Products of families of types in the C-systems defined by a universe in a category[1]

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Abstract

We introduce the notion of a (Π, λ)-structure on a C-system and show that C-systems with (Π, λ)-structures are constructively equivalent to contextual categories with products of families of types. We then show how to construct (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ defined by a universe p in a locally cartesian closed category $\mathcal{C}$ from a simple pull-back square based on p. In the last section we prove a theorem that asserts that our construction is functorial.

1 Introduction

The concept of a C-system in its present form was introduced in [7]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [3] and [2] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In this paper we consider what might be the most important structure on C-systems - the structure that corresponds, for the syntactic C-systems, to the operations of dependent product, λ-abstraction and application. A C-system formulation of this structure was introduced by John Cartmell in [2, pp. 3.37 and 3.41] as a part of what he called a strong M.L. structure. It was studied further by Thomas Streicher in [4, p.71] who called a C-system (contextual category) together with such a structure a "contextual category with products of families of types".

We first show that the structure that Cartmell defined is equivalent to another structure, which we call a (Π, λ)-structure. The proof of this equivalence consists of Constructions 2.5 and 2.6 (of mappings in both directions) and Lemmas 2.7 and 2.8 showing that these mappings are mutually inverse.

Then we consider the case of C-systems of the form $CC(\mathcal{C}, p)$ introduced in [6]. They are defined, in a functorial way, by a category $\mathcal{C}$ with a final object and a morphism $p : \widetilde{U} \to U$ in $\mathcal{C}$ together with the choice of pull-backs of p along all morphisms in $\mathcal{C}$. A morphism with such choices is called a universe in $\mathcal{C}$. An important feature of this construction is that the C-systems $CC(\mathcal{C}, p)$ corresponding to different choices of pull-backs and different choices of final objects are canonically isomorphic. This fact makes it possible to say that $CC(\mathcal{C}, p)$ is defined by $\mathcal{C}$ and p.

We provide several intermediate results about $CC(\mathcal{C}, p)$ when $\mathcal{C}$ is a locally cartesian closed category leading to the main result of this paper - Construction 3.7 that produces a (Π, λ)-structure on $CC(\mathcal{C}, p)$ from a simple pull-back square based on p. This construction was first announced in [5]. It and the ideas that it is based on are among the most important ingredients of the construction of the univalent model of the Martin-Lof type theory.

In this paper we continue to use the diagrammatic order of writing composition of morphisms, i.e., for $f : X \to Y$ and $g : Y \to Z$ the composition of f and g is denoted by $f \circ g$.

2 Products of families of types and (Π, λ)-structures

Let CC be a C-system. Recall that we let $\widetilde{Ob}(CC)$, or simply $\widetilde{Ob}$, denote the set:

$$\widetilde{Ob} = \{s : ft(X) \to X \mid l(X) > 0 \text{ and } s \circ p_X = Id_{ft(X)}\}$$

For $n \in \mathbf{N}$ denote by $Ob_{\geq n}$ the set of objects of CC of length $\geq n$ and by $\widetilde{Ob}_{\geq n}$ the subset of $\widetilde{Ob}(CC)$ that consists of elements $s : ft(X) \to X$ such that $l(X) \geq n$.

Let further $Ob_n(\Gamma)$ be the set of elements Δ in Ob such that $ft^n(\Delta) = \Gamma$ and $\widetilde{Ob}_n(\Gamma)$ the set of elements $s \in \widetilde{Ob}$ such that $s : ft(\Delta) \to \Delta$ where $\Delta \in Ob_n(\Gamma)$. For $n = 0$ we will abbreviate $\widetilde{Ob}_0(\Gamma)$ as $\widetilde{Ob}(\Gamma)$. Note that in view of the definition of $\widetilde{Ob}$ we have $\widetilde{Ob}(X) = \emptyset$ if $l(X) = 0$.

For $f : \Gamma' \to \Gamma$ the functions $\Delta \mapsto f^*(\Delta, n)$ and $s \mapsto f^*(s, n)$, defined in [7] as iterated canonical pull-backs of objects and sections respectively, give us functions:

$$Ob_n(\Gamma) \to Ob_n(\Gamma')$$

$$\widetilde{Ob}_n(\Gamma) \to \widetilde{Ob}_n(\Gamma')$$

which we will write simply as f^*.

Let us note also that if $\Delta, \Delta' \in Ob(\Gamma)$, $u : \Delta \to \Delta'$ is a morphism over Γ and $f : \Gamma' \to \Gamma$ is a morphism then, using the fact the the canonical squares are pull-back, we get a morphism $f^*(\Delta) \to f^*(\Delta')$ that we denote by $f^*(u)$.

The structure of "products of families of types" is defined in [2, pp.3.37 and 3.41] and also considered in [4, p.71]. Let us remind this definition here.

