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A possible new approach to the motivic spectral sequence for algebraic K-theory

Vladimir Voevodsky

ABSTRACT. We describe a simple construction of the spectral sequence relating algebraic K-theory and motivic cohomology modulo two general conjectures on the structure of the motivic homotopy category. The first conjecture is the motivic analog of the fact that the zero stage of the Postnikoff tower for the (topological) sphere spectrum is the Eilenberg-Maclane spectrum $H_{\mathbf{Z}}$. The second is the motivic analog of the fact that the functor $\Omega^1\Sigma^1$ takes n-connected spaces to n-connected spaces.

1. Introduction. Despite the considerable progress in motivic cohomology and motivic homotopy theory achieved in recent years we still do not have a simple construction of the spectral sequence relating motivic cohomology and algebraic K-theory. The construction invented by Dan Grayson (see [**3**]) is simple and elegant but we are still unable to identify the E_2-term of the resulting spectral sequence with the motivic cohomology groups. The approach pioneered by Spencer Bloch and Steven Lichtenbaum in [**1**] and further developed by Eric Friedlander and Andrei Suslin in [**2**] gives a spectral sequence of the required form but is technically and conceptually very involved.

In [**9**] we suggested a different approach to this problem. Its first ingredient is a construction of a canonical Postnikov tower for any motivic spectrum E. The quotients of this tower $s_i(E)$ are called the slices of E and, by construction, there is a spectral sequence whose E_2-term is given by the cohomology theories represented by the slices and which attempts to converge to the theory represented by E. For KGL, the spectrum representing algebraic K-theory, the main problem is to identify the slices with the motivic cohomology spectra $H_{\mathbf{Z}}$ i.e. to prove the following conjecture:

CONJECTURE 1. $s_n(KGL) \cong \Sigma^n_T H_{\mathbf{Z}}$

Since algebraic K-theory is periodic i.e. $\Sigma^1_T KGL = KGL$, it is sufficient to prove this conjecture for $n = 0$.

At the end of [**9**] we outlined a possible approach to such an identification. It depends on two conjectures:

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CONJECTURE 2. *Let* $\mathbf{1}$ *be the sphere spectrum. Then* $s_0(\mathbf{1}) = H_{\mathbf{Z}}$.

CONJECTURE 3. $\Omega_t^\infty(\Sigma_T^n SH^{eff}) \subset \Sigma_t^n SH_s$

The notations used in the second conjecture are explained below. Note that these conjectures concern only general properties of the motivic stable homotopy categories and do not refer to any specifics of the spectrum representing algebraic K-theory. The main goal of this short paper is to give a complete proof that Conjecture 2 and Conjecture 3 imply Conjecture 1. We will use freely the formalism of slices, and refer the reader to [**9**] for the corresponding discussion.

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2. The s-stable motivic homotopy category. We denote by $HS_s(S)$ the stable $\mathbf{A}^1$-homotopy category of S_s^1-spectra over a base scheme S and by $SH(S)$ the stable $\mathbf{A}^1$-homotopy category of T-spectra over S. These categories are considered in detail in [**4**]. Given an S_s^1-spectrum $(E_i, S_s^1 \wedge E_i \to E_{i+1})$, the sequence of pointed sheaves $S_t^i \wedge E_i$ together with the morphisms

$$S_s^1 \wedge S_t^1 \wedge (S_t^i \wedge E_i) \to S_t^{i+1} \wedge S_s^1 \wedge E_i \to S_t^{i+1} \wedge E_{i+1}$$

form an $S_t^1 \wedge S_s^1$-spectrum. Since the homotopy category of $S_s^1 \wedge S_t^1$-spectra is equivalent to the homotopy category of T-spectra we get a functor

$$\Sigma_t^\infty : SH_s(S) \to SH(S)$$

This functor has the usual properties of a suspension spectrum functor. In particular, it has a right adjoint

$$\Omega_t^\infty : SH(S) \to SH_s(S)$$

which takes a fibrant T-spectrum $(E_i, T \wedge E_i \to E_{i+1})$ to the s-spectrum with terms $\Omega_t^i(E_i)$ and the structure maps adjoint to the composition

$$S_t^1 \wedge S_s^1 \wedge \Omega_t^i(E_i) \to T \wedge \Omega_t^i(E_i) \to \Omega_t^i(T \wedge E_i) \to \Omega_t^i(E_{i+1})$$

Let

$$\Sigma_s^\infty : H_\bullet(S) \to SH_s(S)$$

be the usual suspension spectrum functor from the pointed unstable homotopy category to SH_s. Then $\Sigma_t^\infty \Sigma_s^\infty = \Sigma_T^\infty$ where Σ_T^∞ is the suspension spectrum functor from $H_\bullet$ to $SH(S)$.

