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Voevodsky's Seattle Lectures: K-theory and Motivic Cohomology

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These notes are based on a series of talks given by V. Voevodsky at the AMS Joint Summer Research Conference on algebraic K-theory, held in Seattle during July 1997. The purpose of these talks was to outline the proof of "Milnor's conjecture" [**V**]: if F is a field of characteristic $\neq 2$, then the Milnor K-theory of F, reduced modulo two, is isomorphic to the étale cohomology of F with coefficients $\mathbb{Z}/2$: $K_*^M(F)/2 \cong H_{et}^*(F, \mathbb{Z}/2)$.

The first two parts develop a motivic homotopy theory, and are joint work with Fabien Morel. In the third part we define the motivic cohomology of a variety in this setting, a setting where it carries cohomology operations. The final part uses a series of reductions to reduce Milnor's conjecture to a vanishing assertion in motivic cohomology, which can be checked using these operations and two results of Rost.

§1. Unstable Motivic Homotopy Theory (joint with Fabien Morel)

In these lectures, we will sketch the foundations of a homotopy theory for algebraic varieties over a fixed field k. Morally, we want to replace the category **Top** of topological spaces by the category of algebraic varieties over k. The actual details are, of course, more involved, since the category of varieties will need to be enlarged in order to perform standard homotopy-theoretic constructions.

First we recall how topologists construct stable homotopy theory. Using a subscript '$*$' to denote pointed categories, the topologists' homotopy category $\mathbf{Hot}_*$ is obtained from $\mathbf{Top}_*$ by inverting weak homotopy equivalences. Then the stable homotopy category $\mathbf{SHot}_*$ is obtained from $\mathbf{Hot}_*$ by inverting the suspension, *i.e.*, the smash product with the circle S^1. If we view the singular chain complex $C_*(X)$ of a topological space X as a functor to the derived category $D(\mathbf{Ab})$ of abelian

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groups, then this functor factors through $\mathbf{SHot}_*$:

$$\begin{array}{ccc} \mathbf{Top}_* & \xrightarrow{\text{invert w.e.}} & \mathbf{Hot}_* \\ {\scriptstyle C_*}\downarrow & & \downarrow{\scriptstyle \text{invert } S^1\wedge} \\ D(\mathbf{Ab}) & \longleftarrow & \mathbf{SHot}_* \end{array}$$

REMARK 1.1. Thinking of $\mathbf{SHot}_*$ as the category of S^0-modules, the bottom arrow is the base-change functor induced by the morphism $S^0 \to K(\mathbb{Z}, 0)$ of ring spectra. This is only a special case of a more general construction. For example, the morphisms from S^0 to the cobordism spectrum $\mathbf{MU}$ and the topological complex K-theory spectrum $\mathbf{KU}$ give rise to functors from $\mathbf{SHot}_*$ to the derived categories of $\mathbf{MU}$-modules and $\mathbf{KU}$-modules.

Now let us begin our construction of a homotopy theory for algebraic varieties. Our approach is motivated by an analysis of what operations we would like to perform.

Let *SmAff* = *SmAff*/k denote the category of smooth affine algebraic varieties of finite type over k. As nice as it is, this category has some undesirable properties, such as the fact that quotients do not always exist. As a first step in correcting this defect, we replace varieties by presheaves. This has the advantage that we can construct arbitrary colimits of presheaves, and in particular quotients of presheaves.

DEFINITION 1.2. Let $\mathbf{PreShv} = \mathbf{PreShv}(\mathit{SmAff}/k)$ denote the category of contravariant functors from *SmAff*/k to the category of sets; we call such functors *presheaves*. Any scheme Y over k gives a *representable* presheaf h_Y, defined by $h_Y(X) = \mathrm{Hom}(X, Y)$. Thus we have Yoneda embeddings:

$$\mathit{SmAff} \subset \mathit{Schemes} \subset \mathbf{PreShv}.$$

We will abuse notation and write Y for the presheaf h_Y.

COLIMITS 1.3. Any presheaf F is a colimit of representable functors. This well-known observation is due to Kan: F is the colimit of the h_Y over an indexing category of pairs (Y, α), where $\alpha \in F(Y)$. The maps $\alpha_*\colon h_Y \to F$ are determined from (Y, α) by the Yoneda lemma; a morphism from (Y', α') to (Y, α) is a morphism $f\colon Y' \to Y$ sending α' to α, and f induces a factorization $h_{Y'} \xrightarrow{f} h_Y \xrightarrow{\alpha_*} F$ of α'_*.

If Z is a subscheme of Y then we shall write Y/Z for the quotient presheaf h_Y/h_Z. This presheaf is not usually representable by a scheme. However, Y/Z is always a pushout of representable presheaves, in the sense that the following diagram is a cocartesian square of presheaves:

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & Y/Z. \end{array}$$

EXAMPLES 1.4.

1. The *0-sphere* S^0 is the disjoint union of two copies of $\mathrm{Spec}(k)$.
2. Let S^1_s denote the simplicial circle $\mathbb{A}^1/\{0, 1\}$, which is the quotient presheaf obtained from the affine line $\mathbb{A}^1$ by identifying the two points 0 and 1. The

presheaf S^1_s is represented by the affine nodal cubic ($y^2 = x^3 - x$), which by coincidence is the corresponding pushout in the category of schemes.

3. The *Tate object* T is obtained as the quotient of the affine line $\mathbb{A}^1$ by $\mathbb{A}^1 - \{0\}$. This makes sense as a cocartesian square as above.
4. The *Thom space* of a vector bundle $E \to X$ is the quotient presheaf $Th(E) = E/\big(E - s(X)\big)$. Here $s(X)$ is the zero-section of E. For example, the Tate object T is the Thom space of the vector bundle $\mathbb{A}^1 \to \mathrm{Spec}(k)$.

   For comparison, the Thom space of a topological vector bundle E is usually described as the one-point compactification of E. But it can also be described as the quotient of the unit disk bundle by the unit sphere bundle. The complement of the zero-section is clearly the algebraic analogue of the sphere bundle.
5. If (X, x) and (Y, y) are pointed presheaves, their smash product is the pointed presheaf

$$X \wedge Y = X \times Y/\big((X \times \{y\}) \cup (\{x\} \times Y)\big).$$

   Unfortunately, the presheaf $Th(E_1 \times E_2)$ need not equal $Th(E_1) \wedge Th(E_2)$; the case $E_1 = E_2 = \mathbb{A}^1$ is given in the following exercise. The root of the problem is that the Yoneda embedding from schemes to presheaves does not commute with colimits. For example, the scheme colimit of the diagram defining $Th(E)$ is just $\mathrm{Spec}(k)$.

EXERCISE 1.5. a) Show that if $\mathcal{O}_X$ is the trivial line bundle on X, and X_+ denotes the union of X and a disjoint basepoint, then $Th(\mathcal{O}_X) \cong X_+ \wedge T$.

b) If $U = \mathrm{Spec}(R)$, show that $(T \wedge T)(U) = R^2/L_1$ and $Th(\mathbb{A}^2)(U) = R^2/L_2$, where $L_1 = \{(x, y) : x \in R^\times \text{ or } x \in R^\times\}$ and $L_1 = \{(x, y) : xR + yR = R\}$.

The limitations of the presheaf approach become apparent when we consider the Mayer-Vietoris construction, starting with a covering of X by two open subschemes U and V. In order to study this we consider the pushout (in **PreShv**) of U and V along $U \cap V$; it is *not* represented by the pushout scheme $X = U \cup V$. Indeed, the canonical element $1_X \in h_X(X)$ cannot come from an element of $h_U(X)$ unless $U = X$.

To fix this problem, we pass to *sheaves.* Although the Zariski topology suffices for the pushout example above, we will eventually want to use the Nisnevich topology for our sheaves. Here is a quick way to define Nisnevich sheaves, which suffices for our purposes.

DEFINITION 1.6. An *elementary distinguished square* is a cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ i\ } & X \end{array}$$

in which i is an open embedding, p is an étale morphism, and p induces an isomorphism of closed subschemes: $p^{-1}(X - U)_{red} \xrightarrow{\cong} (X - U)_{red}$.

A presheaf is called a *Nisnevich sheaf* if it takes elementary distinguished squares into cartesian squares of sets. We write **Spc** for the category of Nisnevich sheaves on $SmAff/k$. We will often refer to objects of **Spc** as "spaces."†

Every representable scheme h_X is a Nisnevich sheaf. This is equivalent to the fact that the family $\{U \to X, V \to X\}$ is a universal effective epimorphism for every elementary distinguished square.

THE NISNEVICH TOPOLOGY 1.7. A Nisnevich sheaf is the same as a sheaf for the Nisnevich topology, which is a Grothendieck topology defined as follows.

A family $\{U_\alpha \to X\}$ of étale morphisms is called a *Nisnevich cover* if, for every point x of X, there is an α and a point $u \in U_\alpha$ such that $U_\alpha \to X$ sends u to x and induces an isomorphism of residue fields: $k(x) \cong k(u)$. This is equivalent to the requirement that for every field K the function $\coprod U_\alpha(K) \to X(K)$ is surjective.

Any sheaf for the Nisnevich topology satisfies the cartesian condition above, because $\{U, V\}$ is a Nisnevich cover in any elementary distinguished square. A proof that Nisnevich sheaves are sheaves for the Nisnevich topology is given in [**MV, 3.1**], and is based upon the following observation. If $\{U_\alpha \to X\}$ is a Nisnevich cover, then for each generic point x of X there is an α such that $U_\alpha \to X$ is a birational isomorphism.

