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Braided monoidal 2-categories and Manin–Schechtman higher braid groups

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Abstract

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We study a certain coherence problem for braided monoidal 2-categories. For ordinary braided monoidal categories such a problem is well known to lead to braid groups: If we denote by $T(n)$ the pure braid group on n strands then this group acts naturally on each product $A_1 \otimes \cdots \otimes A_n$. It turns out that in the 2-categorical case we have to consider the so-called higher braid group $T(2, n)$ introduced by Manin and Schechtman. The main result is that $T(2, n)$ naturally acts by 2-automorphisms on the canonical 1-morphism $A_1 \otimes \cdots \otimes A_n \to A_n \otimes \cdots \otimes A_1$ for any objects $A_1, \ldots, A_n$.

Introduction

The notion of a braided monoidal category, introduced in [6, 10] serves as an algebraic framework for the theory of the Yang–Baxter equation. For any n objects $A_1, \ldots, A_n$ of such a category the structure data define the action on the product $A_1 \otimes \cdots \otimes A_n$ of pure braid group $T(n)$.

In [12] we have introduced the notion of a braided monoidal 2-category and shown how such structures are related to the Zamolodchikov tetrahedra equation—a 2-dimensional generalization of the Yang–Baxter equation, see [20, 21]. An important

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role in the formalism of [12] is played by special polytopes called resultohedra which were introduced in [7]. These polytopes generalize the two commutative triangles in the axioms of usual braiding, see [6, 10].

In the present paper we show that the coherence questions for braided monoidal 2-categories lead naturally to the so-called higher braid groups $T(2, n)$ introduced by Manin and Schechtman in [14]. The group $T(2, n)$ is defined as the fundamental group of the space of configurations of n affine complex lines in the complex plane $\mathbb{C}^2$ which are in general position and have fixed distinct (real) slopes. This configuration can be obtained from the affine space $\mathbb{C}^n$ by deleting $\binom{n}{3}$ hyperplanes H_{ijk} corresponding to configurations $(L_1, \ldots, L_n)$ such that $L_i \cap L_j \cap L_k \neq \emptyset$.

Note that the usual pure braid group $T(n)$ is the fundamental group of the configuration space of n distinct points on a complex line [4]. In [14] Manin and Schechtman introduced groups $T(k, n)$ for any $k < n$ by considering configurations of n hyperplanes in $\mathbb{C}^k$. The group $T(n)$ is in this notation $T(1, n)$. In the present paper we do not use higher braid groups for $k > 2$.

Our main result, Theorem 4.1, is as follows. Take n objects $A_1, \ldots, A_n$ of a braided monoidal 2-category $\mathscr{A}$. Let γ be any reduced decomposition of the maximal permutation $(n, n-1, \ldots, 1)$ of n symbols into a product of elementary transpositions. The braiding 1-morphisms define a 1-morphism $R_\gamma: A_1 \otimes \cdots \otimes A_n \to A_n \otimes \cdots \otimes A_1$. We prove that the structure 2-morphisms of the braiding define an action of the group $T(2, n)$ on each 1-morphism R_γ by 2-isomorphisms.

The construction of the $T(2, n)$-action is of some interest so we describe it here. The braid relations for the structure morphisms in a braided monoidal 1-category are usually described by means of the Yang–Baxter hexagon whose vertices are all permuted products of three objects, see [6, 10]. The commutativity of this hexagon is not among the axioms of a braiding. These axioms include some (more fundamental) triangles instead. To prove the commutativity of the hexagon, one decomposes it into two braiding triangles and a square of naturality (cf. Section 2.2). This argument goes back (in a slightly different context) at least to Stasheff [19]. An important fact is that there are two ways of decomposing a hexagon and hence two ways to prove its commutativity. In the 2-categorical context any of the two decompositions gives rise to a 2-morphism between the composite 1-morphisms corresponding to paths constituting the boundary of the hexagon. This apparent ambiguity plays a crucial role in our construction. Namely, we associate to the two 2-morphisms arising in this way two paths in the complex configuration space encircling the hyperplane H_{ijk} in the complex domain from two sides. Here i, j, k are the numbers of some objects among $A_1, \ldots, A_n$.

In fact, we construct the action not of the group $T(2, n)$ but of the fundamental groupoid of the configuration space with respect to a natural choice of base points. To different base points there correspond different 1-morphisms from $A_1 \otimes \cdots \otimes A_n$ to $A_n \otimes \cdots \otimes A_1$.

The description of the above fundamental groupoid by generators and relations can be extracted from the work of Salvetti [17] on the topology of complements of configurations of complex hyperplanes with real equations. In particular, the generators and relations in the groupoid are local: each generator is situated near a wall (chamber of codimension 1) of a natural stratification of the real part $\mathbb{R}^n$ of $\mathbb{C}^n$ and each relation near a chamber of codimension 2. This locality is in fact the main advantage of groupoids over groups, see e.g. [5].

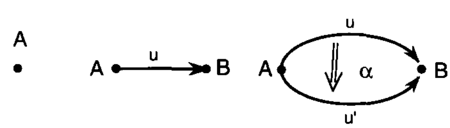
The description of the group $T(2, n)$ itself by generators and relations is more complicated. Such a description (for general $T(k, n)$) was obtained by Lawrence [13]. Still earlier, in 1977, Aomoto [1] has obtained a description by generators and relations of the fundamental group of the full configuration space of n hyperplanes in $\mathbb{C}P^k$ in general position. We do not use these descriptions in the present paper.

The outline of the paper is as follows. In Section 1 we recall the definition of a braided monoidal 2-category from [12]. In Section 2 we construct the Yang–Baxter hexagons in any such category and prove that the corresponding 2-morphisms satisfy certain relations (Zamolodchikov equations). These equations are stated as the commutativity of some 3-dimensional diagrams whose shape is the permutohedron—a certain convex polytope in $\mathbb{R}^3$ with 24 vertices corresponding to all permutations of four letters. Two of these relations (each involving only the hexagons of the same type) were already established in [12]. Section 3 is devoted to higher braid groups and corresponding groupoids. We give the description of these groupoids by (local) generators and relations. In Section 4 we formulate and prove our main result.

1. Braided monoidal 2-categories

1.1. 2-categories

By a 2-category we mean a strict (globular) 2-category in the sense of [8, 18]. Such a category posesses objects (0-morphisms), 1-morphisms and 2-morphisms which are visualized as points, arrows and 2-cells:



So 1-morphisms act between objects (we write $u: A \to B$) and 2-morphisms act between 1-morphisms with the same beginning and end (we write $\alpha: u \Rightarrow u'$). The 1-morphisms can be associatively composed; the 2-morphisms can be composed in two ways: horizontally and vertically. We denote the horizontal composition by $*_0$ and the vertical composition by $*_1$. In particular, for any two objects A,B of a 2-category $\mathscr{A}$ we have an ordinary category $\mathrm{Hom}_{\mathscr{A}}(A, B)$ whose objects are 1-morphisms in $\mathscr{A}$ from A to B and whose morphisms are 2-morphisms in $\mathscr{A}$ between these 1-morphisms.

For examples of 2-categories we refer the reader to [12, 18]. Note, for instance, that any strict monoidal category in the usual sense [6, 10] can be regarded as a 2-category with one object. This was first remarked by Bénabou [2].

We freely use the language of pasting (2-dimensional composition) in 2-categories. For formal treatment of pasting, see [9, 16]. This language permits us to speak about commutative polytopes in 2-categories in a similar way as one speaks of commutative polygons in usual categories.

1.2. Monoidal 2-categories

A (semi-strict) monoidal structure [12] on a 2-category $\mathscr{A}$ is a collection of the following data:

(1) An object **1**.

($\bullet \otimes \bullet$) For any pair of objects, $A,B \in \mathscr{A}$ an object $A \otimes B$, denoted also AB.

($\bullet \otimes \rightarrow$) For any object $A \in \mathscr{A}$ and a 1-morphism $v: B \rightarrow B'$ a 1-morphism $A \otimes v: A \otimes B \rightarrow A \otimes B'$.

($\rightarrow \otimes \bullet$) For any 1-morphism $u: A \rightarrow A'$ and any object B a 1-morphism $u \otimes B: A \otimes B \rightarrow A' \otimes B$.

($\bullet \otimes \overset{\rightarrow}{\underset{\rightarrow}{\downarrow}}$) For any object $A \in \mathscr{A}$ and any 2-morphism $\beta: v \Rightarrow v'$ a 2-morphism $A \otimes \beta: A \otimes v \Rightarrow A \otimes v'$.

($\overset{\rightarrow}{\underset{\rightarrow}{\downarrow}} \otimes \bullet$) For any 2-morphism $\alpha: u \Rightarrow u'$ and any object $B \in \mathscr{A}$ a 2-morphism $\alpha \otimes B: u \otimes B \Rightarrow u' \otimes B$.

