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Introduction

0.1. Let V be a complex vector space. A linear operator $R: V \otimes V \to V \otimes V$ is called a Yang-Baxter operator, if in $\mathrm{End}(V \otimes V \otimes V)$ the equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

holds, where, e.g., $R_{12} = R \otimes 1_V$, etc. This equation is called the Yang-Baxter (or triangle) equation and has many interesting applications in mathematics [D], [Ji], and [RTF].

One expects that the study of higher-dimensional generalizations of the Yang-Baxter equation will eventually lead to a construction of higher-dimensional classical and quantum integrable systems. One such generalization is known and called the *Zamolodchikov tetrahedra equation* [Z1] and [Z2]. This equation can be stated in several forms. In one setting (see 1.5) the unknown of the Zamolodchikov equation is an endomorphism of the triple tensor product $V \otimes V \otimes V$. Still more general *simplex equations* have been defined in [MS1].

The aim of this paper is to give an overview of Zamolodchikov equations and to develop a conceptual framework underlying them. A more or less exhaustive treatment would require a separate book (for reasons to be discussed). Such a book is now being prepared by the authors [KV2]. This chapter, though self-contained, can be regarded as a "digest" of this future book.

0.2. The conceptual framework for Zamolodchikov equations is, in our opinion, provided by the theory of 2-categories [Be], [G], [E], and [KS]. (For Yang-Baxter equations the corresponding conceptual clarification is obtained by means of the "usual" category theory.)

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Informally, 2-categories differ from usual (or 1-) categories in that they posses more structure, namely morphisms between usual morphisms (arrows). These new morphisms are called 2-morphisms and can be thought of as 2-cells or homotopies.

The most important feature of 2-categories is that an algebraic expression in them has the "shape" of a subdivided plane polygon (see 2.13 for more information), whereas the only feasible algebraic expression in a usual category is the composition of a string

$$\bullet \to \bullet \to \bullet \cdots \to \bullet$$

of n consecutive morphisms which has the shape of a subdivided segment. The commutative diagrams in 2-categories have therefore the shape of boundaries of convex polytopes in $\mathbf{R}^3$, etc. Thus 2-categories can be seen as belonging to the realm of a new mathematical discipline which may be called *two-dimensional algebra* and contrasted with the usual one-dimensional algebra dealing with formulas written in lines. This point of view was, to our knowledge first advocated by Brown [**Br**]. A natural generalization of 2-categories— n-categories [**Br**] and [S]—give a way to develop n-dimensional algebra.

The two-dimensional algebra of 2-categories matches, in a nontrivial way, the "spatial complexity" of the Zamolodchikov equations (it is not easy nor illuminating to even write these equations without using some pictorial notation). This is why we believe that the language of 2-categories is useful for the problem at hand.

0.3. We develop a theory of monoidal structures in 2-categories and of braidings (or quasi-symmetries) of these monoidal structures. The axiomatics of braidings involves convex polytopes which we call *resultohedra.* They first appeared in the paper of Gelfand, Kapranov, and Zelevinsky [**GKZ**] as Newton polytopes of the classical resultant of two polynomials.

The relevance of n-categories to n-dimensional generalization of Yang-Baxter equations was suggested by Manin and Schechtman in [**MS2**], devoted to structures motivated by these equations. They also considered the special convex polytope which we call a *permutohedron* (the convex hull of a generic orbit of the symmetric group S_n in $\mathbf{R}^n$).

Our interpretation of Zamolodchikov tetrahedron equation is the commutativity of a three-dimensional diagram in a 2-category having the shape of a permutohedron (§6). (The two-dimensional permutohedron is just the hexagon often used to express the Yang-Baxter equation.) Our main results, Theorems 6.11 and 6.14, say that any braided monoidal 2-category gives rise to a solution (in fact, many solutions) of the (abstract version of) the Zamolodchikov equations, and, conversely, every solution gives rise to a braided monoidal 2-category.

The usual matrix formulation of the Zamolodchikov equation is obtained by choosing a special example of a monoidal 2-category, namely 2-*vector* *spaces*, studied in §5. The idea of 2-vector spaces was induced by the notion of modular functor in conformal field theory [Se] and [**Mo-S**]. They are a convenient means for "domestication" of 2-categories. Another idea leading to 2-vector spaces is that of "*vectorization*" of tensor equations (such as the Yang-Baxter equation, the associativity condition for the tensor of structure constants of an algebra, etc.). The vectorization procedure consists of replacing the entries of a tensor with vector spaces instead of numbers and then mimicking tensor operations by replacing $+$ with $\oplus$ and $\times$ with $\otimes$. Both sides of a tensor equation then become collections of vector spaces, and the condition of coincidence of entries is replaced by isomorphisms of the corresponding vector space entries. These isomorphisms are unknowns of the new, "vectorized" equation. The new equation itself is obtained as a natural compatibility condition for the isomorphisms in question. Thus we show that tetrahedra equations can be obtained as the vectorization of Yang-Baxter equations (6.9).

As one of the applications of our categorical approach to the "usual" Zamolodchikov equation, let us mention Corollary 6.15, which gives for any solution $S \in \mathrm{End}(V \otimes V \otimes V)$ of this equation its "tensor power" S_m which acts in the triple tensor product of $V^{\otimes(m^2)}$ and also satisfies the equation.

0.4. The outline of the paper is as follows. Section 1 is a detailed overview of various versions of Zamolodchikov equations, their interrelations, and known examples of solutions. One reason for doing this is that different authors often mean by tetrahedra (or simplex) equations different things [**Z1**], [**Z2**], [**MS1**], and [**FM**]. We also give in 1.12 an overview of solutions to Zamolodchikov equations.

Preparatory §§2 and 3 present mostly known material on monoidal categories, 2-categories, and abstract Yang-Baxter equations. This material will provide the "1-skeleton" of the theory of §§4–6, so in a sense everything emerges very naturally out of careful rethinking of these well-known concepts. To systematize writing various compatibilities, we introduce certain "hierogliphical" notation (see 2.1) whose advantage increases with the complexity of situation.

Section 4 is devoted to the notion of a monoidal 2-category. In §5 we study the monoidal 2-category of 2-vector spaces. In §6 we give the interpretation of Zamolodchikov equations and prove our main results.

0.5. In order to understand the nature of complications and possibilities offered by 2- and n-categories it is very useful to keep in mind the following intuitive principle.

MAIN PRINCIPLE OF CATEGORY THEORY. *In any category it is unnatural and undesirable to speak about equality of two objects.*

This principle, though rarely formulated explicitly, was the driving force for all studies of coherence problems in categories, starting from the work of Mac Lane [**Macl**]. It can be derived from work with categories most commonly arising in practice, such as the category of all sets or groups, etc. Indeed, what does it mean that two sets (or, say, two topological spaces) are equal? By definition, we can speak about equality of two elements in a given set, but the notion "set of all sets" does not make sense! So the notion of equality of two sets is meaningless.

On the contrary, it is quite legitimate to say that two objects of a category are isomorphic, and we can be so pedantic as to want to specify an isomorphism. For example, for a finite-dimensional vector space V the double dual V^{**} is not equal to V but canonically isomorphic to it.

In a 2-category 1-morphisms between two objects form a usual category. Thus the Main Principle implies that it is unnatural to speak of equality of 1-morphisms. In particular, the associativity of composition should be replaced by canonical 2-morphism (homotopy) between $a(bc)$ and $(ab)c$ (which is also a part of the structure). Such a notion of a lax 2-category was introduced by Benabou [**Be**] and encompasses the usual notion of a monoidal category. To define a similar ("lax") notion of n-category is a daunting task proposed by Grothendieck [**Gro**]. It seems that the appropriately developed formalism of hierogliphs (see 2.1) will be helpful in attacking this problem.

Thus lax n-categories (in situations when it is possible to define them) have a lot of structure which is not all apparent at first glance. This is the origin of many difficulties in the subject and the necessary length of every "honest" exposition.

On the other hand, it is possible to develop a theory of *strict* n-categories in which all necessary relations hold exactly, not up to some connecting polymorphisms. This has been done in various versions [**S**] and [**Br**]. It is possible also to develop the theory of n-dimensional algebraic expressions in such n-categories [**J**] and [**P**]. In some instances the consideration of strict n-categories may be justified by a suitable coherence theorem.

We also follow this trend, but only partly. In fact, in our interpretation of Zamolodchikov equations and in the definition of a 2-braiding it is the laxness which gives rise to the most meaningful data.

0.6. For several reasons we do not go beyond (monoidal) 2-categories and tetrahedra equations. First is the absence of a satisfactory theory of lax n-categories. Second is the possibility of actually drawing pictures of 3-dimensional polytopes, which is lost in higher dimensions. Third is the absence, at the moment, of interesting solutions of higher simplex equations.

We also have tried to reduce to a minimum the parts related to "pure" theory of 2-categories. Thus we refrained from defining (monoidal) 2-functors, etc. Such preparations, however, are necessary for coherence theorems (that any "lax" structure is equivalent, in a suitable sense, to a strict one), and they will be carried out in detail in [**KV2**]. A more interesting development omitted with the "abstract nonsense" is the interpretations of Majid's construction of duals and quantum doubles for monoidal categories **[Maj1]**. The quantum double construction makes a braided monoidal category $D(\mathscr{A})$ from any monoidal category $\mathscr{A}$. From the polycategorical point of view, a monoidal category is a 2-category with one object and a braided monoidal category—a 3-category with one object and one 1-morphism (see 2.10, 4.2). Majid's construction turns out to be a particular case of a general principle that the collection of all n-categories forms an $(n+1)$-category. This leads to certain generalizations of quantum double which will also be studied in **[KV2]**.

0.7. An approach to higher-dimensional algebra different from that of n-categories is being developed by Lawrence [Law]. Her approach is based on a direct axiomatization of the structure existing on the vector space of n-valent tensors (whereas in our theory tensors surface on the very last level as 2-morphisms in the 2-category of 2-vector spaces). It also leads to polytopal pictures and is related to tetrahedra equations.

We have been recently informed by L. Breen that he has some time ago already suggested a definition of a braided monoidal 2-category (in an unpublished letter to P. Deligne). His definition is rather close to ours, although it is restricted to the case of a strict monoidal structure (see §4) on the underlying 2-category and so cannot be directly applied to Zamolodchikov equations. A still more restrictive version of the notion of a braided monoidal 2-category was considered in [JS2].

0.8. We are grateful to Shahn Majid who turned our attention to the fact that triangular diagrams for usual braidings are more fundamental than the Yang-Baxter hexagon and inquired about higher-dimensional generalizations of this fact, thus giving an initial impetus to this work. We would also like to thank I. Frenkel, D. Kazhdan, S. Mac Lane, V. Schechtman and J. Stasheff for discussions of the work at various stages. To Professor Mac Lane our special thanks are due for suggesting numerous improvements and corrections to the previous version of this paper. We also are indebted to J. Fisher for pointing out several misprints and grammatical mistakes.

