Drawing Curves Over Number Fields

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"Lucky we know the Forest so well, or we might get lost"— said Rabbit, and he gave the careless laugh you give when you know the Forest so well that you can't get lost.

A.A. Milne,

The world of Winnie-the-Pooh.

Introduction

0.0. This paper develops some of the ideas outlined by Alexander Grothendieck in his unpublished *Esquisse d'un programme* [0] in 1984.

We draw our curves by means of what Grothendieck called "dessins d'enfant" on the topological Riemann surfaces. In the sequel we shall call them simply "dessins." By definition, a desssin D on a compact oriented connected surface X is a pair

$$D = (K(D), [\iota]),$$

where

$K(D)$ is a connected 1-complex ;

$[\iota]$ is an isotopical class of inclusions $\iota : K(D) \hookrightarrow X$.

We denote by $K_0(D)$ the set of vertices of $K(D)$.

It is supposed that

(a) the complement of $\iota(K(D))$ in X is a disjoint union of open cells ;

(b) the complement of $K_0(D)$ in $K(D)$ is a disjoint union of open segments.

The main construction we work with is based on the theorem of Gennady Belyi [1]. To a pair $D = (K, [\iota])$ it assigns a smooth algebraic curve together with some non-constant rational function on it over some number

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field. Throughout the paper we denote this curve by X_D and this function by β_D. We called them the *Belyi pair* associated to the dessin D.

According to [0], the realization of the possibility of such an assignment was one of the most striking events in Grothendieck's mathematical life. The only one he could compare it with was the following: "... vers l'âge de douze ans, j'étais interné au camp de concentration de Riecros (près de Mende). C'est là que j'ai appris, par une détenue, Maria, qui me donnait des leçons particulières bénévoles, la définition du cercle. Celle-ci m'avait impressionné par sa simplicité et son évidence, alors que la propriété de "rotondité parfaite" du cercle m'apparaissait auparavant comme une réalité mystérieuse au delà des mots".

The correspondence between the curves over number fields and dessins indeed seems to be very fundamental; in the end of this introduction we outline this construction in both directions.

We are interested in the constructive aspects of this correspondence. In the spirit of D. Mumford's monograph [2] we consider 5 ways of defining complex algebraic curves :

(1) Writing an equation.

(2) Defining generators of the uniformizing fuchsian groups.

(3) Specifying a point in the moduli space.

(4) Introducing a metric.

(5) Defining jacobian.

Our general approach is: given D, can we say anything about X_D and β_D?

Overview Of The Main Results

0.1. We use the following terminology. Let D be a dessin on a surface X. When $K(D)$ has no loops and each edge of $K(D)$ lies in the closure of exactly 2 components of $X\backslash K(D)$, we call

(a) a *valency of a vertex* $V \in K_0(D)$ the number of edges from $K(D)\backslash K_0(D)$, whose closures contain V.

(b) a *valency of a component* W of $X\backslash K(D)$ the number of edges from $K(D)\backslash K_0(D)$ that lie in the closure of W.

For the general dessins, see 1.4 below.

For a dessin D on a surface X we call its *0-valency* the least common multiple of the valencies of all the vertices from $K_0(D)$ and its *2-valency* the least common multiple of the valencies of all the components of $X\backslash K(D)$. Sometimes we denote them $v_0(D)$ and $v_2(D)$, respectively.

Call the dessin D *trigonal* if the valencies of all the components of $X\backslash K(D)$ are 3. The trigonal dessins with some regularity assumptions define the triangulations of X.

We call the dessin D on X *balanced* if all the valencies of the vertices of $K_0(D)$ are equal (to $v_0(D)$) and all the valencies of all the components of $X \backslash K(D)$ are equal (to $v_2(D)$).

Similar objects were explored in the classical topology from the combinatorial point of view (see e.g., [3], [15]). We do not pretend any terminological compatibility with this line of research because the technique we use is completely different. (Threlfall, e.g., called our balanced dessins the "regelmässig Zellsystem").

0.1.1. *Equations.* Here we have no general theory and only give a number of examples. We also consider the opposite question: can we actually draw a given curve? The answer is yes for some famous ones: Fermat curves, Klein quartic, and some modular curves are among them.

The completeness of our results decrease rapidly with growing genus; we are able to give some complete lists (of non-trivial experimental material) for genus 0, but for genera exceeding 3 we are able to give only some general remarks.

0.1.2. *Uniformization.* Let $p = v_0(D)$, $q = v_2(D)$. Consider the subgroup $\Gamma_{p,q}$ of $PSL_2(\mathbf{R})$ consisting of those transformations of the Poincaré upper half-plane $\mathcal{H} = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$ that respect the tesselation obtained by the reflections of the regular p-gon with the angles $2\pi/q$ (see Coxeter [3]). We show that there exists a subgroup Γ of finite index in $\Gamma_{p,q}$ such that the quotient $\mathcal{H}/\Gamma$ is isomorphic to X_D. We describe this Γ explicitly by the combinatorics of D. For the balanced dessins D this construction describes the universal covering of X_D.

0.1.3. *Moduli.* Here we consider only the trigonal dessins D and use the results from R.C. Penner's preprint [4], where one finds several constructions equivalent to ours (without reference to the Grothendieck program).

Penner introduced the extended Teichmuller spaces $\tilde{T}_{g,n}$, consisting of marked Riemann surfaces together with the horocycles about each puncture. Using dessins, Penner coordinates $\tilde{T}_{g,n}$'s, essentially by considering the metric on $X \backslash K_0(D)$ of constant curvature -1 and by assigning to it some functions of the lengths of parts of edges of $K(D)$ lying between the horocycles. By this construction $\tilde{T}_{g,n}$ turns out to be homeomorphic to $\mathbf{R}_{>0}^{6g-6+3n}$. We deduce from Penner's results, that under this coordinatisation X_D corresponds to $(\sqrt{2}, \ldots, \sqrt{2})$.

Penner builds the cell decomposition

$$\tilde{T}_{g,n} = \bigcup_{\substack{\text{dessins} \\ D \text{ on curves} \\ \text{of genus } g \\ \text{with } n \text{ vertices}}} C_D$$

where the trigonal dessins D correspond to the open cells. For $n = 1$ it gives the cell decomposition of the space $T_{g,1}$ itself. We deduce from Penner that in some sense $X_D \in C_D$.

