Referee report on
"Martin-Lof identity types in the C-systems defined by a universe category"
by Vladimir Voevodsky, submitted for publication in *Publ. Math. IHES*

Summary. The paper is one in a series of several papers in which the author develops a new approach to the study of dependent type theories. This approach is based on the notion of a C-system (originally introduced by Cartmell in the late '70s under the name of a contextual category). This is contrast with most of the existing literature, which is based on other semantical notions (such as categories with families, categories with attributes, type-categories, and comprehension categories).

The goal of this paper is to introduce and study the notion of a J*-structure* on a C-system, which is a semantic counterpart of Martin-Löf's rules for identity types. This topic is of great importance for the ongoing work on relating type theory and homotopy theory, since identity types are the cornerstone of the these connections. Furthermore, the problem of giving a correct semantics for identity types, especially in homotopy-theoretic models, is complex and difficult and some the literature glosses over delicate issues.

The paper is organized in two main parts. The first part of the paper begins by introducing the notion of a J-structure on a C-system. Then the author defines a general method for constructing C-systems with J-structures. The key issue here is that the notion of a C-system (with or without a J-structure) includes some 'strictness' conditions that are not satisfied immediately by examples. To address this problem, the author builds on an earlier paper in the series, in which it was shown how a category with a universe $(\mathcal{C}, p)$ determines a C-system $\mathbf{CC}(\mathcal{C}, p)$. In the current paper, the author first defines what it means for a category with a universe to have a J-structure, and then shows that if $(\mathcal{C}, p)$ is a category with a universe that has a J-structure, then $\mathbf{CC}(\mathcal{C}, p)$ is a C-system with a J-structure.

In the literature, similar questions have been addressed in the study of other notions of model for type theory (e.g. in the work by Lumsdaine and Warren), showing how every 'non-strict' notion of model can be turned into a 'strict' one (sometimes using category-theoretic coherence theorems). However, the author's approach has the considerable advantage of working uniformly for several type constructors (considered in other papers in the series) and of being very explicit. His analysis of functoriality, also, does not seem to have a counterpart for other notions of model.

The first part of the paper includes also important results relating Quillen model categories and C-systems with a J-structure. In particular, the author shows how, for a category with a universe, it possible to define a J-structure from two distinguished classes of maps, satisfying appropriate assumptions (he considers two such possible groups of assumptions). Examples of such situations arise from Quillen model structures satisfying some mild hypotheses. The combination of these results with the earlier work in the paper shows that every such a Quillen model category gives rise to a C-system with a J-structure, thereby providing a very clear, precise link between homotopical algebra and models of type theory.

The second part of the paper defines when a homomorphism of C-systems preserves J-structures and when a functor between categories with universes is compatible with J-structure. The author then shows that if $\Phi\colon (\mathcal{C}, p) \to (\mathcal{C}', p')$ is a functor between categories with universes that is compatible with J-structures, then the induced homomorphism of C-structures $H_\Phi\colon \mathbf{CC}(\mathcal{C}, p) \to \mathbf{CC}(\mathcal{C}', p')$ preserves J-structures.

Recommendation. This is a very interesting paper, dealing with a topic of great interest for researchers in type theory and containing several important results. In fact, the author's series of papers provides the most complete and detailed account of semantics of dependent type theory that exists in the literature. The analysis of identity types in the present paper is possibly the most intricate and complex part of this series. Although is possible that other accounts based on other notions may be developed in the future and lead to simpler treatments, the author's approach is presently to only one that permits such a rigorous and comprehensive treatment.

However, I am not satisfied by the way in which the paper is written. Indeed, some parts of the paper are virtually impenetrable. While I accept that parts of the material are complex, I think that the author should make a greater effort to help readers understand the content of the paper (especially if he wishes for his approach to become more widely used). I do not suggest the inclusion of more details in the proofs (which are given in abundant detail), but rather the use of simpler notation, the addition of explanations, backward and forward references, and reminders about notation and conventions. I give a complete list of suggested changes below. Even if each individual change may seem rather minor, I believe that they are necessary in order to make the paper suitable for publication in such a prestigious journal.

Thus, my recommendation is that the paper is accepted subject to revision.

