CUBICAL APPROACH TO STRAIGHTENING

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Abstract. For a suitable choice of the cube category, we construct a topology on it such that sheaves with respect to this topology are exactly simplicial sets (thus establishing simplicial sets as a subcategory of the category of cubical sets). We then generalize the construction of the homotopy coherent nerve to cubical categories and establish an analog of Lurie's straightening–unstraightening construction.

Introduction

Cubical sets provide a well-studied combinatorial model for spaces. They were considered by Kan [Kan55, Kan56] before the introduction of simplicial sets. However, while there is only one category of simplicial sets, there are several different categories of cubical sets, depending on the choice of morphisms in the indexing category $\square$.

In each case $\square$ is a subcategory of Cat whose objects are posets of the form $\{0 \leq 1\}^n$. The minimalistic choice (considered for example by Jardine [Jar06]) would be to take the smallest category generated by the *face* and *degeneracy* maps. This category is for instance a test category, but its cartesian product is not homotopically well-behaved (e.g. the cartesian product of the interval with itself has the homotopy type of $S^1 \vee S^2$), which is somewhat unsatisfying. Other authors extend the category $\square$ to include also connections [Mal09, Cis06], which fixes some of the problems with the cartesian product and makes $\square$ into a *strict* test category.

In this paper, we consider a new category of combinatorial cubes, taking $\square$ to be the *full* subcategory of Cat with those objects. Until now, this category has not been used in homotopy theory and has only been considered in dependent type theory to give a constructive interpretation of the Univalence Axiom [CCHM15].[1] Many of the standard methods from simplicial homotopy theory are not available in this setting, for instance, the Eilenberg–Zilber Lemma (asserting that every simplex is a degeneracy of a unique non-degenerate one in a unique way, see e.g. [JT08, Prop. 1.2.2]). Our category $\square$ is also not a (generalized) Reedy category.

We show however that this category can be used to gain better understanding of several constructions in higher category theory and simplicial homotopy theory.

In Section 1, we introduce cubical sets and cubical categories. Our first observation is that cubical sets are more general than simplicial sets. To make this statement precise, we equip the cube category with a Grothendieck topology and show that the sheaves for this topology are precisely simplicial sets. Thus we obtain a full embedding $\mathsf{sSet} \hookrightarrow \mathsf{cSet}$ of the category of simplicial sets into the category of cubical sets. The sheafification functor is then given by triangulation.

In Section 2, we generalize the construction of the homotopy coherent nerve from the category of simplicial categories to that of cubical categories. Precisely, we define a functor $\mathrm{N}_\square \colon \mathsf{cCat} \to \mathsf{sSet}$ (here, cCat denotes the category of categories enriched over cubical sets) and show that homotopy coherent nerve functor $\mathrm{N}_\triangle$ is then the composite $\mathsf{sCat} \to \mathsf{cCat} \to \mathsf{sSet}$.

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[1] In fact, [CCHM15] works with yet another variation on the notion of a cubical set, although our cube category is perfectly sufficient for all of their applications.

We moreover show that if the cubical category is locally Kan, then the resulting simplicial set is a quasicategory, which mirrors an analogous result for the homotopy coherent nerve.

In Section 3, we give a construction taking a map $F\colon S \to \mathrm{N}_\square \mathsf{cSet}$ to a map $\int_S F \to S$ (the *category of elements*), and show that this assignment is functorial. Finally, we prove that $\int_S$ is a right adjoint by explicitly constructing its left adjoint and show that by passing between simplicial and cubical categories, this adjunction recovers Lurie's (straightening $\dashv$ unstraightening)-adjunction.

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1. The category of cubical sets

1.1. **Cubical sets.** Let $\square$ denote the full subcategory of the category Cat of small categories (or Pos of small posets) whose objects are posets of the form $[1]^n$, where $[1] = \{0 \leq 1\}$, which can be identified with the posets of subsets of the set $\{1, \ldots, n\}$. We will refer to $\square$ as the **cube category**. The category cSet of **cubical sets** is the category of contravariant functors $\square^{\mathrm{op}} \to \mathsf{Set}$ and natural transformations.

We will write $\square[1]^n$ for the standard n-cube, that is, the representable cubical set, represented by $[1]^n$. If $n = 0$ or $n = 1$, we will write $\square[0]$ and $\square[1]$ respectively. For each $i = 1, \ldots, n$ and $\varepsilon = 0, 1$, there is a map $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_n)$, whose image under the Yoneda embedding will be denoted $\partial_i^\varepsilon\colon \square[1]^{n-1} \to \square[1]^n$. Similarly, for each $i = 1, \ldots, n$, there is a map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$, whose image under the Yoneda embedding will be denoted $\sigma_i\colon \square[1]^n \to \square[1]^{n-1}$. For a cubical set X, the corresponding maps $d_i^\varepsilon\colon X_n \to X_{n-1}$ and $s_i\colon X_{n-1} \to X_n$ are called **face** and **degeneracy** maps.

As the category cSet is a topos, we can speak about images of maps, as well as unions and intersections of subobjects. By a face of a cube, we understand the image of one of ∂_i^ε's. We define the **boundary** $\partial\square[1]^n$ of the n-cube $\square[1]^n$ as the union of all of its faces, and similarly, the (ε, i)-**open box** $\sqcap_i^\varepsilon[1]^n$ as the union of all the faces except the one in the image of ∂_i^ε.

Definition 1.1.

(1) A cubical set X is a (cubical) **Kan complex** if for all $n \in \mathbb{N}$, $i = 1, \ldots, n$, and $\varepsilon = 0, 1$, and any map $\sqcap_i^\varepsilon[1]^n \to X$, there exists an extension

$$\begin{tikzcd} \sqcap_i^\varepsilon[1]^n \arrow[r] \arrow[d, hook] & X \\ \square[1]^n \arrow[ur, dashed] & \end{tikzcd}$$

(2) A cubical set X is a **universal Kan complex** if for any cubical set K, the exponential X^K is a Kan complex.

Taking $K = \square[0]$, we see that every universal Kan complex is also a Kan complex. The full subcategory of cSet consisting of those cubical sets that are universal Kan complexes will be denoted Kan. The reason to work with the category of universal Kan complexes rather than just Kan complexes is that the former has better categorical properties, for instance, it is closed under exponentials.

Remark 1.2. Using [Mal05, Ex. 1.5.9 and 1.6.11], the category $\square$ is easily seen to be a strict test category and thus cSet carries a model structure, in which cofibrations are monomorphisms. However, not every fibrant object in this model structure is a cubical Kan complex and thus this model structure is not helpful from our point of view.

Our next goal is to establish a topology J on $\square$ such that the category of sheaves $\mathsf{Sh}(\square, J)$ is equivalent to the category sSet of simplicial sets. We will obtain it from a more general construction.

Let $\mathcal{C}$ be a small category and consider the topology J_{jef} on the presheaf category $\mathsf{PrSh}(\mathcal{C})$, given by jointly epimorphic families. Let $u\colon \mathcal{T} \hookrightarrow \mathsf{PrSh}(\mathcal{C})$ be a full subcategory with the property that every representable $\widehat{c} \in \mathsf{PrSh}(\mathcal{C})$ admits a cover by the objects from $\mathcal{T}$. Considering $\mathcal{T}$ as a site with the topology given by the restriction of J_{jef}, we obtain a composite map:

$$\mathsf{PrSh}(\mathcal{C}) \hookrightarrow \mathsf{Sh}(\mathsf{PrSh}(\mathcal{C}), J_{\text{jef}}) \xrightarrow{u_*} \mathsf{Sh}(\mathcal{T}, J_{\text{jef}}|\mathcal{T})$$

where the first map is the Yoneda embedding, and the second map is given by precomposition with u.