Recall that SH^{eff} is the smallest triangulated subcategory in SH which contains suspension spectra and is closed with respect to all direct sums.

LEMMA 2.1. *If* $f : E \to F$ *is a morphism in* SH^{eff} *and* $\Omega_t^\infty(f)$ *is an isomorphism then* f *is an isomorphism.*

PROOF. The definition of SH^{eff} implies that a morphism $f : E \to F$ in this category such that for any pointed simplicial sheaf $X_\bullet$ and any $n \in \mathbf{Z}$ the map

$$Hom_{SH}(\Sigma_s^n \Sigma_T^\infty(X_\bullet), E) \to Hom_{SH}(\Sigma_s^n \Sigma_T^\infty(X_\bullet), F)$$

defined by f is bijective, is an isomorphism. Since $\Sigma_T^\infty = \Sigma_t^\infty \Sigma_s^\infty$, this map is isomorphic to

$$Hom_{SH_s}(\Sigma_s^n \Sigma_s^\infty(X_\bullet), \Omega_t^\infty(E)) \to Hom_{SH_s}(\Sigma_s^n \Sigma_s^\infty(X_\bullet), \Omega_t^\infty(F)).$$

□

For $n \geq 0$, let $\Sigma_t^n SH_s$ be the localizing subcategory of $SH_s(S)$ generated by objects of the form $\Sigma_t^n \Sigma_\infty^s(X_\bullet)$ for all pointed simplicial sheaves $X_\bullet$. We get a filtration:

$$\cdots \subset \Sigma_t^n SH_s \subset \Sigma_t^{n-1} SH_s \cdots \subset \ldots \Sigma_t^0 SH_s = SH_s$$

All the categories appearing in this filtration are compactly generated triangulated categories with all direct sums and the functors respect direct sums and distinguished triangles. Therefore, by [**6**], the inclusion functors have right adjoints and we denote the composition

$$SH_s \to \Sigma_t^n SH_s \to SH_s$$

by f_n. For any E in SH_s we get natural distinguished triangles

$$f_{q+1}E \to f_q(E) \to s_q(E) \to \Sigma_s^1 f_{q+1}(E)$$

where $s_q(E)$ belongs to $\Sigma_t^q SH_s$ and is right orthogonal to $\Sigma_t^{q+1} SH_s$. These triangles form the SH_s-analog of the slice tower (see [**9**, Theorem 2.2]) in SH.

It is clear that the functor Σ_t^∞ maps the slice filtration in SH_s to the slice filtration in SH:

$$\Sigma_t^\infty(\Sigma_t^n SH_s) \subset \Sigma_T^n SH^{eff}$$

In particular, the whole SH_s is mapped to SH^{eff}. It is also clear that the functor Ω_t^∞ respects the "adjoint" filtration i.e. that it maps objects in SH which are right orthogonal to $\Sigma_T^n SH^{eff}$ to objects which are right orthogonal to $\Sigma_t^n SH_s$. We can now restate Conjecture 3 from the introduction:

CONJECTURE 4. *The functor Ω_t^∞ respects the slice filtration i.e.*

$$\Omega_t^\infty(\Sigma_T^n SH^{eff}) \subset \Sigma_t^n SH_s$$

Since all the functors involved in Conjecture 4 are triangulated and commute with direct sums it is sufficient to check that for a smooth scheme X over S one has

$$\Omega_t^\infty(\Sigma_s^\infty(T^n \wedge X_+)) \in \Sigma_t^n SH_s$$

The object on the left hand side can be represented by a homotopy colimit of objects of the form $\Omega_t^i \Sigma_t^{i+n}(\Sigma_s^\infty(X_+))$ where both Σ_t and Ω_t are taken in SH_s. A simple inductive argument shows now that Conjecture 4 follows from the following:

CONJECTURE 5. *For any smooth scheme X over S and any $n \geq 0$ one has*

$$\Omega_t^1 \Sigma_t^1 \Sigma_s^\infty(S_t^n \wedge X_+) \in \Sigma_t^n SH_s$$

The topological analog of this statement (where SH is replaced by the stable homotopy category and SH_s by the unstable one) asserts that $\Omega\Sigma$ takes n-connected objects to n-connected objects. One way to see it is to use the fact that $\Omega\Sigma(X)$ has a model ("James construction") possessing a filtration whose quotients are $X^{\wedge i}$. This is the starting point of the operadic theory of loop spaces and it appears that any such theory for t-loop spaces in SH_s would provide a proof of Conjectures 4 and 5.

