, |Tg_{U_2}| \, |Tg_{U_3}|. \tag{4.1}$$

with $|g| < 1$ and $U_1, U_2, U_3 \subset B_1$ of size $\sim \frac{1}{K}$ and not collinear.

Hence, from [BCT]

$$\int_{B_{\delta R}} |Tg_{U_1}| \, |Tg_{U_2}| \, |Tg_{U_3}| \ll R^\varepsilon \tag{4.2}$$

and

$$\int_{B_R} |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}| \ll \delta^2 R^\varepsilon. \tag{4.3}$$

By (3.6) and Hölder

$$\Big[\sum_{\tau \in \mathcal{E}_\delta} \big(\phi_\tau |Tf_{\tau_1}|^{\frac{1}{3}}.|Tf_{\tau_2}|^{\frac{1}{3}}.|Tf_{\tau_3}|^{\frac{1}{3}}\big)^2 \Big]^{\frac{1}{2}} \leq$$

$$|\mathcal{E}_\delta|^{\frac{1}{6}} \Big[\sum_\tau \phi_\tau^3 |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}| \Big]^{\frac{1}{3}} \leq$$

$$\delta^{-\frac{1}{6}} \Big[\sum_\tau \phi_\tau^3 |Tf_{\tau_1}| \, |Tf_{\tau_2}| \, |Tf_{\tau_3}| \Big]^{1/3} \tag{4.4}$$

where in (4.4) τ ranges over a partition in δ-discs (note that (4.4) does not depend on $\mathcal{E}_\delta$ anymore).

We obtain

$$\|(3.9)\|_{L^3(B_R)} \leq \delta^{-\frac{1}{6}} \Big[\sum_\tau \int \phi_\tau^3 |Tf_{\tau_1}|\ |Tf_{\tau_2}|\ |Tf_{\tau_3}|\Big]^{1/3}. \tag{4.5}$$

Consider a partition of B_R in $\overset{o}{\tau}$-boxes B. Since $|Tf_{\tau_i}|$ are $\approx$ constant on $\overset{o}{\tau}_i$-boxes, hence on each B,

$$\begin{aligned}\int \phi_\tau^3 |Tf_{\tau_1}|\ |Tf_{\tau_2}|\ |Tf_{\tau_3}| &\approx \sum_B (|Tf_{\tau_1}|\ |Tf_{\tau_2}|\ |Tf_{\tau_3}|)\Big|_B \Big(\int_B \phi_\tau^3\Big)\\ &\approx \sum_B \Big[\int_B |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|\Big] \fint_B \phi_\tau^3\\ &\overset{(3.8)}{\ll} R^\varepsilon \int_{B_R} |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|\\ &\overset{(4.3)}{<} R^\varepsilon \delta^2.\end{aligned} \tag{4.6}$$

Therefore

$$\|(3.9)\|_{L^3(B_R)} \ll R^\varepsilon \delta^{-\frac{1}{6}} \tag{4.7}$$

which is our first bound.

5. Take $3 \leq p \leq 4$.

By Hölder again

$$(5.1) = \max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_\tau |Tf_{\tau_1}|^{1/3} |Tf_{\tau_2}|^{1/3} |Tf_{\tau_3}|^{1/3})^2\Big]^{1/2} \leq$$
$$\Big(\frac{1}{\delta}\Big)^{\frac{1}{2}-\frac{1}{p}} \Big[\sum_\tau \phi_\tau^p (|Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|)^{\frac{p}{3}}\Big]^{\frac{1}{p}}$$

implying

$$\|(5.1)\|_p \leq \Big(\frac{1}{\delta}\Big)^{\frac{1}{2}-\frac{1}{p}} \Big[\sum_\tau \int_{B_R} \phi_\tau^p (|Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|)^{p/3}\Big]^{\frac{1}{p}}. \tag{5.2}$$

As in (4.6)

$$\int_{B_R} \phi_\tau^p (|Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|)^{p/3} \leq$$

$$\Big[\max_{B\ \overset{o}{\tau}-\text{box}} \fint_B \phi_\tau^p\Big] \Big[\int_{B_R} (|Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|)^{p/3}\Big] \leq$$

$$R^\varepsilon \Big[\int_{B_R} |Tf_{\tau_1}|.|Tf_{\tau_2}|.|Tf_{\tau_3}|\Big].\delta^{6(\frac{p}{3}-1)}$$

$$< R^\varepsilon \delta^{2p-4} \tag{5.3}$$

by (3.8), (4.3) and since $\|Tf_{\tau_1}\|_\infty < \delta^2$.

Substituting (5.3) in (5.2) gives

$$R^\varepsilon \Big(\frac{1}{\delta}\Big)^{\frac{1}{2}-\frac{1}{p}} \Big(\frac{1}{\delta}\Big)^{\frac{2}{p}} \delta^{2-\frac{4}{p}} = R^\varepsilon \delta^{\frac{3}{2}-\frac{5}{p}}. \tag{5.4}$$

Hence

$$\|(3.9)\|_{L^p(B_R)} \ll R^\varepsilon \text{ for } p \geq \frac{10}{3} = p_0. \tag{5.5}$$

Returning to (5.1), let $0 < \lambda < 1$ be a parameter and denote

$$g_\tau = |Tf_{\tau_1}|^{1/3}.|Tf_{\tau_2}|^{1/3}.|Tf_{\tau_3}|^{1/3} \text{ and } \ g_{\tau,\lambda} = g_\tau 1_{[g_\tau \sim \lambda\delta^2]} \tag{5.6}$$

Then by (4.3)

$$\int_{B_R} [g_{\tau,\lambda}]^p < (\lambda\delta^2)^{p-3} \int_{B_R} (g_{\tau,\lambda})^3 \ll R^\varepsilon \lambda^{p-3} \delta^{2p-4} \tag{5.7}$$

and

$$\Big\{\int_{B_R} \max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_\tau g_{\tau,\lambda})^2\Big]^{p_0/2}\Big\}^{1/p_0} \ll R^\varepsilon \lambda^{1-\frac{3}{p_0}} = R^\varepsilon \lambda^{\frac{1}{10}}. \tag{5.8}$$

Let $1 \leq \mu < \infty$ be another parameter and decompose each ϕ_τ as

$$\phi_\tau = \sum_{\mu \text{ dyadic}} \phi_{\tau,\mu} \text{ where}$$

$$\begin{aligned} \phi_{\tau,\mu} &= \phi_\tau 1_{[\phi_\tau \sim \mu]} \\ \phi_{\tau,1} &= \phi_\tau 1_{[\phi_\tau \leq 1]} \end{aligned} \tag{5.9}$$

If B is a $\overset{o}{\tau}$-box, (3.8) implies for $\mu > 1$

$$\fint_B \phi_{\tau,\mu}^{p_0} \leq \mu^{-4+p_0} \fint_B \phi_\tau^4 \ll R^\varepsilon \mu^{-2/3}. \tag{5.10}$$

Hence, instead of (5.8), we obtain

$$\Big\{\int_{B_R}\max_{\mathcal{E}_\delta}\Big[\sum_{\tau\in\mathcal{E}_\delta}(\phi_{\tau,\mu}g_{\tau,\lambda})^2\Big]^{p_0/2}\Big\}^{1/p_0}\ll R^\varepsilon\lambda^{\frac{1}{10}}.\mu^{-\frac{1}{5}}. \tag{5.11}$$

Next, we perform a different type of estimate. Clearly

$$\max_{\mathcal{E}_\delta}\Big[\sum_{\tau\in\mathcal{E}_\delta}(\phi_{\tau,\mu}g_{\tau,\lambda})^2\Big]^{\frac{1}{2}}\le\mu\Big(\sum_\tau g^2_{\tau,\lambda}\Big)^{\frac{1}{2}} \tag{5.12}$$

with τ ranging over a partition in δ-caps.

We apply the usual procedure to bound (5.12) by a Kakeya maximal function.

Writing

$$|Tf_{\tau_i}|\lesssim|Tf_{\tau_i}|*(\delta^4 1_{\overset{\circ}{\tau}})$$

we have

$$\begin{aligned} g_\tau(x)&\lesssim\int\Big\{\prod_{i=1}^3[|Tf_{\tau_i}|*(\delta^4 1_{\overset{\circ}{\tau}})]^{\frac{1}{3}}\Big\}(z)\ (\delta^4 1_{\overset{\circ}{\tau}})(x-z)dz\\ &=\int\omega(z)(\delta^4 1_{\overset{\circ}{\tau}})(x-z)dz \end{aligned}\tag{5.13}$$

and

$$g^2_{\tau,\lambda}(x)\lesssim\delta^4\int(\omega^2 1_{[\omega\gtrsim\lambda\delta^2]})(z)1_{\overset{\circ}{\tau}}(x-z)dz. \tag{5.14}$$

Further

$$\begin{aligned}\int_{B_R}\omega^2\ 1_{[\omega\gtrsim\lambda\delta^2]}&\lesssim\frac{1}{\lambda\delta^2}\int\omega^3\\ &\lesssim\frac{1}{\lambda\delta^2}\int\Big\{\int\Big[\prod_{i=1}^3|Tf_{\tau_i}|(x-\tau_i)\Big]^{\frac{1}{3}}dx\Big\}\Big[\prod_{i=1}^3(\delta^4 1_{\overset{\circ}{\tau}})(\tau_i)\Big]dz_1dz_2dz_3\\ &\ll R^\varepsilon\lambda^{-1}. \end{aligned}\tag{5.15}$$

Hence, from (5.14), (5.15), we obtain a representation

$$g^2_{\tau,\lambda}\ll R^\varepsilon\delta^4\lambda^{-1}\int 1_{\overset{\circ}{\tau}}(\cdot-y)\mathbb{P}_\tau(dy). \tag{5.16}$$

¿From (5.16) and convexity

$$\|(5.12)\|_{L^{p_0}(B_R)} \ll R^\varepsilon \lambda^{-\frac{1}{2}}\mu\delta^2 \Big\| \Big[\sum_\tau 1_{\overset{o}{\tau}}(x-y_\tau)\Big]^{\frac{1}{2}}\Big\|_{L^{p_0}(B_R)}$$
$$= R^\varepsilon \lambda^{-\frac{1}{2}}\mu\delta^2 \Big[\int \Big[\sum_\tau 1_{\overset{o}{\tau}}(x-y_\tau)\big]^{5/3} dx\Big]^{3/10} \tag{5.17}$$

for some choice of $\{y_\tau\}$-points in $\mathbb{R}^3$.

At this point we can invoke the $L^{5/2}$-bound for the $\mathbb{R}^3$-Kakeya maximal function. In its dual formulation, we have

$$\Big\|\sum_{v\in\mathfrak{S}} 1_{T_v}\Big\|_{L^{5/3}} \leq \Big(\frac{1}{\kappa}\Big)^{\frac{1}{5}+} \tag{5.18}$$

where T is a translate of a tube of width κ and length 1 in direction $v \in \mathfrak{S} \subset S_2$, where $\mathfrak{S}$ consists of κ-separated points.

Rescaling by a factor δ^2 and applying (5.18) with $\kappa = \delta$, it follows

$$\Big\|\sum_\tau 1_{\overset{o}{\tau}}(\cdot - y_\tau)\Big\|_{L^{5/3}} \ll \delta^{-\frac{19}{5}-}. \tag{5.19}$$

Hence

$$(5.17) \ll R^\varepsilon \lambda^{-\frac{1}{2}}\mu\delta^{\frac{1}{10}}. \tag{5.20}$$

which is our final estimate.

Summarizing (4.7), (5.11), (5.20), we have

$$\|\max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_\tau g_\tau)^2\Big]^{\frac{1}{2}}\Big\|_{L^3(B_R)} \ll R^\varepsilon \delta^{-\frac{1}{6}} \tag{5.21}$$

and

$$\Big\|\max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_{\tau,\mu} g_{\tau,\lambda})^2\Big]^{\frac{1}{2}}\Big\|_{L^{10/3}(B_R)} \ll R^\varepsilon \min\big(\lambda^{\frac{1}{10}}\mu^{-\frac{1}{5}}, \lambda^{-\frac{1}{2}}\mu\delta^{\frac{1}{10}}\big)$$
$$\ll R^\varepsilon \delta^{\frac{1}{60}} \tag{5.22}$$

Let

$$q = \frac{33}{10}.$$

Interpolating between (5.12), (5.22), it follows that

$$\|(3.4)\|_{L^q(B_R)} \ll R^{\varepsilon}. \tag{5.23}$$

6. Remains to bound $\|(3.5)\|_q$.

Estimate

$$\Big\| \Big[\sum_{\tau \in \mathcal{E}} (\phi_\tau |Tf_\tau|)^2 \Big]^{\frac{1}{2}} \Big\|_{L^3(B_R)} \leq (\sqrt{R})^{\frac{1}{6}} \Big\{ \sum_{\tau} \int_{B_R} \phi_\tau^3 |Tf_\tau|^3 \Big\}^{\frac{1}{3}} \tag{6.1}$$

where in the second sum, τ ranges over a full position in $\frac{1}{\sqrt{R}}$-caps.