Lemma 1.3. *The map* $\mathsf{PrSh}(\mathcal{C}) \to \mathsf{Sh}(\mathcal{T}, J_{\text{jef}}|\mathcal{T})$ *above is an equivalence of categories.*

Proof. By [MR77, Prop. 1.3.14], the inclusion $\mathsf{PrSh}(\mathcal{C}) \hookrightarrow \mathsf{Sh}(\mathsf{PrSh}(\mathcal{C}), J_{\text{jef}})$ is an equivalence of categories. The inclusion $u\colon (\mathcal{T}, J_{\text{jef}}|\mathcal{T}) \hookrightarrow (\mathsf{PrSh}(\mathcal{C}), J_{\text{jef}})$ satisfies the assumptions of the Lemme de Comparaison [AGV71, Thm. 4.1], and thus u_* gives an equivalence $\mathsf{Sh}(\mathsf{PrSh}(\mathcal{C}), J_{\text{jef}}) \xrightarrow{u_*} \mathsf{Sh}(\mathcal{T}, J_{\text{jef}}|\mathcal{T})$. $\square$

Let $\mathrm{N}\colon \mathsf{Cat} \to \mathsf{sSet}$ be the **nerve functor**, taking a category $\mathcal{C}$ to a simplicial set $\mathrm{N}\mathcal{C}$ whose n-simplices are given by $(\mathrm{N}\mathcal{C})_n = \mathsf{Cat}([n], \mathcal{C})$. This functor is full and faithful.

Theorem 1.4. *The category* $\mathcal{C} = \Delta$ *and the inclusion* $u\colon \square \hookrightarrow \mathsf{sSet}$ *given by the restriction of the nerve functor satisfy the assumptions of Lemma 1.3 and thus yielding an equivalence* $\mathsf{sSet} \simeq \mathsf{Sh}(\square, J)$*, where J is the topology given by families that become jointly epimorphic in* sSet*.*

Proof. The symmetric group Σ_n acts on $\Delta[1]^n \cong \mathrm{N}([1]^n)$ by permuting the factors and the standard n-simplex is the quotient of $\Delta[1]^n$ by this action. $\square$

Denote the inclusion $\mathsf{sSet} \simeq \mathsf{Sh}(\square, J) \hookrightarrow \mathsf{cSet}$ by U. By construction, we obtain that for a simplicial set X, $\mathrm{U}X$ is a cubical set whose n-cubes are given by:

$$\mathrm{U}(X)_n = \mathsf{sSet}(\Delta[1]^n, X).$$

The sheafification functor $\mathrm{T}\colon \mathsf{cSet} \to \mathsf{sSet}$ is given by triangulation, that is, the left Kan extension of the inclusion $u\colon \square \hookrightarrow \mathsf{sSet}$ along the Yoneda embedding:

$$\begin{tikzcd}
\mathsf{cSet} \arrow[rr, bend left=10, "\mathrm{T}"] \arrow[rr, phantom, "\perp"] & & \mathsf{sSet} \arrow[ll, bend left=10, "\mathrm{U}"] \\
& \square \arrow[ul, hook] \arrow[ur, hook, "u"'] &
\end{tikzcd}$$

It follows that U is full and faithful, the counit map $\mathrm{TU} \to \mathrm{id}_{\mathsf{sSet}}$ is an isomorphism, and T, as associated sheaf functor, preserves finite limits.

Lemma 1.5. *The functor* $\mathrm{U}\colon \mathsf{sSet} \to \mathsf{cSet}$ *takes (simplicial) Kan complexes to universal Kan complexes.*

Proof. Let X be a simplicial Kan complex, K a cubical set and consider a lifting problem:

$$\begin{tikzcd}
\sqcap_i^{\varepsilon}[1]^n \arrow[r] \arrow[d, hook] & (\mathrm{U}X)^K \\
\square[1]^n \arrow[ur, dashed] &
\end{tikzcd}$$

Since T preserves finite limits, a filler for the open box $\sqcap_i^\varepsilon[1]^n \to (\mathrm{U}X)^K$ corresponds, by adjointness, to a lift in:

$$\begin{tikzcd} \mathrm{T}K \times \mathrm{T}\sqcap_i^\varepsilon[1]^n \arrow[r] \arrow[d, hook] & X \\ \mathrm{T}K \times \mathrm{T}\square[1]^n \arrow[ur, dashed] & \end{tikzcd}$$

Since T preserves monomorphisms, the map $\mathrm{T}\sqcap_i^\varepsilon[1]^n \to \mathrm{T}\square[1]^n = \Delta[1]^n$ is a cofibration. It is moreover a weak equivalence, because both simplicial sets are contractible. Thus it is anodyne and hence the desired lift exists. $\square$

Remark 1.6. The standard n-simplex is a sub-presheaf of a cube. In general, there are $n! = \#\Sigma_n$ inclusions $\Delta[n] \hookrightarrow \square[1]^n$ (for instance, there are two triangles in a square). The family $\{\Delta[n] \to \square[1]^n\}_{\Sigma_n}$ is not an epimorphism since a surjection with representable codomain must necessarily admit a section (which a map $\coprod \Delta[n] \to \square[1]^n$ cannot), however it becomes a cover after applying the triangulation functor T.

We will write cCat for the category of cubical categories (i.e. categories enriched over the cartesian monoidal category cSet) and cubical functors. Similarly, we will write sCat for the category of simplicial categories (categories enriched over the cartesian monoidal category sSet) and simplicial functors.

Given a simplicially/cubically enriched category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, we will write $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ for the mapping simplicial/cubical set. The subscript $\mathcal{C}$ will be omitted whenever no ambiguity is possible.

Let $\mathcal{C}$ be a cubical category and $x, y \in \mathcal{C}$ two objects. We will write $f: x \to y$ to mean that $f \in \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)_0$. Given $f, g: x \to y$, we write $H: f \to g$ for $H \in \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)_1$ such that $d_1^0 H = f$ and $d_1^1 H = g$. The composition in a cubical category will be denoted with '$\cdot$' and will be written in the diagrammatic order. Thus for $f: x \to y$ and $g: y \to z$, their composite will be written $f \cdot g$.

Every (1-)category can be regarded as a cubical category with discrete mapping cubical sets, which defines an inclusion $\mathsf{Cat} \hookrightarrow \mathsf{cCat}$.

Since both T and U preserve finite products, the adjunction $\mathrm{T}: \mathsf{cSet} \rightleftarrows \mathsf{sSet} : \mathrm{U}$ gives rise to

$$\begin{tikzcd} \mathsf{cCat} \arrow[r, bend left, "\mathrm{T}_\bullet"] \arrow[r, phantom, "\perp"] & \mathsf{sCat}. \arrow[l, bend left, "\mathrm{U}_\bullet"] \end{tikzcd}$$

where $\mathrm{T}_\bullet\mathcal{C}$ (respectively, $\mathrm{U}_\bullet\mathcal{C}$) has the same objects as $\mathcal{C}$ and the mapping objects are obtained by applying T (resp. U) to those of $\mathcal{C}$.

We conclude this section with a discussion of homotopies and (homotopy) equivalences in cubical categories. Since the mapping spaces $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ may not be Kan complexes, we need to consider the notion of a zig-zag (cf. [GZ67, Sec. II.2.5.1 and IV.1.1.1]) in order to make homotopy an equivalence relation.

An abstract zig-zag is a cubical set of the form $\square[1] +_{\square[0]} \ldots +_{\square[0]} \square[1]$ with the property that if some $\square[1]$ receives two maps from $\square[0]$ in the above colimit, then these maps must be different (and necessarily be $d_1^0, d_1^1 : \square[0] \to \square[1]$). A zig-zag in a cubical set X is a cubical map from an abstract zig-zag to X.

Definition 1.7. (1) An **elementary homotopy** between two maps $f, g\colon x \to y$ in a cubical category $\mathcal{C}$ is $H\colon f \to g$ (i.e. a 1-cube $H \in \mathrm{Map}_{\mathcal{C}}(x,y)_1$ with $d_1^0 H = f$ and $d_1^1 H = g$). We write $H\colon f \sim_1 g$ to indicate that H is an elementary homotopy from f to g.
(2) A **homotopy** between two maps $f, g\colon x \to y$ in a cubical category $\mathcal{C}$ is a zig-zag of elementary homotopies from f to g. We write $H\colon f \sim g$ to indicate that H is a homotopy from f to g.
(3) A morphism $f\colon x \to y$ in a cubical category $\mathcal{C}$ is an **equivalence** if there exist maps $g_1, g_2\colon y \to x$ and homotopies $H_1\colon f \cdot g_1 \sim \mathrm{id}_x$ and $H_2\colon g_2 \cdot f \sim \mathrm{id}_y$.

Lemma 1.8. *Homotopy defines an equivalence relation on* $\mathrm{Map}_{\mathcal{C}}(x,y)$.

