Categories and functors in mathematics.

Vladimir Voevodsky

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1 *Introduction.*

Contemporary mathematics is probably the most esoteric of all contemporary sciences. Never in my life have I met a non mathematician who would have a slightest idea of what research mathematicians are interested in today. In fact "today" in this context means something like "in the last fifty years" since it seems that the public view of the world of pure mathematics reflects hardly any of the conceptual changes which occured in the second half of the twenties century. This makes the task of explaining a piece of new mathematics especially difficult because somewhere around 1950-60 mathematics lived through a revolution comparable to the revolution which happened in the physics in the first half of the century. Today when I think about how I can explain my own work I find myself in a position of someone who has to talk about the ideas underlying the unified theory of electro-weak interaction to an audience which never heard of either quantum mechanics or relativity theory.

In the first third of the century mathematics went through a period when it was very concerned with its foundations. This period largely ended by 1940 leaving behind among other things a way to deal with the foundational questions which felt rigorous enough to most working mathematicians. According to it all mathematical objects should be defined as sets with structures and all theorems about these objects should be deduced using the standard rules of logical inference from these definitions and the axioms of set theory. The most prominent illustration of both the power and the limitations of this approach can be found in the work of a group of French mathematicians writing together under the name of Nicola Bourbaki who created a series of volumes where the fundamental objects of mathematics were carefully defined in terms of the set theory and the theorems reflecting their major properties were rigorously proved in the context of these definitions. The parts of this work which deal with abstract algebra and topology still remain in my opinion to be the best expositions of the foundations of these subjects.

The success of the set-theoretic approach led to an explosion in the number of different species of structures considered by mathematicians. Together with new species of purely algebraic, topological and analitical structures a lot of composite species appeared such as topological groups for which it was impossible to tell which of the major branches of mathematics they were objects of. Some species of structures did not look like anything previously known at all. For each species constructions producing new structures of this species from already known ones had to be invented and while a lot of similarities between such constructions for different species should have been apparent there was no formal way to use these similarities or to transfer constructions from the world of one species to the world of another.

Together with the number and complexity of constructions producing new structures of given species from old ones grew the number and complexity of constructions which produced structures of one species from structures of another.

The revolution which happened in 50-60 was a result of a discovery of a new organizing principle which eventually led to a development of a new view of the world of mathematical objects. From this new perspective the set theory loses its central position and mathematical objects are viewed not as collections of elements or points together with explicitly defined structures relating this points to each other but as black boxes without predetermined internal structure which can be distinguished from each other only in terms of properties of their interaction with other objects of the same species. This development was made possible by a discovery that all mathematical objects of given species can be organized into a "society" whose basic structure is independent of the type of objects which it consists of. This structure can be formalized and theorems can be proven about the relations between properties of objects defined in terms of their "social position". Since objects of entirely different species can have the same social positions in their respective societies such theorems have universal significance and specialize to results about structures of many unrelated species.

In 1945 there appeared a paper by Samuel Eilenberg and Saunders MacLane called “General theory of natural equivalences” [1]. In this paper the authors defined yet another two new species of structures which they called categories and functors. In several years it became apparent that these two species of structures provide a basis for the most important since the development of the set-theoretic approach organizational innovation in mathematics. Today we can say that the special role of categories and functors is due to two main circumstances. The fisrt one is that to virtually any species of mathematical structure there corresponds a category (i.e. a structure of the species “category”) and to virtually any natural construction which produces a structure of one species from a structure of another there corresponds a functor. Moreover this correspondence is such that questions about structures of a given species can be reformulated as questions about the category corresponding to this species of structures and a similar statement holds for constructions and functors. The second one which is more prosaic but equaly important is that categories and functors can be visualized in a way which is helpful for the solution of problems which arise from questions about different species of structures. This visualization is based on the concept of a commutative diagram which was .... introduced.....

The main goal of my own mathematical work is to transfer the constructions developed in the field of mathematics which is called homotopy theory to another field which is called algebraic geometry and to use them to answer a bunch of interrelated questions known as the motivic conjectures. In what follows I will try to explain what the first half of the previous sentence mean. I was not able to find any way to give an explanation of what motivic conjectures are about. The main goal of this paper however is not to tell about homotopy theory or algebraic geometry but to use the elementary concepts of these two theories to illustrate how the ideas of categories and functors are used in real mathematics to connect seemingly unrelated fields and ultimately to use the techniques based on the intuition peculiar to one field to deal with problems of another.