General comments

(G.1) The numbering of sections should be revised: we currently have subsections numbered 1, 2, 3, 4 inside both section 2 and section 3. Subsections should be numbered within sections, as 2.1, 2.2, etc...

(G.2) The numbering of theorems and definitions is also confusing: this should also be done within subsections (so that one knowns that Theorem 3.2.1 is in subsection 3.2).

(G.3) I would also avoid having separate numbers for "Problems" and "Constructions", and instead have a presentation similar to that of "Theorem" and "Proof". I would also replace "Construction" with "Solution". If a reference to a solution is needed, I would phrase it as "The functor F constructed in the solution to Problem 3.4".

(G.4) Equations should be numbered within sections, so that the reader can find them more easily.

(G.5) The paper would benefit from having more forward and backward references, so that the reader is reminder of what is being done and where it will be used. Several subsections start and end abruptly, without a summary of what has been done or an indication of what will be done.

(G.6) Introduce and use systematically different fonts for different kinds of objects. Just as an example, I would use Id (rather than IdT), refl (instead of *refl*) and J (instead of J) to denote the main components of a J-structure. This would help readers to distinguish more immediately what different symbols refer to. At present, there are several situations where the notation is actually confusing (e.g. you use the same T both as part of $IdxT$ and as the variable T to which it is applied, resulting in a confusing expression such as $IdxTT$). Similar problems occur throughout the paper, cf. $pE\tilde{U}$ (page 11), pFp (page 13), ...

(G.7) Introduce and use the syntactic category associated to Martin-Löf type theory to illustrate the definitions and constructions made in a general C-system. This helps the readers familiar with type theory to understand what is going on.

(G.8) Provide some explanation for the notion of a category with a universe. One way would be to say that it is an abstract counterpart of the logical framework (as presented in the paper "Martin-Löf type theory" by Nordström, Petersson, and Smith), with the object U being the counterpart of the type Set of sets, and the map $p\colon \tilde{U} \to U$ as the counterpart of the judgement $x\colon \mathsf{Set} \vdash \mathsf{El}(x)\colon \mathsf{Type}$. Building on this, one could say that notion of a universe category having a J-structure corresponds to the formulation of identity types in the logical framework.

(G.9) The frequent long chains of equational reasoning, sometimes with very complex expressions, should be reorganized to help readers follow the calculations. For this, you should introduce 'local' notation for the parts of the equations that remain unchanged (so that reader can see clearly what is changing), and provide a brief justification for the various steps (by referring to the lemmas or equations being applied). I understand that each individual step may seem trivial in isolation, but the cumulative effect of all these calculations over a long paper (in combination with the heavy use of new notation) makes the paper very hard to follow. For example, I would suggest rewriting Construction 1.6 on page 6 with a format like the following:

$$
\begin{array}{rcll}
\delta(T)^*(\mathsf{Id}_x(T)) & = & \delta(T)^*(\ldots) & \text{(by \ldots)} \\
 & = & \mathsf{Id}(T)^*(\ldots) & \text{(by \ldots)} \\
 & = & & \text{(by \ldots)}
\end{array}
$$

Other situations of this kind are on pp. 22, 23, 24, 25, 27, 28, 32, 41, 45, 46, 47, 48.

(G.10) Some of this equational reasoning may be explained more effectively by commutative diagrams.

(G.11) The paper should include some tables summarizing the notation being used, since it is extremely challenging for anyone to keep in mind all the notation being introduced. These tables may include references to where the notation is being defined for the first time.

(G.12) I would recommend to specify more frequently the domain and codomain of the maps being considered, so as to save the reader from making the effort of reconstructing what these are. Examples of this situation are frequent in the paper.

(G.13) Since several maps are written p (or variants of it), I would write the projection maps as $\pi_i \colon X_1 \times X_2 \to X_i$.

(G.14) All proofs and constructions should end with the symbol $\square$ to separate them from the rest of the text. Accordingly, delete the sentences of the form 'This concludes the proof of the theorem'.

(G.15) Diagrams can be typeset more satisfactorily using the `xypic` package.

Specific comments

Title

- Fix the spelling of Martin-Löf
- I think the shorter title "Martin-Löf identity types in C-systems" would be better. If you keep the long title, I would delete 'the' from it.