Proof. For reflexivity, we take $fs_1\colon f \sim f$. Symmetry is immediate since zig-zags are symmetric. Finally, we can compose zig-zags by taking the appropriate pushout along $\Box[0]$. □

Lemma 1.9. *In every cubical category $\mathcal{C}$, the class of equivalences is closed under composition and every identity is an equivalence.*

Proof. To see that identities are homotopy equivalences, take $g_1 = g_2 = \mathrm{id}$ and use reflexivity of homotopy. Now, suppose that $x \xrightarrow{f} y \xrightarrow{f'} z$ are both homotopy equivalences with inverses $g_1, g_2\colon y \to x$ and $g_1', g_2'\colon z \to y$. Suppose we wish to show that $g_1' \cdot g_1$ is a one-sided inverse of $f \cdot f'$. Let $H_1\colon f \cdot g_1 \sim \mathrm{id}_x$ and $H_1'\colon f' \cdot g_1' \sim \mathrm{id}_y$. Then $f^*(g_1)_* H_1'$ is a homotopy $f \cdot f' \cdot g_1' \cdot g_1 \sim f \cdot g_1$, so composing it with H_1 gives the desired homotopy $f \cdot f' \cdot g_1' \cdot g_1 \sim \mathrm{id}_x$. Similarly, one verifies that $g_2' \cdot g_2$ is an inverse of $f \cdot f'$ on the other side. □

Definition 1.10. A cubical category $\mathcal{C}$ is **locally Kan** if for every pair $x, y \in \mathrm{Ob}\,\mathcal{C}$, the cubical set $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a (cubical) Kan complex.

Examples 1.11.

(1) The full cubical subcategory Kan (with $\mathrm{Map}_{\mathsf{Kan}}(X,Y) = Y^X$) of cSet spanned by the universal Kan complexes is locally Kan.
(2) By Lemma 1.5, $\mathrm{U}_\bullet\mathcal{C}$ is a locally Kan cubical category for any locally Kan simplicial category $\mathcal{C}$.
(3) The cubical categories arising via the inclusion $\mathsf{Cat} \hookrightarrow \mathsf{cCat}$ are locally Kan since every discrete cubical set is Kan.

Proposition 1.12. *Let $\mathcal{C}$ be a cubical category and $x, y \in \mathcal{C}$ two objects such that $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a cubical Kan complex. Then two maps $f, g\colon x \to y$ are homotopic if and only if they are elementary homotopic (i.e. $\sim = \sim_1$) and hence elementary homotopy is an equivalence relation on $\mathrm{Map}_{\mathcal{C}}(x,y)_0$.*

Proof. It suffices to show that every homotopy of the form $\Box[1] +_{\Box[0]} \Box[1] \to \mathrm{Map}_{\mathcal{C}}(x,y)$ can be replaced by an elementary homotopy. This follows by considering lifting of different $\sqcap_i^{\varepsilon}[1]^2 \hookrightarrow \Box[1]^2$. □

2. The nerve of a cubical category

The goal of this section is to give a construction of the sSet-valued functor $\mathrm{N}_\Box\colon \mathsf{cCat} \to \mathsf{sSet}$ taking the *coherent nerve* a cubical category, analog of the homotopy coherent nerve of simplicial categories. This functor will arise from a cosimplicial object $\mathfrak{C}\colon \Delta \to \mathsf{cCat}$ in the category of cubical categories. We will prove that if all mapping cubical sets of a cubical category $\mathcal{C}$ are Kan complexes, then the resulting simplicial set is a quasicategory. If in addition all maps of $\mathcal{C}$ are equivalence, then $\mathrm{N}_\Box\mathcal{C}$ is a Kan complex.

For $n \in \mathbb{N}$, we define a cubical category $\mathfrak{C}[n]$ as follows:

- the objects are $0, 1, \ldots, n$;

- given $i, j \in \{0, 1, \dots, n\}$, we define:
$$\mathcal{M}\mathrm{ap}(i,j) := \square[1]^{j-i-1},$$
where we assume $\square[1]^{-1}$ is a singleton and $\square[1]^k = \varnothing$ for $k \leq -2$; (For the exposition reasons, we will slightly abuse notation writing $[1]^{\{i+1,\dots,j-1\}}$ for $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[n]}(i,j)$ throughout the definition of $\mathfrak{C}$, thus omitting the Yoneda embedding and identifying the set $\{i+1, \dots, j-1\}$ with its cardinality.)
- the identity morphism is given by the unique element of $\square[1]^{-1}$;
- the composition operation $\cdot\colon \mathcal{M}\mathrm{ap}(i,j) \times \mathcal{M}\mathrm{ap}(j,k) \to \mathcal{M}\mathrm{ap}(i,k)$ is given by:
$$(x_{i+1}, \dots, x_{j-1}) \cdot (y_{j+1}, \dots, y_{k-1}) = (x_{i+1}, \dots, x_{j-1}, 1, y_{j+1}, \dots, y_{k-1}).$$

One then easily verifies the axioms of an enriched category; for instance, for associativity, we have:
$$((x \cdot y) \cdot z) = ((x, 1, y) \cdot z) = (x, 1, y, 1, z) = (x \cdot (y, 1, z)) = (x \cdot (y \cdot z)).$$

Remark 2.1. The category $\mathfrak{C}[n]$ is obtained by freely adding identity morphisms to a cubical non-unital category with the same objects where $\mathcal{M}\mathrm{ap}(i,i) = \varnothing$.

Given a simplicial operator $\varphi\colon [m] \to [n]$, we define $\varphi_*\colon \mathfrak{C}[m] \to \mathfrak{C}[n]$ as follows:

- on objects $\varphi_*(i) = \varphi(i)$;
- for $i, j \in \{0, 1, \dots, m\}$, we have the induced map
$$\varphi_*\colon [1]^{\{i+1,\dots,j-1\}} \to [1]^{\{\varphi(i)+1,\dots,\varphi(j)-1\}}$$
taking a sequence $(x_{i+1}, \dots, x_{j-1})$ to $(\bar{x}_{\varphi(i)+1}, \dots, \bar{x}_{\varphi(j)-1})$ with $\bar{x}_t := \max\{x_s \mid s \in \varphi^{-1}(t)\}$.

Lemma 2.2.

(1) For a simplicial operator $\varphi\colon [m] \to [n]$, the map $\varphi_*\colon \mathfrak{C}[m] \to \mathfrak{C}[n]$ is a cubical functor.
(2) For a composable pair of simplicial operators φ, ψ, we have $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$ and $(\mathrm{id}_{[m]})_* = \mathrm{id}_{\mathfrak{C}[m]}$ and thus $\mathfrak{C}\colon \Delta \to \mathsf{cCat}$ is a functor (a cosimplicial object in cCat).

Proof. For (1), given composable strings $x = (x_{i+1}, \cdots, x_{j-1})$ and $y = (y_{j+1}, \cdots, y_{k-1})$, we have:
$$\begin{aligned}
\varphi(x) \cdot \varphi(y) &= (\bar{x}_{\varphi(i)+1}, \cdots, \bar{x}_{\varphi(j)-1}) \cdot (\bar{y}_{\varphi(j)+1}, \cdots, \bar{y}_{\varphi(k)-1}) \\
&= (\bar{x}_{\varphi(i)+1}, \cdots, \bar{x}_{\varphi(j)-1}, 1, \bar{y}_{\varphi(j)+1}, \cdots, \bar{y}_{\varphi(k)-1}) \\
&= (\bar{x}_{\varphi(i)+1}, \cdots, \bar{x}_{\varphi(j)-1}, \bar{x}_{\varphi(j)}, \bar{y}_{\varphi(j)+1}, \cdots, \bar{y}_{\varphi(k)-1}) \\
&= \varphi(x \cdot y)
\end{aligned}$$
since $\bar{x}_{\varphi(j)} = \max\{x_s \mid s \in \varphi^{-1}(\varphi(j))\} = x_j = 1$.