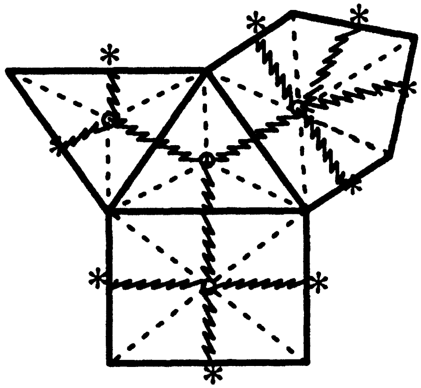
0.1.4. *Metric.* Here we consider only triangulations of X_D. Instead of riemannian metrics we work with "piecewise-euclidean" ones; they also allow us to define a complex structure on X_D. We show that X_D corresponds to the equilateral metrics.

0.1.5. *Jacobians.* Here we also work only with such dessins D that triangulate X. We define "approximate" jacobians $J_D(X)$, about which we think, that their limit, when D becomes finer, is the usual jacobian $J(X)$.

0.2. In the rest of the introduction we explain the essence of the assignment of Belyi pairs to dessins. In both directions we use :

Theorem [1] (Belyi). *A complex non-singular complete complex curve can be defined over some number field if and only if there exists a meromorphic function on it with only three critical values.*

0.2.1. *From dessins to curves.* We are supposed to be given a dessin D on a surface X. Choose a point in each 2-cell, connect the points of the neighbouring 2-cells by an edge of a different type, and connect all these points with the vertices of the original graph inside the 2-cells when it is possible.



Now we have three types of vertices and three types of edges. Since we are going to map the whole picture onto $\mathbf{P}^1(\mathbf{C})$, we mark them :

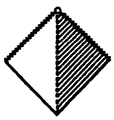
$$\circ \quad \text{over} \quad \text{“}0\text{”},$$
$$\cdot \quad \text{over} \quad \text{“}\infty\text{”},$$
$$* \quad \text{over} \quad \text{“}1\text{”};$$

we find the edges

$$\text{———} \quad \text{over} \quad (0,1),$$
$$\text{over} \quad (1,\infty),$$
$$\text{- - - -} \quad \text{over} \quad (\infty,0).$$

Note that the whole surface has become divided into triangles, each with vertices of all three types. The orientability of the surface results in the possibility of painting it in two colours in such a way that if we move around the black and the white triangles counter-clockwise, the order of vertices is “0”-“1”-“∞” and “0”-“∞”-“1”, respectively.

Now look at the “butterflies”—the pairs of the adjacent triangles of different colours and think of the surface as the union of the butterflies.



If all the butterflies put their wings together to become S^2-like and are identified, we get the desired map

$$\beta_D : X_D \longrightarrow \mathbf{P}^1(\mathbf{C}),$$

ramified only over $\{0, 1, \infty\}$.

After we restore the unique algebraic structure of X_D (by the Riemann existence theorem) in which β_D is rational, we use the easier part of Belyi ("if" in the above formulation) theorem to conclude that X_D and β_D are defined over $\overline{\mathbf{Q}}$. The Belyi pair (X_D, β_D) thus obtained depends only on the original dessin D.

The Belyi function β_D obtained through this construction has all the ramifications over "1" or order 2. We shall call such Belyi functions clean.

The image of $K(D)$ in X can be reconstructed as the β_D-preimage of the segment $[1, \infty]$.

0.2.2. *From curves to dessins.* In this direction we outline the proof of the more striking half of the Belyi theorem ("only if" in the above formulation), closely following [1]. Suppose we are given a curve X over some number field L ; take a non-constant element f of the function field $L(X)$ and consider it as a ramified covering

$$f : X(\mathbf{C}) \longrightarrow \mathbf{P}^1(\mathbf{C});$$

its ramification points belong to $\mathbf{P}^1(\overline{\mathbf{Q}})$.

We are going to transform f to the desired covering by a sequence of replacements of f by $P \circ f$, where P is a polynomial with rational coefficients, considered as a map $\mathbf{P}^1(\mathbf{C}) \longrightarrow \mathbf{P}^1(\mathbf{C})$. After such replacements, infinity will always go to infinity; denote for $F : X(\mathbf{C}) \longrightarrow \mathbf{P}^1(\mathbf{C})$ by W_F the set of finite critical values of F.

By the first step of the construction, we reduce the situation to the case $W_F \subset \mathbf{Q}$. We proceed by induction on $\#(W_F \backslash \mathbf{Q})$. At each inductive step we take for P a generator of the annihilator in $\mathbf{Q}[T]$ of W_F. It is clear that $\#W_{P \circ F} = \#W_F - 1$. So we suppose $W_F \subset \mathbf{Q}$ and proceed to the second step.

Replacing f by $Af + B$ with suitable A, B we can assume $\{0, 1, \infty\} \subset W_F$. Proceed by the induction in $\#(W_F \backslash \{0, 1, \infty\})$. Assuming that there exists $r \in W_F \backslash \{0, 1, \infty\}$ with $0 < r < 1$ (which can again be achieved by a change of f to $Af + B$, denote $r = \frac{m}{m+n}$ with natural n, m. Now use

$$P(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

and check that $P(0) = P(1) = 0$, $P(r) = 1$ and that if $P^1(z) = 0$, then $z \in \{0, 1, r\}$; so replacing f by $P \circ f$ reduces $\#(W_F \backslash \{0, 1, \infty\})$ by 1, which completes the inductive argument.

We have obtained the Belyi function f; to get a clean one put

$$\beta = 4f(1-f).$$

The desired dessin D is defined by setting $K(D) = \beta^{-1}([1,\infty])$ with $K_0(D) = \beta^{-1}(0)$.

0.3. The style of our exposition is far from the standards of modern mathematics; we hope that this flaw is partially compensated for by the explicitness of the results. Besides, we do not prove some of our assertions. This is not only the result of space and time constraints, but rather of the feeling that the proper language for the mathematics of the Grothendieck Program has not yet been found.

The main reasons to publish our results in the present state is our eagerness to invite our colleagues into the world of the divine beauty and simplicity we have been living in since we have been guided by the Esquisse [0]. We emphasize that in the present text we use only a very small part of the ideas one can find in the epoch-making paper.

We are indebted to Yu. I. Manin for the useful discussions and to A.A. Migdal and his colleagues for their interest. We are grateful to I. Gabitov, without whose assistance the preparation of the present text would have been impossible.

Part 1. Generalities

Let D be a dessin on a surface X. We choose a representative of the isotopical class of the inclusions $K(D) \stackrel{\iota}{\hookrightarrow} X$ and in what follows consider $K(D)$ as a subset of X; all our constructions are independent of this choice.