Definition 2.1 *The structure of products of families of types on a C-system CC is a collection of data of the form:*

1. *for every $\Gamma \in Ob$ a function $\Pi^{\Gamma} : Ob_2(\Gamma) \to Ob_1(\Gamma)$, which we write simply as Π,*

2. *for every Γ and $B \in Ob_2(\Gamma)$ a morphism $Ap_B : p_A^*(\Pi(B)) \to B$ over A, where $A = ft(B)$,*

such that:

1. *for any Γ and $B \in Ob_2(\Gamma)$ the map $\lambda inv_{Ap} : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$ defined as:*

$$s \mapsto p_A^*(s) \circ Ap_B$$

   is a bijection,

2. *for any $f : \Gamma' \to \Gamma$ the square*

$$
\begin{array}{ccc}
Ob_2(\Gamma) & \xrightarrow{\Pi^\Gamma} & Ob_1(\Gamma) \\
f^* \downarrow & & \downarrow f^* \\
Ob_2(\Gamma') & \xrightarrow{\Pi^{\Gamma'}} & Ob_1(\Gamma')
\end{array}
$$

   commutes,

3. *for any for any Γ, $B \in Ob_2(\Gamma)$ and $f : \Gamma \to \Gamma'$ one has $f^*(Ap_B) = Ap_{f^*(B)}$.*

We will show in the next section how to construct products of families of types on C-systems of the form $CC(\mathcal{C}, p)$. For this construction we first need to introduce another structure on C-systems and show that this other structure is equivalent to the structure of products of families of types.

Definition 2.2 *Let CC be a C-system. A pre-(Π, λ)-structure on CC is a pair of functions*

$$\Pi : Ob_{\geq 2} \to Ob$$
$$\lambda : \widetilde{Ob}_{\geq 2} \to \widetilde{Ob}$$

such that:

1. $ft(\Pi(\Gamma)) = ft^2(\Gamma)$,

2. $\partial(\lambda(s)) = \Pi(\partial(s))$.

For a pre-(Π, λ)-structure (Π, λ) and $\Gamma \in Ob$ the function Π defines, in view of the first condition of Definition 2.2, a function

$$\Pi^\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

and the function λ defines, in view of the first and the second conditions of Definition 2.2, a function

$$\lambda^\Gamma : \widetilde{Ob}_2(\Gamma) \to \widetilde{Ob}_1(\Gamma)$$

The second condition also implies that the square:

$$
\begin{array}{ccc}
\widetilde{Ob}_2(\Gamma) & \xrightarrow{\lambda^\Gamma} & \widetilde{Ob}_1(\Gamma) \\
\partial \downarrow & & \downarrow \partial \\
Ob_2(\Gamma) & \xrightarrow{\Pi^\Gamma} & Ob_1(\Gamma)
\end{array}
\tag{1}
$$

commutes. One can easily see that the notion of a pre-(Π, λ)-structure could be equally formulated as two families of functions Π^Γ and λ^Γ such that the squares (1) commute.

Definition 2.3 *A pre-(Π, λ)-structure is called a (Π, λ)-structure if the following conditions hold:*

    1. for any $\Gamma \in Ob_{\geq 2}$ the square (1) is a pull-back square,

    2. for any $f : \Gamma' \to \Gamma$ the square

$$
\begin{array}{ccc}
Ob_2(\Gamma) & \xrightarrow{\ \Pi^\Gamma\ } & Ob_1(\Gamma) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
Ob_2(\Gamma') & \xrightarrow{\ \Pi^{\Gamma'}\ } & Ob_1(\Gamma')
\end{array}
\tag{2}
$$

    commutes,

    3. for any $f : \Gamma' \to \Gamma$ the square

$$
\begin{array}{ccc}
\widetilde{Ob}_2(\Gamma) & \xrightarrow{\ \lambda^\Gamma\ } & \widetilde{Ob}_1(\Gamma) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\widetilde{Ob}_2(\Gamma') & \xrightarrow{\ \lambda^{\Gamma'}\ } & \widetilde{Ob}_1(\Gamma')
\end{array}
\tag{3}
$$

    commutes.

Note that the first condition can be equivalently formulated by saying that the functions

$$
\lambda_\Gamma : \widetilde{Ob}(\Gamma) \to \widetilde{Ob}(\Pi(\Gamma))
$$

defined by λ are bijections.

We are going to show that, for a given family of functions Π^Γ, the type of (Π, λ)-structures over Π^Γ is equivalent to the type of products of families of types over the same Π^Γ.

We first reformulate the structure of products of families slightly. Instead of considering $p_A^*(\Pi(B))$ we will consider an object that is isomorphic (but not equal!) to it, namely $p_{\Pi(B)}^*(A)$. Our structure will then be a family of maps Π as before together with, for every Γ and $B \in Ob_2(\Gamma)$, a morphism $Ap'_B : p_{\Pi(B)}^*(A) \to B$ over A such that the map $\lambda inv'_{Ap'} : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$ defined as:

$$
s \mapsto q(s, p_{\Pi(B)}^*(A)) \circ Ap'_B
$$

is a bijection. This can be seen on the following diagram that also contains other elements that will be needed in the construction below.

$$
\begin{array}{ccccc}
B & \xrightarrow{q(s, p_{\Pi(B)}^*(B,2),2)} & p_{\Pi(B)}^*(B,2) & \xrightarrow{q(p_{\Pi(B)}, B, 2)} & B \\
{\scriptstyle p_B}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_B} \\
A & \xrightarrow{q(s, p_{\Pi(B)}^*(A))} & p_{\Pi(B)}^*(A) & \xrightarrow{q(p_{\Pi(B)}, A)} & A \\
{\scriptstyle p_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_A} \\
\Gamma & \xrightarrow{\ \ s\ \ } & \Pi(B) & \xrightarrow{\ p_{\Pi(B)}\ } & \Gamma
\end{array}
\tag{4}
$$