LEMMA 2.2. *Assume that Conjecture 4 holds. Then for any E in SH one has:*

$$\Omega_t^\infty(f_n(E)) = f_n(\Omega_t^\infty(E))$$

$$\Omega_t^\infty(s_n(E)) = s_n(\Omega_t^\infty(E))$$

PROOF. To prove the first equality it is sufficient to show that $\Omega_t^\infty(f_n(E))$ is in $\Sigma_t^n SH_s$ and that the cones of the morphism $\Omega_t^\infty(f_n(E) \to E)$ is right orthogonal to $\Sigma_t^n SH_s$. The former is Conjecture 4. The later is clear because Ω_t^∞ is adjoint to Σ_t^∞. The second equality follows from the first one since s_n is the cone of the morphism $f_{n+1} \to f_n$ and Ω_t^∞ commutes with cones. □

3. A connectivity result. The goal of this section is to prove Theorem 3.2. Its corollary 3.4 will be used below to prove the convergence of the slice spectral sequence for algebraic K-theory. *Everywhere in this section we assume that S is the spectrum of a field.* We need this assumption in order to have the following lemma.

LEMMA 3.1. *Let E be an s-spectrum such that $\pi_{<0}(E) = 0$. Then one has,*

$$\pi_{<0}(\Sigma_t^1 E) = 0$$
$$\pi_{<0}(\Omega_t^1 E) = 0$$

PROOF. See [**4**]. □

THEOREM 3.2. *Let E be an object of $\Sigma_t^q SH_s$ such that $\pi_{<0}(\Omega_t^q E) = 0$. Then $\pi_{<0}(E) = 0$.*

PROOF. Consider the adjunction morphism $\Sigma_t^q \Omega_t^q E \to E$ and let $E^{(1)}$ be its cone. Applying this construction inductively we get a sequence of distinguished triangles

$$\Sigma_t^q \Omega_t^q E^{(n)} \to E^{(n)} \to E^{(n+1)} \to \dots \tag{1}$$

and therefore a sequence of morphisms

$$E \to E^{(1)} \to \dots \to E^{(n)} \to \dots$$

Let $E^{(\infty)}$ be the homotopy colimit of this sequence. This object is in $\Sigma_t^q SH_s$. Applying the functor Ω_t^q to the triangles (1) we get split triangles. Therefore, since Ω_t^q commutes with the homotopy colimits of sequences, $\Omega_t^q E^{(\infty)}$ is zero as the homotopy colimit of a sequence of zero morphisms. The following straightforward lemma implies that $E^{(\infty)} = 0$.

LEMMA 3.3. *Let E be an object in $\Sigma_t^q SH_s$ such that $\Omega_t^q E = 0$. Then $E = 0$.*

PROOF. Under the assumptions of the lemma the class $\mathcal{E}$ of objects F in $\Sigma_t^q SH_s$ such that $Hom(\Sigma_s^n F, E) = 0$ for all n, contains objects of the form $\Sigma_t^q \Sigma_s^\infty(X_+)$ and is closed under triangles and direct sums. Therefore, $\mathcal{E}$ coincides with $\Sigma_t^q SH_s$. □

Since objects of the form $\Sigma_s^i \Sigma_s^\infty(X_+)$ are compact, the fact that $E^{(\infty)}$ is zero implies that for any i one has

$$colim_n\ \pi_i(E^{(n)}) = 0$$

It remains to show that for $i < 0$ the maps

$$\pi_i(E^{(n)}) \to \pi_i(E^{(n+1)})$$

are monomorphisms. The long exact sequence defined by (1) implies that it is sufficient to check that $\pi_{<0}(\Sigma_t^q \Omega_t^q E^{(n)}) = 0$. Proceed by induction on n. By Lemma 3.1 it is sufficient to check that $\pi_{<0}(\Omega_t^q E^{(n)}) = 0$. For $n = 0$ this follows from our assumption on E. Assume that $\pi_{<0}(\Omega_t^q E^{(n-1)}) = 0$. Applying $\pi_i(\Omega_t^q(-))$ to the triangle (1) we get a short exact sequence

$$0 \to \pi_i(\Omega_t^q(E^{(n)})) \to \pi_{i-1}(\Omega_t^q \Sigma_t^q \Omega_t^q(E^{(n-1)})) \to \pi_{i-1}(\Omega_t^q(E^{(n-1)})) \to 0$$