2 *Homotopy theory*

Homotopy theory is a branch of mathematics which originated as a part of topology. From the very beginning topologists were interested in classification problems and in order to classify any kind of objects one first of all needs to be able to distinguish different objects from each other by some well defined properties. Every visualizable object has two obvious numerical properties. One is its dimension - a point is different from an interval because the point has dimension zero and the interval has dimension one. Another one is what mathematicians call the number of connected components i.e. the number of separate pieces which the object consists of. For example a triangle has one connected component while the object formed by two concentric circles has two.

The key idea of homotopy theory is that there are “higher” analogs of the number of connected components and the information which these higher analogs provide is often sufficient to distinguish objects of different shapes. The definition of these higher invariants given below is a good illustration of the set-theoretic method. First we have to introduce a species of sets with structures which will be our formal models for visualizable objects. This can be done in many ways. The structures of the most classical and convenient for us species used for this purpose are called metric spaces. A metric space X is a set together with a function d from the set of pairs of elements of X to the real numbers. Elements of X are often called points of X and the value $d(x, y)$ of the function d on a pair (x, y) is called distance bewteen the points x and y. In order for a set together with such a function to be a metric space d should satisfy some simple axioms which are abstractions of the properties of the usual distance in space (see ....). Any subset of the set of points of a metric space is itself a metric space. Together with the standard structure of the metric space on the Cartesian space $\mathbf{R}^n$ of n-tuples of real numbers defined by the Euclidean distance

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

this gives a natural metric space structure on any subset of $\mathbf{R}^n$. Thus we can talk about metric spaces corresponding to standard subsets such as cubes, spheres, balls etc.

Intuitively we say that two points of a visualizable object belong to the same connected component if there is a “path” inside this object from one point to another. Let us define a path in metric space X as a function γ from the unit interval $[0, 1]$ of the real line to X such that the function (in the usual sense) of two variables x and y given by $d(\gamma(x), \gamma(y))$ is continuous. A “path” in the intuitive sense associated with such a function is given by the set of points of the form $\gamma(x)$ for x in $[0, 1]$. The condition that the function $d(\gamma(x), \gamma(y))$ is continuous guarantees that it does not break anywhere.

For any point x of X define its connected component as the set of all points y such that there exists a path γ in X which starts in x and ends in

y i.e. such that $\gamma(0) = x$ and $\gamma(1) = y$. One can check that the connected components of two different points either do not intersect each other at all or coincide. Thus the set of all points of X gets divided in a unique way into subsets such that any two points inside each subset can be connected by a path but no paths exist between points of different subsets. These subsets are called the connected components of X. If there is finitely many of them as usually happens with the metric spaces we associate to visualizable shapes we can talk about the number of connected components. In general this need not to be the case but we still can consider the set of all connected components. This set which is denoted by $\pi_0(X)$ is the most important homotopy theoretic invariant of X.

So far we defined the notion of connected components for the species of sets with structures which we have choosen as models of visualizable objects. We can now define the higher analogs of $\pi_0(X)$ as follows. Fix a point x of X. Denote by $\Omega^1(X, x)$ the set of all paths in X which begin and end in x. For two paths γ_1, γ_2 in $\Omega^1(X, x)$ consider the function $f_{\gamma_1,\gamma_2}(t) = d(\gamma_1(t), \gamma_2(t))$. This is a function on the unit interval $[0, 1]$ and we may consider its average value $d(\gamma_1, \gamma_2)$ given by the integral

$$d(\gamma_1, \gamma_2) = \int_0^1 f(t)dt.$$

This formula gives us a function on pairs of elements of the set $\Omega^1(X, x)$ and it is easy to verify that it satisfies the axioms of a metric space. Note that this metric space is not really visualizable anymore even if X is a model of a simple shape like a circle but we can still apply our definition of the set of connected component to it. The set $\pi_0(\Omega^1(X, x))$ is denoted by $\pi_1(X, x)$ and it is the first one of the higher analogs of π_0 which homotopy theory studies. To define all the rest of them denote by $*$ the point of $\Omega^1(X, x)$ which is the "trivial" path from x to x i.e. the path γ given by $\gamma(t) = x$ for all t in $[0, 1]$. Now we can define inductively for $n \geq 2$

$$\Omega^n(X, x) = \Omega^1(\Omega^{n-1}(X, x), *)$$

and

$$\pi_n(X, x) = \pi_0(\Omega^n(X, x)).$$

The most famous problem of classical homotopy theory is to compute the number of elements in the sets $\pi_n(S^m, x)$ where S^m is the sphere of dimension m and x is a point on it (the answer does not depend on the choice of x).