1.1.1 The *flag set* $\mathsf{F}(D)$ is, by one of the definitions, a set of triples (U, E, V) such that U is a component of $X \backslash K(D)$, E a component of $K(D) \backslash K_0(D)$, V a vertice from $K_0(D)$ and

(i) E lies in the closure of U;

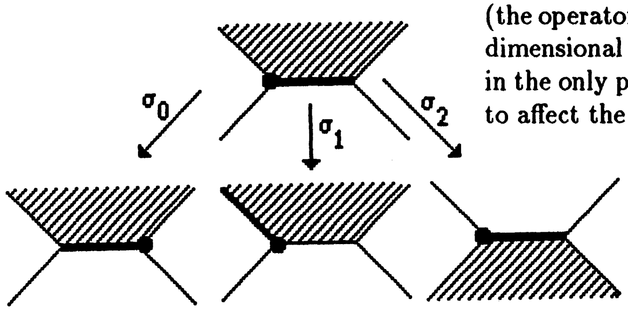
(ii) V lies in the closure of E.

This definition is suitable only for the fine enough dessins; later, we will give a universal definition.

1.1.2. Consider a "*cartographical*" group

$$\mathcal{C}_2 = \langle \sigma_0, \sigma_1, \sigma_2 | \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1 \rangle$$

It acts on the flags in the following way :



(the operator σ_q changes the q-dimensional element of the flag in the only possible way as not to affect the other elements).

Because of the connectedness of X, the flag space $\mathsf{F}(D)$ is C_2-homogeneous.

In analogy with the linear case, we call a (non-oriented) *Borel* subgroup of the flag $F \in \mathsf{F}(D)$ the stationary group of this action:

$$B_{D,F} = \{c \in C_2 | c \cdot F = F\}$$

1.1.3. The *orientability* of X (in the spirit of the Introduction) results in the possibility of defining the map

$$o : \mathsf{F}(D) \longrightarrow \{\pm 1\}$$

satisfying

$$o(\sigma_q \cdot F) = -o(F) \text{ for all } F \in \mathsf{F}(D), q \in \{0, 1, 2\}.$$

We consider the set of *positively oriented flags*

$$\mathsf{F}^+(D) = o^{-1}(1).$$

The *oriented cartographical* group C_2^+ is the one that respects all the sets

$$\mathsf{F}^+(D) \subset \mathsf{F}(D).$$

It is the subgroup of index 2 of C_2 generated by the words in $\sigma_0, \sigma_1, \sigma_2$ of even length. We take the generators

$$\rho_0 = \sigma_2\sigma_1$$
$$\rho_1 = \sigma_0\sigma_2$$
$$\rho_2 = \sigma_1\sigma_0$$

satisfying

$$\rho_2\rho_1\rho_0 = 1$$

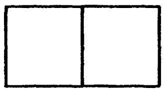
Since all the Borel subgroups corresponding to one dessin are conjugate, in cases where we need only the conjugacy class of $B_{D,F}$, we shall omit the reference to F.

The dessin is called a *Galois* one if the group B_D is normal in C_2^+.

We call an automorphism of the dessin a compatible triple consisting of a permutation $K_0(D) \longrightarrow K_0(D)$ and isotopical classes of homeomorphisms $K(D) \longrightarrow K(D)$, $X \longrightarrow X$.

It is clear, that in the case of a Galois dessin D with Borel subgroup B the factor C_2^+/B acts on D, transitively on the vertices, on the edges and on the components of $X \backslash K(D)$. Therefore, the Galois dessins are balanced.

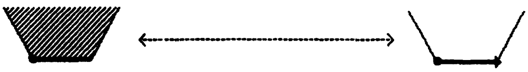
The converse is wrong: the dessin



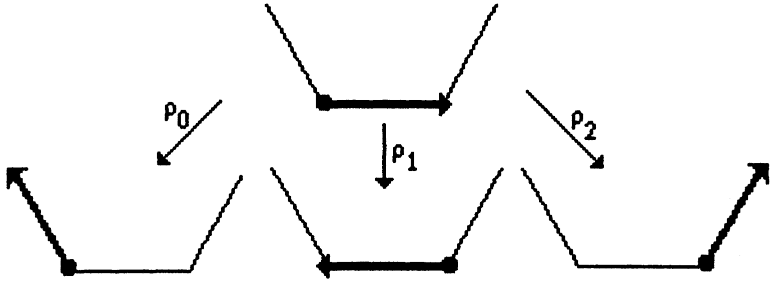
with identified opposite sides gives an example of a balanced non-Galois dessin on the torus.

All the automorphisms of the dessin D on a surface X are realizable as conformal automorphisms of X.

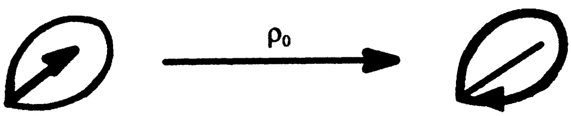
1.1.4. The positively oriented flags are in canonical one-to-one correspondence with the oriented edges of $K(D)$. We use the following convention:

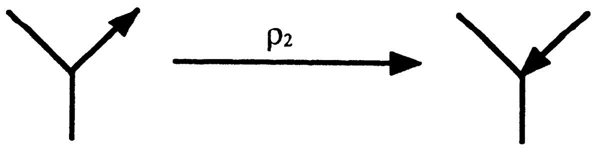


Under this convention the oriented cartographical group acts on the oriented edges in the following way:



In this way the definition of the action of the oriented cartographical group on the oriented flags is naturally extended to all the dessins, with no regularity assumptions. In the degenerate cases this action may look, for instance, like this:





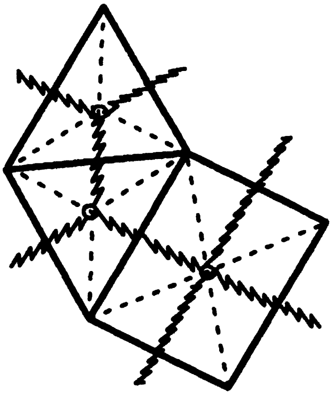
The orders of $F \in \mathsf{F}^+(D)$ with respect to ρ_0, ρ_2 are related with the general valencies (see 0.1) as follows :

$$\#\{\langle\rho_0\rangle\}F = v_0(\text{0-component of } F)$$
$$\#\{\langle\rho_2\rangle\}F = v_2(\text{2-component of } F).$$

In particular, if $v_0(D) = p$, $v_2(D) = q$, then for any $F \in \mathsf{F}^+(D)$

$$\langle\rho_0^p, \rho_2^q\rangle \subset B_{D,F}$$

1.1.5. For a dessin D on a surface X the dual dessin D^* is defined as follows. The set of vertices $K_0(D^*)$ is in canonical one-to-one correspondence with the components of $X \backslash K(D)$, and the set of edges of D—with that of D. The pair of dual dessins looks like



1.1.6. It is very important [0] that the correspondence between the dessins and the Belyi pairs defines the action of the Galois group Gal $(\overline{\mathbf{Q}}/\mathbf{Q})$ on the dessins. In particular, the

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makes sense.