Using again Lemma 3.1 and the inductive assumption we conclude that $\pi_i(\Omega_t^q(E^{(n)}))$ is zero. This finishes the proof of Theorem 3.2. $\square$

Recall that for E in $SH(S)$, $\pi_{p,q}(E)$ denote the sheaf associated to the presheaf

$$X \mapsto Hom_{SH}(\Sigma_t^q \Sigma_s^{p-q} \Sigma_T^\infty(X), E)$$

COROLLARY 3.4. *Assume that Conjecture 4 holds and that $S = Spec(k)$ where k is a field. Let E be an object of $SH(S)$ and $q \geq 0$ an integer such that $\pi_{p,q}(E) = 0$ for $p < q$. Then $\pi_{<0,0}(f_q(E)) = 0$.*

PROOF. By adjunction we have $\pi_{n,0}(f_q(E)) = \pi_n(\Omega_t^\infty(f_q(E)))$. Since $f_q(E)$ is in $\Sigma_T^q SH^{eff}$, Conjecture 4 implies that $\Omega_t^\infty(f_q(E))$ is in $\Sigma_t^q SH_s$. By Theorem 3.2 it is sufficient to show that $\pi_n(\Omega_t^q \Omega_t^\infty(f_q(E))) = 0$ for $n < 0$. By adjunction this group equals

$$\pi_{n,0}(\Omega_t^q \Omega_t^\infty(f_q(E))) = \pi_{n+q,q}(f_q(E))$$

$\square$

4. Computation of $s_0(\Omega_t^\infty(KGL))$. The results of this section do not depend on any conjectures. We assume here that the base scheme S is regular.

Denote the spectrum $\Omega_t^\infty(KGL)$ by KGL_s. Since S is regular

$$KGL^{p,q}(X) = K^Q_{2q-p}(X)$$

where K^Q_* is the usual (Quillen's) K-theory (see [**7**] where we use **BGL** instead of KGL), and therefore,

$$Hom_{SH_s}(\Sigma_s^\infty(X_+), \Sigma_s^n KGL_s) = KGL^{n,0}(X) = K^Q_{-n}(X)$$

For a sheaf of abelian groups $\underline{A}$ denote by $H_{A,s}$ the Eilenberg-MacLane s-spectrum defined by $\underline{A}$, i.e. the sequence of the simplicial sheaves $K(\underline{A}, n) = K(\underline{A}[n])$ together with the obvious structure morphisms. The goal of this section is to prove the following theorem.

THEOREM 4.1. *There exists an isomorphism $s_0(KGL_s) = H_{\mathbf{Z},s}$ which takes the identity map $\mathbf{1} \to s_0(KGL_s)$ to the identity map of $H_{\mathbf{Z},s}$.*

For any s-spectrum E, denote by $\pi_i(E)$, $i \in \mathbf{Z}$ the Nisnevich sheaf associated with the presheaf

$$X \mapsto Hom_{SH_s}(\Sigma_s^\infty(X_+), \Sigma_s^{-i}(E)).$$

If $\pi_i(E) = 0$ for $i < 0$ then we have a canonical morphism $E \to H_{\pi_0(E),s}$. In particular, since $K^{-n}(X) = 0$ for X regular and $n > 0$ and the sheaf associated with K_0 is $\mathbf{Z}$ we get a canonical morphism $\phi : KGL_s \to H_{\mathbf{Z},s}$. To prove the theorem it is sufficient to show that $s_0(H_{\mathbf{Z},s}) = H_{\mathbf{Z},s}$ and that the fiber of ϕ is in $\Sigma_t^1 SH_s$. The former is shown in Lemma 4.2, the later in Lemma 4.6.