1. Yang-Baxter and Zamolodchikov equations

1.1. The constant Yang-Baxter equation. Let V be a complex vector space of finite dimension. The Yang-Baxter (or triangle) equations (see **[Ji]** for general information) are equations on a linear operator $R\colon V\otimes V \to V\otimes V$. Fixing a basis $e_1, \dots, e_n$ in V, we identify R with its set of matrix elements R_{ij}^{kl} given by

$$R(e_i\otimes e_j) = \sum_{k,l} R_{ij}^{kl} e_k \otimes e_l.$$

The equations themselves have the form

$$R_{(12)}R_{(13)}R_{(23)} = R_{(23)}R_{(13)}R_{(12)}. \tag{1-1}$$

Here $R_{(ab)}$ is the operator in $V \otimes V \otimes V$ acting as R on the ath and bth factors and by unity on the remaining factor. For example,

$$R_{(13)}(e_i \otimes e_j \otimes e_k) = \sum_{p,q} R_{ij}^{pq} e_p \otimes e_j \otimes e_q. \tag{1-2}$$

In terms of the matrix elements R_{ij}^{kl} the equations have the form

$$\sum_{i'',j'',k''} R_{jk}^{j''k''} R_{ik''}^{i''k'} R_{i''j''}^{i'j'} = \sum_{i'',j'',k''} R_{ij}^{i''j''} R_{i''k}^{i'k''} R_{j''k''}^{j'k'} \qquad \forall i, j, k, i', j', k'. \tag{1-3}$$

Let $P\colon V \otimes V \to V \otimes V$ be the permutation. Equation (1) is equivalent to the following equation for the operator $\widehat{R} = PR$:

$$\widehat{R}_{(12)}\widehat{R}_{(23)}\widehat{R}_{(12)} = \widehat{R}_{(23)}\widehat{R}_{(12)}\widehat{R}_{(23)}, \tag{1-4}$$

which is sometimes also called the Yang-Baxter equation.

EXAMPLE. The constant Yang-Baxter matrix corresponding to the quantum group $GL(n)$ is the endomorphism of $\mathbf{C}^n \otimes \mathbf{C}^n$ given by

$$R = q\sum_i E_{ii} \otimes E_{ii} + \sum_{i\neq j} E_{ii} \otimes E_{jj} + (q - q^{-1})\sum_{j>i} E_{ij} \otimes E_{ji}. \tag{1-5}$$

Here $E_{ij} \in \mathrm{End}(\mathbf{C}^n)$ are matrix units and $q \neq 0$ is a complex number. See [Maj2], [RTF], and [T] for more details.

1.2. Geometric interpretation. The name "triangle equations" comes from the following geometric interpretation via statistical models (see, e.g., [T]). Let us consider *unordered* configurations $\Gamma = (l_1, \dots, l_r)$ of oriented lines in the Euclidean plane $\mathbf{R}^2$. We restrict ourselves to configurations of general positions, i.e., such that no three lines have a common point and no two lines are parallel. Each line is divided by the points of intersection with other lines into several *segments* of which two are unbounded and others are bounded. The points of intersection of the lines will be called the *vertices* of the configuration, see Figure 1.

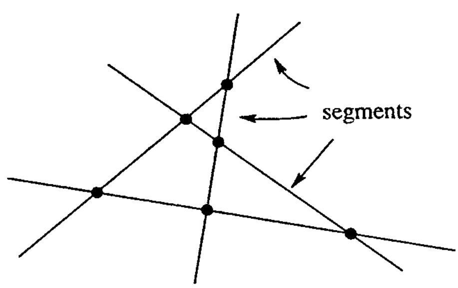


Figure 1

Suppose also that we have a finite set of "colors" which we can think of as numbers $1, 2, \dots, n$. We shall study various coloring of segments in Γ.

More precisely, by a *state* of Γ we mean an assignment of colors to all the segments of Γ, i.e., a map

$$\{\text{segments of } \Gamma\} \to \{1, \dots, n\}.$$

By an *infinite state* we shall mean the coloring of only the unbounded segments. Suppose now that we have a linear operator $R\colon \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$ or, what is the same, a collection of numbers R_{ij}^{kl}, $i, j, k, l = 1, \dots, n$. Let us associate to any state σ a number $E(\sigma)$ which might be loosely called its *energy* (or, rather, Boltzmann weight, i.e., the exponent of the energy) by the following rules:

(1) $E(\sigma) = \prod_v E_v(\sigma)$ where v runs over all the vertices of Γ.
(2) Let v be a vertex of Γ so that $v = l_a \cap l_b$. Then there are four segments adjacent to v, two of them "incoming" and two "outgoing" (see Figure 2).

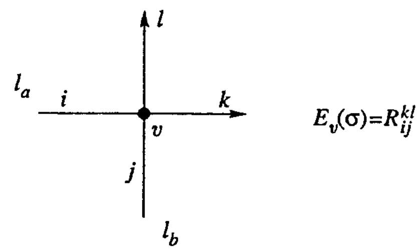


Figure 2

We order the two oriented lines l_a, l_b such that the orientation given by them coincides with the chosen orientation of $\mathbf{R}^2$. Suppose that the colors on the incoming segments are i and j (in the said order) and on the outcoming $-k$ and l (with the same convention about the orders). Then we define $E_v(\sigma) = R_{ij}^{kl}$.

By the general principle of statistical mechanics the probability with which each state σ occurs in the actual behavior of the statistical system equals the contribution of σ into the *partition function*

$$P(\Gamma) = \sum_{\sigma} E(\sigma).$$

We introduce also the conditional partition function corresponding to a given infinite state σ_∞:

$$P(\Gamma, \sigma_\infty) = \sum_{\sigma\ :\ \sigma=\sigma_\infty \text{ at } \infty} E(\sigma).$$

Now the Yang-Baxter equation for $\|R_{ij}^{kl}\|$ can be interpreted as the equality of the following two conditional partition functions depicted in Figure 3 (see next page).

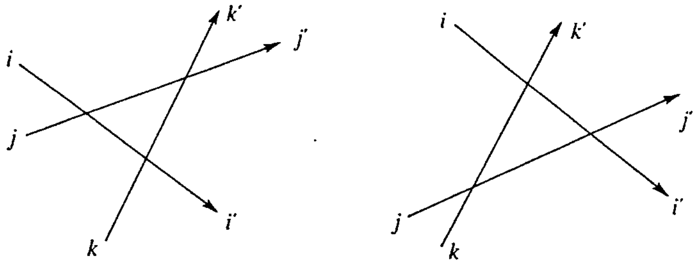


Figure 3

In words, we move one of the lines constituting the triangle in the left-hand side across the point of intersection of two other lines, obtaining the picture on the right. (Note that it does not matter which line is moved—the resulting picture will remain the same!) This movement identifies the sets of infinite states for the old and new configurations, and the conditional partition functions with respect to these two states are required to coincide.

If R satisfies the Yang-Baxter equations, then we can deduce the coincidence of more complicated partition functions. Namely, let Γ be an arbitrary configuration of lines of general position and σ_∞ its infinite state. Consider another configuration Γ' obtained from Γ by moving each line arbitrarily parallel to itself with the only restriction being that Γ' should also be in general position. Then infinite segments of Γ and Γ', and hence the sets of infinite states, are in natural bijection. Let σ'_∞ be the infinite state of Γ' corresponding to σ.

1.3. Proposition. *If R satisfies the Yang-Baxter equations then* $P(\Gamma, \sigma_\infty) = P(\Gamma', \sigma'_\infty)$.

Proof. We can always restrict ourselves to the case when all lines but one are unchanged. Let l be the line which moves. During the process of moving, the topology of the configuration will change when l passes through intersection points of other lines. So it suffices to prove that our partition function remains unchanged after passing such a point, say $l_a \cap l_b$. But this is done by applying the triangle equations to the small triangle formed by l, l_a, l_b.

1.4. Yang-Baxter equations with parameters. Equation (1-1) is known as the *constant* Yang-Baxter equation. Equally important is the setting where the entries of the R-matrix are assumed to be functions of some auxiliary parameters. More precisely, we choose an arbitrary set X of parameters (usually $\mathbf{R}$ or $\mathbf{C}$) and consider the unknown of the equation to be a function $R(x_1, x_2)$ of two variables $x_1, x_2 \in X$ with values in $V \otimes V$. The equation itself has the form

$$\begin{aligned} R_{(12)}(x_1, x_2)R_{(13)}(x_1, x_3)R_{(23)}(x_2, x_3) \\ = R_{(23)}(x_2, x_3)R_{(13)}(x_1, x_3)R_{(12)}(x_1, x_2). \end{aligned} \tag{1-6}$$

A still more general setting is that of a Yang-Baxter system in a monoidal category (see §3).

In most examples, as we have stated, the parameter set X coincides with $\mathbf{R}$ or $\mathbf{C}$, and x_i are thought of as slopes of lines in the geometric interpretation above. The interpretation via statistical models remains the same for variable Yang-Baxter equations, except the local Boltzmann weight $E_v(\sigma)$ at a vertex v depends on the slopes of two lines intersecting at v.

In fact, very often the specialization of the setup goes one step further in that not only the parameters x_i are supposed to be real or complex numbers, but it is assumed that $R(x_1, x_2)$ depends only on one variable $u = x_1 - x_2$. The equation in this case has the familiar form

$$R_{(12)}(u)R_{(13)}(v)R_{(23)}(v-u) = R_{(23)}(v-u)R_{(13)}(v)R_{(12)}(u). \tag{1-7}$$

As an example let us mention the classical Yang solution

$$R(z_1, z_2) = (z_1 - z_2)I + \eta P \tag{1-8}$$

where I is the identity operator of $V \otimes V$ and P is the permutation.

1.5. Statistics on a configuration of planes in $\mathbf{R}^3$. Consider an *ordered* configuration $\Gamma = (H_1, \dots, H_r)$ of planes in the Euclidean 3-space $\mathbf{R}^3$ (note that for lines in the plane we did not use the ordering). We assume that Γ is in general position, i.e., that the intersection of any i planes, $i = 2, 3, 4$, has codimension exactly i (codimension 4 is achieved by only the empty subspace). The planes of configuration are subdivided into *vertices* (points where exactly three planes meet), *segments* (where exactly two planes meet), and *faces*. The space $\mathbf{R}^3$ is subdivided by the configuration to three-dimensional parts called *regions*. By an *orientation* of Γ we will mean a collection of orientations (i.e., choices of directions) of all the lines of intersection $H_i \cap H_j$. Since planes are numbered, lines acquire the natural lexicographic order: $H_i \cap H_j$ precedes $H_k \cap H_l$ if $i < k$ or $i = k$ and $j < l$.

