FLAGS AND GROTHENDIECK CARTOGRAPHIC GROUP[1]

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1. Cartographic group.

1.0. *The Grothendieck cartographic group* is defined by the presentation

$$\mathcal{C}_n := \langle \sigma_0, \ldots, \sigma_n \mid \sigma_i^2 = 1, (\sigma_i \sigma_j)^2 = 1 \text{ if } |i-j| \geq 2 \rangle$$

The *simplicial cartographic group* $\mathcal{C}_{n,3}$ is obtained from $\mathcal{C}_n$ by imposing the additional relations $(\sigma_i \sigma_{i+1})^3 = 1$ for $0 \leq i \leq n-2$. Similarly the *cubic cartographic group* $\mathcal{C}_{n,4}$ can be defined by imposing the relations $(\sigma_0 \sigma_i)^4 = 1$, $(\sigma_i \sigma_{i+1})^3 = 1$ for $0 \leq i \leq n-2$.

We call the *oriented cartographic group* the subgroup $\mathcal{C}_n^+$ the subgroup of index 2 in $\mathcal{C}_n$, generated by the pairwise products of the generators σ_i. In the same way the subgroups $\mathcal{C}_{n,3}^+$ and $\mathcal{C}_{n,4}^+$ are defined.

1.1. The subgroup of $\mathcal{C}_n$ generated by $\sigma_0, \ldots, \sigma_{n-1}$ is isomorphic to the group of automorphisms of the $(n+1)$-dimensional simplex Δ^n, i.e., to the symmetric group S_{n+1}. Under this isomorphism the generator σ_i corresponds to the transposition $(i+1, i+2)$.

The analogous group $\mathcal{C}_n^+$ is isomorphic to the alternating group A_{n+1}.

The subgroup in $\mathcal{C}_{n,4}$ (correspondingly in $\mathcal{C}_{n,4}^+$), generated by $\sigma_0, \ldots, \sigma_{n-1}$ (correspondingly by $\sigma_i \sigma_j$ for $i, j \leq n-1$) is isomorphic to the group of automorphisms of the $(n+1)$-dimensional cube I^{n+1} (correspondingly to the group of its sense-preserving automorphisms).

1.2. Note the isomorphism $\mathcal{C}_{2,3}^+ \cong \mathsf{PSL}_2(\mathbb{Z})$. It has the form

$$\sigma_0 \sigma_2 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_1 \sigma_2 \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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$$\sigma_1\sigma_0 \to \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

1.3. By an *orientation* of a $\mathcal{C}_n$-set X we mean a morphism of $\mathcal{C}_n$-sets $\nu : X \to \mathcal{C}_n/\mathcal{C}_n^+$. A set X is called *orientable* if it admits an orientation. This property is obviously equivalent to the stabilizer of any element of X being contained in $\mathcal{C}_n^+$.

For every oriented $\mathcal{C}_n$-set (X, ν) denote by X^+ the preimage of the distinguished point of $\mathcal{C}_n/\mathcal{C}_n^+$ with respect to ν. The association $(X, \nu) \to X^+$ can be extended to an equivalence between the category of oriented C_n-sets and the category of C_n^+-sets. Denote the inverse equivalence by the rule

$$Y \to Y \times \{1, -1\},$$

where the action of $\mathcal{C}_n$ on $Y \times \{1, -1\}$ has the form

$$\sigma_i(y, 1) = (\sigma_0\sigma_i(y), -1),$$
$$\sigma_i(y, -1) = (\sigma_i\sigma_0(y), 1).$$

2. The simplicial set associated to a $\mathcal{C}_n$-set.

2.0. For each non-degenerate k-dimensional simplex σ in Δ^n (i.e., for each $(k+1)$-element subset of $\{0, \ldots, n\}$) denote by $\mathcal{C}_n(\sigma)$ the subgroup in $\mathcal{C}_n$ generated by all the σ_i with $i \in \sigma$. Obviously for each i with $0 \le i \le \dim\sigma$ the inclusion $\mathcal{C}_n(\sigma) \supset \mathcal{C}_n(\partial_i\sigma)$ holds.

For a $\mathcal{C}_n$-set X define the semi-simplicial set $\tilde{N}(X)$ by the rule

$$\tilde{N}(X)_k = \coprod_{\dim\sigma = k} X/\mathcal{C}_n(\sigma).$$

The face operators are defined in the obvious way. Denote by $N(X)$ the simplicial set associated with $\tilde{N}(X)$.

2.1. Obviously $N(*) \cong \Delta^n$, $N(\mathcal{C}_n/\mathcal{C}_n^+) \cong \Delta^n \coprod_{\partial\Delta_n} \Delta^n$.

2.2. The association $X \to N(X)$ can be extended to a functor from the category $\mathcal{C}_n$-Sets to the category Δ^{op}-Sets of simplicial sets.

2.3. If X is oriented, then the orientation defines a morphism $N(X) \to N(\mathcal{C}_n/\mathcal{C}_n^+)$. The geometric realization $|N(\mathcal{C}_n/\mathcal{C}_n^+)|$ of the simplicial set $N(\mathcal{C}_n/\mathcal{C}_n^+)$ is homeomorphic to the n-dimensional sphere S^n, and the continuous map $|X| \to S^n$ is a covering, non-ramified over the complement of the $(n-2)$-dimensional skeleton of the n-dimensional simplex, embedded into the S^n in the standard way.