Since $|Tf_\tau| \lesssim \frac{1}{R}$, (3.8) implies that

$$(6.1) \ll R^{\frac{1}{12}+\varepsilon}. \tag{6.2}$$

On the other hand, using the decomposition (5.9), we obtain the following estimates on

$$\Big\| \max_{\mathcal{E}_{\frac{1}{\sqrt{R}}}} \Big[\sum_{\tau \in \mathcal{E}} (\phi_{\tau,\mu} |Tf_\tau|)^2 \Big]^{\frac{1}{2}} \Big\|_{L^{p_0}_{B_R}}. \tag{6.3}$$

Using (5.10), we get

$$\begin{aligned} (6.3) &\leq (\sqrt{R})^{\frac{1}{2}-\frac{1}{p_o}} \Big(\sum_{\tau} \|\phi_{\tau,\mu} |Tf_\tau|\|^{p_0}_{L^{p_0}_{B_R}} \Big)^{\frac{1}{p_0}} \\ &\ll (\sqrt{R})^{\frac{1}{2}-\frac{1}{p_0}+\varepsilon} \mu^{-\frac{1}{5}} R^{\frac{1}{p_0}} (\sqrt{R})^{\frac{4}{p_0}-2} \ll R^{\varepsilon} \mu^{-1/5}. \end{aligned} \tag{6.4}$$

Using the bound $\phi_{\tau,\mu} \lesssim \mu$ and the inequality

$$|Tf_\tau|^2 \lesssim \frac{1}{R^2} \int |Tf_\tau|^2(y)\ 1_{\overset{\circ}{\tau}}(x-y) dy \tag{6.5}$$

and

$$\int_{B_R} |Tf_\tau|^2 \lesssim 1 \tag{6.6}$$

for $\tau \subset S_2$ a $\frac{1}{\sqrt{R}}$-cap, we obtain similarly to (5.17)

$$\begin{aligned} (6.3) &\leq \mu \Big\| \Big(\sum_{\tau} |Tf_\tau|^2 \Big)^{1/2} \Big\|_{L^{p_0}_{B_R}} \\ &\ll \frac{\mu}{R} \Big[\Big\| \sum_{\tau} 1_{\overset{\circ}{\tau}}(\cdot - y_\tau) \Big\|_{L^{5/3}_{B_R}} \Big]^{\frac{1}{2}} \\ &\overset{(5.19)}{\ll} \frac{\mu}{R} R^{\frac{19}{20}} = \mu R^{-\frac{1}{20}+\varepsilon}. \end{aligned} \tag{6.7}$$

Hence, from (6.4), (6.7)

$$(6.3) \ll R^\varepsilon \min(\mu^{-\frac{1}{5}}, \mu R^{-\frac{1}{20}}) \ll R^{-\frac{1}{120}+\varepsilon}. \tag{6.8}$$

Interpolation between (6.1), (6.8) implies

$$\|(3.5)\|_{L^q(B_R)} \ll R^\varepsilon. \tag{6.9}$$

Hence, we proved

Theorem 2′.

$$\|Tf\|_{L^q(B_R)} \ll R^\varepsilon \textit{ for } q \geq \frac{33}{10}, |f| \leq 1 \tag{6.10}$$

(implying Theorem 2).

7. One can check how the preceding argument improves if one had the optimal Kakeya maximal function bound at disposal, thus

$$\|\mathcal{M}_\delta\|_{3\to 3} \ll \Big(\frac{1}{\delta}\Big)^\varepsilon \tag{7.1}$$

Recall (5.11)

$$\Big\{\int_{B_R} \max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_{\tau,\mu} g_{\tau,\lambda})^2\Big]^{5/3}\Big\}^{3/10} \ll R^\varepsilon \lambda^{\frac{1}{10}} \mu^{-\frac{1}{5}}. \tag{7.2}$$

Next, apply (5.17) with $p_0 = 3$

$$\Big\|\max_{\mathcal{E}_\delta} \Big[\sum_{\tau\in\mathcal{E}_\delta} (\phi_{\tau,\mu} g_{\tau,\lambda})^2\Big]^{1/2}\Big\|_{L^3(B_R)} \ll$$

$$R^\varepsilon \lambda^{-\frac{1}{2}} \mu \frac{1}{R} \Big[\int \Big[\sum_\tau 1_{\overset{\circ}{\tau}}(x-y_\tau)\Big]^{3/2} dx\Big]^{1/3} \ll R^\varepsilon \lambda^{-\frac{1}{2}} \mu. \tag{7.3}$$

For the (3.5) contribution, recall (6.3), (6.4)

$$\Big\|\max_{\mathcal{E}_{\frac{1}{\sqrt{R}}}} \Big[\sum_{\tau\in\mathcal{E}} (\phi_{\tau,\mu} |Tf_\tau|)^2\Big]^{\frac{1}{2}}\Big\|_{L^{10/3}_{(B_R)}} \ll R^\varepsilon \mu^{-1/5} \tag{7.4}$$

and using (6.5), (6.6), (7.1)

$$\| \cdots \|_{L^3_{(B_R)}} \ll R^\varepsilon \mu. \tag{7.5}$$

Interpolation between (7.2), (7.3) and (7.4), (7.5) gives

$$\|Tf\|_{L^{q_1}(B_R)} \ll R^\varepsilon \quad \text{for} \quad q \geq \frac{36}{11} = 3,27.... \text{ and } \quad |f| \leq 1. \tag{7.6}$$

This leads to an improved Theorem 2 with $3\frac{3}{10}$ replaced by $\frac{36}{11}$.

§5. The Variable Coefficient Case

We consider Hörmander type oscillatory integral operators of the form

$$(T_\lambda f)(x) = \int e^{i\lambda\psi(x,y)} f(y) dy \tag{5.1}$$

with real analytic phase function ψ of the form

$$\psi(x, y) = x_1 y_1 + \cdots + x_{d-1} y_{d-1} + x_d \big(\langle Ay, y\rangle + O(|y|^3)\big) + O(|x|^2 |y|^2) \tag{5.2}$$

and $\langle Ay, y\rangle$ a non-degenerate quadratic form.

Here x (resp. y) are restricted to a neighborhood of $0 \in \mathbb{R}^d$ (resp. $0 \in \mathbb{R}^{d-1}$). In order to bring (5.1) in the format considered earlier, rescale $x \to \frac{x}{\lambda}$ to obtain a phase function

$$\phi(x, y) = x_1 y_1 + \cdots + x_{d-1} y_{d-1} + x_d \big(\langle Ay, y\rangle + O(|y|^3)\big) + \lambda \phi_\nu \Big(\frac{x}{\lambda}, y\Big) \tag{5.3}$$

and ϕ_ν at least quadratic in both x, y. Thus (5.1) becomes

$$(Tf)(x) = \int e^{i\phi(x,y)} f(y) dy \tag{5.4}$$

with x restricted to $|x| < o(\lambda)$. This formulation appears as a perturbation of the restriction problem and preceding analysis can be generalized to this setting.

First recall the [BCT] result in the variable coefficient case (see [BCT], Theorem 6.2 which treats the d-linear case, but generalizes to lower levels of multi-linearity as formulated in [BCT], (40) for ϕ linear in x).

Thus let $1 < k \leq d$ and

$$(T_i f)(x) = \int_{U_i} e^{i\phi_i(x,y)} f(y) dy \quad (1 \leq i \leq k) \tag{5.5}$$

with ϕ_i as in (5.3). We assume the transversality condition

$$|Z_1(x, y^{(1)}) \wedge \cdots \wedge Z_k(x, y^{(k)})| > c \text{ for all } x \text{ and } y^{(i)} \in U_i \tag{5.6}$$

where

$$Z(x, y) = \partial_{y_1}(\nabla_x \phi) \wedge \cdots \wedge \partial_{y_{d-1}}(\nabla_x \phi). \tag{5.7}$$

Then

$$\Big\| \Big(\prod_{i=1}^k |T_i f_i| \Big)^{\frac{1}{k}} \Big\|_q \ll \lambda^\varepsilon \Big(\prod_1^k \|f_i\|_2 \Big)^{\frac{1}{k}} \tag{5.8}$$

with $q = \frac{2k}{k-1}$ and x restricted $|x| < o(|\lambda|)$.

Note that in the restriction problem, $Z(x, y) = Z(y)$ and (5.6) amounts to transversality of the normal vectors at the corresponding hypersurface S which is the graph of $\frac{\partial \phi}{\partial x_d}$.

It turns out that the λ^ε-factor may be removed in (5.8) at the cost of increasing q to $q_1 > \frac{2k}{k-1}$. Thus, as proven in Lemma A3 in the Appendix, under the assumptions (5.5)-(5.7), one has

$$\Big\| \Big(\prod_{i=1}^k |T_i f_i| \Big)^{\frac{1}{k}} \Big\|_{q_1} \le C_{q_1} \Big(\prod_1^k \|f_i\|_2 \Big)^{1/k} \text{ for } q_1 > \frac{2k}{k-1}. \tag{5.8$'$}$$

Using (5.8$'$) instead of (5.8) in §2, §3 to bound global multilinear contributions, will eliminate the R^ε-factors (cf. §3, (4.7) and (4.9) for instance), without the need for an ε-removal at the end (note that the K^ε-factors coming from a local application in §3, (1.1) and (2.2) are harmless).

Remark. We do not claim removal of the λ^ε-factor in Theorem 6.2 from [BCT], but only in its present application to the operators T_i given by (5.5).

Returning to the analysis from §2, §3, also some adjustment is needed with respect to the parabolic rescaling argument that we discuss next.

Note that if we restrict $|y| < \frac{1}{K}$ and rescale, letting $y = \frac{y'}{K}$; $x_1 = Kx'_1, \dots, x_{d-1} = Kx'_{d-1}$ and $x_d = K^2 x'_d$, we obtain

$$\int e^{i\phi'(x', y')} f\Big(\frac{y'}{K}\Big) dy' \text{ where } |x'_1|, \dots, |x'_{d-1}| < \frac{\lambda}{K}, |x'_d| < \frac{\lambda}{K^2} \tag{5.9}$$

and

$$\phi'(x', y') = x'_1 y'_1 + \cdots + x'_{d-1} y'_{d-1} + x'_d \big(\langle Ay', y'\rangle + \frac{1}{K} O(|y'|^3)\big) + \lambda \phi_\nu \Big(\frac{Kx'_1}{\lambda}, \dots, \frac{Kx'_{d-1}}{\lambda}, \frac{K^2 x'_d}{\lambda}; \frac{y'}{K}\Big) \tag{5.10}$$

with x' subject to the restrictions (5.9).

Compared with (5.4), we see that one needs to consider the more general setting of operators

$$(Tf)(x) = \int e^{i\phi(x,y)} f(y)dy \text{ restricting } |x_1|, \dots, |x_{d-1}| < R_1 \text{ and } |x_d| < R \tag{5.11}$$

$(R \le R_1)$, and

$$\phi(x,y) = x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d\big(\langle Ay, y\rangle + 0(|y|^3\big) + R\phi_\nu\Big(\frac{x_1}{R_1}, \dots, \frac{x_{d-1}}{R_1}, \frac{x_d}{R}; y\Big) \tag{5.12}$$

(we use here that ϕ_ν is at least quadratic in y).

It has to be shown that $(5.8')$ remains valid. It turns out that the issue can be reduced to the $R = R_1$ case. We give the details. Let $q > \frac{2k}{k-1}$.

Partition the region

$$Q = [|x_1|, \dots, |x_{d-1}| < R_1] \times [|x_d| < R] = \bigcup_{s \le \frac{R_1}{R}} Q_s$$

in R-cubes and write

$$\int_Q \Big(\prod_1^k |T_i f_i|\Big)^{q/k} dx = \sum_s \int_{Q_s} \Big(\prod_1^k |T_i f_i|\Big)^{q/k} dx. \tag{5.13}$$

Partition the y-domain $\Omega \subset \mathbb{R}^{d-1}$ in cubes Ω_α of size $\sim \frac{1}{R}$ centered at y_α and write

$$(T_i f_i)(x) = \sum_\alpha^{(i)} e^{i\phi(x,y_\alpha)} \Big[\int_{\Omega_\alpha} f_i(y) e^{i[\phi(x,y)-\phi(x,y_\alpha)]} dy\Big].$$

Restricting $x \in Q_s$, the factors [ ] are approximatively constant

$$c_{i,\alpha} = \int_{\Omega_\alpha} f_i(y) e^{i[\varphi(\bar{x},y)-\varphi(\bar{x},y_\alpha)]} dy$$

where $\bar{x}$ is the center of Q_s. For $|z| < R$

$$|T_i f_i|(\bar{x} + z) \approx \Big|\sum_\alpha^{(i)} e^{i\eta(z,y_\alpha)} e^{i\phi(\bar{x},y_\alpha)} c_{i,\alpha}\Big|$$

with $\eta(z, y_\alpha) = \phi(\bar{x} + z, y_\alpha) - \phi(\bar{x}, y_\alpha)$. Hence, defining

$$g_i(y) = c_{i,\alpha} e^{i\phi(\bar{x}, y_\alpha)} \text{ for } y \in \Omega_\alpha$$

we have

$$|T_i f_i|(\bar{x} + z) \approx R^{d-1} \Big| \int e^{i\eta(z,y)} g_i(y) dy \Big|.$$

¿From (5.8′)

$$\int_{B(0,R)} \Big[\prod_1^k |T_i f_i|(\bar{x} + z) \Big]^{\frac{q}{k}} \leq C R^{q(d-1)} \Big(\prod_1^k \|g_i\|_2 \Big)^{\frac{q}{k}}$$

$$\leq C R^{\frac{q(d-1)}{2}} \Big[\prod_1^k \Big(\sum_\alpha^{(i)} |c_{i,\alpha}|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{q}{k}}$$

$$\leq C R^{\frac{q(d-1)}{2}} \Big\{ \prod_1^k \Big[\sum_\alpha^{(i)} \Big| \int_{\Omega_\alpha} f_i(y) e^{i\phi(\bar{x}, y)} dy \Big|^2 \Big]^{\frac{1}{2}} \Big\}^{\frac{q}{k}}.$$

Since $\bar{x}$ is any point in Q_s, we obtain

$$R^{\frac{q(d-1)}{2} - d} \int_{Q_s} \Big\{ \prod_1^k \Big[\sum_\alpha^{(i)} \Big| \int_{\Omega_\alpha} f_i(y) e^{i\phi(x,y)} dy \Big|^2 \Big]^{\frac{1}{2}} \Big\}^{\frac{q}{k}}. \tag{5.14}$$

Summing over s gives

$$\int_Q \Big[\prod_1^k |T_i f_i| \Big]^{\frac{q}{k}} < C R^{\frac{q(d-1)}{2} - d} \int_Q \prod_1^k \Big[\sum_\alpha^{(i)} \Big| \int_{\Omega_\alpha} f_i(y) e^{i\phi(x,y)} dy \Big|^2 \Big]^{\frac{q}{2k}}. \tag{5.15}$$

Note that

$$\int_Q \Big| \int_{\Omega_\alpha} f_i(y) e^{i\phi(x,y)} dy \Big|^2 \leq$$
$$R. \max_{|x_d| < R} \int \Big| \int_{\Omega_\alpha} f_i(y) e^{i[x_1 y_1 + \cdots + x_{d-1} y_{d-1} + R\phi_\nu(\frac{x_1}{R_1}, \cdots, \frac{x_{d-1}}{R_1}, \frac{x_d}{R}; y)]} dy \Big|^2 dx_1 \cdots dx_{d-1}$$
$$\lesssim R \int_{\Omega_\alpha} |f_i|^2 \tag{5.16}$$

using standard orthogonality considerations.