There are several possibilities of defining a statistical model out of an oriented ordered configuration Γ. For example, one can color faces, or segments, or regions. The basic choice for us will be to color the segments (see Figure 4, next page).

Thus, similarly to the above, a state (resp. infinite state) is a map from the set of all segments (resp. only infinite segments) of Γ to $\{1, \dots, n\}$. To every vertex v of configuration Γ, six segments are adjacent, three of them incoming and three outgoing with respect to our chosen orientation of lines.

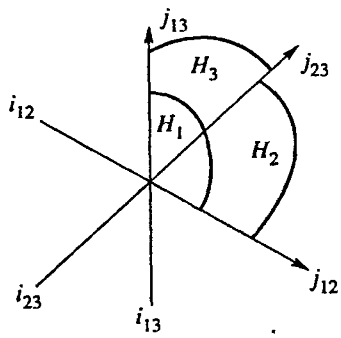


Figure 4

Thus we set the datum (or the set of unknowns) for would-be tetrahedra equations to be a 6-valent tensor $S_{i_1 i_2 i_3}^{j_1 j_2 j_3}$, $i_\nu, j_\nu = 1, \dots, n$, with three upper and three lower indices. In more invariant form this is a linear operator

$$S: V \otimes V \otimes V \to V \otimes V \otimes V$$

in the triple product of the vector space $V = \mathbf{C}^n$. Since we suppose hyperplanes to be numbered, it is convenient to number lines of intersection of hyperplanes by pairs of indices i.e., label $H_i \cap H_j$ by the pair (i, j). Thus it will be more natural to denote indices of S not by i_1, i_2, i_3 and j_1, j_2, j_3 but by i_{12}, i_{13}, i_{23} so that index i_{ab} is being written on the incoming segment of the line $H_a \cap H_b$ in the configuration $\{H_1, H_2, H_3\}$, in Figure 4.

Let $v = H_a \cap H_b \cap H_c, a < b < c$, be a vertex. Suppose we have a state σ. Let the colors given by σ on the incoming (to v) segments of lines $H_a \cap H_b$, $H_a \cap H_c$, $H_b \cap H_c$ be respectively i_{ab}, i_{ac}, i_{bc} and the colors on the outgoing segments of these lines be j_{ab}, j_{ac}, j_{bc} respectively. We define the contribution of the vertex v to the Boltzmann weight to be

$$E_v(\sigma) = S_{i_{ab}, i_{ac}, i_{bc}}^{j_{ab}, j_{ac}, j_{bc}}$$

and the Boltzmann weight itself to be $E(\sigma) = \prod_v E_v(\sigma)$.

Similarly to the case of plane configurations we define the conditional partition function $P(\Gamma, \sigma_\infty)$ with respect to a given infinite state (= coloring of unbounded segments of Γ).

1.6. Zamolodchikov equation (colors on segments). The Zamolodchikov (or tetrahedra) equation is most simply stated in the geometric form in Figure 5.

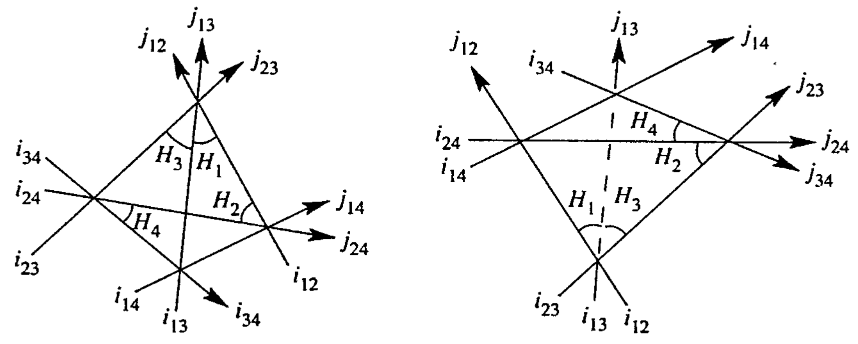


Figure 5

Here on the left we have a configuration of four planes $(H_1, \ldots, H_4)$ (tetrahedron) which has four vertices $v_1 = H_2 \cap H_3 \cap H_4$, $v_2 = H_1 \cap H_3 \cap H_4$, etc. Lines are oriented to be directed from v_i to v_j where $i < j$. On the right we have the configuration obtained by moving one plane across the intersection point of three others (as in the case of triangles, it makes no difference which of the four planes we move). Lines of the right configuration are in natural 1-1 correspondence with lines of the left one, and we orient them by this correspondence. The sets of unbounded segments of both configurations are also naturally identified (by considering the "directions" of segments), and thus we can compare the conditional partition functions. The equation means the coincidence of these functions.

In terms of the matrix elements $S^{j_{12}j_{13}j_{23}}_{i_{12}i_{13}i_{23}}$ the equations are

$$
\begin{aligned}
&\sum_{k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}} S^{k_{23}, k_{24}, k_{34}}_{i_{23}, i_{24}, i_{34}} S^{j_{12}, k_{23}, k_{24}}_{k_{12}, i_{23}, i_{24}} S^{j_{12}, j_{14}, k_{24}}_{k_{12}, k_{14}, i_{24}} S^{j_{12}, j_{13}, j_{23}}_{k_{12}, k_{13}, i_{23}} \\
&= \sum_{k_{12}, k_{13}, k_{14}, k_{23}, k_{24}, k_{34}} S^{k_{12}, k_{13}, k_{23}}_{i_{12}, i_{13}, i_{23}} S^{j_{12}, k_{14}, k_{24}}_{k_{12}, i_{14}, i_{24}} S^{j_{13}, j_{14}, k_{34}}_{k_{13}, k_{14}, i_{34}} S^{j_{23}, j_{24}, j_{34}}_{k_{23}, k_{24}, k_{34}}.
\end{aligned}
\tag{1-9}
$$

This equality should hold for any i_{ab}, j_{ab}, $1 \leq a < b \leq 4$.

1.7. Operator formulation. In both sides of (1-9) we have tensors with six upper and six lower indices. Thus to give an operator interpretation of the equations we should consider operators in the 6-tuple tensor product $V^{\otimes 6}$. To obtain a transparent formula, it is convenient to label factors in this sextuple tensor product by pairs (a, b), $1 \leq a < b \leq 4$. We shall denote the set of such pairs by $\binom{4}{2}$ and our sextuple tensor product by $V^{\otimes\binom{4}{2}}$ to emphasize labelling of factors. Introduce on $\binom{4}{2}$ the lexicographical order $(12) < (13) < (14) < (23) < (24) < (34)$.

We have already used notation such as R_{13} for an operator $R \in \mathrm{End}(V \otimes V)$ which acts on the first and third factors of $V \otimes V \otimes V$ (formula (1-2)). In the same spirit, given an operator $S \in \mathrm{End}(V \otimes V \otimes V)$, for any triple of pairs $((a_1, b_1) < (a_2, b_2) < (a_3, b_3))$ we denote by $S_{(a_1, b_1), (a_2, b_2), (a_3, b_3)}$

the operator in $V^{\otimes\binom{4}{2}}$ which acts on factors with labels (a_i, b_i) as S and on all other factors by unity, for example $S_{(12),(14),(24)}$. To simplify the formulas we introduce the following notation for some 3-element subsets of $\binom{4}{2}$:

$$\hat{1} = \{(23), (24), (34)\}, \qquad \hat{2} = \{(13), (14), (34)\},$$
$$\hat{3} = \{(12), (14), (24)\}, \qquad \hat{4} = \{(12), (13), (23)\}.$$

In other words, $\hat{i}$ is the set of all pairs not containing i. We consider each of these triple of pairs to be equipped with the lexicographic order (in which they are written above). Correspondingly, introduce the notation $S_{\hat{i}}$ for the operator in $V^{\otimes\binom{4}{2}}$ acting on factors with labels from $\hat{i}$ as S and on all other factors as unity. For example, $S_{\hat{2}} = S_{(13),(14),(34)}$, etc.

In this notation the Zamolodchikov equation can be written as the following equality between operators in $V^{\otimes\binom{4}{2}}$:

$$S_{\hat{1}}S_{\hat{2}}S_{\hat{3}}S_{\hat{4}} = S_{\hat{4}}S_{\hat{3}}S_{\hat{2}}S_{\hat{1}}. \tag{1-10}$$

1.8. Equation with parameters. As in the case of Yang-Baxter equations, the above setup for Zamolodchikov equations can be generalized to include a dependence of the tensor $S \in \operatorname{End}(V \otimes V \otimes V)$ on some parameters. As with the placement of colors, the parameters can be associated with vertices, segments, etc. We choose the basic setup to be where parameters are associated with planes. Thus we assume S depends on parameters $z_1, z_2, z_3 \in X$, where z_i is associated with the ith hyperplane in the simplest configuration (H_1, H_2, H_3) of three planes.

The physically interesting choice is $X = \mathbf{R}P^2$, the real projective plane (the set of "directions" of affine hyperplanes in $\mathbf{R}^3$). In this case it is also natural to require that $S(L_1, L_2, L_3)$ remain unchanged under simultaneous rotation of L_i. This is tantamount to the requirement that S depend only on the three plane angles θ_{ij} between planes L_i (cf. [Z1] and [Z2]).

The variable tetrahedra equation can be obtained from (1-9) by considering six parameters z_{ij}, $1 \le i < j \le 4$ and letting each factor S (corresponding to a vertex of the tetrahedron) depend on the three variables associated to planes meeting at this vertex, i.e.,

$$\begin{aligned} &S_{\hat{1}}(z_2, z_3, z_4)S_{\hat{2}}(z_1, z_3, z_4)S_{\hat{3}}(z_1, z_2, z_4)S_{\hat{4}}(z_1, z_2, z_3) \\ &\quad = S_{\hat{4}}(z_1, z_2, z_3)S_{\hat{3}}(z_1, z_2, z_4)S_{\hat{2}}(z_1, z_3, z_4)S_{\hat{1}}(z_2, z_3, z_4). \end{aligned} \tag{1-11}$$

1.9. Zamolodchikov equations (colors on faces). Let us now discuss other options of coloring parts of configurations. By coloring 2-faces we obtain that a collection of Boltzmann weights at a vertex will be a tensor with 12 indices, since three planes intersecting in a point have 12 faces, 4 in each plane. Thus the set of unknowns is a tensor

$$\Sigma \begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \end{pmatrix},$$

where the first index (1, 2, or 3) refers to the number of a plane and the second to the number of a quadrant in the plane. We always number quadrants uniformly. More precisely, let H_a, H_b, H_c, $a < b < c$, be three planes of configuration. Since we assume lines of intersection to be oriented, in each of the three planes two other planes cut a pair of oriented lines which are ordered (lexicographically). Then the numbering is as follows in Figure 6.