In (2), it is clear that $(\varphi \circ \psi)_*$ and $\varphi_* \circ \psi_*$ agree on objects. To see that they also agree on mapping cubical sets, we must show that for each v:
$$\max\left\{ \max\{x_s \mid s \in \psi^{-1}(t)\} \;\middle|\; t \in \varphi^{-1}(v) \right\} = \max\{x_s \mid s \in \psi^{-1}(\varphi^{-1}(v))\}$$
This follows from the fact that the maximum of a finite set can be found by taking a partition of the set, finding the maximum of each element of the partition, and then taking the maximum of those. $\square$

We define the **simplicial nerve** functor $\mathrm{N}_\square\colon \mathsf{cCat} \to \mathsf{sSet}$ by setting:
$$(\mathrm{N}_\square \mathcal{C})_n = \mathsf{cCat}(\mathfrak{C}[n], \mathcal{C}).$$

The category cCat of cubical categories possesses all small colimits (as a category of models for an essentially algebraic theory), and hence we may extend $\mathfrak{C}\colon \Delta \to \mathsf{cCat}$ (by the left Kan extension along the Yoneda embedding) to a functor on sSet:

$$\begin{tikzcd}
\mathsf{sSet} \arrow[rr, bend left=10, "\mathfrak{C}"] \arrow[rr, phantom, "\perp"] & & \mathsf{cCat} \arrow[ll, bend left=10, "\mathrm{N}_\square"] \\
& \Delta \arrow[ul, hook] \arrow[ur, "\mathfrak{C}"'] &
\end{tikzcd}$$

Remark 2.3. Let us point out that in order for $\mathfrak{C}$ to be a cosimplicial object, we need at least face maps, degeneracies, and connections in $\square$. In particular, without connections we would not be able to define one of the degeneracies $s_1 \colon \mathfrak{C}[3] \to \mathfrak{C}[2]$. On the other hand, most of our theorems about $\mathrm{N}_\square$ would remain true for more restrictive choices of morphisms in the category $\square$ (as long as they contain the three classes described above).

Examples 2.4.

(1) If $\mathcal{C}$ is a category, regarded as a cubical category with discrete mapping spaces, then $\mathrm{N}_\square\mathcal{C} \cong \mathrm{N}\mathcal{C}$.
(2) The category cSet is enriched over itself as a presheaf category and one therefore obtains a simplicial set $\mathrm{N}_\square\mathsf{cSet}$. This simplicial set will play an important role in out considerations regarding the Grothendieck construction in Section 3.

Let us try to understand the functor $\mathrm{N}_\square \colon \mathsf{cCat} \to \mathsf{sSet}$ by writing explicitly the 0-, 1-, 2-, and 3-simplices of $\mathrm{N}_\square\mathcal{C}$ for some cubical category $\mathcal{C}$.

Since $\mathfrak{C}[0]$ consists of a single object 0 and a single map id_0, we have that $(\mathrm{N}_\square\mathcal{C})_0 = \operatorname{Ob}\mathcal{C}$.

The category $\mathfrak{C}[1]$ has two objects: 0 and 1, and a single morphism in $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[1]}(0,1)_0$. Thus an element in $(\mathrm{N}_\square\mathcal{C})_1$ consists of two objects $x_0, x_1 \in \mathcal{C}$, together with a map $f_{01} \colon x_0 \to x_1$.

Similarly, an element of $(\mathrm{N}_\square\mathcal{C})_2$ consists of three objects $x_0, x_1, x_2 \in \operatorname{Ob}\mathcal{C}$, three maps $f_{01} \colon x_0 \to x_1$, $f_{12} \colon x_1 \to x_2$, and $f_{02} \colon x_0 \to x_2$, and a homotopy (that is, a 1-cube in $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_2)$) $H_{012} \colon f_{02} \to f_{01} \cdot f_{12}$.

Remark 2.5. The direction of the homotopy H_{012} *towards the composite* is determined by the fact that $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[n]}(0,2) = \{\varnothing \subseteq \{1\}\}$. The map f_{02} is then the value assigned to $\varnothing$ and the composite $f_{01} \cdot f_{12}$ is assigned to $\{1\}$.

A 3-simplex in $\mathrm{N}_\square\mathcal{C}$ consists of the following data:

- four objects $x_0, x_1, x_2, x_3 \in \operatorname{Ob}\mathcal{C}$;
- for each $0 \le i < j \le 3$, a 0-cube $f_{ij} \colon x_i \to x_j$;
- for each triple $0 \le i < j < k \le 3$, a 1-cube $H_{ijk} \colon f_{ik} \to f_{ij} \cdot f_{jk}$;
- a 2-cube:

$$\begin{tikzcd}
f_{03} \arrow[r, "H_{023}"] \arrow[d, "H_{013}"'] \arrow[dr, phantom, "\Theta_{0123}"] & f_{02} \cdot f_{23} \arrow[d, "H_{012} \cdot s_1(f_{23})"] \\
f_{01} \cdot f_{13} \arrow[r, "s_1(f_{01}) \cdot H_{123}"'] & f_{01} \cdot f_{12} \cdot f_{23}
\end{tikzcd}$$

where s_1 is the degeneracy operation.

Intuitively, the n-simplices of $\mathrm{N}_\square\mathcal{C}$ encode the coherence in composing a string of n arrows in $\mathcal{C}$. To see this, let

$$x_0 \to x_1 \to \ldots \to x_n$$

be a composable string in $\mathcal{C}$. The top non-degenerate cell in $\mathrm{N}_\square\mathcal{C}$ is a $(n-1)$-cube, whose $(2n-2)$ faces are accounted for as follows:

- there are $(n+1)$ faces coming from omitting one of the objects $i = 0, 1, \ldots, n$ and considering the possible ways of composing all non-adjacent morphisms;
- there are $(n-3)$ faces obtained by choosing $i \in \{2, 3, \ldots, n-2\}$ and considering separately the strings $x_0 \to \ldots \to x_i$ and $x_i \to \ldots \to x_n$. Thus they are composites of degenerate cells.

In particular, cells of dimension 4 have faces that are composites of degenerate cells of lower dimensions; and 4 is the lowest dimension in which this occurs. In the notation above, one of the faces in a 4-cells has the form:

$$\begin{tikzcd} f_{02}\cdot f_{24} \arrow[r, "H_{012}\cdot s_1(f_{24})"] \arrow[d, "s_1(f_{02})\cdot H_{234}"'] & f_{01}\cdot f_{12}\cdot f_{24} \arrow[d, "s_1(f_{01}\cdot f_{12})\cdot H_{234}"] \\ f_{01}\cdot f_{12}\cdot f_{24} \arrow[r, "H_{012}\cdot s_1(f_{23}\cdot f_{34})"'] & f_{01}\cdot f_{12}\cdot f_{23}\cdot f_{34} \end{tikzcd}$$

We next turn our attention to the question: when is $\mathrm{N}_\square\mathcal{C}$ a quasicategory? Recall that a **quasi-category** is a simplicial set X satisfying the inner horn filling condition; that is for every $n \in \mathbb{N}$, $0 < i < n$, and every map $\Lambda^i[n] \to X$, there exists a filler

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

We will show that if all mapping cubical sets of $\mathcal{C}$ satisfy the Kan condition, then the simplicial nerve of $\mathcal{C}$ is a quasicategory. In our proof, we will only use half of the Kan conditions, namely existence of fillers for $(0,i)$-open boxes, but one can show that if a cubical set X has fillers for $(0,i)$-open boxes, then it must also have fillers for $(1,i)$-open boxes.

Theorem 2.6. *Let $\mathcal{C}$ be a locally Kan cubical category. Then $\mathrm{N}_\square\mathcal{C}$ is a quasicategory.*

Before giving the proof in full generality, we check the cases $n = 2$ and $n = 3$.

When $n = 2$, we need to solve the following lifting problem:

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d, hook] & \mathrm{N}_\square\mathcal{C} \\ \Delta[2] \arrow[ur, dashed] & \end{tikzcd}$$

By $\mathfrak{C} \dashv \mathrm{N}_\square$, this is equivalent to the lifting problem:

$$\begin{tikzcd}
\mathfrak{C}\Lambda^1[2] \arrow[r] \arrow[d] & \mathcal{C} \\
\mathfrak{C}[2] \arrow[ur, dashed] &
\end{tikzcd}$$

in cCat.

A map $\mathfrak{C}\Lambda^1[2] \to \mathcal{C}$ corresponds to a choice of three objects $x_0, x_1, x_2 \in \operatorname{Ob}\mathcal{C}$ along with two maps $f_{01} \colon x_0 \to x_1$ and $f_{12} \colon x_1 \to x_2$. We seek an extension $\mathfrak{C}[2] \to \mathcal{C}$, that is, a map $f_{02} \colon x_0 \to x_2$ together with a homotopy $H_{012} \colon f_{02} \to f_{01} \cdot f_{12}$. This can be expressed as a lifting problem in the category cSet as follows:

$$\begin{tikzcd}
\sqcap^0_1[1] \arrow[r] \arrow[d, hook] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_2) \\
\square[1] \arrow[ur, dashed] &
\end{tikzcd}$$

This problem, however, has a solution since $\mathcal{C}$ was assumed to be locally Kan.

