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Homology of Schemes

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Contents

1. Introduction

In this paper we suggest an approach to the construction of the category of mixed motives. The word "motive" was introduced by A. Grothendieck almost thirty years ago to denote objects of the hypothetical semi-simple $\mathbf{Q}$-linear abelian category where the "universal" cohomology theory on the category of smooth projective algebraic varieties takes values. Some fifteen years later the Grothendieck's idea was developed further by P. Deligne, A. Beilinson, S. Lichtenbaum and others to accommodate all algebraic varieties. These new "motives" were called "mixed motives" after the mixed Hodge structures and old Grothendieck's motives were renamed into "pure motives". For all these years the theory of motives was one of the most important unification concepts in algebraic geometry. In its modern form it gives a very coherent picture of how cohomology theories on the category of algebraic varieties should behave. In particular, it provides a natural "explanation" for many apparently unrelated conjectures such as the Bloch–Kato conjecture in the étale cohomology, the Quillen–Lichtenbaum conjecture and the Beilinson–Soule vanishing conjecture in the algebraic K-theory, the Bloch conjecture on zero cycles and the Grothendieck standard conjectures in the theory of algebraic cycles etc.

Unfortunately, until very recently, the theory of motives and especially the theory of mixed motives remained almost totally hypothetical. While quite a few results which confirmed the feeling that such a theory should exist were obtained no candidates for the category of mixed motives over an arbitrary field were suggested[1] and none of the "standard conjectures" were proved. A good overview of the present state of the theory of motives can be found in [1].

This paper is the first one in a series of related papers where we try to develop techniques necessary to construct the theory of (mixed) motives. The fundamental difference of the approach considered here with the one usually used is that we construct a *triangulated* category of mixed motives instead of the abelian category required by the standard conjectures. This basically means that the original problem is divided into two independent parts — to construct the triangulated category and prove its basic properties and to show that this triangulated category is in fact the derived category of an abelian one. An important feature of this approach is that the construction of a triangulated category of mixed motives is a much more accessible problem that the construction of an abelian one. On the other hand many of the "motivic conjectures" do not require us to pass to the abelian level and can be seen as particular cases of certain basic properties of this triangulated category itself. Moreover, it can be shown that if we are working with integral or finite coefficients instead of the rational ones then the abelian category of mixed motives satisfying "standard conjectures" suggested by A. Beilinson does not exist and therefore the triangulated category is in this case the natural object to work with[2].

In this paper we construct for any noetherian base scheme S a triangulated category $DM(S)$ and a functor $M : Sch/S \longrightarrow DM(S)$ from the category of schemes of finite type over S to $DM(S)$. This functor satisfies the usual properties of homological theories. Denote by $DM_{\mathrm{ft}}(S)$ the full triangulated subcategory of $DM(S)$ generated by the image of the functor M. Then the pair

$$\big(DM_{\mathrm{ft}}(S) \otimes \mathbf{Q}, M_{\mathbf{Q}} : Sch/S \longrightarrow DM_{\mathrm{ft}}(S) \otimes \mathbf{Q}\big)$$

is universal among functors from the category Sch/S to $\mathbf{Q}$-linear triangulated categories which satisfy some analog of the Eilenberg–Steenrod axioms for homological

1 We mean here *all* mixed motives. Some candidates for the category of mixed Tate motives were constructed in [3], [4].

2 It might still be true even for finite coefficients that this triangulated category is a derived category of an exact category, but in any case the classical point of view fails.

theories.

According to Grothendieck's original approach to the theory of motives, it is natural to call $DM_{\mathrm{ft}}(S)$ *the triangulated category of (effective) mixed motives over* S. The subcategory $DM_{\mathrm{ft}}(S)$ is "dense" in the category $DM(S)$, i.e. any object of the last category is a homotopy direct limit of objects of the former category. We call the category $DM(S)$ *the homological category of schemes over* S.

Our construction of $DM(S)$ is based on simple topological intuition. Consider a topological space X (which we assume to be homotopy equivalent to a CW-complex) and suppose that we want to assign to it its "motive" $M(X)$ in the Grothendieck's sense. To do so we will have to specify first the class of cohomology theories with respect to which our motive should be universal. The most obvious choice would be to consider all cohomology theories satisfying Eilenberg–Steenrod axioms. The solution of the corresponding universal problem in topology is well known. Namely, the "category of motives" in this case is the Spanier–Whitehead category and the "motive" of X is its stable homotopy type. However, if we want $M(X)$ to be the "motive" of X in Grothendieck's sense we have to work with a smaller class of theories. The reason for that is that Grothendieck's motive of X is expected to be functorial not only with respect to morphisms in X but also with respect to *correspondences*. Topologically it means that we want to consider theories which have transfer maps with respect to a rather broad class of "coverings". It is known, that the only theories satisfying this property are *ordinary theories*, i.e. the usual cohomology with coefficients in complexes of abelian groups. Thus, the universal category in this situation is the derived category of abelian groups and the "motive" of X is the class of its singular simplicial complex in this category. This reasoning, however contains an element of cheating — namely to describe our universal category we have to know in advance all the theories which factor through it. Properly, one should start with a construction of the universal category with respect to "theories with strong transfers" and then show that it is equivalent to the derived category of abelian groups. The fact that all such theories are ordinary cohomology with coefficients in complexes of abelian groups appears then as a natural corollary of this result. It turns out that if we follow this path carefully then all the topological constructions we have to use to define the required universal category have immediate algebro-geometrical analogs leading to the construction of $DM(S)$ suggested in this paper.

In the first section we describe a general construction which assigns, to any site T equipped with an object I^+ called "interval", a triangulated category $H(T, I^+)$ and a functor

$$M : T \longrightarrow H(T, I^+).$$

In particular if we set $I^+ = \mathbf{A}^1_S$, then for any Grothendieck topology t on the category Sch/S of schemes of finite type over a scheme S there is a "homological theory"

$$M_t : Sch/S \longrightarrow H\big((Sch/S)_t, \mathbf{A}^1_S\big).$$

Unfortunately, when defined for the topologies usually used in algebraic geometry (like Zariski, étale or flat topology) this functor does not satisfy the properties one would expect from the "theory of motives".

In the next section we define two new Grothendieck topologies on the categories Sch/S which are called h- and *qfh*-topologies. We also prove some of their basic properties. We define the *homological category* $DM(S)$ of schemes over S to be the category $H((Sch/S)_h, \mathbf{A}^1_S)$. Though the main object of our interest is the h-topology and the associated theory $M : Sch/S \longrightarrow DM(S)$ we have to use *qfh*-topology as an intermediate step and we mostly consider the theory M_{qfh} in this paper.

In the last section we prove basic properties of the theories M and M_{qfh} associated with the h- and *qfh*-topologies.

The original idea of the present construction appeared as a result of a joint attempt by M. Kapranov and the author to understand the possible role of simplicial sheaves in Beilinson's approach to motives through the idea of "motivic sheaves" and was developed in the author's Ph.D. thesis [16].

The final version of this paper was prepared during my stay at the Institute for Advanced Studies in Princeton[3] in 1993. Since then, a much better understanding of properties of the categories $DM(S)$ was achieved for S being the spectrum of a field (see [17]). In particular it became clear that $DM(S)$ is one of at least two possible categories of motives namely the category of motives "in the étale topology". The corresponding motivic cohomology groups should satisfy Lichtenbaum's axioms and not the Beilinson's axioms which were given for the Zariski topology case. A construction of the Zariski version of DM is given (in the case of a base field) in [17][4]. Some further results on h- and *qfh*-topologies can be found in [13], [14] and [6] where they were used as tools to study algebraic cycles.

I am very grateful to David Kazhdan who was my thesis advisor and to A. Beilinson, A. Goncharov and A. Levin for inspiring discussions.

2. Generalities

2.1 Freely generated sheaves

Let T be a site and $\underline{R}$ a sheaf of commutative rings on T. We will only be interested in the case when $\underline{R}$ is the constant sheaf associated with a ring R.

For any $\underline{R}$ we denote by $\underline{R} - \mathrm{mod}(T)$ the abelian category of sheaves of $\underline{R}$-modules.

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[4] It should be mentioned that the difference between étale and Zariski motives appears only in the torsion effects and that rationally the corresponding categories are equivalent.

Proposition 2.1.1. *Let T be a site and $\underline{R}$ be a sheaf of rings on T. Then there exists a functor $\underline{R}(*) : \mathrm{Sets}(T) \longrightarrow \underline{R}-\mathrm{mod}(T)$ which is left adjoint to the forgetful functor $\underline{R}-\mathrm{mod}(T) \longrightarrow \mathrm{Sets}(T)$.*

Proof. For any sheaf of sets X on T we define the sheaf $\underline{R}(X)$ to be the sheaf associated with the presheaf $U \longrightarrow \underline{R}(U)(X(U))$, where $\underline{R}(U)(X(U))$ is the free $\underline{R}(U)$-module generated by the set $X(U)$. The proof of the adjointness property is trivial.

In the case when $\underline{R}$ is the constant sheaf associated with a ring R, we will denote the functor $\underline{R}(*)$ just by $R(*)$. The sheaf $\mathbf{Z}(X)$ is called the sheaf of abelian groups freely generated by the sheaf of sets X.

We will also use the notation $\tilde{\underline{R}}(*)$ for the functor which takes a sheaf of sets X to the kernel of the morphism $\underline{R}(X) \longrightarrow \underline{R}(\mathrm{pt})$ induced by the canonical morphism from X to the final object of $\mathrm{Sets}(T)$.

The following proposition summarizes the elementary properties of the functors $\underline{R}(*)$.

Proposition 2.1.2.

1. *The functor $\underline{R}$ is right exact, i.e. it takes direct limits in* $\mathrm{Sets}(T)$ *to the direct limits in $\underline{R}-\mathrm{mod}(T)$. In particular it preserves epimorphisms.*
2. *The functor $\underline{R}$ preserves monomorphisms.*
3. *Sheaves of the form $\underline{R}(X)$ are flat.*
4. *For a pair X, Y of sheaves of sets T one has a canonical isomorphism*

$$\underline{R}(X \times Y) \cong \underline{R}(X) \otimes \underline{R}(Y).$$

Proof. 1. It follows from the general properties of adjoint functors.

2. One can easily see, that the functor which takes a sheaf of sets X to the presheaf $U \longrightarrow \underline{R}(U)(X(U))$ preserves monomorphisms. The statement of the proposition now follows from the fact that the functor of the associated sheaf is exact.

3. Easy.

4. It follows directly from the construction of the functor $\underline{R}(*)$ and the definition of tensor products of sheaves of $\underline{R}$-modules.

Proposition is proved.

Proposition 2.1.3. *Let T be a site, $\underline{R}$ a sheaf of rings on T and $L(X)$ the sheaf of sets representable by an object X of T. Then for any sheaf F of $\underline{R}$-modules and any $n \geq 0$ one has canonical isomorphisms*

$$H^n_T(X, F) = \mathrm{Ext}^n_{\underline{R}-\mathrm{mod}}\left(\underline{R}(L(X)), F\right).$$

Proof. It follows immediately from the adjointness property of the functor $\underline{R}$ and the description of cohomological groups in terms of injective resolutions of sheaves.

Let $f : X \longrightarrow Y$ be a morphism in Sets(T). Denote by $\underline{R}(\check{C}(f))$ the complex of $\underline{R}$-modules of the form

$$\ldots \longrightarrow \underline{R}(X \times_Y X) \xrightarrow{\underline{R}(\mathrm{pr}_1)-\underline{R}(\mathrm{pr}_2)} \underline{R}(X) \xrightarrow{\underline{R}(f)} \underline{R}(Y) \longrightarrow 0.$$

Proposition 2.1.4. *For any morphism $f : X \longrightarrow Y$ in the category* Sets(T) *the complex $\underline{R}(\check{C}(f))$ is a resolution of the sheaf* $\mathrm{coker}(\underline{R}(f))$.

Proof. Easy.

There is a different approach to the definition of the functor $\underline{R}$ which is sometimes more convenient than the one described above.

Let U be an object of T. Denote by T/U the site whose underlying category is the category of objects of T over U and the topology is defined in the obvious way. There is a natural morphism of sites $p : T/U \longrightarrow T$ such that the functor p^{-1} takes an object X of T to the object $X \times U \longrightarrow U$ of T/U.

Proposition 2.1.5. *There exists a functor $p_! : p^*(\underline{R})-\mathrm{mod}(T/U) \longrightarrow \underline{R}-\mathrm{mod}(T)$ left adjoint to the functor of the inverse image p^*.*

Proof. Let F be an object of the category $p^*(\underline{R})-\mathrm{mod}(T/U)$. Consider the presheaf $p_\#(F)$ of $\underline{R}$-modules on T of the form

$$p_\#(F)(V) = \bigoplus_{f \in \mathrm{Hom}_T(V,U)} F(f : V \longrightarrow U).$$

We define $p_!(F)$ to be the sheaf associated with the presheaf $p_\#(F)$.

To prove that the functor $p_!$ defined by this construction is indeed left adjoint to the functor of the inverse image, we have to show that for any pair of sheaves $F \in \mathrm{ob}(p^*(\underline{R})(T/U))$, $G \in \mathrm{ob}(\underline{R}(T))$ there exists a natural bijection

$$\mathrm{Hom}_{p^*(\underline{R})(T/U)}\big(F, p^*(G)\big) = \mathrm{Hom}_{\underline{R}(T)}\big(p_!(F), G\big).$$

By the adjointness property of the functor of the associated sheaf, the right-hand side is canonically isomorphic to $\mathrm{Hom}_{\underline{R}(T)}(p_\#(F), G)$. Therefore a morphism $a : p_!(F) \longrightarrow G$ is a natural family of morphisms of the form

$$a_{f:V \longrightarrow U} : F(f : V \longrightarrow U) \longrightarrow G(V).$$

On the other hand one has

$$p^*(G)(f : V \longrightarrow U) = G(V)$$

and therefore a morphism $F \longrightarrow p^*(G)$ is a family of morphisms of exactly the same form. Proposition is proved.

Proposition 2.1.6. *The functor* $p_! : p^*(\underline{R})-\mathrm{mod}(T/U) \longrightarrow \underline{R}-\mathrm{mod}(T)$ *is exact.*

Proof. Since $p_!$ is left adjoint to p_* it is right exact by the general properties of adjoint functors. On the other hand the proof of Proposition 2.1.5 shows that $p_!$ is the composition of the functor $p_{\#}$ with the functor of the associated sheaf. Since both functors are left exact, the same holds for $p_!$.

The connection between the functors $p_!$ and the functors $\underline{R}$ is given by the following proposition.

Proposition 2.1.7. *For any object U of T there is a canonical isomorphism*

$$p_!(p^*(\underline{R})) \cong \underline{R}(L(U))$$

where $p : T/U \longrightarrow T$ is the canonical morphism of sites and $L(U)$ is the sheaf of sets representable by the object U.

Proof. It follows immediately from the explicit constructions of the functors $\underline{R}$ and $p_!$.

2.2 The homological category of a site with interval

Let T be a site. An interval in T is an object I^+, such that there exists a triple of morphisms $(\mu : I^+ \times I^+ \longrightarrow I^+, i_0, i_1 : \mathrm{pt} \longrightarrow I^+)$ satisfying the conditions

$$\mu(i_0 \times \mathrm{Id}) = \mu(\mathrm{Id} \times i_0) = i_0 p$$
$$\mu(i_1 \times \mathrm{Id}) = \mu(\mathrm{Id} \times i_1) = \mathrm{Id},$$

where $p : I^+ \longrightarrow \mathrm{pt}$ is the canonical morphism. We will also assume that the morphism

$$i_0 \coprod i_1 : \mathrm{pt} \coprod \mathrm{pt} \longrightarrow I^+$$

is a monomorphism.