EXAMPLE 1.8. The Nisnevich topology reflects many of the arithmetic properties of k. For example, for each $a \neq 0$ in k, the square

$$\begin{tikzcd} \mathbb{A}^1 - \{0, \pm\sqrt{a}\} \arrow[r, hook] \arrow[d] & \mathbb{A}^1 - \{0, \sqrt{a}\} \arrow[d, "\text{square}"] \\ \mathbb{A}^1 - \{a\} \arrow[r, hook] & \mathbb{A}^1. \end{tikzcd}$$

is an elementary distinguished square if and only if the equation $y^2 = a$ has a solution in k.

Note that colimits of sheaves (e.g., pushouts, quotients and smash products) are the sheafification of the corresponding colimits of presheaves. From now on, when we refer to the Tate object T, the Thom space $Th(E)$ or the smash product $X \wedge Y$, we shall mean the corresponding Nisnevich sheaf, rather than the presheaf.

THOM SPACES 1.9. Let E and E' be vector bundles over X and X', respectively. Then $E \times E'$ is a vector bundle over $X \times X'$. On the level of Nisnevich sheaves we have

$$Th(E \times E') \cong Th(E) \wedge Th(E').$$

In particular, if $\mathcal{O}_X$ is the trivial line bundle on X then $Th(E \oplus \mathcal{O}_X) \cong Th(E) \wedge T$. If Σ_T denotes smashing with T, then this implies that $Th(E \oplus \mathcal{O}_X^n) \cong \Sigma_T^n Th(E)$.

Since $T = Th(\mathbb{A}^1)$, this also implies that $\mathbb{A}^n/\mathbb{A}^n - \{0\} = Th(\mathbb{A}^n) \cong T^{\wedge n}$.

EXERCISE 1.10. Suppose that U, V and X form an elementary distinguished square, and let $U \cup_{U\cap V} V$ denote the pushout of U and V along $U \cap V = U \times_X V$ in the category of Nisnevich sheaves. Show that a) $U \cup_{U\cap V} V \to X$ is an isomorphism of Nisnevich sheaves, and hence that b) X/U is isomorphic to $V/(U \cap V)$. (See [**MV, 3.1.6**].)

†There are alternative approaches to the construction of **Hot** which take a different basic category of spaces. One such construction uses the category of *cdh* sheaves on all varieties, including singular varieties. In the presence of resolution of singularities, the theorem below on Blow-up squares shows that both constructions yield the same stable homotopy category **SHot**.

REMARK. This exercise shows that we could have started with the category Sm/k of all smooth schemes over k, instead of the category $SmAff$ of smooth affine schemes. The resulting category **Spc** of spaces would have been the same.

The homotopy category Hot.

In order to form the homotopy category of spaces, we shall define a model structure on the category **Spc**, in the sense of Quillen [**Q**]. The role of cofibrations will be played by the class C of all monomorphisms.

Since we want the affine line $\mathbb{A}^1$ to be the analogue of the unit interval, we require $X\times\mathbb{A}^1 \to X$ to be a weak equivalence. This leads to the following definition.

DEFINITION 1.11. The class $W = W_{\mathbb{A}^1}$ of ($\mathbb{A}^1$–)*weak equivalences* is the smallest class of morphisms in **Spc** containing all isomorphisms and satisfying the following axioms. A *trivial cofibration* is defined to be a cofibration which is also a weak equivalence.

1. (Homotopy) each projection $X \times \mathbb{A}^1 \to X$ is in W;
2. (Saturation) If two of f, g and fg are in W, so is the third;
3. (Continuity) If $X_\alpha \xrightarrow{f_{\alpha\beta}} X_\beta$ is a filtered system of trivial cofibrations, then each $X_\alpha \to \operatorname{colim}_\beta X_\beta$ is in W.
4. The pushout of a weak equivalence along a cofibration is also a weak equivalence. That is, if c is a cofibration and w a weak equivalence in the left pushout diagram below then w' is also a weak equivalence.

$$\begin{array}{ccc} X & \xrightarrow{w} & Y \\ {\scriptstyle c}\downarrow & & \downarrow \\ Z & \xrightarrow{w'} & Z\cup_X Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{w} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{w'} & Z\cup_X Y \end{array}$$

5. Similarly, the pushout of a trivial cofibration along any map is a weak equivalence. That is, if w is both a cofibration and a weak equivalence in the right pushout diagram above then w' is also a weak equivalence.

We define the $\mathbb{A}^1$-fibrations in **Spc** to be the class $F_{\mathbb{A}^1}$ of all morphisms with the right lifting property relative to trivial cofibrations.

The following result is based upon results Joyal [**Jo**] and Jardine [**J**]. Its proof requires the use of simplicial sheaves, and will be sketched below.

THEOREM 1.12. *The classes C, $W_{\mathbb{A}^1}$ and $F_{\mathbb{A}^1}$ form a proper (left and right) closed model structure on* **Spc**.

DEFINITION 1.13. The homotopy category $\mathbf{Hot} = \mathbf{Hot}_{\mathbb{A}^1}(k)$ of schemes over k is the category obtained from **Spc** by the standard process of inverting the weak equivalences in a model category. In particular, morphisms in **Hot** are of the form fw^{-1} and composition is described by a calculus of fractions.

There is a variant of these constructions, starting from the category $\mathbf{Spc}_*$ of pointed spaces. There is a proper closed model structure on $\mathbf{Spc}_*$; a map in $\mathbf{Spc}_*$ is a cofibration, weak equivalence or fibration if the underlying map in **Spc** is one. This allows us to form the corresponding homotopy category $\mathbf{Hot}_*$ of pointed spaces over k. It is not hard to show that the smash product makes $\mathbf{Hot}_*$ into a symmetric monoidal category with unit S^0.

EXAMPLE OF A FIBRATION 1.14. A scheme X is called *rigid* if $\mathrm{Hom}(Y \times \mathbb{A}^1, X) = \mathrm{Hom}(Y, X)$ for every Y. If X is rigid, then every projection $Y \times X \to Y$ is an $\mathbb{A}^1$-fibration, and $\mathrm{Hom}_{\mathbf{Hot}}(Y, X) = \mathrm{Hom}_{\mathbf{Spc}}(Y, X)$.

Any smooth curve X of genus > 0 is rigid, because any map $\mathbb{A}^1 \to X$ is trivial. Thus there is a full embedding of the category of such curves into **Hot**.

Simplicial Structure.

Let us write $\Delta^{op}\,\mathbf{Spc}$ for the category of simplicial "spaces," *i.e.*, simplicial objects in **Spc**. Since we may regard any set as a scheme (a disjoint union of copies of $\mathrm{Spec}(k)$), and hence as an object of **Spc**, we may regard every simplicial set as a simplicial space. It should not be surprising that the standard n-simplex $\Delta[n]$ is a very useful simplicial space.

Here is another way to obtain simplicial spaces. Let $\Delta^\bullet$ denote the standard cosimplicial object in *SmAff* which in degree n is the scheme

$$\Delta^n = \mathrm{Spec}(k[t_0, \dots, t_n]/(\sum t_i = 1)).$$

There is a "chains" functor $C_*\colon \mathbf{Spc} \to \Delta^{op}\,\mathbf{Spc}$ sending the sheaf X to the simplicial sheaf $X(- \times \Delta^\bullet)$, which in degree n is $Y \mapsto C_n(X)(Y) = X(Y \times \Delta^n)$.

There is also a geometric realization functor $X \mapsto |X|_{\mathbb{A}^1}$, from $\Delta^{op}\,\mathbf{Spc}$ to **Spc**. It is the left adjoint functor to C_*, just as the geometric realization of topological spaces is left adjoint to the singular chain complex. By adjointness, the realization of a constant simplicial space $n \mapsto X$ is just X. Hence the geometric realization functor is characterized by the fact that it sends the n-simplex $\Delta[n]$ to the scheme Δ^n. For example, the realization of the boundary of the n-simplex is the union of the "faces" of the scheme Δ^n, and the realization of the simplicial circle $S^1 = \Delta^1/\{0, 1\}$ is the node S^1_s.

Joyal proved in [**Jo**] that there is a "simplicial" model structure on $\Delta^{op}\,\mathbf{Spc}$, which is closed and proper in the sense of [**BF**]. The "simplicial cofibrations" are the monomorphisms, and the "simplicial weak equivalences" are the maps which induce isomorphisms on all sheaves of homotopy groups. The "simplicial mapping spaces" in this structure are the simplicial sets $\mathrm{Hom}_\bullet(X, Y) = \mathrm{Hom}(X \times \Delta^\bullet, Y)$.

We shall write $\mathbf{Hot}(\Delta^{op}\,\mathbf{Spc})$ for the corresponding "simplicial" homotopy category, obtained from $\Delta^{op}\,\mathbf{Spc}$ by inverting the simplicial weak equivalences.

With this simplicial structure available, it is possible to prove that the classes of monomorphisms, $\mathbb{A}^1$-weak equivalences (defined using $C_*\mathbb{A}^1$ in place of $\mathbb{A}^1$) and $\mathbb{A}^1$-fibrations provide the category $\Delta^{op}\,\mathbf{Spc}$ with the structure of a proper closed model category; see [**MV, 2.2.5**]. The corresponding homotopy category $\mathbf{Hot}_{\mathbb{A}^1}(\Delta^{op}\,\mathbf{Spc})$ may be obtained from the simplicial homotopy category $\mathbf{Hot}(\Delta^{op}\,\mathbf{Spc})$ by classical f-localization, *i.e.*, by inverting all the projections $f\colon C_*\mathbb{A}^1 \times X \to X$. (See [**B**].)