In fact, we did not have to use any Kan condition to produce the required lift. Indeed, we could have simply taken $f_{02} := f_{01} \cdot f_{12}$ and $H_{012} := s_1(f_{01} \cdot f_{12})$. This is because $\mathcal{C}$, as a cubical category, was equipped with composition.

Next, we shall discuss the case $n = 3$ and $i = 1$. The case $i = 2$ is completely analogous and we will comment on it later. As before, the lifting problem:

$$\begin{tikzcd}
\Lambda^1[3] \arrow[r] \arrow[d, hook] & \mathrm{N}_{\square}\mathcal{C} \\
\Delta[3] \arrow[ur, dashed] &
\end{tikzcd}$$

is, by adjointness, equivalent to:

$$\begin{tikzcd}
\mathfrak{C}\Lambda^1[3] \arrow[r] \arrow[d] & \mathcal{C} \\
\mathfrak{C}[3] \arrow[ur, dashed] &
\end{tikzcd}$$

Thus, we are given:

- four objects $x_0, x_1, x_2, x_3 \in \operatorname{Ob}\mathcal{C}$;
- for each $0 \leq i < j \leq 3$, a 0-cube $f_{ij} \colon x_i \to x_j$;
- three 2-simplices $H_{012} \colon f_{02} \to f_{01} \cdot f_{12}$, $H_{013} \colon f_{03} \to f_{01} \cdot f_{13}$, and $H_{123} \colon f_{13} \to f_{12} \cdot f_{23}$;

and we are seeking $H_{023} \colon f_{03} \to f_{02} \cdot f_{23}$ together with $\Theta_{0123} \in \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_3)_2$, i.e. given the solid arrows in the following open box:

$$\begin{tikzcd}
f_{03} \arrow[r, "H_{023}"] \arrow[d, dashed, "H_{013}"'] & f_{02} \cdot f_{23} \arrow[d, "H_{012} \cdot s_1(f_{23})"] \\
f_{01} \cdot f_{13} \arrow[r, "s_1(f_{01}) \cdot H_{123}"'] & f_{01} \cdot f_{12} \cdot f_{23}
\end{tikzcd}$$

we need an extension to a 2-cube. In other words, we need to solve the following lifting problem in cSet:

$$\begin{tikzcd}
\sqcap^0_1[1]^2 \arrow[r] \arrow[d, hook] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_3) \\
\square[1]^2 \arrow[ur, dashed] &
\end{tikzcd}$$

which has a solution since $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_3)$ is a cubical Kan complex.

The above procedure with the inclusion $\Lambda^2[3] \hookrightarrow \Delta[3]$ yields:

$$\begin{tikzcd}
f_{03} \arrow[r, dashed, "H_{023}"] \arrow[d, "H_{013}"'] & f_{02} \cdot f_{23} \arrow[d, "H_{012} \cdot s_1(f_{23})"] \\
f_{01} \cdot f_{13} \arrow[r, "s_1(f_{01}) \cdot H_{123}"'] & f_{01} \cdot f_{12} \cdot f_{23}
\end{tikzcd}$$

that is, we need a lift in:

$$\begin{tikzcd}
\sqcap^0_2[1]^2 \arrow[r] \arrow[d, hook] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_3) \\
\square[1]^2 \arrow[ur, dashed] &
\end{tikzcd}$$

which exists since $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_3)$ is Kan.

We can now give the proof in the general case.

Proof of Theorem 2.6. By adjointness $\mathfrak{C} \dashv \mathrm{N}_{\square}$, we need to solve a family of lifting problems:

$$\begin{tikzcd}
\mathfrak{C}\Lambda^i[n] \arrow[r] \arrow[d] & \mathcal{C} \\
\mathfrak{C}[n] \arrow[ur, dashed] &
\end{tikzcd}$$

where $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n-1$. Writing x_0 and x_n for the value of the horizontal map on 0 and n, respectively, this lifting problem can be in turn reduced to:

$$\begin{tikzcd} \sqcap^0_i[1]^{n-1} \arrow[r] \arrow[d, hook] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x_0, x_n) \\ \square[1]^{n-1} \arrow[ur, dashed] & \end{tikzcd}$$

But since $\mathcal{C}$ is locally Kan, all of these problems admit the required lifts. $\square$

Example 2.7. The cubical category Kan of universal Kan complexes is locally Kan (Examples 1.11), and thus $\mathrm{N}_{\square}\mathsf{Kan}$ is a quasicategory.

Theorem 2.8. *If $\mathcal{C}$ is a locally Kan cubical category in which every morphism is an equivalence, then the simplicial set $\mathrm{N}_{\square}\mathcal{C}$ is a Kan complex.*

The proof of Theorem 2.8 will be preceded by a short discussion, in which we recall Joyal's theorem on existence of lifts for special horns. We begin with preliminary definitions:

Definition 2.9 (Joyal)**.**

(1) The simplicial set K as the pushout:

$$\begin{tikzcd} \Delta[1] + \Delta[1] \arrow[r, "{[02,13]}"] \arrow[d] & \Delta[3] \arrow[d] \\ \Delta[0] \arrow[r] & K \end{tikzcd}$$

(2) Let $\mathcal{C}$ be a quasicategory. A 1-simplex $f\colon \Delta[1] \to \mathcal{C}$ is an **equivalence** if f factors through the inclusion $[12]\colon \Delta[1] \hookrightarrow K$.
(3) Let $\mathcal{C}$ be a quasicategory. A horn $u\colon \Lambda^0[n] \to \mathcal{C}$ (respectively, $v\colon \Lambda^n[n] \to \mathcal{C}$) is **special** if $u|\Delta^{\{0,1\}}$ (respectively, $v|\Delta^{\{n-1,n\}}$) is an equivalence.

Theorem 2.10 (Joyal, [Joy02, Thm. 2.2])**.** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories and consider the following diagram where $i = 0$ or $i = n$:*

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "h"] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta[n] \arrow[r] & \mathcal{D} \end{tikzcd}$$

If h is a special horn, then the lifting problem above admits a solution. In particular, for $\mathcal{D} = \Delta[0]$, quasicategories have fillers for all special horns.

Proof of Theorem 2.8. In a locally Kan cubical category, a 0-cube is an equivalence if and only if the corresponding 1-simplex (the image of $\mathfrak{C}[1] \to \mathcal{C}$) is an equivalence in the quasicategory $\mathrm{N}_{\square}\mathcal{C}$. Thus if every map in $\mathcal{C}$ is an equivalence, every horn is special and hence, by Theorem 2.10, $\mathrm{N}_{\square}\mathcal{C}$ admits fillers for all horns. $\square$

We conclude this section by relating $\mathrm{N}_{\square}$ to the homotopy coherent nerve functor. Let us begin by recalling the construction of the homotopy coherent nerve $\mathrm{N}_{\triangle}\colon \mathsf{sCat} \to \mathsf{sSet}$. It arises from a cosimplicial object in the category sCat of simplicial categories.

Namely, one defines $\mathfrak{C}_{\triangle}\colon \Delta \to \mathsf{sCat}$ by putting $\mathfrak{C}_{\triangle}[n]$ to be the simplicial category with:

- objects: $0, 1, \ldots, n$;

- mapping spaces given by:
$$\mathcal{M}\mathrm{ap}_{\mathfrak{C}_\triangle[n]}(i,j) = \mathrm{N}([1]^{\{i+1,\ldots,j-1\}})$$
- the composition is given by union of subsets.

One then defines $(\mathrm{N}_\triangle \mathcal{C})_n := \mathsf{sCat}(\mathfrak{C}_\triangle[n], \mathcal{C})$ and obtains a pair of adjoint functors:

$$\begin{tikzcd}
\mathsf{sSet} \arrow[rr, bend left, "\mathfrak{C}_\triangle"] \arrow[rr, phantom, "\perp"] & & \mathsf{sCat} \arrow[ll, bend left, "\mathrm{N}_\triangle"] \\
& \Delta \arrow[ul, hook] \arrow[ur, "\mathfrak{C}_\triangle"'] &
\end{tikzcd}$$

where $\mathfrak{C}_\triangle \colon \mathsf{sSet} \to \mathsf{sCat}$ is given, as always, by the left Kan extension of $\mathfrak{C}_\triangle \colon \Delta \to \mathsf{sCat}$ along the Yoneda embedding.