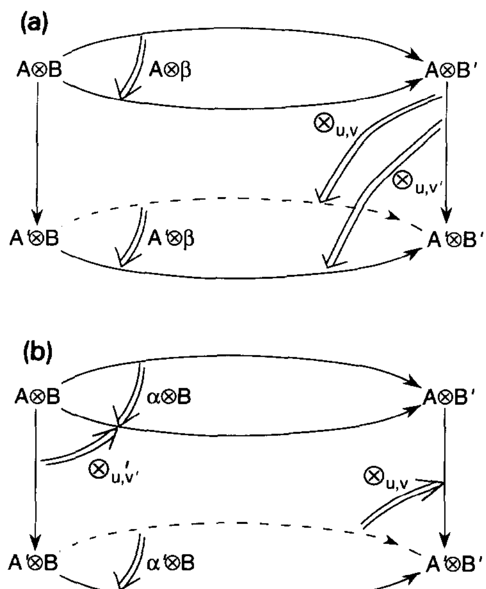
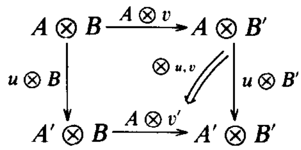


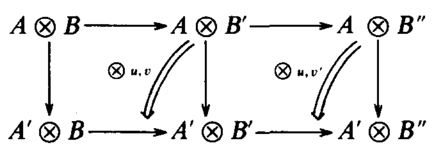
Fig. 1.

$(\to \otimes \to)$ For any two 1-morphisms $u: A \to A'$ and $v: B \to B'$ a 2-isomorphism

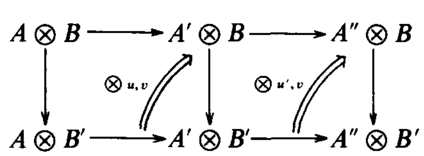


These data should satisfy the following conditions:

(1) For any object $A \in \mathscr{A}$ the correspondences $X \mapsto A \otimes X$ and $X \mapsto X \otimes A$ define (strict) 2-functors $\mathscr{A} \to \mathscr{A}$.
(2) For any i-morphism X, $i = 0,1,2$ one has $X \otimes \mathbf{1} = \mathbf{1} \otimes X = X$.
(3) If A,B are objects of $\mathscr{A}$ and γ is an i-morphism, $i = 0,1,2$, then
$$A\otimes(B\otimes\gamma) = (A\otimes b)\otimes\gamma,\ A\otimes(\gamma\otimes B) = (A\otimes\gamma)\otimes B,\ \gamma\otimes(A\otimes B) = (\gamma\otimes A)\otimes B.$$
(4) $(\to \otimes \overset{\to}{\underset{\to}{\downarrow}})$ For any 1-morphism $u: A \to A'$ and any 2-morphism $\beta: v \Rightarrow \overline{v'}$ where $v,v': B \to B'$, the cylinder in Fig. 1(a) is commutative.
(5) $(\overset{\to}{\underset{\to}{\downarrow}} \otimes \to)$ For any 2-morphism $\alpha: u \Rightarrow u'$ where $u,u': A \to A'$ and any 1-morphism $v: B \to B'$, the cylinder in Fig. 1(b) is commutative.
(6) $(\to \otimes \to\ \to)$ For any 1-morphisms $u: A \to A'$ and $v: B \to B'$, $v': B' \to B''$ the 2-morphism $\otimes_{u,v'v}$ coincides with the pasting of the following diagram:



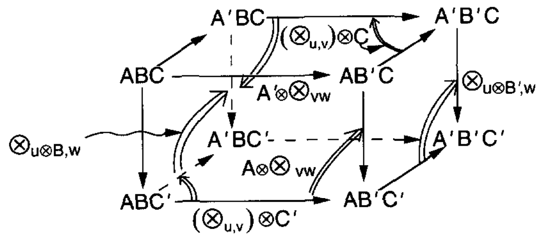
(7) $(\to\ \to \otimes \to)$ For any 1-morphisms $u: A \to A'$, $u': A' \to A''$ and $v: B \to B'$ the 2-morphism $\otimes_{u'u,v}$ coincides with the pasting of the following diagram:



(8) $(\bullet \otimes \to \otimes \to)$ For any object A and any 1-morphisms $v: B \to B', w: C \to C'$ we have $\otimes_{A \otimes v, w} = A \otimes \otimes_{v,w}$;
(9) $(\to \otimes \bullet \otimes \to)$ For any 1-morphism $u: A \to A'$, any object B and any 1-morphisms $w: C \to C'$ we have $\otimes_{u \otimes B, w} = \otimes_{u, B \otimes w}$;
(10) $(\to \otimes \to \otimes \bullet)$ For any 1-morphisms $u: A \to A'$, $v: B \to B'$ and any object C we have $\otimes_{u, v \otimes C} = (\otimes_{u,v}) \otimes C$.

This ends the definition of a semistrict 2-monoidal category. We shall sometimes use the notation $A_1, \ldots, A_n$ for the product $A_1 \otimes \cdots \otimes A_n$ of n objects.

Lemma 1.1. *For any three* 1*-morphisms* $u: A \to A'$, $v: B \to B'$, $w: C \to C'$ *the cube*



is commutative.

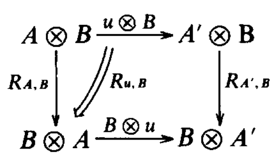
Proof. Obvious. □

1.3. Braided monoidal 2-categories

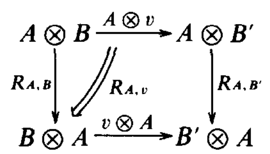
Let $\mathscr{A}$ be a semistrict monoidal 2-category. A braiding in $\mathscr{A}$ is a collection of the following data (cf. [12]):

($\bullet \otimes \bullet$) 1-morphisms (not necessarily isomorphisms or equivalences) $R_{A,B}: A \otimes B \to B \otimes A$ given for any pair A,B of objects of $\mathscr{A}$.

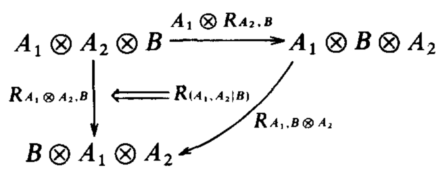
($\to \otimes \bullet$) For any 1-morphism $u: A \to A'$ and any object B, a 2-isomorphism



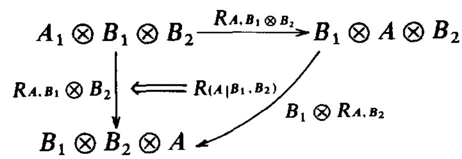
($\bullet \otimes \to$) For any object A and any 1-morphism $v: B \to B'$, a 2-isomorphism



(($\bullet \otimes \bullet$)$\bullet$) For any objects A_1, A_2, B, a 2-isomorphism

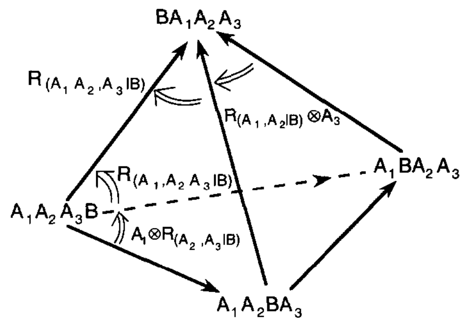


($\bullet \otimes (\bullet \otimes \bullet)$) For any objects A, B_1, B_2, a 2-isomorphism

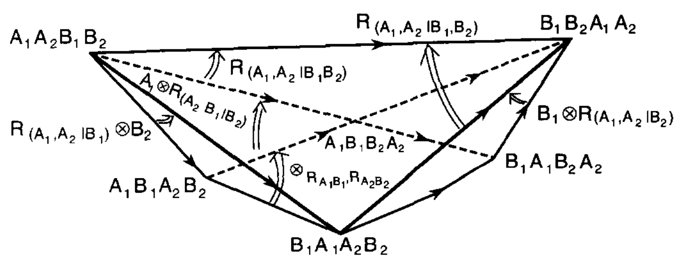


These data should satisfy the following conditions:

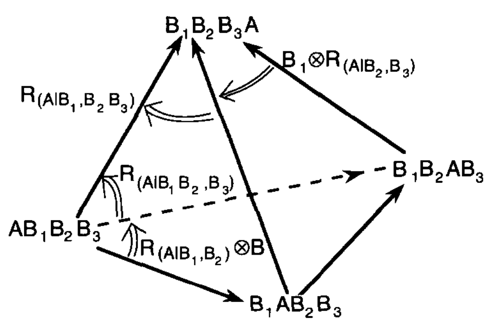
$((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ For any objects A_1, A_2, A_3, B the following tetrahedron is commutative:



