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The f-Vector of the Descent Polytope

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Abstract For a positive integer n and a subset $S \subseteq [n-1]$, the descent polytope DP_S is the set of points $(x_1, \ldots, x_n)$ in the n-dimensional unit cube $[0,1]^n$ such that $x_i \geq x_{i+1}$ if $i \in S$ and $x_i \leq x_{i+1}$ otherwise. First, we express the f-vector as a sum over all subsets of $[n-1]$. Second, we use certain factorizations of the associated word over a two-letter alphabet to describe the f-vector. We show that the f-vector is maximized when the set S is the alternating set $\{1,3,5,\ldots\} \cap [n-1]$. We derive a generating function for $F_S(t)$, written as a formal power series in two non-commuting variables with coefficients in $\mathbb{Z}[t]$. We also obtain the generating function for the Ehrhart polynomials of the descent polytopes.

Keywords Descent set statistics · Maximizing inequalities · Alternating set · Non-commutative rational generating function · Ehrhart polynomial

1 Introduction

A classic topic in combinatorics is the study of descent set statistics of permutations. For a subset S of the set $[n-1] = \{1, \ldots, n-1\}$, the statistic $\beta(S)$ denotes the number of permutations in the symmetric group $\mathfrak{S}_n$ with descent set S. One well-known result is that the descent set statistic $\beta(S)$ is maximized on the alternating sets: $\{1,3,\ldots\} \cap [n-1]$ and $\{2,4,\ldots\} \cap [n-1]$; see [2, 6–8, 12].

In this paper we study a class of polytopes DP_S which we call *descent polytopes*. They are indexed by subsets S of $[n-1]$, and the polytope corresponding to S is the

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closure of the set of points of the unit hypercube $[0, 1]^n$ whose coordinates, viewed as a sequence of n numbers, have descent set S. In general, these polytopes are not simplicial nor simple: the polytope for $n = 3$ and $S = \{1\}$ is the Egyptian pyramid, that is, the square based pyramid. We show how to compute the f-vector of the descent polytope DP_S.

Some invariants of descent polytopes are directly related to the descent set statistic, and others exhibit analogous behavior. For example, the volume of the descent polytope DP_S is given by the descent set statistic $\beta(S)/n!$. We also show in the paper that the f-vector of the descent polytope is entrywise maximized on the alternating set. In order to prove this result we show that the entries of the f-vector obey certain inequalities analogous to those satisfied by descent set statistics; see Theorem 2.4.

One way to encode a subset S of $[n-1]$ is by a word $\mathbf{v}_S$ in two letters, in our case $\mathbf{x}$ and $\mathbf{y}$. Since $\mathbf{v}_S$ can be viewed as a non-commutative monomial in the two variables $\mathbf{x}$ and $\mathbf{y}$, this encoding suggests that one should work with non-commutative generating functions. That is, to study a polytope invariant ϕ of descent polytopes, one has to determine the generating function

$$\sum_{n\geq 1}\sum_{S\subseteq[n-1]} \phi(\mathrm{DP}_S)\cdot \mathbf{v}_S = \sum_{\mathbf{v}} \phi(\mathrm{DP}_{\mathbf{v}})\cdot \mathbf{v}, \tag{1.1}$$

where we tacitly allow the descent polytopes to be indexed by monomials.

In the case of the f-polynomial, an encoding of the f-vector, the generating function in (1.1) is a rational generating function; see Theorem 3.2. Furthermore, by expanding this rational function we obtain a more concise expression for the f-polynomial of the descent polytope DP_S. This expression is in terms of a particular type of factorizations of the monomial $\mathbf{v}_S$; see Corollary 3.3.

Descent polytopes are also lattice polytopes and hence their Ehrhart polynomials are also of interest. We also determine the non-commutative generating function for Ehrhart polynomials of the descent polytopes; see Theorem 4.2. We note that even this power series is rational.

We end the paper with a few open questions and directions for further research.

2 An Expression for the f-Polynomial $F_{\mathbf{v}}$

For a set $S \subseteq [n-1] = \{1, 2, \ldots, n-1\}$, define the *descent polytope* DP_S to be the set of points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $0 \leq x_i \leq 1$, and

$$\begin{cases} x_i \geq x_{i+1} & \text{if } i \in S, \\ x_i \leq x_{i+1} & \text{if } i \notin S. \end{cases}$$

Thus DP_S is the *order polytope* of the ribbon poset $Z_S = \{z_1, z_2, \ldots, z_n\}$ defined by the cover relations $z_i \gtrdot z_{i+1}$ if $i \in S$ and $z_i \lessdot z_{i+1}$ if $i \notin S$; see [9]. It is clear that the set S and its complement $\overline{S} = [n-1] - S$ yields the same descent polytope up to an affine transformation. Also the reverse set $S^{rev} = \{n - i : i \in S\}$ give the same polytope.

Let $\mathbf{x}$ and $\mathbf{y}$ be two non-commuting variables. For $S \subseteq [m]$, define $\mathbf{v}_S = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_m$ where

$$\mathbf{v}_i = \begin{cases} \mathbf{x} & \text{if } i \notin S, \\ \mathbf{y} & \text{if } i \in S. \end{cases}$$

Since pairs (n, S), where $S \subseteq [n-1]$, are in bijective correspondence with $\mathbf{xy}$-words via $(n, S) \mapsto \mathbf{v}_S$, it is natural to parameterize the descent polytopes and their f-polynomials by $\mathbf{xy}$-words. That is, we write $\mathrm{DP}_{\mathbf{v}}$ and $F_{\mathbf{v}}$, where $\mathbf{v} = \mathbf{v}_S$ for some $S \subseteq [|\mathbf{v}|]$, and $|\mathbf{v}|$ denotes the length of the word $\mathbf{v}$. This notation has the advantage, as the $\mathbf{xy}$-word not only encodes the subset but also the dimension n.

For an $\mathbf{xy}$-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, define the statistic $\kappa(\mathbf{v})$ by $\kappa(\mathbf{v}) = 2 + |\{i : \mathbf{v}_i \neq \mathbf{v}_{i+1}\}|$ for $\mathbf{v} \neq 1$, and $\kappa(1) = 1$. A direct observation is that the number of facets of the descent polytope is described by κ.