The usual arguments in topology show that the adjoint functors C_* and $|{-}|_{\mathbb{A}^1}$ take $\mathbb{A}^1$-weak equivalences to $\mathbb{A}^1$-weak equivalences, and $\mathbb{A}^1$-fibrations to $\mathbb{A}^1$-fibrations. It follows that the $\mathbb{A}^1$-structure also makes **Spc** into a proper closed model category, and we can identify the homotopy categories **Hot** of **Spc** and $\mathbf{Hot}_{\mathbb{A}^1}(\Delta^{op}\,\mathbf{Spc})$.

Up to this point, everything we have said has been equally true for Zariski sheaves. Here are two results whose Zariski analogues are false. Recall that a *smooth pair* is a closed immersion $Z \subset X$ in which both X and Z are smooth over k; this condition ensures that the normal bundle $N_X Z$ exists on Z.

THEOREM 1.15 ("HOMOTOPY PURITY"). [**MV, 3.2.23**] *Let $Z \subset X$ be a smooth pair, with normal bundle $N_X Z$. Then the Thom space $Th(N_X Z)$ is $\mathbb{A}^1$-weak equivalent to $X/(X-Z)$, i.e., there is a homotopy cocartesian square*

$$\begin{tikzcd} X - Z \arrow[r] \arrow[d] & X \arrow[d] \\ \mathit{point} \arrow[r] & Th(N_X Z) \end{tikzcd}$$

THEOREM 1.16 (BLOW-UPS). [**MV, 3.2.30**] *Let $Z \subset X$ be a smooth pair, and $\tilde{X}$ the blowup of X along Z. Then the first S^1_s-suspension of the blowup square*

$$\begin{tikzcd} p^{-1}Z \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ Z \arrow[r] & X \end{tikzcd}$$

is homotopy cocartesian.

We do not know if the suspension is necessary in this theorem. The first step in the proof is to show that the homotopy cofiber of the left column is the same as the homotopy cofiber of $\tilde{X}/U \to X/U$, where $U = X - Z$.

Classifying spaces of algebraic groups.

We will write $B_{gm}(GL_n)$ for the infinite Grassmannian $G(n, \infty)$. This is a geometric substitute for the classifying space of GL_n in our setting.

Here is a geometric construction which works more generally for any linear algebraic group G defined over k. The *geometric classifying space* $B_{gm}(G)$ *of* G is an object of **Spc** which is well-defined up to $\mathbb{A}^1$-weak equivalence. To construct it, we fix a faithful representation $\rho\colon G \to GL_n$. In the diagonal representation ρ^i of G on $\mathbb{A}^{ni} = (\mathbb{A}^n)^i$, let $U_i \subset \mathbb{A}^{ni}$ be the maximal open subscheme where G acts freely. The geometric quotient $V_i = U_i/G$ is smooth over k. We define $B_{gm}(G)$ to be the colimit V_∞ of the spaces V_i along the system of closed embeddings $V_i \hookrightarrow V_{i+1}$ induced by the $\mathbb{A}^{ni} \subset \mathbb{A}^{n(i+1)}$.

For example, if $G = GL_n$ and we consider $\mathbb{A}^{ni}$ as parametrizing $n \times i$ matrices, then V_i is the classical Grassmann variety of n-dimensional linear subspaces of $\mathbb{A}^i$, and V_∞ is the usual infinite Grassmannian $G(n, \infty)$ classifying n-dimensional vector bundles. Thus our two definitions of $B_{gm}(GL_n)$ agree.

LEMMA 1.17. *Up to isomorphism in* $\mathbf{Hot}(k)$, *the space $B_{gm}(G)$ doesn't depend upon ρ.*

In topology, the classifying space BG of a group G is a simplicial set. This provides another way to construct a classifying space for a group scheme G. Since each $G(U)$ is a group, the simplicial sets $B\big(G(U)\big)$ assemble to form the simplicial (nisnevich) sheaf $B_{nis}G\colon U \mapsto B\big(G(U)\big)$. By abuse of notation, we shall write BG for the geometric realization $|B_{nis}G|_{\mathbb{A}^1}$. The following is a restatement of [**MV, 4.1.18 and 4.2.7**].

LEMMA 1.18. *There is a natural map $BG \to B_{gm}G$. This map is an $\mathbb{A}^1$-weak equivalence if and only if "Hilbert's Theorem 90" holds for G, i.e., $H^1_{et}(K, G) = 0$ for every finitely generated field extension K of k.*

EXAMPLE 1.19. Consider the algebraic group GL_n. The usual proof of Hilbert's Theorem 90 [**Milne, III(4.10)**] shows that $H^1_{nis}(K, GL_n) = H^1_{et}(K, GL_n) = 0$. Thus BGL_n is weakly equivalent to $B_{gm}(GL_n) = G(n, \infty)$.

EXAMPLE 1.20. Consider the algebraic group μ_2 with $\frac{1}{2} \in k$. The spaces $B\mu_2$ and $B_{gm}(\mu_2)$ are not weak equivalent, because $H^1_{et}(K, \mu_2) = K^\times / K^{\times 2}$ is nonzero for $K = k(t)$.

§2. The stable motivic homotopy category

The pointed category $\mathbf{Spc}_*$ has two circles, the simplicial circle S_s^1 (or node) and the "Tate circle" $S_t^1 = \mathbb{A}^1 - \{0\}$, with 1 as basepoint. The smash product with these two circles gives two different types of suspension (Σ_s and Σ_t) on the category $\mathbf{Spc}_*$, as well as on the homotopy category $\mathbf{Hot}_*$.

The *simplicial suspension* $\Sigma_s F$ of a pointed simplicial sheaf F is the smash product of S_s^1 with F. This is the sheaf associated to the presheaf $U \mapsto \Sigma F(U)$, where Σ denotes the usual suspension of a pointed simplicial set. If X is a space then $|\Sigma_s C_* X|_{\mathbb{A}^1} \simeq |S_s^1| \wedge X$. If we only invert the simplicial suspension on $\mathbf{Hot}_*$ we obtain the *stable simplicial homotopy category*, which we write as $\mathbf{Hot}_*^s$.

DEFINITION 2.1. The *stable homotopy category* $\mathbf{SHot}(k)$ of schemes over k is defined to be the homotopy category $\mathbf{Hot}_*$ with T-suspension $\Sigma_T(X) = T \wedge X$ inverted.

The next exercise shows that T-suspension is $\Sigma_t \circ \Sigma_s$. Hence $\mathbf{SHot}(k)$ may also be obtained from $\mathbf{Hot}_*^s$ by inverting only the "Tate" suspension Σ_t.

EXERCISE 2.2. Show that the Tate object T is weak equivalent to both $S_s^1 \wedge S_t^1$ and the pointed projective line $\mathbb{P}_*^1$. *Hint:* Construct weak equivalences $S_s^1 \wedge S_t^1 \leftarrow P \rightarrow T$, where P is the pushout of $\mathbb{A}^1$ and $\mathbb{A}^1 \wedge S_t^1$ along S_t^1. For the second assertion, show that $T \simeq \mathbb{P}^1/\mathbb{A}^1 \simeq \mathbb{P}^1/\text{point} = \mathbb{P}_*^1$.

THEOREM 2.3. $\mathbf{SHot}(k)$ *is a triangulated category, where the translation functor is simplicial suspension:* $E[1] = \Sigma_s E$. *The smash product* $\wedge$ *provides* $\mathbf{SHot}(k)$ *with a tensor structure compatible with the triangulated structure.*

The proof of this result is long but straightforward; see [**V, 3.10**]. It uses the fact that the cyclic permutation on $S^1 \wedge S^1 \wedge S^1$ is equivalent to the identity. However the reader should beware that, although the transpose map on $S^2 \wedge S^2$ is (homotopic to) the identity in topology, the transpose map τ_T on $T \wedge T$ is not the identity in general. However, τ_T is the identity whenever k contains $\sqrt{-1}$.

Here is another approach to constructing $\mathbf{SHot}(k)$, the stable homotopy category: instead of inverting $\mathbb{A}^1$-weak equivalences first and T-suspensions second, we could first invert T-suspensions. This leads to the notion of T-spectra.

DEFINITION 2.4. A *T-spectrum* $\mathbf{E}$ is a sequence of pointed spaces E_i, together with bonding maps $T \wedge E_i \rightarrow E_{i+1}$. We write T-**Spectra** for the category of T-spectra; a *morphism* $\mathbf{E} \rightarrow \mathbf{F}$ of T-spectra is just a sequence of maps $E_n \rightarrow F_n$ which commute with the bonding maps.

The category of T-spectra has an evident notion of stable $\mathbb{A}^1$-weak equivalence. If we localize the category of T-spectra by inverting stable $\mathbb{A}^1$-weak equivalences à la Bousfield-Friedlander [**BF**], we get a homotopy category of T-spectra, $\mathbf{SHot}^T(k)$.