4

We now state the problem which we will provide a construction for:

Problem 2.4 *Let CC be a C-system and let Π be a family of functions*

$$\Pi^{\Gamma} : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

given for all $\Gamma \in Ob$ such that the corresponding squares of the form (2) commute. To construct a bijection between the following two types of structure:

1. *for every Γ and $B \in Ob_2(\Gamma)$ a bijection*

$$\lambda_B : \widetilde{Ob}(B) \to \widetilde{Ob}(\Pi(B))$$

   such that for every morphism $f : \Gamma' \to \Gamma$ the square

$$
\begin{array}{ccc}
\widetilde{Ob}(B) & \xrightarrow{\ \lambda_B\ } & \widetilde{Ob}(\Pi(B)) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\widetilde{Ob}(f^*(B)) & \xrightarrow{\ \lambda_{f^*(B)}\ } & \widetilde{Ob}(\Pi(f^*(B)))
\end{array}
$$

   defined by f, commutes.

2. *for every $\Gamma \in Ob$ and $B \in Ob_2(\Gamma)$ a morphism $Ap'_B : p^*_{\Pi(B)}(A) \to B$ over A, where $A = ft(B)$, such that the map*

$$\lambda inv'_{Ap'} : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$$

   defined as:

$$s \mapsto q(s, p^*_{\Pi(B)}(A)) \circ Ap'_B$$

   is a bijection and such that for every morphism $f : \Gamma' \to \Gamma$ and $B \in Ob_2(\Gamma)$ one has $f^*(Ap'_B) = Ap'_{f^*(B)}$.

We will construct the solution in four steps - first a function from structures of the first kind to structures of the second, then a function in the opposite direction and the two lemmas proving that the first function is a left and a right inverse to the second.

Construction 2.5 Let us show how to construct a structure of the second kind from a structure of the first kind. To define Ap' consider the digram of Π's defined by the diagram (4):

$$
\begin{array}{ccccc}
\Pi(B) & & \Pi(p^*_{\Pi(B)}(B,2)) & & \Pi(B) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma & \xrightarrow{\ s\ } & \Pi(B) & \xrightarrow{\ p_{\Pi(B)}\ } & \Gamma
\end{array}
\tag{5}
$$

Note that since Π is stable under pull-backs we have

$$\Pi(p^*_{\Pi(B)}(B,2)) = p^*_{\Pi(B)}(\Pi(B))$$

and therefore the diagonal $\delta_{\Pi(B)}$ gives us an element in $\widetilde{Ob}(\Pi(p^*_{\Pi(B)}(B,2)))$. Applying to it the inverse of our λ we get an element $ap : \widetilde{Ob}(p^*_{\Pi(B)}(B,2))$. Define:

$$Ap'_B = ap \circ q(p_{\Pi(B)}, B, 2)$$

Let us prove that these morphisms satisfy the conditions of bijectivity and the stability under pull-backs. We need to show that the mappings $\lambda inv'_{Ap'} : \widetilde{Ob}(\Pi(B)) \to \widetilde{Ob}(B)$ defined as:

$$s \mapsto q(s, p^*_{\Pi(B)}(A)) \circ Ap'_B$$

are bijective. We already have bijective mappings $\Lambda_B : \widetilde{Ob}(B) \to \widetilde{Ob}(\Pi(B))$ given by our λ. It is sufficient to show that the mappings $\lambda inv'_{Ap'}$ are inverse to the ones given by λ from at least one side as any inverse to a bijection is a bijection.

We do it in two steps. First let

$$\lambda inv''(s) = s^*(ap, 2) = q(s, p^*_{\Pi(B)}(A))^*(ap)$$

Let us show that $\lambda inv'' = \lambda inv'_{Ap'}$. Indeed:

$$q(s, p^*_{\Pi(B)}(A))^*(ap) = q(s, p^*_{\Pi(B)}(A))^*(ap) \circ q(s, p^*_{\Pi(B)}(B,2), 2) \circ q(p_{\Pi(B)}, B, 2) =$$

$$q(s, p^*_{\Pi(B)}(A)) \circ ap \circ q(p_{\Pi(B)}, B, 2) = q(s, p^*_{\Pi(B)}(A)) \circ Ap'_B$$

Now we have:

$$\lambda(\lambda inv''(s)) = \lambda(s^*(ap, 2)) = s^*(\lambda(ap), 1) = s^*(\delta_{\Pi(B)}, 1) = s.$$

It remains to check that the mappings Ap' are stable under the base change. Since the base change of morphisms commutes with compositions this follows if we know that ap is stable and $q(-, -, 2)$ is stable. The second fact is verified easily from the axioms of a C-system and the first follows from the stability of δ and the pull-back and the assumption that λ is stable under pull-back.

Construction 2.6 Let us now construct a structure of the first kind from a structure of the second. This is straightforward since a construction of the second kind gives is bijections $\lambda inv'_{Ap'}$ and the inverse to these bijections are bijections required for the structure of the first kind. The fact that the bijections that we obtain in this way are stable under the pull-backs follows from the fact that the pull-backs commute with compositions, that they take morphisms of the form $q(-, -, 1)$ to morphisms of the same form and from our assumption that morphisms Ap' are stable under composition.

Let us denote the map of Construction 2.5 by $C1$ and the map of Construction 2.6 by $C2$.