Lemma 2.1 *The number of* $(n-1)$*-dimensional faces of the* n*-dimensional descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *is given by*

$$f_{n-1}(\mathrm{DP}_{\mathbf{v}}) = n - 1 + \kappa(\mathbf{v}).$$

Proof There are $n-1$ supporting hyperplanes of the form $x_i = x_{i+1}$ that each intersect the polytope in a facet. The hyperplane $x_i = 1$ intersects the polytope in a facet if one of the following three cases holds: $\mathbf{v}_{i-1}\mathbf{v}_i = \mathbf{xy}$; $i = 1$ and $\mathbf{v}_1 = \mathbf{y}$; or $i = n$ and $\mathbf{v}_n = \mathbf{x}$. A similar statement holds for the hyperplane $x_i = 0$. The lemma follows by adding these three statements. □

For an $\mathbf{xy}$-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_{n-1}$ and a subset T of $[n-1]$, define $\mathbf{v}^T$ to be the subword $\mathbf{v}^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$, where $T = \{j_1 < j_2 < \cdots < j_k\}$. The following theorem provides a way to compute the f-polynomial $F_{\mathbf{v}}$.

Theorem 2.2 *Let* $\mathbf{v}$ *be an* $\mathbf{xy}$*-word of length* $n-1$. *Then the* f*-polynomial of the descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *is given by*

$$F_{\mathbf{v}} = 1 + \sum_{T \subseteq [n-1]} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}.$$

Proof For a face $\mathcal{F}$ of a polytope, let $\mathcal{F}^I$ denote the relative interior of $\mathcal{F}$. Then the polytope is the disjoint union of $\mathcal{F}^I$ taken over all faces $\mathcal{F}$, including the polytope itself.

Recall that the descent polytope $\mathrm{DP}_{\mathbf{v}}$ consists of all points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ belonging simultaneously to the half spaces $x_i \geq 0$, $x_i \leq 1$ $(1 \leq i \leq n)$, $x_i \leq x_{i+1}$ $(\mathbf{v}_i = \mathbf{x})$, and $x_i \geq x_{i+1}$ $(\mathbf{v}_i = \mathbf{y})$. A face $\mathcal{F}$ of $\mathrm{DP}_{\mathbf{v}}$ can be uniquely identified by specifying which of these half spaces contain $\mathcal{F}$ on their boundary hyperplanes, as long as the intersection of the whole polytope and the specified boundary hyperplanes is non-empty. Forming the specification just for the half spaces of the form $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$ restricts the location of $\mathcal{F}^I$ in $\mathbb{R}^n$ to the region defined by the relations

$$\begin{aligned} x_1 = x_2 = \cdots = x_{j_1} \lessgtr x_{j_1+1} = x_{j_1+2} = \cdots = x_{j_2} \\ \lessgtr \cdots \lessgtr x_{j_k+1} = x_{j_k+2} = \cdots = x_n \end{aligned} \tag{2.1}$$

for some $T = \{j_1 < j_2 < \cdots < j_k\} \subseteq [n-1]$, where the symbol $\lessgtr$ denotes strict inequality: $x_{j_i} < x_{j_i+1}$ if $\mathbf{v}_{j_i} = \mathbf{x}$, or $x_{j_i} > x_{j_i+1}$ if $\mathbf{v}_{j_i} = \mathbf{y}$. Then T is the set of indexes j for which $\mathcal{F}$ does *not* lie entirely on the boundary hyperplane $x_j = x_{j+1}$ and thus the relative interior $\mathcal{F}^I$ is contained in the interior of the corresponding half space. Let $\mathcal{R}(T)$ denote the intersection of the region defined by (2.1) and the hypercube $[0,1]^n$. Each point $(x_1, \ldots, x_n)$ of $\mathrm{DP}_\mathbf{v}$ belongs to exactly one such region $\mathcal{R}(T)$, namely, the one for $T = \{j \mid x_j \neq x_{j+1}\}$. Thus we have the disjoint union

$$\mathrm{DP}_\mathbf{v} = \bigsqcup_{T \subseteq [n-1]} \mathcal{R}(T).$$

Let us show that the term corresponding to $T \neq \varnothing$ in the expression in the statement of the theorem is the contribution to $F_\mathbf{v}$ of the faces $\mathcal{F}$ of $\mathrm{DP}_\mathbf{v}$ for which $\mathcal{F}^I$ is contained in the region $\mathcal{R}(T)$. In other words, we claim that for $T \neq \varnothing$ we have

$$\sum_{\mathcal{F}:\mathcal{F}^I \subseteq \mathcal{R}(T)} t^{\dim \mathcal{F}} = \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}. \tag{2.2}$$

Fix $\varnothing \neq T \subseteq [n-1]$. To select a particular face $\mathcal{F}$ from the set of all faces with the property $\mathcal{F}^I \subseteq \mathcal{R}(T)$, we need to complete the specification started above, that is, we must specify which of the hyperplanes $x_i = 0, 1$ contain $\mathcal{F}$, and we must make sure that the intersection of the set of the specified hyperplanes and $\mathcal{R}(T)$ is non-empty. In terms of defining relations (2.1), this task is equivalent to setting the common value of some of the "blocks" of coordinates $(x_1, \ldots, x_{j_1})$, $(x_{j_1+1}, \ldots, x_{j_2}), \ldots, (x_{j_k+1}, \ldots, x_n)$ to 0 or 1. Since the relations must remain satisfiable by at least one point in $[0,1]^n$, only the blocks preceded in (2.1) by $>$ (or nothing) and succeeded by $<$ (or nothing) can be set to 0. Similarly, only the blocks preceded by $<$ (or nothing) and succeeded by $>$ (or nothing) can be set to 1. Thus each block can be set to at most one of 0 and 1. The letters of the **xy**-word $\mathbf{v}^T = \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_k}$ encode the inequality signs in (2.1) ($\mathbf{x}$ stands for $<$, and $\mathbf{y}$ stands for $>$), so the number of blocks that can be set to 0 or 1 is the total number of occurrences of $\mathbf{x}$ followed by $\mathbf{y}$, or $\mathbf{y}$ followed by $\mathbf{x}$, in $\mathbf{v}^T$, plus 2, as we also need to count the first and the last blocks. In other words, the number of such blocks is $\kappa(\mathbf{v}^T)$.