As in topology, there is a *suspension T-spectrum* functor Σ_T^∞ from $\mathbf{Spc}_*$ to T-**Spectra**; the ith space of $\Sigma_T^\infty X$ is $T^{\wedge i} \wedge X$, and the bonding maps are the associativity maps. The functor Σ_T^∞ preserves $\mathbb{A}^1$-weak equivalences, and smash products: $\Sigma_T^\infty(X \wedge Y)$ is equivalent to $\Sigma_T^\infty(X) \wedge \Sigma_T^\infty(Y)$. Since the smash product with T is a self-equivalence on the category of T-spectra, Σ_T^∞ induces a functor from $\mathbf{SHot}$ to $\mathbf{SHot}^T(k)$. This functor is an equivalence of homotopy categories.

THEOREM 2.5. *([**V, 3.10**]) The functor $\Sigma_T^\infty\colon \mathbf{Spc}_* \to T$-**Spectra** takes cofibration sequences to distinguished triangles.*

We shall write S_s^m and S_t^m for the suspension T-spectra $\Sigma_T^\infty S_s^m$ and $\Sigma_T^\infty S_t^m$, respectively. Note that if $m < 0$ the object S_s^m belongs to $\mathbf{Hot}_*^s$ but not $\mathbf{Hot}_*$, while the object S_t^m of $\mathbf{SHot}$ does not even belong to $\mathbf{Hot}_*^s$; it exists as the formal object $\Sigma_t^{-m}(S^0)$.

DEFINITION 2.6. For integers p, q the bigraded sphere T-spectrum is

$$S^{p,q} = S_s^{p-q} \wedge S_t^q.$$

If $\mathbf{E}$ is a T-spectrum, we write $\mathbf{E}(q)[p]$ for $S^{p,q} \wedge \mathbf{E}$.

For example, applying Σ_s^{-1} to $S^{2n,n} \cong T^{\wedge n} \cong \mathbb{A}^n/\mathbb{A}^n - \{0\}$ yields a canonical isomorphism in $\mathbf{Hot}_*^s$:

$$\mathbb{A}^n - \{0\} \cong S^{2n-1,n} = (S_s^1)^{n-1} \wedge (S_t^1)^n.$$

E-COHOMOLOGY 2.7. If $\mathbf{E}$ is a T-spectrum, we can define a bigraded cohomology theory on $\mathbf{Hot}(k)$:

$$E^{p,q}(X) = \mathrm{Hom}_{\mathbf{SHot}}\Big(\Sigma_T^\infty(X_+), \mathbf{E}(q)[p]\Big).$$

This definition is parallel to the usual definition in topology, except for the fact that we need to allow for two kinds of suspension. The same sort of bigrading occurs in Atiyah's KR-theory, for example. The (p, q) indexing convention we use is chosen to enhance the comparison with étale cohomology (which is also bigraded).

In the next part, we will introduce the Eilenberg-Mac Lane T-spectra $\mathbf{H}_R$. Here are some other important examples of T-spectra, and their associated cohomology theories.

EXAMPLE 2.8. Let BGL denote the infinite Grassmannian, the union over N of the Grassmannians $G(N,\infty)$. Consider the T-spectrum $\mathbf{K} = (BGL, T \wedge BGL \to BGL)$. This is the analogue of the BU-spectrum in topology, where the bonding maps come from Bott periodicity. If we take $T = \mathbb{P}_*^1$, as in the exercise above, the bonding map $\mathbb{P}_*^1 \wedge BGL \to BGL$ is the classifying map for the canonical "Bott" element of $K_0(\mathbb{P}_*^1 \wedge BGL)$.

THEOREM 2.9. *If X is smooth over k, then $K^{pq}(X)$ is isomorphic to $K_{2q-p}^{alg}(X)$, the algebraic K-theory of X. In particular, $K^{pq}(X) = 0$ if $p > 2q$.*

Note that this bigraded cohomology theory has a $(2,1)$-periodicity: $K^{p+2,q+1} \cong K^{p,q}$. The periodicity is induced by the identifications $K^{2p,p}(k) \cong KU^{2p}(\text{point})$.

EXAMPLE 2.10. Consider the T-spectrum $\mathbf{MGL} = \{Th(E_n \to BGL_n)\}_n$, with bonding maps coming from $E_n \oplus \mathcal{O} \to E_{n+1}$ via $Th(E \oplus \mathcal{O}) \cong \Sigma_T Th(E)$.

The T-spectrum $\mathbf{MGL}$ represents algebraic cobordism, once we impose étale descent and pass to finite coefficients. It is not known what $MGL^{pq}(X)$ is, but it can be proven that $MGL^{2p,p}(k)$ is the topological cobordism group $MU^{2p}(\text{point})$.

§3. Motivic Cohomology and the Motivic Steenrod Algebra

If X is a scheme of finite type over k, there is a sheaf $L(X)$, the "free sheaf with transfers generated by X," defined as follows.

DEFINITION 3.1. $L(X)\colon \mathit{SmAff}/k^{op} \to \mathbf{Ab}$ is the presheaf sending a connected U to the free abelian group on the set of all closed irreducible $W \subset U \times X$ such that the projection $W \to U$ is finite and surjective.

If R is a ring, we define $L(X,R)$ to be the presheaf sending U to the free R-module on the above set. Both $L(X)$ and $L(X,R)$ are Nisnevich sheaves, *i.e.*, objects of **Spc**. (See [**SV2, 6.6 and 5.18**] or [**SV, 4.2.9**].)

The graph of a function gives a map $\mathrm{Hom}(U,X) \to L(X)(U)$, inducing a canonical map $X \to L(X)$ in **Spc**. This is the analogue of the Dold-Thom construction: Let S^nX denote the nth symmetric product of X, *i.e.*, $S^nX = X^n/\Sigma_n$, and let $S^{\cdot}X$ denote $\coprod_{n\geq 0} S^nX$. In topology $S^{\cdot}X$ is the free abelian monoid on X, and it represents homology by the Dold-Thom theorem [**DT**]: $H_n(X,\mathbb{Z}) = \pi_n(S^{\cdot}X)$ for all n. In algebraic geometry, we need to group complete the Hom monoid:

LEMMA 3.2. ([**SV2, 6.8**]) *If U is smooth over k, then $L(X)(U)$ is the group completion of the abelian monoid* $\mathrm{Hom}_{\mathit{Varieties}}(U, S^{\cdot}X)$.

DEFINITION 3.3 ($\mathbf{H}_R$). Given a ring R, we define $K(R(n),2n)$ to be the sheaf of abelian groups (considered as a sheaf of sets, *i.e.*, as an object of **Spc**):

$$K(R(n),2n) = L(\mathbb{A}^n,R)/L(\mathbb{A}^n - \{0\},R).$$

There are product maps $K(R(m),2m) \wedge K(R(n),2n) \to K(R(m+n),2m+2n)$, induced from the external product of cycles. Bonding maps $T \wedge K(R(n),2n) \to K(R(n+1),2n+2)$ are obtained by composing with the natural map from $T = \mathbb{A}^1/\mathbb{A}^1 - \{0\}$ to $L(\mathbb{A}^1)/L(\mathbb{A}^1 - \{0\}) = K(R(1),2)$.

We write $\mathbf{H}_R$ for the resulting T-spectrum, formed from the $K(R(n),2n)$, and call it the "Eilenberg-Mac Lane T-spectrum associated to R."

DEFINITION 3.4. The *motivic cohomology* of X over k is defined to be its $\mathbf{H}_R$-cohomology:

$$H^{pq}(X,R) = H^{pq}_R(X) = \mathrm{Hom}_{\mathbf{SHot}}\big(\Sigma^\infty_T(X_+), \mathbf{H}_R(q)[p]\big).$$

Here are some of the main properties of motivic cohomology. We assume that k admits resolution of singularities, and that X is smooth, so that our motivic cohomology agrees with the one in [**FV**]; see 4.2 below.

1. $H^{2p,p}(X,\mathbb{Z}) = CH^p(X)$, the classical Chow group of p-cycles on X;
2. $H^{pp}(\mathrm{Spec}\,k,\mathbb{Z}) \cong K^M_p(k)$, the Milnor K-groups of the field k;
3. $H^{pq}(X,R) = 0$ when $p > q + \dim(X)$;
4. $H^{pq}(X,R) = 0$ when $p > 2q$;
5. $H^{pq}(X,R) = 0$ when $q < 0$;
6. $H^{pq}(X,\mathbb{Z}) = CH^q(X, 2q-p)$, the higher Chow groups of Bloch;
7. $H^{**}(X,R)$ is a bigraded ring, natural in X.

BEILINSON-SOULÉ VANISHING CONJECTURE. For smooth X, is $H^{pq}(X,\mathbb{Z}) = 0$ for $p < 0$?

EXAMPLE 3.5. For each integer $\ell > 0$ we have $H^{pp}(k, \mathbb{Z}/\ell) = K_p^M(k)/\ell$. This follows from the above properties and the exact sequence

$$H^{pp}(k,\mathbb{Z}) \xrightarrow{\ell} H^{pp}(k,\mathbb{Z}) \longrightarrow H^{pp}(k,\mathbb{Z}/\ell) \longrightarrow H^{p+1,p}(k,\mathbb{Z}).$$

EXERCISE 3.6. Let ℓ be a prime.1. Show that $H^{pq}(X, \mathbb{Z}_{(\ell)})$ is the localization of the abelian group $H^{pq}(X,\mathbb{Z})$ at the prime ℓ.

2. Show that $K_p^M(k) \otimes \mathbb{Q}/\mathbb{Z}_{(\ell)} \cong H^{pp}(\operatorname{Spec} k, \mathbb{Q}/\mathbb{Z}_{(\ell)})$.