Also there is the trivial bound

$$\Big|\int_{\Omega_\alpha} f_i(y)e^{i\phi(x,y)}dy\Big| \le |\Omega_\alpha|^{\frac{1}{2}}\Big(\int_{\Omega_\alpha}|f_i|^2\Big)^{\frac{1}{2}}$$
$$\le R^{-\frac{d-1}{2}}\Big(\int_{\Omega_\alpha}|f_i|^2\Big)^{\frac{1}{2}}$$

implying

$$\sum_\alpha\Big|\int_{\Omega_\alpha} f_i(y)e^{i\phi(x,y)}dy\Big|^2 \lesssim R^{-(d-1)}\|f_i\|_2^2. \tag{5.17}$$

¿From (5.15), (5.16), (5.17) and Hölder's inequality, it follows that

$$(5.15) \le CR^{q\frac{d-1}{2}-d}\Big\{\prod_1^k (R^{1-\frac{d-1}{2}(q-2)}\|f_i\|_2^q)\Big\}^{1/k}$$
$$\le C\Big(\prod_1^k\|f_i\|_2^q)^{1/k} \tag{5.18}$$

as claimed.

We also observe that at suitable local scale, the phase function $\phi(x,y)$ given by (5.12) may be linearized in x, reducing to the restriction setting. Let $x=a+z\in B(a,\rho)$ and write

$$\phi(x,y)=\phi(a,y)+\psi(z,y)+\Omega(z,y) \tag{5.19}$$

denoting

$$\psi(z,y)=z_1y_1+\cdots+z_{d-1}y_{d-1}+z_d\big(\langle Ay,y\rangle+0(|y|^3)\big)+$$
$$\frac{R}{R_1}\Big\langle z', \nabla_{x'}\phi_\nu\Big(\frac{a'}{R_1},\frac{a_d}{R};y\Big)\Big\rangle + z_d\partial_{x_d}\phi_\nu\Big(\frac{a'}{R_1},\frac{a_d}{R};y\Big) \tag{5.20}$$

with $x=(x',x_d)$ and where

$$|\Omega(z,y)|=o(1) \text{ provided } \rho=o(\sqrt{R}). \tag{5.21}$$

Since Ω does not oscillate on $B(a,\rho)$, it may be ignored in the phase function.

A suitable coordinate change in y brings ψ in the form

$$\psi(z,y)=z_1y_1+\cdots+z_{d-1}y_{d-1}+z_d\big(\langle A'y,y\rangle+O(|y|^3)\big) \tag{5.22}$$

with A' a perturbation of A, hence A' non-degenerate (and positive definite if A is positive definite).

Using previous considerations, it is essentially straightforward to carry out the analysis from §2, §3 in the setting (5.11), (5.12), assuming again that A is positive definite and using (5.8′) to bound the global multilinear contributions.

Hence we obtain

Theorem 3. *Consider the operator (5.1) with ψ as in (5.2) and A positive definite. Then*

$$\|T_\lambda f\|_{L^p_{\text{loc}}} \leq C_p \lambda^{-\frac{d}{p}} \|f\|_\infty \tag{5.23}$$

provided

$$\begin{aligned} p &> 2\frac{4d+3}{4d-3} \quad \text{if } d \equiv 0 (\text{mod}\, 3) \\ p &> \frac{2d+1}{d-1} \quad \text{if } d \equiv 1 (\text{mod}\, 3) \\ p &> \frac{4(d+1)}{2d-1} \quad \text{if } d \equiv 2 (\text{mod}\, 3). \end{aligned}$$

In particular, for $d = 3$, we obtain the condition $p > \frac{10}{3}$. Interestingly, it turns out that this is the optimal exponent (as we will explain in the next section).

Without assuming A positive definite, it is well-known that the condition

$$p \geq \frac{2(d+1)}{d-1} \tag{5.24}$$

may be optimal range of validity for the inequality (5.19), when d is *odd* (cf. [B]).

It was shown also in [B] that for d even, there is some $p(d) < \frac{2(d+1)}{d-1}$ such that

$$\|T_\lambda f\|_{L^p_{\text{loc}}} \lesssim \lambda^{-\frac{d}{p}} \|F\|_\infty. \tag{5.25}$$

The following statement makes this more precise

Theorem 4. *Consider the operator (5.1) with ψ as in (5.2) and A non-degenerate. For d even, one has the inequality*

$$\|T_\lambda f\|_{L^p_{\text{loc}}} \leq C_p \lambda^{-\frac{d}{p}} \|f\|_\infty \text{ for } p > \frac{2(d+2)}{d}. \tag{5.26}$$

(the exponent $\frac{2(d+2)}{d}$ was already known to be optimal).

Proof. (sketch)

We consider the setting (5.11), (5.12). Define the integer

$$k = \frac{d}{2} + 1.$$

Thus the condition on the exponent q in (5.8′) becomes $q > \frac{2(d+2)}{d}$.

Following the procedure from §2, §3, we fix a large parameter K and restrict x to a K-ball $B_K = B(a, K)$. Subdividing the y-domain Ω in balls Ω_α of size $\frac{1}{K}$ and considering the operators

$$(T_\alpha f)(x) = \int_{\Omega_\alpha} e^{i\phi(x,y)} f(y) dy$$

we consider the following two alternatives.

Case 1. On B_K, we may estimate

$$|Tf| < C(K)\ |T_{\alpha_i} f| \tag{5.27}$$

for some $\alpha_1, \dots, \alpha_k$ such that (5.6) holds for $y^{(1)} \in \Omega_{\alpha_1}, \dots, y^{(k)} \in \Omega_{\alpha_k}$ (with constant $c \sim \frac{1}{K}$).

Case 2. Failure of Case 1. This implies that on B_K

$$|Tf| \lesssim \Big| \sum_{\alpha \in A} T_\alpha f \Big| + \max_\alpha |T_\alpha f| \tag{5.28}$$

where $\bigcup_{\alpha \in A} \Omega_\alpha$ is contained in an $\sim \frac{1}{K}$-neighborhood of the $(k-2)$-manifold, obtained by requiring $Z(a, y)$ given by (5.7) to belong to some $(k-1)$-dim linear space.

In particular,

$$\# A \lesssim K^{k-2}. \tag{5.29}$$

In Case 1, write on B_K

$$|Tf| \leq C(K) \sum_{\substack{\alpha_1, \dots, \alpha_k \\ (5.6) \text{ holds}}} \Big(\prod_1^k |T_{\alpha_i} f| \Big)^{\frac{1}{k}}. \tag{5.30}$$

The collected contribution may then be estimated using the k-linear bound and gives the estimate

$$\ll C(K). \tag{5.31}$$

In Case 2, we proceed more crudely than in §3 (note that lower dimensional restriction of the y-variable may lead to degenerate phase functions if the quadratic form $\langle Ay, y \rangle$ is not assumed definite.)

¿From (5.28)

$$\Big(\fint_{B_K} |Tf|^q\Big)^{\frac{1}{q}} \le \Big(\fint_{B_K} \Big|\sum_{\alpha\in A} T_\alpha f\Big|^q\Big)^{\frac{1}{q}} + \Big(\sum_\alpha |T_\alpha f|^q\Big)^{\frac{1}{q}}$$
$$= (5.32) + (5.33)$$

Estimate

$$\begin{aligned}(5.32)^q &\le \Big[\fint_{B_K}\Big|\sum_{\alpha\in A} T_\alpha f\Big|^2\Big]\ \Big[\sum_{\alpha\in A}|T_\alpha f|\Big]^{q-2}\\ &\sim \Big[\sum_{\alpha\in A}|T_\alpha f|^2\Big]\ \Big[\sum_{\alpha\in A}|T_\alpha f|\Big]^{q-2} \quad \text{(using simple orthogonality)}\\ &< |A|^{1-\frac{2}{q}+(q-2)(1-\frac{1}{q})}\sum_\alpha |T_\alpha f|^q.\end{aligned}$$

Recalling (5.29)

$$(5.32) \le K^{(k-2)(1-\frac{2}{q})}\Big(\sum_\alpha \fint_{B_K}|T_\alpha f|^q\Big)^{1/q} \tag{5.34}$$

(that also captures (5.33)).

Thus the collected contribution over the B_K is bounded by

$$\begin{aligned}&K^{(k-2)(1-\frac{2}{q})}\Big(\sum_\alpha \|T_\alpha f\|_q^q\Big)^{\frac{1}{q}}\\ &\le K^{(k-2)(1-\frac{2}{q})+\frac{d-1}{q}}\max_\alpha \|T_\alpha f\|_q.\end{aligned} \tag{5.35}$$

Rescaling gives the estimate

$$< K^{(k-2)(1-\frac{2}{q})+\frac{d-1}{q}-(d-1)+\frac{d+1}{q}}Q^{(q)}_{\frac{R_1}{K},\frac{R}{K^2}} = K^{\frac{d+2}{q}-\frac{d}{2}}Q. \tag{5.36}$$

(denoting $Q^{(p)}_{R_1,R}$ a bound on $T: L^\infty \to L^p_{|x'|<R_1,|x_d|<R}$ given by (5.11)).

Since $q > \frac{2(d+2)}{d}$, this concludes the argument.

§6. Some Examples

We present in this section an example for $n=3$ that will illustrate the optimality of the exponent $\frac{10}{3}$ in Theorem 3. It will also explain the differences between the elliptic and hyperbolic cases.

Consider the following phase function

$$\phi(x,y) = -x_1y_1 - x_2y_2 + \frac{1}{2}x_3y_1^2 + x_3^2y_1y_2 + \frac{1}{2}(x_3+x_3^3)y_2^2. \tag{6.1}$$

First analyze the [BCT] transversality condition. Thus

$$\nabla_x\phi = \Big(-y_1, -y_2, \frac{1}{2}(y_1^2+y_2^2) + 2x_3y_1y_2 + \frac{3}{2}x_3^2y_2^2\Big)$$

$$\begin{cases} \partial_{y_1}\nabla_x\phi = (-1, 0, y_1 + 2x_3y_2) \\ \partial_{y_2}\nabla_x\phi = (0, -1, y_2 + 2x_3y_1 + 3x_3^2y_2) \end{cases}$$

$$\begin{aligned} Z(\Phi)(y,x) = \partial_{y_1}\nabla_x\phi \wedge \partial_{y_2}\nabla_x\phi &= (y_1 + 2x_3y_2, y_2 + 2x_3y_1 + 3x_3^2y_2, 1) \\ &= \left(A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, 1\right) \end{aligned}$$

where $A = A_x = \begin{pmatrix} 1 & 2x_3 \\ 2x_3 & 1+3x_3^2 \end{pmatrix}$ is a perturbation of identity.

Concerning condition (40) in [BCT], if one fixes x and restrict $y=(y_1,y_2)$ to non-collinear discs $U_1, U_2, U_3 \subset \mathbb{R}^2$, clearly

$$\det\big(Z(\phi)(y^{(1)},x), Z(\phi)(y^{(2)},x), Z(\phi)(y^{(3)},x)\big) \neq 0$$

for $y^{(i)} \subset V_i$.

Next, consider the Kakeya type sets associated with (6.1).

$$\begin{cases} \partial_{y_1}\phi = -x_1 + x_3y_1 + x_3^2y_2 \\ \partial_{y_2}\phi = -x_2 + x_3^2y_1 + (x_3+x_3^3)y_2 \end{cases} \tag{6.2}$$

and

$$\Gamma_y \text{ is parametrized by } \begin{cases} x_1 = y_1x_3 + y_2x_3^2 \\ x_2 = y_1x_3^2 + y_2(x_3+x_3^3). \end{cases} \tag{6.3}$$

If we shift Γ_y by $(y_2,0,0)$, the tubes

$$\begin{cases} x_1 = y_1x_3 + y_2x_3^2 + y_2 \\ x_2 = y_1x_3^2 + y_2(x_3+x_3^3) \end{cases} \tag{6.4}$$

are contained in the surface

$$S : x_1 x_3 = x_2.$$

Thus one gets again 2D-compression, similar to the hyperbolic example

$$\psi(x, y) = -x_1 y_1 - x_2 y_2 + 2x_3 y_1 y_2 + x_3^2 y_2^2. \tag{6.5}$$

See also [Wi].

We try to exploit this compression as well as possible to make the oscillatory integral

$$\int e^{i\lambda\phi(x,y)} \ f(y)dy \tag{6.6}$$

(with an appropriate f) large.

