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Singular homology of abstract algebraic varieties

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1. Introduction

The main objective of the present paper is to construct a reasonable singular homology theory on the category of schemes of finite type over an arbitrary field k.

Let X be a CW-complex. The theorem of Dold and Thom [3] shows that $H_i(X, \mathbf{Z})$ coincide with π_i of the simplicial abelian group

$$Hom_{top}\left(\Delta^{\bullet}_{top}, \coprod_{d=0}^{\infty} S^d(X)\right)^+,$$

where $S^d(X)$ is the d-th symmetric power of X, Δ^i_{top} is the usual i-dimensional topological simplex and for any abelian monoid M we denote by M^+ the associated abelian group.

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Let now X be any scheme of finite type over a field k. Define $H_i^{sing}(X)$ as π_i of the simplicial abelian group

$$Hom\left(\Delta^\bullet, \coprod_{d=0}^{\infty} S^d(X)\right)^+,$$

where this time Δ^i denotes the linear subvariety of $\mathbf{A}^{i+1}$ given be the equation

$$t_0 + \ldots + t_i = 1.$$

Note further that for any normal connected scheme S the abelian monoid $Hom(S, \coprod_{d=0}^{\infty} S^d(X))$ coincides (after localization by $p =$ exponential characteristic of k) with the monoid of effective cycles in $S \times X$ each component of which is finite and surjective over S. Thus, (after localization by p) $H_*^{sing}(X)$ coincide with

$$\pi_*(C_*(X)) = H_*(C_*(X), d = \sum(-1)^i \partial_i),$$

where $C_*(X)$ is the simplicial abelian group generated by closed integral subschemes $Z \subset \Delta^i \times X$ such that the projection $Z \longrightarrow \Delta^i$ is finite and surjective.

This construction appeared first in the talk given by A.Suslin at the Luminy conference on algebraic K-theory (1987) as a part of a program towards the computation of higher algebraic K-theory of varieties over $\mathbf{C}$. Two main conjectures relating singular algebraic homology to other homology theories were made.

Conjecture 1.1. *If X is a variety over $\mathbf{C}$ then the evident homomorphism*

$$Hom\left(\Delta^\bullet, \coprod_{d=0}^{\infty} S^d(X)\right)^+ \longrightarrow Hom_{top}\left(\Delta^\bullet_{top}, \coprod_{d=0}^{\infty} S^d(X)\right)^+$$

induces isomorphisms

$$H_i^{sing}(X, \mathbf{Z}/n) \cong H_i(X(\mathbf{C}), \mathbf{Z}/n).$$

Conjecture 1.2. *Let X be a complete irreducible variety of dimension n over a field k. Then the evident embedding of complexes*

$$C_*(X) \longrightarrow \mathscr{Z}^n(X, *)$$

is a quasi-isomorphism, i.e. $H_i^{sing}(X) \cong CH^n(X, i)$, where $CH^*(X, *)$ are the higher Chow groups introduced by S.Bloch [1].

Here we prove that the first conjecture is true.[3] The method of the proof is close to the methods developed previously by A. Suslin [21], [22], O. Gabber (unpublished), H. Gillet and R. Thomason [7] and R. Jardine [12] for the computation of algebraic K-theory of algebraically closed fields. In section 4 we prove a rather general version of the rigidity theorem of Suslin, Gabber,

[3] It was proven after this paper was written that the second conjecture also holds, see [23].

Gillet and Thomason (theorem (4.4) below). The crucial role in the application of this rigidity theorem to the problem in question is played by h-topology of Voevodsky [24]. One of the main results of the paper (theorem (7.6) and corollary (7.7) below) shows that if F is any qfh-sheaf on the category of schemes of finite type over an algebraically closed field k of characteristic zero, then

$$H^*_{sing}(F, \mathbf{Z}/n) = Ext^*_{qfh}(F, \mathbf{Z}/n).$$

Applying this theorem to the sheaf $\mathbf{Z}_{qfh}(X)$ and using some topological computations parallel to theorem (7.6) we conclude the proof of conjecture 1 in section 7. The last section contains a generalization of the main theorem to cycles of positive dimension: we prove that for projective varieties over $\mathbf{C}$ algebraic Lawson homology with finite coefficients coincides with the usual one [14]. This answers a question raised by E. Friedlander (see also [4]).

For reader's convenience we have included an appendix where we discuss briefly the definition and some of the main properties of h-cohomology (for more details see ([24]). There we give in particular a corrected version of the proof of the comparison theorem relating h-, qfh- and etale cohomology (the proof of this theorem in ([24]) contains an error).

2. The relative Picard group

Let X be a scheme and let Y be a closed subscheme of X. Set $U = X - Y$ and denote by $i : Y \longrightarrow X$, $j : U \longrightarrow X$ the corresponding closed and open embeddings.

Denote by $Pic(X, Y)$ the group, whose elements are isomorphism classes of pairs of the form (L, ϕ), where L is a linear bundle on X and $\phi : L_{|Y} \cong \mathcal{O}_Y$ is a trivialization of L over Y and the operation is given by the tensor product. There is an evident exact sequence:

$$\Gamma(X, \mathcal{O}^*_X) \longrightarrow \Gamma(Y, \mathcal{O}^*_Y) \longrightarrow Pic(X, Y) \longrightarrow Pic(X) \longrightarrow Pic(Y) \qquad (1)$$

Let $\mathbf{G}_X$ be the sheaf of invertible functions on X. The homomorphism $\mathbf{G}_X \longrightarrow i_*(\mathbf{G}_Y)$ is surjective both in etale and Zariski topologies and we will denote by $\mathbf{G}_{X,Y}$ its kernel.

Lemma 2.1.

$$Pic(X, Y) = H^1_{Zar}(X, \mathbf{G}_{X,Y}) = H^1_{et}(X, \mathbf{G}_{X,Y}).$$

Proof. Any relative linear bundle (L, ϕ) is locally trivial in the Zariski topology and the automorphism group of the trivial relative bundle $(\mathcal{O}_U, Id : (\mathcal{O}_U)_{|Y} \cong \mathcal{O}_{Y \cap U})$ is canonically isomorphic to $\Gamma(U, \mathbf{G}_{X,Y})$. This implies the first formula. The second follows from the Hilbert theorem 90 ([17],[Ch.3]) and the five lemma.

Corollary 2.2. *Assume that n is invertible on X, then*

$$Pic(X,Y)/nPic(X,Y) \hookrightarrow H^2_{et}(X, j_!(\mu_n))$$

(here $j_!$ is the extension by zero functor – see [17]*).*

Proof. This follows from the lemma 2.1 in view of the exact sequence of etale sheaves:

$$0 \longrightarrow j_!(\mu_n) \longrightarrow \mathbf{G}_{X,Y} \xrightarrow{n} \mathbf{G}_{X,Y} \longrightarrow 0.$$

Assume that X is integral and denote by K the field of rational functions on X. A relative Cartier divisor on X is a Cartier divisor D such that $Supp(D) \cap Y = \emptyset$. If D is a relative divisor and $Z = Supp(D)$, then $\mathcal{O}_X(D)_{|X-Z} = \mathcal{O}_{X-Z}$. Thus D defines an element in $Pic(X,Y)$. Denoting the group of relative Cartier divisors by $Div(X,Y)$ we get a homomorphism $cl : Div(X,Y) \longrightarrow Pic(X,Y)$. The image of this homomorphism consists of pairs (L, ϕ) such that ϕ admits an extension to a trivialization of L over an open neighbourhood of Y. In particular this map is surjective provided that Y has an affine open neighbourhood. Finally set

$$\begin{aligned} G &= \{f \in K^* : f \in ker((\mathcal{O}_X^*)_y \longrightarrow (\mathcal{O}_Y^*)_y) \text{ for any } y \in Y\} \\ &= \{f \in K^* : f \text{ is defined and equal to one at each point of } Y\}. \end{aligned}$$

The following two lemmas are straightforward from definitions

Lemma 2.3. *Assume that Y has an affine open neighbourhood in X. Then the following sequence is exact:*

$$0 \longrightarrow \Gamma(X, \mathbf{G}_{X,Y}) \longrightarrow G \longrightarrow Div(X,Y) \longrightarrow Pic(X,Y) \longrightarrow 0.$$

Lemma 2.4. *Assume that U is normal and every closed integral subscheme of U of codimension one which is closed in X is a Cartier divisor (this happens for example when U is factorial) . Then $Div(X,Y)$ is a free abelian group generated by closed integral subschemes $T \subset U$ of codimension one which are closed in X.*

Finally we will mention the homotopy invariance of the relative Picard group.

Lemma 2.5. *Assume that X is normal and Y is reduced. Then $Pic(X,Y) \cong Pic(X \times \mathbf{A}^1, Y \times \mathbf{A}^1)$.*

Proof. It follows from the exact sequence (1), the five lemmas and the homotopy invariance of the Picard group for normal schemes.

3. Singular homology of curves

Let k be a field. All schemes considered in this section are assumed to be of finite type over k. Denote by Δ^n the closed subscheme of $\mathbf{A}^{n+1}$ given by the equation $t_0 + \cdots + t_n = 1$. There are obvious coface and codegeneracy maps

$$\Delta^i : \Delta^{n-1} \longrightarrow \Delta^n$$

$$\sigma^i : \Delta^{n+1} \longrightarrow \Delta^n$$

making $\Delta^\bullet$ a cosimplicial scheme.

Suppose, that S is irreducible and X is any scheme over S. Denote by $C_n(X/S)$ the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^n$ such that the projection $Z \longrightarrow S \times \Delta^n$ is finite and surjective. One verifies immediately that if Z is as above, then each component of $(\Delta^i)^{-1}(Z) \subset X \times \Delta^{n-1}$ is finite and surjective over $S \times \Delta^{n-1}$ and hence has the "correct" dimension, so that the cycle-theoretic inverse image $(\Delta^i)^*(Z)$ is well defined and lies in $C_{n-1}(X/S)$. This gives us the face operators

$$\partial_i = (\Delta^i)^* : C_n(X/S) \longrightarrow C_{n-1}(X/S).$$

In the same way one defines degeneracy operators

$$s_i = (\sigma^i)^* : C_n(X/S) \longrightarrow C_{n+1}(X/S),$$

thus making $C_*(X/S)$ a simplicial abelian group. The homotopy groups of this simplicial abelian group, i.e. the homology of the complex $(C_*(X/S), d = \sum(-1)^i \partial^i)$ will be denoted by $H_*^{sing}(X/S)$.

Assume that S is a normal affine scheme and X is a smooth affine irreducible scheme of relative dimension one over S. By a good compactification of X/S we'll mean an open embedding of schemes over S, $X \hookrightarrow \bar{X}$ such that the following conditions hold:

1. The scheme $Y = \bar{X} - X$ has an affine open neighbourhood in $\bar{X}$ (here and in the sequel we consider Y as a closed reduced subscheme of $\bar{X}$).
2. $\bar{X}$ is normal,
3. $\bar{X} \longrightarrow S$ is a proper morphism with fibers of dimension one.

Theorem 3.1. *Let $S, X, \bar{X}, Y$ be as above, then*

$$\begin{aligned} H_0^{sing}(X/S) &= Pic(\bar{X}, Y) \\ H_i^{sing}(X/S) &= 0 \text{ for } i > 0. \end{aligned}$$

Proof. Let $Z \subset X \times \Delta^n$ be a closed integral subscheme. The projection $Z \longrightarrow S \times \Delta^n$ is finite and surjective if and only if Z is closed and of codimension one in $\bar{X} \times \Delta^n$. Since S is normal and X is smooth over S any such Z is in fact a Cartier divisor on $\bar{X} \times \Delta^n$ (see [11, 21.14.3]). Thus $C_n(X/S) = Div(\bar{X} \times \Delta^n, Y \times \Delta^n)$ (see lemma 2.4). Let U be an an affine open neighbourhood of

Y in $\bar{X}$. Then $U \times \Delta^n$ is an affine open neighbourhood of $Y \times \Delta^n$ in $\bar{X} \times \Delta^n$. According to the lemma 2.3 we have an exact sequence of simplicial abelian groups:

$$0 \longrightarrow A_. \longrightarrow G_. \longrightarrow C_.(X/S) \longrightarrow Pic(\bar{X} \times \Delta^\bullet, Y \times \Delta^\bullet) \longrightarrow 0 \qquad (2)$$

where

$$A_n = \Gamma(\bar{X} \times \Delta^n, \mathbf{G}_{\bar{X} \times \Delta^n, Y \times \Delta^n})$$
$$G_n = \{f \in k(\bar{X} \times \Delta^n)^* : f \text{ is defined and equal to 1 at each point of } Y \times \Delta^n\}$$

and $k(\bar{X} \times \Delta^n)^*$ is the multiplicative group of the field of rational functions on the scheme $\bar{X} \times \Delta^n$.

Let us show that $A_n = 0$ for all n. Consider an element $f \in \Gamma(\bar{X} \times \Delta^n, \mathbf{G}_{\bar{X} \times \Delta^n, Y \times \Delta^n})$. The restriction of f to any geometric fiber of $\bar{X} \times \Delta^n \longrightarrow S \times \Delta^n$ is a regular function on a complete curve and hence has to be a constant on each component of the fiber. On the other hand each component contains at least one point of $Y \times \Delta^n$ where f is equal to one. This shows, that $f(x) = 1$ for any $x \in \bar{X} \times \Delta^n$ and hence $f = 1$ since $\bar{X} \times \Delta^n$ is reduced.