Observe that the dimension of the face of $\mathrm{DP}_\mathbf{v}$ obtained by this specification procedure equals the number of blocks that have not been set to 0 or 1: the common values of the coordinates in those blocks form the "degrees of freedom" that constitute the dimension. Let us call such blocks *free*. The number of faces $\mathcal{F}$ with $\mathcal{F}^I \subseteq \mathcal{R}(T)$ for which the specification procedure results in m free blocks is

$$\binom{\kappa(\mathbf{v}^T)}{|T|+1-m},$$

the number of ways to choose $|T|+1-m$ blocks that are *not* free out of $\kappa(\mathbf{v}^T)$ possibilities. Hence we have

$$\begin{aligned}\sum_{\mathcal{F}:\mathcal{F}^I\subseteq\mathcal{R}(T)} t^{\dim\mathcal{F}} &= \sum_{m=|T|+1-\kappa(\mathbf{v}^T)}^{|T|+1} \binom{\kappa(\mathbf{v}^T)}{|T|+1-m}\cdot t^m \\ &= t^{|T|+1-\kappa(\mathbf{v}^T)}\cdot\sum_{\ell=0}^{\kappa(\mathbf{v}^T)}\binom{\kappa(\mathbf{v}^T)}{\ell}\cdot t^{\ell} \\ &= t^{|T|+1-\kappa(\mathbf{v}^T)}\cdot(t+1)^{\kappa(\mathbf{v}^T)},\end{aligned}$$

proving (2.2).

Finally, for $T=\varnothing$, we have $\mathcal{R}(T)=\{0\le x_1=\cdots=x_n\le 1\}$, which is just the line segment joining the two vertices $(0,\ldots,0)$ and $(1,\ldots,1)$ of $\mathrm{DP}_{\mathbf{v}}$. Thus the contribution of $\mathcal{R}(T)$ to $F_{\mathbf{v}}$ is

$$t+2=1+\left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^{\varnothing})}\cdot t.$$

Adding this equation to the sum of (2.2) taken over the non-empty T proves the theorem. □

Theorem 2.2 yields a combinatorial interpretation of the number of vertices of the polytope $\mathrm{DP}_{\mathbf{v}}$. Call an **xy**-word $\mathbf{v}=\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k$ *alternating* if $\mathbf{v}_i\ne\mathbf{v}_{i+1}$ for all $1\le i\le k-1$. Then we have the following corollary.

Corollary 2.3 *For* $\mathbf{v}$ *an* **xy**-*word of length* $n-1$, *the number of vertices of the descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *is one greater than the number of subsets* $T\subseteq[n-1]$ *for which the word* $\mathbf{v}^T$ *is alternating.*

Proof The number of vertices of $\mathrm{DP}_{\mathbf{v}}$ is the constant term of $F_{\mathbf{v}}$. For the summand corresponding to a subset $T\subseteq[n-1]$ in the formula of Theorem 2.2, the constant term is either 0 or 1, the latter being the case if and only if $|T|+1-\kappa(\mathbf{v}^T)=0$. This condition is equivalent to $\mathbf{v}^T$ being alternating, proving the corollary. □

As we mention in the introduction, the descent set statistic $\beta(S)$ is maximized when S is the alternating set. The most elegant proof of this fact uses the **cd**-index of the simplex; see [7]. For an **xy**-word $\mathbf{v}$, let $\overline{\mathbf{v}}$ denote the word obtained from $\mathbf{v}$ by replacing **x**'s with **y**'s and vice versa. Then the following inequality holds:

$$\beta(\mathbf{uyxv})>\beta(\mathbf{uyy}\overline{\mathbf{v}}), \tag{2.3}$$

where we use **xy**-words to encode the sets. In each of the proofs [2, 6, 8, 12] that the alternating word maximizes the descent set, the arguments rely on proving the inequality (2.3). However, the **cd**-index proof gives a quick way to verify this inequality. We now state a similar inequality for the f-vectors of descent polytopes.

Theorem 2.4 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two* $\mathbf{xy}$*-words such that the sum of their lengths is* $n-3$, *that is*, $|\mathbf{u}|+|\mathbf{v}|=n-3$. *Then the difference*

$$F_{\mathbf{uyxv}}(t) - F_{\mathbf{uyy\overline{v}}}(t) \tag{2.4}$$

has positive coefficients at $1, t, \dots, t^{n-1}$. *That is, for* $0 \le i \le n-1$ *the descent polytope* $\mathrm{DP}_{\mathbf{uyxv}}$ *has more faces of dimension* i *than the descent polytope* $\mathrm{DP}_{\mathbf{uyy\overline{v}}}$.

Proof Let $|\mathbf{u}|=m$ and $|\mathbf{v}|=n-m-3$. For $T \subseteq [m]$ and $U \subseteq [n-m-3]$, define

$$Q_{T,U}(t) := \sum_{E\subseteq\{1,2\}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{u}^T(\mathbf{yx})^E\mathbf{v}^U)} \cdot t^{|T|+|U|+|E|+1}. \tag{2.5}$$

Thus $Q_{T,U}(t)$ is the sum of four of the terms in the summation formula for $F_{\mathbf{uyxv}}(t)$ given by Theorem 2.2, corresponding to fixed choices of letters drawn from $\mathbf{u}$ and from $\mathbf{v}$. Similarly, let us define

$$\overline{Q}_{T,U}(t) := \sum_{E\subseteq\{1,2\}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{u}^T(\mathbf{yy})^E\overline{\mathbf{v}}^U)} \cdot t^{|T|+|U|+|E|+1}.$$

Note that $Q_{T,U}(t)$ depends only on $|T|$, $|U|$, $\kappa(\mathbf{u}^T\mathbf{v}^U)$, the last letter of $\mathbf{u}^T$, and the first letter of $\mathbf{v}^U$, and not on the particular choice of the remaining letters of $\mathbf{u}^T$ and $\mathbf{v}^U$. Thus to show that the difference $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ is a polynomial with non-negative coefficients, it suffices to consider nine cases corresponding to $\mathbf{u}^T$ (respectively, $\mathbf{v}^U$) ending (respectively, beginning) with $\mathbf{x}$ or $\mathbf{y}$, or being equal to the empty word 1.

We summarize our calculations in Table 1. We denote $\kappa(\mathbf{u}^T\mathbf{v}^U)$ by k, and we divide each polynomial by the common factor $(t+1)^k \cdot t^{|T|+|U|+1-k}$. In the fourth column, which corresponds to $Q_{T,U}$, the four summands represent the results of inserting 1, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{yx}$ between $\mathbf{u}^T$ and $\mathbf{v}^U$. For example, if $\mathbf{u}^T$ ends with an $\mathbf{x}$ and $\mathbf{v}^U$ begins with an $\mathbf{x}$, then inserting $\mathbf{y}$ increases the value of the statistic κ by 2, thus contributing a factor of

$$\left(\frac{t+1}{t}\right)^2 \cdot t = (t+1)^2 \cdot t^{-1}$$

to the corresponding term of (2.5). Similarly, the entries in the fifth column consist of a factor resulting from a different value of κ for the word $\mathbf{u}^T\overline{\mathbf{v}}^U$ times the contributions of inserting 1, $\mathbf{y}$ (counted twice), and $\mathbf{y}^2$ between $\mathbf{u}^T$ and $\overline{\mathbf{v}}^U$.