These sets are known to be finite for .... and it is also known that for any $m \geq 2$ there is infinitely many n's for which $\pi_n(S^m, x)$ has more than one element. As of today the number of elements in $\pi_n(S^m, x)$ is computed for all reasonably small n but no good answer for general m and n is known.

3 *Categories and functors*

The word "category" in mathematics is used to refer to a particular species of structures and not to the Aristotelian categories of philosophy. Because of the importance of this species I will give its precise set-theoretic definition. This definition is rather involved and may be hard to comprehend but the last fifty years of mathematics has shown that it is well worth the effort.

A category $\mathcal{C}$ is a structure which consists of a class $ob(\mathcal{C})$ whose elements are called objects of $\mathcal{C}$ together with a set $Mor(X, Y)$ given for each pair of objects X, Y of $\mathcal{C}$ together with a function given for any three objects X, Y, Z of $\mathcal{C}$ from the set of pairs (f, g) where f is in $Mor(X, Y)$ and g is in $Mor(Y, Z)$ to the set $Mor(X, Z)$. Elements of $Mor(X, Y)$ are called morphisms from X to Y and the value of this function on a pair (f, g) is denoted by $g \circ f$ and called the composition of f and g. In order for these data to give a category the composition should satisfy two conditions. The first one says that for any three morphisms f, g, h such that the compositions $h \circ g$ and $g \circ f$ are defined the two possible triple compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ coincide. This condition is called the associativity axiom. The second one says that for any object X there is a morphism i_X from X to itself such that for any Y and any f from Y to X one has $i_X \circ f = f$ and for any Z and any g from X to Z one has $g \circ i_X = f$. This condition is called the identity axiom and the morphisms i_X are called identity morphisms. One can prove that for each X there is exactly one identity morphism i_X.

As the reader might have noticed I used the word "class" for the collection of all objects of a category and the word "set" for the collection of morphisms from one object to another. In the orthodox set-theoretic tradition the word "set" is a technical term which refers to an object whose existence can be logically inferred from the axioms of the set theory. There are situations when it turns out to be impossible to associate a set in this sense to an intuitively accessible collection of objects. For example the classical "Russel paradox" is a proof of the fact that the there is no set corresponding to the collection of all possible sets. To refer to general collections of objects without claiming them to be sets mathematicians use the word "class". One can thus talk about the class of all sets or the class of all metric spaces but not about the set of all sets or the set of all metric spaces.

Each category structure on a class of objects defines a notion of isomorphism for objects of this class. An isomorphism from X to Y is a morphism f such that there exists a morphism g from Y to X inverse to f i.e. such that $g \circ f = i_X$ and $f \circ g = i_Y$. Two objects of a category which are isomorphic (i.e. such that there exists an isomorphism between them) are totaly indistinguishable in categorical terms. Any property defined using only the language of objects, morphisms and compositions which holds for one object will also hold for the other. In a sense the converse is also true. Any property which depends only on the isomorphism class of an object can be expressed in purely categorical terms.

Functors are structure preserving functions from one category to another. More precisely a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function from the class of objects of $\mathcal{C}$ to the class of objects of $\mathcal{D}$ which is usually also denoted by F together with functions $F_{X,Y}$ from $Mor(X,Y)$ to $Mor(F(X), F(Y))$ given for each pair of objects X, Y of $\mathcal{C}$ such that $F_{Y,Z}(g) \circ F_{X,Y}(f) = F_{X,Z}(g \circ f)$ and $F_{X,X}(i_X) = i_{F(X)}$. It follows immediately from this definition that functors take isomorphisms to isomorphisms and thus if X and Y are isomorphic in $\mathcal{C}$ then $F(X)$ and $F(Y)$ are isomorphic in $\mathcal{D}$.