Lemma 2.7 *For a structure of the first kind λ one has $C2(C1(\lambda)) = \lambda$.*

Proof: This is immediate since in Construction 2.5 we proved that the $\lambda inv'_{Ap'}$ that we have constructed are bijections by showing that they are inverses to the λ's that we started with and in Construction 2.6 we defined λ's as inverses to $\lambda inv'_{Ap'}$.

Lemma 2.8 *For a structure of the second kind Ap' one has $C1(C2(Ap')) = Ap'$.*

Proof: This amounts to checking that

$$\lambda inv'_{Ap'}(\Delta_{\Pi(B)}) \circ q(p_{\Pi(B)}, B, 2) = Ap'_B$$

Opening up the definition of $\lambda inv'$ we get the equation

$$q(\delta_{\Pi(B)}, p^*_{p_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(A))) \circ Ap'_{p^*_{\Pi(B)}(B,2)} q(p_{\Pi(B)}, B, 2) = Ap'_B$$

We have for any $f : \Gamma' \to \Gamma$:

$$Ap'_{f*(B,2)} \circ q(f, B, 2) = q(q(f, \Pi(B)), p^*_{\Pi(B)}(A)) \circ Ap'_B$$

and our equation becomes

$$q(\delta_{\Pi(B)}, p^*_{p_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(A))) \circ q(q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(A)) \circ Ap'_B = Ap'_B$$

Which follows from:

$$q(\delta_{\Pi(B)}, p^*_{p_{\Pi(B)}(\Pi(B))}(p^*_{\Pi(B)}(A))) \circ q(q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(A)) =$$

$$q(\delta_{\Pi(B)} \circ q(p_{\Pi(B)}, \Pi(B)), p^*_{\Pi(B)}(A)) = q(Id, p^*_{\Pi(B)}(A)) = Id.$$

This completes our construction for Problem 2.4.

3 (Π, λ)-structures on the C-systems $CC(\mathcal{C}, p)$

We will show now how to construct (Π, λ)-structures on C-systems of the form $CC(\mathcal{C}, p)$ for cartesian closed (pre-)categories[4] $\mathcal{C}$.

We will say that a cartesian closed structure on a (pre-)category consists of the choices of a final object, binary products and for every X, Y in C of a pair $(\underline{Hom}(X, Y), coev)$ where $\underline{Hom}(X, Y)$ is an object and $coev_Y : Y \to \underline{Hom}(X, Y \times X)$ is a morphism such that for every Z the map

$$Hom(Y \times X, Z) \to Hom(Y, \underline{Hom}(X, Z))$$

given by $f \mapsto coev_Y \circ \underline{Hom}(X, f)$, is a bijection. A cartesian closed (pre-)category is a (pre-)category together with a Cartesian closed structure on it[5].

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[4]For the discussion of the difference between a category and a pre-category see the introduction to [7] and [1].

[5]One can also define cartesian closed structures in terms of morphisms $eval : X \times \underline{Hom}(X, Y) \to Y$ but the *coev* formulation will be more convenient for our computations below.

Remark 3.1 On a general pre-category there can be many non-isomorphic cartesian closed structures, i.e., there are can be pairs of cartesian closed structures S_1, S_2 such that the cartesian closed categories (C, S_1), (C, S_2) are not isomorphic. However any two Cartesian closed categories of this form will be equivalent under an appropriate definition of an equivalence of cartesian closed categories. Below we will make sure that our construction are invariant with respect to *equivalences* of cartesian and locally cartesian closed categories so that the particular choices of the cartesian closed structures do not affect their outcome.

A (pre-)category $\mathcal{C}$ is called a lcc (locally cartesian closed) (pre-)category if all its over-categories $\mathcal{C}/X$ are cartesian closed categories.

We will not use a special notation for the forgetting functor from $\mathcal{C}/X$ to $\mathcal{C}$ and in particular for $Y, Y' \in \mathcal{C}/X$ we will write $\underline{Hom}_U(Y, Y')$ both for the internal Hom-object from Y to Y' in $\mathcal{C}/X$ and for its image in $\mathcal{C}$.

Recall from [6] that for $X \in \mathcal{C}$ and $F : X \to U$ we let $(X; F)$ denote the pull-back of p along F and by $p_{(X,F)} : (X; F) \to X$ the projection. Iterating this construction we get sets Ob_n of sequences of the form $(F_1, \ldots, F_n)$ where $F_1 : pt \to U$ and $F_{i+1} : ((pt; F_1); \ldots; F_i) \to U$. One defines $Ob(CC(\mathcal{C}, p)) := \amalg Ob_n$ and

$$Mor_{CC(\mathcal{C},p)}((F_1, \ldots, F_n), (G_1, \ldots, G_m)) := Mor_{\mathcal{C}}(((pt; F_1); \ldots; F_n), ((pt; G_1); \ldots; G_m))$$

For $\Gamma = (F_1, \ldots, F_n)$ we write $int(\Gamma)$ for the object $((pt; F_1); \ldots; F_i)$ of $\mathcal{C}$. Together with the obvious maps on the sets of morphisms the map int defines a full embedding of the category underlying the C-system $CC(\mathcal{C}, p)$ to $\mathcal{C}$.