At this stage, there seems to be quite a difference between (6.1) and (6.5). For (6.5), just take

$$f(y) = e^{iy_1^2}. \tag{6.7}$$

Then

$$\int e^{i\lambda\psi(x,y)} f(y)dy = \int_{\text{loc}} e^{i\lambda[(y_1+x_3y_2)^2-(x_1y_1+x_2y_2)]} dy \tag{6.8}$$

and restricting x to a $\frac{1}{\lambda}$-neighborhood of S

$$(6.8) \approx \int_{\text{loc}} e^{i\lambda[(y_1+x_3y_2)^2-x_1(y_1+x_3y_2)]} dy.$$

Setting $u = y_1 + x_3 y_2$, stationary phase implies

$$|(6.8)| \sim \frac{1}{\sqrt{\lambda}}.$$

and hence

$$\|(6.8)\|_{L^q_x} \sim \frac{1}{\sqrt{\lambda}} \Big(\frac{1}{\lambda}\Big)^{\frac{1}{q}} \lesssim \Big(\frac{1}{\lambda}\Big)^{\frac{3}{q}} \text{ for } q \geq 4.$$

In the elliptic case, this type of construction seems impossible.

But one can make the following one, which will explain where the condition $q \geq \frac{10}{3}$ comes from.

Instead of (6.7), take in (6.6)

$$f(y) = \sum_{s<\sqrt{\lambda}} \sigma_s 1_{[\frac{s}{\sqrt{\lambda}}, \frac{s+c}{\sqrt{\lambda}}]}(y_2) e^{i\lambda \frac{s}{\sqrt{\lambda}} y_1}. \tag{6.9}$$

where $\sigma_s = \pm 1$ and $c > 0$ is a small constant.

Hence

$$(6.6) = \sum_{s<\sqrt{\lambda}} \sigma_s \Big\{ \int_{\frac{s}{\sqrt{\lambda}}<y_2<\frac{s+c}{\sqrt{\lambda}}} e^{i\lambda[\phi(x,y)+\frac{s}{\sqrt{\lambda}}y_1]} dy \Big\}. \tag{6.10}$$

Denoting R the region

$$R = \Big[x_3 \sim 1 \text{ and } |x_2 - x_1 x_3| = o\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big] \tag{6.11}$$

write

$$\int_{\text{loc}} |(6.6)|^q dx \geq \int_R |(6.10)|^q dx. \tag{6.12}$$

Averaging the right side of (6.12) over signs $\sigma_s = \pm 1$, we obtain clearly

$$\int_R \Big\{ \sum_{s<\sqrt{\lambda}} \Big| \int_{\frac{s}{\sqrt{\lambda}}<y_2<\frac{s+c}{\sqrt{\lambda}}} e^{i\lambda[\phi(x,y)+\frac{s}{\sqrt{\lambda}}y_1]} dy \Big|^2 \Big\}^{\frac{q}{2}} dx. \tag{6.13}$$

Since

$$\phi(x,y) = \frac{1}{2}x_3 \Big[\Big(y_1 + x_3 y_2 - \frac{x_1}{x_3}\Big)^2 + \Big(y_2 + \frac{x_1 x_3 - x_2}{x_3}\Big)^2 \Big] - \frac{1}{2}\Big[\frac{x_1^2}{x_3} + \frac{(x_1x_3 - x_2)^2}{x_3}\Big]$$

we have

$$\phi(x,y) + \frac{s}{\sqrt{\lambda}} y_1 = \frac{1}{2} x_3 \Big[\Big(y_1 + x_3 y_2 - \frac{x_1}{x_3} + \frac{s}{\sqrt{\lambda}} \frac{1}{x_3}\Big)^2 + \Big(y_2 - \frac{s}{\sqrt{\lambda}} + \frac{x_1 x_3 - x_2}{x_3}\Big)^2 \Big] + \eta(x,s) \tag{6.14}$$

Therefore, from definition of R

$$(6.13) \sim \int_R \Big\{ \sum_{s<\sqrt{\lambda}} \Big| \int_{\frac{s}{\sqrt{\lambda}}<y_2<\frac{s+c}{\sqrt{\lambda}}} e^{i\frac{\lambda}{2}x_3(y_1+x_3y_2-\frac{x_1}{x_3}+\frac{s}{\sqrt{\lambda}}\frac{1}{x_3})^2} dy \Big|^2 \Big\}^{\frac{q}{2}} dx. \tag{6.15}$$

Stationary phase shows that for $|x_1| = o(x_3)$ and $s = o(\sqrt{\lambda})$, the inner integral in (6.15) is $O\Big(\frac{1}{\lambda}\Big)$.

Hence

$$(6.13) \sim \Big(\frac{1}{\lambda}\Big)^{\frac{3q}{4}} |R| \sim \Big(\frac{1}{\lambda}\Big)^{\frac{3q+2}{4}}$$

by (6.11), and

$$\|(6.6)\|_q \gtrsim \Big(\frac{1}{\lambda}\Big)^{\frac{3}{4}+\frac{1}{2q}}. \tag{6.16}$$

Clearly (6.1) can only hold provided $q \geq \frac{10}{3}$.

§7. Curved Kakeya Estimates

1. Let's begin by describing curved Kakeya problems in $\mathbb{R}^n$. We have a collection of tubes T_i. Each tube T_i is the δ-neighborhood of a curve Γ_i in the unit ball in $\mathbb{R}^n$. The goal of the curved Kakeya problem is to assume some geometric information about the tubes T_i and use it to prove estimates for the L^p norms of $\sum_i \chi_{T_i}$ and/or for the volume of the union of tubes $\cup T_i$. Either kind of estimate is a way of measuring how much the tubes T_i overlap.

Let $\delta > 0$ be a small number.

We assume that each curve has C^2 norm $\lesssim 1$, and that each curve is an algebraic curve of degree $\lesssim 1$. We assume that each curve is contained in the unit ball. (I.e., Γ_i is the restriction of an algebraic curve to the unit ball.) (i.e. Γ_i is the restriction of an algebraic curve to the unit ball.)

We define T_i to be the δ-neighborhood of Γ_i. At each point $x \in T_i$, we can approximately define the tangent direction to the tube T_i at x. Namely, pick any point $x' \in \Gamma_i \cap B(x, \delta)$ and define $v_i(x)$ to be the unit tangent vector to Γ_i at x'. Since Γ_i has C^2-norm $\lesssim 1$, choosing different points x' in $B(x, \delta)$ will lead to an ambiguity of size $\lesssim \delta$. So the function $v_i(x)$ is well-defined up to $O(\delta)$ on the tube T_i.

2. Assuming the Γ_i algebraic, we prove the following slightly stronger version of the multilinear Kakeya estimate for curved tubes due to [BCT]. The next statement deals with the 3-linear setting in $\mathbb{R}^4$ (for simplicity), but can be generalized to k-linear in $\mathbb{R}^n$.

Theorem 6.

Suppose Γ_i are algebraic curves restricted to the unit 4-ball with degree $\lesssim 1$ and C^2 norm $\lesssim 1$. Let T_i denote the δ-neighborhood of Γ_i. Define approximate tangent vectors $v_i(x)$ for $x \in T_i$ as above. Suppose that the number of tubes T_i is N. Then the following estimate holds:

$$\int_{B^4} \Big[\sum_{i=1}^N \chi_{T_i} \sum_{j=1}^N \chi_{T_j} \sum_{k=1}^N \chi_{T_k} \Big| v_i \wedge v_j \wedge v_k \Big| \Big]^{1/2} \lesssim \delta^4 N^{3/2}. \tag{2.1}$$

Choosing the curves Γ_i in the subspace $[e_1, e_2, e_3]$ implies immediately the same statement in $\mathbb{R}^3$ with δ^4 replaced by δ^3 in (2.1).

Since we may repeat tubes T_i, we obtain also the weighted version from Theorem 6.

The proof of the multilinear estimate follows the Dvir polynomial method, introduced for problems over finite fields in [D]. The polynomial method was applied to multilinear Kakeya problems in $\mathbb{R}^n$ in [G], and we will use results from there.

We will build an algebraic hypersurface Z of controlled degree which is concentrated where the tubes T_i overlap heavily, and we will study the intersections between Z and the curves Γ_i.

Recall the definition of directed volume $V_S(v) := \int_S |v \cdot N| dvol_S$, where N denotes the normal vector to S. We need a curved version of the cylinder estimate, Lemma 2.1 in [G].

Lemma 2.2. *If Z is an algebraic surface in $\mathbb{R}^4$ of degree D, and if Γ_i is a curve of degree d, and if Q_α are disjoint cubes of side length $\sim \delta$ which cover T_i, and if x_α is the center point of Q_α, then the following inequality holds:*

$$\sum_\alpha \delta^{-3} V_{Z \cap Q_\alpha}(v_i(x_\alpha)) \lesssim dD. \tag{2.3}$$

Proof. The idea of the proof is to interpret $\delta^{-3} V_{Z \cap Q_\alpha}(v_i(x_\alpha))$ in a nice way: this quantity is roughly the average number of intersections of $Z \cap Q_\alpha$ with a translation of Γ_i by a random vector v of length $\lesssim \delta$. The total number of intersections of Z with (almost every) translate of Γ_I is at most dD by Bezout's theorem.

The errors caused by $v_i(x)$ varying by $\sim \delta$ as x varies in Q_α contribute about δD per cube and so at most D to the final answer.

In the paper [G], tubes had thickness 1. Our tubes have thickness δ, so it's convenient to re-normalize certain quantities. If $Q \subset \mathbb{R}^4$ is a cube of side length δ, then

$$V^{ren}_{Z \cap Q}(v) := \delta^{-3} V_{Z \cap Q}(v). \tag{2.4}$$

We recall the notion of 'visibility' that plays a crucial role in [G].

The visibility of $Z \cap Q$ measures the directed volume of $Z \cap Q$ in various directions, and if there is even one direction where $Z \cap Q$ has low directed volume, the visibility goes down a lot. The renormalized visibility has the following definition.

$$Vis^{\mathrm{ren}}[Z \cap Q] := Vol\Big(\{v \text{ such that } |v| \leq 1 \text{ and } V^{ren}_{Z \cap Q}(v) \leq 1\}\Big)^{-1}. \tag{2.5}$$

As in [G], one needs to introduce modified versions $\bar{V}is$ and $\bar{V}$ of Vis and V, obtained by a suitable averaging over Z. They have all good properties of the originals and moreover depend continuously on Z. See [G] for details.

Next, we state a key result from [G] (see §5, p14), in our renormalized setting.

Lemma 2.6. *Consider the standard δ-lattice in $\mathbb{R}^4$. Let M be a function from the set of 4-cubes Q in this lattice to $\mathbb{Z}_+ \cup \{o\}$. Then there is an algebraic hypersurface of degree D such that*

$$\overline{V}is^{\text{ren}}[Z \cap Q] \geq M(Q) \quad \textit{for all } Q \tag{2.7}$$

and

$$D < C\Big[\sum_Q M(Q)\Big]^{1/4}. \tag{2.8}$$

Let Q_α be a set of δ-cubes that cover the unit 4-ball. For each cube, define

$$F(Q_\alpha) := \sum_{T_i, T_j, \text{ and } T_k \text{ intersect } Q_\alpha} |v_i \wedge v_j \wedge v_k|.$$

Here v_i, v_j, v_k are evaluated at x_α, the center of Q_α.

Lemma 2.9. *The sum $\sum_\alpha \delta^4 F(Q_\alpha)^{1/2} \lesssim d^{3/2}\delta^4 N^{3/2}$.*

The sum on the left-hand side is very close to the integral over the 4-ball we want to estimate:

$$\int_{B^4} \Big[\sum_{i=1}^N \chi_{T_i} \sum_{j=1}^N \chi_{T_j} \sum_{k=1}^N \chi_{T_k} |v_i \wedge v_j \wedge v_k|\Big]^{1/2} \sim \sum_\alpha \delta^4 F(Q_\alpha)^{1/2}. \tag{2.10}$$

We compare our discrete sum and the integral below. First we prove the lemma.

Proof. We construct a surface of degree $\lesssim D$ (for a large D) so that for all α

$$\bar{V}is^{\text{ren}}[Z \cap Q_\alpha] \geq D^4 F(Q_\alpha)^{1/2}\Big[\sum_\alpha F(Q_\alpha)^{1/2}\Big]^{-1}. \tag{2.11}$$

(We can use any D, but we need D big enough so that the RHS is at least 1 for all α.)

The existence of Z follows indeed from Lemma 2.6, taking for $M(Q_\alpha)$ the RHS of (2.11).

We show that

$$D\Big[\sum_\alpha F(Q_\alpha)^{1/2}\Big]^{2/3} \lesssim dDN \tag{2.12}$$

which is equivalent with (2.9). Write using (2.11).

$$D\Big[\sum F(Q_\alpha)^{1/2}\Big]^{2/3} \lesssim \sum F(Q_\alpha)^{1/3} \bar{V}is^{ren}(Q_\alpha)^{1/3} D^{-1/3} \lesssim$$

$$= \sum_\alpha \left[D^{-1} \bar{V}is^{ren}(Q_\alpha) \sum_{T_i, T_j, T_k \text{ meet } Q_\alpha} |v_i \wedge v_j \wedge v_k(x_\alpha)| \right]^{1/3}. \tag{2.13}$$

Linear algebra lemma. For any three vectors v_i, v_j, v_k, the following inequality holds

$$\overline{Vis}^{ren}[Z \cap Q_\alpha] |v_i \wedge v_j \wedge v_k| \lesssim D \bar{V}^{ren}_{Z\cap Q_\alpha}(v_i) \bar{V}^{ren}_{Z\cap Q_\alpha}(v_j) \bar{V}^{ren}_{Z\cap Q_\alpha}(v_k) \tag{2.14}$$

Proof. We abbreviate $\bar{V}^{ren}_{Z\cap Q_\alpha}$ by $\bar{V}$ and $\overline{Vis}^{ren}$ by $\overline{Vis}$.

We use the following facts. The function $\bar{V}$ maps $\mathbb{R}^4$ to $\mathbb{R}$. It is non-negative. It scales by the formula $\bar{V}(\lambda v) = \lambda \bar{V}(v)$ for any $\lambda > 0$ and $v \in \mathbb{R}^4$. It is convex. And finally $|v| \le \bar{V}(v) \lesssim D|v|$ (where the lower bound is ensured by enlarging Z with $\sim \frac{1}{\delta}$ hyperplanes.)