Consider now the simplicial abelian group $G_.$. Let us show, that it is acyclic, i.e. $\pi_*(G_.) = 0$. It suffices to check that for any $f \in G_n$ such that $\partial_i(f) = 1$ for $i = 0, \dots, n$ there exists $g \in G_{n+1}$ such, that $\partial_i(g) = 1$ for $i = 0, \dots, n$ and $\partial_{n+1}(g) = f$. Define functions $g_i \in G_{n+1}$ for $i = 0, \dots, n$ by means of the formula:

$$g_i = (t_{i+1} + \dots + t_{n+1}) + (t_0 + \dots + t_i)s_i(f)$$

These functions satisfy the following equations:

$$\partial_j(g_i) = \begin{cases} 1 & \text{if } j \neq i, i+1 \\ (t_i + \cdots + t_n) + (t_0 + \cdots + t_{i-1})f & \text{if } j = i \\ (t_{i+1} + \cdots + t_n) + (t_0 + \cdots + t_i)f & \text{if } j = i+1 \end{cases}$$

In particular $\partial_0(g_0) = 1$, $\partial_{n+1}(g_n) = f$. Finally we set

$$g = g_n g_{n-1}^{-1} g_{n-2} \dots g_0^{(-1)^n}.$$

This function obviously satisfies the conditions we need. Note now that the lemma (2.5) implies that

$$Pic(\bar{X} \times \Delta^n, Y \times \Delta^n) = Pic(\bar{X}, Y)$$

and we conclude from the exact sequence (2) that

$$H_i^{sing}(X/S) = \begin{cases} Pic(\bar{X}, Y) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Remark. A result similar to our theorem (3.1) was proved independently by S. Lichtenbaum [15].

4. The rigidity theorem

Denote by Sch/k the category of schemes of finite type over k. Let $\mathscr{F} : Sch/k \longrightarrow Ab$ be a presheaf of abelian groups on Sch/k, i.e. a contravariant functor from Sch/k to the category of abelian groups.

Definition 4.1. *A presheaf $\mathscr{F}$ is said to admit transfer maps if for any finite surjective morphism $p : X \longrightarrow S$ in Sch/k, where X is reduced and irreducible and S is irreducible and regular we are given a homomorphism:*

$$Tr_{X/S} : \mathscr{F}(X) \longrightarrow \mathscr{F}(S)$$

such that the following conditions hold:

1. *If p is an isomorphism than $Tr_{X/S} \circ p^* = Id$.*
2. *Let $V \subset S$ be a closed irreducible regular subscheme. Denote by W_i the components of $p^{-1}(V)$ and let*

$$n_i = \sum_{k=0}^{\infty} (-1)^k l_{\mathcal{O}_{X,W_i}} (Tor_k^{\mathcal{O}_{S,V}} (\mathcal{O}_{X,W_i}, k(V)))$$

be the multiplicity of W_i. Then the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{Tr_{X/S}} & \mathscr{F}(S) \\ \downarrow & & \downarrow \\ \oplus \mathscr{F}(W_i) & \xrightarrow{\sum n_i Tr_{W_i/V}} & \mathscr{F}(V). \end{array}$$

Let S be a regular irreducible scheme and let X be any scheme over S. We have a pairing

$$C_0(X/S) \otimes \mathscr{F}(X) \xrightarrow{\langle , \rangle} \mathscr{F}(S)$$

given by the formula

$$\langle Z, \phi \rangle = Tr_{Z/S}(\phi_{|Z})$$

(here $\phi \in \mathscr{F}(X)$ and Z is a reduced irreducible subscheme of X which is finite and surjective over S).

Proposition 4.2. *Assume in addition that $\mathscr{F}$ is homotopy invariant, i.e. $\mathscr{F}(X \times \mathbf{A}^1) = \mathscr{F}(X)$ for any $X \in Sch/k$ Then the above pairing factors through $H_0^{sing}(X/S) \otimes \mathscr{F}(X)$.*

Proof. Let $W \subset X \times \mathbf{A}^1$ be a closed integral subscheme such, that W is finite and surjective over $S \times \mathbf{A}^1$. Denote by W^i the cycle $W \cap (X \times i)$. We have to show, that $\langle W^1 - W^0, \phi \rangle = 0$ for any $\phi \in \mathscr{F}(X)$. Let ψ denote the inverse image of ϕ in $\mathscr{F}(X \times \mathbf{A}^1)$. Consider $\langle W, \psi \rangle \in \mathscr{F}(S \times \mathbf{A}^1)$. The homotopy invariance of $\mathscr{F}$ shows, that

$$\langle W, \psi \rangle_{|S \times 0} = \langle W, \psi \rangle_{|S \times 1}.$$

On the other hand the properties of the transfer map imply that

$$\langle W, \psi \rangle_{|S \times i} = \langle W^i, \psi_{|X \times i} \rangle = \langle W_i, \phi \rangle.$$

From now on we will assume that $\mathscr{F}$ is homotopy invariant. Extend $\mathscr{F}$ to pro-objects in Sch/k setting

$$\mathscr{F}(\lim_{\leftarrow} X_i) = \lim_{\rightarrow}(\mathscr{F}(X_i)).$$

Theorem 4.3. *Assume that $n\mathscr{F} = 0$ where n is prime to $char(k)$. Let S be a henselization of a smooth variety over k in a closed point. Let further X/S be a smooth affine scheme over S of relative dimension one and assume, that X admits a good compactification $\bar{X} \longrightarrow S$.*

If $g_1, g_2 : S \longrightarrow X$ are two sections which coincide in the closed point of S then $g_1^* = g_2^* : \mathscr{F}(X) \longrightarrow \mathscr{F}(S)$.

Proof. We may suppose, that X is irreducible. Let Y denote the closed reduced subscheme $\bar{X} - X$ of $\bar{X}$. The sections g_i are closed embeddings and we will denote by W_i the corresponding subschemes of X. The properties of the transfer map imply that $g_i^*(\phi) = \langle W_i, \phi \rangle$ for any $\phi \in \mathscr{F}(X)$. Thus our statement is equivalent to the fact that $W_1 - W_2 \in C_0(X/S)$ is in the kernel of the above pairing. In view of the proposition (4.2) it is sufficient to show, that image of $W_1 - W_2$ in $H_0^{sing}(X/S)/n$ is zero. According to (2.2) and (3.1) we have:

$$H_0^{sing}(X/S)/n = Pic(\bar{X}, Y)/n \hookrightarrow H_{et}^2(\bar{X}, j_!(\mu_n)).$$

The proper base change theorem implies that

$$H_{et}^2(\bar{X}, j_!(\mu_n)) = H_{et}^2(\bar{X}_0, (j_0)_!(\mu_n)),$$

where $\bar{X}_0$ is the closed fiber of $\bar{X}$ and $j_0 : X_0 \longrightarrow \bar{X}_0$ is the corresponding open embedding. The diagram

$$\begin{array}{ccc} Pic(\bar{X}, Y)/n & \longrightarrow & H_{et}^2(\bar{X}, j_!(\mu_n)) \\ \downarrow & & \downarrow \\ Pic(\bar{X}_0, Y_0)/n & \longrightarrow & H_{et}^2(\bar{X}_0, (j_0)_!(\mu_n)) \end{array}$$

commutes. This shows that $Pic(\bar{X}, Y)/n \hookrightarrow Pic(\bar{X}_0, Y_0)/n$. On the other hand the image of $W_1 - W_2$ in $Pic(\bar{X}_0, Y_0)$ is trivial since g_1, g_2 coincide in the closed point.

Theorem 4.4. *Let S_l be the henselization of $\mathbf{A}^l$ in 0. Then*

$$\mathscr{F}(S_l) = \mathscr{F}(Spec(k))$$

(*still assuming that $n\mathscr{F} = 0$, where n is prime to $char(k)$*) .

Proof. We will proceed by induction on l. For $l = 0$ the statement is evident. In the general case it is sufficient to show, that the homomorphism $\mathscr{F}(Spec(k)) \longrightarrow \mathscr{F}(S_l)$ is surjective. According to our definition $\mathscr{F}(S_l) = \lim_{\longrightarrow (X,x_0)} \mathscr{F}(X)$, where (X, x_0) runs through all etale neighbourhoods of $(\mathbf{A}^l, 0)$. So it is sufficient to show, that if $f : (X, x_0) \longrightarrow (\mathbf{A}^l, 0)$ is an affine etale neighbourhood, then the image of $\mathscr{F}(X)$ in $\mathscr{F}(S_l)$ lies in $\mathscr{F}(Spec(k))$. Let us show that there exists a linear projection $r : \mathbf{A}^l \longrightarrow \mathbf{A}^{l-1}$ such that the relative curve $p' : X' \longrightarrow S_l$ defined by the Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ p\downarrow & & \downarrow r\circ f \\ S_l & \longrightarrow \mathbf{A}^l \xrightarrow{f} & \mathbf{A}^{l-1} \end{array}$$

admits a good compactification. It clearly suffices to construct r such that $r \circ f : X \longrightarrow \mathbf{A}^{l-1}$ admits a good compactification. Let $Z \subset \mathbf{A}^l$ be a divisor such that $f : X \longrightarrow \mathbf{A}^l$ is finite over $\mathbf{A}^l - Z$. There exists a linear projection $r : \mathbf{A}^l \longrightarrow \mathbf{A}^{l-1}$ such that $r_{|Z} : Z \longrightarrow \mathbf{A}^{l-1}$ is finite. We are going to show that any such r has the desired property. To simplify notations assume that r is the standard coordinate projection. The morphi sm $X \longrightarrow \mathbf{A}^l = \mathbf{A}^{l-1} \times \mathbf{A}^1 \hookrightarrow \mathbf{A}^{l-1} \times \mathbf{P}^1$ is quasifinite and according to the Zariski Main Theorem it may be factored in the form $X \hookrightarrow \bar{X} \longrightarrow \mathbf{A}^{l-1} \times \mathbf{P}^1$ where the first arrow is an open embedding and the second one is a finite morphism. Moreover we may assume $\bar{X}$ to be normal. Set $Y = \bar{X} - X$. One checks easily that the image of Y in $\mathbf{A}^{l-1} \times \mathbf{P}^1$ is contained in $Z \cup \mathbf{A}^{l-1} \times \infty$ and that the last scheme admits an open affine neibourhood in $\mathbf{A}^{l-1} \times \mathbf{P}^1$.

Let now $s_1 : S_l \longrightarrow X'$ be the section defined by the canonical morphism $S_l \longrightarrow X$ and $s_2 : S_l \longrightarrow X'$ be the section defined by the composition

$$S_l \longrightarrow S_{l-1} \longrightarrow S_l \longrightarrow X.$$

They obviously coincide in the closed point of S_l and to finish the proof it is sufficient now to apply theorem 4.3.

Theorem 4.5. *Assume that k is an algebraically closed field of characteristic zero. Assume further that $\mathcal{F}$ is a homotopy invariant presheaf on Sch/k equipped with transfer maps. Denote by $\mathcal{F}_h^{\sim}$ (resp. $\mathcal{F}_{qfh}^{\sim}, \mathcal{F}_{et}^{\sim}$) the sheaf associated with $\mathcal{F}$ in the h-topology (resp. in the qfh-topology, etale topology). Then for any $n > 0$ we have canonical isomorphisms:*

$$Ext_{et}^*(\mathcal{F}_{et}^{\sim}, \mathbf{Z}/n) = Ext_{qfh}^*(\mathcal{F}_{qfh}^{\sim}, \mathbf{Z}/n) = Ext_h^*(\mathcal{F}_h^{\sim}, \mathbf{Z}/n)$$

$$= Ext_{Ab}^*(\mathcal{F}(Spec(k)), \mathbf{Z}/n).$$

Proof. The first two isomorphisms follow from the corollary (10.10). To prove the last one denote $\mathcal{F}(Spec(k))$ by $\mathcal{F}_0$. We will use the same notation $\mathcal{F}_0$ for the corresponding constant presheaf and constant h-sheaf. Consider the natural monomorphism of presheaves $\mathcal{F}_0 \longrightarrow \mathcal{F}$ and denote by $\mathcal{F}'$ its cokernel. Applying theorem (4.4) to the presheaves $\mathcal{F}/n$ and ${}_n\mathcal{F}$ we conclude that

$$(\mathcal{F}'/n)(X_x^h) = 0 = ({}_n\mathcal{F}')(X_x^h)$$

for any smooth scheme X and any closed point x on X (here X_x^h denotes the henselization of X in x). Let now X be any object of Sch/k. Resolving singularities we construct a proper surjective morphism $Y \longrightarrow X$ with Y smooth.

If $u \in (\mathscr{F}'/n)(X)$ is any section, then vanishing of the groups $(\mathscr{F}'/n)(Y_y^h)$ implies that we can find an etale covering $\{Y_i \longrightarrow Y\}_{i=1}^k$ such that $u_{|Y_i} = 0$. Since $\{Y_i \longrightarrow X\}$ is an h-covering of X this shows, that $(\mathscr{F}'/n)_h^{\sim} = 0$. The same argument shows that $({}_n\mathscr{F}')_h^{\sim} = 0$. Thus the sheaf $(\mathscr{F}')_h^{\sim}$ is uniquely n-divisible and hence $Ext_h^*((\mathscr{F}')_h^{\sim}, \mathbf{Z}/n) = 0$. Now the exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{F}_h^{\sim} \longrightarrow (\mathscr{F}')_h^{\sim} \longrightarrow 0$$

shows, that

$$Ext_h^*(\mathscr{F}_h^{\sim}, \mathbf{Z}/n) = Ext_h^*(\mathscr{F}_0, \mathbf{Z}/n) = Ext_{Ab}^*(\mathscr{F}_0, \mathbf{Z}/n).$$

Remark. Resolution of singularities for varieties over a field of characteristic $p > 0$ was announced recently by M. Spivakovsky. This would imply that one can drop the restriction on $char(k)$ in theorem (4.5) (demanding instead that n is prime to $char(k)$).

5. qfh-sheaves

The main objective of this section is to show that any qfh-sheaf admits transfer maps satisfying the conditions of definition 4.1. Let $\mathscr{F}$ be a qfh-sheaf of abelian groups and let $X \longrightarrow S$ be a finite surjective morphism such that X is integral and S is normal. Let further Y be the normalization of X in a normal extension of $k(S)$ which contains $k(X)$. Denote the Galois group $Gal(k(Y)/k(S))$ by G. One checks immediately (lemma 5.16) that $\mathscr{F}(S)$ coincides with $\mathscr{F}(Y)^G$. This enables us to define the transfer map

$$Tr_{X/S} : \mathscr{F}(X) \longrightarrow \mathscr{F}(S)$$

by the formula

$$Tr_{X/S}(a) = [F(X) : F(S)]_{insep} \cdot \sum_{q \in Hom_S(Y,X)} q^*(a)$$

(here $[F(X) : F(S)]_{insep}$ is the inseparable degree of the field extension $F(S) \subset F(X)$).