Here is the only theorem in this part that cannot presently be proven without using resolution of singularities. Let Ω_T be the right adjoint of Σ_T, so that the bonding maps $\Sigma_T E_i \to E_{i+1}$ of an T-spectrum $\mathbf{E}$ induce maps $E_i \to \Omega_T E_{i+1}$.

THEOREM 3.7. *Assume that k admits resolution of singularities. Then $\mathbf{H}_R$ is an "Ω_T-spectrum," i.e., the spaces $K(R(n), 2n)$ satisfy:*

$$K(R(n),2n) \simeq \Omega_T K(R(n+1), 2n+2) \qquad \textit{in } \mathbf{Hot}_*.$$

In particular, the unstable group $\operatorname{Hom}_{\mathbf{Hot}}\big(X, K(R(n),2n)\big)$ equals the stable group $H^{2n,n}(X,R) = \operatorname{Hom}_{\mathbf{SHot}}\big(\Sigma_T^\infty(X_+), K(R(n),2n)\big)$.

As in algebraic topology, we can interpret a natural motivic cohomology operation $\phi_X\colon E^{p,q}(X) \to E^{p+i,q+j}(X)$ as a (stable) map $\phi\colon \mathbf{E} \to \mathbf{E}(j)[i] = S^{i,j} \wedge \mathbf{E}$ in $\mathbf{SHot}$. Indeed, ϕ_E sends the identity element in $E^{00}(\mathbf{E}) = \operatorname{Hom}(\mathbf{E},\mathbf{E})$ to ϕ, considered as an element of $E^{ij}(\mathbf{E}) = \operatorname{Hom}(\mathbf{E}, \mathbf{E}(j)[i])$.

Conversely, the map ϕ determines the operations ϕ_X. For each x in $E^{pq}(X)$, the identification $E^{pq}(X) = \operatorname{Hom}_{\mathbf{SHot}}(\Sigma_T^\infty(X_+), \mathbf{E}(p)[q])$ allows us to compose x with

$$\mathbf{E}(q)[p] = S^{pq} \wedge \mathbf{E} \xrightarrow{1\wedge\phi} S^{pq} \wedge S^{ij} \wedge \mathbf{E} = S^{p+i,q+j} \wedge \mathbf{E} = \mathbf{E}(q+j)[p+i]$$

to get the element $\phi_X(x)$ of $E^{p+i,q+j}(X)$. As in topology, the fact that ϕ is stable implies that each ϕ_X is additive: $\phi_X(x+y) = \phi_X(x) + \phi_X(y)$.

Restricting our attention to the T-spectrum $\mathbf{H}_{\mathbb{Z}/\ell}$ which yields the motivic cohomology groups $H^{pq}_{\mathbb{Z}/\ell}(X) = H^{pq}(X, \mathbb{Z}/\ell)$, we see that natural motivic cohomology operations $\phi_X\colon H^{p,q}(X,\mathbb{Z}/\ell) \to H^{p+i,q+j}(X,\mathbb{Z}/\ell)$ correspond to elements of $H^{ij}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbb{Z}/\ell)$.

DEFINITION 3.8. The (mod ℓ) *motivic Steenrod algebra* $\mathcal{A}^{pq} = \mathcal{A}^{pq}(k,\mathbb{Z}/\ell)$ is the algebra of endomorphisms of the T-spectrum $\mathbf{H}_{\mathbb{Z}/\ell}$ in the stable category $\mathbf{SHot}(k)$. That is,

$$\mathcal{A}^{pq} = \operatorname{Hom}_{\mathbf{SHot}}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbf{H}_{\mathbb{Z}/\ell}(q)[p]) = H^{pq}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbb{Z}/\ell).$$

EXAMPLES 3.9. 1. The bigraded ring $H^{**} = H^{**}(k,\mathbb{Z}/\ell)$ is a subring of the bigraded $\mathcal{A}^{**}$. Indeed, if $a \in H^{ij}$ then left multiplication by a induces a natural map from $H^{pq}(X,\mathbb{Z}/\ell)$ to $H^{p+i,q+j}(X,\mathbb{Z}/\ell)$, and is nonzero on $H^{00}(k,\mathbb{Z}/\ell)$ if $a \neq 0$.

When $p = q$, we see by example 3.5 that the group $K_p^M(k)/\ell$ is a subgroup of $\mathcal{A}^{pp}$. In particular, we will be interested in the cohomology operation $\rho \in H^{11}(k,\mathbb{Z}/\ell)$ corresponding to the class of $-1 \in k^\times/k^{\times\ell}$. When ℓ is odd, or when $\ell = 2$ and $\sqrt{-1} \in k$, then clearly $\rho = 0$.

2. ([**V, 3.14**]) If $\operatorname{char}(k) = 0$, then $\mathcal{A}^{pq} = \mathcal{A}^{pq}(k,\mathbb{Z}/\ell)$ is zero for $q < 0$, and $\mathcal{A}^{00} \cong \mathbb{Z}/\ell$ is generated by the identity, 1.

3. The Bockstein $\beta \in \mathcal{A}^{10}$ is the connecting homomorphism in the distinguished triangle

$$\mathbf{H}_{\mathbb{Z}/\ell} \to \mathbf{H}_{\mathbb{Z}/\ell^2} \to \mathbf{H}_{\mathbb{Z}/\ell} \xrightarrow{\beta} \mathbf{H}_{\mathbb{Z}/\ell}.$$

THEOREM 3.10. *Fix a field k of characteristic zero, and a prime ℓ. Then there are motivic cohomology operations $Q_i \in \mathcal{A}^{2\ell^i-1,\ell^i-1}$ satisfying the following properties:*

(i) $Q_iQ_j = -Q_jQ_i$ *and* $Q_i^2 = 0$*;*

(ii) Q_0 *is the Bockstein* β*, and for* $i > 0$ *there are operations* q_i *so that* Q_i *is the commutator* $[\beta, q_i]$*.*

(iii) $\Delta\colon H^{**}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbb{Z}/\ell) \to H^{**}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbb{Z}/\ell) \otimes_{H^{**}} H^{**}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbb{Z}/\ell)$ *satisfies*

$$\Delta(Q_i) = 1 \otimes Q_i + Q_i \otimes 1 + \sum (\rho^{n_j}\phi_j) \otimes \psi_j$$

where the operations ϕ_j and ψ_j have bidegrees (p, q) with $p > 2q$.

In particular, Q_i is primitive if ℓ is odd, or if $\ell = 2$ and $\sqrt{-1} \in k$.

The case $i = 1$ of formula (iii) is $\Delta(Q_1) = 1 \otimes Q_1 + Q_1 \otimes 1 + \rho Q_0 \otimes Q_0$.

REALIZATION 3.11. Any embedding of k in $\mathbb{C}$ gives a functor from Sm/k to complex manifolds, and hence to topological spaces. Using the fact that any sheaf is a colimit of representable sheaves, this functor can be extended to a *topological realization functor* $t_{\mathbb{C}}$ from **Spc** to topological spaces. This sends $\mathbf{H}_{\mathbb{Z}/\ell}$ to the usual Eilenberg-MacLane spectrum with coefficients $\mathbb{Z}/\ell$, and maps $\mathcal{A}^{**}$ to the usual Steenrod algebra. Under this realization, $t_{\mathbb{C}}(Q_i)$ is the usual cohomology operation Q_i (see [**M58**] or [**Mar, ch. 15**]). For example, when $\ell = 2$ we have $Q_1 = [Sq^2, Sq^1] = Sq^3 + Sq^2Sq^1$.

REMARK 3.12. There should be a spectral sequence converging to $K_*(X; \mathbb{Z}/2)$:

$$E_2^{p,q} = H^{p-q}(X; \mathbb{Z}/2(-q)) \Longrightarrow K_{-p-q}(X; \mathbb{Z}/2).$$

There is such a spectral sequence when $X = \mathrm{Spec}(k)$. Its construction in [**RW**] is based upon the spectral sequence for coefficients $\mathbb{Z}$ discovered by Bloch and Lichtenbaum [**BL**]. The differential d_2 goes from $H^{p,q}(X, \mathbb{Z}/2)$ to $H^{p+3,q+1}(X, \mathbb{Z}/2)$, and it is reasonable to ask if d_2 equals the motivic operation Q_1.

For motivation, consider the topological analogue. If X is a topological space then there is an Atiyah-Hirzebruch spectral sequence converging to $KU^*(X; \mathbb{Z}/2)$; the first nontrivial differential $d_3\colon H^p(X, \mathbb{Z}/2) \to H^{p+3}(X, \mathbb{Z}/2)$ is a cohomology operation, so d_3 is a degree 3 element of the usual Steenrod algebra $\mathcal{A}$. Now $\mathcal{A}^3$ is spanned by Sq^2Sq^1 and Sq^3, and simple computations with $X = \mathbb{RP}^4$ and $X = \mathbb{RP}^6$ show that d_3 must equal $Q_1 = Sq^2Sq^1 + Sq^3$. (This folklore result dates back to the calculations in [**St, p. 542**].)

§4. Motivic Cohomology of Quadrics and the Milnor Conjecture

Recall (from §3) that the motivic cohomology of X with coefficients in a commutative ring R was defined as

$$H^{pq}(X,R) = \mathrm{Hom}_{\mathbf{SHot}}(\Sigma_T^\infty X_+, \Sigma^{pq}\mathbf{H}_R),$$

where the bigraded suspension Σ^{pq} is the smash product with $S^{p,q}$ and $\mathbf{H}_R$ is the T-spectrum with nth space $K(R(n),2n) = L(\mathbb{A}^n,R)/L(\mathbb{A}^n - \{0\},R)$.