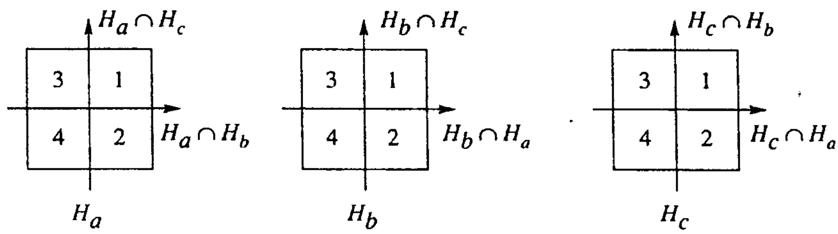


Figure 6

This is exactly the setup considered by Zamolodchikov **[Z1]** and **[Z2]**. To write the tetrahedron equation for Σ, we introduce a labelling of the plane regions cut out in $\mathbf{R}^2$ by three ordered lines l_1, l_2, l_3. This labelling is depicted in Figure 7.

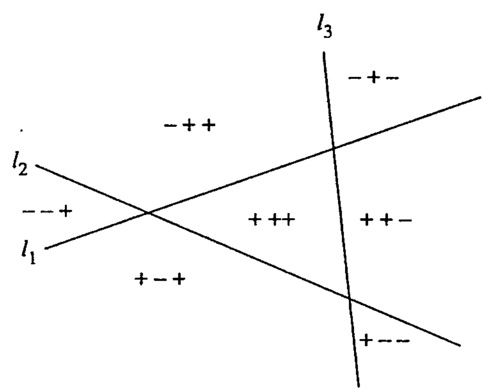


Figure 7

In other words, we choose affine-linear functionals f_i defining l_i in such a way that the value of f_i on the point of intersection of two other lines $l_j \cap l_k$ equals 1 and the sequence of signs associated with a region is given by the values of f_1, f_2, f_3. The sequence $(---)$ does not occur. For a configuration $H_1, \ldots, H_4$ of four planes in $\mathbf{R}^3$ we label the 2-faces of the plane H_i by sequences lie $a+++$, $a++-$, etc., always referring to the lexicographic ordering on the three lines cut out by the other planes on H_a. Correspondingly we denote by, say, i_{1+++} the color associated to the face $1+++$ (the interior triangle in H_1).

The tetrahedron equation for this setup has the form

$$
\begin{aligned}
&\sum_{i_{1+++},\, i_{2+++},\, i_{3+++},\, i_{4+++}} \Sigma \begin{pmatrix} i_{2+++} & i_{2++-} & i_{2+-+} & i_{2+--} \\ i_{3+++} & i_{3++-} & i_{3+-+} & i_{3+--} \\ i_{4+++} & i_{4++-} & i_{4+-+} & i_{4+--} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{1+++} & i_{1++-} & i_{1+-+} & i_{1+--} \\ i_{3-++} & i_{3+++} & i_{3-+-} & i_{3++-} \\ i_{4-++} & i_{4+++} & i_{4-+-} & i_{4++-} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{1-++} & i_{1+++} & i_{-+-} & i_{1++-} \\ i_{2-++} & i_{2+++} & i_{2-+-} & i_{2++-} \\ i_{4--+} & i_{4+-+} & i_{4-++} & i_{4+++} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{1--+} & i_{1+-+} & i_{1-++} & i_{1+++} \\ i_{2--+} & i_{2+-+} & i_{2-++} & i_{2+++} \\ i_{3--+} & i_{3+-+} & i_{3-++} & i_{3+++} \end{pmatrix} \\
&= \sum_{i_{1+++},\, i_{2+++},\, i_{3+++},\, i_{4+++}} \Sigma \begin{pmatrix} i_{1+++} & i_{1-++} & i_{1+-+} & i_{1--+} \\ i_{2+++} & i_{2-++} & i_{2+-+} & i_{2--+} \\ i_{3+++} & i_{3-++} & i_{3+-+} & i_{3--+} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{i++-} & i_{1-+-} & i_{1+++} & i_{1-++} \\ i_{2++-} & i_{2-+-} & i_{2+++} & i_{2-++} \\ i_{4+++} & i_{4-++} & i_{4+-+} & i_{4--+} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{1+--} & i_{1++-} & i_{1+-+} & i_{1+++} \\ i_{3++-} & i_{3-+-} & i_{3+++} & i_{3-++} \\ i_{4++-} & i_{4-+-} & i_{4+++} & i_{4-++} \end{pmatrix} \\
&\quad \times \Sigma \begin{pmatrix} i_{2+--} & i_{2++-} & i_{2+-+} & i_{2+++} \\ i_{3+--} & i_{3++-} & i_{3+-+} & i_{3+++} \\ i_{4+--} & i_{4++-} & i_{4+-+} & i_{4+++} \end{pmatrix}.
\end{aligned}
\tag{1-12}
$$

1.10. Zamolodchikov equation (colors in regions). In a similar fashion we can describe the version of tetrahedra equation where colors are associated with three-dimensional regions cut out by a configuration. Three planes H_1, H_2, H_3 cut the space into eight octants. Since we assume planes numbered and the lines of their intersection oriented, we can label each octant unambigiously by a sequence of three signs $(+)$ or $(-)$. Namely, let K be any octant. We set the first of the three signs associated with K as $(+)$ if K lies on the same side of H_1 as the direction of line $H_2 \cap H_3$, and $(-)$ otherwise. Similarly for the second and third components, see Figure 8.

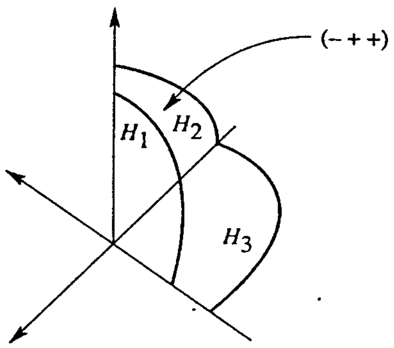


Figure 8

Thus the unknown in the region-colored Zamolodchikov equation is a collection of n^8 numbers:

$$W(i_{+++}, i_{++-}, i_{+-+}, i_{+--}, i_{-++}, i_{-+-}, i_{--+}, i_{---}).$$

We always consider the combinations of $(+)$ and $(-)$ in the order specified above. To write the equation itself we label the regions of the complement to four planes $H_1, \dots, H_4$ by sequences of four signs $(+)$ or $(-)$. We choose an affine functional f_a vanishing on H_a and equal to 1 at the intersection of three other planes and associate with a region K the sequence of signs of f_1, f_2, f_3, f_4 on K. Correspondingly we denote by, say, i_{++-+} the index (color) associated with the region $(++-+)$. In this notation the tetrahedron equation has the form

(1-13)

$$\begin{aligned}
&\sum_{i_{++++}} W(i_{++++}, i_{+++-}, i_{++-+}, i_{++--}, i_{+-++}, i_{+-+-}, i_{+--+}, i_{+---}) \\
&\qquad \times W(i_{-+++}, i_{-++-}, i_{-+-+}, i_{-+--}, i_{-+++}, i_{+++-}, i_{++-+}, i_{++--}) \\
&\qquad \times W(i_{--++}, i_{--+-}, i_{-+++}, i_{-++-}, i_{+-++}, i_{+-+-}, i_{++++}, i_{+++-}) \\
&\qquad \times W(i_{---+}, i_{--++}, i_{-+-+}, i_{-+++}, i_{+--+}, i_{+-++}, i_{+-++}, i_{++-+}, i_{++++}) \\
&\quad = \sum_{i_{++++}} W(i_{++++}, i_{++-+}, i_{+-++}, i_{+--+}, i_{-+++}, i_{-+-+}, i_{--++}, i_{---+}) \\
&\qquad \times W(i_{+++-}, i_{++++}, i_{+-+-}, i_{+-++}, i_{-++-}, i_{-+++}, i_{--+-}, i_{--++}) \\
&\qquad \times W(i_{++--}, i_{++-+}, i_{+++-}, i_{++++}, i_{-+--}, i_{-+-+}, i_{-++-}, i_{-+++}) \\
&\qquad \times W(i_{+---}, i_{+--+}, i_{+-+-}, i_{+-++}, i_{++--}, i_{++-+}, i_{+++-}, i_{++++}).
\end{aligned}$$

This equation was called by Baxter [**Bax**] the *interaction round the cube* equation since the decomposition of $\mathbf{R}^3$ dual to the decompostion given by vertices, segments, faces, and regions of four planes can be viewed as consisting of four cubes. In fact, the two groups of four cubes each corresponding to two tetrahedra in the left- and right-hand sides of the equation form together the boundary of a four-dimensional cube.

1.11. Relations between various settings. The scheme of interrelations between three versions of the tetrahedra equations defined above can be expressed as follows:

$$\text{region version} \subset \text{face version} \subset \text{segment version}.$$

More precisely, given a solution of a region-colored equation, we can construct out of it a solution of face-colored and segment-colored equations as follows.

Note that the colors in our statistical model can be elements of an arbitrary set which should not necessarily be identified with $\{1, \dots, n\}$ for some n. Suppose we have a solution

$$\Sigma\begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \end{pmatrix}$$

of face-colored Zamolodchikov equations with the set of colors I. Define a new set of colors $J = I^4 = I \times I \times I \times I$. Elements of this set are just 4-tuples of old colors. Let us associate these new colors with segments in such a way that the (new) color of a segment is equal to the collection of the (old) colors of four faces containing this segment. Consider three planes H_1, H_2, H_3. Define the Boltzmann weight $S^{\beta_1, \beta_2, \beta_3}_{\alpha_1, \alpha_2, \alpha_3}$ for $\alpha_\nu, \beta_\nu \in J$ to be 0 if the combination of new colors on segments does not come from a combination

$$\begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \end{pmatrix}$$

of old colors on faces of (H_1, H_2, H_3). In the case when the new coloring of segment does come from such a coloring we set

$$S^{\beta_1, \beta_2, \beta_3}_{\alpha_1, \alpha_2, \alpha_3} = \Sigma\begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \end{pmatrix}.$$

It is straightforward to see that the face-colored Zamolodchikov equation (1-13) for

$$\Sigma\begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \end{pmatrix}$$

implies the segment-colored equation (1-9) for $S^{\beta_1, \beta_2, \beta_3}_{\alpha_1, \alpha_2, \alpha_3}$,

1.12. Known solutions. (a) Zamolodchikov has constructed [Z1] and [Z2] a highly nontrivial solution of a face-colored equation including parameters which are angles between planes. In fact, as remarked in [Bax] this solution comes from a solution of a region-colored equation (1-13) by using the rewriting procedure described above. Matrix elements of Zamolodchikov's solution are trigonometric functions of plane angles.