LEMMA 4.2. *One has $s_0(H_{\mathbf{Z},s}) = H_{\mathbf{Z},s}$.*

PROOF. We need to show that $H_{\mathbf{Z},s}$ is orthogonal to $\Sigma_t^1 SH_s$, i.e. that for a smooth scheme X over S and any $n \in \mathbf{Z}$ we have

$$Hom_{SH_s}(\Sigma_t^1 \Sigma_s^\infty(X_+), \Sigma_s^n H_{\mathbf{Z},s}) = 0$$

Since Nisnevich cohomology with coefficients in $\mathbf{Z}$ are homotopy invariant, $H_{\mathbf{Z},s}$ is $\mathbf{A}^1$-local and this group equals $ker(H^n_{Nis}(X \times \mathbf{G}_m, \mathbf{Z}) \to H^n_{Nis}(X, \mathbf{Z}))$. Since S is regular, X and $X \times \mathbf{G}_m$ are smooth and $H^n_{Nis}(-, \mathbf{Z})$ can only be non zero for $n = 0$.

For $n = 0$ the kernel is zero because for a henselian local X the scheme $X \times \mathbf{G}_m$ is connected. $\square$

REMARK 4.3. The assumption that S is regular is not necessary for the proof of Lemma 4.2. In the proof, for a general S, the reduction to H^0_{Nis} should be replaced by the fact that for a local henselian S one has $H^i_{Nis}(S \times \mathbf{G}_m, \mathbf{Z}) = 0$ for $i > 0$.

PROPOSITION 4.4. *Let $E = (E_n)$ be a fibrant s-spectrum such that $\pi_i(E) = 0$ for $i < 0$. Then E belongs to the smallest subcategory closed under distinguished triangles, direct sums and smash products containing $\Sigma_s^\infty(E_0)$.*

PROOF. Denote the smallest subcategory satisfying the conditions listed above by C. Note that $E = hocolim_n\, \Sigma_s^{-n}\Sigma_s^\infty(E_n)$ and, since homotopy colimits of sequences can be expressed in terms of cones and direct sums, it is sufficient to show that $\Sigma_s^\infty(E_n)$ is in C. By induction we may assume that $\Sigma_s^\infty(E_{n-1})$ is in C. Since E is fibrant we have $E_{n-1} = \Omega_s^1(E_n)$. Using an appropriate model for loop spaces we may assume that E_{n-1} has a monoid structure. Let $B(E_{n-1})$ be the classifying space of E_{n-1} defined as the diagonal of the bisimplicial sheaf $B_{\bullet\bullet}(E_{n-1})$ whose rows are E_{n-1}^m. Our connectivity assumption on E implies that $E_n = B(E_{n-1})$. The formula

$$\Sigma_s^\infty(X_\bullet \times Y_\bullet) = (\Sigma_s^\infty(X_\bullet) \wedge \Sigma_s^\infty(Y_\bullet)) \vee \Sigma_s^\infty(X_\bullet) \vee \Sigma_s^\infty(Y_\bullet)$$

implies that the suspension spectra of the rows of $B_{\bullet\bullet}(E_{n-1})$ are in C. By Lemma 4.5 we conclude that the suspension spectrum of E_n is in C. $\square$

LEMMA 4.5. *Let $B_{\bullet\bullet}$ be a pointed bisimplicial sheaf with rows B_i. Then the spectrum $\Sigma_s^\infty(\Delta(B_{\bullet\bullet}))$ belongs to the localizing subcategory C generated by $\Sigma_s^\infty(B_i)$.*

PROOF. Consider $B = B_{\bullet\bullet}$ as a simplicial object over $\Delta^{op}Shv_\bullet$ with terms B_i. Let $Wr(B)$ be the degeneracy free simplicial object obtained by first forgetting the degeneracies of B and then adding new ones freely (see [**8**, p.45]) such that

$$Wr(B)_i = \bigvee_{[i]\to[j]} B_j$$

where $[i] \to [j]$ runs through all monomorphisms in Δ. We have a canonical map $Wr(B) \to B$ which is a weak equivalence column-wise because for a pointed simplicial set X the map $Wr(X) \to X$ is a weak equivalence. Therefore, it remains to check that the suspension spectrum of $\Delta(Wr(B))$ is in C. Since $\Delta(Wr(B)) = colim_n\, \Delta(sk_n(Wr(B)))$ and C is closed under triangles and direct sums it is sufficient to prove that the suspension spectrum of $\Delta(sk_n(Wr(B)))$ is in C for each n. The simplicial object $Wr(B)$ is a degeneracy free object based on the sequence B_i. Therefore, for each n we have a push-out square

$$\begin{array}{ccc} B_n \otimes \partial\Delta^n & \longrightarrow & sk_{n-1}(Wr(B)) \\ \downarrow & & \downarrow \\ B_n \otimes \Delta^n & \longrightarrow & sk_n(Wr(B)) \end{array}$$