The main difficulty is to check that these transfer maps are compatible with the base change homomorphisms, i.e. satisfy the property 4.1(2). To do so we have to interpret the multiplicities n_i appearing in 4.1(2) in similar terms. Actually we do it in more general situation. Suppose that S'/S is any integral scheme over S. We ascribe using elementary Galois theory, certain multiplicities n_i to all irreducible components X_i' of $X' = X \times_S S'$ which, in the case of a regular scheme S coincide with multiplicities given by the Tor-formula and (in the case when S' is also normal) make the following diagram commutative

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \coprod \mathscr{F}(X_i') \\ {\scriptstyle Tr_{X/S}} \downarrow & & \downarrow {\scriptstyle \sum n_i Tr_{X_i'/S'}} \\ \mathscr{F}(S) & \longrightarrow & \mathscr{F}(S'). \end{array}$$

The explicite formula for multiplicities n_i shows that they belong to $\mathbf{Z}[1/p]$ where p is the exponential characteristic of k. Moreover $n_i \in \mathbf{Z}$ if S is a regular scheme. However, A.S. Merkurjev produced recently an example showing that for an arbitrary normal S the multiplicities need not be integers.

In the next section this construction is used to show that if X is any scheme of finite type over k and S is a normal irreducible scheme then $C_0(X \times S/S) \otimes \mathbf{Z}[1/p]$ coincides with the group of sections of the free qfh-sheaf of $\mathbf{Z}[1/p]$-modules $\mathbf{Z}[1/p]_{qfh}(X)$ generated by X.

Fix a field k. All schemes considered in this section are assumed to be of finite type over k. We denote by p the exponential characteristic of k.

We start with some elementary results concerning finite morphisms.

Lemma 5.1. *Let $q : X \longrightarrow S$ be a finite morpism and let G be a finite group acting on X/S. The following conditions are equivalent*

1. *For any $s \in S$ the action of G on $q^{-1}(s)$ is transitive, for any $x \in q^{-1}(s)$ the field extension $k(x)/k(s)$ is normal and the natural homomorphism $G_x = Stab_G(x) \longrightarrow Gal(k(x)/k(s))$ is surjective.*
2. *For any algebraically closed field Ω and any geometric point $\zeta : Spec\ \Omega \longrightarrow S$ the action of G on the geometric fiber $X_\zeta = X \times_S Spec\ \Omega$ is transitive.*

Corollary 5.2. *Assume that the equivalent conditions of lemma 5.1 are fulfilled and assume further that S is irreducible and q is surjective. Then G acts transitively on the set of irreducible components of X and each component of X maps surjectively onto S.*

Lemma 5.3. *Let $X \longrightarrow S$ and $Y \longrightarrow S$ be finite surjective morphisms of integral schemes. Then the canonical map $Hom_S(Y,X) \longrightarrow Hom_{k(S)}(k(X),k(Y))$ is injective. If further the scheme Y is normal then this map is bijective.*

Corollary 5.4. *In assumptions and notations of lemma 5.3 the canonical homomorphism $Aut_S(Y) \longrightarrow Gal(k(Y)/k(S))$ is injective. It is bijective if the scheme Y is normal.*

Definition 5.5. *Let $q : Y \longrightarrow S$ be a finite surjective morphism of integral schemes. We'll be saying that q is a pseudo-Galois covering (or that Y is a pseudo-Galois covering of S) if the field extension $k(Y)/k(S)$ is normal and the natural homomorphism $Aut_S(Y) \longrightarrow Gal(k(Y)/k(S))$ is an isomorphism.*

We derive immediately from lemma 5.3 the following result:

Lemma 5.6. *Let S be an integral scheme and let Y be the normalization of S in a finite normal extension of the field $k(S)$. Then $Y \longrightarrow S$ is a pseudo-Galois covering.*

Lemma 5.7. *Let $Y \longrightarrow S$ be a pseudo-Galois covering of an integral scheme S. Further, let X be an integral scheme and $X \longrightarrow S$ be a finite surjective morphism. Assume that $Hom_S(Y,X) \neq \emptyset$. Then the canonical map $Hom_S(Y,X) \longrightarrow Hom_{k(S)}(k(X),k(Y))$ is bijective and, in particular, $|Hom_S(Y,X)| = [k(X) : k(S)]_{sep}$.*

Proof. Denote the group $Aut_S(Y) = Gal(k(Y)/k(S))$ by G. The map $Hom_S(Y,X) \longrightarrow Hom_{k(S)}(k(X),k(Y))$ is G-equivariant and the action of G on $Hom_{k(S)}(k(X),k(Y))$ is transitive according to elementary Galois theory. This shows that the above map is surjective. Injectivity of this map follows from lemma 5.3

Corollary 5.8. *In conditions and notations of lemma 5.7, let Z be another integral scheme and let $Z \longrightarrow Y$ be a finite surjective morphism. Then the induced map $Hom_S(Y,X) \longrightarrow Hom_S(Z,X)$ is bijective.*

Proof. This follows immediately from the commutativity of the following diagram.

$$\begin{array}{ccc} Hom_S(Y,X) & \longrightarrow & Hom_S(Z,X) \\ \sim\downarrow & & \downarrow \\ Hom_{k(S)}(k(X),k(Y)) & \xrightarrow{\sim} & Hom_{k(S)}(k(X),k(Z)) \end{array}$$

Lemma 5.9. *Let $Y \longrightarrow S$ be a pseudo-Galois covering of a normal integral scheme S and let G be the group $Aut_S(Y) = Gal(k(Y)/k(S))$. Then the pair $(Y \longrightarrow S, G)$ satisfies the equivalent conditions of lemma 5.1.*

Proof. Let $\tilde{Y}$ denote the normalization of the scheme Y. Lemma 5.6 shows that $Aut_S(\tilde{Y}) = Gal(k(\tilde{Y})/k(S)) = Gal(k(Y)/k(S)) = G$. Thus we have a canonical action of G on $\tilde{Y}$ and the projection $\tilde{Y} \longrightarrow Y$ is G-equivariant. It is well-known (see [2, Ch.5, Section 2, n.3]) that the pair $(\tilde{Y} \longrightarrow S, G)$ satisfies the condition (1) of lemma 5.1 and and hence also satisfies the condition (2). Let now $\xi : Spec\ \Omega \longrightarrow S$ be any geometric point of S. Since the morphism $\tilde{Y}_\xi \longrightarrow Y_\xi$ is surjective and the action of G on $\tilde{Y}_\xi$ is transitive, we conclude that the action of G on Y_ξ is transitive as well.

Corollary 5.10. *In conditions and notations of* 5.9 *let S' be any integral scheme over S. Set $Y' = Y \times_S S'$, then*

(1) *The action of G on the set of irreducible components of Y' is transitive and each of these components maps surjectively onto S'.*

(2) *If Y'_0 is a component of Y' (considered as a closed integral subscheme of Y') then $Y'_0 \longrightarrow S'$ is a pseudo-Galois covering.*

Proof. Let $\xi : Spec\ \Omega \longrightarrow S'$ be a geometric point of S'. Lemma 5.9 shows that the action of G on $Y'_\xi = Y' \times_{S'} Spec\ \Omega = Y \times_S Spec\ \Omega$ is transitive, i.e., the pair $(Y' \longrightarrow S', G)$ satisfies the equivalent conditions of lemma 5.1. The first statement follows now from corollary 5.2. Furthermore validity of condition (1) of lemma 5.1 shows that $k(Y'_0)/k(S')$ is a finite normal extension and the homomorphism $Stab_G(Y'_0) \longrightarrow Gal(k(Y'_0)/k(S'))$ is surjective. Since this homomorphism factors through $Aut_{S'}(Y'_0) \longrightarrow Gal(k(Y'_0)/k(S'))$ we conclude that the last homomorphism is surjective as well.

Assume that $X \longrightarrow S$ is a finite surjective morphism from an integral scheme X to a normal integral scheme S. Choose a pseudo-Galois covering $Y \longrightarrow S$ such that $Hom_S(Y,X) \neq \emptyset$ (for example one can take Y to be the normalization of X in any finite normal extension $E/k(S)$ containing $k(X)$). Let further S' be any integral scheme over S. Set $X' = X \times_S S'$ and denote by X_i' the irreducible components of X' (considered as closed integral subschemes of X'). Let further Y_0' be any irreducible component of $Y' = Y \times_S S'$. Any S-morphism $q : Y \longrightarrow X$ defines an S'-morphism $q' : Y' \longrightarrow X'$. The image under q' of Y_0' coincides with one of the closed subschemes X_i' of X'. In this way we get a canonical map

$$c : Hom_S(Y,X) \longrightarrow \coprod_i Hom_{S'}(Y_0', X_i').$$

Fix an S-morphism $q_0 : Y \longrightarrow X$ and for any i denote by $l(i)$ the number of irreducible components of Y' mapped onto X_i' by q_0'. Finally denote by l the total number of irreducible components of Y'.

Lemma 5.11. a) *For any i $Hom_{S'}(Y_0', X_i') \neq \emptyset$ (and in particular the morphism $X_i' \longrightarrow S'$ is surjective).*
b) *The number of elements in the fiber of the map c over any element of $Hom_{S'}(Y_0', X_i')$ is equal to*

$$\frac{[k(X) : k(S)]_{sep} \cdot l(i)}{[k(X_i') : k(S')]_{sep} \cdot l}$$

Proof. Set $G = Aut_S(Y)$, $H = Stab_G(Y_0')$, $V_i = c^{-1}(Hom_{S'}(Y_0', X_i')) = \{q : Y \longrightarrow X : q'(Y_0') = X_i'\}$. The morphism $q_0' : Y' \longrightarrow X'$ is surjective and hence $X_i' = q_0'(Y_1')$ for a certain component Y_1' of Y'. Corollary 5.10 shows that there exists $\sigma \in G$ such that $\sigma(Y_0') = Y_1'$. Thus $(q_0\sigma)'(Y_0') = X_i'$, i.e. $q_0\sigma \in V_i$ and hence $V_i \neq \emptyset$.

To prove the second statement note that the map $c_i = c|_{V_i} : V_i \longrightarrow Hom_{S'}(Y_0', X_i')$ is H-equivariant and the action of H on $Hom_{S'}(Y_0', X_i') = Hom_{k(S')}(k(X_i'), k(Y_0'))$ is transitive since H maps onto $Gal(k(Y_0')/k(S'))$. This shows that all fibers of c_i have the same cardinality equal to $|V_i|/[k(X_i') : k(S')]_{sep}$. To compute $|V_i|$ consider the G-equivariant map $G \longrightarrow Hom_S(Y,X)$ $(\sigma \mapsto q_0\sigma)$. Since the action of G on $Hom_S(Y,X) = Hom_{k(S)}(k(X), k(Y))$ is transitive we see that all fibers of this map consist of $|G|/[k(X) : k(S)]_{sep}$ elements. The inverse image of V_i in G is equal to $\{\sigma \in G : q_0'(\sigma(Y_0')) = X_i'\}$, i.e. consists of those $\sigma \in G$ which take Y_0' to one of $l(i)$ components of Y' lying over X_i'. To compute the number of such σ use the same trick once again. The map $G \longrightarrow \{\text{components of } Y'\}$ $(\sigma \mapsto \sigma(Y_0'))$ is G-equivariant and the action of G on the set of components is transitive. Hence all fibers of this map consist $|G|/l$ elements. Thus cardinality of the inverse image of V_i in G is equal to $|G| \cdot l(i)/l$ and hence

$$|V_i| = \frac{[k(X) : k(S)]_{sep} \cdot l(i)}{l}$$

Definition 5.12. *In the above notations set*

$$(n_i)_{sep} = |\textit{fiber of } c_i| = \frac{[k(X):k(S)]_{sep}\cdot l(i)}{[k(X_i'):k(S')]_{sep}\cdot l},$$

$$(n_i)_{insep} = \frac{[k(X):k(S)]_{insep}}{[k(X_i'):k(S')]_{insep}}$$

$$n_i = (n_i)_{sep}\cdot(n_i)_{insep} = \frac{[k(X):k(S)]\cdot l(i)}{[k(X_i'):k(S')]\cdot l}.$$

The number $n_i \in Z[1/p]$ is called the multiplicity of the component X_i'.

Lemma 5.13. *The number $(n_i)_{sep}$ and hence also n_i is independent of the choice of the pseudo-Galois covering $Y \longrightarrow S$ (such that $Hom_S(Y,X) \neq \emptyset$) and the component Y_0' of Y'.*

Proof. Let $Z \longrightarrow S$ be another pseudo-Galois covering such that $Hom_S(Z,X) \neq \emptyset$ and let Z_0' be a component of $Z' = Z \times_S S'$. Assume first that there exists an S-morphism $f: Z \longrightarrow Y$ such that $f'(Z_0') = Y_0'$. In this case our assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} Hom_S(Y,X) & \longrightarrow & \coprod_i Hom_{S'}(Y_0',X_i') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \coprod_i (f_0')^*} \\ Hom_S(Z,X) & \longrightarrow & \coprod_i Hom_{S'}(Z_0',X_i') \end{array}$$

and corollary 5.8. In the general case let T be the normalization of S in a finite normal extension of $k(S)$ containing both $k(Y)$ and $k(Z)$ and let T_0' be any component of $T' = T \times_S S'$. Lemma 5.3 shows that $Hom_S(T,Y) \neq \emptyset$, $Hom_S(T,Z) \neq \emptyset$. Moreover corollary 5.10 shows that there exist S-morphisms $f: T \longrightarrow Y$, $g: T \longrightarrow Z$ such that $f'(T_0') = Y_0'$, $g'(T_0') = Z_0'$. Thus multiplicities computed using (Y, Y_0'), (T, T_0') or (Z, Z_0') are all the same.