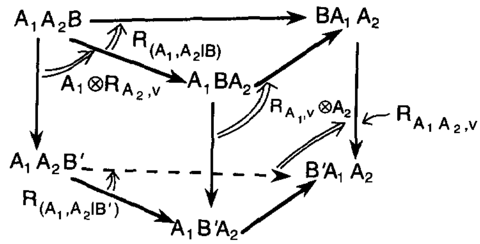
$((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ For any objects A_1, A_2, B_1, B_2 the following polytope is commutative:



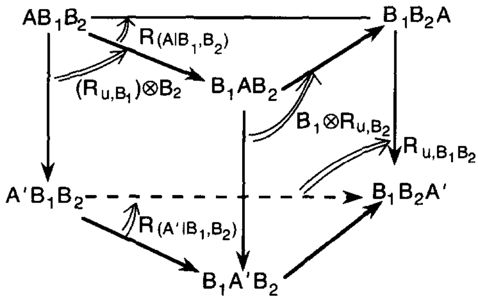
$(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ For any objects A, B_1, B_2, B_3 the following tetrahedron is commutative:



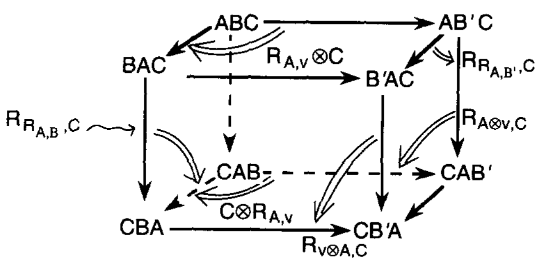
$((\bullet \otimes \bullet) \otimes \rightarrow)$ For any two objects A_1, A_2 and a 1-morphism $v: B \rightarrow B'$ the following triangular prism is commutative:



($\rightarrow \otimes (\bullet \otimes \bullet)$) For any 1-morphism $u: A \rightarrow A'$ and two objects B_1, B_2 the following diagram is commutative:

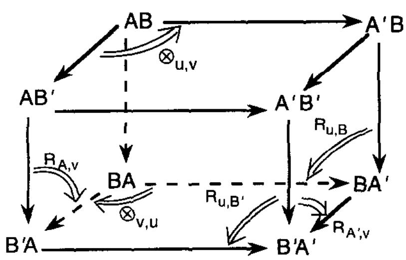


($(\bullet \otimes \rightarrow) \otimes \bullet$) For any object A, a 1-morphism $v: B \rightarrow B'$ and an object C the diagram

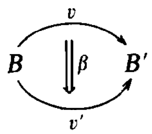


($((\rightarrow \otimes \bullet) \otimes \bullet)$, $(\bullet \otimes (\bullet \otimes \rightarrow))$, $(\bullet \otimes (\rightarrow \otimes \bullet))$) Similarly to the above, left to the reader.

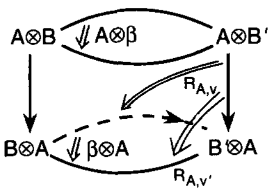
($\rightarrow \otimes \rightarrow$) For any two 1-morphisms $u: A \rightarrow A'$, $v: B \rightarrow B'$ the following cube is commutative:



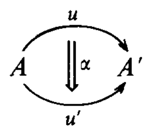
($\bullet \otimes \overset{\rightarrow}{\underset{\rightarrow}{\Downarrow}}$) For any object A and any 2-morphism



the following cylinder is commutative:

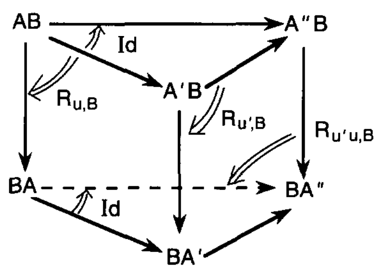


($\Downarrow \otimes \bullet$) A similar cylinder for a 2-morphism



and an object B.

($\to \to \otimes \bullet$) For any composable pair $A \xrightarrow{u} A' \xrightarrow{u'} A''$ of 1-morphisms and an object B the following diagram is commutative:



($\bullet \otimes \to \to$) Similar.

2. Permutohedral diagrams

2.1. The permutohedron

We want to consider a convex polytope whose vertices correspond to permutations of n letters, $n \geq 2$.

By definition, the $(n-1)$-dimensional *permutohedron* P_n (see [3, 15]) is the convex hull of $n!$ points $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$, where σ runs over all the permutations of $\{1, \ldots, n\}$.

It is clear from this definition that P_n lies in the hyperplane

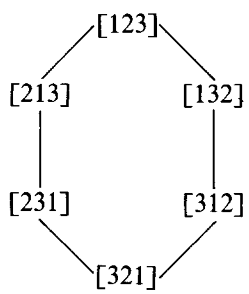
$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum x_i = n(n-1)/2\}$$

and its dimension equals $(n-1)$.

Let S_n be the symmetric group of all permutations of $\{1, \ldots, n\}$. To define a permutation $\sigma \in S_n$ it suffices to specify a sequence $(\sigma(1), \ldots, \sigma(n))$. We shall write $\sigma = (\sigma(1), \ldots, \sigma(n))$. For example, (312) is the permutation of $\{1, 2, 3\}$ sending 1 to 3, 2 to 1 and 3 to 2.

For any $\sigma \in S_n$ let σ^{-1} denote the inverse permutation. We shall denote by $[\sigma]$ the point $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \in P_n$. The advantage of this notation will be seen from the examples.

The 2-dimensional permutohedron P_3 is the hexagon



and the permutohedron P_4 will be drawn in Section 2.3.

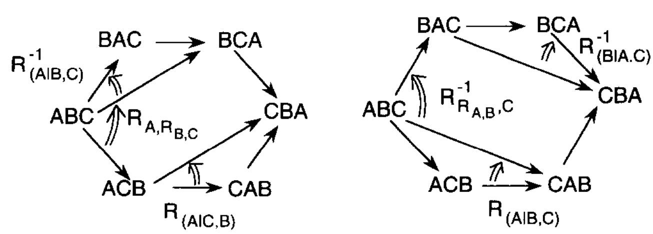
We can notice that two vertices of P_n are connected by an edge if and only if the corresponding permutations are obtained from each other by interchanging two numbers in consecutive positions. This fact is true for any permutohedron P_n, and follows from the description of all the faces of P_n given in [3, 15].

2.2. Two Yang–Baxter hexagons

Let A, B, C be three objects of a braided monoidal 2-category $\mathscr{A}$. Define the 2-morphisms

$$S^{+}_{A,B,C}, S^{-}_{A,B,C} : (C \otimes R_{A,B}) *_0 (R_{A,C} \otimes B) *_0 (A \otimes R_{B,C})$$
$$\to (R_{B,C} \otimes A) *_0 (B \otimes R_{A,C}) *_0 (R_{A,B} \otimes C)$$

by the following pasting diagrams:



These diagrams are 2-categorical analogs of the two *ways of proof* of the Yang–Baxter relations in a braided monoidal 1-category. They have been first considered by Stasheff [19].

2.3. *Eight Zamolodchikov equations*

Let A, B, C, D be objects of a braided monoidal 2-category. Consider a pasting diagram of the form given in Fig. 2 where S means either S^+ or S^-—the 2-morphisms defined in Section 2.2. We claim that there are eight choices of signs $+$ and $-$ for the Ss which make the diagram commutative.

The boundary of the permutohedron consists of two "halves": the front one and the back one. The commutativity means that the 2-morphisms given by the pasting of the front half equals the 2-morphism given by the pasting of the back half.