Section 1

(1.1) Page 2, paragraph -2: I do not think that the sequent notation for general C-systems is widespread as the author claims, since most authors use it only for syntactic C-systems. Of course, there are also some papers where dependent type theories are used as internal languages for certain kinds of categories (such as locally cartesian closed categories). In this case, I think that the claim that this use 'is not reflected in any mathematical statement that one can refer to' is an overstatement, since there are results ensuring the soundness of the interpretation of dependent type theories in appropriate categories. Of course, you may not want to use these results (and hence avoid any use of logical notation outside examples) because the aim of the paper is to set up a theory from scratch. In any case, this should be clarified.

(1.2) Page 3, Line 19-20: I cannot understand the sentence "its form shows that we assume ...".

(1.3) Add a short discussion of the fact that other approaches (most notably the one of Lumsdaine and Warren) have been suggested to address the semantics of identity types.

Section 2

Subsection 1

(2.1.1) I would recommend starting with a brief review of the notion of a C-system, so as to make the paper more self-contained. This could also be a good place for defining the syntactic C-system associated to Martin-Löf type theory. It would be useful to spell out the rules for identity types, for reference.

(2.1.2) Page 5, Definition 1.1: explain that this is the function taking $x, y \colon A$ and returning $\mathsf{Id}_A(x, y)$.

(2.1.3) Page 5, Problem 1.3: Explain what is this in the syntactic C-system.

(2.1.4) Page 6, Problem 1.5: Say that, in the syntactic C-system, this map is the one sending x to $(x, x, \mathsf{refl}(x))$.

(2.1.5) Page 6, Definition 1.7: why use $s0$ instead of s_0? Terminology 'ι-rule' is common for Coq's inductive types, but others refer to it as the 'computation rule'. At least mention this.

Subsection 2

(2.2.1) Page 7: The title of the subsection should be 'J-structures on a universe in a category'.

(2.2.2) Page 7, first diagram: since the maps $p_{X,F}$ are chosen as part of the data of a universe, I would make this more explicit by writing $\mathsf{p}_{X,F}$, showing that p is an external operation. Similarly, I would then write $\mathsf{q}(f)$ rather than $Q(F)$. Also, it would be best to be consistent and always use upper-case letters for objects and lower-case letters for maps.

(2.2.3) Page 8, after second display: write domain and codomain of the functor $D_p(-, V)$.

(2.2.4) Page 8, 5th display: explain notation $\underline{Hom}$. Also, why write $\eta^!$ and η instead of η and η^{-1} (or similar).

(2.2.5) Page 9, before Lemma 2.2: explain what will follow in the subsection and motivation for it (e.g. where it will be used).

(2.2.6) Page 9, display -3: I think there is a typo in $\eta^{\prime,!}$. If correct, an alternative (more readable) notation should be chosen.

(2.2.7) Page 11, 1st display: notation such as $E\tilde{U}$ and $pE\tilde{U}$ is quite verbose, so I would suggest to use something simpler, e.g. (E, p_E), leaving implicit that they depend on U and p anyway?

(2.2.8) Page 11, displays in the mid-page: paste these diagrams together.

(2.2.9) Page 12, Definition 2.8: $Jp \circ coJ = Id$ is unclear. Maybe mis-typed? What is Id here?

(2.2.10) Page 13, Line 6: notation $\diamond$ needs to be defined (or recalled, if defined earlier).

(2.2.11) Page 13, Problem 2.9: explain the notation adj.

(2.2.12) Page 13, Line -3: what is coJ?

Subsection 3

(2.3.1) I think it would be best to swap the order between the current subsection 3 and subsection 4. This is because the material in subsection 4 is very closely related to the material in subsection 2. Section 2 would then end with Corollary 3.9 and (to be added) some comments about how the combination of results in the last 2 subsections allow us to construct C-systems from appropriate categories with two classes of maps. This is one of the key points of the paper and it would provide a nice, strong conclusion for the first part of the paper.

(2.3.2) Page 14 (bottom) and page 15 (top): mention that a similar notions (maps with additional structure, providing a choice of diagonal fillers) is being considered by Coquand and his collaborators in the development of the cubical model.

(2.3.3) Page 15, Line 9: I think the notation TC and FB is too heavy (and forces readers to constantly remind themselves that this is not F applied to B, T applied to C). Why not just $(\mathcal{L}, \mathcal{R})$ (in a different font for emphasis)?