Theorem 2.11. *There is a natural isomorphism of functors* $\mathrm{N}_\triangle \cong \mathrm{N}_\square \circ \mathrm{U}_\bullet$.

Proof. Since the mapping cubical sets $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[n]}(i,j)$ are representable, $\mathrm{T}_\bullet$ acts as the nerve functor N on them and hence $\mathrm{T}_\bullet \circ \mathfrak{C} \cong \mathfrak{C}_\triangle$. Thus, by $\mathrm{T}_\bullet \dashv \mathrm{U}_\bullet$, we have the following sequence of natural isomorphisms:
$$\mathrm{N}_\triangle = \mathsf{sCat}(\mathfrak{C}_\triangle, -) \cong \mathsf{sCat}(\mathrm{T}_\bullet \circ \mathfrak{C}, -) \cong \mathsf{sCat}(\mathfrak{C}, \mathrm{U}_\bullet(-)) = \mathrm{N}_\square \circ \mathrm{U}_\bullet.$$
$\square$

Putting together Theorems 2.6 and 2.11 and Examples 1.11, we obtain the following corollary.

Corollary 2.12. *If $\mathcal{C}$ is a locally Kan simplicial category, then $\mathrm{N}_\triangle\mathcal{C}$ is a quasicategory.*

3. The Grothendieck construction

In this section, we consider the construction taking a simplicial map $F \colon S \to \mathrm{N}_\square(\mathsf{cSet})$ to an object in the slice category $\int_S F \in \mathsf{sSet} \downarrow S$. This is analogous to the Grothendieck construction of the category of elements. We then show that this assignment is functorial and admits a left adjoint. Finally, we will relate this construction to Lurie's (straightening $\dashv$ unstraightening)-adjunction (cf. [Lur09, Sec. 2.2.1]).

We begin by defining, given $F \colon \Delta[n] \to \mathrm{N}_\square(\mathsf{cSet})$, the set
$$\mathrm{Sect}F = \left\{ \Delta[n+1] \xrightarrow{G} \mathrm{N}_\square(\mathsf{cSet}) \;\middle|\; G|\Delta^{\{0\}} = \square[0] \text{ and } G|\Delta^{\{1,\ldots,n+1\}} = F \right\}.$$

Here, we write $F|\Delta^{\{i,\ldots,j\}}$ for the restriction of F to the simplicial subset of $\Delta[n]$ spanned by the vertices $i,\ldots,j$. A map $F \colon \Delta[n] \to \mathrm{N}_\square(\mathsf{cSet})$ should be thought of as a *homotopy coherent family* of cubical sets, indexed by $\Delta[n]$, and the set $\mathrm{Sect}F$ as the set of its *homotopy cohrent sections*. Let us illustrate these intuitions with examples of such families and their sections for small values of n.

Examples 3.1.

(1) If $n = 0$, a simplex $F \colon \Delta[0] \to \mathrm{N}_\square\mathsf{cSet}$ corresponds to a choice of a cubical set X, and the set $\mathrm{Sect}F$ is simply the set X_0 of 0-cubes of X.
(2) For $n = 1$, a map $F \colon \Delta[1] \to \mathrm{N}_\square\mathsf{cSet}$ gives a pair of cubical sets X_0 and X_1, along with a map $f_{01} \colon X_0 \to X_1$. The set $\mathrm{Sect}F$ consists then of triples:
$$(x_0 \in X_0, x_1 \in X_1, p_{01} \in (X_1)_1),$$
where $p_{01} \colon x_1 \to f_{01}(x_0) \in X_1$.

(3) For $n = 2$, a map $F\colon \Delta[2] \to \mathrm{N}_\square\mathsf{cSet}$ corresponds to a choice of three cubical sets X_0, X_1, and X_2, together with maps $f_{01}\colon X_0 \to X_1$, $f_{12}\colon X_1 \to X_2$, and $f_{02}\colon X_0 \to X_2$, and a homotopy $\alpha_{012}\colon f_{02} \to f_{12} \cdot f_{01}$. An element in $\mathrm{Sect}F$ is a septuple:

$$(x_0 \in X_0, x_1 \in X_1, x_2 \in X_2, p_{01} \in (X_1)_1, p_{12} \in (X_2)_1, p_{02} \in (X_2)_1, H_{012} \in (X_2)_2)$$

where $p_{01}\colon x_1 \to f_{01}(x_0)$, $p_{12}\colon x_2 \to f_{12}(x_1)$, $p_{02}\colon x_2 \to f_{02}(x_0)$, and H_{012} is a 2-cube:

$$\begin{tikzcd}
x_2 \arrow[r, "p_{02}"] \arrow[d, "p_{12}"'] & f_{02}(x_0) \arrow[d, "\alpha_{012}(x_0)"] \\
f_{12}(x_1) \arrow[r, "f_{12}(p_{01})"] & f_{12}f_{01}(x_0)
\end{tikzcd}$$

Definition 3.2. Let S be a simplicial set and $F\colon S \to \mathrm{N}_\square\mathsf{cSet}$ a simplicial map. Define the **Grothendieck construction** of F to be the simplicial set $\int_S F$ whose n-simplices are given by:

$$\left(\int_S F\right)_n = \big\{(s\colon \Delta[n] \to S, G \in \mathrm{Sect}(Fs))\big\}.$$

The simplicial set $\int_S F$ is equipped with a canonical projection $P_F\colon \int_S F \to S$, given by $P_F(s, G) = s$. Let us now establish the universal case of this construction. We will write $(\mathrm{N}_\square\mathsf{cSet})_*$ for the simplicial set $\int\limits_{\mathrm{N}_\square\mathsf{cSet}} \mathrm{id}$ and P for the associated projection $(\mathrm{N}_\square\mathsf{cSet})_* \to \mathrm{N}_\square\mathsf{cSet}$. Given a simplicial set S and a map $F\colon S \to \mathrm{N}_\square\mathsf{cSet}$, define $Q_F\colon \int_S F \to (\mathrm{N}_\square\mathsf{cSet})_*$ by $Q_F(s, G) = (Fs, G)$.

Proposition 3.3. *For any simplicial map $F\colon S \to \mathrm{N}_\square\mathsf{cSet}$, the square:*

$$\begin{tikzcd}
\int_S F \arrow[r, "Q_F"] \arrow[d, "P_F"'] & (\mathrm{N}_\square\mathsf{cSet})_* \arrow[d, "P"] \\
S \arrow[r, "F"] & \mathrm{N}_\square\mathsf{cSet}
\end{tikzcd}$$

is a pullback.

Proof. The square is easily seen to commute. Consider a simplicial set K with maps $f_1\colon K \to S$ and $f_2\colon K \to (\mathrm{N}_\square\mathsf{cSet})_*$ with $F \circ f_1 = P \circ f_2$. Define $\overline{f}\colon K \to \int_S F$ by putting for $x\colon \Delta[n] \to K$, $\overline{f}(x) = (f_1 \circ k, \mathrm{pr}_2(f_2 \circ x))$. ☐

In the situation of Proposition 3.3, given a map $f\colon S' \to S$, by the universal property of the pullback, we obtain a map $Q_{F,f}\colon \int_{S'} F \circ f \to \int_S F$. Explicitly, this map is given by:

$$Q_{F,f}(s', G) = (fs', G).$$

Combining the previous proposition with the two pullback lemma, we obtain:

Corollary 3.4. *Given any map $f\colon S' \to S$ of simplicial set, the following square:*

$$\begin{tikzcd}
\int_{S'} F \arrow[r, "Q_{F,f}"] \arrow[d, "P_{F\circ f}"'] & \int_S F \arrow[d, "P_F"] \\
S' \arrow[r, "f"] & S
\end{tikzcd}$$

is a pullback. ☐

So far, we have defined an assignment taking a simplicial map $F\colon S \to (\mathrm{N}_\Box \mathsf{cSet})$, or equivalently a cubical functor $F\colon \mathfrak{C}[S] \to \mathsf{cSet}$, to a map $\int_S F \to S$. We now wish to extend it to a functor $\int_S\colon \mathsf{cSet}^{\mathfrak{C}[S]} \to \mathsf{sSet}\downarrow S$, where $\mathsf{cSet}^{\mathfrak{C}[S]}$ is the category of cubical functors $\mathfrak{C}[S]$. To this end let $F, F'\colon \mathfrak{C}[S] \to \mathsf{cSet}$ and let $\varphi\colon S \to \mathsf{cSet}$ be a morphism from F to F' in $(\mathrm{N}_\Box \mathsf{cSet})^S$.