Part 2. Results

2.1. *Equations.* We start with some of the most simple dessins and try to determine which Belyi pairs over what number fields they define. Grothendieck [0] doubts that the problem can be solved by a uniform method. We can confirm from our experience that if such a method exists, it is quite involved.

We adopt the following drawing conventions:

(a) for genus 0 we put the dessins in the euclidian plane; if we have a vertex at infinity, we put arrow marks on the edges going there;

(b) for genus 1 we draw everything in the period parallelogram;

(c) for genus 2 and higher we realize the surfaces as polygons with identified boundary edges and draw our dessins on these polygons; the pairs of identified edges are directed controversially (which means that one edge goes clockwise while another one goes counterclockwise).

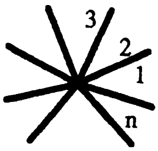
2.1.0. *Genus* 0. Since the curves of genus 0 have no moduli, the only thing to calculate is the Belyi function, which in this case is just a rational map $\mathbf{P}^1(\mathbf{C}) \longrightarrow \mathbf{P}^1(\mathbf{C})$ with three critical values. It is defined up to composition with $PSL_2(\mathbf{C})$-transformations from the left and from the right. The proper choice of the representatives seems to be connected with delicate arithmetical questions which we do not discuss here; in the examples below we try to choose them in the shortest form. As a result, the critical values of the Belyi functions vary.

We consider first the simplest case : suppose that $X\backslash K(D)$ is connected. Then $K(D)$ is just a tree inside an oriented 2-sphere S^2, and we shall draw it in the plane (where the orientation is essential!).

Using our right to choose the $PSL_2(\mathbb{C})$-representatives as we like, we normalize the Belyi maps in such a way that the "centre" of the only 2-cell lies at ∞ and goes to ∞ and the intersections of $K(D)$ and $K(D^*)$ will go to 0. Then the Belyi map is represented by a polynomial, all the zeroes of which are double; therefore it is a square of a polynomial, which we denote h_D or simply h. We are going to present a number of h for some simple dessins D.

We present three infinite series of tree-like dessins for which h can be specified:

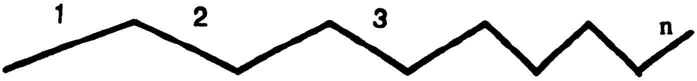
1.



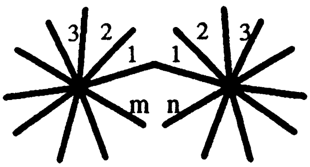
$h(z) = z^n$

2.

$h(z)$ is the n-th Chebyshev polynomial: $\cos(nt) = h(\cos t)$



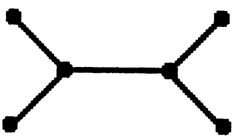
3.



$h(z) = z^m(1-z)^n$
(up to multiplication by constant the Belyi function from the introduction)

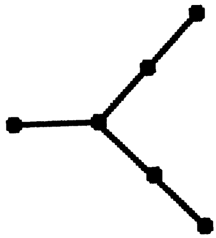
To cover all the trees with the number of edges not exceeding 6, we should add eight individual dessins (three of them equivalent under the Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$-action).

1.

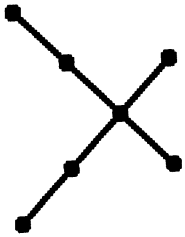


$h(z) = (z+1)^3(3z^2 - 9z + 8)$

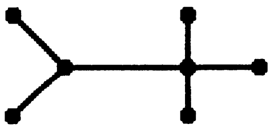
2. $h(z) = z^3(9z^2 - 15z + 40)$



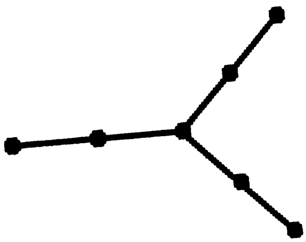
3. $h(z) = z^4(36z^2 + 36z + 25)$



4. $h(z) = z^3(6z^2 + 96z + 25)$



5. $h(z) = z^3(z^3 + 1)$



Note that all these dessins, as well as the infinite series above, are defined over the rationals $\mathbb{Q}$.

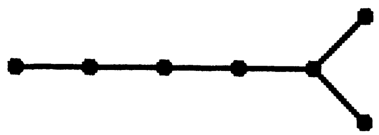
The remaining three dessins are defined over three cubic fields, permutable by the Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$. They lie in the field of decomposition of the polynomial

$$25t^3 - 12t^2 - 24t - 16 = 25(t-a)(t-a_+)(t-a_-)$$

(we agree that $a \in \mathbb{R}$, Im $(a_+) > 0$, Im $(a_-) < 0$).

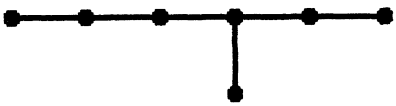
The dessin

D_a:



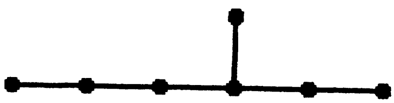
is defined over $\mathbb{Q}(a)$,

D_{a_+}:



over $\mathbb{Q}(a_+)$,

D_{a_-}:



over $\mathbb{Q}(a_-)$.