The following property of multiplicities is evident from definition 5.12.

Lemma 5.14. *In the above notations and assumptions the following formula holds*

$$\sum_i n_i \cdot [k(X_i'):k(S')] = [k(X):k(S)].$$

Let $S'' \longrightarrow S' \longrightarrow S$ be a tower of morphisms of integral schemes. Assume that the schemes S and S' are normal. Let further $X \longrightarrow S$ be a finite surjective morphism from an integral scheme X to S. Set $X' = X \times_S S'$, $X'' = X \times_S S'' = X' \times_{S'} S''$ and denote by X_i' (resp. X_j'') the irreducible components of X' (resp. X''). Denote by n_i (resp. n_j) the multiplicity of X_i' (resp. X_j''). Finally denote by n^i_j the multiplicity of X''_j considered as a component of $X'_i \times_{S'} S''$ (setting $n^i_j = 0$ if X''_j is not a component of $X'_i \times_{S'} S''$).

Lemma 5.15 (*Transitivity of multiplicities*). *In the above notations we have the following formulae*:

$$(n_j)_{sep} = \sum_i (n^i_j)_{sep} \cdot (n_i)_{sep}; \qquad n_j = \sum_i n^i_j \cdot n_i.$$

Proof. It suffices to establish the first formula. Choose a pseudo-Galois covering $Y \longrightarrow S$ such that $Hom_S(Y,X) \neq \emptyset$, let Y'_0 be a component of $Y' = Y \times_S S'$ and let Y''_0 be a component of $Y'_0 \times_{S'} S''$. Note that Y'_0 is a pseudo-Galois covering of S' and $Hom_{S'}(Y'_0, X'_i) \neq \emptyset$ for all i (see 5.10 5.11). Thus the scheme Y'_0 may be used to compute the multiplicities n^i_j. Our statement follows now from the commutativity of the diagram

$$\begin{array}{ccc} Hom_S(Y,X) & \longrightarrow & \coprod_i Hom_{S'}(Y'_0, X'_i) \\ & \searrow \quad \swarrow & \\ & \coprod_j Hom_{S''}(Y''_0, X''_j) & \end{array}$$

Lemma 5.16. *Assume that $q : Y \longrightarrow S$ is a pseudo-Galois covering of an integral normal scheme S. Set $G = Aut_S(Y)$. For any qfh-sheaf of sets $\mathscr{F}$ the map $q^* : \mathscr{F}(S) \longrightarrow \mathscr{F}(Y)$ gives a bijection $\mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}(Y)^G$.*

Proof. Consider the morphism $f : Y \times G \longrightarrow Y \times_S Y$ such that the restriction of f to $Y \times \sigma$ coincides with $(1_Y, \sigma)$. The morphism f is finite and it follows easily from 5.9 and 5.1 that it's also surjective. Thus f is a qfh-covering and hence $f^* : \mathscr{F}(Y \times_S Y) \longrightarrow \mathscr{F}(Y \times G) = \mathscr{F}(Y) \times G$ is an injective map. Since $q : Y \longrightarrow S$ is a qfh-covering we have an exact sequence

$$\mathscr{F}(S) \longrightarrow \mathscr{F}(Y) \rightrightarrows \mathscr{F}(Y \times_S Y)$$

We conclude immediately from the previous discussion that the kernel of the pair of maps (pr_1^*, pr_2^*) coincides with $\mathscr{F}(Y)^G$.

Assume that $X \longrightarrow S$ is a finite surjective morphism from an integral scheme X to an integral normal scheme S. Choose a pseudo-Galois covering $Y \longrightarrow S$ such that $Hom_S(Y,X) \neq \emptyset$ and set $G = Aut_S(Y)$. Let finally $\mathscr{F}$ be a qfh-sheaf of abelian groups. The homomorphism

$$[k(X) : k(S)]_{insep} \cdot \left(\sum_{q \in Hom_S(Y,X)} q^* \right) : \mathscr{F}(X) \longrightarrow \mathscr{F}(Y)$$

is G-invariant and according to lemma 5.16 defines a homomorphism $\mathscr{F}(X) \to \mathscr{F}(S)$ which we denote $Tr_{X/S}$. The same procedure as in the proof of lemma 5.13 shows easily that this homomorphism is independent of the choice of Y.

Assume now that $S' \longrightarrow S$ is a morphism of integral normal schemes. Set $X' = X \times_S S'$ and denote by X'_i and n_i the components of X' and their multiplicities. Assume finally that $\mathscr{F}$ is a qfh-sheaf of $\mathbf{Z}[1/p]$-modules.

Lemma 5.17. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{Tr_{X/S}} & \mathcal{F}(S) \\ \downarrow & & \downarrow \\ \coprod_i \mathcal{F}(X_i') & \xrightarrow{\sum n_i Tr_{X_i'/S'}} & \mathcal{F}(S') \end{array}$$

Proof. Choose a pseudo-Galois covering $Y \longrightarrow S$ such that $Hom_S(Y,X) \neq \emptyset$. Let Y_0' be any irreducible component of $Y' = Y \times_S S'$ and denote the canonical morphism $Y_0' \longrightarrow Y$ by f. For any $a \in \mathcal{F}(X)$ we have:

$$\begin{aligned}(Tr_{X/S}(a)|_{S'})|_{Y_0'} &= (Tr_{X/S}(a)|_Y)|_{Y_0'} \\ &= f^*\Big([k(X):k(S)]_{insep} \cdot \sum_{q \in Hom_S(Y,X)} q^*(a)\Big) \\ &= [k(X):k(S)]_{insep} \cdot \sum_i \Big((n_i)_{sep} \cdot \sum_{r \in Hom_S(Y_0',X_i')} r^*(a|_{X_i'})\Big) \\ &= \sum_i (n_i) \cdot [k(X_i'):k(S')]_{insep} \cdot \sum_{r \in Hom_S(Y_0',X_i')} r^*(a|_{X_i'}) \\ &= \sum_i (n_i) \cdot Tr_{X_i'/S'}(a|_{X_i'})|_{Y_0'}\end{aligned}$$

Since the restriction homomorphism $\mathcal{F}(S') \longrightarrow \mathcal{F}(Y_0')$ is injective we conclude that

$$Tr_{X/S}(a)|_{S'} = \sum n_i \cdot Tr_{X_i'/S'}(a|_{X_i'}).$$

Lemma 5.18. *Let $f : X \longrightarrow S$ be a finite surjective morphism from an integral scheme X to a regular scheme S. Let further $S' \hookrightarrow S$ be a closed integral subscheme of S. Set $X' = X \times_S S'$ and denote by X_i' and n_i the components of X' and their multiplicities. Then n_i coincides with the multiplicity with which $[X_i']$ appears in the cycle $f^*([S'])$ ((computed for example using the Tor-formula – see* [19]) .

Proof. Choose a pseudo-Galois covering $g : Y \longrightarrow S$, an S-morphism $q : Y \longrightarrow X$ and a component Y_0' of $Y' = Y \times_X X'$. All components of Y' appear in the cycle $g^*([S'])$ with the same multiplicity (see 5.10). Using the projection formula (see [19, Chap. 5]) $g_*g^*([S']) = [k(Y):k(S)] \cdot [S']$ we see that this common multiplicity is equal to $\frac{[k(Y):k(S)]}{[k(Y_0'):k(S')] \cdot l}$, where l is the total number of components of Y'. Using now the projection formula $q_*g^*([S']) = [k(Y):k(X)] \cdot f^*([S'])$ we see that multiplicity with which $[X_i']$ appears in $f^*([S'])$ is equal to

$$\frac{1}{[k(Y):k(X)]} \cdot l(i) \cdot \frac{[k(Y_0'):k(S')]}{[k(X_i'):k(S')]} \cdot \frac{[k(Y):k(S)]}{[k(Y_0'):k(S')] \cdot l} = \frac{[k(X):k(S)] \cdot l(i)}{[k(X_i'):k(S')] \cdot l} = n_i$$

The previous lemma shows, in particular, that the multiplicities n_i are integers provided that the scheme S is regular. In general however multiplicities need not be integers as one can see from the following example due to A.S. Merkurjev:

Assume that *char* $k = p > 0$ and let $a, b \in k^*$ be two elements independent modulo $(k^*)^p$. Set $A = k[T_0, T_1, T_2]/(aT_0^p + bT_1^p - T_2^p)$, $S = Spec\ A$. One verifies easily that A is an integrally closed domain so that S is a normal integral scheme. Let X be the normalization of S in the field $k(S)(\gamma)$, where $\gamma^p = b/a$. It's easy to check that $X = Spec\ k(\alpha, \beta)[T_1, T_2](\alpha^p = a, \beta^p = b)$ and the image of T_0 in $k(\alpha, \beta)[T_1, T_2]$ is equal to $\alpha^{-1}T_2 - \gamma T_1$. Set finally $S' = Spec\ k$ and let $S' \to S$ be the only singular point of S ($T_0 = T_1 = T_2 = 0$). The scheme $X' = Spec\ k(\alpha, \beta)$ is irreducible and the multiplicity of the only component of X' is given by the formula

$$n = (n)_{insep} = \frac{[k(X) : k(S)]_{insep}}{[k(X') : k(S')]_{insep}} = \frac{p}{p^2} = p^{-1}$$

6. Free qfh-sheaves

We preserve the notations and assumptions of the previous section. Let Sch/k denote the category of all schemes of finite type over k. In this section we consider Sch/k as a site in qfh-topology and we denote by $(Sch/k)^\wedge$ (resp. $(Sch/k)^\sim$) the category of presheaves of sets on Sch/k (resp. the category of qfh-scheaves of sets on Sch/k). For any presheaf of sets $\mathcal{F}$ we denote by $\mathcal{F}^\sim$ the associated qfh-sheaf of sets.

Let Nor/k denote the full subcategory of Sch/k consisting of integral normal schemes and let $(Nor/k)^\wedge$ denote the category of presheaves of sets on Nor/k. The restriction functor $(Sch/k)^\wedge \longrightarrow (Nor/k)^\wedge$ has a right adjoint $e : (Nor/k)^\wedge \longrightarrow (Sch/k)^\wedge$ defined by the following formula, Here h_T is a presheaf on the category Nor/k given by the formula $h_T(S) = Hom_{Sch/k}(S, T)$ and Nor/T is a category whose objects are integral normal schemes S together with a morphism $S \longrightarrow T$

Definition 6.1. *We'll be saying that $\mathcal{H} \in (Nor/k)^\wedge$ is a qfh-sheaf of sets on the category Nor/k if it satisfies the following two properties*

(6.1.1) *For any $S \in Nor/k$ $\mathcal{H}$ defines a sheaf in the Zariski topology of S.*

(6.1.2) *If S' is the normalization of S in a finite normal extension of the field $k(S)$ with Galois group G (so that $Aut_S(S') = G$ – see 5.4) then the canonical map $\mathcal{H}(S) \longrightarrow \mathcal{H}(S')^G$ is bijective.*

We denote by $(Nor/k)^\sim$ the category of qfh-sheaves of sets on Nor/k.

Theorem 6.2. *The restriction functor defines an equivalence of categories $(Sch/k)^\sim \longrightarrow (Nor/k)^\sim$. The quasiinverse equivalence is defined by the functor e.*

Proof. Note first of all that if $\mathcal{F} \in (Sch/k)^\sim$ is a qfh-sheaf of sets then its restriction to the category Nor/k is again a qfh-sheaf of sets (see lemma 5.16). Note further that for any $\mathcal{H} \in (Nor/k)^\wedge$ the restriction of the presheaf $\mathcal{H}^e \in (Sch/k)^\wedge$ to Nor/k coincides with $\mathcal{H}$ (Ioneda lemma).

Lemma 6.3. *Assume that $\mathcal{F} \in (Sch/k)^{\sim}$ and denote by $\mathcal{H}$ the restriction of $\mathcal{F}$ to the category Nor/k. Then the adjunction map $\mathcal{F} \longrightarrow \mathcal{H}^e$ is an isomorphism.*

Proof. Let $T \in Sch/k$ be any scheme. Denote by T_i the irreducible components of T (considered as closed integral subschemes of T) and denote by $\tilde{T}_i$ the normalization of T_i. The family $\{\tilde{T}_i \longrightarrow T\}_i$ is a qfh-covering of T and hence the map $\mathcal{F}(T) \longrightarrow \prod_i \mathcal{F}(\tilde{T}_i)$ is injective. This implies that the map $\mathcal{F}(T) \longrightarrow \mathcal{H}^e(T) = \varprojlim_{Nor/T} \mathcal{F}(S)$ is injective as well. Let further $\phi = \{\phi_S\}_{S \longrightarrow T} \in \varprojlim_{Nor/T} \mathcal{F}(S)$ be any element of $\mathcal{H}^e(T)$

For any pair of indices i, j the images of

$$(\phi_{\tilde{T}_i})|_{\tilde{T}_i \times_T \tilde{T}_j}, (\phi_{\tilde{T}_j})|_{\tilde{T}_i \times_T \tilde{T}_j} \in \mathcal{F}(\tilde{T}_i \times_T \tilde{T}_j)$$

in $\mathcal{H}^e(\tilde{T}_i \times_T \tilde{T}_j)$ are both equal to $\phi|_{\tilde{T}_i \times_T \tilde{T}_j}$

Since the map $\mathcal{F}(\tilde{T}_i \times_T \tilde{T}_j) \longrightarrow \mathcal{H}^e(\tilde{T}_i \times_T \tilde{T}_j)$ is injective we conclude that $(\phi_{\tilde{T}_i})|_{\tilde{T}_i \times_T \tilde{T}_j} = (\phi_{\tilde{T}_j})|_{\tilde{T}_i \times_T \tilde{T}_j}$. Since the family $\{\tilde{T}_i \longrightarrow T\}$ is a qfh-covering of T this shows that there exists a unique element $\psi \in \mathcal{F}(T)$ with the property $\psi|_{\tilde{T}_i} = \phi_{\tilde{T}_i}$ for each i. A direct inspection shows that moreover $\psi|_S = \phi_S$ for any normal scheme S over T and hence $\phi \in \mathcal{H}^e(T)$ is the image of $\psi \in \mathcal{F}(T)$, i.e. the map $\mathcal{F}(T) \longrightarrow \mathcal{H}^e(T)$ is surjective.