Each of the two halves contains four hexagons. Let us number the hexagons on each half in the order in which they are taken in the evaluation of pasting. So the front hexagons will be denoted $H_1, \ldots, H_4$ and the back ones $H'_1, \ldots, H'_4$, where H_1 is the right bottom hexagon, H_2 the right top one, H_3 the left top one, H_4 the left bottom

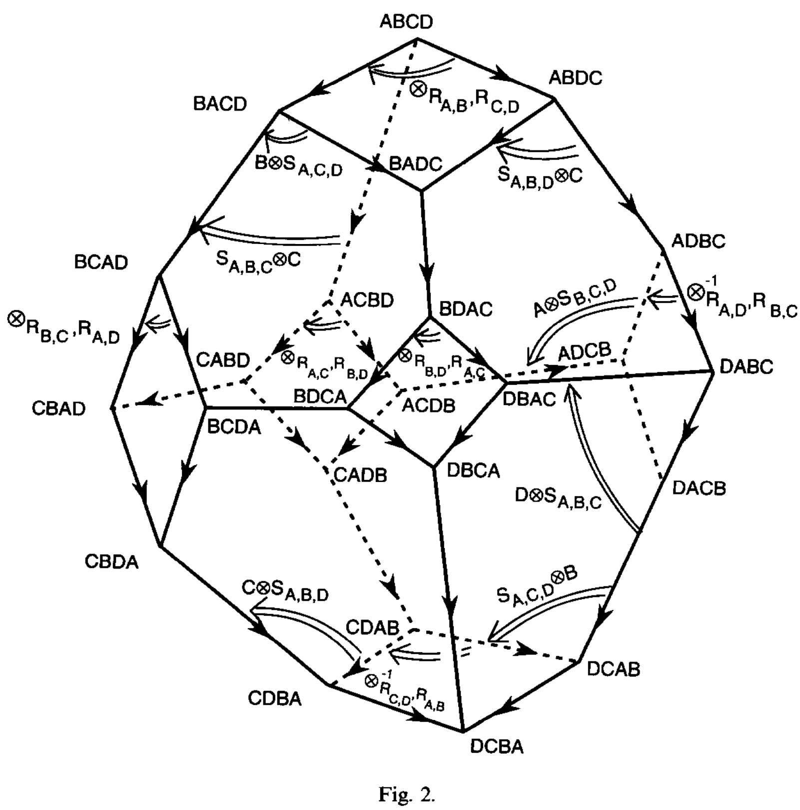


Fig. 2.

one. The hexagon H_i' is diametrally opposite to H_{5-i}. The commutativity of the permutohedron can be symbolically expressed by the equation

$$H_4 H_3 H_2 H_1 = H_4' H_3' H_2' H_1'.$$

In this equality we suppress, therefore, the notations for square-shaped 2-morphisms and for 1-morphisms.

Theorem 2.1. *In the above symbolic notation we have the following eight equalities between 2-morphisms S^+ and S^-:*

$$\text{(Z1)} \quad S^+S^+S^+S^+ = S^+S^+S^+S^+,$$

$$\text{(Z2)} \quad S^-S^+S^+S^+ = S^+S^+S^+S^-,$$

$$\text{(Z3)} \quad S^-S^-S^+S^+ = S^+S^+S^-S^-,$$

$$\text{(Z4)} \quad S^-S^-S^-S^+ = S^+S^-S^-S^-,$$

$$\text{(Z5)} \quad S^-S^-S^-S^- = S^-S^-S^-S^-,$$

$$\text{(Z6)} \quad S^+S^-S^-S^- = S^-S^-S^-S^+,$$

$$\text{(Z7)} \quad S^+S^+S^-S^- = S^-S^-S^+S^+,$$

$$\text{(Z8)} \quad S^+S^+S^+S^- = S^-S^+S^+S^+,$$

Here the factors on the left-hand sides of any equality relate to the front part of the diagram and those on the right-hand side to the back part of the diagram.

For example, (Z3) means that if we put the 2-morphisms of the type S^- into the hexagons H_4, H_3, H_2', H_1' and the 2-morphisms of the type S^+ into the other four hexagons then we get a commutative permutohedron.

Proof. To prove the commutativity of the permutohedral diagram corresponding to any given equation, we shall decompose the permutohedron into smaller polytopes whose commutativity will be implied by the axioms of braided monoidal 2-category.

In [12] we have constructed, for any codimension-1 face Γ of the permutohedron P_n a polyhedral decomposition $D(\Gamma)$ of P_n whose polyhedra of maximal dimension correspond to nonempty subfaces (of all dimensions) of Γ. The decompositions of the hexagon exhibited in Section 2.2 are examples of this construction. In general, for opposite faces Γ, Γ' the corresponding decompositions coincide.

For the 3-dimensional permutohedron there are two kinds of faces: squares and hexagons. The corresponding decompositions are shown in Figs. 3 and 4.

Taking Γ to be the bottom (or top) square we get a decomposition into 9 polytopes corresponding to subfaces of this square. To vertices of the square there correspond polytopes appearing in the axiom $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ in Section 1.3. The commutativity of these polytopes follows from this axiom. To edges of the square there correspond triangular prisms whose commutativity follows from the naturality of the tensor product with respect to 1- and 2-morphisms (axioms 4 and 5 of Section 1.2). To the single 2-face of the square there corresponds the cubical diagram whose

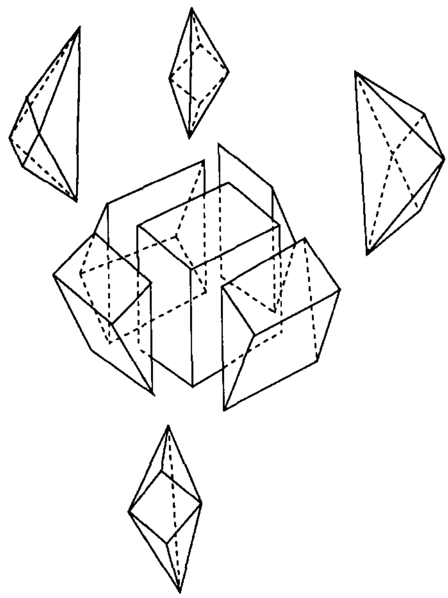


Fig. 3.

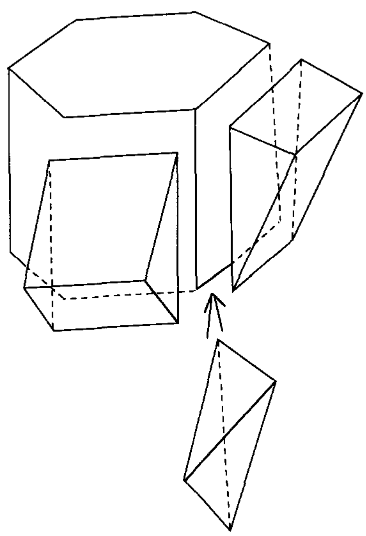


Fig. 4.

commutativity follows from the axiom ($\rightarrow \otimes \rightarrow$) of Section 1.3. This decomposition induces the decomposition of each hexagon and thus assigns to the hexagon a 2-morphism S^+ or S^-. By reading them from the diagram we deduce the validity of the equation (Z3).

The equation (Z6) is obtained by a similar decomposition associated to any of the two squares on the sides of the permutohedron.

Taking Γ to be the hexagon H_1, we get a decomposition into:

- 6 tetrahedra (corresponding to vertices of H_1) which occur in the axiom $((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ of Section 1.3;
- 6 triangular prisms (corresponding to edges of H_1) whose commutativity follows from naturality of $\otimes$;
- one hexagonal prism (corresponding to the only 2-face of H_1). We are free to decompose both hexagons of this prism in a way corresponding either to S^+ or to S^- and the resulting diagram will be commutative.

This gives two equations, (Z4) and (Z5):

$$S^- S^- S^- S^\pm = S^\pm S^- S^- S^- .$$

Other equations from (Z1)–(Z8) are obtained in the similar way, using the other hexagonal faces of the permutohedron. □

3. Higher braid groups

3.1. Modular configurations and higher braid groups

The (pure) braid group $T(n)$ (see [4]) is the fundamental group of the space

$$\mathbb{C}^n_* = \{(x_1, \ldots, x_n \in \mathbb{C}^n | x_i \neq x_j \text{ for } i \neq j\},$$

which is the complement of the configuration of hyperplanes $\{x_i = x_j\}$ in $\mathbb{C}^n$. We are going to describe the generalization of the groups $T(n)$ due to Manin and Schechtman [14].

Let $l_1, \ldots, l_n$ be n distinct lines in $\mathbb{C}^2$ containing the point 0. Let $f_i(x, y) = 0$ be linear equations of l_i. Consider the coordinate space $\mathbb{C}^n$. For any triple $1 \leq j < k < m \leq n$ we define the three affine lines $f_j(x, y) = c_j$, $f_k(x, y) = c_k$, $f_m(x, y) = c_m$ have a common point. If $f_j(x, y) = a_j x + b_j y$ then H_{jkm} is given by the linear equation

$$\phi(c_1, \ldots, c_n) = \det \begin{pmatrix} a_j & b_j & c_j \\ a_k & b_k & c_k \\ a_m & b_m & c_m \end{pmatrix} = 0$$

on $(c_1, \ldots, c_n)$.

The union of the hyperplanes H_{ijk} is called the *modular configuration* for $(l_1, \ldots, l_n)$ and denoted by $\Lambda(l_1, \ldots, l_n)$. Clearly the topology of the complement $\mathbb{C}^n - \Lambda(l_1, \ldots, l_n)$ does not depend on the choice of lines l_i provided the lines are distinct. The fundamental group $\pi_1(\Lambda(l_1, \ldots, l_n))$ is called the *higher braid group* and denoted $T(2, n)$.