By definition of $Ob(CC(\mathcal{C}, p))$ we have, for any Γ, a bijection $Ob_1(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), U)$ and by definition of the canonical pull-back squares in $CC(\mathcal{C}, p)$ these bijections are natural in Γ i.e. for any $f : \Gamma' \to \Gamma$ the square

$$
\begin{array}{ccc}
Ob_1(\Gamma) & \longrightarrow & Hom_{\mathcal{C}}(int(\Gamma), U) \\
{\scriptstyle f^*}\downarrow & & \downarrow \\
Ob_1(\Gamma') & \longrightarrow & Hom_{\mathcal{C}}(int(\Gamma'), U)
\end{array}
$$

where the right hand side vertical map is given by the composition with f, commutes. Similarly, we have bijections $\widetilde{Ob}_1(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), \widetilde{U})$ and again one verifies easily that these bijections are natural in Γ.

In the case when $\mathcal{C}$ is an lcc category we can also describe $Ob_2(\Gamma)$ and $\widetilde{Ob}_2(\Gamma)$ in similar terms.

We first present a more general construction. For $X \in \mathcal{C}$ let $D_p(X, V)$ be the set of pairs of the form $(F_1 : X \to U, F_2 : (X; F_1) \to V)$.

The sets $D_p(X, V)$ form a covariant functor in V in the obvious way. They also form a contravariant functor in X if one defines for $f : X' \to X$:

$$D_p(f, V)(F_1, F_2) = (f \circ F_1, q \circ F_2)$$

8

where q is the unique morphism that makes the following diagram commute:

$$
\begin{array}{ccccc}
(X'; f \circ F_1) & \xrightarrow{\;q\;} & (X, F_1) & \xrightarrow{Q(F_1)} & \widetilde{U} \\
{\scriptstyle p_{(X', f \circ F_1)}}\downarrow & & {\scriptstyle p_{(X, F_1)}}\downarrow & & \downarrow{\scriptstyle p} \\
X' & \xrightarrow[\;f\;]{} & X & \xrightarrow{\;F_1\;} & U
\end{array}
\tag{6}
$$

For $V \in \mathcal{C}$ denote by $\underline{U} \times V$ the product considered as an object over U. Denote the functor $\underline{Hom}_U(\widetilde{U}, -)$ from $\mathcal{C}/U$ to itself by R_p and the functor $R_p(\underline{U} \times -)$ by I_p.

Problem 3.2 *To construct bijections*

$$
\eta : D_p(X, V) \to Hom_{\mathcal{C}}(X, I_p(V))
$$

that are natural in X and V.

Construction 3.3 We have

$$
D(X, V) = \amalg_{F_1 : X \to U} Hom_U(X \times_U \widetilde{U}, \underline{U} \times V).
$$

On the other hand, by definition of $\underline{Hom}_U$ we have that for each $F_1 : X \to U$ the map from $Hom_U(X \times \widetilde{U}, \underline{U} \times V)$ to $Hom_U(X, \underline{Hom}_U(\widetilde{U}, \underline{U} \times V))$ given by

$$
f \mapsto coev_X \circ R_p(f)
$$

is a bijection. But also

$$
Hom_{\mathcal{C}}(X, R_p(\underline{U} \times V)) = \amalg_{F_1 : X \to U} Hom_U(X, R_p(\underline{U} \times V))
$$

This gives us isomorphisms of sets:

$$
D(X, V) \to Hom_{\mathcal{C}}(X, R_p(\underline{U} \times V))
$$

to verify that they are isomorphisms of functors in X we need to check for each $f : X' \to X$, $F_1 : X \to U$ and $F_2 : (X; F_1) \to V$ the equation:

$$
f \circ coev_{F_1} \circ R_p((p_{(X, F_1)} \circ F_1) \boxtimes F_2) = coev_{f \circ F_1} \circ R_p((p_{(X', f \circ F_1)} \circ f \circ F_1) \boxtimes (q \circ F_2))
$$

where q is the morphism defined by the diagram (6) above and where for $a : A \to U$ and $b : A \to B$ we let $a \boxtimes b$ denote the morphism $a \times b : A \to \underline{U} \times B$ as a morphism over U.

We have $f \circ coev_{F_1} = coev_{f \circ F_1} \circ R_p(q)$ and since R_p is functorial it remains to check that

$$
R_p(q \circ ((p_{(X, F_1)} \circ F_1) \boxtimes_U F_2)) = R_p((p_{(X', f \circ F_1)} \circ f \circ F_1) \boxtimes_U (q \circ F_2))
$$

for which it is sufficient to check that

$$
q \circ ((p_{(X, F_1)} \circ F_1) \boxtimes_U F_2) = (p_{(X', f \circ F_1)} \circ f \circ F_1) \boxtimes_U (q \circ F_2)
$$

which follows from the equality $p_{(X', f \circ F_1)} \circ f = q \circ p_{(X; F_1)}$ and the property that $a \circ (b \boxtimes_U G) = (a \circ b) \boxtimes_U (a \circ G)$.

9

Problem 3.4 *For a locally cartesian closed closed $\mathcal{C}$ and a universe $p : \widetilde{U} \to U$ in $\mathcal{C}$ to construct for any $\Gamma \in Ob(CC(\mathcal{C}, p))$ bijections*

$$\eta : Ob_2(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), I_p(U))$$

and

$$\widetilde{\eta} : \widetilde{Ob_2}(\Gamma) \to Hom_{\mathcal{C}}(int(\Gamma), I_p(\widetilde{U}))$$

that are natural in Γ and compatible with the function ∂.