Now $\overline{Vis}$ is defined as $Vol\{v \in B^4 | \bar{V}(v) \le 1\}^{-1}$. So we have to prove that

$$Vol\{v \in B^4 | \bar{V}(v) \le 1\} \gtrsim |v_i \wedge v_j \wedge v_k| D^{-1} \bar{V}(v_i)^{-1} \bar{V}(v_j)^{-1} \bar{V}(v_k)^{-1}. \tag{2.15}$$

Let v_0 be a unit vector perpendicular to the plane spanned by v_i, v_j, v_k. Let $e_0 = v_0/D$. Then $\bar{V}(e_0) \le 1$. Also, let $e_i := v_i/\bar{V}(v_i)$, so that $\bar{V}(e_i) = 1$. Define e_j, e_k similarly. Since $\bar{V}(v) \ge |v|$, it follows that $|e_i| \le 1$. Since $\bar{V}$ is convex, $\bar{V} \le 1$ on the convex hull of the eight points $\pm e_0, \pm e_i, \pm e_j, \pm e_k$. This convex hull lies in B^4. Its volume is approximately $|e_0 \wedge e_i \wedge e_j \wedge e_k|$. Since e_0 is perpendicular to the other vectors, this wedge is equal to $|e_0||e_i \wedge e_j \wedge e_k| = D^{-1}|v_i \wedge v_j \wedge v_k| \bar{V}(v_i)^{-1} \bar{V}(v_j)^{-1} \bar{V}(v_k)^{-1}$. proving (2.15).

¿From (2.14)

$$(2.13) \lesssim \sum_\alpha \Big[\sum_{T_i, T_j, T_k meet Q_\alpha} \bar{V}^{ren}_{Z\cap Q_\alpha}(v_i) \bar{V}^{ren}_{Z\cap Q_\alpha}(v_j) \bar{V}^{ren}_{Z\cap Q_\alpha}(v_k) \Big]^{1/3} =$$

$$= \sum_\alpha \sum_{T_i \text{ meets } Q_\alpha} \bar{V}^{ren}_{Z\cap Q_\alpha}(v_i) = \sum_{i=1}^N \sum_{Q_\alpha \text{ meets } T_i} \bar{V}^{ren}_{Z\cap Q_\alpha}(v_i).$$

By the cylinder estimate, the last line is bounded $\lesssim NdD$ as required.

This proves Lemma 2.9.

Finally, we return to the integral and show that the error in our discrete approximation is not too big:

$$\int_{B^4}\Big[\sum_{i=1}^N \chi_{T_i}\sum_{j=1}^N \chi_{T_j}\sum_{k=1}^N \chi_{T_k}\Big|v_i(x)\wedge v_j(x)\wedge v_k(x)\Big|\Big]^{1/2} dx =$$

$$= \sum_\alpha \int_{Q_\alpha}\Big[\sum_{i=1}^N \chi_{T_i}\sum_{j=1}^N \chi_{T_j}\sum_{k=1}^N \chi_{T_k}\,|v_i\wedge v_j\wedge v_k|\,\Big]^{1/2} dx$$

$$\leq \sum_\alpha \int_{Q_\alpha}\Big[\sum_{T_i,T_j,T_k \text{ meet } Q_\alpha} |v_i(x)\wedge v_j(x)\wedge v_k(x)|\Big]^{1/2} dx$$

$$\leq \sum_\alpha \int_{Q_\alpha}\Big[\sum_{T_i,T_j,T_k \text{ meet } Q_\alpha} |v_i(x_\alpha)\wedge v_j(x_\alpha)\wedge v_k(x_\alpha)|\Big]^{1/2} + \text{ Error} \tag{2.16}$$

where

$$\text{Error } \lesssim \sum_\alpha \int_{Q_\alpha}\Big[\sum_{i,j,k} \chi_{\tilde{T}_i}\chi_{\tilde{T}_j}\chi_{\tilde{T}_k}|v_i\wedge v_j|\delta\Big]^{1/2} \lesssim$$

$$\lesssim \delta^{1/2}\Big(\int_{B^4}\sum_{i,j}\chi_{T_i}\chi_{T_j}|v_i\wedge v_j|dx\Big)^{1/2}\Big(\int_{B^4}\sum \chi_{T_k}\Big)^{1/2} \sim$$

$$N^{1/2}\delta^2\Big(\int_{B^4}\sum_{i,j}\chi_{T_i}\chi_{T_j}|v_i\wedge v_j|dx\Big)^{1/2}. \tag{2.17}$$

By Lemma 2.9, the first term in (2.16) is bounded by $C.\delta^4 d^{3/2}N^{3/2}$.

In (2.17) we encounter a 2-linear version of our original 3-linear integral.

This can be estimated by a much easier argument in the same spirit.

We show that

$$\int_{B^4}\mathcal{X}_{T_i}\mathcal{X}_{T_j}|v_i\wedge v_j| < C\delta^4 d^2. \tag{2.18}$$

Hence (2.17) $< CN^{3/2}\delta^4 d$ and this completes the proof of Theorem 8.

It remains to justify (2.18). Thus

Lemma 2.19. *Suppose that Γ_1 and Γ_2 are degree d algebraic curves in B^4 and C^2 curves of norm $\lesssim 1$, and T_i are δ tubes around Γ_i.*

Then

$$\int_{B^4} \chi_{T_1}\chi_{T_2}|v_1(x) \wedge v_2(x)|dx \lesssim d^2\delta^4. \tag{2.19}$$

Proof. (sketch) (This is an easier version of the 3-linear estimate (2.1)).

Cut the unit ball into δ cubes Q_α.

Pick D a large degree. Choose Z a degree D hypersurface so that $\bar{V}^{ren}_{Z\cap Q_\alpha}(x) \geq |x|$ and

$$\overline{Vis}^{ren}[Z \cap Q_\alpha] \gtrsim D^4|v_1 \wedge v_2(x_\alpha)|\Big[\sum_\alpha |v_1 \wedge v_2(x_\alpha)|\Big]^{-1}. \tag{2.20}$$

Now our integral is roughly

$$\delta^4 \sum_{Q_\alpha \subset T_1 \cap T_2} |v_1 \wedge v_2(x_\alpha)|. \tag{2.21}$$

The error in this approximation is $\delta Vol(T_1 \cap T_2) \lesssim d\delta^4$ which is not larger than the main term.

It suffices to prove

$$\sum_\alpha |v_1 \wedge v_2| \lesssim d^2. \tag{2.22}$$

Manipulating (2.20), we see that

$$\sum_\alpha |v_1 \wedge v_2| \lesssim D^{-4}[\sum_\alpha \overline{Vis}^{ren}[Z \cap Q_\alpha]^{1/2}|v_1 \wedge v_2|^{1/2}]^2 \leq$$

(by a linear algebra lemma like the one above)

$$\lesssim D^{-2}[\sum \bar{V}^{ren}(v_1)^{1/2}\bar{V}^{ren}(v_2)^{1/2}]^2 \leq$$

$$\leq D^{-2}\Big(\sum_\alpha \bar{V}^{ren}(v_1)\Big)\Big(\sum_\alpha \bar{V}^{ren}(v_2)\Big).$$

Now the first term in parentheses is essentially the average number of intersections between Z and Γ_i after translating Γ_i by a random vector of length $\lesssim \delta$, and so it has size at most dD by Bezout's theorem. (Compare the cylinder estimate above.) The same applies to the second term. So the whole expression is bounded $\lesssim d^2$.

3. Application to curved Kakeya sets

Again we restrict ourselves to $n = 4$ but the result generalize to even dimension n (the exponent $\frac{3}{2}$ in Theorem 7 below is then replaced by $1 + \frac{2}{n}$.)

Let the curves $\{\Gamma_i\}$ be as specified in the beginning of §7. We also make an 'angle assumption' for pairs of curves, as follows.

The index set $\{i\}$ is given a geometric structure. For each curve i, we associate a point y_i in $B^{n-1}(1)$. We assume that the points y_i are δ-separated. We make the following crucial geometric assumption. If a point x lies in T_i and in T_j, then the angle between $v_i(x)$ and $v_j(x)$ is $\gtrsim |y_i - y_j|$. This assumption prevents too many near-tangencies in the overlaps of the tubes.

Theorem 7. *Under the hypotheses above, for all $p > 3/2$,*

$$\Big\| \sum_i \chi_{T_i} \Big\|_p \lesssim \delta^{-3+4/p}. \tag{3.1}$$

Hence, any curved Kakeya set in $\mathbb{R}^4$ (defined from algebraic curves of controlled degree and controlled C^2 norm) has Minkowski dimension at least 3.(*)

Examples (cf. [B2]) show that the statement in Theorem 7 is best possible.

The proof of Theorem 7 uses an inductive argument, where we assume that a good estimate holds for a partial sums $\sum_{y_i \in \text{ small ball}} \chi_{T_i}$ and then we prove that a good estimate holds for a partial sum on y_i in a larger ball.

Theorem 7′. *Let T_i obey the hypotheses from Theorem 7 . Suppose that $p > 3/2$. Suppose that ρ is a scale in the range $\delta \le \rho \le 1$. Let B_ρ denote any ball of radius ρ in $B^3(1)$. Then the following estimate holds.*

$$\Big\| \sum_{y_i \in B_\rho} \chi_{T_i} \Big\|_p \lesssim \delta^{-3+\frac{4}{p}} \rho^{3-\frac{1}{p}}. \tag{3.2}$$

When $\rho = 1$, Theorem 7′ implies Theorem 7. When $\rho = \delta$, Theorem 7′ is trivial. We will prove Theorem 7′ by induction on ρ. So we are allowed to assume that Theorem 7′ holds for all $\bar{\rho} < \rho/2$. In other words, we know

$$\Big\| \sum_{y_i \in B_{\bar{\rho}}} \chi_{T_i} \Big\|_p \le \alpha \delta^{-3+4/p} \bar{\rho}^{3-1/p}. \tag{3.3}$$

(*)We will indicate later on in this section how to generalize this last claim to C^∞-curves.

In this equation α is a large constant that we will choose later. Assuming (3.3), we will prove that the same estimate holds for balls of radius ρ, *with the same constant* α. In other words, we will prove

$$\Big\| \sum_{y_i \in B_\rho} \chi_{T_i} \Big\|_p \leq \alpha \delta^{-3+4/p} \rho^{3-1/p}. \tag{3.4}$$

Once we have proven (3.4), the inductive argument shows that Theorem 7′ holds for all ρ, and we are done. The idea of the proof is as follows. We cover B_ρ with smaller balls, and then write $\sum_{y_i \in B_\rho}$ as a sum of contributions from the smaller balls. To bound the L^p norm of this sum, we use a combination of two tools. First, (3.3) bounds the L^p norms of the contributions from each smaller ball. By itself, this is not enough, but it shows that for (3.4) to fail, we need to have points where many smaller balls are contributing. The size of this effect is controlled by the multilinear estimate.

Let K be a large constant to be determined later. We cover B_ρ by K^3 smaller balls, each of radius at most $10\rho/K$. We call each of these smaller balls a "clump". Hence our set of tubes is divided into $\sim K^3$ clumps.

We divide B^4 into two regions, depending on how the tubes through x are divided among the clumps. We call a point $x \in B^4$ "narrow" if there exist $< 10^4 K$ clumps which contain half of the tubes through the point x. We call x "broad" if it is not narrow. Let $N \subset B^4$ be the set of narrow points, and $N^c \subset B^4$ the set of broad points.

Our inductive hypothesis directly controls $\| \sum_{y_i \in B_\rho} \chi_{T_i} \|_{L^p(N)}$.

Lemma 3.5. *Let $p > 3/2$. Assuming (3.3), and assuming that $K = K(p)$ is sufficiently large, the following estimate holds:*

$$\int_{Narrow} \Big[\sum_{y_i \in B_\rho} \chi_{T_i} \Big]^p dx \leq (1/2) \alpha^p \delta^{4-3p} \rho^{3p-1}. \tag{3.6}$$

More explicitly, we say that K is sufficiently large if $[2 \cdot 10^7]^p K^{-2p+3} < 1/2$. Notice that this condition depends only on p.

Proof. Fix $x \in Narrow$. We divide the sum $\sum_{y_i \in B_\rho} \chi_{T_i}(x)$ into clumps:

$$\sum_{y_i \in B_\rho} \chi_{T_i}(x) \leq \sum_{j=1}^{K^3} \Big[\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big]. \tag{3.7}$$

Now since x is narrow, the sum on the right-hand side is controlled by the sum from only $10^4 K$ clumps. In other words, we can pick a set $C(x)$ of at most $10^4 K$ clumps so that

$$(3.7) \leq 2 \sum_{j \in C(x)} \Big[\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big]. \tag{3.8}$$

Now by Holder's inequality, this last sum is dominated by

$$(3.8) \leq 2 \Big[\sum_{j \in C(x)} \Big(\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big)^p \Big]^{1/p} [10^4 K]^{\frac{p-1}{p}}.$$

Putting together the string of inequalities we just proved, we see that for each $x \in Narrow$,

$$\Big[\sum_{y_i \in B_\rho} \chi_{T_i}(x) \Big]^p \leq 2^p [10^4 K]^{p-1} \sum_{j=1}^{K^3} \Big(\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big)^p.$$

Now integrating over the narrow set, we get

$$\int_{Narrow} \Big[\sum_{y_i \in B_\rho} \chi_{T_i}(x) \Big]^p dx \leq 2^p [10^4 K]^{p-1} \sum_{j=1}^{K^3} \int_{B^4} \Big(\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big)^p dx. \tag{3.9}$$

But by induction (3.3), the integral involving each smaller clump in (3.9) is controlled

$$\int_{B^4} \Big(\sum_{y_i \in clump(j)} \chi_{T_i}(x) \Big)^p dx \leq \alpha^p \delta^{4-3p} (10\rho/K)^{3p-1}.$$

Plugging this estimate into (3.8), we get

$$\int_{Narrow} \Big[\sum_{y_i \in B_\rho} \chi_{T_i}(x) \Big]^p dx \leq 2^p [10^4 K]^{p-1} K^3 \alpha^p \delta^{4-3p} (10\rho/K)^{3p-1}. \tag{3.10}$$

Grouping terms in the right-hand side, we get

$$\leq [2 \cdot 10^4 \cdot 10^3]^p K^{-2p+3} \alpha^p \delta^{4-3p} \rho^{3p-1}.$$

We choose $K = K(p)$ sufficiently large so that

$$[2 \cdot 10^7]^p K^{-2p+3} < 1/2.$$

Since $p > 3/2$, we can choose K sufficiently large to make this inequality hold. This proves Lemma 3.5.