The goal of this section is, to assign to any site with interval, a tensor triangulated category $H(T, I^+)$ (or just $H(T)$) which is called the homological category of T and to prove its elementary properties.

Let I^1 be the kernel of the canonical morphism $\mathbf{Z}(I^+) \longrightarrow \mathbf{Z}$. Denote by $D(T)$ the derived category of the category $\mathrm{Ab}(T)$ of sheaves of abelian groups on T constructed by means of bounded complexes. It is known to be a tensor triangulated category.

We are going to define the homological category $H(T)$ of (T, I^+) as a localization of the category $D(T)$ with respect to the class of "contractible" objects.

Consider the morphism

$$i = \mathbf{Z}(i_0) - \mathbf{Z}(i_1) : \mathbf{Z} \longrightarrow I^1.$$

Since $i_0 \coprod i_1$ is a monomorphism the morphism i is a monomorphism. Denote its cokernel by S^1. We define I^n (resp. S^n) to be the n-th tensor power of I^1 (resp. S^1). Note that there is a canonical morphism

$$\partial : S^1 \longrightarrow \mathbf{Z}[1]$$

in $D(T)$ which corresponds to the extension of the sheaf S^1 by means of $\mathbf{Z}$ defined by the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{i} I^1 \longrightarrow S^1 \longrightarrow 0.$$

Definition 2.2.1. A sheaf of abelian groups F on T is called strictly contractible if there exists a morphism

$$\phi : F \otimes I^1 \longrightarrow F$$

such that the composition $\mathrm{Id}_F \otimes i : F \longrightarrow F$ is the identity morphism. A sheaf of abelian groups F on T is called contractible if it has a left resolution which consists of strictly contractible sheaves.

Denote by $\mathrm{Contr}(T)$ the thick subcategory (see [5, Appendix]) of the category $D(T)$ generated by contractible sheaves.

Definition 2.2.2. The homological category $H(T)$ of a site with interval (T, I^+) is the localization of the category $D(T)$ with respect to the subcategory $\mathrm{Contr}(T)$.

The following lemma provides us with some trivial examples of strictly contractible sheaves.

Lemma 2.2.3.

1. *The sheaf* $\ker(\mathbf{Z}((I^+)^n) \longrightarrow \mathbf{Z})$ *is strictly contractible for any* $n \geq 0$.
2. *For any sheaf F and any strictly contractible sheaf G the sheaves $F \otimes G$,* $\underline{\mathrm{Hom}}(G, F)$ *are strictly contractible.*

Proposition 2.2.4. *Let X be an object of the category $D(T)$ and Y an object of* $\mathrm{Contr}(T)$. *Then $X \otimes Y$ belongs to the category* $\mathrm{Contr}(T)$.

Proof. It follows easily from our definitions and Lemma 2.2.3(2).

To get more sophisticated examples of contractible sheaves we need the following construction.

Let $f : (0, \ldots, n) \longrightarrow (0, \ldots, m)$ be a morphism in the standard simplicial category Δ. We define a morphism of sets $\phi(f) : \{1, \ldots, m\} \longrightarrow \{0, \ldots, n+1\}$ as follows:

$$\phi(f)(i) = \begin{cases} \min\{l \in \{0, \ldots, n\} | f(l) \geq i\} & \text{if this set is not empty} \\ n+1 & \text{otherwise.} \end{cases}$$

Denote by $\mathrm{pr}_k : (I^+)^n \longrightarrow (I^+)^1$ the k-th projection and by $p : (I^+)^n \longrightarrow \mathrm{pt}$ the canonical morphism from $(I^+)^n$ to the finial object of T. We define the morphism $a(f) : (I^+)^m \longrightarrow (I^+)^n$ setting

$$\mathrm{pr}_k \circ a(f) = \begin{cases} pr_{\phi(f)(k)} & \text{if } \phi(f)(k) \in \{1, \dots, n\} \\ i_0 \circ p & \text{if } \phi(f)(k) = n+1 \\ i_1 \circ p & \text{if } \phi(f)(k) = 0. \end{cases}$$

One can easily see that for a composable pair of morphisms f, g in Δ we have $a(f \circ g) = a(f) \circ a(g)$ and hence our construction gives a cosimplicial object $a : \Delta \longrightarrow \mathrm{T}$ in T. To be more specific we will denote it by a_{I^+}.

Let F be a sheaf of abelian groups on T. Denote by $C_*(F)$ the complex of sheaves whose terms are the sheaves $\underline{\mathrm{Hom}}(\mathbf{Z}((I^+)^n), F)$ and the differentials are the alternated sums of the morphisms induced by the coface morphisms of the cosimplicial object a_{I^+}.

Lemma 2.2.5. *Let F be a sheaf of abelian groups on T such that the complex $C_*(F)$ is exact. Then F is contractible.*

Proof. It follows easily from our definitions and Lemma 2.2.3.

Denote by $H_0(T)$ the localization of the category $D(T)$ with respect to the thick subcategory generated by objects of the form $X \otimes I^1$, $X \in \mathrm{ob}(D(T))$.

For any object X of $\mathrm{Sets}(T)$ denote by $\tilde{\mathbf{Z}}(X)$ the kernel of the natural morphism $\mathbf{Z}(X) \longrightarrow \mathbf{Z}$. We define the functor $M : \mathrm{Sets}(T) \longrightarrow H(T)$ (resp. $\tilde{M}$: $\mathrm{Sets}(T) \longrightarrow H(T)$) as the composition of the functor $\mathbf{Z}(-)$ (resp. $\tilde{\mathbf{Z}}(-)$) with the canonical functor $\mathrm{Ab}(T) \longrightarrow D(T)$. We will also use the notations M_0, $\tilde{M}_0$ for the corresponding functors to the category $H_0(T)$.

Proposition 2.2.6. *Let $X, Y \in \mathrm{ob}\, D(T)$. Then one has*

$$\mathrm{Hom}_{H_0(T)}(X, Y) = \lim_{n \to \infty} \mathrm{Hom}_{D(T)} \left(X \otimes S^n, Y[n]\right)$$

where the direct system on the right-hand side is defined by tensor multiplication of morphisms with $\partial : S^1 \longrightarrow \mathbf{Z}[1]$.

Proof. Note first of all that the morphism $\partial : S^1 \longrightarrow \mathbf{Z}[1]$ is an isomorphism in $H_0(T)$ and therefore there is a canonical morphism

$$\lim_{n \to \infty} \mathrm{Hom}_{D(T)} \left(X \otimes S^n, Y[n]\right) \longrightarrow \mathrm{Hom}_{H_0(T)}(X, Y).$$

Let $F : D(T) \longrightarrow D'$ be an exact functor from $D(T)$ to a triangulated category D' such that $F(g)$ is an isomorphism for any morphism g whose cone lies in the thick subcategory generated by objects of the form $X \otimes I^1$. Then there exists the unique extension of the map $\mathrm{Hom}_{D(T)}(X, Y) \longrightarrow \mathrm{Hom}_{D'}(F(X), F(Y))$ to the map

$$\lim_{n\to\infty} \mathrm{Hom}_{D(T)}\big(X \otimes S^n, Y[n]\big) \longrightarrow \mathrm{Hom}_{D'}\big(F(X), F(Y)\big).$$

The universal property of localization implies that to prove our theorem it is sufficient to show that, for any object Y of the thick subcategory generated by objects of the form $X \otimes I^1$, there exists n such that $\mathrm{Id}_Y \otimes \partial^{\otimes n} = 0$. It is sufficient to show that the class of objects satisfying this property contains objects of the form $X \otimes I^1$ and is thick.

Let $Y = X \otimes I^1$. Then $\mathrm{Id}_Y \otimes \partial : Y \otimes S^1 \longrightarrow Y[1]$ can be included in the exact triangle

$$Y \longrightarrow Y \otimes I^1 \longrightarrow Y \otimes S^1 \longrightarrow Y[1].$$

The morphism $\mu : I^1 \otimes I^1 \longrightarrow I^1$ gives us a splitting of the morphism $Y \longrightarrow Y \otimes I^1$ and, therefore $\mathrm{Id}_Y \otimes \partial = 0$.

Let us show now that our class of objects is indeed thick ([5, Appendix]). Let $X \longrightarrow Y \longrightarrow Z \xrightarrow{f} X[1]$ be an exact triangle such that for some m and n one has $\mathrm{Id}_X \otimes \partial^{\otimes m} = 0$ and $\mathrm{Id}_Y \otimes \partial^{\otimes n} = 0$ (we can restrict ourself to this case because if $\mathrm{Id}_U \otimes \partial^{\otimes n} = 0$ for some n then the same holds for any U[k]). Let us show that $\mathrm{Id}_Z \otimes \partial^{\otimes (m+n)} = 0$. Consider the diagram:

$$\begin{tikzcd}
Y \otimes S^n \arrow[r] \arrow[d] & Z \otimes S^n \arrow[r] \arrow[d] & x[1] \otimes S^n \arrow[dl] \\
Y[n] \arrow[r] & Z[n] &
\end{tikzcd}$$

The dotted arrow exists because the upper string is a part of an exact triangle and $Y \otimes S^n \longrightarrow Y[n]$ is equal to zero. Denote it by α. One obviously has

$$\mathrm{Id}_Z \otimes \partial^{\otimes (m+n)} = \big(\mathrm{Id}_Z \otimes \partial^{\otimes n}\big) \otimes \partial^{\otimes m} = \big(\alpha \otimes \partial^{\otimes m}\big)\,(f \otimes \mathrm{Id}_{S^m})$$

and

$$\alpha \otimes \partial^{\otimes m} = \alpha[m]\big(\mathrm{Id}_{X[1]\otimes S^n} \otimes \partial^m\big) = 0.$$

The proof of the second axiom of thick classes is similar to this one.

Corollary 2.2.7. *Let X, Y be a pair of objects of the $D(T)$ such that for any n and m one has*

$$\mathrm{Hom}_{D(T)}\big(X \otimes I^n, Y[m]\big) = 0.$$

Then

$$\mathrm{Hom}_{H_0(T)}\big(X, Y[m]\big) = \mathrm{Hom}_{D(T)}\big(X, Y[m]\big).$$

Proof. We have to show that the morphisms

$$\mathrm{Hom}_{D(T)}\left(X, Y[m]\right) \longrightarrow \mathrm{Hom}\left(X \otimes S^n, Y[m+n]\right)$$

are isomorphisms for all n. We will prove it by the induction on n. For $n = 0$ our statement is trivial. To make the inductive step consider the exact triangle

$$X \otimes S^{n-1} \longrightarrow X \otimes I^1 \otimes S^{n-1} \longrightarrow X \otimes S^n \longrightarrow X \otimes S^{n-1}[1].$$

It is sufficient to show that $\mathrm{Hom}_{D(T)}(X \otimes I^1 \otimes S^{n-1}, Y[m]) = 0$. Obviously, if X satisfies the conditions of our proposition so does $X \otimes I^1$. Therefore, by induction we have

$$\mathrm{Hom}_{D(T)}\left(X \otimes I^1 \otimes S^{n-1}, Y[m]\right) = \mathrm{Hom}_{D(T)}\left(X \otimes I^1, Y[m-n]\right) = 0.$$

Definition 2.2.8. An object $Y \in \mathrm{ob}(D(T))$ is called strictly homotopy invariant if for any $X \in \mathrm{ob}(D(T))$ one has $\mathrm{Hom}(X \otimes I^1, Y) = 0$.

Proposition 2.2.9. *Let $Y \in \mathrm{ob}(D(T))$ be a strictly homotopy invariant object. Then for any X one has*

$$\mathrm{Hom}_{H(T)}(X, Y) = \mathrm{Hom}_{D(T)}(X, Y).$$

Proof. Obvious.

An object X of $D(T)$ is called an object of finite dimension if there exists N such that for any $F \in \mathrm{ob}(\mathrm{Ab}(T))$ and any $n > N$ one has

$$\mathrm{Hom}_{D(T)}\left(X, F[n]\right) = 0.$$

Proposition 2.2.10. *Let (T, I^+) be a site with interval and X be an object of $D(T)$ such that the objects $\mathbf{Z}, X \otimes I^n$ are of finite dimension. Then for any $Z \in \mathrm{ob}(D(T))$ one has*

$$\mathrm{Hom}_{H(T)}(X, Z) = \mathrm{Hom}_{H_0(T)}(X, Z).$$

Proof. It follows easily from our definitions and Proposition 2.2.6.

Let (T_1, I_1^+), (T_2, I_2^+) be a pair of sites with interval. A morphism $F : (T_1, I_1^+) \longrightarrow (T_2, I_2^+)$ is by definition a morphism of sites $F : T_1 \longrightarrow T_2$ such that $F^{-1}(I_2^+)$ is isomorphic to I_1^+. For example, if T_1, T_2 have the same underlying categories and the topology of T_1 is stronger than that of T_2 and $I_1^+ \cong I_2^+$, then an identity functor is a morphism of sites with interval.

Proposition 2.2.11. *Let $F : (T_1, I_1^+) \longrightarrow (T_2, I_2^+)$ be a morphism of sites with interval. Then it induces an exact tensor functor*

$$H(F) : H(T_2) \longrightarrow H(T_1).$$

Proof. There is a functor $F^* : D(T_2) \longrightarrow D(T_1)$ which is induced by the functor of inverse image of sheaves. One can easily see, using the universal property of localization, that the composition

$$D(T_2) \longrightarrow D(T_1) \longrightarrow H(T_1)$$

factors through a functor $H(F) : H(T_2) \longrightarrow H(T_1)$ which obviously satisfies all the properties we need.

There is an obvious analogue of this proposition for the categories $H_0(T_1)$, $H_0(T_2)$. We denote the corresponding functor by $H_0(F)$.

3. The h-topology on the category of schemes

3.1 The h-topology

Definition 3.1.1. A morphism of schemes $p : X \longrightarrow Y$ is called *a topological epimorphism* if the underlying topological space of Y is a quotient space of the underlying topological space of X, i.e. if p is surjective and a subset A in Y is open if and only if the subset $p^{-1}(A)$ is open in X.

A topological epimorphism $p : X \longrightarrow Y$ is called *a universal topological epimorphism* if for any morphism $f : Z \longrightarrow Y$ the projection $Z \times_Y X \longrightarrow Y$ is a topological epimorphism.

One can easily see that any open or closed surjective morphism is a topological epimorphism in this sense and any surjective proper or flat morphism as well as any composition of such morphisms is a universal topological epimorphism.

Definition 3.1.2. The h-topology on the category of schemes is the Grothendieck topology associated with the pretopology whose coverings are of the form $\{p_i : U_i \longrightarrow X\}$, where $\{p_i\}$ is a finite family of morphisms of finite type such that the morphism $\coprod p_i : \coprod U_i \longrightarrow X$ is a universal topological epimorphism.

We will also use *qfh*-topology, which corresponds to coverings of the same type such that the morphisms p_i are quasi-finite.

Examples.

1. Any flat covering is an h-covering. Moreover, since any flat surjective morphism of finite type admits a section over a quasi-finite surjective flat morphism, even the *qfh*-topology is stronger than the flat one.