For each $b \in \{a, a_\pm\}$ we have

$$h_{D_c}(z) = z^3(z+1)^2(z+b).$$

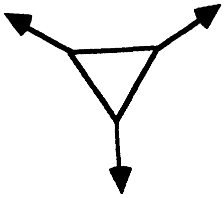
Now we turn our attention to the Galois dessins. They constitute two families: corresponding to the plane polygons and to the Platonic solids. As for the plane n-gons, in one of the normalizations they are described by the rational function

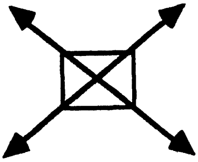
$$z^n + (1/z)^n$$

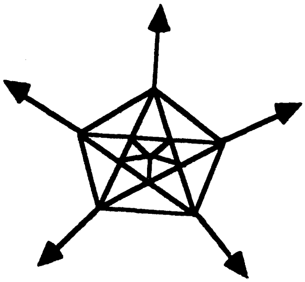
To discuss the platonic solids, introduce the notation

$$\dot{X}_D = X_D \backslash K_0(D)$$

and denote by Y_n the affine modular curve $\mathcal{H}/\Gamma(n)$, where $\Gamma(n)$ is a principal congruence subgroup of $PSL_2(\mathbb{Z})$ (the kernel of the natural homomorphism $PSL_2(\mathbb{Z} \longrightarrow PSL_2(\mathbb{Z}/n\mathbb{Z}))$. We claim that the platonic solids with the triangle 2-cells correspond to the modular curves in the following way:

—for tetrahedral $D =$  $\dot{X}_D = Y_3$

—for octahedral $D =$  $\dot{X}_D = Y_4$

—for icosahedral $D =$  $\dot{X}_D = Y_5$.

For all of them, the Belyi map corresponds to the canonical projection

$$Y_n = \mathcal{H}/\Gamma(n) \longrightarrow \mathbf{P}^1(\mathbf{C}) \simeq \mathcal{H}/PSL_2(\mathbf{Z}).$$

The curve Y_2 did not enter this list because the corresponding

$D =$ 

is not a platonic solid. But this dessin is a remarkable one: the corresponding rational function in one of the normalizations is

$$\frac{27(z^3 - z + 1)}{4z^2(z-1)},$$

which is exactly the expression for the J-invariant of the elliptic curve

$$y^2 = x(x-1)(x-2).$$

Left composition of this function with any canonically normalized Belyi function corresponds to the barycentric subdivision of the corresponding dessin.

2.1.1. *Genus* 1. On the curves of genus 1 there exist balanced dessins of valencies $v_0 = v_2 = 4$, $v_0 = 6$, $v_2 = 3$ and $v_0 = 3$, $v_2 = 6$.

The corresponding curves are isogenic to the elliptic curve

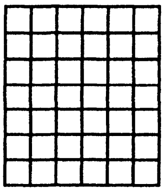
$$y^2 = x^3 - x$$

in the first case and to the curve

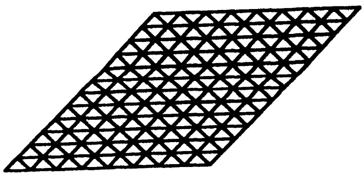
$$y^2 = x^3 - 1$$

in the second and in the third cases.

The Galois dessins represent exactly these curves. (See [3] for the proofs of the classification results.) In the period parallelogram these dessins look like

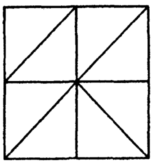


$\{v_0, v_2\} = \{4, 4\}$

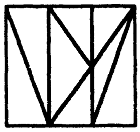


$\{v_0, v_2\} = \{6, 3\}$

The problem of determining the J-invariant of the curves drawn by a dessin on the torus seems to be rather hard. For instance, consider the dessin on the torus which in the fundamental parallelogram looks like

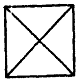


The corresponding curve is defined over $\mathbf{Q}(\sqrt{7})$. The $(\sqrt{7} \to -\sqrt{7})$-conjugated dessin looks like



The J-invariants are $-\frac{1}{16\,27}(8 \mp 3\sqrt{7})^2(2 \pm \sqrt{7})^6(10 \pm 3\sqrt{7})^3$. This calculation answers the question of A. A. Migdal, 1986.

Sometimes the curve can be determined by the additional symmetries of the dessin. For instance, for

D= 

X_D is determined by the equation

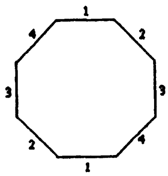
$$y^2 = x^3 - x$$

with Belyi function $-27x^4/(x^2-4)^3$ because of the symmetry of 4th order.

2.1.2. *Genus* 2. We are able to give some results only for Galois dessins. They are listed in [3]; there are 10 of them, but we choose only one from each pair of the dual ones.

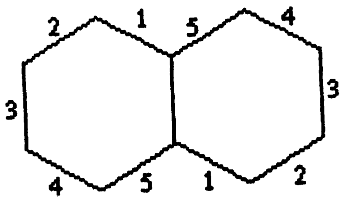
DESSIN | BELYI PAIR

1.



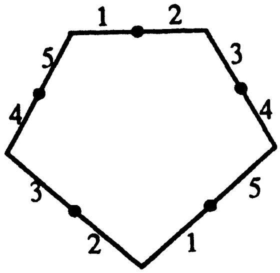
$$y^2 = x^5 - x$$
$$\beta = x^4$$

2.



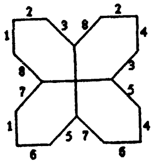
$$y^2 = x^5 - 1$$
$$\beta = x^5$$

3.



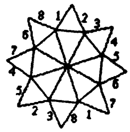
$$y^2 = x^6 - 1$$
$$\beta = x^6$$

4.



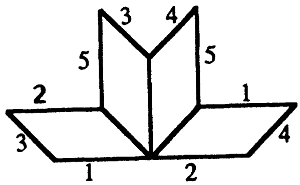
$$y^2 = x^6 - 1$$
$$\beta = (1+x^6)^2/4x^6$$

5.



$$y^2 = x^5 - x$$
$$\beta = 31^3 \frac{4x^2(1+x^4)^4}{[27(1+x^4)^2 - 8x^4]^3}$$

6.



$$y^2 = x^5 - x$$
$$\beta = 4x^4/(1+x^4)^2$$

2.1.3. *Genus* 3. Here we discuss only three curves. The Klein quartic is defined in the homogeneous coordinates by the equation

$$x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0$$

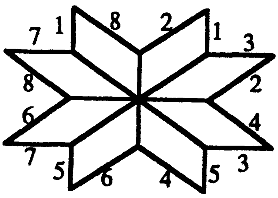
Its full automorphism group has 168 elements; the quotient by this group is isomorphic to $\mathbf{P}^1(\mathbf{C})$, and the natural projection defines a clean Belyi map.

We do not attempt to draw the corresponding dessin and refer the reader to Klein's paper [5], where one finds a very beautiful triangular tesselation of the fundamental domain of the Klein quartic on the universal covering (this was the figure from which the uniformization started). After suitable identifications, we get some triangular dessin D on a curve of genus 3.

Thus the Klein quartic itself has been drawn; indeed, the automorphism group of D and consequently of the curve X has 168 elements, and Klein quartic is the only curve of genus 3 with this (highest possible) number of automorphisms [6].