Also recall from §1 that if F is a presheaf then $C_*(F)$ is the simplicial presheaf $F(-\times\Delta^\bullet)$. If F is a presheaf of abelian groups then we shall also write $C_*(F)$ for the chain complex of presheaves (of abelian groups) corresponding to this simplicial presheaf under the Dold-Kan correspondence.

DEFINITION 4.1. ([**V2, 4.1.8**]) If R is a ring, the *motivic complex* $R(n)$ over k is the chain complex of presheaves

$$R(n) = C_*\Big(L(\mathbb{A}^n,R)/L(\mathbb{A}^n - \{0\},R)\Big)[-2n]$$

INDEXING CONVENTION. Here we use the shifting convention that if F_* is a chain complex then the degree i part of the chain complex $F[n]$ is F_{i-n}. In order to take the hypercohomology of F_*, we reindex as a cochain complex ($F^n = F_{-n}$), so that $\mathbb{H}^i(X,F_*[n]) = \mathbb{H}^{i+n}(X,F_*)$.

THEOREM 4.2. $H^{pq}(X,R) \cong \mathbb{H}^p_{zar}(X,R(q))$.

In particular, this shows that the groups $H^{pq}(X,R)$ do not depend upon the choice of the ground field k, since the presheaves $R(q)$ do not.

This theorem follows by combining the following lemma, partly due to K. Brown [**Br, p. 426**], with the (hard) theorem 3.7 which states that the stable groups $H^{pq}(X,R)$ coincide with the unstable Hom groups in the lemma.

MAIN LEMMA 4.3. $\mathbb{H}^{2n}_{zar}(X,R(n)) \cong \mathbb{H}^{2n}_{Nis}(X,R(n)) \cong \mathrm{Hom}_{\mathbf{Hot}}(X,K(R(n),2n))$.

DEFINITION 4.4 (H_L^{**}). The *Lichtenbaum motivic cohomology* of X is defined to be the étale hypercohomology of $R(q)$: $H_L^{pq}(X,R) = \mathbb{H}^p_{et}(X,R(q))$.

WARNING. There is a subtlety in this definition. There is a stable étale homotopy category $\mathbf{SHot}_{et}$, obtained by replacing Nisnevich sheaves by étale sheaves in the definitions, and "étale motivic cohomology" groups $\mathrm{Hom}_{\mathbf{SHot}_{et}}(\Sigma_T^\infty X_+, \Sigma^{pq}\mathbf{H}_R)$ of X. In characteristic zero, they agree with $H_L^{pq}(X,R)$. However, they are *not* the same as $H_L^{pq}(X,R)$ in characteristic p, unless we replace the coefficients R by $R[\frac{1}{p}]$.

To see why, consider the Frobenius morphism F in characteristic p. Let $\mathbb{G}_a$ denote the sheaf of abelian groups $X\mapsto H^0(X,\mathcal{O}_X)$. Since $\mathbb{G}_a\simeq 0$ in the derived category DM_k of [**V2**], the Artin-Schreier sequence $0\to\mathbb{Z}/p\to\mathbb{G}_a\xrightarrow{1-F}\mathbb{G}_a\to 0$ implies that the constant sheaf $\mathbb{Z}/p$ is zero in DM_k, and that DM_k is $\mathbb{Z}[1/p]$-linear in characteristic p. (See [**V1, 4.1.7**].) Applying $X\mapsto L(X)$ to $\mathbb{Z}/p$ yields $\mathbf{H}_{\mathbb{Z}/p}$. Hence $\mathbf{H}_{\mathbb{Z}/p}$ is $\mathbb{A}^1$-weak equivalent to zero in $\mathbf{SHot}_{et}$.

We can identify Lichtenbaum motivic cohomology with finite coefficients.

THEOREM 4.5. *(Suslin-Voevodsky) If $1/\ell \in k$ then for all X*

$$H_L^{p,q}(X, \mathbb{Z}/\ell) \cong H_{et}^p(X, \mu_\ell^{\otimes q}).$$

The proof doesn't require resolution of singularities, provided that we use the above definition of H_L^{pq}. The case $q = 1$ follows from the fact that $\mathbb{Z}(1)_{et} = \mathbb{G}_m[-1]$ in the étale topology which implies that $\mathbb{Z}/\ell(1)_{et} = \mu_\ell$.

We are interested in the following fundamental conjecture, made independently by Beilinson ([**Bei, p. 22**]) and Lichtenbaum ([**L, p. 130**]). It effectively connects motivic cohomology to étale cohomology.

BEILINSON-LICHTENBAUM CONJECTURE 4.6. *The map $H^{pq}(X, \mathbb{Z}) \to H_L^{pq}(X, \mathbb{Z})$ is an isomorphism for all $p \leq q + 1$.*

If true, this would imply as a corollary that $H^{pq}(X, \mathbb{Z}/\ell) \cong H_{et}^p(X, \mu_\ell^{\otimes q})$ for $p \leq q$. Since $H^{pp}(k, \mathbb{Z}/\ell) = K_p^M(k)/\ell$, the case $p = q$ of Lichtenbaum's Conjecture includes the:

NORM RESIDUE HOMOMORPHISM CONJECTURE 4.7. *([**BK, p. 118**]) If $\frac{1}{\ell} \in k$, the Norm Residue homomorphism*

$$K_p^M(k)/\ell \to H_{et}^p(k, \mu_\ell^{\otimes p})$$

is an isomorphism for all p.

This in turn is a generalization to all primes ℓ of *Milnor's conjecture*, that if $\mathrm{char}(k) \neq 2$ then $K_p^M(k)/2 \cong H_{et}^p(k, \mathbb{Z}/2)$. (Milnor originally posed this as a question in [**M70**].)

The analogue when $\mathrm{char}(k) = \ell$ is $K_p^M(k)/\ell \cong \nu(p)_F$; this was proven for $\ell = 2$ by Kato in [**K**], and for all ℓ by Gabber and Bloch-Kato [**BK, 2.1**].

It is helpful to localize the Lichtenbaum Conjecture at a prime ℓ. Since the sheaf $\mathbb{Z}_{(\ell)}(q)$ is the localization of $\mathbb{Z}(q)$ at ℓ, we see that the groups $H^{p,q}(X, \mathbb{Z}_{(\ell)})$ and $H_L^{p,q}(X, \mathbb{Z}_{(\ell)})$ are the localizations of $H^{pq}(X, \mathbb{Z})$ and $H_L^{pq}(X, \mathbb{Z})$ at the prime ℓ, respectively.

EXAMPLE 4.8 ($R = \mathbb{Q}$). We have $H^{pq}(X, \mathbb{Q}) \cong H_L^{pq}(X, \mathbb{Q})$ for all p and q. This follows from the fact that étale and Nisnevich hypercohomology agree for any complex of étale sheaves of $\mathbb{Q}$-vector spaces, such as $\mathbb{Q}(q)$, which is true in turn because all higher Galois cohomology groups vanish for uniquely divisible coefficients.

Hence the kernel and cokernel of $H^{pq}(X, \mathbb{Z}) \to H_L^{pq}(X, \mathbb{Z})$ are torsion groups.

This leads to the following definition, based on the Beilinson-Lichtenbaum Conjecture for q; a modified version of this was discussed in the Suslin-Voevodsky paper [**SV1, 5.6**].

DEFINITION 4.9 (BL CONDITION). We say that *$BL(q, \ell)$ holds for k* if for every $p \leq q + 1$ and every X of finite type over k we have

$$H^{pq}(X, \mathbb{Z}_{(\ell)}) \cong H_L^{pq}(X, \mathbb{Z}_{(\ell)}).$$

The reduction above shows that if $BL(q, \ell)$ holds for k then the Norm Residue Homomorphism Conjecture holds for q and ℓ.

The special case $p = q + 1$, $X = \mathrm{Spec}(K)$ of the $BL(q, \ell)$ condition turns out to be critical; since $H^{pq}(K, \mathbb{Z}_{(\ell)}) = 0$ for $p > q$, it amounts to asking whether

$H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)}) = 0$ for every field K over k. Lichtenbaum called this vanishing condition the (generalized) *Hilbert's Theorem 90* in [**L**], because when $q = 1$ it reduces to the classical version of Hilbert's Theorem 90 (see below). In fact, this vanishing condition implies the rest of the BL condition:

THEOREM 4.10. *(Suslin-Voevodsky* [**SV1, 5.9**]*) Assume that k admits resolution of singularities. Then $BL(q, \ell)$ holds for k if and only*

$$H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)}) = 0 \quad \textit{for every field extension } K \textit{ of } k.$$

If k has characteristic $p > 0$, and $\ell \neq p$, Geisser and Levine proved in [**GL**] that the analogue of this theorem for $BL(q, \ell)$ holds for all q, provided that the groups $H^{pq}(K, \mathbb{Z}_{(\ell)})$ are replaced by Bloch's higher Chow groups.

LOW DEGREE CASES 4.11. We know that $BL(q, \ell)$ holds for $q \leq 1$, because we know $\mathbb{Z}(q)$ in this range. $BL(q, \ell)$ is trivial for $q < 0$ because then $\mathbb{Z}(q) = 0$. It holds for $q = 0$ because $\mathbb{Z}(0) = \mathbb{Z}$ and $H_{et}^1(K, \mathbb{Z}) = \mathrm{Hom}(Gal(\bar{K}/K), \mathbb{Z}) = 0$. Finally, it holds for $q = 1$ because $\mathbb{Z}(1)_{et} \cong \mathbb{G}_m[-1]$ and $H_L^2(K, \mathbb{Z}(1)) = H_{et}^1(K, \mathbb{G}_m) = 0$.