To conclude the proof of the theorem 6.2 it suffices now to establish the following lemma.

Lemma 6.4. *If $\mathcal{H} \in (Nor/k)^{\sim}$ then $\mathcal{H}^e \in (Sch/k)^{\sim}$.*

Proof. In view of lemma 6.3 it suffices to show that $\mathcal{H} \cong \mathcal{F}|_{Nor/k}$ for an appropriate $\mathcal{F} \in (Sch/k)^{\sim}$. Let $\mathcal{F}$ be the sheaf associated with the presheaf $\mathcal{H}^e$. The sheaf $\mathcal{F}$ may be constructed in two steps. On the first step one constructs a separated presheaf $\mathcal{F}_1$ setting $\mathcal{F}_1(T) = \mathcal{H}^e(T)/\sim$, where $\sim$ is the following equivalence relation: $\phi \sim \phi'$ iff there exists a qfh-covering $\{T_i \longrightarrow T\}$ such that $\phi|_{T_i} = \phi'|_{T_i}$. Finally one defines $\mathcal{F}(T)$ to be $\check{H}^0(T, \mathcal{F}_1) = \varinjlim_{\mathcal{V}} \check{H}^0(\mathcal{V}, \mathcal{F}_1)$, where $\mathcal{V}$ runs through the filtered ordered (by the relation "refinement") set of equivalence classes of qfh-coverings of T. Any qfh-covering of a normal scheme S admits a refinement of the form $\mathcal{V} = \{Y_i \longrightarrow S\}_{i \in I}$ where Y is the normalization of S in a finite normal extension of the field $k(S)$ and $\{Y_i\}_{i \in I}$ is a Zariski open covering of the scheme Y (see lemma 10.4). We conclude immediately from definition 6.1 that the map $\mathcal{H}(S) \longrightarrow \mathcal{H}(Y) \longrightarrow \prod \mathcal{H}(Y_i)$ is injective and hence $\mathcal{F}_1(S) = \mathcal{H}^e(S) = \mathcal{H}(S)$. Furthermore

$$\check{H}^0(\mathcal{V}, \mathcal{F}_1) = Ker\left(\prod_i \mathcal{F}_1(Y_i) \rightrightarrows \prod_{i,j} \mathcal{F}_1(Y_i \times_S Y_j)\right)$$

Denote by G the group $Gal(k(Y)/k(S)) = Aut_S(Y)$. One sees immediately that the family of closed embeddings $f_\sigma = (1, \sigma) : Y \longrightarrow Y \times_S Y$ $(\sigma \in G)$ is a qfh-covering of $Y \times_S Y$. Since $f_\sigma^{-1}(Y_i \times_S Y_j) = Y_i \cap \sigma^{-1}(Y_j)$ we conclude that the family $\{f_\sigma : Y_i \cap \sigma^{-1}(Y_j) \longrightarrow Y_i \times_S Y_j\}_{\sigma \in G}$ is a qfh-covering of $Y_i \times_S Y_j$ and hence $\mathscr{F}_1(Y_i \times_S Y_j) \hookrightarrow \prod_\sigma \mathscr{F}_1(Y_i \cap \sigma^{-1}(Y_j)) = \prod_\sigma \mathscr{H}(Y_i \cap \sigma^{-1}(Y_j))$. Thus to give an element in $\check{H}^0(\mathscr{V}, \mathscr{F}_1)$ is the same as to give a family $\phi_i \in \mathscr{F}_1(Y_i) = \mathscr{H}(Y_i)$ with the following property:

$$\phi_i|_{Y_i \cap \sigma^{-1}(Y_j)} = \sigma^*(\phi_j)|_{Y_i \cap \sigma^{-1}(Y_j)}$$

for any i, j and $\sigma \in G$. Taking σ to be the identity of G we see that $\phi_i|_{Y_i \cap Y_j} = \phi_j|_{Y_i \cap Y_j}$ and hence there exists a unique $\phi \in \mathscr{H}(Y)$ such that $\phi_i = \phi|_{Y_i}$. Furthermore varying σ we check easily that $\phi \in \mathscr{H}(Y)^G = \mathscr{H}(S)$. Thus $\check{H}^0(\mathscr{V}, \mathscr{F}_1) = \mathscr{H}(S)$ and hence the canonical map $\mathscr{H}(S) \longrightarrow \mathscr{F}(S) = \varinjlim_{\mathscr{V}} \check{H}^0(\mathscr{V}, \mathscr{F}_1)$ is a bijection.

Let $X \in Sch/k$ be any scheme. For a normal integral scheme S set $z_0^c(X)(S) = C_0(X \times S/S)[1/p] =$ {a free $\mathbf{Z}[1/p]$-module generated by closed integral subschemes $Z \subset X \times S$ for which the projection $p_2|_Z : Z \longrightarrow S$ is a finite surjective morphism}. Let $f : S' \longrightarrow S$ be a morphism in the category Nor/k. For Z as above set $Z' = Z \times_S S'$ and let Z_i', n_i be the components of Z' and their multiplicities. Lemma 5.11(a) shows that the projection $p_2|_{Z_i'} : Z_i' \longrightarrow S'$ is finite and surjective. Now we define a homomorphism $f^* : z_0^c(X)(S) \longrightarrow z_0^c(X)(S')$ by setting $f^*(Z) = \sum n_i Z_i'$. Lemma 5.15 implies that if $g : S'' \longrightarrow S'$ is another morphism then $(fg)^* = g^* f^*$. Thus we have made $z_0^c(X)$ into a presheaf of $\mathbf{Z}[1/p]$-modules on the category Nor/k. We want to show that actually $z_0^c(X)$ is a sheaf. Validity of (6.1.1) for $z_0^c(X)$ is straightforward. To prove that (6.1.2) also holds we need the following lemma.

Lemma 6.5. *Let S be a normal integral scheme and let S' be the normalization of S in a finite normal extension of the field $k(S)$ with the Galois group G. Let further $Z \longrightarrow S$ be a finite surjective morphism from an integral scheme Z to S. Set $Z' = Z \times_S S'$ and let Z_i', n_i $(i = 1, \ldots, l)$, be the components of Z' and their multiplicities. Then*

1. *G acts transitively on the set $\{Z_i'\}_{i=1}^l$*
2. $l = \frac{[k(S'):k(S)]_{sep}}{[k(Z_1'):k(Z)]_{sep}} = \frac{[k(Z):k(S)]_{sep}}{[k(Z_1'):k(S')]_{sep}}$
3. $n_i = \frac{[k(S'):k(S)]_{insep}}{[k(Z_1'):k(Z)]_{insep}} = \frac{[k(Z):k(S)]_{insep}}{[k(Z_1'):k(S')]_{insep}}$

Proof. The first statement was proved earlier (see corollary 5.10). Since the components Z_i' map surjectively onto S' we conclude that they are in one to one correspondence with components of $Spec\ k(S') \times_{S'} Z' = Spec\ k(S') \times_S Z = Spec\ (k(S') \otimes_{k(S)} k(Z))$. This makes evaluation of the number of components an easy exercise in the Galois theory, which we leave to the reader. Since the group G acts transitively on the set of components we conclude further that multiplicities n_i are all the same. Denoting their common value by n we derive from lemma 5.14 the following formula:

$$[k(S') : k(S)] = n \cdot l \cdot [k(Z_1') : k(Z)].$$

Corollary 6.6. $z_0^c(X) \in (Nor/k)^\sim$.

Proof. Let S be an integral normal scheme and let S' be the normalization of S in a finite normal extension of the field $k(S)$ with the Galois group G. Denote the canonical morphism $S' \longrightarrow S$ by f. Since f is a finite surjective morphism we have a well-defined direct image homomorphism $f_* : z_0^c(X)(S') \longrightarrow z_0^c(X)(S)$. Lemma 5.14 shows that the composition $f_* f^*$ coincides with the multiplication by $[k(S') : k(S)]$ and, in particular, f^* is injective. Note further that $z_0^c(X)(S')^G$ is generated by expressions of the form $\sum_{\sigma \in G/H} \sigma(V)$, where $V \subset X \times S'$ is a closed integral subscheme such that the projection $V \longrightarrow S'$ is finite and surjective and $H = Stab_G(V)$. Denote the image of V in $X \times S$ by Z. Then $Z \in z_0^c(X)(S)$ and lemma 6.5 shows that

$$\sum_{\sigma \in G/H} \sigma(V) = f^* \left(\frac{[k(V) : k(Z)]_{insep}}{[k(S') : k(S)]_{insep}} \cdot Z \right)$$

According to theorem 6.2 the sheaf $z_0^c(X) \in (Nor/k)^\sim$ extends uniquely to a qfh-sheaf on the category Sch/k. We'll use the same notation $z_0^c(X)$ for this extension. On the other hand we may consider the qfh-sheaf of $\mathbf{Z}[1/p]$-modules freely generated by X. More precisely $\mathbf{Z}[1/p]_{qfh}(X)$ is a qfh-sheaf associated with the presheaf $T \mapsto \mathbf{Z}[1/p](Hom_{Sch/k}(T,X))$.

Theorem 6.7. *Assume that the scheme X is separated. Then* $\mathbf{Z}[1/p]_{qfh}(X) = z_0^c(X)$.

Proof. Associating to a morphism $f : S \longrightarrow X$ its graph $\Gamma_f \in z_0^c(X)(S)$ we get a canonical homomorphism from the restriction of the presheaf $\{T \mapsto \mathbf{Z}[1/p](Hom_{Sch/k}(T,X))\}$ to the category Nor/k to $z_0^c(X) \in (Nor/k)^\sim$, which defines (since the functor e is right adjoint to the restriction functor) a homomorphism $\{T \mapsto \mathbf{Z}[1/p](Hom_{Sch/k}(T,X))\} \longrightarrow z_0^c(X)$ of presheaves on the category Sch/k. Finally, since $z_0^c(X)$ is a qfh-sheaf, we get a homomorphism of qfh-sheaves $\mathbf{Z}[1/p]_{qfh}(X) \longrightarrow z_0^c(X)$. Assume further that S is an integral normal scheme and $Z \subset X \times S$ is a closed integral subscheme finite and surjective over S. Associating to Z the element $Tr_{Z/S}(p_1|_Z) \in \mathbf{Z}[1/p]_{qfh}(X)(S)$ we get a homomorphism $z_0^c(X)(S) \longrightarrow \mathbf{Z}[1/p]_{qfh}(X)(S)$. Lemma 5.17 implies immediately that these homomorphisms are compatible with the base change. Thus we get, using once again theorem 6.2, a homomorphism of qfh-sheaves $z_0^c(X) \longrightarrow \mathbf{Z}[1/p]_{qfh}(X)$. Verification that the composition $\mathbf{Z}[1/p]_{qfh}(X) \longrightarrow z_0^c(X) \longrightarrow \mathbf{Z}[1/p]_{qfh}(X)$ coincides with the identity map is straightforward. Let's check that the composition $z_0^c(X) \longrightarrow \mathbf{Z}[1/p]_{qfh}(X) \longrightarrow z_0^c(X)$ is also the identity map. In view of theorem 6.2 we may work with integral normal schemes only. So let S be an integral normal scheme and let $Z \subset X \times S$ be a closed integral subscheme finite and surjective over S. Let Y be the normalization of S in a finite normal extension of the field $k(S)$ containing $k(Z)$ (so that $Hom_S(Y,Z) \neq \emptyset$). Denote the Galois group $Gal(k(Y)/k(S))$ by G. The image of Z under the above composition is equal to $Tr_{Z/S}(\Gamma_{p_1}) \in z_0^c(X)(S)$.

The pull-back of this section to Y is equal (according to the definition of the transfer maps) to

$$[k(Z):k(S)]_{insep} \cdot \sum_{q \in Hom_S(Y,Z)} \Gamma_{p_1 q}$$

Note further that identifying $Z \times_S Y$ with the closed subscheme of $X \times Y$ we identify $\Gamma_q \subset Z \times_S Y$ with $\Gamma_{p_1 q} \subset X \times Y$ and finally that the graphs Γ_q are precisely the components of $Z \times_S Y$. Thus lemma 6.5 shows that the above expression coincides with the pull-back of $Z \in z_0^c(X)(S)$ to Y. Since the homomorphism $z_0^c(X)(S) \longrightarrow z_0^c(X)(Y)$ is injective, we conclude that $Tr_{Z/S}(\Gamma_{p_1}) = Z \in z_0^c(X)(S)$.

Finally we want to establish a relation between cycles and morphisms to symmetric powers. Assume that $X \longrightarrow Y$ is a finite flat morphism of constant degree d. Thus $X = Spec\ \mathscr{A}$ where $\mathscr{A}$ is a locally free sheaf of $\mathcal{O}_Y$-algebras of constant rank d. Multiplication in $\mathscr{A}^{\otimes d}$ defines an $\mathcal{O}_Y$-bilinear pairing

$$(\mathscr{A}^{\otimes d})^{\Sigma_d} \otimes_{\mathcal{O}_Y} \wedge^d \mathscr{A} \longrightarrow \wedge^d \mathscr{A},$$

making $\wedge^d \mathscr{A}$ a module over an $\mathcal{O}_Y$-algebra $(\mathscr{A}^{\otimes d})^{\Sigma_d}$. In this way we get a homomorphism of $\mathcal{O}_Y$-algebras

$$(\mathscr{A}^{\otimes d})^{\Sigma_d} \longrightarrow End_{\mathcal{O}_Y}(\wedge^d \mathscr{A}) = \mathcal{O}_Y$$

and hence a canonical section for the projection $S^d(X/Y) \longrightarrow Y$. Assume now, that $X \longrightarrow Y$ is a finite surjective morphism, X is integral and Y is normal connected. Set $d = [k(X):k(Y)]$ and $X = Spec\ \mathscr{A}$. According to the above construction we get a canonical homomorphism of $k(Y)$-algebras $(k(X)^{\otimes d})^{\Sigma_d} \longrightarrow k(Y)$. For any $y \in Y$ the image of $(\mathscr{A}_y^{\otimes d})^{\Sigma_d}$ in $k(Y)$ is integral over $\mathcal{O}_Y$ and hence is contained in $\mathcal{O}_Y$. This shows that in this situation we still have a canonical homomorphism of $\mathcal{O}_Y$-algebras $(\mathscr{A}^{\otimes d})^{\Sigma_d} \longrightarrow \mathcal{O}_Y$ and hence a canonical section for the projection $S^d(X/Y) \longrightarrow Y$.