(2.3.4) Page 15, Theorem 3.2: should be capital Ω in item (2).

(2.3.5) Page 16, top ("attempts to localize"): do you mean that the localization is not necessarily a Quillen model category?

(2.3.6) Page 16, Conditions 3.3, part (3): correct "iis".

(2.3.7) Page 16, first line after Conditions 3.3: make "fibrant" in italics.

(2.3.8) Page 17, just before Lemma 3.6: explain what is going to happen and that you are going to consider again the objects $I_p(V)$, giving backward reference.

(2.3.9) Page 17, lines -2 and -1: explain Δ.

(2.3.10) Page 18, Theorem 3.8, item (3): I think this should be capital Ω.

(2.3.11) Page 19, Corollary 3.9: I would split condition (2) in two separate conditions: (2) U is fibrant, (3) pullback of a trivial cofibration along fibrations is a trivial cofibration. Also, delete 'full' from final sentence (it suggests that there is a special kind of J-structures called full). Where is the condition of pullback of trivial cofibrations used? Mention examples satisfying these conditions.

(2.3.12) Page 20, line 2, "but not the univalent universes": which univalent universes? The ones in the simplicial model?

(2.3.13) Page 20, Problem 3.10, item (3): do you mean that there exists a factorisation (in which case you should simply write i and q, to avoid using the axiom of choice) or that a choice of factorisations is given.

(2.3.14) Page 21, after 2nd display: explain the notation $\langle in_{n+1} \rangle_n$.

Subsection 4

(2.4.1) Page 21, line -7: use a different font for $CC(\mathcal{C}, p)$ for emphasis (and avoid confusions with the C's), say $\mathbf{CC}(\mathcal{C}, p)$. In fact, the notation $\mathbf{CC}$ is not ideal, since it was probably suggested by 'contextual category', which the author does not use.

(2.4.2) Page 21, line -5: say "is equipped with a functor $int\colon \mathbf{CC}(\mathcal{C}, p) \to \mathcal{C}$.

(2.4.3) Page 21, display -3: I do not understand how you can apply int to (Γ, F).

(2.4.4) Page 22, display -4: chain of equations should be displayed as suggested in general comments and steps justified.

(2.4.5) Page 23, line 3, 4: enumerate equations in statement and refer to them with those numbers rather than by 'first', 'second', 'third'.

Section 3

Subsection 1

(3.1.1) Page 29, Line -6: Provide some forward references for where this material will be used and some comments on what will be done in this subsection. Warn the reader that this could be skipped, provided one accepts the existence of a certain map (the map at the bottom of page 39).

(3.1.2) Page 30, 3rd display: give backward reference for notation $f * g$.

(3.1.3) Page 30, last display: add number to this equation for future reference and recall where the notation $I_p(V)$ was introduced.

(3.1.4) Page 31, 3rd display: refer back to the display of page 30.

(3.1.5) Page 31, Lemma 1.2: what is $D_{p_2}(X, V)$?

(3.1.6) Page 32, Line -3: replace 'proofs' with 'proves'.

(3.1.7) Page 33, 3rd display: this is an instance of one of my general comments; rewrite the chain of equalities in an aligned format, with side-comments to justify the steps. This helps the reader to do pattern-matching and see what is going on.

Subsection 2

(3.2.1) Page 36, Lemma 2.2: again, recall what Δ is.

(3.2.2) Page 40, Definition 2.10: it is a bit ackward to end a section with a definition. It would be best to add some comments regarding how the definition will be used in the subsequent pages.

Subsection 3

(3.3.1) Page 40: Start the section by explaining what will be done.

Subsection 4

(3.4.1) Page 41, Line -6: give domain and codomain of Φ.

(3.4.2) Page 43, lemma 4.4: the notation makes this almost unreadable. Please adopt some shorthands or simplifications.

(3.4.3) Page 44, Remark 4.5: move the remark after the proof or before the statement.

(3.4.4) Page 44, 5th and 6th display: I think a diagram makes the situation much easier to understand.

(3.4.5) Page 46, Line -2: reference needs to be fixed.

(3.4.6) Pages 46-47: these equational proofs are virtually unreadable.

References

(R.1) [1] has been published, and superceded by other papers.

(R.2) Correct author name so that [17, 18, 19] are uniform.