We conclude that in every case the quotient $\frac{Q_{T,U}(t)-\overline{Q}_{T,U}(t)}{(t+1)^{k-1}\cdot t^{|T|+|U|+1-k}}$ is a polynomial of degree 2 with non-negative coefficients. Hence the difference $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ is a polynomial with non-negative coefficients. Summing over all possible pairs (T, U) yields that the difference in (2.4) has non-negative coefficients. More specifically, the polynomial $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ has degree $(k-1) + (|T|+|U|+1-k) + 2 = |T|+|U|+2$. This degree can attain any integer value between 2 and $n-1$.

Table 1 Calculations for the proof of Theorem 2.4

$\mathbf{u}^T$	$\mathbf{v}^U$	$\kappa(\mathbf{u}^T\overline{\mathbf{v}}^U)$	$\frac{Q_{T,U}(t)}{(t+1)^k\, t^{\lvert T\rvert+\lvert U\rvert+1-k}}$	$\frac{\overline{Q}_{T,U}(t)}{(t+1)^k t^{\lvert T\rvert+\lvert U\rvert+1-k}}$	$\frac{Q_{T,U}(t)-\overline{Q}_{T,U}(t)}{(t+1)^{k-1}\, t^{\lvert T\rvert+\lvert U\rvert+1-k}}$
$\cdots\mathbf{x}$	$\mathbf{x}\cdots$	$k+1$	$1+t+(t+1)^2t^{-1}$ $+(t+1)^2$	$(t+1)t^{-1}$ $\cdot(1+2t+t^2)$	$(t+1)^2$
$\cdots\mathbf{x}$	$\mathbf{y}\cdots$	$k-1$	$1+t+t$ $+(t+1)^2$	$(t+1)^{-1}t$ $\cdot\big(1+2(t+1)^2\,t^{-1}$ $+(t+1)^2\big)$	t^2
$\cdots\mathbf{y}$	$\mathbf{x}\cdots$	$k-1$	$1+t+t+t^2$	$(t+1)^{-1}t$ $\cdot(1+2t+t^2)$	$(t+1)^2$
$\cdots\mathbf{y}$	$\mathbf{y}\cdots$	$k+1$	$1+(t+1)^2t^{-1}$ $+t+(t+1)^2$	$(t+1)\,t^{-1}$ $\cdot(1+2t+t^2)$	$(t+1)^2$
1	$\mathbf{x}\cdots$	k	$1+t+(t+1)$ $+(t+1)t$	$1+2t+t^2$	$(t+1)^2$
1	$\mathbf{y}\cdots$	k	$1+(t+1)$ $+t+(t+1)^2$	$1+2(t+1)$ $+(t+1)t$	t^2+t
$\cdots\mathbf{x}$	1	k	$1+t+(t+1)$ $+(t+1)^2$	$1+2(t+1)$ $+(t+1)t$	t^2+t
$\cdots\mathbf{y}$	1	k	$1+(t+1)$ $+t+(t+1)t$	$1+2t+t^2$	$(t+1)^2$
1	1	$k=1$	$1+(t+1)$ $+(t+1)+(t+1)^2$	$1+2(t+1)$ $+(t+1)t$	$(t+1)^2$

Thus the leading terms of these differences contribute positively to the coefficients of $t^2, t^3, \dots, t^{n-1}$ in the difference (2.4). Furthermore, in the case $T = U = \varnothing$ we have $Q_{T,U}(t) - \overline{Q}_{T,U}(t) = (t+1)^2$, which yields a positive contribution to the constant and the linear terms of the overall difference. The proof is now complete. □

Let $\mathbf{z}_n$ be the alternating word of length n starting with the letter $\mathbf{x}$. Then $\overline{\mathbf{z}_n}$ is the alternating word beginning with $\mathbf{y}$. That is, the two alternating words are

$$\mathbf{z}_n = \underbrace{\mathbf{xyx}\cdots}_{n} \quad \text{and} \quad \overline{\mathbf{z}_n} = \underbrace{\mathbf{yxy}\cdots}_{n}.$$

We now have the maximization result for the f-vector of descent polytopes.

Corollary 2.5 *The f-vector of the two descent polytopes* $\mathrm{DP}_{\mathbf{z}_{n-1}}$ *and* $\mathrm{DP}_{\overline{\mathbf{z}_{n-1}}}$ *is maximal among the f-vectors of all descent polytopes of dimension n. That is, for each $0 \le i \le n-1$, the polytope* $\mathrm{DP}_{\mathbf{z}_{n-1}}$ *has more faces of dimension i than the descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *of dimension n for a non-alternating word* $\mathbf{v}$.

3 The Power Series $\Phi(\mathbf{x}, \mathbf{y})$

We now derive a non-commutative generating function $\Phi(\mathbf{x}, \mathbf{y})$ for the f-polynomial $F_{\mathbf{v}}$, which belongs to the ring $\Phi(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[t]\langle\langle \mathbf{x}, \mathbf{y}\rangle\rangle$. We define the power series $\Phi(\mathbf{x}, \mathbf{y})$ by

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v},$$

where the sum is over all $\mathbf{xy}$-words $\mathbf{v}$. Since we have the symmetry $F_{\mathbf{v}} = F_{\overline{\mathbf{v}}}$, we obtain that $\Phi(\mathbf{x}, \mathbf{y})$ is symmetric with respect to $\mathbf{x}$ and $\mathbf{y}$, that is,

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x}).$$

Let $\mathbf{v}$ be an $\mathbf{xy}$-word $\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$. Consider the following polynomials:

$$K_{\mathbf{v}}(t) := \sum_{T\subseteq[n-1]:\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1},$$

$$L_{\mathbf{v}}(t) := \sum_{T\subseteq[n-1]:\mathbf{v}_{j_1}=\mathbf{y}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1},$$

where $\mathbf{v}_{j_1}$ denotes the first letter of the word $\mathbf{v}^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$, as in the notation of Theorem 2.2. Since $\mathbf{v}^T$ begins with either $\mathbf{x}$ or $\mathbf{y}$ unless $T = \varnothing$, we have

$$F_{\mathbf{v}} = K_{\mathbf{v}} + L_{\mathbf{v}} + t + 2, \tag{3.1}$$

where $t+2$ is the f-polynomial of DP_1, the line segment. We continue with a lemma that relates the two polynomials $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$.