2. Any surjective proper morphism of finite type is an h-covering.
3. Let X be a scheme and G a finite group acting on X. Suppose that there exist a categorical quotient X/G (see [7, ex.5 n.1]). Then the canonical projection $p : X \longrightarrow X/G$ is a qfh-covering.
4. Consider the blowup $p : X_x \longrightarrow X$ of a surface X with center in a closed point $x \in X$ and let $U = X_x - \{x_0\}$ where x_0 is a closed point over x. Then the natural morphism $p_U : U \longrightarrow X$ *is not* an h-covering. In fact, let us consider a curve C in X such that $p^{-1}(C) = p^{-1}(\{x\}) \cup \tilde{C}$ and $\tilde{C} \cap p^{-1}(\{x\}) = \{x_0\}$. Obviously, $p_U^{-1}(C - \{x\})$ is closed in U but $C - \{x\}$ is not. Therefore p_U is not a topological epimorphism.

We are going to define now a special class of h-coverings which are called coverings of normal form. The main result of this section is the theorem which says that any h-covering of an excellent noetherian scheme admits a refinement which is an h-covering of normal form.

Proposition 3.1.3. *Let $\{U_i \xrightarrow{p_i} X\}$ be an h-covering of a noetherian scheme X. Denote by $\coprod V_j$ the disjoint union of irreducible components of $\coprod U_i$ such that for any j there exists an irreducible component of X_i of X which is dominated by V_j. Then the morphism $q : \coprod V_j \longrightarrow X$ is surjective.*

Proof. Suppose first that X is irreducible. Let $x \in X$ be a point of X. We want to prove that x lies in the image of q. Considering the base change along the natural morphism $\mathrm{Spec}(\mathcal{O}_x) \longrightarrow X$ we may suppose that X is the spectrum of a local ring and x is the closed point of X.

Denote by Z the closure of the image of those irreducible components of $\coprod U_i$ which are not dominant over X. Since this image is a constructible set which does not contain the generic point of X one has $Z \neq X$. It follows from [9, 10.5.5 and 10.5.3] that the set of points of dimension one is dense in X. Therefore there exists a point $y \in X$ of dimension one which does not belong to Z. If x does not lie in the image of q then the preimage $q^{-1}(y)$ is closed which implies that $p_i^{-1}(y)$ are closed as well, giving us a contradiction with the condition that $\{p_i\}$ is an h-covering since y is not closed in $X = \mathrm{Spec}(\mathcal{O}_x)$.

Suppose now that X is an arbitrary scheme and let $X_{\mathrm{red}} = \cup X_k$ be the decomposition of the maximal reduced subscheme of X into the union of its irreducible components. Consider the natural morphisms $X_k \longrightarrow X$ and let $\{U_i \times_X X_i \longrightarrow X_i\}$ be the preimages of our h-covering. Then the morphisms $\coprod V_{jk} \longrightarrow X_k$, where V_{jk} are the irreducible components of $\coprod U_i \times_X X_k$ which are dominant over X_k are surjective, implying that $\coprod V_j \longrightarrow X$ is surjective since $\coprod V_j = \coprod \coprod V_{jk}$.

Remark. This proposition leads to the following generalization of example 4 above. Let Z be a closed subscheme of an integral scheme X and $X_Z \longrightarrow X$ the blowup with the center in Z. Suppose that, for an open subscheme $U \subset X_Z$, the composition $U \longrightarrow X_Z \longrightarrow X$ is an h-covering. Then $U = X_Z$. To show this, let us consider the base change along the projection $X_Z \longrightarrow X$. Then $U \times_X X_Z$ is an open subscheme in $X_Z \times_X X_Z$. This last scheme is a union of the diagonal Δ and a component, which is not dominant over X_Z. According to our proposition $(U \times_X X_Z) \cap \Delta \longrightarrow X_Z$ is a surjection, which implies that $U = X_Z$.

Proposition 3.1.4. *Let $\{p_i : U_i \longrightarrow X\}$ be a finite family of quasi-finite morphisms over a normal connected noetherian scheme X. Then $\{p_i\}$ is a qfh-covering if and only if the subfamily $\{q_j\}$ consisting of those p_i which are dominant over X is surjective. In that case $\{q_j\}$ is also a qfh-covering of X.*

Proof. The "only if" part follows immediately from the previous proposition.

To prove the "if" part it is sufficient to notice that in the case of a normal connected noetherian scheme X a dominant quasi-finite morphism is universally open [7, p. 24] and therefore a surjective family of such morphisms is an h-covering.

Remark. The statement of the proposition above is false for schemes which are not normal. To show this, consider a surface X over an algebraically closed field and let $x, y \in X$ be two different closed points of X. Let Y be the scheme obtained from X by gluing the point x, y together. Let $U = X - \{x\}$. The natural morphism $p : U \longrightarrow Y$ is dominant and surjective but it is not a *qfh*-covering. In fact, let us consider a curve $C \subset X$ in X, which contains x and does not contain y. Then the subscheme $p^{-1}(C - \{x\})$ is closed in U, while $C - \{x\}$ is not closed in Y.

Definition 3.1.5. A finite family of morphisms $\{U_i \xrightarrow{p_i} X\}$ is called an h-covering of normal form if the morphisms p_i admit a factorization of the form $p_i = s \circ f \circ \mathrm{in}_i$, where $\{\mathrm{in}_i : U_i \longrightarrow \bar{U}\}$ is an open covering, $f : \bar{U} \longrightarrow X_Z$ is a finite surjective morphism and $s : X_Z \longrightarrow X$ is the blowup of a closed subscheme in X.

Beginning at this point, we restrict our considerations to excellent noetherian schemes (see [9, 7.8]).

Let us recall several properties of excellent schemes, which we will use below without additional references. Any scheme of the form $X = \mathrm{Spec}(A)$ where A is a field or a Dedekind domain with the field of fractions of characteristic zero is excellent. If a scheme X is excellent and $Y \longrightarrow X$ is a morphism of finite type, then Y is excellent. Any localization of an excellent scheme is excellent. For any excellent integral scheme X and any finite extension L of the field of functions on X, the normalization of X in L is finite over X.

Lemma 3.1.6. *Let $f : Y \longrightarrow X$ be a finite morphism such that Y is an irreducible scheme. Then the underlying topological space of the diagonal $Y \subset Y \times_X Y$ is an irreducible component of $Y \times_X Y$.*

Proof. Obvious.

Lemma 3.1.7. *Let X be an excellent normal connected noetherian scheme and let L be a finite purely inseparable extension of the field of functions $K(X)$ of X.*

Then the normalization $f : Y \longrightarrow X$ *of* X *in* L *is a universal homeomorphism* (*see Definition* 3.2.4).

Proof. Since X is excellent, the morphism f is finite and surjective, which implies that it is universally surjective. It is sufficient to show that f is universally injective. According to [10, 3.7.1] it is equivalent to the surjectivity of the diagonal morphism $\Delta : Y \longrightarrow Y \times_X Y$. Since X is normal the morphism f is universally open ([7, p. 24]). In particular, considering the base change along f we see that the projection $Y \times_X Y \longrightarrow Y$ is an open morphism. It implies that each irreducible component of $Y \times_X Y$ is dominant over Y. According to the previous lemma our statment would follow if we prove that the general fiber of the projection $Y \times_X Y \longrightarrow Y$ is connected. This fiber is the scheme $Z = \mathrm{Spec}(L) \times_{\mathrm{Spec}(K(X))} \mathrm{Spec}(L)$ and, since our extension is purely inseparable, one has $Z_{\mathrm{red}} = \mathrm{Spec}(L)$ which finishes the proof.

Let Z be a closed subscheme of a scheme X. We denote by $p_Z : X_Z \longrightarrow X$ the blowup of X with center in Z. For a scheme $Y \longrightarrow X$ over X, denote by $\tilde{p}_Z(Y)$ the closure in $Y \times_X X_Z$ of the open subscheme $Y \times_X X_Z - \mathrm{pr}_2^{-1}(p_Z^{-1}(Z))$. The scheme $\tilde{p}_Z(Y)$ over X_Z is called the strict transform of Y with respect to p_Z.

Theorem 3.1.8 platification by blowup. *Let* $f : Y \longrightarrow X$ *be a morphism of finite type, which is flat over an open subset* $U \subset X$. *Then there exists a closed subscheme* Z *disjoint with* U *such that the strict transform* $\tilde{p}_Z(Y)$ *is flat over* X_Z.

Proof. See [12, 5.2].

Theorem 3.1.9. *Let* $\{U_i \xrightarrow{p_i} X\}$ *be an h-covering of an excellent reduced noetherian scheme* X. *Then there exists an h-covering of normal form, which is a refinement of* $\{p_i\}$.

Proof. Suppose first, that X is a normal connected scheme and all the morphisms p_i are dominant and quasi-finite. Considering the normalizations of the schemes U_i we may suppose that U_i are normal and connected as well. Let $\bar{p}_i : \bar{U}_i \longrightarrow X$ be the finite morphisms such that $\bar{U}_i$ are normal and connected and there exist factorizations of the form $U_i \xrightarrow{\mathrm{in}_i} \bar{U}_i \xrightarrow{\bar{p}_i} X$, where in_i are open immersions ([11, 1.1.8]).

There exists a connected normal scheme $\bar{V}$ and a finite surjective morphism $\bar{q} : \bar{V} \longrightarrow X$ such that it can be factorized through all the morphisms $\bar{p}_i$ and there exists a factorization of $\bar{q}$ of the form $\bar{V} \xrightarrow{\bar{g}} \bar{W} \xrightarrow{\bar{r}} X$ where $\bar{W}$ is a connected normal scheme and $\bar{r}$,$\bar{g}$ correspond to purely inseparable and Galois extensions of the fields of functions respectively. Let $V_i = V \times_{\bar{U}_i} U$. The compositions $\{q_i : V_i \longrightarrow \bar{V} \longrightarrow X\}$ define an h-covering which is a refinement of the initial one. Let G be the Galois group of the extension of the fields which corresponds to the morphism $\bar{g}$. The group G acts on $\bar{V}$. Consider the open subsets $\sigma(V_i)$ for $\sigma \in G$. Since $\cup q_i(V_i) = X$ and the morphism $\bar{r}$ defines a homeomorphism of the underlying topological spaces (Lemma 3.1.7), we have $\cup\sigma(V_i) = V$. The covering $\{\sigma(V_i) \longrightarrow X\}$ is of normal form and we claim that it is a refinement of the covering

$\{V_i \longrightarrow X\}$. To see it it is sufficient to define a morphism from one to another as the family of morphisms $\sigma^{-1} : \sigma(V_i) \longrightarrow V_i$.

Let now X be a noetherian exellent reduced scheme and p_i be flat quasi-finite morphisms. Consider the normalization $X_{\mathrm{norm}} \longrightarrow X$ of X. It is a finite morphism and X_{norm} is a disjoint union of connected normal schemes X_j. Applying the above construction to the covering $U_i \times_X X_j \longrightarrow X_j$ we obtain in this case the refinement we need.

Consider now the case of the general h-covering $\{p_i : U_i \longrightarrow X\}$ of a noetherian exellent reduced scheme X. It follows from [9, 11.1.1] that there exists a dense open subscheme X_0 of X such that all the morphisms p_i are flat over X_0. Let Z be a closed subscheme disjoint with X_0 such that the morphism $f : \tilde{p}_Z(\coprod U_i) \longrightarrow X_Z$ is flat (Theorem 3.1.8). Since $X_Z \times_X (\coprod U_i) \longrightarrow X_Z$ is an h-covering and the closure of the complement $X_Z \times_X (\coprod U_i) - \tilde{p}_Z(\coprod U_i)$ lies over $p_Z^{-1}(Z)$ and therefore is not dominant over any irreducible component of X_Z, Proposition 3.1.3 implies that f is a surjection. There exists then a quasi-finite flat surjective morphism $U' \longrightarrow X_Z$ which can be factorized through f. The normal refinement for such type of coverings was constructed above.

3.2 Representable sheaves

Denote by Sch/S the category of separated schemes of finite type over a noetherian exellent scheme S. All through this section *a scheme* means an object of Sch/S and all morphisms of schemes are morphisms over S.

Let L be a functor $Sch/S \longrightarrow Shv_h(S)$ which takes a scheme X/S to the corresponding representable sheaf, i.e $L(X)$ is the h-sheaf associated with the presheaf $Y \longrightarrow \mathrm{Mor}_S(X,Y)$. We will also use the notation L_{qfh} for the corresponding functor with respect to the *qfh*-topology.

Since both the h-topology and the *qfh*-topology are not subcanonical, the functors L and L_{qfh} are not full embeddings. The question we are interested in in this section is what can be said about the set of morphisms $L(X) \longrightarrow L(Y)$? Since this set coincides with the set of sections of the sheaf $L(Y)$ over X to answer our question, we have to describe the sheaf $L(Y)$ associated with the presheaf representable by Y.

Let us recall first the general construction of the sheaf associated with a presheaf [11, 2.2], [2]. Let P be a presheaf. For any scheme X define an equivalence relation on the set $P(X)$, setting sections $a, b \in P(X)$ to be equivalent if there exists a covering $\{p_i : U_i \longrightarrow X\}$ of X such that for any i one has $p_i^*(a) = p_i^*(b)$. Denote by P' the presheaf such that $P'(X)$ is the set of equivalence classes of elements of $P(X)$.

For any covering $\mathcal{U} = \{p_i : U_i \longrightarrow X\}$ denote by $H^0(\mathcal{U}, P')$ the equalizer of the maps $\coprod P'(U_i) \rightrightarrows \coprod P'(U_i \times_X U_j)$ which are induced by the projections. For any refinement $\mathcal{U}'$ of $\mathcal{U}$ there is defined an obvious map $H^0(\mathcal{U}, P') \longrightarrow H^0(\mathcal{U}', P')$.

We set

$$aP(X) = \varinjlim H^0(\mathcal{U}, P').$$

It can be shown that aP is indeed a sheaf associated with P and the natural morphism of presheaves $P' \longrightarrow aP$ is injective.

We are going to apply this construction to the representable presheaves.

Lemma 3.2.1. *Let X be a scheme and X_{red} its maximal reduced subscheme. Then the natural morphism $L_{qfh}(i) : L_{qfh}(X_{\mathrm{red}}) \longrightarrow L_{qfh}(X)$ is an isomorphism.*

Proof. Since the morphism $i : X_{\mathrm{red}} \longrightarrow X$ is a monomorphism in the category of schemes and the functor L is left exact, so is $L(i)$. From the other hand, i is a *qfh*-covering which implies that $L(i)$ is an epimorphism. Therefore $L(i)$ is an isomorphism.

Lemma 3.2.2. *Let X be a reduced scheme and $U \longrightarrow X$ an h-covering. Then it is epimorphism in the category of schemes. In particular for any reduced X and any Y the natural map $\mathrm{Mor}_S(X, Y) \longrightarrow \mathrm{Mor}(L(X), L(Y))$ is injective.*

Proof. It follows immediately from the fact that h-coverings are surjective on the underlying topological spaces of schemes.

For a scheme X denote by $L_0(X)$ the presheaf obtained on the first step of the construction of the sheaf $L(X)$ which was described above. Two previous lemmas shows that for any scheme Y one has $L_0(X)(Y) = \mathrm{Mor}_S(Y_{\mathrm{red}}, X)$.

Lemma 3.2.3. *Let $X = \mathrm{Spec}(K)$, where K is a field. Then for any scheme Y one has $\mathrm{Mor}(L(X), L(Y)) = \mathrm{Mor}(L_{qfh}(X), L_{qfh}(Y)) = Y(K')$, where K' is a maximal purely inseparable extension of the field K.*

Proof. It follows immediately from the previous lemma and the remark that the extension L of K is purely inseparable if and only if the diagonal $\Delta : \mathrm{Spec}(L) \longrightarrow \mathrm{Spec}\,(L) \times_{\mathrm{Spec}\,(K)} \mathrm{Spec}\,(L)$ induces an isomorphism of $\mathrm{Spec}\,(L)$ with $(\mathrm{Spec}\,(L) \times_{\mathrm{Spec}\,(K)} \mathrm{Spec}\,(L))_{\mathrm{red}}$.