In [14], Manin and Schechtman have defined a series of groups $T(k, n)$ for any $k < n$ by using modular configurations for n hyperplanes in the k-dimensional space (instead of $\mathbb{C}^2$). The group $T(1, n)$ is the pure braid group $T(n)$. In the present paper we will not need the higher braid groups for $k > 2$.

3.2. *Higher braid groupoids (coarse version)*

In the definition of the higher braid group $T(2, n)$ the choice of lines $l_1, \ldots, l_n \subset \mathbb{C}^2$ was inessential. We want to choose them in a special way. Namely, we take l_m to be real and more precisely to be defined by the real equation $mx - y = 0$. So the lines become ordered by their slopes. The hyperplanes $H_{klm} \subset \mathbb{C}^n$ forming the modular configuration will also have real equations. Denote the complement $\mathbb{C}^n - \bigcup H_{klm}$ by $\mathbb{C}^n_{**}$ and its real part $\mathbb{R}^n - \bigcup H_{klm}$ by $\mathbb{R}^n_{**}$.

Connected components of $\mathbb{R}^n_{**}$ are open unbounded convex polyhedra. Let us take one point x_K inside each component K. Now define the *coarse higher braid groupoid* $\mathcal{TC}(2, n)$ to be the fundamental groupoid of $\mathbb{C}^n_{**}$ with respect to the system of base points $x_K \in \mathbb{R}^n_{**}$.

3.3. *Higher braid groupoid (fine version)*

If $p = (x, y)$ is a point of $\mathbb{R}^2$, we shall write $x = x(p)$, $y = y(p)$ thus regarding x and y as coordinate functions on $\mathbb{R}^2$.

Let $(L_1, \ldots, L_n)$ be a configuration of (real) lines in $\mathbb{R}^2$. We shall say that this configuration is *super-generic* if, first of all, no three lines of L_k intersect and no two are parallel (i.e. they are in general position) and, second, all the $\binom{n}{2}$ numbers $y(L_k \cap L_m)$ are distinct.

We shall consider lines L_k given by the equations $y = kx - c_k$ where $(c_1, \ldots, c_n) \in \mathbb{R}^n$. The ordinate $y(L_k \cap L_m)$ is, under this assumption, a linear function of $(c_1, \ldots, c_n)$,

$$y(L_k \cap L_m) = \frac{mc_k - kc_m}{k - m}.$$

Let $\mathbb{R}^n_{***} \subset \mathbb{R}^n_{**}$ be the space of those $(c_1, \ldots, c_n)$ for which the configuration of $L_1, \ldots, L_n$ is super-generic. Clearly, the space $\mathbb{R}^n_{***}$ is the complement in $\mathbb{R}^n$ to a configuration of hyperplanes which contains the modular configuration formed by hyperplanes H_{klm} defined in Section 3.1 and also the hyperplanes $W_{k,m,p,q}$ defined by the condition $y(L_k \cap L_m) = y(L_p \cap L_q)$. The hyperplanes H_{klm} will be called *essential* since they are deleted and the hyperplanes $W_{k,m,p,q}$ will be called *dummy* since they are not deleted and are only used to define new base points.

We denote this new configuration of hyperplanes by Ξ or $\Xi(l_1, \ldots, l_n)$, where l_m is the line $y = mx$.

Each connected component of $\mathbb{R}^n_{**}$ becomes subdivided into several connected components of $\mathbb{R}^n_{***}$ which also are convex polyhedra.

Let us take one point y_E inside each component E of $\mathbb{R}^n_{***}$. We define the *fine higher braid groupoid* $\mathcal{TF}(2, n)$ to be the fundamental groupoid of the space $\mathbb{C}^n_{**}$ with respect to the system of base points y_E. Thus $\mathcal{TC}(2, n)$ and $\mathcal{TF}(2, n)$ are fundamental groupoids of the same space and differ only in the choice of the set of base points. More precisely, $\mathcal{TF}(2, n)$ differs from $\mathcal{TC}(2, n)$ only by putting several base points inside each component of $\mathbb{R}^n_{**}$ instead of one. In particular, the group of automorphisms of any object of any of these groupoids is the higher braid group $T(2, n)$.

We are paying considerable attention to this version of braid groupoid since it is this version which allows us to give a nice formulation of Theorem 4.1.

3.4. Generators and relations in fundamental groupoids

The space $\mathbb{C}^n_{**}$ whose fundamental group (or groupoid) we are interested in, is the complement in $\mathbb{C}^n$ of a configuration of hyperplanes with real equations. Such spaces were studied by Salvetti [17] who constructed a combinatorial model for their homotopy types. This gives a description of the fundamental groupoid by generators and relations. Let us recall Salvetti's construction.

Let $H = (H_1, \ldots, H_m)$ be a configuration of real hyperplanes in $\mathbb{R}^n$ given by affine equation $\phi_i = 0$. Let $H_\mathbb{C}$ be the complexification of H, i.e. the configuration of hyperplanes $H_{i,\mathbb{C}} \subset \mathbb{C}^n$ given by the same equations ϕ_i. We are interested in the fundamental groupoid of $\mathbb{C}^n - H_\mathbb{C}$ with respect to the natural system of base points: one point x_K inside each connected component of $\mathbb{R}^n - H$.

Define *chambers* of H to be equivalence classes of points in $\mathbb{R}^n$ by the following equivalence relation: $x \approx y$ if the set of H_i containing x is the same as the set of H_i containing y. Thus chambers are convex polyhedra forming a cellular decomposition of $\mathbb{R}^n$ which we call the *chamber complex.*

Let now M be a chamber of codimension 1 (a wall) and H_i the hyperplane from H in which M is open. The chamber M lies in the boundary of two n-dimensional chambers, say, K and K'. Let $h_{KK'}$ be the path in $\mathbb{R}^n$ which joins base points x_K and $x_{K'}$ and intersects only the wall M. This path does not lie in $\mathbb{C}^n - H_\mathbb{C}$ (since it intersects M) and we define a new path $g_{KK'}$ by perturbing $h_{KK'}$ near the intersection point $h_{KK'} \cap M$ in order to go around $H_{i,\mathbb{C}}$ in the complex domain. More precisely, suppose that f_i is negative in K and positive and K'. Then $g_{KK'}$ goes around $H_{i,\mathbb{C}}$ in such a way that $\mathrm{Im}(f_i)$ (the imaginary part) remains nonnegative (see Fig. 5.).

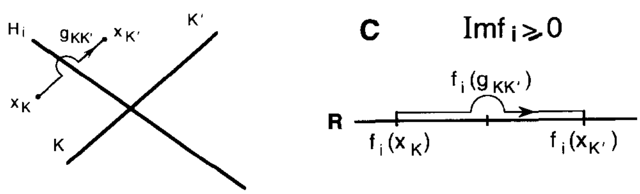
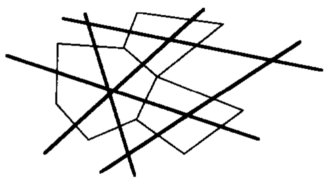


Fig. 5.

Note that the definition of $g_{KK'}$ does not depend on the choice of sign of f_i. On the other hand, $g_{K'K}$ does not coincide with $g_{KK'}$ and always goes around the same hyperplane from the other side so that $g_{KK'}g_{K'K}$ is a loop encircling this hyperplane. The paths $g_{KK'}$ will be the generators in our description of the fundamental groupoid. Thus each chamber of codimension 1 gives rise to two generators.

Relations among the generators will correspond to chambers of codimension 2. It is convenient to consider the *dual complex* $\hat{C}$ of the chamber complex. By definition, i-dimensional cells of $\hat{C}$ are in bijection with $(n-i)$-dimensional chambers; in particular, chambers of dimension n correspond to vertices of $\hat{C}$:



If M is a chamber of H we shall denote by X_M the corresponding cell of $\hat{C}$. This is in accord with the notation x_K for the chosen base point inside a maximal chamber K. It is convenient to change the labelling of the generators $g_{KK'}$ constructed above. Namely, we shall think that the generators correspond to pairs consisting of a 1-cell of $\hat{C}$ and one of its two vertices. More precisely, if (Y, y) is such a pair, $y = x_K$ and the other end of Y is $x_{K'}$ then we shall define the generator $g(Y, y)$ to be $g_{KK'}$.