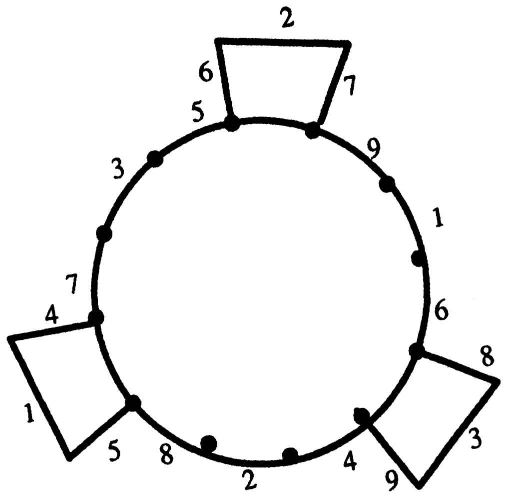
Here is the dessin for the Fermat quartic

$$x^4 + y^4 = 1 :$$



with the Belyi function $\beta = (x^4+1)^2/(4x^4)$.

Here is the dessin for the Picard curve [7] $y^3 = x^4 - 1$



with the Belyi function of the same form $\beta = (x^4+1)^2/(4x^4)$.

2.1.4. *Higher Genera.* We propose two infinite families of curves for which something can be done.

One is the family of the generalized Fermat curves

$$x^m + y^n = 1$$

It is easy to check that the functions x^m and y^n are the Belyi ones. Taking $n = 2$, we get a drawable curve of any given genus. The other series is the family of modular curves $X_0(n)$ (the factors $\mathcal{H}/\Gamma_0(n)$, where $\Gamma_0(n)$ is the preimage of the upper triangular matrices under the canonical projection $PSL_2(\mathbf{Z}) \longrightarrow PSL_2(\mathbf{Z}/n\mathbf{Z})$). The dessins on these curves are the projections of the $PSL_2(\mathbf{Z})$-orbits of the arc from exp $(2\pi i/3)$ to exp $(\pi i/3)$ on the boundary of the modular figure. For the discussion of an explicit description of modular curves see [8], [9].

2.2 Uniformization

Let D be an arbitrary dessin on a surface X. Let $p = v_0(D)$, $q = v_2(D)$. We canonically associate to D a conjugacy class of discrete subgroups

$$\Gamma_D \subset PSL_2(\mathbf{R})$$

and then show that

$$X_D \simeq \mathcal{H}/\Gamma_D$$

2.2.0. *Construction.* The reference for the material below is [3].

Consider in the Poincaré upper half-plane $\mathcal{H}$ the regular q-gon $\Pi_{p,q}$ with angles $2\pi/p$. The half-plane is tesselated by its reflections through its sides.

Consider the group $\Gamma_{p,q} \subset PSL_2(\mathbf{R})$ that respects this tesselation. It is generated by two elliptic elements

$$\begin{aligned} \beta = &\text{ the rotation of } \Pi_{p,q} \text{ about the centre of} \\ &\text{ angle } 2\pi/q. \\ \gamma = &\text{ the rotation of } \mathcal{H} \text{ about one of the vertices} \\ &\text{ of } \Pi_{p,q} \text{ of angle } 2\pi/p. \end{aligned}$$

It is geometrically obvious that these elements satisfy the relations

$$\beta^q = \gamma^p = 1,$$
$$(\beta\gamma)^2 = 1$$

(the transformation $\beta\gamma$ being a symmetry around the centre of the side of $\Pi_{p,q}$).

These relations allow us to define the homomorphism

$$\begin{aligned} \mathcal{H}_{p,q} : C_2^+ &\longrightarrow \Gamma_{p,q} \subset PSL_2(\mathbf{R}) \\ \rho_0 &\longmapsto \gamma \\ \rho_2 &\longmapsto \beta \end{aligned}$$

Fix some $F \in \mathbf{F}(D)$. The desired group is

$$\Gamma_{D,F} = \mathcal{H}_{p,q}(B_{D,F})$$

Its conjugacy class is independent of all the choices involved. Denote it by Γ_D.

2.2.1. Theorem.

$$\mathcal{H}/\Gamma_D \simeq X_D$$

Sketch of the proof. If we conect the centre of $\Pi_{p,q}$ by geodesics with all the vertices and all the centres of the sides and then paint (as Klein did in [5]) these triangles in black and white this way

we realize that for every subgroup $\Gamma \subset \Gamma_{p,q}$ of finite index these triangles are in one-to-one correspondence with the flags of the dessin on $\mathcal{H}/\Gamma$ obtained by the projection of the above infinite dessin on $\mathcal{H}$. The orientation of the flags corresponds to the colour of the triangles.

It remains to realize that this dessin is isomorphic to the original one and that the natural projection

$$\mathcal{H}/\Gamma_D \longrightarrow \mathcal{H}/\Gamma_{p,q} \simeq \mathbf{P}^1(\mathbf{C})$$

is the same as the one defined by this dessin by our construction (see Introduction).

2.2.2. Theorem. *If D is balanced, then the natural map*

$$\mathcal{H} \longrightarrow \mathcal{H}/\Gamma_D$$

is isomorphic to the universal covering of X.

Indeed, for general D the ramification index of the map $\mathcal{H} \longrightarrow \mathcal{H}/\Gamma$ equals $p/v_0(V)$ over $V \in K_0(D)$ and $q/v_2(S)$ over the "centre" of the component S of $X \setminus K(D)$.

Therefore, this map is unramified for the balanced D.

In this way we can effectively describe the universal coverings of all the curves that were drawn by the balanced dessins in the previous section. For the Klein quartic it was done by Klein [5].

2.2.3. Now we turn to the universal coverings of the curves $\dot{X}_D$ with trigonal D.

Proposition. *The group C_2^+ with the additional relation $\rho_2^3 = 1$ is isomorphic to $PSL_2(\mathbf{Z})$.*

Using [3], *define this isomorphism by the assignment*

$$\rho_2 \longmapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_1 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho_0 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Theorem. *For trigonal dessins D the curve $\dot{X}_D$ is isomorphic to $\mathcal{H}/\Gamma$, where $\Gamma \subset PSL_2(\mathbf{Z})$ corresponds to the Borel subgroup B_D under the above isomorphism.*

2.3 Moduli

2.3.0. The results of this part are based on Penner's paper [4]. Now we work with the above curve $\dot{X}_D$.

Introduce the notation $[D] = \#(K_0(D))$. The curves $\dot{X}_D$ after a suitable marking turn into the points of the Teichmuller space $T_{g,n}$ (see, e.g., [16]). We always assume $2g - 2 + [D] > 0$ and interpret the points of $T_{g,n}$ as metrics with constant curvature -1 on $\dot{X}_D$.