Construction 3.5 When we take $X = int(\Gamma)$ we get $D_p(X, U) = Ob_2(\Gamma)$ and $D_p(X, \widetilde{U}) = \widetilde{Ob_2}(\Gamma)$ with the functoriality in X corresponding to the functions f^* and the functoriality for the projection $\widetilde{Ob_2}(\Gamma) \to Ob_2(\Gamma)$ corresponding to the operation ∂. Therefore we can define the required bijections using Construction 3.3.

In the following p_2 is the morphism defined by the projection $p : \widetilde{U} \to U$.

Problem 3.6 *Let $\mathcal{C}$ be a locally cartesian closed category with a final object. Let $p : \widetilde{U} \to U$ be a morphism with a universe structure on it. Let $P, \widetilde{P}$ be a pair of morphisms that make the square:*

$$
\begin{array}{ccc}
\underline{Hom}_U(\widetilde{U}, \underline{U} \times \widetilde{U}) & \xrightarrow{\widetilde{P}} & \widetilde{U} \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p} \\
\underline{Hom}_U(\widetilde{U}, \underline{U} \times U) & \xrightarrow{P} & U
\end{array}
\tag{7}
$$

a pull-back square.

To construct a (Π, λ)-structure on $CC(\mathcal{C}, p)$.

Construction 3.7 In view of Construction 3.5 any pair $(P, \widetilde{P})$ that makes square (7) commutative defines a pre-(Π, λ)-structure on $CC(\mathcal{C}, p)$ that also satisfies the second and the third condition of the definition of a (Π, λ)-structure. If this square is a pull-back square then this pre-(Π, λ)-structure satisfies the first condition of Definition 2.3 and therefore it is a (Π, λ)-structure.

4 Functoriality properties of the (Π, λ)-structures arising from universes

Let us outline now the functoriality properties of the (Π, λ) structures of Construction 3.7.

Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}, p', pt')$ be two (pre-)categories with universes. Recall from [6] that a functor of categories with universes from $(\mathcal{C}, p, pt)$ to $(\mathcal{C}, p', pt')$ is a triple $(\Phi, \phi, \widetilde{\phi})$ where Φ is a functor $\mathcal{C} \to \mathcal{C}'$ and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \widetilde{U} \to \widetilde{U}'$ are two morphisms such that F takes

the final object to a final object, pull-back squares based on p to pull-back squares and such that the square

$$
\begin{array}{ccc}
\Phi(\widetilde{U}) & \xrightarrow{\ \widetilde{\phi}\ } & \widetilde{U}' \\
{\scriptstyle \Phi(p)} \downarrow & & \downarrow {\scriptstyle p'} \\
\Phi(U) & \xrightarrow{\ \phi\ } & U'
\end{array}
\tag{8}
$$

is a pull-back square. By [6] any such functor defines a homomorphism of C-systems

$$H : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

In order to prove our main functoriality Theorem 4.5 we need describe in more detail the maps

$$Ob_1(\Gamma) \to Ob_1(H(\Gamma))$$
$$Ob_2(\Gamma) \to Ob_2(H(\Gamma))$$

and the similar maps on $\widetilde{Ob}_1$ and $\widetilde{Ob}_2$. We will be doing it with respect to the identifications:

$$
\begin{aligned}
Ob_1(\Gamma) &= Hom(int(\Gamma), U) \\
\widetilde{Ob}_1(\Gamma) &= Hom(int(\Gamma), \widetilde{U}) \\
Ob_2(\Gamma) &= D_p(int(\Gamma), U) \\
\widetilde{Ob}_2(\Gamma) &= D_p(int(\Gamma), \widetilde{U})
\end{aligned}
\tag{9}
$$

For X, V in $\mathcal{C}$ we have the functoriality map

$$\Phi : Hom(X, V) \to Hom(\Phi(X), \Phi(V))$$

If $(\Phi, \phi, \widetilde{\phi})$ is a functor of categories with universes we also have maps

$$\Phi_2 : D_p(X, V) \to D_{p'}(\Phi(X), \Phi(V))$$

defined as follows. Let $(F_1 : X \to U, F_2 : (X; F_1) \to V)$ be an element in $D_p(X, V)$. Consider $(\Phi(X); \Phi(F_1) \circ \phi)$. Since the square (8) is a pull-back square there is a unique morphism q that makes the following diagram commutative:

$$
\begin{array}{ccccc}
(\Phi(X); \Phi(F_1) \circ \phi) & \xrightarrow{\ q\ } & \Phi(\widetilde{U}) & \xrightarrow{\ \widetilde{\phi}\ } & \widetilde{U}' \\
{\scriptstyle p_{(\Phi(X);\Phi(F_1)\circ\phi)}} \downarrow & & {\scriptstyle \Phi(p)} \downarrow & & \downarrow {\scriptstyle p'} \\
\Phi(X) & \xrightarrow{\ \Phi(F_1)\ } & \Phi(U) & \xrightarrow{\ \phi\ } & U'
\end{array}
$$

and then the corresponding left hand side square is a pull-back square. Together with the fact that Φ takes pull-back squares based on p to pull-back squares we obtain a canonical isomorphism

$$\iota : (\Phi(X); \Phi(F_1) \circ \phi) \to \Phi(X; F_1)$$

and we define:

$$\Phi_2(F_1, F_2) := (\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2))$$

We will need the following property of these maps below.