Definition 3.2.4. Let $f : X \longrightarrow Y$ be a morphism of finite type. It is called radicial (resp. universal homeomorphism) if for any scheme $Z \longrightarrow Y$ over Y the morphism $X \times_Y Z \longrightarrow Z$ induces an immersion (resp. homeomorphism) of the underlying topological spaces.

Proposition 3.2.5. *Let $f : X \longrightarrow Y$ be a morphism of finite type. Then one has*

1. *The morphism $L(f)$ (resp. $L_{qfh}(f)$) is a monomorphism if and only is f is radicial.*
2. *The morphism $L(f)$ is an epimorphism if and only if f is a universal topological epimorphism.*
3. *The morphism $L(f)$ (resp. $L_{qfh}(f)$) is an isomorphism if and only if f is a universal homeomorphism.*

Proof. It follows from Lemma 3.2.1 that we may suppose X, Y to be reduced schemes.

1. The "if" part follows from the trivial observation that any radicial morphism with the reduced source is a monomorphism in the category of schemes and left exactness of the functor L. The "only if" part follows from Proposition 3.2.2 and the criterion that the morphism is radicial if and only if it induces monomorphisms on the sets of geometrical points (see [10]).

2. It is easy to show that a morphism of schemes $f : X \longrightarrow Y$ induces an epimorphism on the corresponding representable sheaves if and only if there exists a covering $U \longrightarrow Y$ which can be factorized through f. It implies the result we need, since if there exists a universal topological epimorphism which can be factorized through f, then f itself is a universal topological epimorphism.

3. Suppose that f is a universal homeomorphism. Then it is a *qfh*-covering, and, therefore, $L_{qfh}(f)$ is a surjection. On the other hand, any universal homeomorphism is a radicial morphism which implies, according to (1), that $L(f)$ is a monomorphism as well. Suppose now, that $L(f)$ is an isomorphism. Then by (1) and (2), f is a radicial universal topological epimorphism, which obviously implies that f is a universal homeomorphism.

Let X, Y be a pair of schemes and $f \in \mathrm{Mor}(L(X), L(Y))$. We say that an h-covering $\{p_i : U_i \longrightarrow X\}$ *realizes* f if there exist morphisms $f_i : U_i \longrightarrow Y$ such that $L(f_i) = f \circ L(p_i)$. It follows from Lemma 3.2.2 that in that case one has $f_i \circ \mathrm{pr}_1^{\mathrm{red}} = f_j \circ \mathrm{pr}_2^{\mathrm{red}}$, where $\mathrm{pr}_i^{\mathrm{red}}$ are the restrictions of the projections $U_i \times_X U_j \longrightarrow U_i$ and $U_i \times_X U_j \longrightarrow U_j$ to the maximal reduced subscheme $(U_i \times_X U_j)_{\mathrm{red}}$ of the scheme $U_i \times_X U_j$. Note that if $\{V_{ij} \longrightarrow U_i \longrightarrow X\}$ is a refinement of the h-covering $\{p_i : U_i \longrightarrow X\}$ and $\{V_{ij} \longrightarrow X\}$ realizes f, then the coverings $\{V_{ij} \longrightarrow U_i\}$ realizes $f \circ L(p_i)$.

Lemma 3.2.6. *Let X be a reduced scheme and $f \in \mathrm{Mor}(L(X), L(Y))$ be such a morphism that it can be realized on the open covering of X. Then there exists a morphism $\tilde{f} \in \mathrm{Mor}_S(X, Y)$ such that $L(\tilde{f}) = f$.*

Proof. Let $X = \cup U_i$ be the open covering in question and $f_i : U_i \to Y$ the morphisms which realize f. Since for open subschemes U_i, U_j of a reduced scheme X, one has

$$U_i \times_X U_j = U_i \cap U_j = (U_i \times_X U_j)_{\mathrm{red}}$$

and open coverings are effective epimorphisms in the category of schemes, there exists a morphism $\tilde{f} : X \to Y$ whose restriction to $\coprod U_i$ equal $\coprod f_i$ and therefore $L(\tilde{f}) = f$.

Lemma 3.2.7. *Let $p : X' \longrightarrow X$ be an h-covering such that $p_*(\mathcal{O}_{X'}) = \mathcal{O}_X$. Then for any $f \in \mathrm{Mor}(L(X), L(Y))$ which can be realized by p, there exists a morphism $\tilde{f} \in \mathrm{Mor}_S(X, Y)$ such that $L(\tilde{f}) = f$.*

Proof. Denote by $f' : X' \longrightarrow Y$ the morphism such that $L(f') = f \circ L(p)$. Then, since p is a topological epimorphism, there exists a *continuous map* $\tilde{f}$ from the underlying topological space of X to the underlying topological space of Y such that as a continuous map f' equals $\tilde{f} \circ p$. Since $p_*(\mathcal{O}_{X'}) = \mathcal{O}_X$, the morphism of sheaves $\mathcal{O}_Y \longrightarrow f'_*(\mathcal{O}_{X'})$ defines a morphism of sheaves $\mathcal{O}_Y \longrightarrow f'_*(\mathcal{O}_{X'}) = \tilde{f}_*(p_*(\mathcal{O}_{X'})) = \tilde{f}_*(\mathcal{O}_X)$, and, therefore $\tilde{f}$ corresponds to a morphism of schemes, which obviously satisfies the condition we need.

Proposition 3.2.8. *Let $\tilde{f} \in \mathrm{Mor}(L(X), L(Y))$ be a morphism of representable h-sheaves. Then there exists a finite surjective morphism $p : X' \longrightarrow X$ such that $\tilde{f} \circ L(p) = L(f')$ for a morphism $f' : X' \longrightarrow Y$.*

Proof. Let $\{p_i : U_i \longrightarrow X\}$ be an h-covering which realizes $\tilde{f}$ and let $f_i : U_i \longrightarrow Y$ be the corresponding morphisms. According to Theorem 3.1.9 we may suppose that our covering is a covering of normal form. Let $U_i \xrightarrow{\mathrm{in}_i} \bar{U} \xrightarrow{s} X_Z \xrightarrow{r} X$ be the normal decomposition of p_i. Consider the morphism $r \circ s$. Since it is proper there exists the Stein decomposition of the form $r \circ s = r' \circ s'$ where s' is a proper surjective morphism $\bar{U} \longrightarrow X'$ such that $s'_*(\mathcal{O}_{\bar{U}}) = \mathcal{O}_{X'}$ and r' is a finite surjective morphism. Our proposition follows now from Lemmas 3.2.7 and 3.2.6.

Theorem 3.2.9. *The category $L(Sch/S)$ (resp. $L_{qhf}(Sch/S)$) of representable h-sheaves (resp. qhf-sheaves) is a localization of the category Sch/S of schemes over S with respect to the class of universal homeomorphisms.*

Proof. It follows from Proposition 3.2.5(3) that it is sufficient to show that for any schemes X, Y and a morphism $f \in \mathrm{Mor}(L(X), L(Y))$, there exists a universal homeomorphism $X_0 \longrightarrow X$ which realizes f. Let $p : X' \longrightarrow X$ be a finite morphism such that there exists a morphism $f' : X' \longrightarrow Y$ satisfying $L(f') = f \circ L(p)$. Let us define a sheaf $\mathcal{R}$ of finite $\mathcal{O}_X$-algebras over X as follows. Let U be an open subset of X. Then $\mathcal{R}(U)$ is a subalgebra in $\mathcal{O}_{X'}({f'}^{-1}(U))$ which consists of functions $g \in \mathcal{O}_{X'}({f'}^{-1}(U))$ such that there exists an element $\tilde{g} \in \mathrm{Mor}(L(X), L(\mathbf{A}^1))$ satisfying $L(g) = \tilde{g} \circ L(p)$. One can easily see that the morphism $\mathrm{Spec}(\mathcal{R}) \longrightarrow X$ is a finite surjective morphism, which realizes f. To finish the proof it is sufficient to show that it is a universal homeomorphism. It is almost obvious.

Proposition 3.2.10. *Let S be a scheme of characteristic zero. Then there exists a functor $R : L(Sch/S) \longrightarrow Sch/S$ left adjoint to L. For a scheme X, the scheme $R(L(X))$ is a semi-normalization of X (see [15]).*

In particular for any seminormal scheme X and any scheme Y one has

$$\mathrm{Mor}(L(X), L(Y)) = \mathrm{Mor}_S(X, Y).$$

Proof. Let X be a normal scheme of characteristic zero. Suppose that $p : Y \longrightarrow X$ is a universal homeomorphism. Considering the base change along the immersion of the generic point of X, we conclude that p is birational. On the other hand p is universally closed and quasi-finite which implies that it is finite. Then p is an isomorphism by [8, 4.4.9].

Therefore, for any scheme X of characteristic zero and any $f \in \mathrm{Mor}(L(X), L(Y))$ there exists a finite morphism $p : X' \longrightarrow X$ which realises f such that p is a universal homeomorphism and the normalization of X can be factorized through p. It follows easily from the results of [15] that the seminormalization of X is exactly the universal morphism satisfying this property, which finishes the proof.

The situation in positive characteristic is a bit more complicated. Roughly speaking, there exists an analog of the functor R in that case. Namely $R(L(X))$ for an integral scheme X should be a seminormalization of X in the maximal purely inseparable extension of its field of functions. The problem is that this scheme is not in general a noetherian scheme, and, therefore we can not construct R in the category of noetherian schemes.

The following proposition provides us all the information we really need about the sets $\mathrm{Mor}(L(X), L(Y))$ in the general case.

Proposition 3.2.11. *Let X be a normal connected scheme. Then for any scheme Y one has*

$$\mathrm{Mor}(L(X), L(Y)) = \varinjlim_{L} \mathrm{Mor}_S(X_L, Y)$$

where the limit is defined over the category of purely inseparable extensions of the field of functions of X and X_L denotes the normalization of X in the extension L.

Proof. It follows almost automatically from the above results.

Proposition 3.2.12. *Let Y be a scheme of finite type over S. Then the natural morphism*

$$\mathrm{Mor}_S(X, Y) \longrightarrow \mathrm{Mor}(L(X), L(Y))$$

is a bijection for any X if and only if Y is étale over S.

Proof. It follows from the valuative criterion for étale morphisms (see [9, ex.17])

3.3 Sheaves $\mathbf{Z}(X)$ in h-topology

Let X be a scheme over S. We denote by $\mathbf{Z}(X)$ (resp. $\mathbf{Z}_{qfh}(X)$) the h-sheaf (resp. the *qfh*-sheaf) of abelian groups freely generated by the sheaf of sets $L(X)$. We will also use notations $\mathbf{N}(X)$, $\mathbf{N}_{qfh}(X)$ for the corresponding freely generated sheaves of abelian monoids.

For an abelian monoid A, we denote by A^+ the abelian group associated with A in the obvious way.

Proposition 3.3.1. *For any schemes X, Y over S and a section $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a finite surjective morphism $\bar{p} : \bar{U} \longrightarrow Y$ such that $\bar{p}^*(a) = \sum a_j^+ - \sum a_k^-$, where a_j^+, a_k^- correspond to morphisms $\bar{U} \longrightarrow X$.*

Proof. According to the construction of the associated sheaf and Theorem 3.1.9 above, for any $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a covering

$$\{U_i \xrightarrow{\mathrm{in}_i} \bar{U}\} \xrightarrow{\bar{p}} Y$$

of normal form such that

$$\mathrm{in}_i^* \, \bar{p}^*(a) = \sum a_{ij}^+ - \sum a_{ik}^-$$

where $a_{ij}^+, a_{ik}^- \in \mathrm{Mor}_S(U_i, X)$ are elements such that $a_{ij}^+ \neq a_{ik}^-$ for any j, k.

For a pair i_1, i_2 of indices we have

$$\mathrm{pr}_1^* \left(\sum a_{i_1 j}^+ - \sum a_{i_1 k}^-\right) = \mathrm{pr}_2^* \left(\sum a_{i_2 j}^+ - \sum a_{i_2 k}^-\right)$$

in $\mathbf{Z}_{qfh}(X)(U_{i_1} \times_U U_{i_2})$. Since $U_{i_1} \times_U U_{i_2} = U_{i_1} \cap U_{i_2}$ is reduced it implies that this equality also holds on the level of formal sums of morphisms $U_i \longrightarrow X$. It means that with respect to some order on the set of indices one has

$$\mathrm{pr}_1^* \, a_{i_1 j}^+ = \mathrm{pr}_2^* \, a_{i_2 j}^+$$
$$\mathrm{pr}_1^* \, a_{i_1 k}^- = \mathrm{pr}_2^* \, a_{i_2 k}^-.$$

There exists then a family of morphisms $a_{ij}^+, a_{ik}^- \in \mathrm{Mor}_S(U_{i_1} \cup U_{i_2}, X)$ such that

$$a_{ij|U_{i_1}}^+ = a_{i_1 j}^+$$
$$a_{ij|U_{i_2}}^+ = a_{i_2 j}^+$$
$$a_{ik|U_{i_1}}^- = a_{i_1 k}^-$$
$$a_{ik|U_{i_2}}^- = a_{i_2 k}^-.$$

The statment of our proposition follows now by the induction by the number of open subschemes U_i of $\bar{U}$.

Proposition 3.3.2. *Let X be a normal connected scheme and let $p : Y \longrightarrow X$ be the normalization of X in a Galois extension of its field of functions. Then for any qfh-sheaf F of abelian monoids the image of $p^* : F(X) \longrightarrow F(Y)$ coincides with the submonoid $F(Y)^G$ of Galois invariant elements in $F(Y)$.*

Proof. Obviously $\mathrm{Im}(p^*)$ lies in $F(Y)^G$. Let $a \in F(Y)^G$ be a Galois invariant element of $F(Y)$. Consider the scheme $Y \times_X Y$. It is a union of irreducible components of the form

$$Y \times_X Y = \bigcup_{g \in G} Y_g$$

and Y_g can be identified with Y in such a way that the restriction of the first projection $Y \times_X Y \longrightarrow Y$ becomes an identity and the restriction of the second one is the isomorphism $Y \longrightarrow Y$ induced by $g \in G$. To prove, that $a \in \mathrm{Im}(p^*)$ it is sufficient to show that $\mathrm{pr}_1^*(a) = \mathrm{pr}_2^*(a)$ in $F(Y \times_X Y)$. Since the decomposition of $Y \times_X Y$ in the union of its irreducible components is a *qfh*-covering, it is sufficient to show that for any $g \in G$ one has $\mathrm{pr}_1^*(a)_{|Y_g} = \mathrm{pr}_2^*(a)_{|Y_g}$, which means exactly that a is a Galois invariant.