Lemma 3.1 *For an* $\mathbf{xy}$-*word* $\mathbf{v}$ *the following four equalities hold*:

$$K_{\mathbf{yv}} = K_{\mathbf{v}},$$
$$L_{\mathbf{xv}} = L_{\mathbf{v}},$$
$$K_{\mathbf{xv}} = L_{\mathbf{yv}} = (t+1)\cdot(K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1).$$

Proof For an integer i and a set $U \subseteq \mathbb{Z}$, let $U + i$ denote the set obtained by adding i to each element of U. Also let $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$, where each $\mathbf{v}_i$ is either $\mathbf{x}$ or $\mathbf{y}$.

Clearly, $(\mathbf{yv})^T$ begins with $\mathbf{x}$ if and only if $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$, in which case $(\mathbf{yv})^T = \mathbf{v}^{T-1}$. Hence $K_{\mathbf{yv}} = K_{\mathbf{v}}$.

Now, $(\mathbf{xv})^T$ begins with $\mathbf{x}$ if and only if either $1 \in T$, or else $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$. In the former case, we have $T = \{1 < j_1 + 1 < j_2 + 1 < \cdots < j_k + 1\}$, and $(\mathbf{xv})^T = \mathbf{x}\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$. Set $U = (T - \{1\}) - 1 = \{j_1 < \cdots < j_k\}$. Then $\kappa((\mathbf{xv})^T) = \kappa(\mathbf{v}^U)$ if $\mathbf{v}_{j_1} = \mathbf{x}$, and $\kappa((\mathbf{xv})^T) = \kappa(\mathbf{v}^U) + 1$ if $\mathbf{v}_{j_1} = \mathbf{y}$. Hence

$$\sum_{1\in T\subseteq[n]} \left(\frac{t+1}{t}\right)^{\kappa((\mathbf{xv})^T)} \cdot t^{|T|+1} = (t+1)^2 + t \cdot \sum_{U:\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^U)} \cdot t^{|U|+1}$$

$$+ (t+1) \cdot \sum_{U:\mathbf{v}_{j_1}=\mathbf{y}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^U)} \cdot t^{|U|+1}$$

$$= (t+1)^2 + t \cdot K_{\mathbf{v}} + (t+1) \cdot L_{\mathbf{v}}, \qquad (3.2)$$

where the first term $(t+1)^2$ corresponds to $T = \{1\}$ and $U = \varnothing$. In the case where $1 \notin T$ and $\mathbf{v}^{T-1}$ begins with $\mathbf{x}$ we have, as before, $(\mathbf{xv})^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k} = \mathbf{v}^{T-1}$, and hence

$$\sum_{T:\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa((\mathbf{xv})^T)} \cdot t^{|T|+1} = \sum_{\mathbf{v}_{j_1}=\mathbf{x}} \left(\frac{t+1}{t}\right)^{\kappa(\mathbf{v}^{T-1})} \cdot t^{|T-1|+1} = K_{\mathbf{v}}. \qquad (3.3)$$

Adding (3.2) and (3.3) yields

$$K_{\mathbf{xv}} = (t+1) \cdot (K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1).$$

The relations for $L_{\mathbf{xv}}$ and $L_{\mathbf{yv}}$ follow from symmetry that arises from exchanging the variables $\mathbf{x}$ and $\mathbf{y}$. □

Starting with $K_1 = L_1 = 0$, one can use Lemma 3.1 to recursively compute $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$, and hence $F_{\mathbf{v}}$, from (3.1). Recall the generating power series

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v},$$

where the sum is over all $\mathbf{xy}$-words, including the empty word $\mathbf{v} = \mathbf{v}_{\varnothing} = 1$. Define the two generating power series

$$\mathrm{K}(\mathbf{x},\mathbf{y}) := \sum_{\mathbf{v}} K_{\mathbf{v}} \cdot \mathbf{v},$$

$$\Lambda(\mathbf{x},\mathbf{y}) := \sum_{\mathbf{v}} L_{\mathbf{v}} \cdot \mathbf{v}.$$

From the definitions of $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$ it follows that $K_{\mathbf{v}} = L_{\overline{\mathbf{v}}}$. By the symmetry in the two variables $\mathbf{x}$ and $\mathbf{y}$ we have

$$\Lambda(\mathbf{x},\mathbf{y}) = \mathrm{K}(\mathbf{y},\mathbf{x}).$$

Then, by (3.1), we have

$$\Phi(\mathbf{x},\mathbf{y}) = \mathrm{K}(\mathbf{x},\mathbf{y}) + \Lambda(\mathbf{x},\mathbf{y}) + (t+2) \cdot \sum_{\mathbf{v}} \mathbf{v}$$

$$= \mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) + (t+2) \cdot \sum_{r \ge 0} (\mathbf{x}+\mathbf{y})^r$$

$$= \mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) + (t+2) \cdot \frac{1}{1-\mathbf{x}-\mathbf{y}}. \qquad (3.4)$$

Using the equations in Lemma 3.1 and recalling that $K_1 = 0$ we obtain

$$\begin{aligned}
\mathrm{K}(\mathbf{x},\mathbf{y}) &= \sum_{\mathbf{v}} K_{\mathbf{xv}} \cdot \mathbf{xv} + \sum_{\mathbf{v}} K_{\mathbf{yv}} \cdot \mathbf{yv} \\
&= (t+1)\cdot \mathbf{x} \cdot \sum_{\mathbf{v}} (K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1) \cdot \mathbf{v} + \mathbf{y} \cdot \sum_{\mathbf{v}} K_{\mathbf{v}} \cdot \mathbf{v} \\
&= (t+1)\cdot \mathbf{x} \cdot \left(\mathrm{K}(\mathbf{x},\mathbf{y}) + \Lambda(\mathbf{x},\mathbf{y}) + (t+1) \cdot \frac{1}{1-\mathbf{x}-\mathbf{y}} \right) + \mathbf{y} \cdot \mathrm{K}(\mathbf{x},\mathbf{y}) \\
&= (t+1)\cdot \mathbf{x} \cdot \left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}} \right) + \mathbf{y} \cdot \mathrm{K}(\mathbf{x},\mathbf{y}),
\end{aligned}$$

where the last step is by (3.4). Rearranging terms we have

$$\mathrm{K}(\mathbf{x},\mathbf{y}) = (t+1)\cdot (1-\mathbf{y})^{-1} \cdot \mathbf{x} \cdot \left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}} \right).$$

Adding this equation and its symmetric version obtained by exchanging **x** and **y** one has

$$\begin{aligned}
\mathrm{K}(\mathbf{x},\mathbf{y}) + \mathrm{K}(\mathbf{y},\mathbf{x}) = (t+1) \cdot \big((1-\mathbf{y})^{-1} \cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y}\big) \\
\cdot \left(\Phi(\mathbf{x},\mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}} \right),
\end{aligned}$$

using the symmetry $\Phi(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{y},\mathbf{x})$. Now using (3.4) we can solve for $\Phi(\mathbf{x},\mathbf{y})$ and arrive at the following theorem.