(b) Manin and Schechtman constructed [**MS1**] a constant solution of tetrahedra equations with colors in regions. Their construction is as follows. Suppose that the set I of colors forms a finite Abelian group. The operation in I will be denoted $+$. Let $h \in I$ be such that $2h = 0$. Define the matrix element

$$W(i_{+++}, i_{++-}, i_{+-+}, i_{+--}, i_{-++}, i_{-+-}, i_{--+}, i_{---})$$

to be equal to zero unless in the group I we have the equality

$$i_{---} = h + i_{+++} + i_{++-} + i_{+-+} + i_{+--} - i_{-++} - i_{-+-} - i_{--+} - i_{---}.$$

When this equality holds, the matrix element is equal to 1.

(c) It was remarked by one of the authors (M. K.) that an infinite-dimensional solution of the segment-colored equation can be obtained from the representation theory of the ring of functions on the quantum group $GL(n)$ developed by Vaksman and Soibelman [**VSo**], [**So**]. By definition, this ring, denoted $\mathbf{C}[GL(n)_q]$, has generators x_{ij}, $1 \leq i, j \leq n$, subject to relations

$$x_{ij}x_{il} = q \cdot x_{il}x_{ij}, \qquad j < l,$$

$$\begin{aligned} x_{ij}x_{kj} &= q \cdot x_{kj}x_{ij}, && i < k, \\ x_{kj}x_{il} &= x_{il}x_{kj}, && i < k, j < l, \\ [x_{ij}, x_{kl}] &= (q^{-1} - q) \cdot x_{kj}x_{ik}, && i < k, j < l. \end{aligned}$$

This ring is a Hopf algebra under the comultiplication $\Delta(x_{ij}) = \sum x_{ip} \otimes x_{pj}$.

Vaksman and Soibelman have constructed in [**VSo**] an infinite-dimensional representation of $\mathbf{C}[GL(2)_q]$ in the vector space V with basis e_i, $i = 0, 1, 2, \dots$, which can be given by

$$x_{11}e_i = (1 - q^{2i})e_{i-1}, \qquad x_{22}e_i = e_{i+1},$$
$$x_{21}e_i = q^i e_i, \qquad x_{12}e_i = q^{i+1} e_i.$$

There are two ring homomorphisms $\phi_{12}, \phi_{23}\colon \mathbf{C}[GL(3)_q] \to \mathbf{C}[GL(2)_q]$ defined by

$$\phi_{12}(x_{ij}) = \begin{cases} x_{ij} & \text{if } i, j \leq 2, \\ 0 & \text{otherwise}, \end{cases}$$
$$\phi_{23}(x_{ij}) = \begin{cases} x_{i-1,j-1} & \text{if } i, j \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

They correspond to the two embeddings $GL(2) \subset GL(3)$ in the classical case. By means of these homomorphisms one constructs, out of V, two representations V_{12}, V_{23} of $\mathbf{C}[GL(3)_q]$. It is the basic fact of the theory of [**So**] that the two representations $V_{12} \otimes V_{23} \otimes V_{12}$ and $V_{23} \otimes V_{12} \otimes V_{23}$ of $\mathbf{C}[GL(3)_q]$ (with $\otimes$ defined by means of the Hopf algebra structure) are irreducible and isomorphic. Denote by e_i the basis in V_{12} and by f_i the basis in V_{23}. Let

$$S: V_{12}\otimes V_{23}\otimes V_{12}\to V_{23}\otimes V_{12}\otimes V_{23}, \qquad S(e_i\otimes f_j\otimes e_k)=\sum_{a,b,c} S_{ijk}^{abc} f_a\otimes e_b\otimes f_c,$$

be the intertwining operator taking $e_0\otimes f_0\otimes e_0$ to $f_0\otimes e_0\otimes f_0$ (the "highest weight vectors"). Consider the tensor $S\circ P_{13}=\|S_{kji}^{abc}\|$, where P_{13} is the permutation of the first and third factors. Then this tensor satisfies the Zamolodchikov equation (1-9) or (1-10). This follows from the interpretation (via Soibelman theory) of operators in both sides of equation (1-10) as isomorphisms between two irreducible representations of $\mathbf{C}[GL(4)_q]$. An explicit formula for S is as follows:

$$S_{ijk}^{abc}=\sum_{\substack{0\le\lambda\le i\\ 0\le\mu\le j\\ \lambda-\mu=i-b}} q^{(\lambda-\mu)^2+(a-c-i)\lambda+(a+c+i+2j-1)\mu+i(c-j)}\begin{bmatrix} i\\ \lambda\end{bmatrix}_{q^2}\begin{bmatrix} j\\ \mu\end{bmatrix}_{q^2}[a+1,\, a+i-\lambda],$$

where

$$[m,n]=(1-q^{2m})(1-q^{2m+2})+\cdots+(1-q^{2n}),$$

and

$$\begin{bmatrix} N\\ a\end{bmatrix}_{q^2}=\frac{[1,N]}{[1,a][1,N-a]}$$

is the q^2-binomial coefficient. The solution (1-14) acts in an infinite-dimensional vector space V. Detailed study of this solution and the finite-dimensional solution arising from it when q is a root of unity will be carried out in another paper.

(d) Frenkel and Moore have defined [FM] a still different version of the tetrahedra equation. The unknown in the Frenkel-Moore equation is, similarly to the segment-colored version of Zamolodchikov equation, a linear operator $S: V\otimes V\otimes V\to V\otimes V\otimes V$, but the equation has the form of equality of two operators in $V^{\otimes 4}$,

$$S_{123}S_{124}S_{134}S_{234}=S_{234}S_{134}S_{124}S_{123}, \tag{1-15}$$

whereas the Zamolodchikov equation requires equality of two operators in $V^{\otimes 6}$. We do not know how to interpret (1-15) by using a tetrahedron.

Frenkel and Moore have constructed in [FM] a solution of their equation depending on a complex parameter q, similar to (1-5).

2. Monoidal categories and 2-categories

2.1. Hierogliphic notation. We shall encounter monoidal categories, (monoidal) 2-categories, etc. The structure data and axioms for them involve a lot of diagrams. We find it convenient to label them by certain pictures, which we call "hierogliphs". Each notion requires its own system of hierogliphs. With a hierogliph it is easy to write the corresponding axiom.

Let us illustrate this approach first on the simplest example of the notion of category. The hierogliphs will be constructed of three basic elements (which are also hierogliphs): $\bullet$, $\to$, and I (from "identity"). The hierogliph $(\bullet)$ means that a category has a set (class) of objects, $(\to)$ that it has a class of morphisms, and (I) that each object has the associated identity morphism. Composite hierogliphs encode the composition structure in a category. Thus $(\to\to)$ means the axiom that morphisms can be composed, $(I\to)$ and $(\to I)$ that the identity morphisms are left and right units with respect to this composition, and $(\to\to\to)$ that the composition of morphisms is associative. More complicated hierogliphs such as $(\to\to\to\to)$ will not occur in the definition of category (since the associativity of the product of four morphisms follows from that for three morphisms, etc.), but they will surface in the definition of a 2-category.

For monoidal categories the hierogliphs will be constructed from symbols $\bullet$, 1, $\to$, $\to\to$ as their unbracketed tensor products. For example, $\bullet \otimes 1 \otimes \to$ is a hierogliph. The following definition of a monoidal category is, of course, equivalent to the standard one [JS2], [FY], [Mac1], and [Mac2]. The level of detail and authomatism provided by hierogliphs will be indispensable in working with (braided) monoidal 2-categories in §§4–6.

2.2. Definition. Let $\mathscr{A}$ be a category. A *monoidal structure* on $\mathscr{A}$ is a collection of the following data:

(1) An object $\mathbf{1} = \mathbf{1}_{\mathscr{A}} \in \mathrm{Ob}\,\mathscr{A}$

$(\bullet\otimes\bullet)$ For any two objects $A, B \in \mathscr{A}$, a new object $A\otimes B$

$(\to\otimes\bullet)$ For a morphism $u\colon A\to A'$ and an object B, a morphism $u\otimes B\colon A\otimes B\to A'\otimes B$

$(\bullet\otimes\to)$ For an object A and a morphism $v\colon B\to B'$, a morphism $A\otimes v\colon A\otimes B\to A\otimes B'$

$(\bullet\otimes\bullet\otimes\bullet)$ For any three objects A, B, C, an isomorphism $a_{A,B,C}\colon A\otimes(B\otimes C)\to(A\otimes B)\otimes C$

$(1\otimes\bullet)$ For any object A, an isomorphism $l_A\colon \mathbf{1}\otimes A\to A$

$(\bullet\otimes 1)$ For any object A, an isomorphism $r_A\colon A\otimes\mathbf{1}\to A$

These data should satisfy the following conditions:

$(\to\otimes\to)$ For any two morphisms $u\colon A\to A'$, $v\colon B\to B'$ we have $(u\otimes B')(A\otimes v) = (A'\otimes v)\otimes(u\otimes B)$.

$(\to\to\otimes\bullet)$ For any pair of composable morphisms $A\xrightarrow{u}A'\xrightarrow{u'}A''$ and an object B we have $(u'u)\otimes B = (u'\otimes B)(u\otimes B)$.

$(\bullet\otimes\to\to)$ For any object A and any pair of composable morphisms $B\xrightarrow{v}B'\xrightarrow{v'}B''$ we have $A\otimes(vv') = (A\otimes v)(A\otimes v')$.

(The first three conditions just express in detailed form the fact that $\otimes$ is a functor from the Cartesian product $\mathscr{A}\times\mathscr{A}$ to $\mathscr{A}$.)

$(\bullet\otimes\bullet\otimes\bullet\otimes\bullet)$ For any four objects A, B, C, D the diagram

$$\begin{tikzcd}[column sep=small]
 & A\otimes(B\otimes(C\otimes D)) \arrow[dl, "a_{A,B,C\otimes D}"'] \arrow[dr, "A\otimes a_{B,C,D}"] & \\
(A\otimes B)\otimes(C\otimes D) \arrow[d, "a_{A\otimes B,C,D}"'] & & A\otimes((B\otimes C)\otimes D) \arrow[d, "a_{A,B\otimes C,D}"] \\
((A\otimes B)\otimes C)\otimes D & & (A\otimes(B\otimes C))\otimes D \arrow[ll, "a_{A,B,C}\otimes D"']
\end{tikzcd}$$

is commutative.

$(\to\otimes\bullet\ \otimes\bullet)$, $(\bullet\otimes\to\otimes\bullet)$, $(\bullet\otimes\bullet\otimes\to)$ The isomorphisms $a_{A,B,C}$ are natural in A, B, C.

$(\bullet\otimes\bullet\otimes 1)$ For any two objects A, B the diagram

$$\begin{tikzcd}
A\otimes(B\otimes\mathbf{1}) \arrow[r, "A\otimes r_B"] \arrow[d, "a_{A,B,1}"'] & A\otimes B \\
(A\otimes B)\otimes\mathbf{1} \arrow[ur, "r_{A\otimes B}"'] &
\end{tikzcd}$$

is commutative.