Given an n-simplex $(s\colon \Delta[n] \to S, G\colon \Delta[1+n] \to \mathrm{N}_\Box \mathsf{cSet})$ in $\int_S F$, define $\int_S \varphi(s, G) := (s, G')$, where $G'\colon \mathfrak{C}[1+n] \to \mathsf{cSet}$ is defined:

- on objects $G'_0 = \Box[0]$ and $G'_{1+i} = F'_i$ for $i = 0, 1, \ldots, n$;
- on mapping cubical sets $G'_{i,j}\colon \mathcal{M}\mathrm{ap}_{\mathfrak{C}[1+n]}(i,j) \to {F'_j}^{F'_i}$ is given by: $G'_{1+i,1+j} = F'_{i,j}$ for $i, j = 0, \ldots n$ and $G'_{0,j} = \varphi_j \circ G_{0,j}$.

It follows, by naturality of φ, that $G'\colon \mathfrak{C}[1+n] \to \mathsf{cSet}$ is a cubical functor and moreover, by construction of G', we obtain the following:

Proposition 3.5. *With the definition above $\int_S$ defines a functor $\mathsf{cSet}^{\mathfrak{C}[S]} \to \mathsf{sSet}\downarrow S$.* $\Box$

One can also look at $\int_S$ as a functor defined on a slightly different (but isomorphic, not only equivalent!) category which we describe below.

The cosimplicial object $\mathfrak{C}\colon \Delta \to \mathsf{cCat}$ defines the simplicial enrichment on the category cCat of cubical categories. Indeed, the simplicial set $\mathcal{M}\mathrm{ap}^\Delta(\mathcal{C}, \mathcal{D})$ can be defined by:

$$\mathcal{M}\mathrm{ap}^\Delta(\mathcal{C}, \mathcal{D})_n = \mathsf{cCat}(\mathcal{C} \times \mathfrak{C}[n], \mathcal{D}).$$

In particular, with this definition $\mathrm{N}_\Box \mathcal{C} \cong \mathcal{M}\mathrm{ap}^\Delta([0], \mathcal{C})$ for any cubical category $\mathcal{C}$. Moreover, we may define the *morphism part of* $\mathrm{N}_\Box$ by putting $\overrightarrow{\mathrm{N}}_\triangle \mathcal{C} = \mathcal{M}\mathrm{ap}^\Delta([1], \mathcal{C})$, which yields the following description of its n-simplices:

$$(\overrightarrow{\mathrm{N}}_\triangle \mathcal{C})_n = \mathsf{cCat}([1] \times \mathfrak{C}[n], \mathcal{C}),$$

The diagram $\delta_0, \delta_1\colon [0] \rightrightarrows [1]$ defines a cocategory object in Cat and hence for any cubical category $\mathcal{C}$, we obtain a category object in sSet:

$$\overrightarrow{\mathrm{N}}_\triangle \mathcal{C} \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} \mathrm{N}_\Box \mathcal{C}.$$

Given a cubical category $\mathcal{C}$ and a simplicial set S, we define a category $(\mathrm{N}_\Box \mathcal{C})^S$ as follows:

- the objects are simplicial maps $S \to \mathrm{N}_\Box \mathcal{C}$;
- the morphisms are simplicial maps $S \to \overrightarrow{\mathrm{N}}_\triangle \mathcal{C}$;
- the domain and codomain operations are given by postcomposition with δ_0^* and δ_1^*.

Let $\mathcal{C}$ and S be as above and let $\mathcal{C}^{\mathfrak{C}[S]}$ denote the category whose objects are given by cubical functors $\mathfrak{C}[S] \to \mathcal{C}$ and whose maps are cubical natural transformation.

Cubical natural transformations between functors $\mathfrak{C}[S] \to \mathcal{C}$ correspond naturally to cubical functors $\mathfrak{C}[S] \times [1] \to \mathcal{C}$. Thus, given such a natural transformation and an n-simplex of S, we obtain a map $\mathfrak{C}[n] \times [1] \to \mathcal{C}$, hence an n-simplex in $\overrightarrow{\mathrm{N}}_\triangle \mathcal{C}$. This defines a functor $\mathcal{C}^{\mathfrak{C}[S]} \to (\mathrm{N}_\Box \mathcal{C})^S$, which is easily seen to be an equivalence (and, in fact, an isomorphism) of categories.

The classical version of the Grothendieck construction (i.e. for functors $\mathcal{C} \to \mathsf{Set}$) admits a left adjoint. The same is true in our setting and we next show that the functor $\int_S\colon \mathsf{cSet}^{\mathfrak{C}[S]} \to \mathsf{sSet}\downarrow S$ constructed above also admits a left adjoint $L_S\colon \mathsf{sSet}\downarrow S \to \mathsf{cSet}^{\mathfrak{C}[S]}$.

Recall that the **join** is a functor $\star\colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$ together with two natural transformations $X \to X \star Y \leftarrow Y$ such that $\Delta[m] \star \Delta[n] \cong \Delta[m+n+1]$, naturally in both m and n, and for all

$X, Y \in \mathsf{sSet}$ the functors $X \star -\colon \mathsf{sSet} \to X \downarrow \mathsf{sSet}$ and $- \star Y\colon \mathsf{sSet} \to Y \downarrow \mathsf{sSet}$ preserve colimits. Explicitly, $X \star Y$ is given by:

$$(X \star Y)_n = \coprod_{i+j=n-1} X_i \times Y_j,$$

where $X_{-1} = Y_{-1} = \{*\}$. We will write $X^{\triangleleft}$ for the join $\{*\} \star X$.

To construct L_S, take $p\colon X \to S$ and consider the pushout:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "p"'] & X^{\triangleleft} \arrow[d] \\ S \arrow[r] & X^{\triangleleft} +_X S \end{tikzcd}$$

We define $L_S p\colon \mathfrak{C}[S] \to \mathsf{cSet}$ as $L_S p := \mathcal{M}\mathrm{ap}_{\mathfrak{C}[X^{\triangleleft}+_X S]}(*, -)$.

Theorem 3.6. *For any simplicial set S, the functors $L_S\colon \mathsf{sSet}\downarrow S \rightleftarrows \mathsf{cSet}^{\mathfrak{C}[S]} : \int_S$ form an adjoint pair.*

Proof. The functor L_S preserves colimits, thus it suffices to construct a natural bijection between maps $s \to P_F$ in $\mathsf{sSet} \downarrow S$ and $L_S s \to F$ in $\mathsf{cSet}^{\mathfrak{C}[n]}$, where $s\colon \Delta[n] \to S$ is a simplicial map and $F\colon \mathfrak{C}[S] \to \mathsf{cSet}$ is a cubical functor.

A map $s \to \int_S F$ corresponds naturally to an n-simplex in $\int_S F$ whose first component is s and the second component is $G\colon \Delta[1+n] \to \mathrm{N}_{\square}(\mathsf{cSet})$ such that $G|\Delta^{\{0\}} = \square[0]$ and $G|\Delta^{\{1,\ldots,1+n\}} = Fs$.

Such a map determines therefore an extension $\overline{F}\colon \Delta[1+n] +_{\Delta[n]} S \to \mathrm{N}_{\square}(\mathsf{cSet})$ of F:

$$\begin{tikzcd} \Delta[n] \arrow[r, hook] \arrow[d] & \Delta[1+n] \arrow[d] \arrow[ddr, bend left, "G"] & \\ S \arrow[r] \arrow[drr, bend right, "F"'] & \Delta[1+n] +_{\Delta[n]} S \arrow[dr, dashed, "\overline{F}"] & \\ & & \mathrm{N}_{\square}\mathsf{cSet} \end{tikzcd}$$

which, by adjointness, gives $\overline{F}\colon \mathfrak{C}[1+n] +_{\mathfrak{C}[n]} \mathfrak{C}[S] \to \mathsf{cSet}$. But since $\overline{F}(0) = G(0) = \square[0]$, by the Enriched Yoneda Lemma, this gives a unique map $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[1+n]+_{\mathfrak{C}[n]}\mathfrak{C}[S]}(0, -) \to \overline{F}$ in $\mathsf{cSet}^{\mathfrak{C}[1+n]+_{\mathfrak{C}[n]}\mathfrak{C}[S]}$, whose restriction to $\mathfrak{C}[S]$ gives the required natural transformation.