Lemma 4.1 *Let $f : X' \to X$ be a morphism and $d \in D_p(X, V)$, then one has*

$$D_{p'}(\Phi(f), \Phi(V))(\Phi_2(d)) = \Phi_2(D_p(f, V)(d))$$

Proof: Let $d = (F_1, F_2)$. Then

$$D_{p'}(\Phi(f), \Phi(V))(\Phi_2(d)) = D_{p'}(\Phi(f), \Phi(V))(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2)) =$$

$$(\Phi(f) \circ \Phi(F_1) \circ \phi, q' \circ \iota \circ \Phi(F_2))$$

where

$$\iota : (\Phi(X); \Phi(F_1) \circ \phi) \to \Phi(X; F_1)$$
$$q' : (\Phi(X'); \Phi(f) \circ \Phi(F_1) \circ \phi) \to (\Phi(X); \Phi(F_1) \circ \phi)$$

are the canonical morphisms and

$$\Phi_2(D_p(f, V)(F_1, F_2)) = \Phi_2(f \circ F_1, q \circ F_2) =$$

$$(\Phi(f \circ F_1) \circ \phi, \iota' \circ \Phi(q \circ F_2))$$

where

$$\iota' : (\Phi(X'); \Phi(f \circ F_1) \circ \phi) \to \Phi(X'; f \circ F_1)$$
$$q : (X'; f \circ F_1) \to (X; F_1)$$

are canonical morphisms. We have

$$\Phi(f) \circ \Phi(F_1) \circ \phi = \Phi(f \circ F_1) \circ \phi$$

and it remains to check that

$$q' \circ \iota \circ \Phi(F_2) = \iota' \circ \Phi(q \circ F_2)$$

or that $q' \circ \iota = \iota' \circ \Phi(q)$. The codomain of both morphisms is $\Phi(X; F_1)$ that by our assumption on Φ is a pull-back of p' and $\Phi(F_1) \circ \phi$. Therefore it is sufficient to verify that the compositions of these two morphisms with the projections to $\widetilde{U}'$ and $\Phi(X)$ coincide.

This is done by a direct computation from definitions.

Recall from [6, Construction 3.3] that for every Γ we have a canonical isomorphism

$$\psi_\Gamma : int'(H(\Gamma)) \to \Phi(int(\Gamma))$$

Lemma 4.2 *Let $(\Phi, \phi, \widetilde{\phi})$ be a functor between categories with universes. Then, with respect to the identifications (9) the maps defined by H are of the form:*

12

1. on Ob_1:

$$Hom(int(\Gamma), U) \overset{\Phi}{\to} Hom(\Phi(int(\Gamma)), \Phi(U)) \overset{\phi}{\to} Hom(\Phi(int(\Gamma)), U')$$

$$\overset{\psi_\Gamma}{\to} Hom(int(H(\Gamma)), U')$$

2. on $\widetilde{Ob}_1$:

$$Hom(int(\Gamma), \widetilde{U}) \overset{\Phi}{\to} Hom(\Phi(int(\Gamma)), \Phi(\widetilde{U})) \overset{\phi}{\to} Hom(\Phi(int(\Gamma)), \widetilde{U}')$$

$$\overset{\psi_\Gamma}{\to} Hom(int(H(\Gamma)), \widetilde{U}')$$

3. on Ob_2:

$$D_p(int(\Gamma), U) \overset{\Phi_2}{\to} D_p(\Phi(int(\Gamma)), \Phi(U)) \overset{\phi}{\to} D_p(\Phi(int(\Gamma)), U') \overset{\psi_\Gamma}{\to} D_p(int(H(\Gamma)), U')$$

4. on $\widetilde{Ob}_2$

$$D_p(int(\Gamma), \widetilde{U}) \overset{\Phi_2}{\to} D_p(\Phi(int(\Gamma)), \Phi(\widetilde{U})) \overset{\phi}{\to} D_p(\Phi(int(\Gamma)), \widetilde{U}') \overset{\psi_\Gamma}{\to} D_p(int(H(\Gamma)), \widetilde{U}')$$

Proof: It follows immediately from the construction of H given in [6].

Problem 4.3 *Assume that $\mathcal{C}$ and $\mathcal{C}'$ are locally cartesian closed categories with universes. For $(\Phi, \phi, \widetilde{\phi})$ as above and $V \in \mathcal{C}$ to construct a morphism*

$$\phi_{2,V} : \Phi(I_p(V)) \to I_{p'}(\Phi(V))$$

Construction 4.4 Let

$$\eta : D_p(X, V) \to Hom(X, I_p(V))$$
$$\eta' : D_{p'}(X', V') \to Hom(X', I_{p'}(V'))$$

be bijections from Construction 3.3. We define:

$$\phi_{2,V} := \eta'(\Phi_2(\eta^{-1}(Id_{I_p(V)})))$$

For $(\Phi, \phi, \widetilde{\phi})$ as above let us denote by

$$\phi_2 : \Phi(I_p(U)) \to I_{p'}(U')$$

the composition of $\phi_{2,U}$ with the morphism defined by $\phi : \Phi(U) \to U'$ and by

$$\widetilde{\phi}_2 : \Phi(I_p(\widetilde{U})) \to I_{p'}(\widetilde{U}')$$

the composition of $\phi_{2,\widetilde{U}}$ with the morphism defined by $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$.