Theorem 3.2 *The generating power series* $\Phi(\mathbf{x},\mathbf{y})$ *is given by*

$$\Phi(\mathbf{x},\mathbf{y}) = \left(1 + \frac{t+1}{1-(t+1)\cdot((1-\mathbf{y})^{-1}\cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y})} \right) \cdot \frac{1}{1-\mathbf{x}-\mathbf{y}}.$$

Corollary 3.3 *For an* **xy**-*word* **v** *the* f-*vector of the descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *is given by the sum*

$$F_{\mathbf{v}}(t) = 1 + \sum_{(\mathbf{u}_1,\ldots,\mathbf{u}_{k-1},\mathbf{u}_k)} (t+1)^k,$$

where the sum ranges over all factorizations of the word $\mathbf{v} = \mathbf{u}_1 \cdots \mathbf{u}_{k-1} \cdot \mathbf{u}_k$ *such that each of the factors* $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ *are of the form* $\mathbf{x}^i\mathbf{y}$ *or* $\mathbf{y}^i\mathbf{x}$, *where* $i \geq 0$, *and there is no condition on the last factor* $\mathbf{u}_k$.

Proof Rewrite Theorem 3.2 as

$$\begin{aligned}
\Phi(\mathbf{x},\mathbf{y}) &= \frac{1}{1-\mathbf{x}-\mathbf{y}} + \frac{1}{1-(t+1)\cdot((1-\mathbf{y})^{-1}\cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y})} \cdot \frac{t+1}{1-\mathbf{x}-\mathbf{y}} \\
&= \sum_{\mathbf{v}} \mathbf{v} + \sum_{j\geq 0} \left((t+1) \cdot \sum_{i\geq 0} (\mathbf{y}^i\mathbf{x} + \mathbf{x}^i\mathbf{y}) \right)^j \cdot (t+1) \cdot \sum_{\mathbf{v}} \mathbf{v},
\end{aligned}$$

where in both sums $\mathbf{v}$ ranges over all $\mathbf{xy}$-words. The corollary follows by reading the generating function. □

Example 3.4 Consider the 5-dimensional descent polytope $\mathrm{DP}_{\mathbf{v}}$ where $\mathbf{v} = \mathbf{xyyx}$. We have the following list of 11 factorizations:

$$\begin{aligned}\mathbf{v} &= \mathbf{xyyx} \\ &= \mathbf{x}\cdot\mathbf{yyx} \quad = \mathbf{xy}\cdot\mathbf{yx} \\ &= \mathbf{x}\cdot\mathbf{y}\cdot\mathbf{yx} \quad = \mathbf{x}\cdot\mathbf{yyx}\cdot 1 \quad = \mathbf{xy}\cdot\mathbf{y}\cdot\mathbf{x} \quad = \mathbf{xy}\cdot\mathbf{yx}\cdot 1 \\ &= \mathbf{x}\cdot\mathbf{y}\cdot\mathbf{y}\cdot\mathbf{x} \quad = \mathbf{x}\cdot\mathbf{y}\cdot\mathbf{yx}\cdot 1 = \mathbf{xy}\cdot\mathbf{y}\cdot\mathbf{x}\cdot 1 \\ &= \mathbf{x}\cdot\mathbf{y}\cdot\mathbf{y}\cdot\mathbf{x}\cdot 1\end{aligned}$$

Hence the f-polynomial of the polytope $\mathrm{DP}_{\mathbf{xyyx}}$ is given by

$$\begin{aligned}F_{\mathbf{xyyx}} &= 1 + (t+1) + 2\cdot(t+1)^2 + 4\cdot(t+1)^3 + 3\cdot(t+1)^4 + (t+1)^5 \\ &= 12 + 34\cdot t + 42\cdot t^2 + 26\cdot t^3 + 8\cdot t^4 + t^5.\end{aligned}$$

For the alternating word $\mathbf{z}_{n-1}$ we can say more about the associated descent polytope. The number of vertices of $\mathrm{DP}_{\mathbf{z}_{n-1}}$ is the Fibonacci number F_{n+2}; see for instance [10, Exercise 1.14e]. More generally, the f-vector of $\mathrm{DP}_{\mathbf{z}_{n-1}}$ is given by the next result.

Corollary 3.5 *The f-polynomial of the n-dimensional descent polytope* $\mathrm{DP}_{\mathbf{z}_{n-1}}$ *is described by*

$$F_{\mathbf{z}_{n-1}} = 1 + \sum_{(c_1,c_2,\ldots,c_k)} (t+1)^k,$$

where the sum is over all compositions of n such that all but the last part is less than or equal to 2, *that is*, $c_1, \ldots, c_{k-1} \in \{1,2\}$.

Proof The only factors of the alternating word $\mathbf{z}_{n-1}$ of the form $\mathbf{x}^i\mathbf{y}$ or $\mathbf{y}^i\mathbf{x}$ have $i = 0, 1$. Hence it is enough to record the length of each factor $\mathbf{u}_i$, that is, $d_i = |\mathbf{u}_i|$. Thus we are summing over vectors of non-negative integers $(d_1, \ldots, d_k)$ such that the sum of the entries is $n-1$ and $d_1, \ldots, d_{k-1} \in \{1,2\}$ and $d_k \geq 0$. By adding one to the last entry d_k we have a composition of n. □

This corollary yields the generating function

$$\sum_{n\geq 1} F_{\mathbf{z}_{n-1}}\cdot x^n = \frac{x}{1-x} + \frac{1}{1-(t+1)\cdot(x+x^2)}\cdot(t+1)\cdot\frac{x}{1-x}. \tag{3.5}$$

Setting $t = 0$ in this generating function and adding constant 1 yields $(1+x)/(1-x-x^2)$, the generating function for the Fibonacci numbers as expected.

4 A Generating Power Series for the Ehrhart Polynomials of DP_S

Besides the f-vector, another geometric invariant of a polytope is the *Ehrhart polynomial*. As a function of a non-negative integer r, the Ehrhart polynomial of a lattice polytope P is the number of lattice points in the dilation $r \cdot P$. Ehrhart's fundamental result is that this function is a polynomial in r. In the case of the n-dimensional descent polytope DP_S, the Ehrhart polynomial $\iota_S(r)$ counts the number of lattice points satisfying the inequalities $0 \le x_1, x_2, \ldots, x_n \le r$, $x_i \ge x_{i+1}$ for $i \in S$, and $x_i \le x_{i+1}$ for $i \notin S$. In this section we derive the generating power series

$$I(r; \mathbf{x}, \mathbf{y}) := \sum_{\mathbf{v}} \iota_{\mathbf{v}}(r) \cdot \mathbf{v}.$$

As before, we adopt the shorthand $\iota_{\mathbf{v}_S} = \iota_S$.