We start with a review of Penner's approach to the Teichmuller spaces $T_{g,n}$ with $n > 0$. Penner introduces the augmented Teichmuller spaces $\tilde{T}_{g,n}$, whose points correspond to the points of the usual Teichmuller space T together with the horocycles about each puncture (a closed curve, orthogonal to all the geodesics, going to the cusp).

Denote for any dessin $D = (K, [\iota])$ by $K_1(D)$ the set of (nonoriented!) edges of K, i.e., the set of the connected components of $K(D)\backslash K_0(D)$.

Also denote by $K_2(D)$ the set of connected components of $X\backslash K(D)$.

In this section we consider only trigonal dessins D. In what follows, $n = [D]$.

Lemma. $\dim_{\mathbf{R}} \tilde{T}_{g,n} = \#K_1(D) = 6g - 6 + 3[D]$.

Proof. For the Euler characteristic we have

$$\#K_0(D) - \#K_1(D) + \#K_2(D) = 2 - 2g,$$

and the number of "1, 2-flags" in the pair $(X, K(D))$ equals

$$2\#K_2(D) = 3\#K_1(D).$$

From these two equalities we have

$$\#K_1(D) = \#K_0(D) + \#K_2(D) + 2g - 2 = \#K_0(D) + \frac{2}{3}\#K_1(D) + 2g - 2,$$

and

$$\begin{aligned}\#K_1(D) = 3([D] + 2g - 2) &= [D] + 2(3g - 3 + [D]) \\ &= [D] + \dim_{\mathbf{R}} T_{g,n} = \dim_{\mathbf{R}} \tilde{T}_{g,n}\end{aligned}$$

So it is natural to try to coordinatize $\tilde{T}_{g,n}$ by the functions on the set of edges $K(D)$. On the part of $\tilde{T}_{g,n}$, on which the horocycles are so small that they do not intersect, there exists a natural function: think of every edge from $K(D)$ as an (infinite) line from puncture to puncture in $X(D)$, deform it to the geodesic and take the lengths of the part between the horocycles. It turns out that the analytic continuation of this construction gives the global coordinatization of $\tilde{T}_{g,n}$!

To establish it, Penner uses the Minkowski space $\mathbf{M}^3$ with the coordinates $x = (x_0, x_1, x_2)$ and with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2$$

induced from the scalar product $\langle\ ,\ \rangle$ of signature $(- + +)$. It follows from the local isomorphity of $SL\ (\mathbf{R})$ and $SO\ (1,2)$ and some easy Lie group considerations, that the Poincaré upper half-plane is isometric to a connected component of the hyperboloid

$$\langle x, x\rangle = -1,$$

and the space of horocycles on it to the future light cone

$$\langle x, x\rangle = 0, x > 0.$$

Denote by ℓ the Poincaré length of the above part of geodesic; let the intersections of this geodesic with the horocycles be represented by the Minkowski space points u, v. Then (see Lemma 2.1 from [4])

$$-\langle u, v\rangle = 2\exp(\ell),$$

and this formula allows the length coordinatization to be continued to the whole $\tilde{T}_{g,n}$.

Denote the length map thus described by

$$\text{Pen} : \tilde{T}_{g,n} \longrightarrow \mathbf{R}_{>0}^{6g-6+3n}$$

Theorem 3.1 from [4] says, that this map is a real-analytic homeomorphism.

Now we can state our results.

2.3.1. Theorem. *There exists a set of horocycles on $\dot{X}_D$, endowed with the metric of the constant curvature, such that the above map Pen sends the corresponding point of $\tilde{T}_{g,n}$ to*

$$(\sqrt{2}, \ldots, \sqrt{2}) \in \mathbf{R}_{>0}^{6g-6+3n}.$$

The proof follows from Penner's proposition 6.5 from [4], which states that the point

$$\mathrm{Pen}^{-1}(\sqrt{2}, \ldots, \sqrt{2})$$

is uniformizable by a subgroup of $PSL_2(\mathbf{Z})$, and from our Theorem 2.3 from the previous section.

2.3.2. Next, we describe Penner's universal cell decomposition of the space $\tilde{T}_{g,n}$. Its points are interpreted as the conjugacy classes of the fuchsian groups $\Gamma \subset SO^+(2,1)$ together with the Γ-invariant set B on the light cone (the only point of B on each light ray corresponds to the choice of the horcycle). To such a pair (Γ, B) the convex hull of the discrete set $\Gamma \cdot B$ is associated. The cells of $\tilde{T}_{g,n}$'s correspond to the pairs (Γ, B) with the fixed combinatorics of the boundary of this convex hull. Projecting the edges of this boundary to the hyperboloid $\langle x, x\rangle = -1$, we get a Γ-invariant tesselation of it. Dividing then by Γ, we get the dessins (with fixed number $n = [D]$ of vertices) that parametrize the cells of the decomposition we are describing.

From now on, we suppose $[D] = 1$; then the above construction defines a decomposition of the Teichmuller space $T_{g,1}$ itself. Since this decomposition is invariant under the Teichmuller modular group, it induces a finite cell decomposition of the moduli space $M_{g,1}$. For a dessin D, denote by C_D the cell corresponding to it. The cells C_D of maximal dimension correspond to the trigonal dessins D.

2.3.3. Theorem. *For $[D] = 1$ the point of $M_{g,1}$ corresponding to X_D lies inside C_D.*

The proof follows from Penner's reasoning on the non-emptiness of C_D (Corollary 6.3 of [4]), combined with Proposition 6.5.

2.3.4. Without proof we state one more result concerning the position of the curves X_D with the Galois D's in the moduli spaces. Since they have many automorphisms, they correspond to some strong singularities of the moduli spaces.

Theorem. *The curve X over $\overline{\mathbb{Q}}$ can be realized as X_D with a Galois dessin D if and only if it is a projection of an isolated fixed point of some finite subgroup of the Teichmuller modular group.*

2.4 Metrics

In this section, we consider only piecewise-euclidean metrics—the ones that are formed by putting together compatible flat polygons; the resulting metric is flat away from the isolated points where the vertices meet and where the discrete curvature occurs (as a difference of 2π and the sum of the flat angles).