As I said in the introduction to all or almost all species of mathematical objects one can associate categories. A typical example of a category associated to a species is the category of sets (denoted **Sets**). Its objects are sets, morphisms from X to Y are functions from X to Y and the composition is the usual composition of functions i.e. $g \circ f$ is the function from X to Z given on an element x of X by $g(f(x))$. A function from one set to another is an isomorphism in this category if and only if it is a bijection. In particular two finite sets are isomorphic if and only if they have the same number of elements.

Another such example is the category of metric spaces and continuous functions. The class of objects of this category is the class of all metric spaces. The set of morphisms in this category from a metric space X to a metric space Y is the set of all continuous functions from X to Y. A continuous function f from X to Y is a function from the set of points of X to the set of points of Y such that for any point x of X and any $\epsilon > 0$ there exists $\delta > 0$ such that for any point x' whose distance to x is not greater than δ the distance from its image $f(x')$ in Y to the image $f(x)$ of x is not greater than ϵ. This is of course just the usual (ϵ, δ)-formulation of continuity adapted to general metric spaces. If f is a continuous function from X to Y and g is a continuous function from Y to Z then the function $g \circ f$ from X to Z is again continuous. This gives us composition of continuous functions and one checks easily that the category axioms are satisfied in this case. Two metric spaces which are isomorphic in this catgeory are called homeomorphic. We denote this category by **MSp**.

A structure of a category on the class of all objects of given species is not determined by the species itself. In the case of metric spaces we could define another category with the same class of objects taking morphisms to be the distance preserving functions instead of the continuous ones. A distance presering function from X to Y is a function f from the set of points of X to the set of points of Y such that $d(x, x') = d(f(x), f(x'))$. Such functions can also be composed and the category axioms hold in this case as well. Two metric spaces isomorphic in this catgeory are called isometric. For example two circles of different radius are homeomorphic but not isometric which shows that our categories are substantially diffrent. Everywhere below talking about the catgeory of metric spaces I will refer to the category of metric spaces and continuous functions.

The construction of the homotopy set $\pi_0(X)$ described in the previous section associates a set to any metric space X. As we now know both metric spaces and sets can be considered as objects of categories and it turns out that this construction gives an example of a functor from **MSp** to **Sets**. To check that π_0 is a functor one has to verify that for any continuous map $f : X \to Y$ of metric spaces and any two points x, x' in X which can be connected by a path the points $f(x)$ and $f(x')$ can be connected by a path in Y. This follows from the fact that for a path γ in X the function from the unit interval to Y given by $t \mapsto f(\gamma(t))$ is a path in Y.

Similarly the higher homotopy sets $\pi_n(X, x)$ give examples of functors but since their construction starts with a metric space together with a point they are functors not from the category of metric spaces but from a slightly different category of metric spaces with a distinguished point $\mathbf{MSp}_\bullet$ whose objects are pairs (X, x) where X is a metric space and x is a point of X and morphisms from (X, x) to (Y, y) are continuous functions f from X to Y such that $f(x) = y$.

Because π_n are functors they take homeomorphic spaces to isomorphic sets and therefore the number of elements in the homotopy sets of two homeomorphic spaces is the same. For example one can see that the boundary of a cube (or more generally of any convex polyhedron) is homeomorphic to the sphere of the same dimension and therefore they have the same number of elements in their homotopy sets. This inference is one of the classic examples of categorical thinking.

As I said in the introduction one of the most important factors which contributed to the widespread use of categories in mathematics is the possibility to use visualize them. The trick which allows us to do it is based on the notion of a commuative diagram in a category. Consider the picture:

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array} \tag{1}$$

This picture is an example of what in mathematics is called a diagram. A general diagram is a picture which consists of vertices connected by arrows. If I have a category $\mathcal{C}$ a diagram *in* $\mathcal{C}$ is a diagram whose vertices are marked by objects of $\mathcal{C}$ and whose arrows are marked by morphisms between the corresponding objects. For example a diagram of the form (1) in a category $\mathcal{C}$ may be written as follows

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[g']{} & D \end{array} \tag{2}$$

For any path in a diagram in $\mathcal{C}$ i.e. for any sequence of arrows where the end of one is the beginning of the next the composition of morphisms which mark these arrows is a morphism from the object marking the beginning of the path to the object marking the end. If the compositions of morphisms corresponding to all paths between two vertices coincide a diagram is called commutative. To say that a diagram of the form (2) is commutative is the same as to say that $g \circ f = g' \circ f'$.