Let now X be a 2-dimensional cell of $\hat{C}$ i.e. $X = X_M$ for some chamber M of codimension 2. If M is contained in p hyperplanes from H then X is a $2p$-gon. Let $x = x_K$ be one of the $2p$ vertices of X. To the pair (X, x) we associate a relation $R(X, x)$ among generators $g(Y, y)$ as follows. Let x' be the vertex in the $2p$-gon X opposite to x. Then we have two paths $(Y_1, \ldots, Y_p)$ and $(Y'_1, \ldots, Y'_p)$ on the boundary of X from x to x'. We denote by y_i the vertex of the segment Y_i closest to x in the sense of minimal length of an edge path (the notion of a closest vertex makes sense in a polygon with an even number of vertices, which is the case). Similarly let y'_i be the vertex of Y'_i closest to x. We define the relation $R(X, x)$ to have the form:

$$g(Y_p, y_p)g(Y_{p-1}, y_{p-1}) \ldots g(Y_1, y_1)$$

$$= g(Y'_p, y'_p)g(Y'_{p-1}, y'_{p-1}) \ldots g(Y'_1, y'_1).$$

Summarizing, we have the following structure data:

(i) Objects correspond to vertices x of $\hat{C}$.

(ii) Generators correspond to pairs (Y, y) where Y is a 1-cell of $\hat{C}$ and y is one of two vertices of Y.

(iii) Relations correspond to pairs (X, x) where X is a 2-cell of $\hat{C}$ and x is one of its vertices.

3.5. Generators and relations in coarse higher braid groupoids

The coarse higher braid groupoid $\mathscr{TC}(2, n)$ is a particular case of the groupoids considered in Section 3.4, when the configuration in question is the modular configuration $\Lambda = \Lambda(l_1, \ldots, l_n)$, the line l_m being given by $y = mx$. Thus generators of $\mathscr{TC}(2, n)$ are paths $g_{KK'}$ for all pairs of adjacent components (codimension 0 chambers) of $\mathbb{R}^n_{**}$. The relations come from codimension-2 chambers of the configuration Λ. By definition of the modular configuration, there can be only two types of behavior of the configuration Λ near a chamber M of codimension 2:

(I) The chamber M lies on four hyperplanes from Λ namely on H_{ijk}, H_{ijl}, H_{ikl}, H_{jkl} for some $i < j < k < l$.

(II) The chamber M lies on just two hyperplanes from Λ namely H_{ijk} and H_{pqr} where $(i, j, k\} \cap \{p, q, r\}$ contains no more than one element.

Thus the relations in $\mathscr{TC}(2, n)$ will have the form of commutative octagons and commutative squares.

3.6. Generators and relations in fine higher braid groupoids

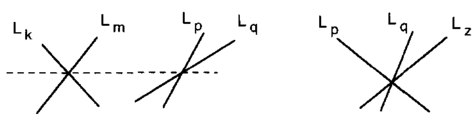
The fine higher braid groupoid $\mathscr{TF}(2, n)$ differs from $\mathscr{TC}(2, n)$ only by putting several base points inside each component of $\mathbb{R}^n_{**}$ instead of one. It is, therefore, easy to compare these two groupoids using the modular configuration Λ and the bigger configuration Ξ (see Section 3.3). We have only to take into account the presence of dummy hyperplanes $W_{k,l,p,q}$.

The description of generators is as follows.

G1 For every two adjacent components E,E' of $\mathbb{R}^n_{***}$ lying in the same component K of $\mathbb{R}^n_{**}$ there is one ("dummy") generator $h_{EE'}: y_E \to y_{E'}$ which coincides with $h^{-1}_{E'E}$. It corresponds to the path joining y_E and $y_{E'}$ which lies entirely inside K.

G2 For every two adjacent components E,E' of $\mathbb{R}^n_{***}$ lying in two different components K,K' of $\mathbb{R}^n_{**}$ there are two generators $g_{EE'}$ and $g_{E'E}$ which go around the hyperplane of the modular configuration separating E from E' and K from K'.

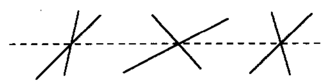
Instead of pairs of adjacent components of $\mathbb{R}^n_{***}$ we can as well speak about walls (chambers of codimension 1) separating such pairs of components. Any wall corresponding to G1 consists of line configurations $(L_1, \ldots, L_n)$ such that $y(L_k \cap L_m) = y(L_p \cap L_q)$ for some k, m, p, q. Any wall corresponding to G2 consists of $(L_1, \ldots, L_n)$ such that $L_p \cap L_q \cap L_r \neq \emptyset$ for some p, q, r. These situation are depicted below:



G1 G2

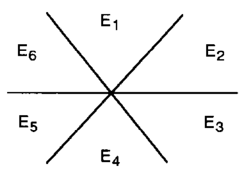
Relations between generators exhibited in G1 and G2 correspond to codimension-2 chambers of the configuration Ξ. Any chamber is defined by prescribing intersections of $L_1, \ldots, L_n$ and relative positions of heights of intersection points. Let us list all possible types of chambers and write down the corresponding relations.

R1. Suppose that our codimension-2 chamber (denote it by N) has the type



This means that three pairs of lines have intersections at the same height.

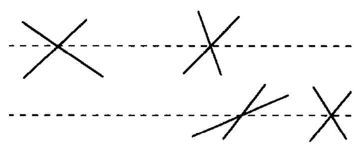
Such a chamber N lies in three hyperplanes of the configuration Ξ, all of them dummy. So there are 6 chambers of codimension 0 adjacent to N:



Denote them $E_1, \ldots, E_6$ (in a cyclic order). Then we have only one relation between dummy generators:

$$h_{E_3E_4}h_{E_2E_3}h_{E_1E_2} = h_{E_5E_4}h_{E_6E_5}h_{E_1E_6}.$$

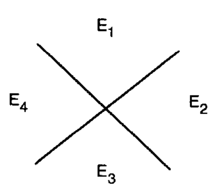
R2. Suppose that the chamber N has the type



This means that there are two pairs of lines, say (L_a, L_b), (L_c, L_d) and $(L_{a'}, L_{b'})$, $(L_{c'}, L_{d'})$ such that

$$y(L_a \cap L_b) = y(L_c \cap L_d), \qquad y(L_{a'} \cap L_{b'}) = y(L_{c'} \cap L_{d'}).$$

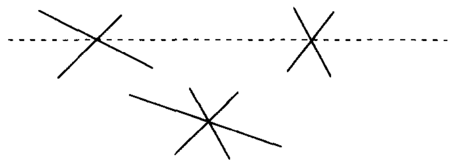
Such a chamber N lies in two hyperplanes, both of them dummy. Hence it is adjacent to 4 chambers of codimension 0. Denoting these chambers $E_1, \ldots, E_4$:



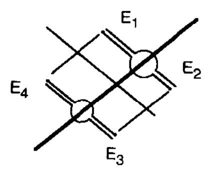
we get one relation

$$h_{E_2E_3}h_{E_1E_2} = h_{E_4E_3}h_{E_1E_4}.$$

R3. The chamber N has the type



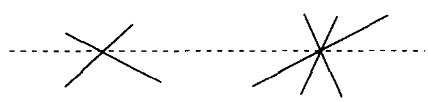
This means that two pairs of lines have intersections at the same height and there is another triple of lines which has a non-empty intersection. (note that one to two or three lines from the said triple may actually belong to the first group of four lines). Such a chamber lies in two hyperplanes from Ξ, one dummy and one essential:



There are 4 chambers of codimension 0 adjacent to N. Denoting them $E_1, \ldots, E_4$ we get two relations:

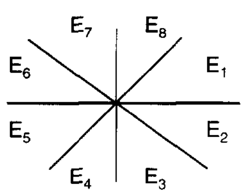
$$h_{E_2E_3} g_{E_1E_2} = g_{E_4E_3} h_{E_1E_4}, \qquad h_{E_2E_3} g^{-1}_{E_2E_1} = g^{-1}_{E_3E_4} h_{E_1E_4}.$$

R4. The chamber N has the type



This means that a triple of lines has a non-empty intersection at the same height as the intersection point of some other pair of lines.