Theorem 3.3.3. *Let X be a scheme and Y a normal scheme. Then one has*

$$\mathbf{Z}_{qfh}(X)(Y) = \mathbf{N}_{qfh}(X)(Y)^+.$$

Proof. Denote by F the presheaf of the form

$$Y \longrightarrow \mathbf{N}_{qfh}(X)(Y)^+.$$

Obviously the *qfh*-sheaf associated with F is isomorphic to $\mathbf{Z}_{qfh}(X)$. In particular, there is a natural map

$$\phi : \mathbf{N}_{qfh}(X)(Y)^+ \longrightarrow \mathbf{Z}_{qfh}(X)(Y)$$

and we have to prove that it is a bijection for normal Y. Let us show first that ϕ is an injection. It follows immediately from the construction of the associated sheaf, that it is sufficient to show that for any *qfh*-covering $\{U_i \longrightarrow Y\}$ the natural map

$$F(Y) \longrightarrow \bigoplus_i F(U_i)$$

is injective. Note that according to the axioms of sheaf the map

$$\mathbf{N}_{qfh}(X)(Y) \longrightarrow \bigoplus_i \mathbf{N}_{qfh}(X)(U_i)$$

is injective. Our statment now follows easily from the following lemma:

Lemma 3.3.4. *Let $a, b \in \mathbf{N}_{qfh}(X)(Y)$ be a pair of sections such that $a + x = b + x$ for some $x \in \mathbf{N}_{qfh}(X)(Y)$. Then $a = b$.*

Proof. There exists a covering $\{p_i : U_i \longrightarrow Y\}$ of Y such that

$$p_i^*(x) = \sum x_{ij}$$
$$p_i^*(a) = \sum a_{ik}$$
$$p_i^*(b) = \sum x_{il}$$

where $x_{ij}, a_{ik}, b_{il} \in L(X)(U_i)$.

Since $\mathbf{N}_{qfh}(X)$ is a sheaf, it is sufficient to show that $p_i^*(a) = p_i^*(b)$. An equality

$$\sum a_{ik} + \sum x_{ij} = \sum b_{il} + \sum x_{ij}$$

in $\mathbf{N}_{qfh}(X)(U_i)$ means that there is a covering $\{q_{im} : V_{im} \longrightarrow U_i\}$ such that for any m one has the equality

$$\sum q_{im}^* a_{ik} + \sum q_{im}^* x_{ij} = \sum q_{im}^* b_{il} + \sum q_{im}^* x_{ij}$$

which holds on the level of formal sums of sections of the sheaf $L(X)$ over V_{im}. It implies that

$$\sum q_{im}^* a_{ik} = \sum q_{im}^* b_{il}$$

and, therefore, $p_i^*(a) = p_i^*(b)$.

Let us prove now that in our case the map ϕ is also surjective. By Proposition 3.3.1, for any $a \in \mathbf{Z}_{qfh}(X)(Y)$ there exists a finite surjective morphism $\bar{p} : \bar{U} \longrightarrow Y$ such that $\bar{p}^*(a) = \sum a_j^+ - \sum a_k^-$. We may suppose that Y is connected. Since Y is normal we may suppose that $\bar{p}$ admits a decomposition of the form

$$\bar{U} \xrightarrow{\bar{p}_0} \bar{U}_0 \xrightarrow{\bar{p}_1} Y$$

where $\bar{p}_1$ is the normalization of Y in a purely inseparable extension of its field of functions and $\bar{p}_0$ is the normalization of $\bar{U}_0$ in a Galois extension of its field of functions with a Galois group G. For any $g \in G$ we have

$$\sum a_j^+ - \sum a_k^- = \sum g a_j^+ - \sum g a_k^-$$

in $\mathbf{Z}_{qfh}(X)(\bar{U})$ and, since $\bar{U}$ is reduced, the same equality holds on the level of the formal sums of morphisms $\bar{U} \longrightarrow X$. It implies that

$$\sum a_j^+ = \sum g a_j^+$$
$$\sum a_k^- = \sum g a_k^-$$

in $\mathbf{N}_{qfh}(X)(\bar{U})$ and, according to Proposition 3.3.2, there exist a pair ak, a^- of elements of $\mathbf{N}_{qfh}(X)(\bar{U}_0)$ such that $\bar{p}_0^*(a^+) = \sum a_j^+$ and $\bar{p}_0^*(a^-) = \sum a_k^-$. By Lemma 3.1.7 we have $\mathbf{N}_{qfh}(X)(\bar{U}_0) = \mathbf{N}_{qfh}(X)(Y)$ which finishes the proof.

Theorem 3.3.5. *Let X be an affine scheme over S. Then one has*

$$\mathbf{Z}(X) = \mathbf{Z}_{qfh}(X).$$

Proof. It is sufficient to show that for an affine scheme X the *qfh*-sheaf $\mathbf{Z}_{qfh}(X)$ is an h-sheaf. By Theorem 3.1.9 we have to prove only that $\mathbf{Z}_{qfh}(X)$ satisfies the axioms of sheaf for h-coverings of normal form. Let Y be a scheme over S and $\{U_i \longrightarrow \bar{U} \longrightarrow Y_Z \longrightarrow Y\}$ its covering of normal form. Let us show first that the map $u : \mathbf{Z}_{qfh}(X)(Y) \longrightarrow \oplus_i \mathbf{Z}_{qfh}(X)(U_i)$ is injective. Let $a \in \mathbf{Z}_{qfh}(X)(Y)$ be an element such that $u(a) = 0$. By Proposition 3.3.1 there exists a finite surjective morphism $\bar{q} : \bar{V} \longrightarrow Y$ such that $\bar{q}^*(a) = \sum a_j^+ - \sum a_k^-$ where a_j^+, a_k^- correspond to morphisms $\bar{V} \longrightarrow X$. Denote the morphism $Y_Z \longrightarrow Y$ by s. Since $\{U_i \longrightarrow \bar{U} \longrightarrow Y_Z\}$ is a *qfh*-covering an equality $u(a) = 0$ implies that $s^*(a) = 0$ in $\mathbf{Z}_{qfh}(X)(Y_Z)$. Consider the fiber product $Y_Z \times_Y \bar{V}$ and let $\mathrm{pr}_1, \mathrm{pr}_2$ be the projection to Y_Z and $\bar{V}$, respectively. We have $\mathrm{pr}_2^* \bar{q}^*(a) = \mathrm{pr}_1^* s^*(a) = 0$ in $Y_Z \times_Y \bar{V}$. It implies that with respect to a suitable order on the index set we have $a_j^+ \circ \mathrm{pr}_2 = a_j^- \circ \mathrm{pr}_2$ as morphisms $(Y_Z \times_Y \bar{V})_{\mathrm{red}} \longrightarrow X$. Therefore, since $(Y_Z \times_Y \bar{V})_{\mathrm{red}} \longrightarrow \bar{V}_{\mathrm{red}}$ is an epimorphism in the category of schemes, we have $a_j^+ = a_j^-$ on $\bar{V}_{\mathrm{red}}$ which implies, that $a = 0$.

Now let $a_i \in \mathbf{Z}_{qfh}(X)(U_i)$ be a family of sections such that $\mathrm{pr}_1^*(a_i) = \mathrm{pr}_2^*(a_j)$ in $\mathbf{Z}_{qfh}(X)(U_i \times_Y U_j)$ where $\mathrm{pr}_1 : U_i \times_Y U_j \longrightarrow U_i$, $\mathrm{pr}_2 : U_i \times_Y U_j \longrightarrow U_j$ are the projections. We have to prove that there exists an element $a \in \mathbf{Z}_{qfh}(X)(Y)$ such that its restriction on U_i is equal to a_i. Passing to a refinement we may suppose that $a_i = \sum a_{ij}^+ - \sum a_{ik}^-$ where a_{ij}^+, a_{ik}^- correspond to morphisms $U_i \longrightarrow X$. As in the proof of Proposition 3.3.1 we see that there exists a family of morphisms $a_j^+, a_k^- \in Mor_S(\bar{U}, X)$ such that

$$a^+_{j|U_i} = a^+_{ij}$$

$$a^-_{k|U_i} = a^-_{ik}.$$

Consider the Stein decomposition $\bar{U} \xrightarrow{f} W \xrightarrow{g} Y$ of the morphism $\bar{U} \longrightarrow Y_Z \longrightarrow Y$. Since $f_*\mathcal{O}_{\bar{U}} = \mathcal{O}_W$ and X is affine over S, one has $\mathrm{Mor}_S(\bar{U}, X) = \mathrm{Mor}_S(W, X)$. Therefore there exists a family of morphisms $b_j^+, b_k^- : W \longrightarrow X$ such that

$$f \circ b_j^+ = a_j^+$$

$$f \circ b_k^- = a_k^-.$$

Since

$$\mathrm{pr}_1^* \left(\sum a_j^+ - \sum a_k^-\right) = \mathrm{pr}_2^* \left(\sum a_j^+ - \sum a_k^-\right)$$

in $\mathbf{Z}_{qfh}(\bar{U} \times_Y \bar{U})$ and the natural morphism $\bar{U} \times_Y \bar{U} \longrightarrow W \times_Y W$ is an h-covering it follows from the injectivity result proved above that the same equality holds in $\mathbf{Z}_{qfh}(W \times_Y W)$. Since $W \longrightarrow Y$ is a finite surjective morphism and, therefore a *qfh*-covering, it implies that there exists an element $a \in \mathbf{Z}_{qfh}(X)(Y)$ such that $g^*(a) = \sum b_j^+ - \sum b_k^-$ in $\mathbf{Z}_{qfh}(X)(W)$, which finishes the proof.

Proposition 3.3.6. *Let X be a scheme over S such that there exist symmetric powers S^nX of X over S. Then the sheaves $\mathbf{N}(X)$, $\mathbf{N}_{qfh}(X)$ are representable by the (ind-) scheme $\coprod_{n\geq 0} S^nX$.*

Proof. It is obviously sufficient to prove our proposition in the case of *qfh*-topology. Note first that the sheaf representable by $\coprod_{n\geq 0} S^nX$ is a sheaf of abelian monoids. To prove the proposition, it is sufficient to show that it satisfies the universal property of $\mathbf{N}_{qfh}(X)$. It means that for any *qfh*-sheaf of abelian monoids G and any section $a \in G(X)$ of G over X, there should exist a unique element $f \in \mathrm{Hom}(L(\coprod_{n\geq 0} S^nX), G) = G(\coprod_{n\geq 0} S^nX)$ which is a homomorphism of sheaves of abelian monoids and whose restriction on $X = S^1X$ is equal to a.

Consider the natural morphism $q : X^n \longrightarrow S^nX$ and let $y_n = \sum pr_i^*(a) \in G(X^n)$. This element is obviously invariant with respect to the action of the symmetric group S_n. Exactly in the same way as in the proof of Proposition 3.3.2 one can show that there exists an element $f_n \in G(S^nX)$ such that $q^*(f_n) = y_n$.

It is easy to see now that an element $1 \oplus y_1 \oplus \cdots \oplus y_n \in \oplus_{n\geq 0} G(S^nX) = G(\coprod_{n\geq 0} S^nX)$ satisfies our conditions.

Proposition 3.3.7. *Let Z be a closed subscheme of a scheme X and $p : Y \longrightarrow Z$ be a proper surjective morphism of finite type which is an isomorphism outside Z. Then the kernel of the morphism of qfh-sheaves*

$$\mathbf{Z}_{qfh}(p) : \mathbf{Z}_{qfh}(Y) \longrightarrow \mathbf{Z}_{qfh}(X)$$

is canonically isomorphic to the kernel of the morphism

$$\mathbf{Z}_{qfh}(p_{|Z}) : \mathbf{Z}_{qfh}(p^{-1}(Z)) \longrightarrow \mathbf{Z}_{qfh}(Z).$$

Proof. The inclusion of schemes $p^{-1}(Z) \longrightarrow Y$ induces a morphism of sheaves

$$\ker\left(\mathbf{Z}_{qfh}(p_{|Z})\right) \longrightarrow \ker\left(\mathbf{Z}_{qfh}(p)\right),$$

which is obviously a monomorphism. It is sufficient to show that it is an epimorphism. By Proposition 2.1.4 we have epimorphisms of sheaves

$$\mathbf{Z}_{qfh}(Y \times_X Y) \longrightarrow \ker\left(\mathbf{Z}_{qfh}(p)\right)$$

$$\mathbf{Z}_{qfh}\left(p^{-1}(Z) \times_Z p^{-1}(Z)\right) \longrightarrow \ker\left(\mathbf{Z}_{qfh}(p_{|Z})\right).$$

The last morphism is obviously zero on the diagonal $Y \subset Y \times_X Y$ and the statement of our proposition follows from the fact that the morphism

$$\Delta \coprod i : Y \coprod p^{-1}(Z) \times_Z p^{-1}(Z) \longrightarrow Y \times_X Y$$

is a *qfh*-covering and hence induces an epimorphism of the corresponding freely generated sheaves of abelian groups.

Theorem 3.3.8. *Let X be a normal connected scheme and let $f : Y \longrightarrow X$ be a finite surjective morphism of the separable degree d. Then there is a morphism*

$$\mathrm{tr}(f) : \mathbf{Z}_{qfh}(X) \longrightarrow \mathbf{Z}_{qfh}(Y)$$

such that $\mathbf{Z}_{qfh}(f)\,\mathrm{tr}(f) = d\,\mathrm{Id}_{\mathbf{Z}_{qfh}(X)}$

Proof. We may suppose that Y is the normalization of X in a finite extension of the field of functions on X. There is a decomposition $f = f_0 f_1$, where f_1 corresponds to a separable and f_0 to a purely inseparable extension, respectively. By Lemma 3.1.7 and Proposition 3.2.5, the morphism f_0 induces an isomorphism on the *qfh*-sheaves. It implies that we may restrict our considerations to the case $f_0 = \mathrm{Id}$. Let $\tilde{f} : \tilde{Y} \longrightarrow X$ be the normalization of X in a Galois extension which contains $K(Y)$. The morphism $\mathbf{Z}_{qfh}(X) \longrightarrow \mathbf{Z}_{qfh}(Y)$ is a section of the sheaf $\mathbf{Z}_{qfh}(Y)$ over X. Let $G = \mathrm{Gal}(\tilde{Y}/X)$ be the Galois group of $\tilde{Y}$ over X and $H = \mathrm{Gal}(\tilde{Y}/Y)$ its subgroup which corresponds to Y. By Proposition 3.3.2, to construct such a section, it is sufficient to find a section a of $\mathbf{Z}_{qfh}(Y)$ over $\mathbf{Z}_{qfh}(\tilde{Y})$ which is G-invariant. We set

$$a = \sum_{x \in G/H} x(g),$$

where $g : \tilde{Y} \longrightarrow Y$ is the natural morphism. It is easy to see that the corresponding section of $\mathbf{Z}_{qfh}(Y)$ over X satisfies all the properties we need.

3.4 Comparison results and cohomological dimension

Theorem 3.4.1. *Let X be a normal scheme and F be a qfh-sheaf of $\mathbf{Q}$-vector spaces. Then one has*

$$H^i_{qfh}(X, F) = H^i_{\mathrm{et}}(X, F).$$

Proof. It follows from the Leray spectral sequence that to prove our theorem it is sufficient to show that, for any normal strictly local ring R, one has

$$H^i_{qfh}(\mathrm{Spec}(R), F) = 0$$

for $i > 0$. It is easy to see that we actually need only to consider the case $i = 1$. Let $a \in H^1_{qfh}(\mathrm{Spec}(R), F)$ be a cohomological class. Then there exists a *qfh*-covering $\{U_i \longrightarrow \mathrm{Spec}(R)\}$ and a Čech cocycle $\{a_i j\} \in \oplus F(U_i \times_{\mathrm{Spec}(R)} U_j)$ which represents a. To prove that $a = 0$, it is sufficient to show that the natural surjection of sheaves of $\mathbf{Q}$-vector spaces $\mathbf{Z}(\coprod U_i) \otimes \mathbf{Q} \longrightarrow \mathbf{Z}(\mathrm{Spec}(R)) \otimes \mathbf{Q}$ splits. It follows from Theorem 3.3.8 above and the next lemma.