In a sense which can be made precise a category is completely determined by its commuative diagrams. Thus even though I can not visualize the whole category of sets or of metric spaces I can easily visualize "parts" of it in the form of diagrams. If I have four sets A, B , C and D and four morphisms $f : A \to B$, $g : B \to D$ and $f' : A \to C$, $g' : C \to D$ such that $g \circ f = g' \circ f'$ I can visualize the part of the category of sets span by these four objects and four morphisms as a square of the form (2). In general if I have finitely many objects and morphisms between them with some realtions it is often possible to imagine the part of the category span by these objects and morphisms as a diagram of some form.

A functor can also be visualizied using the imagery of commutative diagrams. If we have a commutative diagram in $\mathcal{C}$ say of the form (2) and F is a functor from $\mathcal{C}$ to $\mathcal{D}$ then

$$\begin{array}{ccc} F(A) & \xrightarrow{F_{A,B}(f)} & F(B) \\ {\scriptstyle F_{A,C}(f')}\downarrow & & \downarrow{\scriptstyle F_{B,D}(g)} \\ F(C) & \xrightarrow[F_{C,D}(g')]{} & F(D) \end{array} \qquad (3)$$

is a commutative diagram in $\mathcal{D}$. The same thing happens with commutative diagrams of all other types so that we can see a functor from one category to another as a mapping from commuative diagrams in one category to commutative diagrams in another.

Given a functor F from $\mathcal{C}$ to $\mathcal{D}$ and a functor G from $\mathcal{D}$ to $\mathcal{E}$ one can easily define their composition $G \circ F$ which is a functor from $\mathcal{C}$ to $\mathcal{E}$. All categories and functors between them almost form a category but not quite because the collection of all functors bewteen two categories need not be a set in the sense explained above. As one may guess it is possible to define the notion of a 2-category such that all categories will form a 2-category in the same manner as all sets form a category. The ideas associated with the concept of 2-categories and related concepts of higher categories play considerable role in contemporary mathematics. However the passage from categories to higher categories is more of a quantitative than of a qualitative nature and its significance can not be comapred to the significance of the leap from set-theoretic to categorical thinking.

4 *The category of algebraic equations*

The category of metric spaces which we defined in the previous section is one of many possible models of the world of visualizable objects. In this section we will define categories of a completely different sort whose objects are systems of algebraic equations [1]. These categories form the world of

[1] I believe that I learned the definition which is given below from a book by Yu. I. Manin called "Affine schemes" which was published by Moscow University in the early

algebraic geometry.

¿From school we know two ways of changing a system of equations in such a way that there is a one-to-one correspondence bewteen the solutions of the old system and the solutions of the new one. The first of these methods is to add to one of the equations of the system a multiple of another one. This procedure does not change the solutions at all. The second one is to replace the old variables x_i by new variables x'_j in such a way that x'_j's can be expressed in terms of x_i's and vice versa. For example the equations $x_1^2 + x_2^2 - 1 = 0$ and $(x'_1)^2 + 2x'_1 + (x'_2)^2 = 0$ can be obtained from each other by such a replacement. In this case the set of solutions changes but there is a bijection bewteen the solutions of the new system and the solutions of the old one. In this particular example the sets of real solutions of both equations are circles of radius 1 but the center of the first one is in $(0, 0)$ and the center of the second in $(-1, 0)$. If we are only interested in the "geometry" of the set of solutions which is what algebraic geometry is about it is natural to consider the systems which can be obtained from each other by one of these procedures to be equivalent. It turns out that one can construct a category whose objects are systems of equations such that the transformations of these two types give isomorphisms in this category and a functor from it to the category of sets which takes a system of equations to the set of its solution. Because of these two properties this category is ideally suited for the study of geometry of the solutions.