Such a chamber N lies in one essential hyperplanes and 3 dummy hyperplanes and is thus adjacent to 8 chambers of codimension 0 which we number as follows:

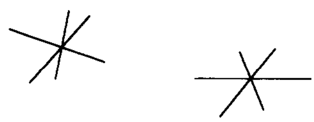


We get two relations:

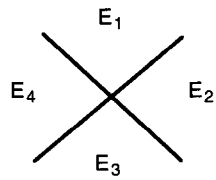
$$g_{E_6,E_5} h_{E_7E_6} h_{E_8E_7} h_{E_1E_8} = h_{E_4E_5} h_{E_3E_4} h_{E_2E_3} g_{E_1E_2},$$

$$g^{-1}_{E_5,E_6} h_{E_7E_6} h_{E_8E_7} h_{E_1E_8} = h_{E_4E_5} h_{E_3E_4} h_{E_2E_3} g_{E_2E_1}.$$

R5. The chamber N has the type



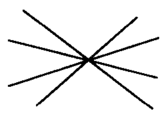
This means that there are two triples of lines having each a nonempty intersection. Such a chamber N lies in two hyperplanes of Ξ, both essential and is adjacent to 4 chambers of codimension 0, which we denote $E_1, \ldots, E_4$:



By Salvetti's recipe we get four relations. To write them down it is convenient to use periodic notation for E_i by agreeing that $E_{i+4} = E_{i-4} = E_i$, for $i = 1, \ldots, 4$. In this notations the relations have the form:

$$g_{E_{i+1}E_{i+2}} g_{E_iE_{i+1}} = g_{E_{i-1}E_{i-2}} g_{E_iE_{i-1}}, \quad i = 1, \ldots, 4.$$

R6. The chamber N has the type



This means that some four lines intersect in one point. Such a chamber lies in 4 hyperplanes of Ξ, all of them essential and hence it is adjacent to 8 chambers of codimension 0. Denoting these chambers cyclically by $E_1, \ldots, E_8$ and extending the notation with period 8 (similarly to R5) we get the eight relations:

$$g_{E_{i+3}E_{i+4}} g_{E_{i+2}E_{i+3}} g_{E_{i+1}E_{i+2}} g_{E_iE_{i+1}} = g_{E_{i-3}E_{i-4}} g_{E_{i-2}E_{i-3}} g_{E_{i-1}E_{i-2}} g_{E_iE_{i-1}}.$$

4. Main Theorem

4.1. Coherence problem for 2-braidings

Let $\mathscr{A}$ be a braided monoidal 2-category and $A_1, \ldots, A_n$ be objects of $\mathscr{A}$. We can construct a diagram $\mathbf{P}(A_1, \ldots, A_n)$ in the 2-category $\mathscr{A}$ whose vertices are the $n!$ permuted products $A_\sigma = A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$, $\sigma \in S_n$ (so even when two products coincide for some reason, we still consider them as distinct vertices of the diagram). For any two σ,τ such that σ,τ with respect to the weak Bruhat order and the vertices $[\sigma],[\tau]$ of the permutohedron P_n are joined by an edge e, we have a 1-morphism $U_e: A_\sigma \to A_\tau$ in $\mathscr{A}$ induced by the braiding. We associate this morphism to the edge e.

2-faces of the permutohedron P_n are of two types: squares and hexagons. Any square corresponds to interchanges of two independent pairs of objects, say A_i, A_j and A_k, A_l such that the object of each pair are adjacent. More precisely, vertices of such a square have the form

$$XA_iA_jYA_kA_lZ, \qquad XA_jA_iYA_k,A_lZ,$$

$$XA_iA_jYA_lA_kZ, \qquad XA_jA_iYA_lA_kZ,$$

where $i<j$, $k<l$ and X, Y, Z are permuted products of some of As. We fill such a square with the 2-morphism

$$\otimes_{X\otimes R_{A_i,A_j},Y\otimes R_{A_k,A_l}\otimes Z}. \tag{4.1}$$

Note that this 2-morphism coincides with the 2-morphism $\otimes_{X\otimes R_{A_i,A_j},Y\otimes R_{A_k,A_l}\otimes Z}$ by the axiom $(\to\otimes\bullet\otimes\to)$ of a monoidal 2-category.

Any hexagon of P_n corresponds to all permutations of a triple of adjacent objects. More precisely, vertices of such a hexagon have the form

$$XA_iA_jA_kY,\ XA_jA_iA_kY,\ \dots,\ XA_kA_jA_iY, \quad i<j<k, \tag{4.2}$$

where X and Y are above. Such a hexagon can be filled in two ways, namely by 2-morphisms $X\otimes S^{\pm}_{A_i,A_j,A_k}\otimes Y$, where $S^{\pm}$ were defined in Section 2.2.

Let us now glue to the 1-skeleton of the permutohedron two hexagons in place of each hexagonal face of P_n and one square in place of each square-shaped 2-face. All 0-, 1- and 2-faces of the obtained CW-complex are now filled by 0-, 1- and 2-morphisms of $\mathscr{A}$. This is, by definition, the diagram $\mathbf{P}(A_1,\dots,A_n)$. It is a natural question to analyze "how commutative" this diagram is. It turns out that the answer involves higher braid groups.

Consider the 1-category $\mathrm{Hom}_{\mathscr{A}}(A_1\otimes\cdots\otimes A_n,\ A_n\otimes\cdots\otimes A_1)$. Any monotone edge path γ in the permutohedron P_n from $[1,2,\dots,n]$ to $[n,n-1,\dots,1]$ defines an object R_γ of this 1-category–the composition of braiding 1-morphisms corresponding to edges of γ. If γ and γ' differ by a modification on a square (resp. hexagon), we have a 2-morphism (resp. two 2-morphisms) $R_\gamma\to R_{\gamma'}$. So we get a system (groupoid) of 2-morphisms between various R_γ generated by these elementary 2-morphisms and their inverses. In particular, for any given γ we get a group of 2-automorphisms of $R\gamma$. These groups are isomorphic (non-canonically) for different γ, since any two edge paths can be connected by a chain of modifications.

4.2. Main theorem

Theorem 4.1. *Let $A_1,\dots,A_n$ by any objects of a braided monoidal 2-category $\mathscr{A}$. The 2-morphisms in the permutohedral diagram $\mathbf{P}(A_1,\dots,A_n)$ define a functor from the fine higher braid groupoid $\mathscr{TF}(2,n)$ to the 1-category $\mathrm{Hom}_{\mathscr{A}}(A_1\otimes\cdots\otimes A_n,$ $A_n\otimes\cdots\otimes A_1)$. In particular, we get an action of the higher braid group $T(2,n)$ on any object of this category of the form R_γ, γ being a monotone edge path in the permutohedron.*

The analogous statement for a braided monoidal 1-category is that we have an action of the usual pure braid group $T(n) = T(1, n)$ on any product $A_1 \otimes \cdots \otimes A_n$.

The proof of this theorem will occupy the rest of this section. To construct the functor we need some combinatorial preliminaries.

4.3. Line configurations in $\mathbb{R}^2$ and edge paths in the permutohedron

The correspondence recalled below is a particular case of a more general construction in [14].

Let $L_1, \ldots, L_n$ be a configuration of lines in $\mathbb{R}^2$ such that L_m is given by the equation $y = mx - c_m$, $c_m \in \mathbb{R}$. Suppose that this configuration of lines is super-generic, i.e. the vector $(c_1, \ldots, c_n)$ lies in $\mathbb{R}^n_{***}$. Let us interest the lines L_i by a horizontal line $y = t$ with varying t. Let $x_i(t)$ be the point of intersection of $\{y = t\}$ with L_i. For $t \ll 0$ we have $x_1(t) < \cdots < x_n(t)$ and for $t \gg 0$ we have $x_1(t) > \cdots > x_n(t)$. By varying t from $-\infty$ to $+\infty$ we get a sequence of permutations of $\{1, \ldots, n\}$ given by the orders of $x_i(T)$. It is immediate to see that this sequence will be a monotone edge path in the permutohedron P_n, which we denote by $\gamma(L_1, \ldots, L_n)$, see Fig. 6.

Clearly, the edge path $\gamma(L_1, \ldots, L_n)$ depends only on the component of the space $\mathbb{R}^n_{***}$ where the vector $(c_1, \ldots, c_n)$ lies. We shall, for any component E of $\mathbb{R}^n_{***}$ denote the corresponding path by $\gamma(E)$.

Let us call two monotone edge paths γ, δ in P_n going from $[1 \ldots n]$ to $[n \ldots 1]$ *square-equivalent* if the following condition holds:

There exists a sequence $\gamma_1, \ldots, \gamma_r$ of monotone edge paths such that $\gamma_1 = \gamma$, $\gamma_r = \delta$ and each γ_{i+1} is obtained from γ_i by replacing two consecutive edges forming half of the boundary of the same face.

It was remarked in [14] that if E, E' are two components of $\mathbb{R}^n_{***}$ belonging to the same component of $\mathbb{R}^n_{**}$ then the edge paths $\gamma(E)$, $\gamma(E')$ are square-equivalent. This can

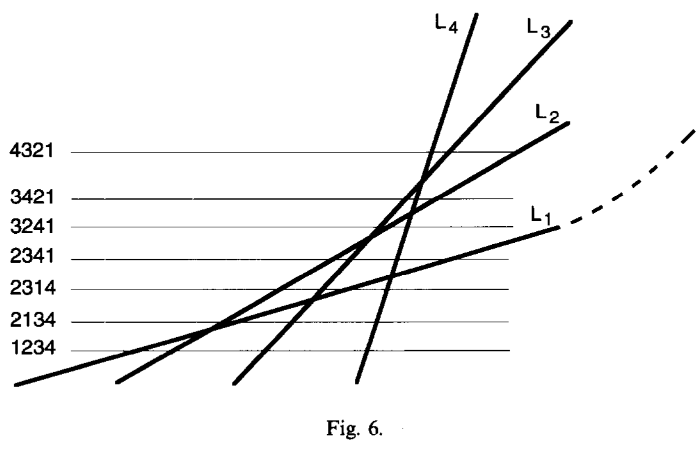


Fig. 6.

be seen by moving $(c_1, \ldots, c_n) \in \mathbb{R}^n$ inside a component K of $\mathbb{R}^n_{**}$ is such a way that at every moment of time no more than two of $\binom{n}{2}$ ordinates $y(L_j \cap L_m)$ coincide. The passage through a position when $y(L_j \cap L_m) = y(L_p \cap L_q)$ gives an elementary modification of the path using the boundary of a square.