Regarding the case $q = 2$, Lichtenbaum has shown in [**L**] that the vanishing of $H_L^{3,2}(K, \mathbb{Z})$ amounts to the "Hilbert Theorem 90 for K_2" of [**MS, 14.1**].

MAIN THEOREM 4.12 FOR $\ell = 2$. *Let k be a field of characteristic zero. Then:*

$$H_L^{q+1,q}(k, \mathbb{Z}_{(2)}) = 0 \qquad \textit{for all } q.$$

This implies Milnor's Conjecture for fields of characteristic zero. We can use this case to verify Milnor's Conjecture for all fields of characteristic $\neq 2$.

COROLLARY 4.13 (MILNOR'S CONJECTURE). *If $\frac{1}{2} \in k$, the mod 2 Norm Residue homomorphism is an isomorphism for all p:*

$$K_p^M(k)/2 \xrightarrow{\cong} H_{et}^p(k, \mathbb{Z}/2).$$

Our proof of the Main Theorem will proceed by induction on q, following the outline of the proof of the Merkurjev-Suslin theorem [**MS**].

Here is the first step, which works for any prime ℓ, using the inductive assumption that $BL(q-1, \ell)$ holds. (The induction is not needed when $\ell = 2$.)

LEMMA 4.14. *If $K_q^M(K)/\ell = 0$ then $H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)}) = 0$.*

PROOF. The exact sequence of coefficients $0 \to \mathbb{Z}_{(\ell)} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(\ell)} \to 0$ induces an exact sequence

$$H_{et}^q(K, \mathbb{Q}/\mathbb{Z}_{(\ell)}(q)) \to H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)}) \to H_L^{q+1,q}(K, \mathbb{Q}).$$

We have seen that the last term is zero. So it suffices to show that $H_{et}^q(K, \mathbb{Z}/\ell(q))$ is zero, since this will imply that the (ℓ-torsion) first term is also zero.

To prove this, we may reduce to the case in which K has no extensions of degree prime to ℓ. Using the arguments of [**Su**] for $q = 2, 3$, a Galois cohomology calculation shows that $K_q^M(E)/\ell = 0$ for all finite extensions E of K.

An element β of $H^q(K, \mathbb{Z}/\ell(q))$ vanishes on some finite extension E, which we may assume has the form $E = K(\sqrt[\ell]{b})$. A calculation shows that this implies that $\beta = [b] \cup \alpha$ for some α in $H^{q-1}(K, \mathbb{Z}/\ell(q-1))$, where $[b]$ is the class of b in $K^\times/K^{\times\ell} = H^1(K, \mathbb{Z}/\ell(1))$. Since $BL(q-1, \ell)$ holds, $H^{q-1}(K, \mathbb{Z}/\ell(q-1)) = K_{q-1}^M(K)/\ell = 0$. Hence $\alpha = 0$, and we have $\beta = 0$.

EXERCISE 4.15. Let K be a field extension of k. Compare the motivic and étale sequences for $0 \to \mathbb{Z}_{(\ell)}(q) \to \mathbb{Q}(q) \to \mathbb{Q}/\mathbb{Z}_{(\ell)}(q) \to 0$, using 3.6, 4.5 and 4.8 to show that there is an exact sequence:

$$H_L^{q,q}(K, \mathbb{Z}_{(\ell)}) \to K_q^M(K) \otimes \mathbb{Q}/\mathbb{Z}_{(\ell)} \to H_{et}^q(K, \mathbb{Q}/\mathbb{Z}_{(\ell)}(q)) \to H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)}) \to 0.$$

The second step in the proof of the Main Theorem 4.12, outlined below, uses a geometric construction. For every $a = (a_1, \dots, a_q) \in (k^\times)^q$, we must find an extension K_a/k such that: 1) $\{a_1, \dots, a_q\}$ is in $\ell K_q^M(K_a)$, and 2) the map $H_L^{q+1,q}(k, \mathbb{Z}_{(\ell)}) \to H_L^{q+1,q}(K_a, \mathbb{Z}_{(\ell)})$ is injective.

The Main Theorem follows from these two steps. Indeed, by repeatedly taking maximal prime-to-ℓ extensions and then the composite of the fields K_a in step two, we form an extension K of k so that 1) $K_q^M(K)/\ell = 0$ and 2) $H_L^{q+1,q}(k, \mathbb{Z}_{(\ell)})$ embeds into the group $H_L^{q+1,q}(K, \mathbb{Z}_{(\ell)})$, which is zero by step one. Thus $H_L^{q+1,q}(k, \mathbb{Z}_{(\ell)}) = 0$.

DEFINITION 4.16. Let $a = (a_1, \dots, a_q)$ be a sequence of nonzero elements of k, and let ℓ be a prime. We say that a variety X_a over k is a *generic splitting variety* for (a, ℓ), and that its function field $K_a = k(X_a)$ is a *generic splitting field*, if the following condition holds for every extension field K of k:

$$X_a \text{ has a rational point over } K \Leftrightarrow \{a_1, \dots, a_q\} \in \ell K_q^M(K).$$

We don't know how to construct generic splitting fields unless $\ell = 2$ or q is small. For example, when $q = 1$ it is easy to see that the generic splitting field is the finite field extension $K_a = k(\sqrt[\ell]{a_1})$ of k. The use of Brauer-Severi varieties as generic splitting varieties for $q = 2$ was central to the approach in [**MS**].

BRAUER-SEVERI VARIETIES. When $q = 2$, the generic splitting variety for (a, b) is the Severi-Brauer Variety $X_{(a,b)}$ associated to the division algebra $A(a, b) = k\{x, y\}/(x^\ell - a, y^\ell - b, yx - \zeta xy)$, $\zeta^\ell = 1$. It is classical that $X_{(a,b)}$ has a rational point over k if and only if $A(a, b)$ is a matrix algebra. Bass and Tate proved (see [**M71**], 15.7 and 15.12) that this holds if and only if $\{a, b\}$ vanishes in $K_2(k)/\ell$.

PFISTER QUADRICS. When $\ell = 2$, we use the following construction. For $a_i \in k$, let $\langle 1, -a_i \rangle$ denote the quadratic form $x^2 - a_i y^2$. Now fix a sequence $a = (a_1, \dots, a_q)$ of units in k. The *Pfister form* of dimension 2^q is the quadratic form

$$\ll a_1, \dots, a_q \gg = \langle 1, -a_1 \rangle \otimes \cdots \langle 1, -a_q \rangle.$$

The *Pfister quadric* X_a is the smooth projective quadric of dimension $2^{q-1} - 1$ given by the quadratic equation $\ll a_1, \dots, a_{q-1} \gg = a_q t^2$. It is a classical fact that X_a has a K-rational point if and only if $\ll a_1, \dots, a_q \gg$ represents zero over a field K.

For example, if $q = 2$ then X_a is the plane curve $x^2 = a_1 y^2 + a_2 t^2$, which is the Brauer-Severi Variety for $\ell = 2$.

THEOREM 4.17. *The Pfister quadric X_a is a generic splitting variety for $(a, 2)$, $q \geq 2$.*

IDEA OF PROOF. If X_a has a K-rational point with $t \neq 0$ then the quadratic form $\beta = \ll a_1, \dots, a_{q-1} \gg$ represents a_q over K. This means that there is a quadratic extension $E = K(\sqrt{b})$ so that β represents zero over E, and $a_q = N(e)$ is the norm of some $e \in E$. Since β represents zero, the Pfister quadric for $(a_1, \dots, a_{q-1})$ has an E-rational point; by induction on q, this implies that $\{a_1, \dots, a_{q-1}\} = 0$ in $K^M_{q-1}(E)/2$. But $\{a_1, \dots, a_q\}$ is the norm of $\{a_1, \dots, a_{q-1}, e\}$, so it vanishes in $K^M_q(k)/2$.

To complete step two, we need to prove that $H^{q+1,q}_L(k, \mathbb{Z}_{(2)}) \to H^{q+1,q}_L(K_a, \mathbb{Z}_{(2)})$ is a monomorphism, where the generic splitting field K_a is the function field of X_a.

For this, we use a reduction borrowed from descent theory. Given a scheme X over k, consider the simplicial scheme $\check{C}(X)$ defined by

$$\check{C}(X): \quad X \leftleftarrows X \times X \,\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\, X \times X \times X \ \cdots .$$

If $X(k) \neq \emptyset$, it is easy to see that the augmented simplicial scheme $\check{C}(X) \to \mathrm{Spec}(k)$ is aspherical. If $X(k) = \emptyset$ and we consider $\check{C}(X)$ as a simplicial Nisnevich sheaf then $\check{C}(X)(k)$ is empty. This proves most of the following lemma; the rest follows by "étale descent" from an E/k with $X(E) \neq \emptyset$, which shows that $\check{C}(X) \to \mathrm{Spec}(k)$ induces an isomorphism in étale cohomology.