Let us call an element w of the set $\{0, 1, \ldots, r\}^n$ an r-*word* of length n. Define the descent set $D(w)$ of $w = (w_1, w_2, \ldots, w_n)$ to be the set of positions i such that $w_i > w_{i+1}$. For an $\mathbf{xy}$-word $\mathbf{v}$ of length $n-1$, let $\beta(r, \mathbf{v})$ be the number of r-words w of length n such that $D(w)$ is encoded by $\mathbf{v}$ (that is, $\mathbf{v} = \mathbf{v}_{D(w)}$). Note that $\beta(r, \mathbf{v})$ is not quite the Ehrhart polynomial $\iota_{\mathbf{v}}$, as it only counts those integer points of $r\,\mathrm{DP}_{\mathbf{v}}$ with strict descents. Still, a generating power series for $\beta(r, \mathbf{v})$ is the first step in our computation of $I(r; \mathbf{x}, \mathbf{y})$.

For an $\mathbf{xy}$-word of length $n-1$, let $\alpha(r, \mathbf{v})$ be the number of r-words of length n such that $D(w)$ is contained in the subset of $[n-1]$ encoded by $\mathbf{v}$. Fix $r \ge 0$, $n > 0$, and $S \subseteq [n-1]$. Write the word $\mathbf{v}_S$ as

$$\mathbf{v}_S = \mathbf{x}^{g_1 - 1}\mathbf{y}\mathbf{x}^{g_2 - 1}\mathbf{y} \cdots \mathbf{y}\mathbf{x}^{g_k - 1},$$

so that $g = (g_1, g_2, \ldots, g_k)$ is the composition $\mathrm{co}(S)$ of n associated to S. To construct a word counted by $\alpha(r, \mathbf{v})$, one needs to choose, for every i, a multiset of elements of $\{0, 1, \ldots, r\}$ that go into the "block" corresponding to the part g_i of g, put them in (weakly) increasing order, and concatenate the blocks. There are $\binom{r+g_i}{g_i}$ ways of choosing the elements for the ith block. Thus if we define

$$Q_r(x) := \sum_{j \ge 1} \binom{r+j}{j} \cdot x^{j-1} = x^{-1} \cdot \bigl((1-x)^{-r-1} - 1\bigr)$$

then the generating power series for $\alpha(r, \mathbf{v})$ is

$$\begin{aligned} A(r; \mathbf{x}, \mathbf{y}) &:= \sum_{\mathbf{v}} \alpha(r, \mathbf{v}) \cdot \mathbf{v} \\ &= \sum_{k \ge 1} Q_r(\mathbf{x}) \cdot \bigl(\mathbf{y} \cdot Q_r(\mathbf{x})\bigr)^{k-1} \\ &= Q_r(\mathbf{x}) \cdot \bigl(1 - \mathbf{y} \cdot Q_r(\mathbf{x})\bigr)^{-1}, \end{aligned} \tag{4.1}$$

where k runs through all possible numbers of parts of the composition $g = \mathrm{co}(S)$, and the index j in the definition of Q_r corresponds to the choice of each part size.

A standard application of the inclusion-exclusion principle yields

$$B(r;\mathbf{x},\mathbf{y}) := \sum_{\mathbf{v}} \beta(r;\mathbf{x},\mathbf{y})\mathbf{v} = A(r;\mathbf{x}-\mathbf{y},\mathbf{y}). \tag{4.2}$$

The final step is the following claim:

Lemma 4.1 *The generating function of the Ehrhart polynomials of the descent polytopes is expressed in terms of* $B(r;\mathbf{x},\mathbf{y})$ *as*

$$I(r;\mathbf{x},\mathbf{y}) = (1-\mathbf{y})^{-1} \cdot B\big(r;\ \mathbf{x}(1-\mathbf{y})^{-1}, \mathbf{y}(1-\mathbf{y})^{-1}\big).$$

Proof For an $\mathbf{xy}$-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$ and an integer point $p = (p_1, p_2, \ldots, p_n)$ in $r\,\mathrm{DP}_{\mathbf{v}}$, define $\varphi(\mathbf{v}, p) = (\mathbf{u}, q)$, where the $\mathbf{xy}$-word $\mathbf{u}$ and the r-word q are obtained from $\mathbf{v}$ and p as follows: for each index $i \in [n-1]$ such that $p_i = p_{i+1}$ and $\mathbf{v}_i = \mathbf{y}$, remove the letter at the ith position from $\mathbf{v}$ as well as the ith coordinate from p. For example, if $p = (2,3,1,1,1,1,1,4)$ and $\mathbf{v} = \mathbf{xyyyxyx}$, then the removal should be done for $i = 3,4,6$, so $q = (2,3,1,1,4)$ and $\mathbf{u} = \mathbf{xyxx}$.

Let $B(r,\mathbf{u})$ be the set of r-words with descent set encoded by $\mathbf{u}$, so that $\beta(r,\mathbf{u}) = |B(r,\mathbf{u})|$. Note that if $\varphi(\mathbf{v}, p) = (\mathbf{u}, q)$ then $q \in B(r,\mathbf{u})$. For a fixed $\mathbf{u}$ and $q \in B(r,\mathbf{u})$, the inverse image $\varphi^{-1}(\mathbf{u}, q)$ can be obtained by performing the following operation in all possible ways: start with $\mathbf{u}$, insert an arbitrary number of $\mathbf{y}$'s (maybe none) in each of the gaps between consecutive letters of $\mathbf{u}$, before the first letter of $\mathbf{u}$, and after the last letter of $\mathbf{u}$, and for each coordinate q_i of q, insert as many copies of q_i before that coordinate as the number of $\mathbf{y}$'s that were inserted before the ith letter of $\mathbf{u}$ (for $i = n$, use the number of $\mathbf{y}$'s inserted after the last letter of $\mathbf{u}$). The resulting $\mathbf{xy}$-word $\mathbf{v}$ and integer point p satisfy $\varphi(\mathbf{v}, p) = (\mathbf{u}, q)$. In terms generating functions we have