Let us call the set of all square-equivalence classes of monotone edge paths in P_n from $[1 \ldots n]$ to $[n \ldots 1]$ *the higher Bruhat order* and denote it by $B(2, n)$. (This definition is one of the characterizations of $B(2, n)$ from [14]; the name "order" comes from the fact that $B(2, n)$ is equipped with a natural partial order relation generalizing the weak Bruhat order on the symmetric group.) The above reasoning gives a map of sets

$$\pi_0(\mathbb{R}^n_{**}) \to B(2, n).$$

This map is injective but in general not surjective: as shown in [11] every element in $B(2, n)$ comes from a so-called pseudo-line arrangement (oriented matroid) and there are (rather old) examples of arrangements of pseudo-lines which cannot be straightened.

This means, in particular, that not every monotone edge path in the permutohedron from $[1 \ldots n]$ to $[n \ldots 1]$ comes from a component of $\mathbb{R}^n_{***}$.

4.4. The construction of the functor

Let $A_1, \ldots, A_n$ be objects of a braided monoidal 2-category $\mathscr{A}$. We shall construct a functor

$$F: \mathscr{T}\mathscr{F}(2, n) \to \mathrm{Hom}_{\mathscr{A}}(A_1 \otimes \cdots \otimes A_n, A_n \otimes \cdots \otimes A_1)$$

whose existence is claimed in Theorem 4.1.

On objects we define F to be the correspondence between connected components of $\mathbb{R}^n_{***}$ and monotone edge paths in the permutohedron recalled in Section 4.3.

By definition, objects of $\mathscr{T}\mathscr{F}(2, n)$ are marked points of $\mathbb{R}^n_{***}$, one in each connected component of this space. Let $(L_1, \ldots, L_n)$ by any such point and let $\gamma(L_1, \ldots, L_n)$ be the corresponding edge path in the permutohedron (see Section 4.3). We define $F(L_1, \ldots, L_n)$ to be the 1-morphism

$$R_{\gamma(L_1 \ldots, L_n)}: A_1 \ldots A_n \to A_n \ldots A_1,$$

i.e. the composition of braiding 1-morphisms corresponding to edges of the path $\gamma(L_1, \ldots, L_n)$. This defines F on objects of $\mathscr{T}\mathscr{F}(2, n)$.

On generators of 1-morphisms we define F as follows.

(1) Let E, E' be two adjacent components of $\mathbb{R}^n_{***}$ lying in the same component of K of $\mathbb{R}^n_{**}$. Let $\gamma(E), \gamma(E')$ be the corresponding edge paths in P_n. These paths coincide everywhere except a segment of length 2 which on one of the paths reads

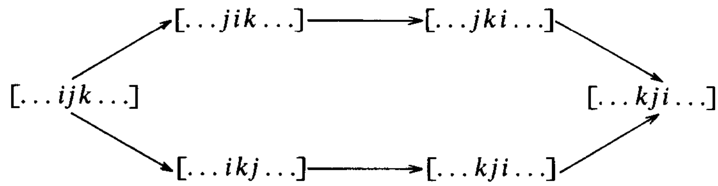
$$[\ldots ij \ldots kl \ldots] \to [\ldots ji \ldots kl \ldots] \to [\ldots ji \ldots lk \ldots]$$

and on the other path

$$[\ldots ij \ldots kl \ldots] \to [\ldots ij \ldots lk \ldots] \to [\ldots ji \ldots lk \ldots]$$

If the second variant holds for $\gamma(E)$ then we define the 1-morphism $F(h_{EE'})$ (the value of F on the generator $h_{EE'}$) to correspond to the 2-morphism in $\mathscr{A}$ given by (4.1). If the first variant holds for $\gamma(E)$ then we define $F(h_{EE'})$ to correspond to the inverse of the 2-morphism (4.1).

(2) Let E, E' be two adjacent components of $\mathbb{R}^n_{***}$ lying in different components of $\mathbb{R}^n_{**}$. The edge paths $\gamma(E)$, $\gamma(E')$ in the permutohedron differ by a modification on a hexagon

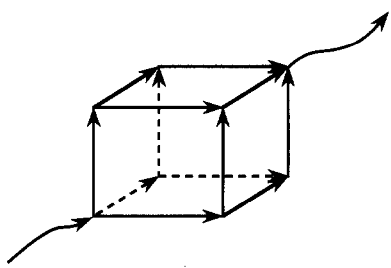


(4.3)

for some $i<j<k$. The corresponding objects in the permutohedral diagram $\mathbf{P}(A_1, \ldots, A_n)$ (see Section 4.1) have the form given in (4.2), i.e. $XA_iA_jA_kY$ and similar products for all the permutations of A_i, A_j, A_k. Suppose now that γ goes along the upper part of the hexagon (4.3). In this case define the 1-morphism $F(g_{EE'})$ (the value of F on the generator $g_{EE'}$ going around the hyperplane H_{ijk} in the complex domain, see Section 3.4) to correspond to the 2-morphism $X \otimes S^+_{A_i,A_j,A_k} \otimes Y$ in $\mathscr{A}$. If $\gamma(E)$ goes along the lower part of the above hexagon, we define $F(g_{EE'})$ to correspond to the 2-morphism $X \otimes S^+_{A_i,A_j,A_k} \otimes Y$.

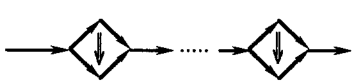
To finish the proof of Theorem 4.1 it suffices to prove that F preserves the relations R1–R6 in the groupoid $\mathscr{I}\mathscr{F}(2, n)$ exhibited in Section 3.5.

R1. The hexagonal diagram in $\mathrm{Hom}_{\mathscr{A}}(A_1 \otimes \cdots \otimes A_n,\ A_n \otimes \cdots \otimes A_1)$ whose commutativity we have to prove, corresponds to a cubical diagram in $\mathscr{A}$ which is a part of the permutohedral diagram $\mathbf{P}(A_1, \ldots, A_n)$. More precisely, six paths $\gamma(E_1), \ldots, \gamma(E_6)$ in P_n lie on a subdiagram of the form:



for some cubical 3-face Q of P_n. So our relation expresses just the commutativity of the cube Q. This commutativity follows from Lemma 1.1.

R2. The four paths $\gamma(E_i)$ lie on a subdiagram in $\mathbf{P}(A_1, \ldots, A_n)$ of the form

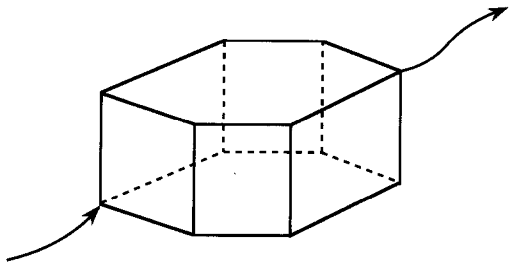


and our relation follows from the 2-dimensional associativity in $\mathscr{A}$.

R3. Follows similarly to R2 by consideration of subdiagrams of the form

(the square may precede or follow the hexagon.)

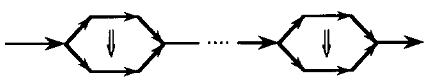
R4. The paths $\gamma(E_i)$, $i = 1,, \ldots, 8$, lie on a subdiagram of the form



for some hexagonal prism Π in the permutohedron. The two relations correspond to the fillings of the two hexagons of Π by S^+ or S^-. So it suffices to prove the commutativity of the prism with respect to each of the fillings. This follows from the axioms

$$(\overset{\rightarrow}{\underset{\rightarrow}{\downarrow}} \otimes \rightarrow) \text{ and } (\rightarrow \otimes \overset{\rightarrow}{\underset{\rightarrow}{\downarrow}})$$

R5. Follows similarly to R2 and R3 by consideration of subdiagrams of the form



(there are 4 choices of fillings).

R6. These eight relations are exactly the eight Zamolodchikov equations proven in Section 2.3.

Theorem 4.1 is proven.

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