Applying the diagonal functor we get a push-out square in pointed simplicial objects of the form

$$\begin{array}{ccc} B_n \wedge \partial\Delta^n_+ & \longrightarrow & \Delta(sk_{n-1}(Wr(B))) \\ \downarrow & & \downarrow \\ B_n \wedge \Delta^n_+ & \longrightarrow & \Delta(sk_n(Wr(B))) \end{array}$$

which implies that the cone of the morphism

$$\Sigma^\infty_s \Delta(sk_{n-1}(Wr(B))) \to \Sigma^\infty_s \Delta(sk_n(Wr(B)))$$

is isomorphic to the cone of the morphism

$$\Sigma^\infty_s(B_n \wedge \partial\Delta^n_+) \to \Sigma^\infty_s(B_n \wedge \Delta^n_+)$$

i.e. to $\Sigma^n_s \Sigma^\infty_s(B_n)$. This finishes the proof of the lemma. □

LEMMA 4.6. *The fiber of ϕ is in $\Sigma^1_t SH_s$.*

PROOF. The fiber of ϕ satisfies the connectivity assumption of Proposition 4.4. It remains to see that $\Sigma^\infty_s(fiber(KGL_{s,0} \to \mathbf{Z}))$ is in $\Sigma^1_t SH_s$. Since $KGL_{s,0}$ represents K_0 it is $\mathbf{A}^1$-weakly equivalent, by [**5**, Theorem 4.3.13], to $BGL \times \mathbf{Z}$ where BGL is the geometric infinite Grassmannian. Our result follows now from Lemma 4.7 and the fact that $\Sigma^1_t SH_s$ is closed under homotopy colimits of sequences. □

LEMMA 4.7. *Let $BGL(n,m)$ be the Grassmannian of rank n submodules in $\mathcal{O}^m$ which we consider as a pointed scheme by means of any point. Then $\Sigma^\infty_s(BGL(n,m))$ is in $\Sigma^1_t SH_s$.*

PROOF. Let X be a smooth scheme over S and U be a dense open subscheme in X such that $X - U$ is a divisor with normal crossings. A simple inductive argument together with the homotopy purity theorem ([**5**, Theorem 3.2.23]) shows that then the cone of the map $\Sigma^\infty_s(U_+) \to \Sigma^\infty_s(X_+)$ is in $\Sigma^1_t SH_s$. Let U be an open subscheme in $BGL(n,m)$ such that:

1. U contains the distinguished point
2. $U \cong \mathbf{A}^N$
3. $BGL(n,m) - U$ is a divisor with normal crossings.

Applying the previous remark we conclude that $\Sigma^\infty_s(BGL(n,m))$ is in $\Sigma^1_t SH_s$. □

5. The slice spectral sequence for algebraic K-theory. Consider the slice tower

$$f_{q+1}KGL \to f_q(KGL) \to s_q(KGL) \to \Sigma^1_s f_{q+1}(KGL) \tag{2}$$

By construction, the algebraic K-theory spectrum KGL is periodic i.e. we have an isomorphism $\Sigma^1_T \wedge KGL = KGL$. This isomorphism defines isomorphisms

$$s_q(KGL) = \Sigma^q_T s_0(KGL)$$

$$f_q(KGL) = \Sigma^q_T f_0(KGL)$$

For any smooth scheme X over S consider the spectral sequence for

$$KGL^{p,q}(X) = Hom_{SH}(\Sigma^\infty_T(X_+), \Sigma^q_t \Sigma^{p-q}_s KGL)$$

defined by the tower (2). If we index this spectral sequence such that it starts with the E_2-term then we have

$$E^{p,q}_2 = s_0(KGL)^{p-q,-q}(X) \tag{3}$$

where $s_0(KGL)^{*,*}$ denotes the cohomology theory defined by the spectrum $s_0(KGL)$. Theorem 5.1 below identifies $s_0(KGL)$ with the integral motivic Eilenberg-MacLane spectrum and (3) becomes

$$E_2^{p,q} = H^{p-q,-q}(X, \mathbf{Z})$$

Proposition 5.5 implies that our spectral sequence strongly converges to

$$KGL^{p+q,0}(X) = K_{-p-q}(X).$$

THEOREM 5.1. *Assume that $S = Spec(k)$ where k is a field and that Conjectures 2 and 3 hold over S. Then there is an isomorphism $s_0(KGL) = H_{\mathbf{Z}}$ in $SH(S)$ which takes the unit of $s_0(KGL)$ to the unit of $H_{\mathbf{Z}}$.*