The notion of a homomorphism of C-systems with (Π, λ)-structures used in the theorem below is defined in the obvious way.

Theorem 4.5 *Let $(\Phi, \phi, \widetilde{\phi})$ be as above and let $(P, \widetilde{P})$, $(P', \widetilde{P}')$ be as in Problem 3.6 for $\mathcal{C}$ and $\mathcal{C}'$ respectively.*

Assume that the squares

$$
\begin{array}{ccc}
\Phi(I_p(U)) & \xrightarrow{\phi_2} & I_{p'}(U') \\
{\scriptstyle \Phi(P)}\downarrow & & \downarrow{\scriptstyle P'} \\
\Phi(U) & \xrightarrow{\phi} & U
\end{array}
\tag{10}
$$

and

$$
\begin{array}{ccc}
\Phi(I_p(\widetilde{U})) & \xrightarrow{\widetilde{\phi}_2} & I_{p'}(\widetilde{U}') \\
{\scriptstyle \Phi(\widetilde{P})}\downarrow & & \downarrow{\scriptstyle \widetilde{P}'} \\
\Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}
\end{array}
\tag{11}
$$

commute. Then the homomorphism

$$
H(\Phi, \phi, \widetilde{\phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')
$$

is a homomorphism of C-systems with (Π, λ)-structures.

Proof: We will show that the square

$$
\begin{array}{ccc}
Ob_2(\Gamma) & \xrightarrow{H} & Ob_2(H(\Gamma)) \\
{\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi'} \\
Ob_1(\Gamma) & \xrightarrow{H} & Ob_1(\Gamma)
\end{array}
\tag{12}
$$

commutes. The proof of commutativity of a similar square for $\widetilde{Ob}$ and $\widetilde{P}$ is obtained by replacing ϕ with $\widetilde{\phi}$, ϕ_2 with $\widetilde{\phi}_2$ and the corresponding replacements of U with $\widetilde{U}$.

Consider the map A defined as the composition

$$
Hom(int(\Gamma), I_p(U)) \xrightarrow{\Phi} Hom(\Phi(int(\Gamma)), \Phi(I_p(U))) \xrightarrow{\phi_2} Hom(\Phi(int(\Gamma)), I_{p'}(U'))
$$

$$
\xrightarrow{\psi_\Gamma} Hom(int(H(\Gamma)), I_{p'}(U'))
$$

Since $\Pi = \eta \circ P$ and $\Pi' = \eta' \circ P'$ it is sufficient to show that the squares

$$
\begin{array}{ccc}
D_p(int(\Gamma), U) & \longrightarrow & D_{p'}(int(H(\Gamma)), U') \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\
Hom(int(\Gamma), I_p(U)) & \xrightarrow{A} & Hom(int(H(\Gamma)), I_{p'}(U')) \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\
Hom(int(\Gamma), U) & \longrightarrow & Hom(int(H(\Gamma)), U')
\end{array}
$$

where the top and bottom arrows are from Lemma 4.2, commute. The commutativity of the lower square follows immediately from the assumption that Φ is a functor and from the commutativity of (10).

To prove the commutativity of the upper square it is sufficient (in view of the naturality of η' in the first and second arguments) to prove commutativity of the diagram

$$
\begin{array}{ccccc}
D_p(int(\Gamma), U) & \xrightarrow{\Phi_2} & D_{p'}(\Phi(int(\Gamma)), \Phi(U)) & \xrightarrow{\eta'} & Hom(\Phi(int(\Gamma)), I_{p'}(\Phi(U))) \\
\eta \downarrow & & & & \uparrow{\phi_{2,U}} \\
Hom(int(\Gamma), I_p(U)) & = & Hom(int(\Gamma), I_p(U)) & \xrightarrow{\Phi} & Hom(\Phi(int(\Gamma)), \Phi(I_p(U)))
\end{array}
$$

The upper arrow is actually the composition with the morphism $\phi_{2,U} : \Phi(I_p(U)) \to I_{p'}(\Phi(U))$. Therefore we need to verify, for all $a \in D_p(int(\Gamma), U)$, the equation:

$$\Phi(\eta(a)) \circ \phi_{2,U} = \eta'(\Phi_2(a))$$

By definition of $\phi_{2,U}$ and contravariant functoriality of η' we have

$$\Phi(\eta(a)) \circ \phi_{2,U} = \Phi(\eta(a)) \circ \eta'(\Phi_2(\eta^{-1}(Id))) = \eta'(D_{p'}(\Phi(\eta(a)), \Phi(U))(\Phi_2(\eta^{-1}(Id))))$$

By Lemma 4.1 we further have:

$$\eta'(D_{p'}(\Phi(\eta(a)), \Phi(U))(\Phi_2(\eta^{-1}(Id)))) = \eta'(\Phi_2(D_p(\eta(a), U)(\eta^{-1}(Id))))$$

It remains to show that $D_p(\eta(a), U)(\eta^{-1}(Id)) = f$. Since η is a bijection we may apply it on both sides and by functoriality of η we get

$$\eta(D_p(\eta(a), U)(\eta^{-1}(Id))) = \eta(f) \circ \eta(\eta^{-1}(Id)) = \eta(f) \circ Id = \eta(f).$$

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