The piecewise-euclidean metrics define complex structures as well as the riemannian ones; any complex structure on a Riemann surface can be obtained in this way. A nice proof of this fact can be obtained using Strebel differentials; for a modern exposition of this theory see Douady-Hubbard [11] (though they do not formulate explicitly the result we need).

2.4.0. We are going to work only with the piecewise-euclidean metrics, in which all the polygons are the equilateral triangles; we call them the equilateral metrics.

The equilateral metrics have no continuous moduli and depend only on the combinatorics of the triangles; so the trigonal dessins D appear.

Denote the corresponding curves Y_D.

They define the countable set of points in the moduli spaces.

2.4.1. Theorem. *For any trigonal dessin D the curve Y_D is isomorphic to X_D.*

2.4.2. Theorem. *The set of curves Y_D for all the dessins D on the surfaces of genus g "is" exactly the set $M_g(\overline{\mathbb{Q}})$ of curves over all the number fields.*

For the proof, see our paper [10].

2.4.3. This result can be interpreted in terms of string physics (see, for instance, [12]; in fact, we were influenced by the authors of this paper). It shows that integration over all the metrics on Riemann surfaces using the lattice-like method of approximation of Riemann metrics uncovers the arithmetical nature of the subject.

In the spirit of fashionable ideas of modern theoretical physics, it is natural to suggest that the non-archimedian components would also be taken into account in this approach. For the discussion of these ideas see Manin [13].

We suppose, that the dessins on the Riemann surfaces may turn to be quite fundamental in quantum physics, being the natural analogues of Feynman graphs in the pre-string theories.

2.5 Jacobians

In this section we work with the trigonal dessins D on the Riemann surfaces of positive genus g.

2.5.0. The dessin D defines on X the structure of a cell complex, which will be used in the realisation of the cohomology classes. Denote by $\mathcal{O}$ the structure sheaf, Ω the sheaf of germs of holomorphic differentials on the curve X_D.

Using the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & H^1(X_D,\mathbf{Z}) & = & H^1(X_D,\mathbf{Z}) & & \\
 & & & & \cap\downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X_D,\Omega) & \xrightarrow{\quad I \quad} & H^1(X_D,\mathbf{C}) & \longrightarrow & H^1(X,\mathcal{O}) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & J(X_D) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where the horizontal exact sequence comes from the exact sheaf sequence

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathcal{O} \xrightarrow{\quad d \quad} \Omega \longrightarrow 0,$$

we realize the jacobian $J(X)$ as the double coset space

$$J(X_D) = I(H^D(X,\Omega))\backslash H^1(X_D,\mathbf{C})/H^1(X_D,\mathbf{Z}).$$

2.5.1. The steps of our construction are:

(i) Construct a piecewise-linear analogue of the abelian differentials on X and as a result obtain a g-dimensional space

$$L_D \subset H^1(X_D, \mathbf{C}),$$

where the RHS is interpreted as the space of the cell cohomologies of X_D.

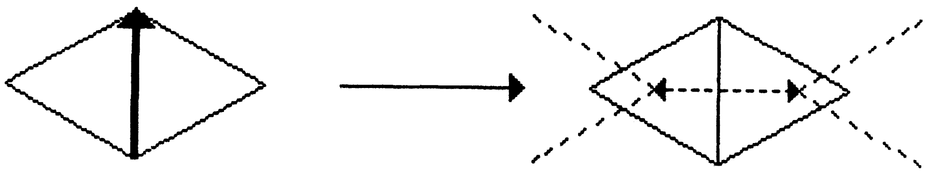
(ii) Iterate the refinements

and use the canonical isomorphisms of X_D and $X_{\alpha D}$, which results in the canonical isomorphism

$$H^1(X_D, \mathbf{C}) = H^1(X_{\alpha D}, \mathbf{C}).$$

Under this isomorphism the spaces $L_{\alpha^n D}$ form a sequence of the g-dimensional subspaces of $H^1(X_D, \mathbf{C})$, whose limit is (we hope)* the I-image of the space of abelian differentials in $H^1(X_D, \mathbf{C})$.

2.5.2. The space L_D is constructed in the following way: Denote by $C^1(D)$ the space of the cell cochains of X_D with complex coefficients. To a 1-cochain on D we associate the 1-cochain on D as it is shown below:



Thus we get an operator

$$\star_D : C^1(D) \longrightarrow C^1(D^*)$$

which, we suppose, is the proper analogue of the harmonic Hodge operator (see also [14]). It enjoys the following properties:

(a) $$\star_D \star_{D^*} = -1.$$

* (Added in proof). It is really so. The demonstration will be published in a forthcoming paper.

(b) If $\mathcal{H}^1(D) = Z^1(D) \cap (\star_D^{-1}(Z^1(D^*)))$

(Z^1 denoting the cocycles), then the projection

$$\mathcal{H}^1(D) \longrightarrow H^1(X_D, \mathbf{C})$$

is an isomorphism.

Thus we have obtained the diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{H}^1(D) & \underset{\star_{D^*}}{\overset{\star_D}{\rightleftarrows}} & \mathcal{H}^1(D^*) \\ \downarrow & & \downarrow \\ H^1(X_D, \mathbf{C}) & = & H^1(X_{D^*}, \mathbf{C}) \end{array}$$

which defines the operators on $H^1(X_D, \mathbf{C})$ for which we use the notations $\underline{\star}_D$ and $\underline{\star}_{D^*}$; they also satisfy

$$\underline{\star}_D \underline{\star}_{D^*} = -1.$$

Denote by $\wedge$ the cup-product on $H^1(X_D, \mathbf{C})$.

For x, y from $H^1(X_D, \mathbf{C})$, define

$$(x, y) = \star_D(x) \wedge y,$$

Proposition. *(,) is symmetric and positively defined. Denote by $\{\ell_\alpha | \alpha = 1, \ldots, 2g\}$ the set of eigenvalues of $\underline{\star}_D$ and $\{L_\alpha\}$ the corresponding eigenspaces.*

Proposition.

(a) $\forall \alpha, \ell_\alpha \notin \mathbf{R}$.

(b) *For any $\ell \in \{\ell_\alpha\}$ also $-\ell \in \{\ell_\alpha\}$; the corresponding eigenspaces are equidimensional.*

Set $L_D = \oplus_{\mathrm{Im} L_\alpha < 0} L_\alpha$. It follows from the last proposition that $\dim (L_D) = g$. Thus the approximate Jacobians can be defined as

$$L_D \backslash H^1(X_D, \mathbf{C}) / H^1(X_D, \mathbf{Z}).$$

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