$(1\otimes\bullet\otimes\bullet)$ For any two objects A, B the diagram

$$\begin{tikzcd}
\mathbf{1}\otimes(A\otimes B) \arrow[r, "l_{A\otimes B}"] \arrow[d, "a_{1,A,B}"'] & A\otimes B \\
(\mathbf{1}\otimes A)\otimes B \arrow[ur, "l_A\otimes B"'] &
\end{tikzcd}$$

is commutative.

$(\bullet\otimes 1\otimes\bullet)$ For any objects A, B the diagram

$$\begin{tikzcd}
A\otimes(\mathbf{1}\otimes B) \arrow[r, "A\otimes l_B"] \arrow[d, "a_{A,1,B}"'] & A\otimes B \\
(A\otimes\mathbf{1})\otimes B \arrow[ur, "r_A\otimes B"'] &
\end{tikzcd}$$

is commutative.

REMARK. In fact, axioms $(1\otimes\bullet\otimes\bullet)$ and $(\bullet\otimes\bullet\otimes 1)$ follow from $(\bullet\otimes 1\otimes\bullet)$. This was shown by Kelly [Ke2].

$(1\otimes\to)$ For any 1-morphism $u\colon A\to A'$ the diagram

$$\begin{tikzcd}
\mathbf{1}\otimes A \arrow[r, "1\otimes u"] \arrow[d, "l_A"'] & \mathbf{1}\otimes A' \arrow[d, "l_{A'}"] \\
A \arrow[r, "u"] & A'
\end{tikzcd}$$

is commutative.

($\to \otimes 1$) For any 1-morphism $u\colon A \to A'$ the diagram

$$\begin{array}{ccc} A\otimes \mathbf{1} & \xrightarrow{u\otimes 1} & A'\otimes \mathbf{1} \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle r_{A'}} \\ A & \xrightarrow{u} & A' \end{array}$$

is commutative.

($1\otimes 1$) The morphisms $l_1, r_1\colon \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}$ coincide.

2.3. Examples. (a) A standard example of a monoidal category is provided by the category Vect of finite-dimensional vector spaces (over the field $\mathbf{C}$ of complex numbers). The operation $\otimes$ is given by the usual tensor product of vector spaces. The associativity isomorphism $a_{A,B,C}\colon A\otimes(B\otimes C) \to (A\otimes B)\otimes C$ is defined on decomposable tensors by the rule $a\otimes(b\otimes c) \mapsto (a\otimes b)\otimes c$. The unit object $\mathbf{1}$ is the field $\mathbf{C}$ considered as a vector space over itself. The isomorphism $l_A\colon \mathbf{C}\otimes A \to A$ takes $1\otimes a$ to a; similarly, the isomorphism $r_A\colon A\otimes \mathbf{C} \to A$ takes $a\otimes 1$ to a.

(b) Let H be a (possibly noncommutative) algebra over $\mathbf{C}$ and H-mod the category of finite-dimensional left H-modules. The tensor product over $\mathbf{C}$ of two H-modules M and N will be an $H\otimes_{\mathbf{C}} H$-module and does not a priori have any H-module structure. However, suppose that H is equipped with a *comultiplication*, i.e., a linear map $\Delta\colon H \to H\otimes H$ satisfying two properties:

($*$) The operator Δ is a homomorphism of algebras.

($**$) The operator Δ is coassociative; i.e., the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H\otimes H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta\otimes 1} \\ H\otimes H & \xrightarrow{1\otimes\Delta} & H\otimes H\otimes H \end{array}$$

is commutative.

In this case we can, for any two left H-modules M, N equip the vector space $M\otimes_{\mathbf{C}} N$ with the structure of a left H-module by means of Δ, i.e., by setting $a.(m\otimes n) = \Delta(A)(m\otimes n)$ for $a\in H$. This will indeed be an H-action if condition ($*$) is satisfied. If, moreover, ($**$) holds, then the standard isomorphism $M\otimes(N\otimes P) \to (M\otimes N)\otimes P$ defined in n. (a), will be an H-module isomorphism. An algebra equipped with a comultiplication is called a *bialgebra* (or Hopf algebra). Thus we have seen that H-mod for a bialgebra H is equipped with almost all the data of monoidal structure except the unit object. In order for H-mod to have a unit object, H should possess a *counit*, i.e., a homomorphism $\varepsilon\colon H \to \mathbf{C}$ such that the compositions $H \xrightarrow{\Delta} H\otimes H \xrightarrow{1\otimes\varepsilon} H$, $H \xrightarrow{\Delta} H\otimes H \xrightarrow{\varepsilon\otimes 1} H$ are identities. In this case $\mathbf{C}$ is equipped with the structure of H-module (via ε) and is a unit object.

(c) Let $\mathscr{C}$ be any category. Let $\mathscr{A} = \underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ be the category of all functors from $\mathscr{C}$ to itself. By definition, objects of $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ are all functors $\mathscr{C} \to \mathscr{C}$. A morphism in $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ between two functors $F, G: \mathscr{C} \to \mathscr{C}$ is, by definition, a natural transformation of functors $T: F \Rightarrow G$, i.e., a collection of morphisms $T_A: F(A) \to G(A)$ given for any object $A \in \mathscr{C}$ such that for any morphism $u: A \to B$ in $\mathscr{C}$ we have $T_B F(u) = G(u) T_A$ (see [Mac2]). The category $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ has a natural monoidal structure $\otimes$ defined as follows.

($\bullet \otimes \bullet$) On objects the structure $\otimes$ is given by the composition of functors: $F \otimes G = F \circ G$.

($\bullet \otimes \to$) Let $F: \mathscr{C} \to \mathscr{C}$ be a functor and $T: G \Rightarrow G'$ be a natural transformation. We define the morphism $F \otimes T$ of $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ to be the natural transformation having $(F \otimes T)_A = F(T_A)$.

($\to \otimes \bullet$) Let $S: F \Rightarrow F'$ be a natural transformation and $G: \mathscr{C} \to \mathscr{C}$ a functor. We define the transformation $S \otimes G$ to have $(S \otimes G)_A = S_{G(A)}$.

($\bullet \otimes \bullet \otimes \bullet$) The composition of functors is well known to be strictly associative, and we define the associativity isomorphism to be the identity.

The rest of the data (involving unities) are trivial. We leave the verification of the axioms to the reader.

2.4. Definition. A monoidal category $\mathscr{A}$ is called *strict* if we have equalities of objects

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C, \qquad \mathbf{1} \otimes A = A \otimes \mathbf{1} = A$$

for any objects A, B, C and the morphisms $a_{A,B,C}, l_A, r_A$ are identities.

Of course, "large" monoidal categories like Vect arising in practice are never strict: the vector spaces $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are, strictly speaking, different. However, working with strict categories is convenient. It is justified by the coherence theorem of Mac Lane [Mac1] and [Mac2], which implies that every monoidal category can be replaced by an "equivalent" strict one. (For a formulation exactly in this form, see §1 of [JS1].)

To give a precise meaning to "equivalent" one should introduce the notion of a monoidal functor and monoidal natural transformation. We shall refrain from doing this here, referring the reader to [Saa]. Instead, we shall give some examples of strict monoidal categories.

2.5. Examples. (a) The monoidal category $\underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$ defined in Example 2.3(c) is strict.

(b) We shall construct a coordinatized version Vect_c of the monoidal category Vect which is strict.

($\bullet$) Objects of Vect_c are formal symbols $[n]$, where $n = 0, 1, 2, \ldots$.

($\to$) $\mathrm{Hom}_{\mathrm{Vect}_c}([m], [n])$ to be the set of all $m \times n$ complex matrices $F = \|f_{ij}\|$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

($\to\to$) The composition of morphisms is given by matrix multiplication.

(1) The unit object is the symbol $[1]$.

$(\bullet \otimes \bullet)$ The tensor product of objects is defined by $[m] \otimes [n] = [mn]$.

$(\bullet\otimes \rightarrow)$ The tensor product of an object $[m]$ and a morphism $G = \|g_{ij}\| : [n] \rightarrow [n']$ is the following (block) matrix of format $(mn) \times (mn')$:

$$[m] \otimes G = \begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & G \end{pmatrix}.$$

$(\rightarrow \otimes\bullet)$ The tensor product of morphism $F = \|f_{ij}\| : [m] \rightarrow [m']$ and an object $[n]$ is the following block matrix of format $(mn) \times (m'n)$:

$$F \otimes [n] = \begin{pmatrix} f_{11} \cdot \mathbf{1}_n & f_{12} \cdot \mathbf{1}_n & \cdots & f_{1m} \cdot \mathbf{1}_n \\ f_{21} \cdot \mathbf{1}_n & f_{22} \cdot \mathbf{1}_n & \cdots & f_{2m} \cdot \mathbf{1}_n \\ \vdots & \vdots & \vdots & \vdots \\ f_{m'1} \cdot \mathbf{1}_n & f_{m'2} \cdot \mathbf{1}_n & \cdots & f_{m'm} \cdot \mathbf{1}_n \end{pmatrix}.$$

$(\bullet\otimes\bullet\otimes\bullet)$, $(1\otimes\bullet)$, $(\bullet\otimes 1)$ The morphisms $a_{[m],[n],[p]}$, $l_{[m]}$, $r_{[n]}$ are identity matrices.

Let us verify that these data indeed define a (strict) monoidal category. The only thing that needs checking is the condition $(\rightarrow \otimes \rightarrow)$. Let $F: [m] \rightarrow [m']$, $G: [n] \rightarrow [n']$ be two morphisms in Vect_c, i.e., matrices. We need to show the commutativity of the diagram

$$\begin{tikzcd} {[mn]} \arrow[r, "{[m]\otimes G}"] \arrow[d, "{F\otimes[n]}"'] & {[mn']} \arrow[d, "{F\otimes[n']}"] \\ {[m'n]} \arrow[r, "{[m']\otimes G}"] & {[m'n']}. \end{tikzcd}$$

The composition $(F \otimes [n])([m] \otimes G)$ is represented by the following block matrix of size $(mn) \times (m'n')$:

$$\begin{pmatrix} Gf_{11} & Gf_{12} & \cdots & Gf_{1m} \\ Gf_{21} & Gf_{22} & \cdots & Gf_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ Gf_{m'1} & Gf_{m'2} & \cdots & Gf_{m'm} \end{pmatrix}.$$

In each (i, j)th block, which is an $(n \times n')$ matrix, we have the scalar multiple $Gf_{ij} = \|g_{kl}f_{ij}\|$, $k = 1, \ldots, n$, $l = 1, \ldots, n'$. The other possible composition will give a similar matrix but will $f_{ij}G$ in blocks. These give the same answer by the commutativity of our field $\mathbf{C}$ of scalars (so an attempt to construct in this fashion the tensor product for free modules over noncommutative ring would fail in exactly this place).