Lemma 3.4.2. *Let X be the spectrum of a strictly local ring and let $\{p_i : U_i \longrightarrow X\}$ be a qfh-covering. Then there exists a finite surjective morphism $p : V \longrightarrow X$ and a morphism $s : V \longrightarrow \coprod U_i$ such that*

$$p = \left(\coprod p_i\right) \circ s.$$

Proof. We may assume that $U_1 \longrightarrow X$ is finite and the image of all other U_i does not contain the closed point of X (see [11, I.4.2]. We should prove that if our family of morphisms is a *qfh*-covering then $U_1 \longrightarrow X$ is surjective. Let us do it by the induction by dimension of X. The result is obvious for $\dim X < 2$. Let $x \in X$ be a point of dimension one. Considering the base change along the embedding $Z_x \longrightarrow X$, where Z_x is the closure of x we conclude that x lies in the image of U_1. Therefore the image of U_1 contains all points of dimension 1 in X. Since it is closed it implies that it coincide with X.

Our theorem is proved.

Lemma 3.4.3. *Let k be a separably closed field. Then for any qfh-sheaf of abelian groups F and any $i > 0$, one has*

$$H^i_{qfh}\big(\mathrm{Spec}(k), F\big) = 0.$$

Proof. Obvious.

Theorem 3.4.4. *Let X be a scheme and F a locally constant in the étale topology sheaf on Sch/X. Then F is a qfh-sheaf and one has*

$$H^i_{qfh}(X, F) = H^i_{\mathrm{et}}(X, F).$$

Proof. The fact that F is a *qfh*-sheaf is obvious. To prove the comparison statment it is sufficient to show that if X is a strictly henselian scheme then $H^q_{qfh}(X, F) = 0$ for $q > 0$.

Denote by Finite(X) the site which objects are schemes finite over X and coverings are surjective families of morphisms. We have an obvious morphism of sites

$$\gamma : (Sch/X)_{qfh} \longrightarrow \mathrm{Finite}(X).$$

Lemma 3.4.2 implies that for any *qfh*-sheaf of abelian groups $\mathcal{G}$ on Sch/X this morphism of sites induces isomorphisms

$$H^i_{\mathrm{finite}}(X, \gamma_*(\mathcal{G})) = H^i_{qfh}(X, \mathcal{G}).$$

Hence it is sufficient to show that $H^i_{\mathrm{finite}}(X, \gamma_*(F)) = 0$ for $i > 0$.

Let $x : \mathrm{Spec}(k) \longrightarrow X$ be the closed point of X. For any finite morphism $Y \longrightarrow X$, the scheme Y is a disjoint union of strictly henselian schemes (see ([11])) and hence the number of connected components of Y coincides with the number of conected components of the fiber $Y_x \longrightarrow \mathrm{Spec}(k)$. This implies that the canonical morphism

$$\gamma_*(F) \longrightarrow x_*\big(\gamma_*(F)\big)$$

of sheaves on the finite sites is an isomorphism. Lemma 3.4.3 implies now that one has

$$H^i_{\mathrm{finite}}\big(X, \gamma_*(F)\big) = H^i_{\mathrm{finite}}\big(\mathrm{Spec}(k), \gamma_*(F)\big) = H^i_{qfh}\big(\mathrm{Spec}(k), F\big) = 0$$

for any $i > 0$.

Theorem 3.4.5. *Let X be a scheme and F a locally constant torsion sheaf in étale topology on Sch/X. Then F is an h-sheaf and for any $i \geq 0$, one has a canonical isomorphism*

$$H^i_h(X, F) = H^i_{\mathrm{et}}(X, F).$$

Proof. See [13].

Remark. The theorem above is false for sheaves which are not torsion sheaves, but it can be shown that it is still valid for arbitrary locally constant sheaves if X is a *smooth* scheme of finite type over a field of characteristic zero (we need this condition only to be able to use the resolution of singularities).

Theorem 3.4.6. *Let X be a scheme of the (absolute) dimension N. Then for any h-sheaf of abelian groups and any $i > n$ one has*

$$H^i_h(X, F) \otimes \mathbf{Q} = 0.$$

Proof. We need first the following lemma.

Lemma 3.4.7. *Let X be a scheme of the absolute dimension N. Then for any étale sheaf of abelian groups F and any $i > N$ one has*

$$H^i_{\mathrm{et}}(X, F) \otimes \mathbf{Q} = 0.$$

Proof. (cf. [11, p. 221]) We use an induction by N. For $N = 0$ our statment is obvious. Let $x_1, \ldots, x_k$ be the set of general points of X and $\mathrm{in}_j : \mathrm{Spec}(K_j) \longrightarrow X$ the corresponding inclusions. Consider the natural morphism of sheaves on the small étale site over X:

$$F \longrightarrow \bigoplus_{j=1}^{k} (\mathrm{in}_j)_*(\mathrm{in}_j)^*(F).$$

Then kernel and cokernel of this morphism have the support in codimension at least one and, therefore, their cohomology vanish in the dimension greater than $N-1$ by inductive assumption. To finish the proof it is sufficient now to notice that $H^i(X, (\mathrm{in}_j)_*(\mathrm{in}_j)^*(F)) \otimes \mathbf{Q} = 0$ by the Leray spectral sequence of the inclusions in_j.

It follows from this lemma and Theorem 3.4.1 above, that for a normal scheme X of the dimension N and any $i > 1$, one has $H^i_{qfh}(X, F) \otimes \mathbf{Q} = 0$.

According to the spectral sequence which connects Čech and usual cohomology, to prove our theorem it is sufficient to show that $\check{H}^i_h(X, F) \otimes \mathbf{Q} = 0$ for $i > N$. Let $a \in \check{H}^i_h(X, F) \otimes \mathbf{Q}$ be a cohomology class and $\{U_i \longrightarrow \bar{U} \longrightarrow X_Z \longrightarrow X\}$ an h-covering of normal form which realizes a. Passing to a refinement we may suppose that X_Z is normal. Since $\{U_i \longrightarrow \bar{U} \longrightarrow X_Z\}$ is a qfh-covering the restriction of a to X_Z is equal to zero. It follows from Propositions 3.3.7 and 2.1.3, that there are two long exact sequences:

$$\ldots \longrightarrow \mathrm{Ext}^{i-1}(G, F) \longrightarrow H^i_h(X, F) \longrightarrow H^i_h(X_Z, F) \longrightarrow \mathrm{Ext}^i(G, F) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow \mathrm{Ext}^{i-1}(G, F) \longrightarrow H^i_h(Z, F) \longrightarrow H^i_h(PN_Z, F) \longrightarrow \mathrm{Ext}^i(G, F) \longrightarrow \ldots$$

and, since $\dim(PN_Z) < \dim(X)$ our result follows by the induction by $\dim(X)$.

Corollary 3.4.8. *Let X be a scheme of absolute dimension N. Then for any qfh-sheaf of abelian groups F on Sch/X and any $i > N$ one has*

$$H^i_{qfh}(X, F) = 0.$$

4. Categories $DM(S)$

4.1 Definition and general properties

Consider the category Sch/S of schemes over a base S as a site with either h- or qfh-topology. It has a structure of a site with interval if we set $I^+ = \mathbf{A}^1_S$. Morphisms (μ, i_0, i_1) from the definition of a site with interval are the multiplication morphism and the points 0, 1, respectively.

Denote by Δ^n_S the scheme $S \times_S \mathrm{Spec}\ \mathbf{Z}[x_0, \ldots, x_n]/\sum x_i = 1$. One can easily see that Δ^n_S is (noncanonically) isomorphic to $\mathbf{A}^n_S$. For any morphism $f : [n] \longrightarrow [m]$ in the standard simplicial category Δ we denote by $a'(f) : \Delta^n_S \longrightarrow \Delta^m_S$ the morphism which corresponds to the homomorphism of rings $a'(f)^*$ of the form

$$a'(f)^*(x_i) = \begin{cases} \sum x_j & \text{such that } f(j) = i \\ & \text{if } f^{-1}(i) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

This constructions defines a cosimplicial object $a' : \Delta \longrightarrow \mathrm{Aff}/S$.

Proposition 4.1.1. *The cosimplicial object a' is isomorphic to the cosimplicial a_{I^+} of the site with interval $((\mathrm{Aff}/S)_{\mathrm{cl}}, \mathbf{A}^1_S)$.*

Proof. Denote the functor a_{I^+} by a. We have to construct for any $n \geq 0$ an isomorphism

$$\phi_n : \Delta^n_S \longrightarrow \mathbf{A}^n_S$$

such that for any morphism $\alpha : [n] \longrightarrow [m]$ in Δ, one has

$$\phi_m \circ a'(\alpha) = a(\alpha) \circ \phi_n.$$

Denote by $\psi^n_i : [n] \longrightarrow [1], i = 0, \ldots, n+1$ the morphisms of the form

$$\psi^n_i(k) = \begin{cases} 0 & \text{for } k < i \\ 1 & \text{for } k \geq i. \end{cases}$$

One can easily see that the morphisms

$$a\left(\prod_{i=0}^{n+1} \psi^n_i\right) : (I^+)^n_S \longrightarrow (I^+)^{n+2}$$

$$a'\left(\prod_{i=0}^{n+1} \psi^n_i\right) : \Delta^n_S \longrightarrow (\Delta^1)^{n+2}$$

are closed embeddings. This implies easily that it is sufficient to construct isomorphisms ϕ_n such that

$$\phi_m \circ a'(\psi^n_i) = a(\psi^n_i) \circ \phi_1$$

for all $n \geq 0$ and $i = 0, \ldots, n+1$. We obviously have

$$a(\psi^n_i)(x_1, \ldots, x_n) = \begin{cases} x_i & \text{for } i \in \{1, \ldots, n\} \\ 1 & \text{for } i = 0 \\ 0 & \text{for } i = n+1 \end{cases}$$

$$a'(\psi^n_i)(z_0, \ldots, z_n) = \begin{cases} \left(\sum_{j=0}^{i-1} z_j, \sum_{j=i}^{n} z_j\right) & \text{for } i \in \{1, \ldots, n\} \\ (0,1) & \text{for } i = 0 \\ (1,0) & \text{for } i = n+1. \end{cases}$$

We can define ϕ_n by

$$\phi^n(z_0 \ldots, z_n) = \left(\sum_{k=1}^{n} z_k, \sum_{k=2}^{n} z_k, \ldots, z_n\right).$$

Proposition is proved.

We define the category $DM_h(S)$ (resp. $DM_{qfh}(S)$) to be the homological category of the site with interval $((Sch/S)_h, \mathbf{A}^1_S)$ (resp. $((Sch/S)_h, \mathbf{A}^1_S)$). Let $M_h, \tilde{M}_h : Sch/S \longrightarrow DM_h(S)$ (resp. $M_{qfh}, \tilde{M}_{qfh}$) be the corresponding functors. We identify sheaves of abelian groups on Sch/S with the corresponding objects of $DM(S)$ and schemes with the corresponding representable sheaves of sets. We also omit the specification of topology in all the statements below which hold for both h- and qfh-topologies.

It follows immediately from our construction that the categories $DM(S)$ are tensor triangulated categories, and for any morphism of schemes $f : S_1 \longrightarrow S_2$ there is defined an exact, tensor functor $f^* : DM(S_2) \longrightarrow DM(S_1)$ such that for a scheme X over S_2 one has $f^*(M(X)) = M(X \times_{S_2} S_1)$. The properties of the functor $\mathbf{Z}(-)$ imply that for any schemes X, Y over S one has

$$M\left(X \coprod Y\right) = M(X) \oplus M(Y)$$
$$M\left(X \times_S Y\right) = M(X) \otimes M(Y).$$

Proposition 4.1.2. *Let $X = U \cup V$ be an open or closed covering of X. Then there is a natural exact triangle in $DM(S)$ of the form*

$$M(U \cap V) \longrightarrow M(U) \oplus M(V) \longrightarrow M(X) \longrightarrow M(U \cap V)[1].$$

Proof. It follows from Proposition 2.1.4.

Proposition 4.1.3. *Let $p : Y \longrightarrow X$ be a locally trivial (in Zariski topology) fibration whose fibers are affine spaces. Then the morphism $M(p) : M(Y) \longrightarrow M(X)$ is an isomorphism.*

Proof. It follows from Proposition 4.1.2 and the obvious fact that for any scheme X the morphism $M(\mathrm{pr}_1) : M(X \times \mathbf{A}^n) \longrightarrow M(X)$ is an isomorphism.

Proposition 4.1.4. *Let $f : Y \longrightarrow X$ be a finite surjective morphism of normal connected schemes of the separable degree d. Then there is a morphism $\mathrm{tr}(f) : M(X) \longrightarrow M(Y)$ such that $M(f)\,\mathrm{tr}(f) = d\,\mathrm{Id}_{M(X)}$.*

Proof. It follows from Theorem 3.3.8.

Proposition 4.1.5. *Let Z be a closed subscheme of a scheme X and $p : Y \longrightarrow X$ a proper surjective morphism of finite type which is an isomorphism outside Z. Then there is an exact triangle in $DM_h(S)$ of the form*

$$M_h(X)[1] \longrightarrow M_h(p^{-1}(Z)) \longrightarrow M_h(Z) \oplus M_h(Y) \longrightarrow M_h(X).$$

Proof. It follows from the fact that p is an h-covering and from Proposition 2.1.4.

Remark. The above proposition is *false* for the *qfh*-topology.

It follows easily from our construction that for any sheaf F on Sch/S and any object X of this category we have canonical morphisms

$$H^i(X,F) \longrightarrow DM\big(M(X), F[i]\big).$$

Proposition 4.1.6. *Let F be a locally free in étale topology sheaf of torsion prime to the characteristic of S. Then for any scheme X one has a natural isomorphism*

$$DM\big(M(X), F[n]\big) = H^n_{\mathrm{et}}(X,F).$$

Proof. It follows from Proposition 2.2.9, Theorems 3.4.4, 3.4.5 and the homotopy invariance of étale cohomologies with locally constant coefficients (see [11, p. 240]).

Proposition 4.1.7. *Let S be a scheme of characteristic $p>0$. Then the category $DM(S)$ is $\mathbf{Z}[1/p]$-linear.*

Proof. It is sufficient to show that the sheaf $\mathbf{Z}/p$ is isomorphic to zero in the category $DM(S)$. Consider the Artin–Shrier exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{G}_a \xrightarrow{F-1} \mathbf{G}_a \longrightarrow 0$$

where $\mathbf{G}_a$ is the sheaf of abelian groups represented by $\mathbf{A}^1$ and F is the geometrical Frobenius morphism. Since $\mathbf{G}_a$ is obviously a strictly contractible sheaf the existence of this sequence implies the result we need.

The following two theorems follow easily from the results of [13].

Theorem 4.1.8. *Let X be a scheme of finite type over $\mathbf{C}$. Then one has canonical isomorphisms of abelian groups*

$$DM_h\big(\mathbf{Z}, M(X)\otimes \mathbf{Z}/n[k]\big) = H_k\big(X(\mathbf{C}), \mathbf{Z}/n\big).$$

Let us call an object X of the category $DM(S)$ a torsion object if there exists $N>0$ such that $N\ \mathrm{Id}_X = 0$.

Theorem 4.1.9. *Let k be a field of characteristic zero. Denote by D_k the derived category of the category of torsion sheaves of abelian groups on the small étale site of $\mathrm{Spec}(k)$. Then the canonical functor*

$$\tau : D_k \longrightarrow DM_h\big(\mathrm{Spec}(k)\big)$$

is a full embedding and any torsion object in $DM_h(\mathrm{Spec}(k))$ is isomorphic to an object of the form $\tau(K)$ for $K\in \mathrm{ob}(D_k)$.