To give the precise definition of this category I need to say more about *coefficients* of the systems which we consider. If I take a system with rational coefficients and transform it to another system with rational coefficients by a change of variables which uses with irrational coefficients there is no reason to expect that the rational solutions of the new system will be in one to one correspondence with the rational solution of the old one. For example the equations $x^2 - 2 = 0$ and $y^2 - 1 = 0$ can be obtained from each other by the change $x = \sqrt{2}y$ but the first of them has no rational solutions and the second one has two. It means that some systems of equations may be isomorphic when considered with say real coefficients but not isomorphic when considered with the rational ones. Similar things may happen bewteen real and complex coefficients or rational and intergral coefficients. This means that to define "the category of systems of algebraic equations" we have to explicitly specify what coefficients we consider. We could define separately

70-ies but unfortunately I can not check this since I have no way to get hold of it now.

the categories for integral, rational, real and complex coefficients but this is not very elegant. To deal with this problem mathematicians invented a species of sets with structures called commutative rings.

A commutative ring is a set with two operations which are customary denoted by $x+y$ and xy and called addition and multiplication which satisfy a collection of axioms mimicing the usual properties of addition and multiplication of numbers such as associativity, distributivity and commutativity (**??**). Strictly speaking one should denote a ring by a triple $(R, +, \bullet)$ making explicit refernce to the operations but usually one just writes R. There is, of course, the category of rings. Its objects are rings and morphisms are homomorphisms of rings i.e. functions $f : R \to S$ such that $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$. The standard examples of rings are given by different types of numbers. The sets of integers, rational numbers, real numbers and complex numbers with their usual addition and multiplication operations all are commutative rings. In mathematics they are denoted by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ respectively. There are infinitely many other commuative rings. For example the set of all polynomials in variables $x_1, \ldots, x_n$ over any commuative ring R with the usual operations of addition and multiplication of polynomials is a commutative ring denoted by $R[x_1, \ldots, x_n]$.

Define a system of algebraic equations over R as a pair $(n, (f_i)_{i=1,\ldots,m})$ where n is a non negative integer which signifies the number of variables our system depends on and f_i are polynomials with coefficients in R in variables $x_1, \ldots, x_n$. Note that n may be equal to zero in which case f_i are just elements of R.

For two systems $X = (n, (f_i)_{i=1,\ldots,m})$ and $Y = (n', (g_j)_{j=1,\ldots,m'})$ consider the set $A(X, Y)$ of families of polynomials $\phi_1, \ldots, \phi_{n'}$ in variables $x_1, \ldots, x_n$ such that the polynomials $g_j(\phi_1(x_1, \ldots, x_n), \ldots, \phi_{n'}(x_1, \ldots, x_n))$ can be expressed as linear combinations of polynomials f_i i.e. such that there exist polynomials u_{ij} in variables $x_1, \ldots, x_n$ satisfying

$$g_j(\phi_1(x_1, \ldots, x_n), \ldots, \phi_{n'}(x_1, \ldots, x_n)) = \sum_i u_{ij}(x_1, \ldots, x_n) f_i(x_1, \ldots, x_n)$$

Two elements $(\phi_l)_{l=1,\ldots,n'}$, $(\phi'_l)_{l=1,\ldots,n'}$ of this set are said to be equivalent if the polynomials $\phi_l - \phi'_l$ are linear combinations of polynomials f_i. The set of equivalence classes with respect to this equivalence is the set of morphisms from X to Y.

If we have a third system $Z = (n'', (h_k)_{k=1,\ldots,m''})$ and a morphism from Y to Z given by a collection of polynomials $\psi_1, \ldots, \psi_{n''}$ then the polynomials

$\psi_l(\phi_1(x_1, \ldots, x_r), \ldots, \phi_{n'}(x_1, \ldots, x_r))$, $l = 1, \ldots, n''$ give a morphism from X to Z which is the composition of ϕ and ψ. Denote the category which we defined by $Algeq_R$. In contemporary algebraic geometry it is known as the category of affine schemes of finite type over R.

If $\phi_l(x_1, \ldots, x_n)$ is an element in $A(X, Y)$ and $(a_i)_{i=1,\ldots,n}$ are elements in R such that $x_i = a_i$ is a solution of X then $x_j = \phi_j(a_1, \ldots, a_n)$ is a solution of the system Y. Thus an element in $A(X, Y)$ gives us a function from the set of solutions of X to the set of solutions of Y and one can check that two equivalent elements give the same function. This gives us a functor from the category $Algeq_R$ to the category of sets which takes a system to its set of solutions. In particular we conclude that two systems which are isomorphic as object of this category have isomorphic sets of solutions.