2.6. Hierogliphs for 2-categories. In a category one has objects and morphisms. In a 2-category, a new class of entities is present: 2-morphisms or "morphisms between morphisms" or "homotopies". A 2-morphism can act between two usual morphisms (1-morphisms) if their sources and targets coincide, see Figure 9.

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

Figure 9

Here A, B are objects, $u, v : A \to B$ are 1-morphisms, and α is a 2-morphism from u to v.

There is a different notion of a two-dimensional category in which 2-morphisms have the space of squares (instead of "globes"), as above. Such structures are called double categories **[Br]**, **[E]**, and **[G]**. We shall not use them here.

The data and axioms for lax 2-categories will be labelled by hierogliphs constructed from the following elementary symbols:

$$\bullet, \quad \to, \quad \overset{\to}{\underset{\to}{\downarrow}}, \quad I, \quad \overset{\to}{\underset{\to}{1}}.$$

The notion of a lax 2-category was introduced by Benabou **[Be]**. Let us give a formal definition.

2.7. Definition. A *lax 2-category* $\mathscr{C}$ is a collection of the following data:

($\bullet$) A set (class) $\mathrm{Ob}\,\mathscr{C}$ whose elements are called objects.

($\to$) A set $1-\mathrm{Mor}(\mathscr{C})$ whose elements are called 1-morphisms and two maps $s_0, t_0 \colon 1-\mathrm{Mor}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C}$ called source and target maps of dimension 0. If $u \in 1-\mathrm{Mor}\,\mathscr{C}$, $A = s_0(u)$, $B = t_0(u)$, then we write $u \colon A \to B$.

$\left(\overset{\to}{\underset{\to}{\downarrow}}\right)$ A set $2-\mathrm{Mor}\,\mathscr{C}$ whose elements are called 2-morphisms and maps $s_1, t_1 \colon 2-\mathrm{Mor}\,\mathscr{C} \to 1-\mathrm{Mor}\,\mathscr{C}$ called source and target maps of dimension 1. It is required that $s_0 s_1 = t_0 t_1$, $t_0 s_1 = t_0 t_1$ as maps $2-\mathrm{Mor}\,\mathscr{C} \to \mathrm{Ob}\,\mathscr{C}$. These maps are denoted s_0, t_0. A 2-morphism α is usually visualized as a 2-cell (Figure 9) where $u = s_1(\alpha)$, $v = t_1(\alpha)$, $A = s_0(\alpha)$, $B = t_0(\alpha)$.

We shall also write $\alpha : u \Rightarrow v$.

($\to\to$) For any two 1-morphisms $u \colon A \to B$, $v \colon B \to C$, a 1-morphism $v *_0 u \colon A \to C$ is called their composition. The composition of 1-morphisms will also be sometimes denoted $v \circ u$ or just vu.

$\left(\to \overset{\to}{\underset{\to}{\downarrow}}\right)$ For any 1-morphism $u \colon A \to B$ and a 2-morphism

$$\begin{tikzcd} B \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\alpha"] & C \end{tikzcd}$$

a 2-morphism (also called 0-composition)

$$\begin{tikzcd} A \arrow[r, bend left=50, "vu"] \arrow[r, bend right=50, "v'u"'] \arrow[r, phantom, "\Downarrow \alpha *_0 u"] & C \end{tikzcd}$$

$\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\ \to\right)$ For any 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left=50, "u"] \arrow[r, bend right=50, "u'"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \end{tikzcd}$$

and a 1-morphism $v: B \to C$ a 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left=50, "vu"] \arrow[r, bend right=50, "vu'"'] \arrow[r, phantom, "\Downarrow v *_0 \alpha"] & C \end{tikzcd}$$

(also called 0-composition)

$(\to\to\to)$ For any triple $A \xrightarrow{w} B \xrightarrow{v} C \xrightarrow{u} D$ of composable 1-morphisms, a 2-isomorphism (i.e., a 2-morphism invertible with respect to $*_1$) $a_{u,v,w}: u(vw) \Rightarrow (uv)w$.

$\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\\ \downarrow\\ \to\end{smallmatrix}\right)$ For any pair α, β of 2-morphisms such that $s_1(\alpha) = t_1(\beta)$ a 2-morphism $\alpha *_1 \beta: s_1(\beta) \Rightarrow t_1(\alpha)$.

(I) For any object A a 1-morphism $\mathrm{Id}_A: A \to A$ called the identity morphism of A.

(1) For any 1-morphism $u: A \to B$ a 2-morphism $\mathbf{1}_u: u \Rightarrow u$ called the identity 2-morphism of u.

$(\to I)$ For any 1-morphism $u: A \to B$ a 2-morphism $l_u: \mathrm{Id}_B \circ u \Rightarrow u$.

$(I \to)$ For any 1-morphism $u: A \to B$ a 2-morphism $r_u: u \circ \mathrm{Id}_A \Rightarrow u$.

These data should satisfy the following conditions.

$(\to\to\to\to)$ For any 4-tuple $A \xrightarrow{w} B \xrightarrow{v} C \xrightarrow{u} D \xrightarrow{x} E$ of composable 1-morphisms the diagram of 2-morphisms

$$\begin{tikzcd} & x(u(vw)) \arrow[dl, Rightarrow, "a_{x,u,vw}"'] \arrow[dr, Rightarrow, "x *_0 a_{u,v,w}"] & \\ (xu)(vw) \arrow[d, Rightarrow, "a_{xu,v,w}"'] & & x((uv)w) \arrow[d, Rightarrow, "a_{x,uv,w}"] \\ ((xu)v)w & & (x(uv))w \arrow[ll, Rightarrow, "a_{x,u,v} *_0 w"] \end{tikzcd}$$

is commutative.

$\left(\begin{smallmatrix}\to & \to\\ \downarrow & \downarrow\\ \to & \to\end{smallmatrix}\right)$ For any pair of 2-morphisms of the form

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

we have the following equality of 2-morphisms:

$$(\beta *_0 u') *_1 (v *_0 \alpha) = (v' *_0 \alpha) *_1 (\beta *_0 u).$$

The 2-morphism given by any of the sides of this equality is denoted $\beta *_0 \alpha \colon v *_0 u \Rightarrow v' *_0 u'$ (and called the 0-composition of α and β).

$\begin{pmatrix}\to\\ \downarrow\\ \to\\ \downarrow\\ \to\\ \downarrow\\ \to\end{pmatrix}$ The composition $*_1$ of 2-morphisms is associative.

$\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\, I\right)$ For any 2-morphism $\alpha\colon u \Rightarrow u'$, $u, u' \colon A \to B$, we have $\mathrm{Id}_B *_0 \alpha = \alpha$.

$\left(I\, \begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\right)$ For any 2-morphism α as above we have $\alpha *_0 \mathrm{Id}_A = \alpha$.

$(\to\to I)$ For any pair $A \xrightarrow{u} B \xrightarrow{v} C$ of composable 1-morphisms the diagram of 2-morphisms

$$\begin{tikzcd} \mathrm{Id}_C \circ (v \circ u) \arrow[r, Rightarrow, "a_{\mathrm{Id}_C, v, u}"] \arrow[d, Rightarrow, "l_{vu}"'] & (\mathrm{Id}_C \circ v) \circ u \arrow[dl, Rightarrow, "l *_0 l_u"] \\ vu & \end{tikzcd}$$

is commutative.

$(\to I \to)$ For any pair $A \xrightarrow{u} B \xrightarrow{v} C$ of composable 1-morphisms the diagram of 2-morphisms

$$\begin{tikzcd} v \circ (\mathrm{Id}_B \circ u) \arrow[r, Rightarrow, "a_{v, \mathrm{Id}_B, u}"] \arrow[d, Rightarrow, "v *_0 l_u"'] & (v \circ \mathrm{Id}_B) \circ u \arrow[dl, Rightarrow, "r_v *_0 u"] \\ vu & \end{tikzcd}$$

is commutative.

$(I \to\to)$ For any pair $A \xrightarrow{u} B \xrightarrow{v} C$ of composable 1-morphisms the diagram of 2-morphisms

$$\begin{tikzcd} \mathrm{Id}_A \circ (v \circ u) \arrow[r, Rightarrow, "a_{\mathrm{Id}_A, v, u}"] \arrow[d, Rightarrow, "l_{uv}"'] & (\mathrm{Id}_A \circ v) \circ u \arrow[dl, Rightarrow, "v *_0 l_u"] \\ vu & \end{tikzcd}$$

is commutative.

$\begin{pmatrix}\to\\ \downarrow\\ \to\\ 1\end{pmatrix}$, $\begin{pmatrix}1\\ \to\\ \downarrow\\ \to\end{pmatrix}$ The 2-morphisms $\mathbf{1}_u$ are right and left units with respect to $*_1$

2.8. The main example: the "category" of all categories. The archetypal example of a 2-category is given by the "category" Cat of all usual categories. It is defined as follows (cf. [Be], [GZ], and [Mac2]):

($\bullet$) Objects if **Cat** are all categories.

($\to$) 1-morphisms in **Cat** are functors between categories.

$\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\right)$ 2-morphisms in **Cat** are natural transformations between functors.

($\to\to$) Composition of functors is defined in the usual way.

($\to\to\to$) The composition of functors is strictly associative and the associativity 2-morphisms are identities.

All the other data are well known and we leave them to the reader.

We deliberately ignore set-theoretic (or, rather, category-theoretic) "difficulties" which are usually mentioned when speaking about the category of all categories. Our opinion is that the introduction of the 2-categorical structure on **Cat** resolves these difficulties for good, just as the consideration of the *categorical* structure on the collection of all sets resolves the difficulties with the "set" of all sets.

By considering categories with some additional structure, one obtains more complicated examples of 2-categories. This is similar to the way of constructing usual categories by considering sets (0-categories) with additional structure.

2.9. Hom-categories in a 2-category. Let $\mathscr{C}$ by any (lax) 2-category and $A, B \in \mathscr{C}$ objects. Define a 1-category $\mathrm{Hom}_{\mathscr{C}}(A, B)$ to have, as objects, 1-morphisms $A \to B$, and as morphisms 2-morphisms between these 1-morphisms with composition defined by $*_1$.

2.10. Monoidal categories as 2-categories: delooping. The reader has certainly noted the similarity between the definitions of a monoidal category and a 2-category. In fact, as was remarked by Benabou [Be], the former notion can be regarded as a particular case of the latter.

More precisely, let $\mathscr{A}$ be a monoidal 1-category. Define a 2-category $\Omega^{-1}\mathscr{A}$ with one object pt, the set of 1-morphisms $\mathrm{Ob}\mathscr{A}$, and the set of 2-morphisms $\mathrm{Mor}\mathscr{A}$, the composition $*_0$ defined by $\otimes$ and $*_1$ by composition of 1-morphisms in $\mathscr{A}$. Axioms of a monoidal category for $\mathscr{A}$ translate into axioms of a 2-category for $\Omega^{-1}\mathscr{A}$. It is easy to see that any 2-category $\mathscr{C}$ with one object comes in this way from some monoidal category, namely $\mathscr{A} = \underline{\mathrm{Hom}}(\mathscr{C}, \mathscr{C})$.