Remarks.

1. We do not know whether or not the analogs of the above two theorems hold for the *qfh*-topology.
2. Using the resolution of singularities in positive characteristic, one can drop the condition $\mathrm{char}(k)=0$ in the last theorem, considering instead objects of torsion prime to $\mathrm{char}(k)$.

4.2 Tate motives

All through this section we are working with the categories $DM(S)$ with respect to *qfh*-topology. All the results below obviously hold for h-topology as well. Since the results of this section do not depend of the base scheme S, we will omit S in all notations below where it is possible.

Definition 4.2.1. The Tate motive $\mathbf{Z}(1)$ is the object of the category DM which corresponds to the sheaf $\mathbf{G}_m$ shifted by minus one, i.e.

$$\mathbf{Z}(1) = \mathbf{G}_m[-1].$$

We denote by $\mathbf{Z}(n)$ the n-tensor power of $\mathbf{Z}(1)$ and for any object X of DM by $X(n)$ the tensor product $X \times \mathbf{Z}(n)$.

Proposition 4.2.2. *For any n and k there exists an exact triangle of the form*

$$\mathbf{Z}(n) \xrightarrow{k} \mathbf{Z}(n) \longrightarrow \mu_k^{\otimes n} \longrightarrow \mathbf{Z}(n)[1]$$

where $\mu_k^{\otimes n}$ denotes the object of the category DM which corresponds to the n-th tensor power of the sheaf μ_k of k-th roots of unit.

Proof. It is sufficient to show that one has an isomorphism $Z(n)\otimes\mathbf{Z}/k\mathbf{Z} \cong \mu_k^{\otimes n}$, i.e. isomorphism $\mathbf{G}_m^{\otimes n} \otimes \mathbf{Z}/k\mathbf{Z} \cong \mu_k^{\otimes n}[n]$ (note that the tensor product on the left-hand side is a tensor product in the category DM which corresponds to the L-tensor product on the level of the derive category of sheaves).

Note first that μ_k is, by definition, the kernel of the morphism of the sheaves $\mathbf{G}_m \longrightarrow \mathbf{G}_m$ which corresponds to the morphism of schemes

$$\mathbf{A}^1 - 0 \longrightarrow \mathbf{A}^1 - 0$$

which takes z to z^k. In h-topology it is a surjection. Therefore one has $\mathbf{G}_m \otimes^L \mathbf{Z}/k\mathbf{Z} \cong \mu_k[1]$. To finish the proof of the proposition one should show that $\mu_k^{\otimes n} \otimes^L \mathbf{G}_m \cong \mu_k^{\otimes(n+1)}[1]$, which is easy.

For any scheme X we define its motivic cohomology to be the groups

$$H^p\big(X, \mathbf{Z}(q)\big) = DM\big(M(X), \mathbf{Z}(q)\big).$$

When it is necessary we will use the notations $H^p_{qfh}(X, \mathbf{Z}(q))$ and $H^p_h(X, \mathbf{Z}(q))$ for these groups defined with respect to *qfh*-and h-topology, respectively.

There is defined an obvious multiplication of the form

$$H^p\big(X, \mathbf{Z}(q)\big) \otimes H^{p'}\big(X, \mathbf{Z}(q')\big) \longrightarrow H^{p+p'}\big(X, \mathbf{Z}(q+q')\big)$$

which satisfies all standard properties. In particular the direct sum

$$\bigoplus_{p,q} H^p\big(X, \mathbf{Z}(q)\big)$$

has a natural structure of a bigraded ring, which is commutative as a bigraded ring.

Proposition 4.2.3. *Let X be a scheme. For any q and any k prime to characteristic of X one has a long exact sequence of the form*

$$\ldots \longrightarrow H^p(X,\mathbf{Z}(q)) \xrightarrow{k} H^p(X,\mathbf{Z}(q)) \longrightarrow H^p_{\mathrm{et}}(X,\mu_k^{\otimes n}) \longrightarrow H^{p+1}(X,\mathbf{Z}(q)) \longrightarrow \ldots$$

Proof. It follows from Proposition 4.2.2 that the only thing we have to prove is that under our assumptions one has an isomorphism

$$DM(M(X),\mu_k^{\otimes n}[p]) \cong H^p_{et}(X,\mu_k^{\otimes n}).$$

It follows from Proposition 4.1.6 and the fact that $\mu_k^{\otimes n}$ is a locally free in étale topology sheaf over $\mathrm{Spec}(\mathbf{Z}[1/k])$.

Proposition 4.1.7 implies that for a schemes X of characteristic $l > 0$ the groups $H^p(X,\mathbf{Z}(q))$ are $\mathbf{Z}[1/l]$ modules.

Proposition 4.2.4. *Let X be a regular scheme of exponential characteristic p. Then for any $i \geq 0$ one has a canonical isomorphism*

$$H^i_{qfh}(X,\mathbf{Z}(1)) = H^{i-1}(X,\mathbf{G}_m) \otimes \mathbf{Z}[1/p].$$

Proof. It follows from our comparison results and homotopy invariance of étale cohomology with coefficients in $\mathbf{G}_m$ over regular schemes.

Theorem 4.2.5. *The tautological section of the sheaf $\mathbf{G}_m$ over $\mathbf{A}^1 - \{0\}$ defines an isomorphism in DM*

$$\tilde{M}(\mathbf{A}^1 - \{0\}) \cong \mathbf{Z}(1)[1].$$

Proof. Note first that the morphism

$$\phi : \tilde{\mathbf{Z}}(\mathbf{A}^1 - 0) \longrightarrow \mathbf{G}_m$$

defined by the tautological section of $\mathbf{G}_m$ over $\mathbf{A}^1 - 0$ is an epimorphism. It is sufficient to show that its kernel is a contractible sheaf.

Let $\Delta^\cdot$ be the cosimplicial scheme over S whose terms are the schemes

$$\Delta^n = \mathrm{Spec}\left(\mathbf{Z}[t_0,\ldots,t_n]\right) \Big/ \left(\sum t_i = 0\right) \times_{\mathrm{Spec}(\mathbf{Z})} S$$

and coface and codegeneracy morphisms are defined in the obvious way.

Theorem 3.3.6 implies that the sheaf $\mathbf{Z}(\mathbf{A}^1 - 0)$ is isomorphic to the sheaf of abelian groups associated with the sheaf of abelian monoids representable by the scheme $\coprod S^n(\mathbf{A}^1 - 0)$.

The scheme $S^n(\mathbf{A}^1 - 0)$ for $n > 0$ is isomorphic to the scheme $(\mathbf{A}^1 - 0) \times \mathbf{A}^{n-1}$ and one can easily see that the sheaf $\ker(\phi)$ is isomorphic to the sheaf of abelian groups associated with the sheaf of abelian monoids representable by the scheme $\mathbf{A}^\infty$. It implies easily that the complex of sheaves $C_*(\ker(\phi))$ is exact. Hence $\ker(\phi)$ is contractible by Lemma 2.2.5.

Corollary 4.2.6. *The morphism $\tilde{M}(\mathbf{P}^1_S) \longrightarrow \mathbf{G}_m[1]$ which corresponds to the cohomological class in $H^1(\mathbf{P}^1, \mathbf{G}_m)$ represented by the line bundle $\mathcal{O}(-1)$ is an isomorphism in $DM_{qfh}(S)$*

Proof. It follows easily from the theorem by consideration of the open covering of $\mathbf{P}^1$ by means of two affine lines.

Theorem 4.2.7. *Let X be a scheme and E be a vector bundle on X. Denote by $P(E) \longrightarrow X$ the projectivization of E. One has a natural isomorphism in DM*

$$M(P(E)) \cong \bigoplus_{i=0}^{\dim E-1} M(X)(i)[2i].$$

Proof. We may suppose X to be our base scheme. Let $\mathcal{O}(-1)$ be the tautological line bundle on $P(E)$ and $a : M(P(E)) \longrightarrow \mathbf{Z}(1)[2]$ the morphism in the category $DM(X)$ which corresponds to the class of this bundle in $H^1(P(E), \mathbf{G}_m)$. Using the morphism $M(P(E)) \longrightarrow M(P(E)) \otimes M(P(E))$ induced by the diagonal, we can define elements $a^i \in DM(M(P(E)), \mathbf{Z}(i)[2i])$ as tensor powers of $a = a^1$. We claim that the direct sum

$$\phi : \bigoplus_{i=0}^{\dim E-1} a^i : M(P(E)) \longrightarrow \bigoplus_{i=0}^{\dim E-1} \mathbf{Z}(i)[2i]$$

is an isomorphism in $DM(X)$.

Consider a trivializing open covering $X = \cup U_i$ of X. Let us suppose for simplicity of notation that this covering consists only of two open subsets. By Proposition 4.1.2 we have an exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z}(U \cap V) \longrightarrow \mathbf{Z}(U) \oplus \mathbf{Z}(V) \longrightarrow \mathbf{Z} = \mathbf{Z}(X) \longrightarrow 0.$$

Since our construction of the map ϕ is natural with respect to restrictions to open subsets, the existence of this exact sequence let us restrict our considerations to the case of a trivial bundle E. In other words we should consider a scheme $\mathbf{P}^n$ over S and to prove that the morphism in $DM(S)$ which is defined as the direct sum

$$\phi = \bigoplus_{i=0}^{n} a_n^i$$

where a corresponds to the line bundle $\mathcal{O}(-1)$ is an isomorphism. We use an induction on n. For $n = 0$ our statement is trivial. Consider the covering of $\mathbf{P}^n$ of the form

$$\mathbf{P}^n = \mathbf{P}^n - \{0\} \cup \mathbf{A}^n$$

where $\{0\}$ is the point with coordinates $[1, 0, \dots, 0]$. We have the following exact triangle in DM

$$M(\mathbf{A}^n - \{0\}) \longrightarrow M(\mathbf{P}^n - \{0\}) \oplus M(\mathbf{A}^n) \longrightarrow M(\mathbf{P}^n) \longrightarrow M(\mathbf{A}^n - \{0\})[1].$$

Let us construct a morphism from this exact triangle to an exact triangle of the form

$$\mathbf{Z}(n)[2n-1] \oplus \mathbf{Z} \longrightarrow \bigoplus_{i=0}^{n-1} \mathbf{Z}(i)[2i] \oplus \mathbf{Z} \longrightarrow \bigoplus_{i=0}^{n} \mathbf{Z}(i)[2i] \longrightarrow \mathbf{Z}(n)[2n] \oplus \mathbf{Z},$$

and show that it is an isomorphism on the first two terms, which would imply that it is an isomorphism of exact triangles. Define a cohomological class $\psi \in H^{n-1}(\mathbf{A}^n - \{0\}, \mathbf{G}_m^{\otimes n})$ as follows. Consider the covering of the scheme $\mathbf{A}^n - \{0\}$ of the form

$$\mathbf{A}^n - \{0\} = \bigcup_{i=1}^{n} \mathbf{A}^n - H_i$$

where H_i is a hyperplane $x_i = 0$. A Čech cocycle in $Z^{n-1}(\mathbf{A}^n - \{0\}, \mathbf{G}_m^{\otimes n})$ with respect to this covering is a section of the sheaf $\mathbf{G}_m^{\otimes n}$ over $\cap_{i=1}^{n} \mathbf{A}^n - H_i$. We set ψ to be the cohomological class which corresponds to the tautological section of the form

$$(x_1, \dots, x_n) \longrightarrow x_1 \otimes \cdots \otimes x_n.$$

Define a morphism $f : M(\mathbf{A}^n - \{0\}) \longrightarrow \mathbf{Z}(n)[2n-1] \oplus \mathbf{Z}$ as the direct sum of the morphism which corresponds to ψ and the structural morphism.

Lemma 4.2.8. *f is an isomorphism.*

Proof. Easy by the induction on n starting with Theorem 4.2.5

Let $p : \mathbf{P}^n - \{0\} \longrightarrow \mathbf{P}^{n-1}$ be a natural projection whose fibers are affine lines. It is obviously an isomorphism in DM. Define now a morphism

$$g : M(\mathbf{P}^n - \{0\}) \oplus M(\mathbf{A}^n) \longrightarrow \bigoplus_{i=0}^{n-1} \mathbf{Z}(i)[2i] \oplus \mathbf{Z}$$

as the direct sum of the morphism

$$\bigoplus_{i=0}^{n-1} M(p) a_{n-1}^i$$

and the structural morphism of $\mathbf{A}^n$. Note that g is an isomorphism according to our inductive assumption. One can easily see that the family of morphisms f, g, ψ, $f[1]$ is indeed a morphism of the exact triangles.

Theorem is proved.

4.3 Monoidal transformations

All through this section we are working with the *qfh*-topology. In particular the notation $DM(S)$ is used for the category $DM_{fqh}(S)$. All the results below obviously hold for the *h*-topology as well.

Let us recall some notations. For a scheme X and its closed subscheme Z we denote by X_Z the blowup of X with center in Z and by $p_Z : X_Z \longrightarrow X$ the corresponding projection.

By PN_Z we denote the projectivization of the normal cone to Z in X and by $p : PN_Z \longrightarrow Z$ the morphism which is the restriction of p_Z. Let $O_X(Z)$ be the kernel of the morphism of *qfh*-sheaves

$$\mathbf{Z}_{qfh}(p) : \mathbf{Z}_{qfh}(PN_Z) \longrightarrow \mathbf{Z}_{qfh}(Z).$$

By Proposition 3.3.7 it is naturally isomorphic to the kernel of the morphism $\mathbf{Z}_{qfh}(p_Z)$.

Theorem 4.3.1. *Let $Z \subset X$ be a smooth pair over S. Then the sequence of sheaves*

$$O_X(Z) \longrightarrow \mathbf{Z}_{qfh}(X_Z) \longrightarrow \mathbf{Z}_{qfh}(X)$$

defines an exact triangle in $DM(S)$ of the form

$$O_X(Z) \longrightarrow M(X_Z) \longrightarrow M(X) \longrightarrow O_X(Z)[1].$$

In other words the cokernel of the morphism $\mathbf{Z}_{qfh}(p_Z)$ is isomorphic to zero in $DM(S)$.

Proof. Let us prove first the following lemma.

Lemma 4.3.2.. *Let $X \cup U_i$ be an open covering of X and $X_Z = \cup V_i$ the corresponding covering of X_Z. Consider the long exact sequences of sheaves which are defined by these coverings and the natural morphism between them*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_{qfh}(\cap V_i) \arrow[r] \arrow[d] & \ldots \arrow[r] & \oplus \mathbf{Z}_{qfh}(V_i) \arrow[r] \arrow[d] & \mathbf{Z}_{qfh}(X_Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}_{qfh}(\cap U_i) \arrow[r] & \ldots \arrow[r] & \oplus \mathbf{Z}_{qfh}(U_i) \arrow[r] & \mathbf{Z}_{qfh}(X) \arrow[r] & 0.
\end{tikzcd}$$

Then the complex which is the cokernel of this morphism is exact.

Proof. The exactness of the cokernel of this morphism is equivalent to the exactness of the kernel of this morphism. By Proposition 3.3.7 this kernel is isomorphic to the kernel of the morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}(\cap V_i \cap PN_Z) \arrow[r] \arrow[d] & \dots \arrow[r] & \oplus\mathbf{Z}(V_i \cap PN_Z) \arrow[r] \arrow[d] & \mathbf{Z}(PN_Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}(\cap U_i \cap Z) \arrow[r] & \dots \arrow[r] & \oplus\mathbf{Z}(U_i \cap Z) \arrow[r] & \mathbf{Z}(Z) \arrow[r] & 0
\end{tikzcd}$$

(here we use the notation $\mathbf{Z}(-)$ instead of $\mathbf{Z}_{qfh}(-)$). These two complexes are obviously exact, since they correspond to the covering of PN_Z and Z respectively which are induced by $\{U_i\}$. On the other hand in our case the normal cone to Z is a vector bundle and, therefore, the morphism $PN_Z \longrightarrow Z$ is flat. In particular it splits over some *qfh*-covering, which implies that the vertical arrows in the diagram above are surjections. Since the kernel of a surjection of exact complexes is exact, our lemma is proved.