At this point we fix $K = K(p)$.

Next we have to control the contribution from the broad points in B^4. We do this using the multilinear estimate.

Lemma 3.11. *Let Broad $\subset B^4$ denote the set of broad points.*

$$\int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^{\frac{3}{2}} \leq C(K)\delta^{-1/2}\rho^{7/2}. \tag{3.12}$$

Proof. Let $x \in B^4$ be a broad point. The broadness of x leads to the following estimate:

$$\Big| \sum_{y_i \in B_\rho} \chi_{T_i}(x) \Big|^3 \leq \rho^{-2} C(K) \sum_{y_i \in B_\rho} \chi_{T_i}(x) \sum_{y_j \in B_\rho} \chi_{T_j}(x) \sum_{y_k \in B_\rho} \chi_{T_k}(x) |v_i(x) \wedge v_j(x) \wedge v_k(x)|. \tag{3.13}$$

This holds because most triples of tubes through a broad point lie in clumps that fail to be coplanar, and so we have $|v_i(x) \wedge v_j(x) \wedge v_k(x)| \geq \rho^2/C(K)$.

Taking the square root of (3.13) and integrating, we get

$$\int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^{\frac{3}{2}} \leq$$

$$\leq C(K)\rho^{-1} \int_{B^4} \Big[\sum_{y_i \in B_\rho} \chi_{T_i}(x) \sum_{y_j \in B_\rho} \chi_{T_j}(x) \sum_{y_k \in B_\rho} \chi_{T_k}(x) |v_i(x) \wedge v_j(x) \wedge v_k(x)| \Big]^{1/2} dx. \tag{3.14}$$

But the right-hand side is controlled by the Multilinear Estimate. The number of points $y_i \in B_\rho$ is $\leq 100[\rho/\delta]^3$. According to Theorem 6, the right-hand side is bounded above by

$$(3.14) \lesssim_K \rho^{-1}\delta^4[\rho/\delta]^{9/2} = \delta^{-1/2}\rho^{7/2}$$

proving Lemma 3.11.

The estimate in Lemma 3.11 controls the $L^{3/2}$ norm of $\sum \chi_{T_i}$ on the broad set. There is an obvious estimate for the L^∞ norm, and by combining them we can estimate the L^p norm for our choice of $p > 3/2$.

We clearly have the L^∞ bound

$$\sup_x \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big| \lesssim \rho^3\delta^{-3}. \tag{3.15}$$

Since our $p > 3/2$, we see that

$$\int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^p dx \lesssim [\rho^3\delta^{-3}]^{p-3/2} \int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^{3/2} dx.$$

Applying Lemma 3.11 to bound the last integral, we see that

$$\int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^p dx \le C(K) \rho^{3p-1} \delta^{4-3p}. \tag{3.16}$$

Now we choose α large enough that $C(K) \le (1/2)\alpha^p$. (So α depends on K and p.) Now we know that

$$\int_{Broad} \Big| \sum_{y_i \in B_\rho} \chi_{T_i} \Big|^p dx \le (1/2) \alpha^p \rho^{3p-1} \delta^{4-3p}. \tag{3.17}$$

and (3.6), (3.17) imply (3.4).

This concludes the proof of Theorem $7'$ and hence Theorem 7.

4. Estimates for C^k curves

We can prove estimates for C^k curved Kakeya sets by approximating the C^k curves using polynomials. This idea was suggested to us by Alex Nabutovsky. He referred us to Jackson's theorem in approximation theory and related results.

The results in this section look far from optimal, but we wanted to show that something can be done for non-algebraic curves as well with these methods.

Jackson type theorem. *If $f : [0,1] \to \mathbb{R}$ has C^k norm 1, then we can approximate f by a degree d polynomial P so that*

$$|f(x) - P(x)| \lesssim d^{-k} \text{ for all } x \in [0,1]. \tag{4.1}$$

In particular, we may approximate a C^k curve Γ_i by a degree d algebraic curve with the same δ-tube and with $d \lesssim \delta^{-1/k}$.

Remark. This algebraic curve will be just the graph of a degree d polynomial. There are many more algebraic curves and so one may hope for a better estimate, but it would take some more sophisticated approximation theory.

Tracking the dependence on degree in Theorem 7, the following estimate is gotten.

Theorem $7''$. *Under the hypotheses in section 3, for all $p > 3/2$,*

$$\Big\| \sum_i \chi_{T_i} \Big\|_p \lesssim d^{\frac{3}{2p}} \delta^{-3+4/p}. \tag{4.2}$$

Hence we get the following estimate for C^k curves Γ_i with $k \ge 2$ obeying the angle condition:

Theorem 8. *Under the hypotheses above, for all $p > 3/2$,*

$$\Big\| \sum_i \chi_{T_i} \Big\|_p \lesssim_k \delta^{-3+\frac{4}{p}-\frac{3}{2pk}}. \tag{4.3}$$

In particular, for C^∞ curves we have essentially the same estimate that we had for algebraic curves.

An immediate consequence of Theorem 8 is the following result on the Minkowski dimension of curved Kakeya sets.

Theorem 9. *Any curved Kakeya set in 4D associated to C^∞-curves obeying the angle condition, has Minkowski dimension at least 3.*

The method described in §7 can be generalized to higher dimension. In particular, for n even, smooth curved Kakeya sets in $\mathbb{R}^n$ have Minkowski dimension at least $\frac{n}{2}+1$. This statement, which in some sense is the companion to Theorem 4, is the sharp version of a phenomenon first observed in [B2]. Note that for n odd, (algebraic) curved Kakeya sets may have Minkowski dimension $\frac{n+1}{2}$ (cf. [B2]).

§8. Further Comments

It is not quite clear at this point what is the exact potential of the method introduced in this paper (when the optimal result is not attained) and we have not tried to push the techniques to their limit. In particular, further improvements in Theorem 2 are not out of question and one could also explore if the more refined strategy used to obtain Theorem 2 in 3D has a higher dimensional counterpart (possibly improving upon Theorem 1).

Returning to inequality (5.8′) in §5, we present next an alternative proof for $n = 3$ of the following statement (which suffices for the application to Theorem 3 when $n = 3$).

Proposition 8.1. *Under the transversality assumption (5.6), (5.7) from §5 one has the 3-linear estimate in $3D$*

$$\Big\| \prod_{i=1}^{3} (T_\lambda^{(i)} f_i) \Big\|_{L^{q/3}} < \lambda^{-\frac{9}{q}} \prod_{i=1}^{3} \|f_i\|_2 \text{ for } q > \frac{10}{3} \tag{8.2}$$

where the operators $T_\lambda^{(i)}$ are given by (5.1), (5.2) with positive definite quadratic form and the phase functions are assumed algebraic of bounded degree.

Proposition 8.1 is weaker then $(5.8')$ in §5, but may be obtained directly without the need for an ε-removal lemma; hence this argument may have some interest.

Returning to the argument in [BCT] (which is similar to the one in [B1]) there are basically two steps, that will be suitably modified.

1. The first step in the approach involves the 'intermediate scale' $|x| < \frac{1}{\sqrt{\lambda}}$. At this scale, as explained in (5.19)-(5.23) from §5, the problem may be linearized in x. This allows to derive a trilinear bound from the bilinear $2\times 2 \to \frac{q}{2}$ estimate for $q > \frac{2(d+1)}{d} = \frac{10}{3}$ due to [T1] in the restriction theory rather than relying on a bootstrap. We point out that the linear result from [T1] for the paraboloid and, more generally, smooth hypersurfaces with positive definite second fundamental form, may fail without this last hypothesis (for instance for a hyperbolic paraboloid, cf. [V]), if no additional assumptions.

2. At the second stage of the argument, the issue is the 3-linear Kakeya estimate (in the curved case), which is Proposition 6.8 in [BCT]. Here another factor λ^ε enters in their argument. However, Theorem 6 of the paper may be used, since it immediately implies (by lowering the dimension from $\mathbb{R}^4$ to $\mathbb{R}^3$).

Proposition 8.3. *Denoting $\{\tau_i\}$ δ-neighborhoods of a family $\{\Gamma_i\}$ of smooth algebraic curves of degree $\lesssim 1$ in $B(0,1)\subset\mathbb{R}^3$ and v_i the tangent vector at a given point $p\in\Gamma_i$, one has*

$$\int \Big[\sum_{i,j,k} \lambda_i\mu_j\eta_k \mathcal{X}_{\tau_i\wedge\tau_i\wedge\tau_k}|v_i\wedge v_j\wedge v_k|\Big]^{1/2} < C\delta^3\Big(\sum|\lambda_i|\Big)^{1/2}\Big(\sum|\mu_j|\Big)^{1/2}\Big(\sum|\eta_k|\Big)^{1/2}$$

Proof of Proposition 8.1

Rescaling $x\to\frac{x}{\lambda}$, we obtain the phase function

$$\phi(x,y) = \lambda\phi\Big(\frac{x}{\lambda}, y\Big) \text{ where } |x| = o(\lambda).$$

Partition the y-domain Ω in boxes Ω_α of size $\frac{1}{\sqrt{\lambda}}$ centered at points y_α. Write for $y\in\Omega_\alpha$

$$\phi(x,y) = \phi(x,y_\alpha) + \langle\nabla_y\phi(x,y_\alpha)\rangle_O(\lambda|y-y_\alpha|^2)$$

where the last term may be dropped.

$$T_\alpha f(x) = \int_{\Omega_\alpha} e^{i\langle\nabla_y\phi(x,y_\alpha), y-y_\alpha\rangle} f(y)dy \tag{8.4}$$

and write

$$Tf(x) = \sum_\alpha e^{i\phi(x,y_\alpha)}(T_\alpha f)(x). \tag{8.5}$$

Next, introduce a variable $z \in B(0, \sqrt{\lambda})$, writing

$$Tf(x+z) \sim \sum_\alpha e^{i\phi(x+z,y_\alpha)}(T_\alpha f)(x). \tag{8.6}$$

Returning to (8.2), write

$$\int_{B(0,\lambda)} \Big[\prod_{i=1}^3 |T^{(i)}f_i|\Big]^{\frac{q}{3}} \sim \lambda^{-\frac{3}{2}} \int_{B(0,\lambda)} \Big\| \prod_{i=1}^3 (T^{(i)}f_i)(x+z)\Big\|^{q/3}_{L^{q/3}(|z|<\sqrt{\lambda})} dx \tag{8.7}$$

with $T^{(i)}f_i(x+z)$ replaced by (8.6).

Estimate $\| \prod_{i=1}^3 \|_{\frac{q}{3}} \leq \| \prod_{i=1,2} \|^{\frac12}_{\frac{q}{2}} \| \prod_{i=2,3} \|^{\frac12}_{\frac{q}{2}} \| \prod_{i=3,1} \|^{\frac12}_{\frac{q}{2}}$.

Denoting

$$\eta(z,y) = \phi(x+z,y) - \phi(x,y) \qquad (x \text{ fixed})$$

we bound

$$\int_{B(0,\sqrt{\lambda})} \Big| \sum_\alpha e^{i\eta(z,y_\alpha)}(T_\alpha^{(1)}f_1)(x)\Big|^{\frac{q}{2}} \ \Big|\sum_\beta e^{i\eta(z,y_\beta)}(T_\beta^{(2)}f_2)(x)\Big|^{\frac{q}{2}} dz. \tag{8.8}$$

Define functions g_1, g_2 by

$$g_1(y) = e^{i\eta(z,y_\alpha)}(T_\alpha^{(1)}f_1)(x) \text{ for } y \in \Omega_\alpha \tag{8.9}$$

and similarly for g_2.

Clearly

$$(8.8) \sim \lambda^q \int_{B(0,\sqrt{\lambda})} \Big| \int e^{i\eta(z,y)} g_1(y)dy\Big|^{\frac{q}{2}} \ \Big| \int e^{i\eta(z,y)} g_2(y)dy\Big|^{\frac{q}{2}} dz. \tag{8.10}$$

Since, following (5.19)-(5.22) in §5, η has the form

$$\eta(z,y) = z_1y_1 + z_2y_2 + z_3\big(\langle Ay,y\rangle + O(|y|^3)\big) + O(|z|\frac{|x|}{\lambda}|y|^2) + O\Big(\frac{|z|^2}{\lambda}|y|^2\Big) \tag{8.11}$$

the last term in (8.11) may be dropped for $|z| < \sqrt{\lambda}$. Hence $\eta(z,y)$ may be viewed as linear in z, of the form

$$z_1y_1 + z_2y_2 + z_3\langle A'y,y\rangle + O(|z| \ |y|^3) \tag{8.12}$$

with A' positive definite.