LEMMA 4.18. (i) *The augmentation $\check{C}(X) \to \mathrm{Spec}(k)$ is a simplicial weak equivalence in* **Spc** *if and only if $X(k) \neq \emptyset$.*

(ii) *In any case, it induces an isomorphism $H^{pq}_L(\check{C}(X), -) \cong H^{pq}_L(\mathrm{Spec}(k), -)$.*

COROLLARY 4.19. *Assume that $BL(q-1, \ell)$ holds for k. Then for all $p \leq q$,*

$$H^{p,q-1}(k, \mathbb{Z}/\ell) \cong H^{p,q-1}(\check{C}(X), \mathbb{Z}/\ell).$$

Indeed, the BL condition allows us to replace both sides by étale cohomology, where we can invoke (ii).

We now specialize X to the Pfister quadric X_a.

PROPOSITION 4.20. *The map $H^{q+1,q}_L(k, \mathbb{Z}_{(2)}) \to H^{q+1,q}_L(K_a, \mathbb{Z}_{(2)})$ is injective if and only if*

$$H^{q+1,q}(\check{C}(X_a), \mathbb{Z}_{(2)}) = 0.$$

We have reduced step two to a vanishing assertion in motivic cohomology. Note that the L subscript has disappeared from our assertion!

SKETCH OF THE PROOF. Pick $u \in H^{q+1,q}_L(k, \mathbb{Z}_{(\ell)})$ vanishing in $H^{q+1,q}_L(K_a, \mathbb{Z}_{(\ell)})$. Then u must vanish on some dense open U in X_a. A Gysin sequence argument and induction on q shows that the image of u in $H^{q+1,q}_L(X_a, \mathbb{Z}_{(\ell)})$ comes from some u_0 in $H^{q+1,q}(X_a, \mathbb{Z}_{(\ell)})$. Because X_a is a quadric with a K_a-rational point, it is rational over K_a. Using the homotopy invariance of hypercohomology, we can show that the image of u in $H^{q+1,q}_L(\check{C}(X_a), \mathbb{Z}_{(\ell)})$ comes from $H^{q+1,q}(\check{C}(X_a), \mathbb{Z}_{(\ell)})$, a group which is zero by assumption.

We shall deduce the vanishing assertion from the following vanishing theorem. It is a combination of two theorems proven by Markus Rost in the late 1980's, using the language of Chow motives.

THEOREM 4.21. *(Rost) If* $s = 2^q - 1$ *and* $t = 2^{q-1}$ *then:* $H^{st}(\check{C}(X_a), \mathbb{Z}_{(2)}) = 0$.

We shall connect Rost's result to our situation by reducing modulo two and using a mod 2 motivic cohomology operation ϕ. For legibility, we write $\check{C}$ for $\check{C}(X_a)$ and set $s = 2^q - 1$, $t = 2^{q-1}$.

Since there is a degree two extension E of k so that $\check{C}(E) \neq \emptyset$, the usual transfer argument shows that $H^{pq}(\check{C}, \mathbb{Z}_{(2)})$ has exponent 2 for $p > q$. In particular, if ρ denotes reduction $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$, then ρ and β fit into an exact sequence

$$0 \to H^{p,q}(\check{C}, \mathbb{Z}_{(2)}) \xrightarrow{\rho} H^{p,q}(\check{C}, \mathbb{Z}/2) \xrightarrow{\beta} H^{p+1,q}(\check{C}, \mathbb{Z}/2).$$

We define the mod 2 cohomology operation ϕ to be the composite of the successive cohomology operations $Q_1, Q_2, Q_3, \dots, Q_{q-2}$. Each operation Q_i maps H^{pq} to $H^{p+2^{i+1}-1, q+2^i-1}$, so their composite ϕ maps $H^{q+1,q}$ to $H^{s,t}$.

Because $Q_0 = \beta$ commutes with ϕ (mod 2), there is a homomorphism Φ making the following diagram commute:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{q+1,q}(\check{C}, \mathbb{Z}_{(2)}) & \xrightarrow{\rho} & H^{q+1,q}(\check{C}, \mathbb{Z}/2) & \xrightarrow{\beta} & H^{q+2,q}(\check{C}, \mathbb{Z}/2) \\
& \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
0 \longrightarrow & H^{st}(\check{C}, \mathbb{Z}_{(2)}) & \xrightarrow{\rho} & H^{st}(\check{C}, \mathbb{Z}/2) & \xrightarrow{\beta} & H^{s+1,t}(\check{C}, \mathbb{Z}/2).
\end{array}$$

By Rost's theorem, the lower left group $H^{st}(\check{C}, \mathbb{Z}_{(2)})$ is zero. The following result, that the middle map ϕ is a monomorphism, implies that the upper left group $H^{q+1,q}(\check{C}, \mathbb{Z}_{(2)})$ is zero. This proves the vanishing assertion, and completes the proof of the Main Theorem 4.12.

THEOREM 4.22. [**V, 4.11**] *For* $s = 2^q - 1$ *and* $t = 2^{q-1}$, *the cohomology operation*

$$\phi \colon H^{q+1,q}(\check{C}, \mathbb{Z}/2) \rightarrowtail H^{s,t}(\check{C}, \mathbb{Z}/2)$$

is a monomorphism.

To prove this, we need some machinery which works for any prime ℓ.

Just as in topology [**Mar**], we can use the fact that $Q_i^2 = 0$ to define the *motivic Margolis cohomology* HM_i^{**} of X with coefficients $\mathbb{Z}/\ell$. Set $s = 2\ell^i - 1$ and $t = \ell^i - 1$. Then $HM_i^{p,q}(X, \mathbb{Z}/\ell)$ is defined to be the cohomology of the complex

$$H^{*-s,*-t}(X, \mathbb{Z}/\ell) \xrightarrow{Q_i} H^{*,*}(X, \mathbb{Z}/\ell) \xrightarrow{Q_i} H^{*+s,*+t}(X, \mathbb{Z}/\ell).$$

EXAMPLE 4.23. If $i = 0$ then Q_0 is the Bockstein. A routine diagram chase shows that $HM_0^{pq}(X, \mathbb{Z}/\ell)$ is the subgroup of elements divisible by ℓ in $H^{pq}(X, \mathbb{Z}/\ell^2)$.

One way to study Margulis cohomology is to introduce Φ_i-cohomology, where Φ_i is the homotopy fiber of $Q_i \colon \mathbf{H}_{\mathbb{Z}/\ell} \to \mathbf{H}_{\mathbb{Z}/\ell}(t)[s]$. From the long exact cohomology sequence

$$\cdots \Phi_i^{pq}(X) \xrightarrow{u} H^{pq}(X, \mathbb{Z}/\ell) \xrightarrow{Q_i} H^{p+s,q+t}(X, \mathbb{Z}/\ell) \xrightarrow{w} \Phi_i^{p+1,q}(X) \cdots$$

and a diagram chase, we see that $HM_i^{p,q}(X, \mathbb{Z}/\ell) = 0$ if and only if the "obstruction map" $wu \colon \Phi_i^{pq}(X) \to \Phi_i^{p+1-s,q-t}(X)$ is zero.

Define the simplicial space $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}(X)$ by the distinguished triangle

$$\tilde{\mathcal{X}}(X) \to \check{C}(X) \to \mathrm{Spec}(k)$$

Since $\mathrm{Hom}(X, X) \neq \emptyset$, it is not hard to see that $\Sigma_T^\infty(X_+) \wedge \tilde{\mathcal{X}}(X) = 0$.

THEOREM 4.24. *Let X be a smooth hypersurface of degree ℓ in $\mathbb{P}^{\ell^n}$. Then*

$$HM_i^{**}(\tilde{\mathcal{X}}(X), \mathbb{Z}/\ell) = 0 \quad \textit{for all } i \leq n.$$

The proof of this theorem is given in [**V, 3.22, 3.25 and 4.11**]. It uses the motivic Thom class $\tau\colon \mathbf{MGL} \to \mathbf{H}_{\mathbb{Z}/\ell}$ and the classes in $MGL_{2p,p}(X)$ of the plane sections of X, which correspond to important bordism classes in $MU_{2p}(\text{point})$. With these tools, and the formula 3.10(iii) above, we can show that the obstruction map wu for $\tilde{\mathcal{X}}$ factors through the group $\Phi_i^{*,*}(\Sigma_T^\infty(X_+) \wedge \tilde{\mathcal{X}}(X), \mathbb{Z}/\ell) = 0$.

Applying Theorem 4.24 to the Pfister quadric hypersurface in $\mathbb{P}^{2^{q-1}}$, we see that for $i < q$ there is an exact sequence

$$H^{q-i-1,q-i}(\tilde{\mathcal{X}}) \xrightarrow{Q_i} H^{q-i+2^{i+1}-2,q-i+2^i-1}(\tilde{\mathcal{X}}) \xrightarrow{Q_i} H^{q-i+2^{i+2}-3,q-i+2^{i+1}-2}(\tilde{\mathcal{X}}).$$

Because we know that $BL(q-i, 2)$ holds by induction, and $H_L^{*,*}$ vanishes on $\tilde{\mathcal{X}}(X_a)$ by (ii) of the previous lemma, the first group vanishes:

$$H^{q-i-1,q-i}(\tilde{\mathcal{X}}(X_a), \mathbb{Z}/2) \cong H_L^{q-i-1,q-i}(\tilde{\mathcal{X}}(X_a), \mathbb{Z}/2) = 0.$$

This proves that Q_i is a monomorphism on the middle group. For $i = 1$ this Q_1 is the beginning $H^{q+1,q} \to H^{q+4,q+1}$ of the operation ϕ. By induction on i, each composite $Q_i \cdots Q_2 Q_1$ (and in particular ϕ) is a monomorphism on $H^{q+1,q}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$. So we are done with the proof of the Main Theorem 4.12.

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