Let Z be a scheme of finite type over k such, that any finite subset of Z is contained in an affine open set, so that one can consider symmetric powers $S^d(Z)$ of Z. Let S be a normal connected scheme and let $X \subset Z \times S$ be a closed integral subscheme for which the projection $X \longrightarrow S$ is finite and surjective. Set $d = [k(X):k(S)]$. The subscheme X defines an S-morphism

$$S \longrightarrow S^d(X/S) \longrightarrow S^d(Z \times S/S) = S^d(Z) \times S/S$$

and hence a morphism $f_X : S \longrightarrow S^d(Z)$. In this way we get a canonical homomorphism of monoids

$$C_0^{eff}(Z \times S/S) \longrightarrow Hom\left(S, \coprod_{d=0}^{\infty} S^d(Z)\right)$$

where $C_0^{eff}(Z \times S/S) \subset C_0(Z \times S/S)$ is the submonoid of effective cycles. Localizing at p we finally get a homomorphism

$$z_0^c(Z)^{eff}(S) \longrightarrow Hom\left(S, \coprod_{d=0}^{\infty} S^d(Z)\right)[1/p].$$

Theorem 6.8. *The above homomorphism*

$$z_0^c(Z)^{eff}(S) \longrightarrow Hom\left(S, \coprod_{d=0}^{\infty} S^d(Z)\right)[1/p]$$

is an isomorphism for any normal connected S.

Proof. Set $Y = Z^d$. The group Σ_d acts admissibly on Y and $Y/\Sigma_d \cong S^d(Z)$. If Ω is an algebraically closed field and $\omega : Spec\ \Omega \longrightarrow S^d(Z)$ is any Ω-point then Σ_d acts transitively on the set of Ω-points of Y over ω. This implies that the natural morphism $Y \times \Sigma_d \longrightarrow Y \times_{S^d(Z)} Y$ is surjective (and finite) and hence defines a qfh-covering of $Y \times_{S^d(Z)} Y$. This shows that for any qfh-sheaf $\mathscr{F}$ one has $\mathscr{F}(S^d(Z)) = \mathscr{F}(Y)^{\Sigma_d}$.

Let $p_i : Y \longrightarrow Z$ be the projection on the i-th factor. The element

$$\sum_{i=1}^{d} p_i \in \Gamma(Y, \mathbf{Z}[1/p]_{qfh}(Z))$$

is Σ_d-invariant and hence defines a canonical element

$$u_d \in \Gamma(S^d(Z), \mathbf{Z}[1/p]_{qfh}(Z)).$$

Let now $f : S \longrightarrow S^d(Z)$ be any morphism. It defines an element $f^*(u_d) \in \Gamma(S, \mathbf{Z}[1/p]_{qfh}(Z)) = z_0^c(Z)(S)$. One verifies easily that $f^*(u_d) \in z_0^c(Z)^{eff}(S)$ and that the resulting map

$$Hom\left(S, \coprod_{d=0}^{\infty} S^d(Z)\right)[1/p] \longrightarrow z_0^c(Z)^{eff}(S)$$

is inverse to the one constructed before.

7. Singular cohomology of qfh-sheaves

In this section we consider Sch/k as a site in qfh-topology. We denote by $\mathscr{F}^{\sim}$ the qfh-sheaf associated with a presheaf $\mathscr{F}$.

Let $\mathscr{F}$ be a presheaf of abelian groups. Applying $\mathscr{F}$ to the cosimplicial scheme $\Delta^\bullet$ we get a simplicial abelian group $C_*(\mathscr{F})$. The homotopy groups of $C_*(\mathscr{F})$ coincide with the homology of the complex $(C_*(\mathscr{F}), d = \sum(-1)^i\partial_i)$ and will be denoted $H_*^{sing}(\mathscr{F})$. For any abelian group A we set further:

$$H_*^{sing}(\mathscr{F}, A) = H_*(C_*(\mathscr{F}) \overset{L}{\otimes} A)$$

$$H^*_{sing}(\mathscr{F}, A) = H^*(RHom(C_*(\mathscr{F}), A)).$$

Note that if $\mathscr{F}$ has no torsion then $C_*(\mathscr{F}) \overset{L}{\otimes} \mathbf{Z}/n = C_*(\mathscr{F}) \otimes \mathbf{Z}/n$ and $RHom(C_*(\mathscr{F}), \mathbf{Z}/n) = Hom(C_*(\mathscr{F}), \mathbf{Z}/n)$ for any $n > 0$.

Define a presheaf $\mathcal{F}_q$ by the formula $\mathcal{F}_q(X) = \mathcal{F}(X \times \Delta^q)$. Thus $\mathcal{F}_*$ is a simplicial presheaf of abelian groups and the complex $C_*(\mathcal{F})$ coincides with the complex of global sections of the corresponding complex of presheaves $\mathcal{F}_*$. Applying the functor of associated sheaf to this complex of presheaves we get a complex of sheaves $(\mathcal{F}_*)^{\sim}$. Note that $\mathcal{F}_q$ is a sheaf provided that $\mathcal{F}$ is, however $(\mathcal{F}_q)^{\sim}$ need not coincides with the sheaf $(\mathcal{F}^{\sim})_q$.

Suppose, that n is prime to $char(k)$ and consider the hypercohomology spectral sequences corresponding to the complex $(\mathcal{F}_*)^{\sim}$

$$I_1^{p,q} = Ext^p((\mathcal{F}_q)^{\sim}, \mathbf{Z}/n) \Rightarrow Ext^{p+q}((\mathcal{F}_*)^{\sim}, \mathbf{Z}/n)$$

$$II_2^{p,q} = Ext^p(H_q((\mathcal{F}_*)^{\sim}), \mathbf{Z}/n) \Rightarrow Ext^{p+q}((\mathcal{F}_*)^{\sim}, \mathbf{Z}/n).$$

The following theorem shows that the first spectral sequence degenerates.

Theorem 7.1. *For any $\mathcal{F}$ and any q the canonical homomorphism $\mathcal{F}^{\sim} \longrightarrow (\mathcal{F}_q)^{\sim}$ induces isomorphisms on $Ext^*(-, \mathbf{Z}/n)$.*

Proof. Using the fact that $\Delta^q \cong \mathbf{A}^q$ and proceeding by induction on q we reduce the general case to the case $q = 1$.

Lemma 7.2. *The natural homomorphism of sheaves $\mathbf{Z}(\mathbf{A}^1) \longrightarrow \mathbf{Z}$ induces isomorphisms*

$$\underline{Ext}^*(\mathbf{Z}, \mathbf{Z}/n) \longrightarrow \underline{Ext}^*(\mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n).$$

In other words $\underline{Hom}(\mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n) = \mathbf{Z}/n$ and $\underline{Ext}^i(\mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n) = 0$ for $i > 0$.

Proof. By definition the sheaf $\underline{Ext}^i(\mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n)$ is the sheaf associated with the presheaf

$$X \longrightarrow Ext^i_{Sch/X}(\mathbf{Z}(\mathbf{A}^1)_{|Sch/X}, \mathbf{Z}/n).$$

The sheaf $\mathbf{Z}(\mathbf{A}^1)_{|Sch/X}$ coincides with $\mathbf{Z}(\mathbf{A}^1_X/X)$ and hence we have

$$Ext^i_{Sch/X}(\mathbf{Z}(\mathbf{A}^1)_{|Sch/X}, \mathbf{Z}/n) = H^i(\mathbf{A}^1_X, \mathbf{Z}/n) = H^i(X, \mathbf{Z}/n).$$

It is clear that the associated sheaf is isomorphic to $\mathbf{Z}/n$ for $i = 0$ and is trivial for $i > 0$.

This lemma implies easily that for any sheaf of abelian groups G the natural homomorphism $G \otimes \mathbf{Z}(\mathbf{A}^1) \longrightarrow G$ induces isomorphisms

$$Ext^*(G, \mathbf{Z}/n) \longrightarrow Ext^*(G \otimes \mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n).$$

Consider the embeddings $i_0, i_1 : G \longrightarrow G \otimes \mathbf{Z}(\mathbf{A}^1)$ defined by the points $0, 1 \in \mathbf{A}^1$ respectively. These morphisms induce the same isomorphism

$$i_0^* = i_1^* : Ext^*(G \otimes \mathbf{Z}(\mathbf{A}^1), \mathbf{Z}/n) \longrightarrow Ext^*(G, \mathbf{Z}/n).$$

We will say that the sheaf G is contractible if there exists a homomorphism $\phi : G \otimes \mathbf{Z}(\mathbf{A}^1) \longrightarrow G$ such, that $\phi \circ i_0 = 0$ and $\phi \circ i_1 = Id_G$. We conclude immediately from previous remarks that for any contractible sheaf G we have $Ext^*(G, \mathbf{Z}/n) = 0$. To finish the proof it is sufficient to note that the kernel of the natural morphism of sheaves $(\mathcal{F}_1)^{\sim} \longrightarrow \mathcal{F}^{\sim}$ is contractible.

Corollary 7.3.

$$I_2^{p,q} = \begin{cases} Ext^p(\mathcal{F}^\sim, \mathbf{Z}/n) & q = 0 \\ 0 & q > 0 \end{cases}$$

and hence $Ext^p((\mathcal{F}_*)^\sim, \mathbf{Z}/n) = Ext^p(\mathcal{F}^\sim, \mathbf{Z}/n)$.

Now let us investigate the second spectral sequence. From now on we will assume that $\mathcal{F}$ is equipped with transfer maps $Tr_{X/S}$ satisfying the properties (4.1). The sheaf $H_q((\mathcal{F}_*)^\sim)$ is associated with the presheaf $X \longrightarrow H_q(\mathcal{F}(X \times \Delta^*))$. We will denote this presheaf by $\mathcal{H}_q$.

Lemma 7.4. *For any scheme X consider the embeddings $i_0, i_1 : X \longrightarrow X \times \mathbf{A}^1$ defined by the points $0, 1$ of $\mathbf{A}^1$. The induced homomorphisms of complexes*

$$i_0^*, i_1^* : \mathcal{F}(X \times \mathbf{A}^1 \times \Delta^\bullet) \longrightarrow \mathcal{F}(X \times \Delta^\bullet)$$

are homotopic.

Proof. One can use the usual topological homotopy operator (see [16], [Ch. 2, Sect. 8]). Define a homomorphism

$$s_p : \mathcal{F}(X \times \mathbf{A}^1 \times \Delta^p) \longrightarrow \mathcal{F}(X \times \Delta^{p+1})$$

by the formula

$$s_p = \sum_{i=0}^{p} (-1)^i (Id_X \times \psi_i)^*$$

where $\psi_i : \Delta^{p+1} \longrightarrow \Delta^p \times \mathbf{A}^1$ is the linear isomorphism taking v_j to $v_j \times 0$ if $j \leqq i$ or to $v_{j-1} \times 1$ if $j > i$ (here $v_j = (0, \ldots, 1, \ldots, 0)$ is the j-th vertex of Δ^{p+1} (resp. Δ^p)). A staraightforward computation shows that $sd + ds = i_1^* - i_0^*$.

The usual reasoning now proves the following:

Corollary 7.5. *The presheaf $\mathcal{H}_q$ is homotopy invariant.*

Assume that $f : X \longrightarrow S$ is a finite surjective map with X integral and S regular. For any i we have a homomorphism

$$Tr_{X \times \Delta^i / S \times \Delta^i} : \mathcal{F}(X \times \Delta^i) \longrightarrow \mathcal{F}(S \times \Delta^i).$$

The properties of transfer maps imply in particular that the resulting map

$$Tr_{X/S} : \mathcal{F}(X \times \Delta^*) \longrightarrow \mathcal{F}(S \times \Delta^*)$$

is a homomorphism of complexes and hence induces homomorphisms on homology groups

$$Tr_{X/S} : \mathcal{H}_q(X) \longrightarrow \mathcal{H}_q(S).$$

It is clear that these maps satisfy the conditions of definition (4.1). Assuming now that our base scheme is the spectrum of an algebraically closed field of characteristic zero we deduce from theorem (4.5) that

$$II_2^{p,q} = Ext^p_{Ab}(H_q^{sing}(\mathcal{F}), \mathbf{Z}/n).$$

Comparing this spectral sequence to the hyperhomology spectral sequence corresponding to the complex of abelian groups $C_*(\mathcal{F})$ we conclude that (identifying $C_*(\mathcal{F})$ with the corresponding complex of constant sheaves) the natural homomorphism $C_*(\mathcal{F}) \longrightarrow (\mathcal{F}_*)^\sim$ induces isomorphisms on $Ext^*(-, \mathbf{Z}/n)$. Thus we have proven the main result of this paper.

Theorem 7.6. *Let k be an algebraically closed field of characteristic zero. Let $\mathcal{F}$ be a presheaf on Sch/k which admits transfer maps. Then both arrows in the diagram*

$$C_*(\mathcal{F}) \longrightarrow (\mathcal{F}_*)^\sim \longleftarrow \mathcal{F}^\sim$$

induce isomorphisms on $Ext^*_{qfh}(-, \mathbf{Z}/n)$. In particular for any $n > 0$ we have canonical isomorphisms

$$H^*_{sing}(\mathcal{F}, \mathbf{Z}/n) = Ext^*_{qfh}(\mathcal{F}^\sim, \mathbf{Z}/n).$$

Corollary 7.7. *For any qfh-sheaf $\mathcal{F}$ both arrows in the diagram*

$$C_*(\mathcal{F}) \longrightarrow \mathcal{F}_* \longleftarrow \mathcal{F}$$

induce isomorphisms on $Ext_{qfh}(-, \mathbf{Z}/n) = Ext_{et}(-, \mathbf{Z}/n)$.