Applying the bilinear $2\times 2\to \frac{q}{2}$ bound from [T1], it follows that

$$
\begin{aligned}
(8.10) &\lesssim \lambda^q \|g_1\|_2^{\frac{q}{2}} \|g_2\|_2^{\frac{q}{2}}\\
&\sim \lambda^{q/2}\Big[\sum_\alpha |(T_\alpha^{(1)} f_1)(x)|^2\Big]^{\frac{q}{4}} \Big[\sum_\beta |(T_\beta^{(2)} f_2)(x)|^2\Big]^{\frac{q}{4}}. \qquad (8.13)
\end{aligned}
$$

¿From (8.13), the following bound on (8.7) is obtained

$$\lambda^{\frac{1}{2}(q-3)} \int_{B(0,\lambda)} \Big\{\prod_{i=1}^3 \Big[\sum_\alpha |(T_\alpha^{(i)} f_i)(x)|^2\Big]^{\frac{q}{6}}\Big\} dx. \tag{8.14}$$

The next step is to capture the factors in (8.14) by curved Kakeya maximal functions. ¿From definition of T_α

$$|T_\alpha f|^2(x) = |\hat{f}_\alpha|^2\big(\nabla_y \phi(x, y_\alpha)\big) \text{ where } f_\alpha = f|_{\Omega_\alpha}. \tag{8.15}$$

Let b be a standard bumpfunction on $\mathbb{R}^{d-1}$. Then $|\hat{f}_\alpha|^2$ may be recovered by an average of translates $b(\frac{\xi-\cdot}{\sqrt{\lambda}})$ with averaging weight $\lambda^{-1}\|f_\alpha\|_2^2$.

Denoting

$$c_\alpha^{(i)} = \|f_{i,\alpha}\|_2^2 \qquad (i=1,2,3)$$

satisfying

$$\sum_\alpha c_\alpha^{(i)} = \|f_i\|_2^2 \tag{8.16}$$

we obtain therefore

$$\sum_\alpha |(T_\alpha^{(i)} f_i)(x)|^2 \lesssim \lambda^{-1} \sum_{\alpha,\nu}^{(i)} b\big(\lambda^{-\frac{1}{2}}(\nabla_y \phi(x, y_\alpha) - \xi_{\alpha,\nu})\big).c_{\alpha,\nu}^{(i)} \tag{8.17}$$

where $c_{\alpha,\nu}^{(i)} > 0, \sum_\nu c_{\alpha,\nu}^{(i)} = c_\alpha^{(i)}$ and

$$\sum_{\alpha,\nu} c_{\alpha,\nu}^{(i)} = \|f_i\|_2^2. \tag{8.18}$$

Substituting (8.17) in (8.14), one gets

$$\lambda^{-\frac{3}{2}} \int_{B(0,\lambda)} \Big\{\prod_{i=1}^3 \Big[\sum_{\alpha,\nu}^{(i)} b\big(\lambda^{-\frac{1}{2}}(\nabla_y\phi(x,y_\alpha) - \xi_{\alpha,\nu})\big) c_{\alpha,\nu}^{(i)}\Big]^{\frac{q}{6}}\Big\} dx$$

$$= \lambda^{\frac{3}{2}} \int_{B(0,1)} \Big\{ \prod_{i=1}^{3} \Big[\sum_{\alpha,\nu}^{(i)} c^{(i)}_{\alpha,\nu} b\big(\lambda^{-\frac{1}{2}} (\nabla_y \phi(\lambda x', y_\alpha) - \xi_{\alpha,\nu})\big)\Big]^{\frac{q}{6}} \Big\} dx' \tag{8.19}$$

We may now apply Proposition 8.3. In the present trilinear setting, $|v_i \wedge v_j \wedge v_k| > c$ and hence

$$\Big\| \prod_{i=1}^{3} \Big[\sum_s \lambda_s^{(i)} \mathcal{X}_{\tau_s(i)} \Big] \Big\|_{L^{1/2}} \leq c\delta^6 \prod_{i=1}^{3} \Big[\sum_s |\lambda_s^{(i)}| \Big] \tag{8.20}$$

where $\delta = \frac{1}{2}$ and the tubes τ of the form

$$\lambda^{-1} |\nabla_y \phi(\lambda x, y) - \xi| < \lambda^{-1/2}. \tag{8.21}$$

Interpolation of (8.20) with the obvious L^∞-bound gives, for $r \geq \frac{1}{2}$

$$\Big\| \prod_{i=1}^{3} \Big[\sum_s \lambda_s^{(i)} \mathcal{X}_{\tau_s^{(i)}} \Big] \Big\|_{L^r} \leq c\delta^{\frac{3}{r}} \prod_{i=1}^{3} \Big[\sum_s |\lambda_s^{(i)}| \Big]. \tag{8.22}$$

Application of (8.22) to (8.19) with $r = \frac{q}{6} > \frac{5}{9}$ implies, by (8.18)

$$(8.7), (8.14), (8.19) < C\lambda^{3/2} \Big(\frac{1}{\sqrt{\lambda}} \Big)^3 \prod_{i=1}^{3} \|f_i\|_2^{q/3} \tag{8.23}$$

and in view of the initial rescaling, (8.2) follows.

Appendix: Upsilon Removal Lemmas

We consider first the restriction (or extension) problem.

What follows is basically a modification of Theorem 1.2 in [T2] on deriving global restriction estimates from local ones. A significant difference is that instead of considering bounds of the type ($\gamma > 0$)

$$\|\hat{f}|_S\|_{L^p(d\sigma)} \lesssim R^\gamma \|f\|_{L^p(B_R)} \tag{1}$$

for $f \in L^p(\mathbb{R}^n), \operatorname{supp} f \subset B_R$, we start from a local inequality of the form

$$\|\hat{f}|_S\|_{L^1(d\sigma)} \lesssim R^\gamma \|f\|_{L^p(B_R)}. \tag{2}$$

Compared with the argument from [T2], this will require additional involvement of the Maurey-Nikishin factorization theorem.

Lemma A1. *Assuming* $1 < p < 2$, $0 < \gamma \ll 1$ *and (2). Then*

$$\|\hat{f}|_S\|_{L^1(d\sigma)} \lesssim \|f\|_{p_1} \tag{3}$$

for $f \in L^{p_1}(\mathbb{R}^n)$ *and*

$$\frac{1}{p_1} > \frac{1}{p} + \frac{C}{\log \frac{1}{\gamma}}. \tag{4}$$

In particular, if (1) holds for arbitrary small $\gamma > 0$, the global inequality (3) will hold for any $p_1 < p$.

We start by dualizing (2), implying that the operator

$$T : L^\infty(S, d\sigma) \to L^{p'}(B_R) : \varphi \to \widehat{\varphi\sigma}|_{B_R} \qquad \Big(p' = \frac{p}{p-1}\Big)$$

satisfies $\|T\| < R^\gamma$. Hence, from the theory of absolutely summary operators, fixing any $r > p' > 2$, there is a probability measure μ on S, such that

$$\|\widehat{\varphi\sigma}\|_{L^{p'}(B_R)} \lesssim R^\gamma \|\varphi\|_{L^r(d\mu)}. \tag{5}$$

There is no harm to assume $\frac{d\mu}{d\sigma} > \frac{1}{2}$.

We first enforce some smoothness for the density. Let $\tau : S \to S$ be any diffeomorphism that is $\frac{1}{R}$-close to the identity. Then, for $|x| < R$, a change of variables gives

$$\begin{aligned}
\widehat{(\varphi \circ \tau)}\sigma(x) = \int \varphi\big(\tau(\xi)\big) e(x.\xi)\sigma(d\xi) = \\
\int \varphi(\xi') e\big(x.\tau^{-1}(\xi')\big)\Delta(\xi')\sigma(dx') \ \text{ where } |1-\Delta| \lesssim \frac{1}{R} \\
= \int (\Delta\varphi)(\xi') e(x.\xi')\sigma(d\xi') \qquad (6)\\
+ O\Big\{\sum_{j\geq 1} \frac{1}{j!}\Big|\int (\Delta\psi_j\varphi)(\xi') e(x\xi')\sigma(d\xi')\Big|\Big\} \qquad (7)
\end{aligned}$$

where (7) is obtained by Taylor expansion of $e\big(x.(\tau^{-1}(\xi') - \xi')\big)$ and $|\psi_j(\xi')| < (R|\tau^{-1}(\xi') - \xi'|)^j < 1$ by assumption on τ. Hence

$$|T(\varphi \circ \tau)| \leq |T(\varphi\Delta)| + \sum_{j\geq 1} \frac{1}{j!} |T(\Delta\psi_j\varphi)|$$

and applying (5)

$$\|T(\varphi\circ\tau)\|_{L^{p'}(B_R)} \lesssim R^\gamma \|\varphi\|_{L^r(d\mu)}.$$

Replacing φ by $\varphi\circ\tau^{-1}$, we obtain

$$\|T\varphi\|_{p'} \lesssim R^\gamma\|\varphi\circ\tau^{-1}\|_{L^r(d\mu)} = R^\gamma\|\varphi\|_{L^r(d\mu_\tau)}$$

with $\mu_\tau = (\tau^{-1})_*[\mu]$. Averaging over τ as above allows us to smoothen out μ at scale $\frac{1}{R}$ and replace μ by a probability measure μ' on S, $\mu \ll \sigma$ and $\frac{d\mu'}{d\sigma} = \rho \geq \frac{1}{2}$ with ρ smooth at scale $\frac{1}{R}$. Thus we have

$$\|\widehat{\varphi\sigma}\|_{L^{p'}(B_R)} \leq R^\gamma\|\varphi\rho^{1/r}\|_{L^r(d\sigma)} \tag{8}$$

and dualizing

$$\|\hat{f}\rho^{-1/r}\big|_S\|_{L^{r'}(d\sigma)} \leq R^\gamma\|f\|_p \text{ if } \operatorname{supp} f\subset B_R. \tag{8$'$}$$

In what preceeds, we fixed $R \geq 1$. Note that ρ depends on R.

Following [T2], define a finite collection of balls $\{B(a_\alpha, R)\}_{\alpha=1}^N$ in $\mathbb{R}^n$ as 'sparse' if for $\alpha \neq \alpha'$

$$|a_\alpha - a_{\alpha'}| > (NR)^C \tag{9}$$

(C some constant to specify).

Let now $\operatorname{supp} f \subset \bigcup_\alpha B(a_\alpha, R)$, i.e.

$$f = \sum_{\alpha=1}^N f_\alpha(x-a_\alpha) \text{ with } \operatorname{supp} f_\alpha\subset B_R.$$

Hence

$$\hat{f}(\xi) = \sum e(a_\alpha.\xi)\hat{f}_\alpha(\xi)$$

and since $\|\varphi\|_1 = 1$

$$\|\hat{f}\big|_S\|_{L^1(d\sigma)} \leq \Big\|\Big[\sum e(a_\alpha.\xi)\hat{f}_\alpha(\xi)\Big]\rho^{-\frac{1}{r}}(\xi)\Big\|_{L^{r'}(d\sigma)}. \tag{10}$$

Note that by our construction of ρ, the function $g_\alpha = \hat{f}_\alpha.\rho^{-\frac{1}{r}}\big|_S$ is smooth at scale $\frac{1}{R}$. The sparsity of $\{a_\alpha\}$ allows then to estimate

$$\Big\|\sum_1^N e(a_\alpha.\xi)g_\alpha(\xi)\Big\|_{L^{r'}(d\sigma)} \leq 2\Big(\sum\|g_\alpha\|_{L^{r'}(d\sigma)}^{r'}\Big)^{1/r'}. \tag{11}$$

This is basically Lemma 3.2 in [T2] and we include the argument for completeness sake.

Establish (11) by interpolation.

More precisely, the claim will follow from an inequality for $1 \leq s \leq 2$

$$\Big\| \sum_1^N e(a_\alpha.\xi)(\tilde{\varphi}_\alpha * P_{\frac{1}{R}})(\xi)\Big|_S \Big\|_{L^s(d\sigma)} \lesssim \Big(\sum \|\varphi_\alpha\|^s_{L^s(d\sigma)}\Big)^{\frac{1}{s}} \tag{12}$$

where $\{\varphi_\alpha\}$ are arbitrary functions in $L^s(S, d\sigma)$, $\sim$ denotes a well-behaved extension operator from $L^*(S) \to L^*(\mathbb{R}^n)$ (take for instance the harmonic extension) and $P_{\frac{1}{R}}$ is an $\frac{1}{R}$-approximate identity.

For $s = 1$, (12) is trivial from triangle inequality and since $\|(\tilde{\varphi} * P_{\frac{1}{R}})\big|_S\|_1 \lesssim \|\varphi\|_1$.

For $s = 2$, we obtain for the square of the left side of (12)

$$\sum_1^N \|\varphi_\alpha\|_2^2 + \sum_{\alpha \neq \alpha'} \Big| \int e\big((a_\alpha - a_{\alpha'}).\xi\big)(\tilde{\varphi}_\alpha * P_{\frac{1}{R}})(\xi).\overline{(\tilde{\varphi}_{\alpha'} * P_{\frac{1}{R}})}(\xi)\sigma(d\xi)\Big| \tag{13}$$

and show that the contribution of the off-diagonal is small.

