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Motivic Cohomology Groups Are Isomorphic to Higher Chow Groups in Any Characteristic

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In this short paper we show that the motivic cohomology groups defined in [3] are isomorphic to the motivic cohomology groups defined in [1] for smooth schemes over any field. In view of [1, Proposition 12.1] this implies that motivic cohomology groups of [3] are isomorphic to higher Chow groups. This fact was previously known only under the resolution of singularities assumption. The new element in the proof is Proposition 4.

The motivic complex $\mathbf{Z}(q)$ of weight q was defined in [3] as $C_*(\mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge q}))[-q]$. In [1, Section 8] Friedlander and Suslin defined complexes, which we will denote $\mathbf{Z}_{tr}^{FS}(q)$, as $C_*(z_{equi}(\mathbf{A}^q, 0))[-2q]$ where $z_{equi}(X, 0)$ is the sheaf of equidimensional cycles on X of relative dimension zero. In this paper we prove the following result.

Theorem 1. For any field k, the complexes of sheaves with transfers $\mathbf{Z}(q)$ and $\mathbf{Z}^{FS}(q)$ on Sm/k are quasi-isomorphic in the Zariski topology. □

Corollary 2. For any field k, any smooth scheme X over k and any $p, q \in \mathbf{Z}$, there is a natural isomorphism

$$H^{p,q}(X, \mathbf{Z}) = CH^q(X, 2q - p) \tag{1}$$

and the same holds for the motivic cohomology and higher Chow groups with coefficients. □

Proof. The hypercohomology groups with coefficients in $\mathbf{Z}_{tr}^{FS}(q)$ are shown in [1, Proposition 12.1] to coincide with the higher Chow groups for smooth varieties over all fields.

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Motivic cohomology groups are defined in [3, Definition 3.1] as hypercohomology groups with coefficients in $\mathbf{Z}(q)$. Therefore, Theorem 1 implies Corollary 2. ■

To prove the theorem we have to show that for any smooth scheme X over k and any point x of X the complexes of abelian groups $\mathbf{Z}(q)(\mathrm{Spec}(\mathcal{O}_{X,x}))$ and $\mathbf{Z}^{FS}(q)(\mathrm{Spec}(\mathcal{O}_{X,x}))$ are quasi-isomorphic. Let k_0 be the subfield of constants in k. Then there exists a smooth variety X_0 over k_0 (possibly of dimension greater than the dimension of X) and a point x_0 on X_0 such that the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X_0,x_0}$ are isomorphic. Therefore, we may always assume that $k = k_0$ and in particular that k is perfect.

Since both complexes are complexes of presheaves with transfers with homotopy invariant cohomology presheaves it is sufficient to show that they are quasi-isomorphic in the Nisnevich topology. Consider the triangulated category of motives $DM = DM_-^{\mathrm{eff}}(k)$ defined in [5]. For our purposes it will be convenient to think of it as of the localization of the derived category of complexes of sheaves with transfers by $\mathbf{A}^1$-contractible objects. Since the complexes we consider are $\mathbf{A}^1$-local it is sufficient to show that they are isomorphic in DM. Since for any sheaf with transfers F the natural morphism $F \to C_*(F)$ is an isomorphism in DM it is sufficient to show that $\mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n})[n] \cong z_{equi}(\mathbf{A}^n, 0)$ in DM. We construct two isomorphisms. The first one is of the form

$$\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \longrightarrow \mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n})[n] \tag{2}$$

and the second one of the form

$$\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \longrightarrow z_{equi}(\mathbf{A}^n, 0). \tag{3}$$

The following construction of (2) is well known. For a family of open embeddings $\mathcal{U} = \{U_i \to X\}_{i=1,\ldots,n}$, let $S(\mathcal{U})$ denote the complex of sheaves with transfers

$$0 \longrightarrow \mathbf{Z}_{tr}\left(\cap_{i=1}^{i=n} U_i\right) \longrightarrow \cdots \longrightarrow \oplus_{i=1}^{i=n} \mathbf{Z}_{tr}(U_i) \longrightarrow 0 \tag{4}$$

with the sum of $\mathbf{Z}_{tr}(U_i)$ placed in degree zero. We have an obvious map $S(\mathcal{U}) \to \mathbf{Z}_{tr}(X)$. The following lemma is a version of [5, Proposition 3.1.3].

Lemma 3. If $\mathcal{U} = \{U_i \to X\}$ is a Zariski covering of X, then the morphism $S(\mathcal{U}) \to \mathbf{Z}_{tr}(X)$ is a quasi-isomorphism in the Nisnevich topology. □

Let $\mathcal{U}_n$ be the standard covering of $\mathbf{P}^n$ by $n+1$ copies of $\mathbf{A}^n$. Let further $\mathcal{V}_n$ be the family of maps $\{U_i \to \mathbf{A}^n\}$ where

$$U_i = \{(x_1, \ldots, x_i) : x_i \neq 0\}. \tag{5}$$

The embedding $\mathbf{P}^{n-1} \to \mathbf{P}^n$ defines a morphism $S(\mathcal{U}_{n-1}) \to S(\mathcal{U}_n)$ and the complementary embedding $\mathbf{A}^n \to \mathbf{P}^n$ defines a morphism $S(\mathcal{V}_n) \to S(\mathcal{U}_n)$. By Lemma 3 the cokernel of the first morphism represents in DM the object $\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1})$. The complex $S(\mathcal{V}_n)$ is the nth tensor power of the complex $\mathbf{Z}_{tr}(\mathbf{A}^1 - \{0\}) \to \mathbf{Z}_{tr}(\mathbf{A}^1)$ and therefore it represents in DM the object

$$\mathbf{Z}_{tr}(\mathbf{G}_m^{\wedge n})[n] = \left((\tilde{\mathbf{Z}}_{tr}(\mathbf{A}^1 - \{0\}; 1))[1]\right)^{\otimes n}. \tag{6}$$

It remains to show that the map

$$S(\mathcal{V}_n) \longrightarrow S(\mathcal{U}_n)/S(\mathcal{U}_{n-1}) \tag{7}$$

is an isomorphism in DM. This can be easily seen term-by-term. The first isomorphism is constructed.