It follows from this lemma that it is sufficient to prove our proposition locally. More precisely, it is sufficient to construct an open covering $X = \cup U_i$ of X such that all the cokernels of the morphisms $\mathbf{Z}_{qfh}(p_{Z\cap U_i})$ are isomorphic to zero in $DM(S)$. Since $Z \subset X$ is a smooth pair, there exists a covering $X = \cup U_i$ such that, for any i, there is an étale morphism $f_i : U_i \longrightarrow \mathbf{A}^N$ satisfying $Z \cap U_i = f^{-1}(\mathbf{A}^k)$, where $N = \dim_S X$ and $k = \dim_S Z$ (see [7, 2.4.9]). Let U be one of those open subschemes. It is sufficient to prove that $\mathrm{coker}(\mathbf{Z}_{qfh}(p_{Z\cap U}))$ is isomorphic to zero in $DM(S)$. Denote the scheme $U \cap Z$ by Y. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_{qfh}(U - Y) \arrow[r] \arrow[d, equal] & \mathbf{Z}_{qfh}(U_Y) \arrow[r] \arrow[d, "a"] & \mathbf{Z}_{qfh}(U_Y)/\mathbf{Z}_{qfh}(U - Y) \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & \mathbf{Z}_{qfh}(U - Y) \arrow[r] & \mathbf{Z}_{qfh}(U) \arrow[r] & \mathbf{Z}_{qfh}(U)/\mathbf{Z}_{qfh}(U - Y) \arrow[r] & 0.
\end{tikzcd}$$

It is easy to see that the morphism $\mathrm{coker}(a) \longrightarrow \mathrm{coker}(b)$ is an isomorphism. It is sufficient, therefore, to prove, that $\mathrm{coker}(b)$ is isomorphic to zero in $DM(S)$. We will need the following lemma.

Lemma 4.3.3. *Let $Z \longrightarrow X$ be a closed embedding and $f : U \longrightarrow X$ an étale surjective morphism such that $U \times_X Z \longrightarrow Z$ is an isomorphism. Then one has a natural isomorphism of sheaves*

$$\mathbf{Z}(U)/\mathbf{Z}(U - f^{-1}(Z)) = \mathbf{Z}(X)/\mathbf{Z}(X - Z).$$

Proof. Consider the diagram of sheaves:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}(U - f^{-1}(Z)) & \xrightarrow{i} & \mathbf{Z}(U) & \longrightarrow & \mathbf{Z}(U)/\mathbf{Z}(U - f^{-1}(Z)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}(X - Z) & \longrightarrow & \mathbf{Z}(X) & \longrightarrow & \mathbf{Z}(X)/\mathbf{Z}(X - Z) & \longrightarrow & 0.
\end{array}$$

We have to prove that the right vertical arrow is an isomorphism. It is obviously an epimorphism, so it is sufficient to prove that $\ker \mathbf{Z}(f)$ lies in $\mathrm{Im}(i)$. Note that it is sufficient to prove this for presheaves of the form $\mathbf{Z}_0(X)(W) = \oplus \mathbf{Z}(\mathrm{Hom}(W_i, X))$, where W_i are the connected components of a scheme W. Let W be a connected scheme. Then $\ker(\mathbf{Z}_0(f))$ is the group of formal sums of the form $\sum_{i \in I} n_i g_i$, where $g_i : W \longrightarrow U$ are morphisms such that there exists a decomposition $I = \coprod I_k$ such that $f \circ g_i = f \circ g_j$ for $i, j \in I_k$ and $\sum_{i \in I_k} n_i = 0$ for any k. Therefore, we have to prove only that if $f \circ g = f \circ h$ for some $g, h : W \longrightarrow U$ then either $g = h$ or g and h can be factorized through $U - f^{-1}(Z)$. Let g, h be such morphisms. Then there exists a morphism $g \times h : W \longrightarrow U \times_X U$, whose compositions with the projections are the morphisms g and h resp. To finish the proof it is sufficient to notice that under the assumptions of our lemma there is a decomposition of the form $U \times_X U = \Delta(U) \coprod U_0$ where Δ is the diagonal embedding and the projections $\mathrm{pr}_1, \mathrm{pr}_2 : U_0 \longrightarrow U$ can be factorized through $U - f^{-1}(Z)$. Lemma is proved.

Let $W = \mathbf{A}^{N-k} \times (\mathbf{A}^k \cap f(Y))$. We may replace U by $f^{-1}(W)$ and suppose that $f(U) \subset W$. Denote by V the product $\mathbf{A}^{N-k} \times Y$. There is an étale morphism of the form

$$\mathrm{Id}_{\mathbf{A}^{N-k}} \times f_{|Y} : V \longrightarrow W.$$

Consider the fiber product $V \times_W U$ and let

$$U' = (V \times_W U) - (\mathrm{pr}_1^{-1}(Z) - \Delta(Z))$$

where $\Delta(Z) \longrightarrow V \times_W U$ is the diagonal. One can easily see that both projections $\mathrm{pr}_1 : U' \longrightarrow V$ and $\mathrm{pr}_2 : U' \longrightarrow W$ satisfy the conditions of the lemma above.

Note now that since our construction is based on étale morphisms, it is natural with respect to blowups. It implies that $\mathrm{coker}(b)$ is isomorphic to the cokernel of the morphism

$$\mathbf{Z}_{qfh}(Y \times (\mathbf{A}^{N-k}_{\{0\}}/(\mathbf{A}^{N-k} - \{0\}))) \longrightarrow \mathbf{Z}_{qfh}(Y \times (\mathbf{A}^{N-k}/(\mathbf{A}^{N-k} - \{0\}))).$$

We reduced our problem, therefore to the case of the blowup of a point on the affine space. It is sufficient to show that the cokernel of the morphism $\mathbf{Z}_{qfh}(\mathbf{A}^n_{\{0\}}) \longrightarrow \mathbf{Z}_{qfh}(\mathbf{A}^n)$ represents zero in $DM(S)$, or, equivalently, that the kernel of this morphism is isomorphic to its cone in $DM(S)$. It follows from Proposition 3.3.7 and the fact that $\mathbf{A}^n_{\{0\}}$ is isomorphic to the total space of the vector bundle $\mathcal{O}[-1]$ on $\mathbf{P}^{n-1}$ and, therefore, $M(\mathbf{A}^n_{\{0\}})$ is isomorphic to $M(\mathbf{P}^{n-1})$. Theorem is proved.

Theorem 4.3.4. *Let $Z \subset X$ be a smooth pair over S. Then one has a natural isomorphism in $DM(S)$:*

$$M(X_Z) = M(X) \oplus \left(\bigoplus_{i=1}^{\operatorname{codim} Z - 1} Z(i)[2i] \right).$$

Proof. By Theorem 4.3.1 we have an exact triangle

$$O_X(Z) \longrightarrow M(X_Z) \longrightarrow M(X) \longrightarrow O_X(Z)[1].$$

By definition $O_X(Z)[1]$ is a cone of the natural morphism $M(PN_Z) \longrightarrow M(Z)$. Since $PN(Z)$ is the projectivization of the normal bundle to Z in X it follows from Theorem 4.2.7 that

$$O_X(Z) \cong \bigoplus_{i=1}^{\operatorname{codim} Z - 1} Z(i)[2i].$$

To prove our theorem it is sufficient to construct a splitting of the exact triangle above. Let $i_0 : X \longrightarrow X \times \mathbf{A}^1$ be the embedding of the form $i_0 = \mathrm{Id}_X \times \{0\}$. Consider the diagram

$$\begin{tikzcd}
O_X(Z) \arrow[r] \arrow[d] & O_{X \times \mathbf{A}^1}(Z \times \{0\}) \arrow[d] \\
M(X_Z) \arrow[r, "\tilde{i}_0"] \arrow[d] & M(X \times \mathbf{A}^1_{Z \times \{0\}}) \arrow[d] \\
M(X) \arrow[r, "i_0"] & M(X \times \mathbf{A}^1).
\end{tikzcd} \qquad (1)$$

There is a canonical splitting of the morphism $M(p_{Z \times \{0\}})$ by the morphism $M(X \times \mathbf{A}^1) \cong M(X) \longrightarrow M(X \times \mathbf{A}^1_{Z \times \{0\}})$ induced by the obvious lifting of the embedding $Id_X \times \{1\} : X \longrightarrow X \times \mathbf{A}^1$. To define a splitting of the projection $M(X_Z) \longrightarrow M(X)$ (or, equivalently, of the embedding $O_X(Z) \longrightarrow M(X_Z)$) it is sufficient to define a splitting of the morphism $O_X(Z) \longrightarrow O_{X \times \mathbf{A}^1}(Z \times \{0\})$. Its existence (and, moreover a canonical choice) follows from Theorem 4.2.7. Theorem is proved.

4.4 Gysin exact triangle

The goal of this section is to prove the following theorem. As in the previous section we denote by DM the category DM_{qfh} and again our results hold for the h-topology as well.

Theorem 4.4.1. *Let $Z \subset X$ be a smooth pair over S and $U = X - Z$. Then there is defined a natural exact triangle in $DM(S)$ of the form*

$$M(U) \longrightarrow M(X) \longrightarrow M(Z)(d)[2d] \longrightarrow M(U)[1]$$

where d is codimension of Z. In other words we have a natural isomorphism $M(X/U) \cong M(Z)(d)[2d]$ in $DM(S)$.

Proof. Let us construct first a morphism $M(X/U) \longrightarrow M(Z)(d)[2d]$ in $DM(S)$. Consider again the diagram (1). The morphism $\mathrm{Id} \times 1 : X \longrightarrow X \times \mathbf{A}^1$ has a natural lifting to $X \times \mathbf{A}^1_{Z\times\{0\}}$, which in the composition with the morphism $M(p_Z) : M(X_Z) \longrightarrow M(X)$, defines a morphism

$$\tilde{i_1} : M(X_Z) \longrightarrow M\big(X \times \mathbf{A}^1_{Z\times\{0\}}\big).$$

One obviously has

$$M\big(p_{Z\times\{0\}}\big)\tilde{i_1} = M(p_Z)\tilde{i_0},$$

which implies that there exists a lifting of $\tilde{i_0} - \tilde{i_1}$ to a morphism $M(X_Z) \longrightarrow O_{X\times\mathbf{A}^1}(Z\times\{0\})$. It follows from Theorem 4.3.4 that this lifting is well defined. Its composition with the natural morphism

$$O_{X\times\mathbf{A}^1}(Z \times \{0\}) \longrightarrow O_{X\times\mathbf{A}^1}(Z \times \{0\})/O_X(Z)$$

factors through a morphism $M(X) \longrightarrow O_{X\times\mathbf{A}^1}(Z\times\{0\})/O_X(Z)$ which is also well defined by Theorem 4.3.4. We have by 4.2.7

$$O_X(Z) \cong \bigoplus_{i=1}^{d-1} M(Z)(i)[2i]$$
$$O_{X\times\mathbf{A}^1}(Z \times \{0\}) \cong \bigoplus_{i=1} dM(Z)(i)[2i]$$

and therefore

$$O_{X\times\mathbf{A}^1}(Z \times \{0\})/O_X(Z) \cong M(Z)(d)[2d].$$

This construction provides us with a morphism $M(X) \longrightarrow M(Z)(d)[2d]$. Considering it more carefully one can easily see that this morphism can, in fact, be factorized through $M(X/U)$. Denote this last morphism

$$M(X/U) \longrightarrow M(Z)(d)[2d]$$

by $G_{(X,Z)}$. To finish the proof of our theorem it is sufficient to show that it is an isomorphism in DM.

Consider the special case $X = \mathbf{P}^n$, $Z = \{x\}$ where x is an S-point of $\mathbf{P}^N$. In this special case the diagram (1) has the following form

$$\begin{tikzcd}
\tilde{M}(\mathbf{P}^{n-1}) \arrow[r] \arrow[d] & \tilde{M}(\mathbf{P}^n) \arrow[d] \\
M(\mathbf{P}^n_{\{x\}}) \arrow[r, "\tilde{i}_0, \tilde{i}_1"] \arrow[d] & M\big((\mathbf{P}^n \times \mathbf{A}^1)_{\{x\}\times\{0\}}\big) \arrow[d] \\
M(\mathbf{P}^n) \arrow[r, "M(i_0), M(i_1)"] & M(\mathbf{P}^n \times \mathbf{A}^1).
\end{tikzcd}$$

By Theorem 4.3.4 we have

$$M\left(\mathbf{P}^n_{\{x\}}\right) \cong \left(\bigoplus_{i=0}^{n} \mathbf{Z}(i)[2i]\right) \oplus \left(\bigoplus_{j=1}^{n-1} \mathbf{Z}(j)[2j]\right) \tag{2}$$

$$M\left((\mathbf{P}^n \times \mathbf{A}^1)_{\{x\}\times\{0\}}\right) \cong \left(\bigoplus_{i=0}^{n} \mathbf{Z}(i)[2i]\right) \oplus \left(\bigoplus_{j=1}^{n} \mathbf{Z}(j)[2j]\right). \tag{3}$$

Let us describe these isomorphisms explicitly. Denote by

$$a, b \in H^1\left((\mathbf{P}^n \times \mathbf{A}^1)_{\{x\}\times\{0\}}, \mathbf{G}_m\right)$$

the classes which correspond to the divisor $p^{-1}_{\{x\}\times\{0\}}(\mathbf{P}^{n-1} \times \mathbf{A}^1)$ and the special divisor respectively. It is easy to see that the isomorphism (3) is of the form $\oplus_{i=0}^n a^i \oplus \oplus_{j=1}^n b^j$. Similarly, if we denote by $a_0, b_0 \in H^1(\mathbf{P}^n_{\{x\}}, \mathbf{G}_m)$ the elements which correspond to $p^{-1}_{\{x\}}(\mathbf{P}^{n-1})$ and the special divisor respectively, the isomorphism (2) can be written as $(\oplus_{i=0}^n a_0^i) \oplus (\oplus_{j=1}^{n-1} b_0^j)$.

One obviously has

$$\tilde{i}_0 a = \tilde{i}_1 a = a_0$$

$$\tilde{i}_1 b = 0, \quad \tilde{i}_0 b = b_0$$

which implies that with respect to the isomorphisms above the morphism $h_{\mathbf{P}^n,\{x\}}$ has the form $h_{\mathbf{P}^n,\{x\}} = b_0^n$. To prove that it is an isomorphism it is sufficient to show that $b_0^n = a_0^n$. Since $a_0 b_0 = 0$ it is equivalent to the equality $(a_0 - b_0)^n = 0$.

Consider the morphism $q : \mathbf{P}^n_{\{x\}} \longrightarrow \mathbf{P}^{n-1}$ which corresponds to the projection from the point x to $\mathbf{P}^{n-1}$. Let $c \in H^1(\mathbf{P}^{n-1}, \mathbf{G}_m)$ be the class of a hyperplane. One can easily see that

$$q^*(c) = a_0 - b_0$$

which implies our result, since c^n is obviously zero.

To prove the theorem in the general case one should use exactly the same localization technique as in the proof of Theorem 4.3.1.

Theorem is proved.

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