Proof. One has only to note that qfh-sheaves admit transfer maps according to results of Sect. 5 (and use the comparison theorem 10.10 for the equality of the *Ext*-groups).

Applying corollary (7.7) to the free qfh-sheaf $\mathbf{Z}(X)$ and using the theorem 6.7 we come to the following result.

Corollary 7.8. *Let X be a separated scheme of finite type over an algebraically closed field k of characteristic zero. Then*

$$H^*_{sing}(X, \mathbf{Z}/n) = H^*_{qfh}(X, \mathbf{Z}/n) = H^*_{et}(X, \mathbf{Z}/n).$$

Remark. Resolution of singularities for varieties over an algebraically closed field of characteristic $p > 0$ would imply that all results of this section are valid in positive characteristic as well (with n prime to p).

8. Singular homology of varieties over C

Denote by CW the category of topological spaces which admit a triangulation. Note that the product in CW is the usual product equipped with the compactly generated topology. We will be considering CW as a site with the Grothendieck topology defined by local homeomorphisms. To distinguish the usual topological simplices from the schemes Δ^i used above we will use the notation Δ^i_{top} for the topological simplices.

The spaces Δ^i_{top} form a cosimplicial topological space. If $\mathscr{F}$ is any presheaf of abelian groups on CW then we will denote by $C_*(\mathscr{F})$ the simplicial abelian group $\mathscr{F}(\Delta^*_{top})$ and for any abelian group A we will denote by $H^{sing}_*(\mathscr{F},A)$ (resp. $H^*_{sing}(\mathscr{F},A)$) the homology of the complex $C_*(\mathscr{F}) \overset{L}{\otimes} A$ (resp. $RHom(C_*(\mathscr{F}),A)$). For any presheaf $\mathscr{F}$ we will be also considering (the same as above) the presheaves $\mathscr{F}_q$ defined by the formula $\mathscr{F}_q(X) = \mathscr{F}(X \times \Delta^q_{top})$. Thus $\mathscr{F}_.$ is a simplicial presheaf of abelian groups and $C_*(\mathscr{F})$ coincides with the complex of global sections of the associated complex of sheaves. Our main theorem (7.6) has the following easy topological version.

Theorem 8.1. *For any presheaf $\mathscr{F}$ on CW and any abelian group A both arrows in the diagram*

$$C_*(\mathscr{F}) \longrightarrow (\mathscr{F}_*)^\sim \longleftarrow \mathscr{F}^\sim$$

induce isomorphisms on $Ext^*(-,A)$, so that

$$Ext^*(\mathscr{F}^\sim,A) = H^*_{sing}(\mathscr{F},A).$$

Proof. The proof of this theorem is strictly parallel to that of theorem (7.6). One considers the hypercohomology spectral sequences

$$I_1^{pq} = Ext^p((\mathscr{F}_q)^\sim,A) \Rightarrow Ext^{p+q}((\mathscr{F}_*)^\sim,A)$$

$$II_2^{pq} = Ext^p(H_q((\mathscr{F}_*)^\sim),A) \Rightarrow Ext^{p+q}((\mathscr{F}_*)^\sim,A).$$

Repeating the argument of (7.1) one shows that $Ext^*((\mathscr{F}_q)^\sim,A) = Ext^*(\mathscr{F}^\sim,A)$ and hence the first spectral sequence degenerates and provides isomorphisms $Ext^*(\mathscr{F}^\sim,A) \longrightarrow Ext^*((\mathscr{F}_*)^\sim,A)$. Furthermore the sheaf $H_q((\mathscr{F}_*)^\sim)$ is associated with the presheaf $\mathscr{H}_q$ of the form $\mathscr{H}_q(X) = H_q(\mathscr{F}(X \times \Delta^*_{top}))$. The presheaf $\mathscr{H}_q$ is homotopy invariant and hence $\mathscr{H}_q(X) = \mathscr{H}_q(pt) = H_q^{sing}(\mathscr{F})$ for any contractible space X. Since any object of CW is locally contractible this implies that the sheaf $H_q((\mathscr{F}_*)^\sim)$ is a constant sheaf equal to $H_q^{sing}(\mathscr{F})$. Thus $C_*(\mathscr{F}) \longrightarrow (\mathscr{F}_*)^\sim$ is a quasi-isomorphism and in particular induces isomorphisms on any Ext-groups.

For any abelian monoid M denote by M^+ the associated abelian group. If $M.$ is a simplicial abelian monoid then applying the functor $^+$ componentwise we get a simplicial abelian group $M.^+$.

Let X be an object of CW. Consider the abelian topological monoid $\coprod_{d=0}^\infty S^d(X)$ and the corresponding simplicial abelian monoid $Sin.(\coprod S^d(X))$. The following result is a variant of the Dold-Thom theorem [3].

Theorem 8.2. *The evident embedding*

$$\mathbf{Z}(Sin.(X)) \longrightarrow Sin.\left(\coprod_{d=0}^{\infty} S^d(X)\right)^{+}$$

is a weak equivalence of simplicial abelian groups, i.e.

$$\pi_*\left(Sin.\left(\coprod_{d=0}^{\infty} S^d(X)\right)^{+}\right) = \pi_*(\mathbf{Z}(Sin.(X))) = H_*(X, \mathbf{Z}).$$

Denote by $j : CW \longrightarrow (Sch/\mathbf{C})$ the canonical morphism of sites such that $j^{-1}(X) = X(\mathbf{C})$ for any object X of $Sch/\mathbf{C}$. The classical comparison theorem [9] states that $j_*(\mathbf{Z}/n) = \mathbf{Z}/n$, $R^q j_*(\mathbf{Z}/n) = 0$ for $q > 0$ and hence the natural map

$$Ext^*(\mathscr{F}, \mathbf{Z}/n) \longrightarrow Ext^*(j^*(\mathscr{F}), \mathbf{Z}/n)$$

is an isomorphism for any etale sheaf of abelian groups $\mathscr{F}$ on $Sch/\mathbf{C}$.

Let Z be an object of $Sch/\mathbf{C}$. Denote by $\mathscr{M}$ (resp. $\mathscr{M}_{top}$) the etale (resp. topological) sheaf represented by the ind-scheme $\coprod_{d=0}^{\infty} S^d(Z)$ (resp. by the ind-topological space $\coprod_{d=0}^{\infty} S^d(X(\mathbf{C}))$). Since j^* commutes with direct limits one sees immediately that $j^*(\mathscr{M}) = \mathscr{M}_{top}$. Let further $\mathscr{F}$ (resp. $\mathscr{F}_{top}$) denote the presheaf of abelian groups $X \mapsto \mathscr{M}(X)^+$ (resp. $X \mapsto \mathscr{M}_{top}(X)^+$). Let finally $\mathscr{M}^+$ (resp. $\mathscr{M}_{top}^+$) be the sheaf of abelian groups associated with the presheaf $\mathscr{F}$ (resp. $\mathscr{F}_{top}$). The homomorphism $\mathscr{M} \longrightarrow \mathscr{M}^+$ (resp. $\mathscr{M}_{top} \longrightarrow \mathscr{M}_{top}^+$) is universal for homomorphisms from $\mathscr{M}$ (resp. $\mathscr{M}_{top}$) to sheaves of abelian groups and the universal property of j^* shows immediately that $j^*(\mathscr{M}^+) = j^*(\mathscr{M})^+ = \mathscr{M}_{top}^+$. The canonical homomorphism of sheaves $M \longrightarrow j_*(\mathscr{M}_{top})$ together with an evident embedding of cosimplicial spaces $\Delta_{top}^{\bullet} \longrightarrow \Delta^{\bullet}(\mathbf{C})$ defines a homomorphism of simplicial abelian monoids

$$\mathscr{M}(\Delta^{\bullet}) \longrightarrow j_*(\mathscr{M}_{top})(\Delta^{\bullet}) = \mathscr{M}_{top}(\Delta^{\bullet}(\mathbf{C})) \longrightarrow \mathscr{M}_{top}(\Delta_{top}^{\bullet}).$$

Applying the functor $^+$ we get further a homomorphism of simplicial abelian groups (and hence of the corresponding complexes of abelian groups)

$$C_*(Z) = \mathscr{M}(\Delta^{\bullet})^+ \longrightarrow Sin.\left(\coprod_{d=0}^{\infty} S^d(Z(\mathbf{C}))\right)^{+}.$$

Thus for any $n \geqq 0$ we have the induced homomorphisms on homology and cohomology with $\mathbf{Z}/n$-coefficients

$$H_*^{sing}(Z, \mathbf{Z}/n) \longrightarrow H_*(Z(\mathbf{C}), \mathbf{Z}/n)$$

$$H^*(Z(\mathbf{C}), \mathbf{Z}/n) \longrightarrow H_{sing}^*(Z, \mathbf{Z}/n).$$

Theorem 8.3. *For any separated scheme $Z \in Sch/\mathbf{C}$ the above homomorphisms are isomorphisms.*

Proof. Note first of all that the presheaf $\mathscr{F}$ coincides (according to theorems (6.7) and (6.8)) with the qfh-sheaf $\mathbf{Z}_{qfh}(Z)$ and hence admits the transfer maps. The theorem (7.6) implies now that both arrows in the diagram

$$C_*(Z) = C_*(\mathscr{F}) \longrightarrow \mathscr{F}_* \longleftarrow \mathscr{F} = \mathscr{M}^+$$

induce isomorphisms on $Ext(-, \mathbf{Z}/n)$. Applying the functor j^* we get a diagram of sheaves on CW which has the same property. Consider finally the commutative diagram

$$\begin{array}{ccccccc} j^*(C_*(Z)) & \longrightarrow & j^*(\mathscr{F}_*) & \longleftarrow & j^*(\mathscr{M}^+) & = & \mathscr{M}^+_{top} \\ \downarrow & & \downarrow & & & & \downarrow = \\ C_*(\mathscr{F}_{top}) & \longrightarrow & (\mathscr{F}_{top})^\sim_* & \longleftarrow & (\mathscr{F}_{top})^\sim & = & \mathscr{M}^+_{top}. \end{array}$$

All horizontal arrows induce isomorphisms on $Ext^*(-, \mathbf{Z}/n)$. The right hand side vertical arrow is an isomorphism. Thus the left hand side vertical arrow induces isomorphisms on $Ext^*(-, \mathbf{Z}/n)$. In other words the homomorphism of complexes of abelian groups

$$C_*(Z) \longrightarrow Sin_* \left(\coprod_{d=0}^{\infty} S^d(Z(\mathbf{C})) \right)^+$$

induces isomorphism on cohomology and hence also on homology with finite coefficients.

9. Algebraic Lawson homology

Let $Z \subset \mathbf{P}^N$ be an integral projective scheme over $\mathbf{C}$. Denote by $C_{r,d}(Z)$ the (not nesessary connected) projective algebraic set of effective cycles of dimension r and degree d on Z [20]. We will be considering $C_{r,d}(Z)$ as a reduced projective scheme. The Chow monoid of effective r-cycles on Z is a disjoint union

$$C_r(Z) = \coprod_{d=0}^{\infty} C_{r,d}(Z)$$

provided with the operation determined by addition of cycles. Thus $C_r(Z)$ is an abelian topological monoid (to simplify notations we do not distinguish between a reduced scheme over $\mathbf{C}$ and the topological space of its $\mathbf{C}$-valued points).

The Lawson homology of Z are defined by means of the formula

$$L_rH_{2r+i}(Z) = \pi_i(C_r(Z)^+)$$

where for any topological monoid M we denote by M^+ its homotopy-theoretic group completion i.e $M^+ = \Omega B(M)$ (for its relation to the "naive" group completion as well as for a more detailed account on Lawson homology see [6]).

Let M be any abelian topological monoid. The canonical monoid map $|Sin.(M)| \longrightarrow M$ is a weak equivalence and hence induces an equivalence of group completions $|Sin.(M)|^+ \longrightarrow M^+$. On the other hand $|Sin.(M)|^+$ is canonically homotopy equivalent to $|Sin.(M)^+|$ where $Sin.(M)^+$ is the componentwise group completion of the simplicial abelian monoid $Sin.(M)$. Thus we get a weak equivalence $|Sin.(M)^+| \longrightarrow M^+$. This shows in particular that the groups $L_rH_{2r+*}(Z)$ coincide with the homotopy groups of a simplicial abelian group $Sin.(C_r(Z))^+$. One can consider also the algebraic version of this construction: consider the simplicial abelian monoid $Hom(\Delta, C_r(Z))$, apply to it the group completion functor and finally set

$$L_rH^{alg}_{2r+i}(Z) = \pi_i(Hom(\Delta^\bullet, C_r(Z))^+).$$

Since every algebraic morphism $\Delta^q \longrightarrow C_r(Z)$ defines by restriction a continious map $\Delta^q_{top} \longrightarrow C_r(Z)$ we get a homomorphism of simplicial abelian groups

$$Hom(\Delta^\bullet, C_r(Z))^+ \longrightarrow Sin.(C_r(Z))^+$$

and hence the induced maps from algebraic to topological Lawson homology with finite coefficients

$$L_rH^{alg}_{2r+i}(Z, \mathbf{Z}/n) \longrightarrow L_rH_{2r+i}(Z, \mathbf{Z}/n).$$

The simplicial abelian groups used to define $L_rH^{alg}_{2r+i}(Z)$ may be given a slightly different description. E. Friedlander and H. Lawson [5] proved that for any normal connected scheme S the abelian monoid $Hom(S, C_r(Z))$ coincides with the monoid of effective cycles on $S \times Z$ every component of which is equidimensional of relative dimension r over S. Thus $Hom(\Delta^q, C_r(Z))^+$ coincides with the group of cycles in $\Delta^q \times Z$ every component of which is equidimensional of relative dimension r over Δ^q.