Conversely, given a cubical natural transformation $\varphi\colon \mathcal{M}\mathrm{ap}_{\mathfrak{C}[1+n]+_{\mathfrak{C}[n]}\mathfrak{C}[S]}(0, -) \to F$ we extend F to $\overline{F}\colon \mathfrak{C}[1+n] +_{\mathfrak{C}[n]} \mathfrak{C}[S] \to \mathsf{cSet}$ by putting $\overline{F}(0) = \square[0]$ and defining $\overline{F}_{0,i}\colon \mathcal{M}\mathrm{ap}_{\mathfrak{C}[1+n]}(0, 1) \to F(i)$ by $\overline{F}_{0,i} = \varphi_i$. Such an $\overline{F}$ determines an n-simplex in $\int_S F$ whose first component is s.

Both of these maps are natural since all the steps involved in the construction were natural. It is moreover immediate to see that these maps mutual inverses, thus yielding the required bijection. $\square$

We next consider a *relative* version of this construction. Fix a simplicial set S, cubical category $\mathcal{C}$, and a cubical functor $\phi\colon \mathfrak{C}[S] \to \mathcal{C}$. Associated with ϕ, there is an adjoint pair:

$$\begin{tikzcd} \mathsf{cSet}^{\mathfrak{C}[S]} \arrow[r, bend left, "\phi_!"] \arrow[r, phantom, "\perp"] & \mathsf{cSet}^{\mathcal{C}}. \arrow[l, bend left, "\phi^*"] \end{tikzcd}$$

where ϕ^* is given by precomposition with ϕ and $\phi_!$ is the left Kan extensions along ϕ.

Thus, we define $\int_\phi = \int_S \phi^*$ and $L_\phi = \phi_! L_S$. The following proposition is immediate by construction.

Proposition 3.7. *For any cubical functor $\phi\colon \mathfrak{C}[S] \to \mathcal{C}$, the functors*

$$L_\phi \colon \mathsf{sSet}{\downarrow}S \rightleftarrows \mathsf{cSet}^{\mathcal{C}} : \int_\phi$$

form an adjoint pair. □

Unwinding the definitions, we see that the n-simplices of $\int_\phi F$ (where $F\colon \mathcal{C} \to \mathsf{cSet}$ is a cubical functor) are give by:

$$\left(\int_\phi F\right)_n = \left\{(s\colon \Delta[n] \to S, G \in \mathrm{Sect}(\overline{F\phi}s)\right\}.$$

Here, $\overline{F\phi}\colon S \to \mathrm{N}_\square(\mathsf{cSet})$ denotes the adjoint transpose of $F\phi\colon \mathfrak{C}[S] \to \mathsf{cSet}$.

Conversely, $L_\phi p$ arises as a restriction of the functor $\mathcal{M}\mathrm{ap}_{\mathfrak{C}[X^\triangleleft]+_{\mathfrak{C}[X]}\mathcal{C}}(*, -)$ on the pushout:

$$\begin{tikzcd} \mathfrak{C}[X] \arrow[r] \arrow[d, "\mathfrak{C}[p]"'] & \mathfrak{C}[X^\triangleleft] \arrow[dd] \\ \mathfrak{C}[S] \arrow[d, "\phi"'] & \\ \mathcal{C} \arrow[r] & \mathfrak{C}[X^\triangleleft] +_{\mathfrak{C}[X]} \mathcal{C} \end{tikzcd}$$

The analogous version of the Grothendieck construction, but for maps $S \to \mathrm{N}_\triangle\mathsf{sSet}$ is described in [Lur09, Sec. 2.2.1] and referred to as the unstraightening functor. We now show how to relate this construction to ours.

To begin, let us recall the construction of Lurie's (straightening ⊣ unstraightening)-adjunction. For the remainder of the section, fix a simplicial set S, a simplicial category $\mathcal{C}$, and a simplicial functor $\phi\colon \mathfrak{C}_\triangle[S] \to \mathcal{C}$ (or equivalently, a simplicial map $S \to \mathrm{N}_\triangle\mathcal{C}$).

Given a map $p\colon X \to S$ of simplicial sets, one first forms the following pushout in sCat:

$$\begin{tikzcd} \mathfrak{C}_\triangle[X] \arrow[r] \arrow[d, "\mathfrak{C}_\triangle[p]"'] & \mathfrak{C}_\triangle[X^\triangleleft] \arrow[dd] \\ \mathfrak{C}_\triangle[S] \arrow[d, "\phi"'] & \\ \mathcal{C} \arrow[r] & \mathfrak{C}_\triangle[X^\triangleleft] +_{\mathfrak{C}_\triangle[X]} \mathcal{C} \end{tikzcd}$$

and then defines $\mathrm{St}_\phi(p) := \mathcal{M}\mathrm{ap}_{\mathfrak{C}_\triangle[X^\triangleleft]+_{\mathfrak{C}_\triangle[X]}\mathcal{C}}(*, -)$. This gives the **straightening** functor $\mathrm{St}_\varphi\colon \mathsf{sSet}{\downarrow} S \to \mathsf{sSet}^{\mathcal{C}}$ (where $\mathsf{sSet}^{\mathcal{C}}$ denotes the category of simplicial functors $\mathcal{C} \to \mathsf{sSet}$ and simplicial natural transformations). It can be shown, either by the Adjoint Functor Theorem or an explicit construction, that St_ϕ admits a right adjoint, called the **unstraightening**, $\mathrm{Un}_\phi\colon \mathsf{sCat}^{\mathcal{C}} \to \mathsf{sSet}{\downarrow}S$.

The category sSet is a simplicial category (as a presheaf category) and therefore $\mathrm{U}_\bullet\mathsf{sSet}$ is a cubical category whose objects are simplicial sets and the mapping cubical set $\mathcal{M}\mathrm{ap}_{\mathrm{U}_\bullet\mathsf{sSet}}(X,Y)$ is given by $\mathrm{U}(Y^X)$, where Y^X is the internal exponential object. Moreover, the adjunction $\mathrm{T} \dashv \mathrm{U}$ yields a pair of cSet-enriched functors $\widetilde{\mathrm{T}}\colon \mathsf{cSet} \rightleftarrows \mathrm{U}_\bullet\mathsf{sSet} : \widetilde{\mathrm{U}}$, which are adjoint since U preserves exponentials.

Returning to the question of expressing the $\mathrm{St}_\phi \dashv \mathrm{Un}_\phi$, we have the following:

Theorem 3.8. *Let $\phi\colon \mathfrak{C}_\triangle[S] \to \mathcal{C}$ be a simplicial functor and let $\overline{\phi}\colon \mathfrak{C}[S] \to \mathrm{U}_\bullet\mathcal{C}$ be its adjoint transpose (under the $\mathrm{T}_\bullet \dashv \mathrm{U}_\bullet$ adjunction). Then St_ϕ and Un_ϕ arise as respectively the upper and the lower composites in the diagram:*

$$\begin{tikzcd} \mathsf{sSet}{\downarrow}S \arrow[r, bend left, "L_{\overline{\phi}}"] \arrow[r, phantom, "\perp"] & \mathsf{cSet}^{\mathrm{U}_\bullet\mathcal{C}} \arrow[l, bend left, "\int_{\overline{\phi}}"] \arrow[r, bend left, "\widetilde{\mathrm{T}}\circ -"] \arrow[r, phantom, "\perp"] & (\mathrm{U}_\bullet\mathsf{sSet})^{\mathrm{U}_\bullet\mathcal{C}} \arrow[l, bend left, "\widetilde{\mathrm{U}}\circ -"] \end{tikzcd}$$

via $(U_\bullet\mathsf{sSet})^{\mathrm{U}_\bullet\mathcal{C}} \simeq \mathsf{sSet}^{\mathcal{C}}$.

Proof. Since U is full and faithful, we obtain the equivalence $(\mathrm{U}_\bullet\mathsf{sSet})^{\mathrm{U}_\bullet\mathcal{C}} \simeq \mathsf{sSet}^{\mathcal{C}}$. The upper composite being St_ϕ is a consequence of the fact that mapping spaces in $\mathfrak{C}_\triangle[S]$ are obtained by applying T to those of $\mathfrak{C}[S]$. The result then follows by uniqueness of adjoints. ☐

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