Denoting $\Phi_\alpha = \tilde{\varphi}_\alpha * P_{\frac{1}{R}}$, we may assume $\operatorname{supp} \hat{\Phi}_\alpha \subset B_R$ so that clearly, invoking the decay of $\hat{\sigma}$ and the fact that $|a_\alpha - a_{\alpha'}| \gg R$

$$\Big| \int e\big((a_\alpha - a_{\alpha'}).\xi\big)\Phi_\alpha(\xi)\overline{\Phi_{\alpha'}}(\xi)\sigma(d\xi)\Big| \lesssim$$

$$\frac{1}{|a_\alpha - a_{\alpha'}|^{\frac{n-1}{2}}} \ \|\hat{\phi}_\alpha\|_1 \ \|\hat{\phi}_{\alpha'}\|_1 \lesssim \frac{R^n}{|a_\alpha - a_{\alpha'}|^{\frac{n-1}{2}}} \|\phi_\alpha\|_2 \ \|\phi_{\alpha'}\|_2$$

$$\lesssim \frac{R^n}{|a_\alpha - a_{\alpha'}|^{\frac{n-1}{2}}} \ \|\varphi_\alpha\|_2 \|\varphi_{\alpha'}\|_2.$$

Consequently, the second term in (13) is bounded by the first, provided

$$\max_\alpha \sum_{\alpha' \neq \alpha} \frac{1}{|a_\alpha - a_{\alpha'}|^{\frac{n-1}{2}}} < R^{-n}.$$

This will be ensured if we require for $\alpha \neq \alpha'$

$$|a_\alpha - a_{\alpha'}| > N^{\frac{n+1}{n(n-1)}} R^{\frac{2n}{n-1}} \tag{14}$$

as implied by (9) for C large enough. Then (12) will hold for $s = 2$ and hence for $1 \le s \le 2$. Thus we proved (11).

Application of (11) with $g_\alpha = \hat f_\alpha . \rho^{-\frac{1}{r}}|_S$ and invoking (8′) implies that

$$\|\hat f|_S\|_{L^1(d\sigma)} \lesssim R^\gamma \Big(\sum_{\alpha=1}^N \|f_\alpha\|_p^{r'}\Big)^{\frac{1}{r'}} \lesssim R^\gamma N^{\frac{1}{r'}-\frac{1}{p}} \|f\|_p \tag{15}$$

(recall that $r > p'$ is arbitrary).

Thus inequality (15) holds provided supp f is contained in a sparse collection of N balls of radius R.

The next ingredient is the following covering lemma (Lemma 3.3) from [T2].

Lemma A2. *Suppose $E \subset \mathbb{R}^n$ is a finite union of 1-cubes and take $0 < \delta < 1$. Then there exist $O\big(\frac{1}{\delta}|E|^\delta\big)$ sparse collections of balls that cover E, such that the balls in each collection have radius at most $O\big(|E|^{C^{1/\delta}}\big)$.*

Of course the number of balls in each collection is trivially bounded by $|E|$.

Assume $\operatorname{supp} f \subset E$ and apply Lemma A2 to E (assumed a union of 1-cubes). Hence

$$E \subset \bigcup_{j \lesssim \frac{1}{\delta}|E|^\delta} \ \bigcup_{a \in \mathcal{E}_j} B(a, R_j)$$

with $R_j \lesssim |E|^{C^{1/\delta}}$ and $\{B(a,R_j); a \in \mathcal{E}_j\}$ sparse for each j; $\#\mathcal{E}_j \le N = |E|$.

Writing $f = \sum f_j, f_j = f\big|_{\bigcup_{a\in\mathcal{E}_j} B(a,R_j)}$, application of inequality (15) to each f_j implies

$$\|\hat f\big|_S\|_{L^1(d\sigma)} \lesssim \frac{1}{\delta}|E|^{\gamma C^{1/\delta}+\delta} N^{\frac{1}{r'}-\frac{1}{p}} \|f\|_p. \tag{16}$$

Taking $\delta \sim \frac{1}{\log\frac{1}{\gamma}}$ and $r < p' + \frac{1}{\log\frac{1}{\gamma}}$, we conclude that

$$\|\hat f\big|_S\|_{L^1(d\sigma)} \lesssim_\gamma |E|^{\frac{C}{\log\frac{1}{\gamma}}} \|f\|_p. \tag{17}$$

Let $p_1 < p$ and $f \in L^{p_1}(\mathbb{R}^n)$, $\|f\|_{p_1} \le 1$, which we assume constant on c-cubes ($c \sim 1$).

Decompose in level sets

$$f = \sum_{k\ge 0} f\big|_{[2^{-k-1} \le |f| < 2^{-k}]} = \sum f_k$$

with supp $f_k = E_k, E_k$ a union of N_k c-cubes and $2^{-kp_1} N_k \lesssim 1$.

¿From (17)

$$\|\hat{f}_k\big|_S\|_{L^1(d\sigma)} \lesssim N_k^{\frac{c}{\log\frac{1}{\gamma}}} \|f_k\|_p \lesssim 2^{k[\frac{cp_1}{\log\frac{1}{\gamma}} + \frac{p_1}{p} - 1]}$$

and therefore

$$\|\hat{f}\big|_S\|_{L^1(d\sigma)} < C_\gamma \tag{18}$$

provided

$$\frac{C}{\log\frac{1}{\gamma}} < 1 - \frac{p_1}{p} \tag{19}$$

which amounts to condition (4).

Arguing like in [T2], we showed that (18) holds for any function $f \in L^{p_1}(\mathbb{R}^n)$ of the form $f = \sum_{\xi\in\mathcal{L}} \lambda_\xi 1_{B(\xi,c)}$ with $\sum |\lambda_\xi|^{p_1} \leq 1$ and $\mathcal{L}$ a (shifted) 1-lattice. Taking $c > 0$ a sufficiently small constant as to ensure that $\widehat{1_{B(0,c)}}$ is positive on S, it follows that

$$\Big\|\Big[\sum_{\xi\in\mathcal{L}} \lambda_\xi e(x.\xi)\Big]\Big|_S\Big\|_{L^1(d\sigma)} < C\Big(\sum |\lambda_\xi|^{p_1}\Big)^{\frac{1}{p_1}}. \tag{20}$$

Another averaging over translates $\mathcal{L}$ of the $\mathbb{Z}^n$-lattice gives (3).

This completes the proof of Lemma A1.

Next, we prove the upsilon-removal lemma in the variable coefficient multilinear case. Recall the setting.

Consider T_λ and in (1.4), (1.5) with fixed, large λ and define

$$(Tf)(x) = \int e^{i\phi(x,y)} f(y)dy \tag{21}$$

with

$$\phi(x,y) = x_1y_1 + \cdots + x_{n-1}y_{n-1} + x_n\big(\langle Ay, y\rangle + O(|y|^3)\big) + \lambda\phi_\nu\Big(\frac{x}{A}, y\Big) \tag{22}$$

as in §5, where $|x| = o(\lambda), |y| = o(1)$ and A non-degenerate.

Let $2 \leq k \leq n$ and $U_1, \dots, U_k$ fixed balls in y-space satisfying the transversality condition (5.6). For $j = 1, \dots k$, denote

$$T_j f = \int_{U_j} e^{i\phi(x,y)} f(y)dy. \tag{23}$$

Clearly the [BCT] result implies that if $1 < R < o(\lambda)$, then

$$\Big\|\Big(\prod_{j=1}^{k}|T_jf_j|\Big)^{\frac{1}{k}}\Big\|_{L^q(B_R)} \ll R^\varepsilon\Big(\prod_{j=1}^{k}\|f_j\|_2\Big)^{1/k} \tag{24}$$

with $q = \frac{2k}{k-1}$ and $B_R = B(0,R)$. This statement is also easily seen to imply (24) with $B_R = B(a,R)$ any R-ball with $|a| = o(\lambda)$.

Our aim is to remove the R^ε-factor at the cost of increasing slightly the exponent q. Thus

Lemma A3. *Under the above assumptions and taking $q_1 > \frac{2k}{k-1}$, we have an inequality*

$$\Big\|\Big(\prod_{j=1}^{k}|T_jf_j|\Big)^{\frac{1}{k}}\Big\|_{q_1} \leq C_{n,k,q_1}\Big(\prod_{j=1}^{k}\|f_j\|_2\Big)^{1/k}. \tag{25}$$

(Note that we do not claim removal of the λ^ε-factor in Theorem 6.2 from [BCT], as the context of our Lemma A3 is more restrictive, since the T_j-operators are given by (22), (23))

Let $\|f_j\|_2 = 1$ and $F = \big(\prod_1^k |T_jf_j|\big)^{\frac{1}{k}}$.

Let $E \subset \mathbb{R}^d$ be obtained as union of a sparse collection of R-balls $B(a_\alpha, R)$, $|a_\alpha| = o(\lambda)$ with $\alpha = 1,\dots,N$. We will show that

$$\|F|_E\|_q < C_\varepsilon R^\varepsilon. \tag{26}$$

Using Lemma A2, this will imply that for $E' \subset \mathbb{R}^n$ any finite union of 1-cubes we have

$$\|F\big|_{E'}\|_q < \frac{1}{\delta}C_\varepsilon|E'|^{\delta+\varepsilon C^{1/\delta}} \tag{27}$$

with $\delta > 0$ a parameter. Hence, for all $\varepsilon < 0$

$$\|F\big|_{E'}\|_q < C'_\varepsilon|E'|^\varepsilon \tag{28}$$

from where one easily deduces that $\|F\|_{q_1} < C_{q_1}$ for $q_1 > q$.

Let $E = \bigcup B(a_\alpha, R)$ be as above and fix α. Write for $x \in B(a_\alpha, R)$

$$(Tf)(x) = \int e^{i[\phi(x,y)-\phi(a_\alpha,y)]}\big(e^{i\phi(a_\alpha,y)}f(y)\big)\ \omega_j(y)dy \tag{29}$$

with ω_j a smooth localization on U_j.

Denoting $g(y) = e^{i\phi(a_\alpha,y)}f(y)$,

$$(29) = \int \Big[\int e^{i[\phi(x,y)-\phi(a_\alpha,y)+\xi y]}\omega_j(y)dy\Big]\hat{g}(\xi)d\xi. \tag{30}$$

Since $|\nabla_y[\phi(x,y)-\phi(a_\alpha,y)]| \lesssim |x-a_\alpha| \lesssim R$, we may clearly replace in (30) the function g by $P_{R_1}g = (\hat{g}\eta_{R_1})^\vee$, denoting $\eta_{R_1}(z) = \eta(\frac{z}{R_1})$ where $0 \le \eta \le 1$ is a smooth bumpfunction with $\eta(0) = 1$, and taking say

$$R_1 = 100NR. \tag{31}$$

The remaining contribution to (30) will then indeed by L^∞-bounded by $0\big((NR)^{-C}\big)$.

Defining

$$f_\alpha = e^{-i\phi(a_\alpha,y)}P_{R_1}\big(e^{i\phi(a_\alpha,y)}f\big)$$

we can thus replace Tf by Tf_α on $B(a_\alpha,R)$. Note that $|f_{j,\alpha}| \le |f_j| * |\overset{\vee}{\eta}_{R_1}|$ may clearly be assumed supported by U_j.

Estimate

$$\begin{aligned} \|F\big|_E\|_q^q &= \sum_\alpha \|F\big|_{B(a_\alpha,R)}\|_q^q \\ &= \sum_\alpha \Big\|\Big(\prod_j |T_j(f_{j,\alpha})|\Big)^{\frac{1}{k}}\Big\|^q_{L^q(B(a_\alpha,R))} + o(1) \\ &\overset{24}{\le} C_\varepsilon R^{q\varepsilon}\sum_\alpha\Big(\prod_j \|f_{j,\alpha}\|_2\Big)^{q/k} + o(1) \\ &< C_\varepsilon R^{q\varepsilon}\max_j\Big[\sum_\alpha \|f_{j,\alpha}\|_2^q\Big] + o(1). \end{aligned} \tag{32}$$

Since $q > 2$,

$$\Big(\sum_\alpha \|f_\alpha\|_1^q\Big)^{\frac{1}{q}} \le \Big(\sum \|f_\alpha\|_2^2\Big)^{\frac{1}{2}} = \Big(\sum_\alpha \|P_{R_1}(e^{i\phi(a_\alpha,y)}f)\|_2^2\Big)^{1/2}. \tag{33}$$

To bound (33), take functions $\{\zeta_\alpha\}$ such that supp $\hat{\zeta}_\alpha \subset B(0,R_1)$ and $\sum_\alpha \|\zeta_\alpha\|_2^2 = 1$ and evaluate

$$\sum_\alpha \langle e^{i\phi(a_\alpha,\cdot)}f,\zeta_\alpha\rangle \le \Big\|\sum_\alpha e^{i\phi(a_\alpha,y)}\zeta_\alpha(y)\Big\|_2 \|f\|_2. \tag{34}$$

For the off-diagonal terms $\alpha \neq \beta$

$$|\langle e^{i\phi(a_\alpha,\cdot)}\zeta_\alpha, e^{i\phi(a_\beta,\cdot)}\zeta_\beta\rangle| = \Big|\int e^{i[\phi(a_\alpha,y)-\phi(a_\beta,y)]}(\zeta_\alpha\bar{\zeta}_\beta)(y)dy\Big|. \tag{35}$$

where

$$\phi(a,y)-\phi(a',y) = (a_1-a_1')y_1+\cdots+(a_{d-1}-a_{d-1}')y_{d-1}+(a_d-a_{d'})\big(\langle Ay,y\rangle+O(|y|^3)\big) + \lambda\Big[\phi_\nu\Big(\frac{a}{\lambda},y\Big)-\phi_\nu\Big(\frac{a'}{\lambda},y\Big)\Big]$$

satisfies either

$$|\nabla_y[\phi(a,y)-\phi(a',y)]| \gtrsim |a-a'|$$

or

$$|\det D_y^2[\phi(a,y)-\phi(a',y)]| \gtrsim |a-a'|^{n-1}.$$

Hence, recalling the sparsity assumption $|a_\alpha-a_\beta|>(NR)^C \gg R_1$, it follows that

$$(34) \lesssim |a_\alpha-a_\beta|^{-\frac{n-1}{2}}\|\hat{\zeta}_\alpha\|_1\ \|\hat{\zeta}_\beta\|_1 \lesssim R_1^{n-1}(NR)^{-C}\|\zeta_\alpha\|_2\ \|\zeta_\beta\|_2. \tag{36}$$

Therefore $(34) \leq 2\big(\sum\|\zeta_\alpha\|_2^2\big)^{1/2} \leq 2$ and (33) is bounded. Inequality (26) now follows from (32), completing the proof of Lemma A3.

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