PROOF. By Conjecture 2, we have $s_0(\mathbf{1}) = H_{\mathbf{Z}}$. Therefore, the unit map $\mathbf{1} \to KGL$ defines a morphism $H_{\mathbf{Z}} \to s_0(KGL)$ which takes the unit morphism $\mathbf{1} \to H_{\mathbf{Z}}$ to the unit morphism $\mathbf{1} \to KGL$. We are going to prove that any such morphism is an isomorphism.

Conjecture 2 implies in particular that $H_{\mathbf{Z}} \in SH^{eff}$. Therefore, by Lemma 2.1 it is sufficient to prove that

$$\Omega_t^{\infty}(H_{\mathbf{Z}} \to s_0(KGL)) \tag{4}$$

is an isomorphism.

LEMMA 5.2. *Let $S = Spec(k)$ where k is a field and let $H_{\mathbf{Z}}$ be the motivic Eilenberg-MacLane spectrum. Then $\Omega_t^{\infty}(H_{\mathbf{Z}}) = H_{\mathbf{Z},s}$.*

PROOF. The spectrum $H_{\mathbf{Z},s}$ can be characterized by the property that

$$\pi_i(H_{\mathbf{Z},s}) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

We have:

$$\pi_i(\Omega_t^{\infty}(H_{\mathbf{Z}})) = \pi_{i,0}(H_{\mathbf{Z}}) = \begin{pmatrix} \text{The sheaf associated} \\ \text{to the presheaf} \\ X \mapsto H^{i,0}(X, \mathbf{Z}) \end{pmatrix}$$

and the lemma follows from the fact that for a smooth connected scheme X over a field one has:

$$H^{i,0}(X, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases} \tag{5}$$

□

By Lemma 2.2 (this is where Conjecture 3 is used) we have $\Omega_t^{\infty}(s_0(KGL)) = s_0(\Omega_t^{\infty}(KGL))$ and by Theorem 4.1 we have $s_0(\Omega_t^{\infty}(KGL)) = H_{\mathbf{Z},s}$. Therefore, (4) is a morphism from $H_{\mathbf{Z},s}$ to $H_{\mathbf{Z},s}$. The fact that it is an isomorphism follows from the lemma below.

LEMMA 5.3. *Let $f : H_{\mathbf{Z},s} \to H_{\mathbf{Z},s}$ be an endomorphism which takes the unit map $\mathbf{1} \to H_{\mathbf{Z},s}$ to itself. Then f is the identity.*

PROOF. Since $H_{\mathbf{Z},s}$ is $\mathbf{A}^1$-local the endomorphisms of this object in SH_s can be computed in the stable homotopy category of simplicial presheaves (without $\mathbf{A}^1$-localization). Our result follows from the fact that in this category the endomorphisms of $H_{\mathbf{Z},s}$ are given by $H^0(S, \mathbf{Z})$. □

□

REMARK 5.4. The only place in the proof of Theorem 5.1 where we used the assumption that S is the spectrum of a field is in the proof of Lemma 5.2. If we knew that the motivic cohomology of weight zero of all regular connected schemes is given by (5) then we could prove Theorem 5.1 for any regular S.

PROPOSITION 5.5. *Let S be the spectrum of a field and assume that Conjecture 4 holds. Then for a smooth scheme X over S of absolute dimension d and $n > d-q$ one has:*

$$Hom_{SH}(\Sigma_T^\infty(X_+), \Sigma_s^n f_q(KGL)) = 0.$$

PROOF. The sheaf $\pi_{p,q}(KGL)$ is the sheaf associated to the presheaf $K_{p-2q}(-)$. In particular it is zero for $p < 2q$. Applying Corollary 3.4 we conclude that $\pi_{<q}(f_q(KGL)) = 0$ for all $q \geq 0$. The statement of the proposition follows from the Connectivity Theorem (see [**4**]). □

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SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON NJ, USA.
E-mail address: `vladimir@ias.edu`