We define (3) as the morphism given by the obvious homomorphism of presheaves with transfers of the same form as (3).

Proposition 4. The morphism $\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \to z_{equi}(\mathbf{A}^n, 0)$ is an isomorphism in DM. □

Proof. Define F_n as the subpresheaf in $z_{equi}(\mathbf{A}^n, 0)$ such that for a smooth connected X the group $F_n(X)$ is generated by closed irreducible subschemes Z of $X \times \mathbf{A}^n$ which are equidimensional over X of relative dimension 0 and which do not intersect $X \times \{0\}$. Consider $\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\})$ as a subpresheaf in $\mathbf{Z}_{tr}(\mathbf{P}^n)$. If U is a smooth scheme and Z is a closed irreducible subset in $U \times \mathbf{P}^n$ which represents an element of $\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\})(U)$, then it does not intersect $U \times \{0\}$ and therefore the image of Z under the map (8) is contained in $F_n(U)$. We get the following diagram of morphisms of presheaves

$$\begin{array}{ccc}
\mathbf{Z}_{tr}(\mathbf{P}^{n-1}) & = & \mathbf{Z}_{tr}(\mathbf{P}^{n-1}) \\
\downarrow & & \downarrow \\
\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\}) & \longrightarrow & \mathbf{Z}_{tr}(\mathbf{P}^n) \\
\downarrow & & \downarrow \\
F_n & \longrightarrow & z_{equi}(\mathbf{A}^n, 0)
\end{array} \tag{8}$$

The statement of the proposition follows from the diagram (8) and Lemmas 5, 6, and 7. ■

Lemma 5. The morphism of Nisnevich sheaves

$$\mathbf{Z}_{tr}(\mathbf{P}^n)/\mathbf{Z}_{tr}(\mathbf{P}^n - \{0\}) \longrightarrow z_{equi}(\mathbf{A}^n, 0)/F_n \tag{9}$$

is an isomorphism. □

Proof. Let S be a henselian local scheme and Z an irreducible closed subset of $S \times \mathbf{A}^n$ which is equidimensional of relative dimension zero over S. If Z does not belong to $F_n(S)$ then the intersection of Z with $S \times \{0\}$ is nonempty. Since it is closed, it must contain the closed point of S and thus the image of Z in S contains the closed point. But then Z is finite over S by [2, Chapter 1, Theorem 4.2(c)] and thus it is closed in $S \times \mathbf{P}^n$. This proves surjectivity. The proof of injectivity is similar. ■

Lemma 6. Let F_n be the sheaf defined in the proof of Proposition 4, then $F_n \cong 0$ in DM. □

Proof. We show that F_n is contactible, that is, that there exists a collection of homomorphisms

$$\phi_X : F_n(X) \longrightarrow F_n(X \times \mathbf{A}^1) \tag{10}$$

naturally in X and such that the composition of ϕ_X with the restriction to $X \times \{0\}$ and $X \times \{1\}$ is zero and the identity, respectively.

Consider the morphism $\pi : X \times \mathbf{A}^n \times \mathbf{A}^1 \to X \times \mathbf{A}^n$ given by the formula $\pi(x, r, t) = (x, rt)$. An element of $F_n(X)$ is a cycle $\mathcal{Z}$ on $X \times \mathbf{A}^n$ which does not intersect $X \times \{0\}$. Since π is flat over $X \times (\mathbf{A}^n - \{0\})$ the cycle $\pi^*(\mathcal{Z})$ is well defined and one checks immediately that it belongs to $F_n(X \times \mathbf{A}^1)$. One further verifies that the homomorphisms we constructed are compatible with the functoriality in X and satisfy the required conditions for the restrictions to $X \times \{0\}$ and $X \times \{1\}$.

The homomorphisms ϕ_X define for any X a homomorphism

$$C_*(F_n)(X) \longrightarrow C_*(F_n)(X \times \mathbf{A}^1). \tag{11}$$

By [4, Proposition 3.6] the restriction maps $C_*(F_n)(X \times \mathbf{A}^1) \to C_*(F_n)(X)$ corresponding to $X \times \{0\}$ and $X \times \{1\}$ are quasi-isomorphisms. This implies that our homomorphism is a quasi-isomorphism which equals zero on cohomology. Therefore $C_*(F_n)$ is exact and we conclude that $F_n \cong 0$ in DM. ■

Lemma 7. The morphism

$$Z_{tr}(\mathbf{P}^{n-1}) \longrightarrow Z_{tr}(\mathbf{P}^{n} - \{0\}) \tag{12}$$

is an isomorphism in DM. □

Proof. This morphism is a section of the morphism $Z_{tr}(p)$ where

$$p : \mathbf{P}^{n} - \{0\} \longrightarrow \mathbf{P}^{n-1} \tag{13}$$

is locally trivial in the Zariski topology bundle whose fibers are affine spaces. Any such p gives an isomorphism of motives because of the homotopy invariance and the Mayer-Vietoris properties. ■

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