Theorem 9.1. *For any Z and any $n > 0$ we have a canonical isomorphism*

$$L_rH^{alg}_{2r+i}(Z, \mathbf{Z}/n) \cong L_rH_{2r+i}(Z, \mathbf{Z}/n).$$

Proof. Let $\mathcal{M}$ (resp. $\mathcal{M}_{top}$) denote the etale (resp. topological) sheaf of abelian monoids represented by $C_r(Z)$. Let further $\mathcal{F}$ denote the presheaf of abelian groups $X \mapsto \mathcal{M}(X)^+$ and let $\mathcal{M}^+$ denote the sheaf associated with $\mathcal{F}$. It is clear that $j^*(\mathcal{M}) = \mathcal{M}_{top}$ and hence $j^*(\mathcal{M}^+) = \mathcal{M}^+_{top}$. Notice further that $\mathcal{M}$ (and hence also $\mathcal{F}$) admits transfer maps: suppose that we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & C_r(Z) \\ {\scriptstyle f}\downarrow & & \\ S & & \end{array}$$

with f finite and surjective, X integral and S normal. Set $d = [k(X) : k(S)]$. According to Sect. 6 f defines a morphism $S \longrightarrow S^d(X/S)$. Define $Tr_{X/S}(g)$ as the composition

$$S \longrightarrow S^d(X/S) \longrightarrow S^d(X) \xrightarrow{S^d(g)} S^d(C_r(Z)) \longrightarrow C_r(Z)$$

where the last morphism is defined by the monoid structure on $C_r(Z)$. It is easy to check that

$$Tr_{X/S} : Hom(X, C_r(Z)) \longrightarrow Hom(S, C_r(Z))$$

is a monoid homomorphism and further that it satisfies the conditions (4.1). Proceeding now exactly as in the proof of the theorem (8.3) we get a commutative diagram

$$\begin{array}{ccccc} C_*(\mathscr{F}) & \longrightarrow & j^*((\mathscr{F}_*)^\sim) & \longleftarrow & j^*(\mathscr{M}^+) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(\mathscr{F}_{top}) & \longrightarrow & (\mathscr{F}_{top})^\sim_* & \longleftarrow & \mathscr{M}^+_{top}. \end{array}$$

All the horizontal arrows and the right hand side vertical arrow induce isomorphisms on $Ext^*(-, \mathbf{Z}/n)$. Thus the homomorphism of complexes of abelian groups

$$Hom(\Delta^\bullet, C_r(Z))^+ = C_*(\mathscr{F}) \longrightarrow C_*(\mathscr{F}_{top}) = Sin.(C_r(Z))^+$$

induces isomorphisms on cohomology with finite coefficients and hence also on homology with finite coefficients.

10. Appendix: h-cohomology

A morphism of schemes $p : X \longrightarrow Y$ is called a topological epimorphism if the underlying Zariski topological space of Y is a quotient space of the underlying Zariski topological space of X (i.e. p is surjective and a subset A of Y is open if and only if $p^{-1}(A)$ is open in X), p is called a universal topological epimorphism if for any Z/Y the morphism $p_Z : X \times_Y Z \longrightarrow Z$ is a topological epimorphism.

An h-covering of a scheme X is a finite family of morphisms of finite type $\{p_i : X_i \longrightarrow X\}$ such that $\coprod p_i : \coprod X_i \longrightarrow X$ is a universal topological epimorphism.

A qfh-covering of a scheme X is an h-covering $\{p_i\}$ such that all the morphisms p_i are quasi-finite.

h-coverings (resp. qfh-coverings) define a pretopology on the category of schemes, h-topology (resp. qfh-topology) is the associated topology.

Using the theorem of Chevalley (see [8, p.24]) one proves easily the following lemma.

Lemma 10.1. *Let X be a normal connected scheme and let $\{f_i : X_i \longrightarrow X\}_{i \in I}$ be a finite family of quasi-finite morphisms of finite type. Assume that all X_i are irreducible and denote by J the set of those i for which X_i dominate X. The above family is a qfh-covering if and only if $X = \bigcup_{i \in J} f_i(X_i)$.*

Let S be a fixed base scheme (which will be assumed noetherian). Denote by Sch/S the category of schemes of finite type over S and consider Sch/S as a site in h- qfh- or etale topology. We have evident morphisms of sites

$$(Sch/S)_h \xrightarrow{\alpha} (Sch/S)_{qfh} \xrightarrow{\beta} (Sch/S)_{et}.$$

Theorem 10.2. *Let A be any abelian group. Denote by the same letter A the corresponding constant sheaf in Zariski topology (which happens to be a sheaf even in h-topology), then $\beta_*(A) = A$, $R^q\beta_*(A) = 0$ for $q > 0$.*

Proof. It suffices to show that if X is a strictly henselian scheme then $H^0_{qfh}(X,A) = A$ and $H^q_{qfh}(X,A) = 0$ for $q > 0$. The first formula is obvious. To prove the second one we proceed by induction on q. Let $u \in H^q_{qfh}(X,A)$ be any cohomology class. There exists a qfh-covering $\{Y_i \longrightarrow X\}_{i\in I}$ such that $u|_{Y_i} = 0$ for all i. The standard properties of quasifinite schemes over a henselian scheme (see [17], [Ch.1]) show that there exists $i \in I$ and a component Y of Y_i which is finite over X and maps surjectively onto X. Note that the scheme Y is also strictly henselian and the field extension $k(x) \subset k(y)$ is purely inseparable (here x and y are closed points of X and Y respectively). This implies immediately that all the schemes $Y \times_X \ldots \times_X Y$ are strictly henselian. Consider now the spectral sequence

$$E_1^{ij} = H^i_{qfh}(Y \times_X \ldots \times_X Y, A) \Rightarrow H^{i+j}_{qfh}(X,A)$$

The terms E_1^{0j} are all equal to A and the differential $d_1 : E_1^{0j} \longrightarrow E_1^{0,j+1}$ is either zero or identity map depending on parity of j. This shows that $E_2^{0j} = 0$ for $j > 0$. On the other hand $E_1^{ij} = 0$ for $0 < i < q$ in view of the induction hypothesis. Thus the edge homomotphism $H^q_{qfh}(X,A) \longrightarrow H^q_{qfh}(Y,A)$ is injective and hence $u = 0$.

Lemma 10.3. *Let X be a normal connected excellent scheme. Any qfh-covering of X admits a refinement of the form $\{V_i \longrightarrow V \longrightarrow X\}_{i\in I}$ where V is the normalization of X in a finite normal extension of its field of functions and $\{V_i \longrightarrow V\}$ is a Zariski open covering of V.*

Proof. Let $\{p_j : U_j \longrightarrow X\}_{j\in J}$ be a qfh-covering of X. Replacing each U_i by the family of its irreducible components we may assume that all U_j are integral. Using lemma (10.1) we may assume that U_j dominate X for any j. According to the Zariski main theorem each p_j admits a factorization of the form $U_j \hookrightarrow \bar{U}_j \xrightarrow{\bar{p}_j} X$, where $\bar{U}_j$ is integral, $U_j \hookrightarrow \bar{U}_j$ is an open embedding and $\bar{p}_j$ is finite and surjective. Let E be the composite of normal closures of the fields $k(\bar{U}_j)$ over $k(X)$. Let V denote the normalization of X in E and let $q : V \longrightarrow X$ be the canonical morphism. For each j the morphism q factors through $\bar{U}_j$ and we will denote by V_j the inverse image of U_j in V.

Finally set $G = Gal(E/k(X))$, $I = J \times G$ and for $i = (j, \sigma) \in I$ set $V_i = V_j^\sigma$. One checks easily (using lemma 5.9) that $\bigcup_{i \in I} V_i = V$ and that $\{V_i \longrightarrow X\}_{i \in I}$ is a refinement of the original covering $\{U_j \longrightarrow X\}_{j \in J}$.

An h-covering $\{p_i : U_i \longrightarrow X\}$ is said to be of normal form if the morphisms p_i admit a factorization of the form $p_i = s \circ f \circ in_i$, where $\{in_i : U_i \hookrightarrow U\}_i$ is a Zariski open covering of U, $f : U \longrightarrow Y$ is a finite surjective morphism and $s : Y \longrightarrow X$ is a blow up of a closed subscheme of X.

Lemma (10.3) and the "platification par eclatement" theorem of [18] imply easily:

Corollary 10.4. *Any h-covering of a reduced noetherian excellent scheme admits a refinement of normal form.*

Lemma 10.5. *Assume that k is a separably closed field and Y/k is a nonempty proper scheme over k. Then for any prime p the following sequence is exact*

$$0 \longrightarrow H_{et}^*(Spec(k), \mathbf{Z}/p) \longrightarrow H_{et}^*(Y, \mathbf{Z}/p) \longrightarrow H_{et}^*(Y \times_k Y, \mathbf{Z}/p) \longrightarrow \dots .$$

Proof. Denote the graded $\mathbf{Z}/p$-algebra $H_{et}^*(Y, \mathbf{Z}/p)$ by R. According to the Kunneth formula ([17], [Ch.6, Sect. 8]) we have $H_{et}^*(Y^k, \mathbf{Z}/p) = R^{\otimes k}$ so that the above sequence takes the form

$$0 \longrightarrow \mathbf{Z}/p \longrightarrow R \xrightarrow{d} R^{\otimes 2} \xrightarrow{d} \cdots$$

where

$$d(r_1 \otimes \ldots \otimes r_k) = \sum_{i=0}^{k} (-1)^i (r_1 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_k).$$

Choose a closed point $y \in Y$ and consider the augmentation

$$e : R = H_{et}^*(Y, \mathbf{Z}/p) \longrightarrow H_{et}^*(k(y), \mathbf{Z}/p) = \mathbf{Z}/p.$$

Finally define $s : R^{\otimes k} \longrightarrow R^{\otimes (k-1)}$ by means of the formula

$$s(r_1 \otimes \ldots \otimes r_k) = e(r_1)(r_2 \otimes \ldots \otimes r_k).$$

A straightforward verification shows that s is a contracting homotopy for the complex in question.

Corollary 10.6. *Assume that X is a strictly henselian scheme and $f : Y \longrightarrow X$ is a surjective proper morphism. Then for any prime p the following sequence is exact*

$$0 \longrightarrow H_{et}^*(X, \mathbf{Z}/p) \longrightarrow H_{et}^*(Y, \mathbf{Z}/p) \longrightarrow H_{et}^*(Y \times_X Y, \mathbf{Z}/p) \longrightarrow \cdots .$$

Proof. This follows immediately from the proper base change theorem ([17], [Ch. 6, Sect. 2]) and lemma (10.5).

Theorem 10.7. *Assume that S is excellent and $n > 0$. Then $(\beta\alpha)_*(\mathbf{Z}/n) = \mathbf{Z}/n$ and $R^q(\beta\alpha)_*(\mathbf{Z}/n) = 0$ for $q > 0$.*

Proof. We may assume that $n = p$ is a prime integer. The first formula is evident, to prove the second one we will use induction on q. Assume that $R^i(\beta\alpha)_*(\mathbf{Z}/n) = 0$ for $1 \leqq i \leqq -1$. The hypercohomology spectral sequence gives (for any $X \in ob(Sch/S)$) an exact sequence

$$0 \longrightarrow H^q_{et}(X, \mathbf{Z}/p) \longrightarrow H^q_h(X, \mathbf{Z}/p) \longrightarrow H^0_{et}(X, R^q(\beta\alpha)_*(\mathbf{Z}/p)).$$

This shows that $X \mapsto H^q_h(X, \mathbf{Z}/p)/H^q_{et}(X, \mathbf{Z}/p)$ is a separated presheaf in etale topology and implies:

Lemma 10.8. *Suppose that $u \in H^q_h(X, \mathbf{Z}/p)$ and there exists an etale covering $\{X_i \longrightarrow X\}$ such, that $u_{|X_i} \in H^q_{et}(X_i, \mathbf{Z}/p)$, then $u \in H^q_{et}(X, \mathbf{Z}/p)$.*

Proof. To prove that $R^q(\beta\alpha)_*(\mathbf{Z}/p) = 0$ it is sufficient to show that if X is an excellent strictly henselian noetherian scheme, then $H^q_h(X, \mathbf{Z}/p) = 0$. Fix an element $u \in H^q_h(X, \mathbf{Z}/p)$ and find (using (10.4)) a proper surjective morphism $Y \longrightarrow X$ and an open covering $Y = \cup Y_i$ such that $u_{|Y_i} = 0$. Lemma (10.8) shows that $u_{|Y} \in H^q_{et}(Y, \mathbf{Z}/p)$. Since

$$pr_1^*(u_{|Y}) - pr_2^*(u|Y) = 0 \in H^q_{et}(Y \times_X Y, \mathbf{Z}/p) \subset H^q_h(Y \times_X Y, \mathbf{Z}/p)$$

corollary (10.6) implies that $u_{|Y} = 0$. Finally consider the spectral sequence

$$E_1^{ij} = H^i_h(Y \times_X \cdots \times_X Y, \mathbf{Z}/p) \Rightarrow H^{i+j}_h(X, \mathbf{Z}/p).$$

Induction hypothesis and corollary (10.6) show that

$$E_2^{ij} = 0 \text{ for } 0 \leqq i \leqq q-1, (i,j) \neq (0,0).$$

This spectral sequence shows that $H^q_h(X, \mathbf{Z}/p) \hookrightarrow H^q_h(Y, \mathbf{Z}/p)$. Thus $u = 0$.

Corollary 10.9.

$$\alpha_*(\mathbf{Z}/n) = \mathbf{Z}/n$$

$$R^q\alpha_*(\mathbf{Z}/n) = 0 \textit{ for } q > 0.$$

Corollary 10.10. *Let $\mathscr{F}$ be an etale sheaf and let $\mathscr{G}$ be a qfh-sheaf, then*

$$Ext^*_{et}(\mathscr{F}, \mathbf{Z}/n) = Ext^*_{qfh}(\beta^*\mathscr{F}, \mathbf{Z}/n)$$

$$Ext^*_{qfh}(\mathscr{G}, \mathbf{Z}/n) = Ext^*_h(\alpha^*\mathscr{G}, \mathbf{Z}/n).$$

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