Denote this complement by U_n. Together with the construction from **1.3** it defines a functor from the category of $\mathcal{C}_n^+$-sets to the category of non-ramified coverings over U_n, which is equivalent to the category of $\pi_1(U_n)$-sets.

2.4. The group $\pi_1(U_n)$ can be presented by generators x_{ij} with $i, j \in \{0, \ldots, n\}$ and relations

$$x_{ij}x_{ji} = 1,$$

$$x_{ij}x_{jk}x_{ki} = 1.$$

The generator x_{ij} corresponds geometrically to the loop around the $(n-2)$-dimensional face $\{0, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$ of the simplex Δ^n. In particular, as an abstract group the group $\pi_1(U_n)$ is a free group with n generators.

2.5. The functor associating to a $\mathcal{C}_n^+$-set X^+ the covering $|N(X)| \to S^n$ is isomorphic to the functor induced by the projection $\pi_1(U_n) \to \mathcal{C}_n^+$ sending x_{ij} to $\sigma_i\sigma_j$. In particular, it is fully faithful.

3. The cartographic group and the flag sets of simplicial pseudomanifolds

3.0. A simplicial set is called a *simplicial pseudomanifold* of dimension n if the following conditions hold:

a. Every non-degenerate simplex of X is a face of some non-degenerate n-dimensional simplex;
b. For each non-degenerate $(n-1)$-simplex σ_{n-1} there are exactly two pairs (i, σ_n) and (j, σ'_n) with $\sigma_n, \sigma'_n \in X_n$,$0 \le i, j \le n$ such that σ_n, σ'_n are non-degenerate and $\sigma_{n-1} = \partial_i\sigma_n = \partial_j\sigma'_n$.
c. If σ, σ' are non-degenerate simplexes of X, then there exists a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ of non-degenerate n-simplexes, such that σ_i and σ_{i+1} have a common non-degenerate $(n-1)$-face.

3.1. If the geometric realization of a simplicial set is a connected manifold of dimension n, then it is a simplicial pseudomanifold. We call such pseudomanifolds *nonsingular*. The converse is false already in dimension 2. In the general case the geometric realization of a pseudomanifold is a "conic space", i.e. a neighborhood of each point is homeomorphic to a cone over the conical space of the preceding dimension.

3.2. For any transitive oriented $\mathcal{C}_n$-set X the simplicial set $N(X)$ is an n-dimensional simplicial pseudomanifold.

3.3. Let X be an n-dimensional simplicial pseudomanifold. Introduce the *flag set of* X: it is the set $F(X)$ of all sequences $\big((\sigma_n, i_n), \dots, (\sigma_1, i_1), \sigma_0\big)$ such that σ_k is a non-degenerate k-dimensional simplex in X and $\partial_{i_k}\sigma_k = \sigma_{k-1}$. For each flag $f = \big((\sigma_n, i_n), \dots, (\sigma_1, i_1), \sigma_0\big) \in F(X)$ and every j with $0 \leq j \leq n$ there exists a unique flag $\big((\sigma'_n, i'_n), \dots, (\sigma'_1, i'_1), \sigma'_0\big)$ such that $(\sigma_k, i_k) = (\sigma'_k, i'_k)$ and $(\sigma_j, i_j) \neq (\sigma'_j, i'_j)$. We let $\sigma_j(f)$ denote it. It defines an action of $\mathcal{C}_{n,3}$ (and hence of $\mathcal{C}_n$) on the set $F(X)$.

3.4. The flag set of the n-dimensional pseudomanifold is oriented as a $\mathcal{C}_n$-set if and only if it is oriented as a pseudomanifold, i.e. if $\mathrm{H}^n(X, \mathbb{Z}) \cong \mathbb{Z}$.

3.5. If X is a nonsingular orientable pseudomanifold, then $N(F(X))$ is isomorphic as a simplicial set to the barycentric subdivision of X. Generally it is not true that $|N(F(X))|$ is homeomorphic to $|X|$, however, if X is orientable there exists a map $N(F(X)) \to |X|$ realizing the "normalization" of $|X|$ in the usual sense. Thus, for example, the only possible singularities in the two-dimensional orientable case are the multiple points, and the map above "resolves" them in the usual sense. In particular, for any oriented X of dimension ≤ 2 the pseudomanifold $N(F(X))$ is nonsingular.

3.6. Let M be a triangulated manifold. Then in a way similar to 3.3 one can define the flag set $F(M)$, which will be a $\mathcal{C}_n$-set. The orientation of M defines an orientation of $F(M)$, and hence the covering $M \cong |N(F(M))| \to S^n$ is non-ramified away from the $(n-2)$-skeleton of Δ^n. Under this covering the preimage of the face of Δ^n corresponding to the subset $\{0, \dots, k\}$ coincides with the k-dimensional skeleton of the initial triangulation of M for $k \leq n-1$.