5 *Homotopies in the world of equations*

Let us try now to find parallels between objects of the category of metric spaces and objects of the category $Algeq_R$. Consider first the space pt which consists of one point. Up to an isomorphism it is the only object of the category of metric spaces such that there is exactly one morphism from any other object to it. In the category $Algeq_R$ there is also a unique up to an isomorphism object with this property namely the system $0 = 0$ in the empty set of variables. The unique morphism from any other system to it is given by the empty family of polynomials. Denote this object by pt_R.

Consider now the morphisms $Mor(pt, X)$ from the point to a metric space X in the category of metric spaces. This set can be identified with the set of all points of X by the map which takes a morphism $f : pt \to X$ to the image of f. Thus the analog of the set of points for an object X in the category $Algeq_R$ is the set of morphisms $Mor(pt_R, X)$. If X is a system of the form (**??**) then a morphism from pt_R to X is given by a collection of polynomials $Z_1, \ldots, Z_t$ in zero variables i.e. a collection of elements of R such that $h_i(Z_1, \ldots, Z_t)$ are all zero. Thus morphisms from pt_R to any system are exactly the solutions of this system. We found that the set of solutions of a system is categorically analogous to the set of points of a space.

Let us try now to define $\pi_0(X)$ for an object X of $Algeq_R$. First we have to find a categorical reformulation of our definition of π_0 for metric spaces. The key ingredient of this definition is the notion of a path. Consider the unit interval $[0, 1]$ as a metric space. A path in X is a function γ from points of $[0, 1]$ to points of X and one can prove that our condition that the function $d(\gamma(t), \gamma(t'))$ in two variables is continuous is exactly equivalent to the condition that γ is a continuous function between metric spaces. The points 0 and 1 of the unit interval can be interpreted as morphisms $pt \to [0, 1]$. Let us denote these two morphisms by i_0 and i_1 respectively. Then $\pi_0(X)$ can be identified with the set of equivalence classes of morphisms from pt to X where two morphisms $x, x' : pt \to X$ are called equivalent if there exists a morphism $\gamma : [0, 1] \to X$ such that $i_0 \circ \gamma = x$ and $i_1 \circ \gamma = x'$. This is our categorical reformulation of the construction of π_0.

To transfer it to the category $Algeq_R$ we need to choose an object I in this category together with two morphisms $i_0, i_1 : pt_R \to I$ which we will use to replace the unit interval with its points 0 and 1 in the definition given above. The candidate for such object was known for many years. It is the object denoted by $\mathbf{A}^1$ and called the affine line which corresponds to the system $0 = 0$ of equations in one variable t. Since any value of t is the solution of this system the set of morphisms from pt_R to $\mathbf{A}^1$ can be identified with the set of all elements of R. In particular since any ring has by definition two distinguished elements called zero and one this set always contains two distinguished morphisms which we denote as above by i_0 and i_1. If can now apply the definition of π_0 given above replacing the unit interval by $\mathbf{A}^1$ and the points 0 and 1 by morphisms i_0, i_1.

Here we encounter the first problem. In order for our definition of π_0 for metric spaces to work we needed to know that if a point x can be connected by a path to a point x' and x' can be connected to x'' then x can be connected to x''. For metric spaces this is true and easy to check. For systems of equations and the paths defined as morphisms from $\mathbf{A}^1$ it is false. For the system $xy = 0$ in variables x, y the point $(0, 1)$ can be connected by a path to the point $(0, 0)$ and $(0, 0)$ can be connected to the point $(1, 0)$ but there is no path connecting $(1, 0)$ and $(0, 1)$.

While all of the above analogies were known for many years this and other more complicated problems prevented us from finding a correct definition of π_0 and of higher homotopy sets in this context. The first acceptable definition appeared only a few years ago. It uses complicated categorical techniques and was made possible by the fact that in the last fifty years definitions of homotopy sets π_n where given for objects of many diffrent categories which were getting progressively further removed from the context of metric or topological spaces. However the main underlying idea of this definition is still the analogy between the unit interval $[0, 1]$ and the affine line $\mathbf{A}^1$ which is explain above.