$$\begin{aligned}\sum_{(\mathbf{v},p)\in\varphi^{-1}(\mathbf{u},q)} \mathbf{v} &= \big(1+\mathbf{y}+\mathbf{y}^2+\cdots\big)\cdot \mathbf{u}_1 \cdot \big(1+\mathbf{y}+\mathbf{y}^2+\cdots\big)\\ &\qquad \cdot \mathbf{u}_2\cdots\mathbf{u}_m \cdot \big(1+\mathbf{y}+\mathbf{y}^2+\cdots\big)\\ &= (1-\mathbf{y})^{-1}\cdot \mathbf{u}_1 \cdot (1-\mathbf{y})^{-1} \cdot \mathbf{u}_2\cdots\mathbf{u}_m \cdot (1-\mathbf{y})^{-1},\end{aligned} \tag{4.3}$$

where $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_m$ and $\mathbf{u}_i \in \{\mathbf{x},\mathbf{y}\}$. Consider the sum of (4.3) over all pairs $(\mathbf{u}, q)$ such that $q \in B(r,\mathbf{u})$. The left-hand side of the resulting identity is

$$\sum_{\{(\mathbf{v},p):p\in r\,\mathrm{DP}_{\mathbf{v}}\}} \mathbf{v} = \sum_{\mathbf{v}} \iota_{\mathbf{v}}(r)\cdot \mathbf{v} = I(r;\mathbf{x},\mathbf{y}).$$

The right-hand side is

$$\begin{aligned}&\sum_{\mathbf{u}} \big|B(r,\mathbf{u})\big| \cdot (1-\mathbf{y})^{-1}\cdot \mathbf{u}_1 \cdot (1-\mathbf{y})^{-1}\cdot \mathbf{u}_2\cdots\mathbf{u}_m \cdot (1-\mathbf{y})^{-1}\\ &\quad = (1-\mathbf{y})^{-1}\cdot B\big(r;\mathbf{x}(1-\mathbf{y})^{-1}, \mathbf{y}(1-\mathbf{y})^{-1}\big).\end{aligned}$$

□

Combining the above results, we obtain our desired theorem:

Theorem 4.2 *The generating function of the Ehrhart polynomials of the descent polytopes is given by*

$$I(r;\mathbf{x},\mathbf{y}) = (1-\mathbf{y})^{-1}\cdot Q_r\big((\mathbf{x}-\mathbf{y})(1-\mathbf{y})^{-1}\big) \cdot \big(1-\mathbf{y}(1-\mathbf{y})^{-1}\cdot Q_r\big((\mathbf{x}-\mathbf{y})(1-\mathbf{y})^{-1}\big)\big)^{-1},$$

where $Q_r(x) = x^{-1}\cdot((1-x)^{-r-1}-1)$.

5 Concluding Remarks

A more general invariant of the descent polytopes to study is the flag f-vector. The flag f-vector is efficiently encoded by the **cd**-index. Is there a way to describe the **cd**-index of the descent polytope DP_S in terms of the **xy**-word $\mathbf{v}_S$? Finding a non-commutative generating function for the **cd**-indices of descent polytopes would be a natural way to extend the results of this paper. The **cd**-indexes of descent polytopes up to dimension 6 can be found in [1, Appendix A.2].

Setting $t=1$ in the polynomial $F_{\mathbf{v}}(t)$ we obtain the number of faces of the descent polytope $\mathrm{DP}_{\mathbf{v}}$. In particular, for the alternating word $\mathbf{z}_n$ we obtain the sequence $\{F_{\mathbf{z}_{n-1}}(1)\}_{n\geq 1} = 3, 7, 19, 51, \ldots$. This sequence has a different combinatorial interpretation, as it matches the sequence A052948 in the Online Encyclopedia of Integer Sequences [11] defined as the number of paths from $(0,0)$ to $(n+1,0)$ with allowed steps $(1,1)$, $(1,0)$ and $(1,-1)$ contained within the region $-2\leq y\leq 2$. The generating function

$$\frac{1-2x^2}{1-3x+2x^3}$$

given in [11] indeed results if $t=1$ is substituted into (3.5) and the constant 1 is added. Is there a bijective proof? A first step to find such a bijective proof would be to find a statistic on these lattice paths with the same distribution as the dimensions of the faces of the descent polytope $\mathrm{DP}_{\mathbf{z}_{n-1}}$.

For an **xy**-word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n$ let $\mathbf{v}^*$ denote the reverse of the word, that is, $\mathbf{v}^* = \mathbf{v}_n\cdots\mathbf{v}_2\mathbf{v}_1$. Note that the two descent polytopes $\mathrm{DP}_{\mathbf{v}}$ and $\mathrm{DP}_{\mathbf{v}^*}$ only differ by a linear transformation and hence their f-polynomials agree, that is, $F_{\mathbf{v}} = F_{\mathbf{v}^*}$. However the expressions for the f-polynomials for $F_{\mathbf{v}}$ and $F_{\mathbf{v}^*}$ in Corollary 3.3 differ. Is there a bijection between the factorizations of $\mathbf{v}$ and $\mathbf{v}^*$? The number of factorizations of $\mathbf{v}$ is also equal the number of alternating subwords of $\mathbf{v}$; see Corollary 2.3. This fact also asks for a bijective proof.

A second way to encode subsets of $[n-1]$ is by compositions. In [1, Chap. 3] this encoding is used to obtain more recurrences to compute the f-polynomial F_S.

More inequalities for the descent statistic have been proved in [3, 4]. Can these inequalities be extended to the f-polynomial $F_{\mathbf{v}}$? For instance, Ira Gessel asked the following question: where does the maximum of the descent set statistic occur when restricting to words $\mathbf{v}$ of length $n-1$ having exactly k runs of **x**'s and **y**'s. He conjectured and it was proved in [3] that the maximum occurs at the composition

$(r, \underbrace{r+1, \ldots, r+1}_{a}, r, \ldots, r)$ where $r = \lfloor (n-1)/k \rfloor$ and $a = (n-1) - r \cdot k$. Would the f-polynomial be maximized at the same composition?

Descent polytopes occur as a subdivision of the n-dimensional unit cube in the work of Ehrenborg, Kitaev, and Perry [5]. They are studying consecutive pattern avoidance with analytic means. When considering descent pattern avoidance they obtain operators on $L^2([0,1]^n)$ whose eigenfunctions only depends on x_1 when restricted to a descent polytope.

Theorem 4.2 gives a rational non-commutative generating function for the Ehrhart polynomials of the descent polytopes. However, we know that this generating function is symmetric in the two variables $\mathbf{x}$ and $\mathbf{y}$. Is there a different rational expression for this generating function that shows this symmetry?

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