6 *Conclusion*

Two elements of modern mathematics distinguish it from other systems of thought such as formal logic or the systems of philosophy. One is the particular manner in which it combines the use of the symbolic and the visual abilities of our minds. Mathematical objects can be vaguely divided into algebraic and geometric ones depending on which of their two faces the symbolic or the visual one is more prominent. The most interesting results in mathematics are always obtained with the use of both of these faces.

The second one is the use of axiomatic systems as means for unambigous communication. To be made known to other people the insights achieved with the use of the symbolic or the visual face of mathematical objects must be translated into an axiomatic form. This translation which is often the most time consuming part of mathematical work is the best way we know to make it possible for people to find mistakes in each other reasoning. Each time people attempted to replace axiomatic method as a means of communication of mathematics by something less rigorous this led to a growing number of mistakes and ultimately to the state when the whole product of their work had to be discarded.

The combination of two types of constructions - the ones originated on the visual side and the ones originated on the symbolic but both expressed in the axiomatic terms to enable unambigous communication is at the heart of the mathematical method. Since the oldest objects of mathematics originated as models of objects of the observable world and most or all of the newer ones are connected with them by sequences of constructions of one of these two types the esoteric world of contemporary mathematics taken as a whole is a direct result of our minds with all their peculiarities trying to make sense of the reality.

The discovery of the categorical approach gave us a completely new way to relate mathematical objects of diffrent species to each other which is based on neither the symbolic nor on the visual faces of these objects but on their "social" faces. While the intuition underlying these ideas must have always been present it is only since the middle of this century that the discovery of the formal definitions of categories, functors and other related concepts gave us a way to express these relations in axiomatic terms and thus to make them a part of mathematics.

In the previous four sections I gave an example of such use. We considered there two main species of mathematical objects. The first one called the metric spaces is one of the classic species used to model the properties of visualizable objects related to their “shape”. It is relatively easy to learn how to translate the insights provided by the visual intuition into the formal language of metric spaces. The definition of the set of connected components of a metric space given above is one example of such a translation. The first definition of π_1 was obtained by translation of another intuitive idea and the first definitions of higher π_n certainly depended on visual intuition as well. Later on the interplay between the visual intuition and the symbolic computations stimulated by the problem of computing the number of elements in the stable homotopy sets of spheres led to a multitude of constructions of both types which today constitute the fields of homotopy theory and algebraic topology.

The objects of another species which we considered are systems of algebraic equations with coefficients in commutative rings. They give an example of objects of algebraic type since they primary ability needed to work with them is the ability of symbolic manipulation. Classical algebraic geometry is a field of mathematics which studies the geometry of the sets of solutions of such systems. It developed a complicated machinery which enables one to use visual intuition based on the geometry of the solutions of equations over real and complex numbers to answer questions about systems of equations over general rings. Two most famous theorems in algebraic geometry proven in the second half of the century the Deligne’s proof of the Weil conjectures and the Faltings’s proof of the .. conjecture both establish a relations between the “shape” of the set of complex solutions of a system and the number of its solution over rings of arithmetical nature.

The homotopy theory of algebraic varieties attempts to transfer the machinery of the homotopy theory to the world of algebraic geometry. Instead of looking for constructions relating individual objects of one world to individual objects of another it attempts to achieve this transfer using the fact that both metric spaces (or any other models for visualizable objects) and algebraic varieties form structures of the same species namely categories. If we discover a proper way to express all the constructions of homotopy theory in purely in terms of the model category we work with (say of metric spaces) we should be able then to apply them directly to any other category with appropriate additional structures on it which should include the catgeory of algebraic varieties. As of today we do not have a full knowledge of how this can be done in general but we know enough to find one by one appropriate algebro-geometric analogs of main constructions of homotopy theory.

As I said in the introduction my interest in the homotopy theory for algberaic varieties is stimulated by a number of questions in algebraic geometry which were posed in the last thirty years and which are collectively refered to as standard motivic conjectures. A few years ago we succeeded in answering one of these questions which is called Milnor's conjecture using this new machinery. Today we begin to perceive new similarities between the rest of motivic conjectures and a group of results proven in the homotopy theory in the 80-ies. If these similarities are not deceiving then the analogy between the homotopy theory of spaces and the homotopy theory of algebraic varieties is tighter that anyone thought. In any even the facts we already know make it plausible that both the homotopy theory of spaces and the homotopy theory of varieties are particular cases of one theory. So far we are unable to invent such a theory but one day we will.

References

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