Thus monoidal categories are essentially the same as 2-categories with just one object. We shall call this observation the *delooping principle* and call the 2-category $\Omega^{-1}\mathscr{A}$ the delooping of the monoidal category $\mathscr{A}$.

Any object A of any 2-category $\mathscr{C}$ defines a sub-2-category in $\mathscr{C}$ with the unique object A. The delooping principle implies the following.

2.11. Proposition. *Let $\mathscr{C}$ be any (lax) 2-category and $A \in \mathscr{C}$ an object. Then the composition $*_0$ defines on* $\mathrm{Hom}_{\mathscr{C}}(A, A)$ *a structure of a lax monoidal category.*

For instance, the monoidal structure on the category of all functors from a category $\mathscr{C}$ to itself (Example 2.3(c)) is just a particular case of the 2-categorical structure on the collection of all categories.

2.12. Strict 2-categories. A 2-category $\mathscr{C}$ is called strict if all the 2-morphisms $a_{u,v,w}$, l_u, r_u (given by the data with labels $(\to\to\to)$, $(\to I)$, and $(I \to)$) are identity 2-morphisms (in particular, their one-dimensional sources and targets coincide). In other words, the composition of 1-morphisms is strictly associative and units are strict. For example, the 2-category **Cat** of all categories is strict.

It is clear that the monoidal categories $\mathrm{Hom}_{\mathscr{C}}(A, A)$ associated to a strict 2-category $\mathscr{C}$ are strict, and, conversely, if $\mathscr{A}$ is a strict monoidal category then the 2-category $\Omega^{-1}\mathscr{A}$ is strict.

A strict 2-category can be seen as a category enriched in the Cartesian category Cat of usual categories (in the sense that Hom's are no more sets but categories), see **[Mac2]** for the definitions.

It follows from the coherence theorem of Mac Lane and Pare **[Mac-P]** that any lax 2-category is equivalent (in a suitable sense) to a strict one. This remark will enable us to consider in the sequel only strict 2-categories.

2.13. Pasting in strict 2-categories. Commutative polytopes. In strict 2-categories it is possible to consider algebraic expressions which have the form of a two-dimensional cell subdivided into several other two-dimensional cells. Such a construction is known as *pasting.* There are several approaches to the theory of pasting **[J]** and **[P]** see also **[KV1]**. Let us give an example.

Suppose that in some strict 2-category $\mathscr{A}$ we have a system of objects, 1- and 2-morphisms of the form depicted in Figure 10.

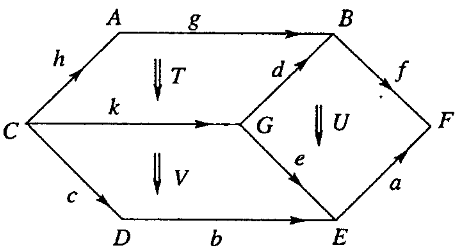


Figure 10

This means that $a: E \to F$, $b: D \to E$, etc., are 1-morphisms whose source and target objects are depicted by the arrows; $T: gh \Rightarrow dk$, $U: fd \Rightarrow ae$, $V: ek \Rightarrow bc$ are 2-morphisms. In this situation it is possible to associate to the above picture a 2-morphism from fgh to abc called the pasting of U, V, W. To define it in terms of the compositions $*_i$, we should gradually move the path fgh, restructuring it at each step by replacing the beginning of some cell with its end:

$$fgh \Rightarrow fdk \Rightarrow aek \Rightarrow abc.$$

This amounts to a definition of pasting (in our situation) as

$$(a *_0 V) *_1 (U *_0 k) *_1 (f *_0 T).$$

In general, one can consider a subdivided polygon together with its *realization* in the 2-category $\mathscr{C}$. A realization is the association, to every vertex of the subdivision, of an object of C, to every edge a 1-morphism, and to every 2-cell a 2-morphism. We shall not give here a formal definition of what are admissible patterns for pasting (pasting schemes), referring the reader to [J] and [P].

In our concrete examples it will always be clear how to formalize a given pasting expression.

Thus a formula in a 2-category has the shape of a subdivided polygon. Suppose we have two such subdivided polygons P, P' realized in our 2-category $\mathscr{C}$ such that their boundaries are identified with each other and the 1-morphisms associated to corresponding edges of the boundaries are the same. Then we can glue P, P' together, obtaining a three-dimensional *polytope*. The commutativity of such a polytope means that the results of pasting the two parts of its boundary coincide. For example, the commutativity of the cube in Figure 11

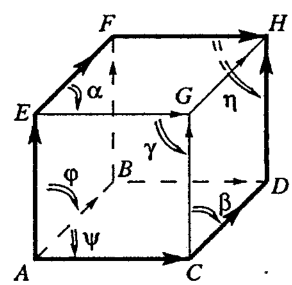


Figure 11

means that we have the equality in Figure 12.

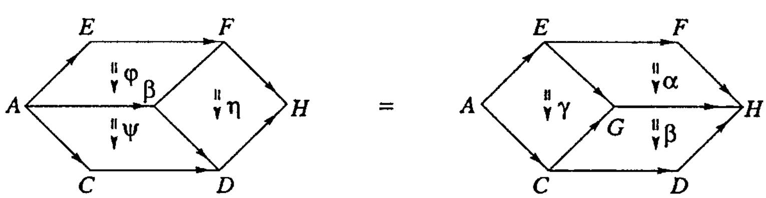


Figure 12

2.14. Convention. To faciliate the deciphering of polytopal diagrams, we always denote by thick arrows the common boundary of two composable polygons constituting a diagram. Note that such a decomposition of the boundary of a polytope into two composable polygons is always unique.

We use commutative polytopes often. In fact, most of our examples come from genuine convex polytopes in three-dimensional space. (Recall that a convex polytope is the convex hull of a finite set of points.) A convenient way to prove the commutativity of some polytope is to construct its decomposition into smaller polytopes (also realized in our 2-category) which are known to be commutative.

3. Braided monoidal categories and Yang-Baxter equation

From now on we consider only strict monoidal categories. This is justified by the Mac Lane coherence theorem [**Macl** and **Mac2**]. The following definition was (up to minor modifications) introduced in [**FY**] and [**JS2**].

3.1. Definition. Let $\mathscr{A}$ be a strict monoidal category. A *braiding* (resp. an isobraiding) on $\mathscr{A}$ is a family of morphisms (resp. isomorphisms) $R_{A,B}$: $A\otimes B \to B\otimes A$ given for any pair (A, B) of objects of $\mathscr{A}$ which satisfy the following conditions:

$(\to\otimes\bullet)$, $(\bullet\otimes\to)$ $R_{A,B}$ are natural in A and B.

$(\bullet\otimes(\bullet\otimes\bullet))$ For any objects A, B_1, B_2 the diagram

$$\begin{tikzcd} A\otimes B_1\otimes B_2 \arrow[r, "R_{A,B_1}\otimes B_2"] \arrow[d, "R_{A,B_1\otimes B_2}"'] & B_1\otimes A\otimes B_2 \arrow[dl, "B_1\otimes R_{A,B_2}"] \\ B_1\otimes B_2\otimes A & \end{tikzcd}$$

is commutative.

$((\bullet\otimes\bullet)\otimes\bullet)$ For any objects A_1, A_2, B the diagram

$$\begin{tikzcd} A_1\otimes A_2\otimes B \arrow[r, "A_1\otimes R_{A_2,B}"] \arrow[d, "R_{A_1\otimes A_2,B}"'] & A_1\otimes B\otimes A_2 \arrow[dl, "R_{A_1,B}\otimes A_2"] \\ B\otimes A_1\otimes A_2 & \end{tikzcd}$$

is commutative.

$(\mathbf{1}\otimes\bullet)$, $(\bullet\otimes\mathbf{1})$ For any object A the diagrams

$$\begin{tikzcd} \mathbf{1}\otimes A \arrow[r, "R_{\mathbf{1},A}"] \arrow[d, "l_A"'] & A\otimes\mathbf{1} \arrow[dl, "r_A"] \\ A & \end{tikzcd} \qquad \begin{tikzcd} A\otimes\mathbf{1} \arrow[r, "R_{A,\mathbf{1}}"] \arrow[d, "r_A"'] & \mathbf{1}\otimes A \arrow[dl, "l_A"] \\ A & \end{tikzcd}$$

are commutative.

3.2. Braidings and isobraidings. Usually the requirement that morphisms $R_{A,B}$ are isomorphisms is included in the definition of a braiding. This property holds for most practical examples. However, in the definition of 2-braiding (§6) of which the present notion will be the 1-skeleton, it will be reasonable not to require the isomorphicity of $R_{A,B}$. Moreover, solutions of

Yang-Baxter equation (1-7) depending on a parameter u usually have poles for some u and are degenerate for some u. As was pointed out to us by D. Kazhdan, this does not permit us to construct a genuine braided monoidal category from such a solution.

3.3. Examples. (a) The category Vect of $\mathbf{C}$-vector spaces with the monoidal structure $\otimes$ given by the tensor product has a natural braiding

$$R_{A,B}: A \otimes B \to B \otimes A, \quad a \otimes b \mapsto b \otimes a.$$

Here we have, in addition, the identity $R_{A,B}R_{B,A} = \mathrm{Id}$. Braided monoidal categories satisfying this additional property are called *symmetric* and were studied by Mac Lane [Macl].

(b) Let $g\,\mathrm{Vect}$ be the category of $\mathbf{Z}$-graded finite-dimensional vector spaces $V = \bigoplus_{i \in \mathbf{Z}} V_i$, $\dim(V) < \infty$, and grading-preserving linear operators. For $a \in V_i$ we write $i = |a|$. Equip $g\,\mathrm{Vect}$ with the usual tensor product

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j.$$

Let $q \in \mathbf{C}^*$ be a nonzero complex number. Define a system of operators $R_{V,W}: V \otimes W \to W \otimes V$ setting on homogeneous elements $R(v \otimes w) = q^{|v|\cdot|w|}(w \otimes v)$. Clearly these operators are natural in V and W. To show that they form a braiding we have to check only the triangles $((\bullet \otimes \bullet) \otimes \bullet)$ and $(\bullet \otimes (\bullet \otimes \bullet))$ from Definition 3.1. This is left to the reader.

(c) Let H be a bialgebra (see Example 2.3(b)) and H-mod the category of finite-dimensional left H-modules. As we have seen in the cited example, this category is equipped with a monoidal structure $\otimes$, where $M \otimes N$ for $M, N \in H$-mod is the usual tensor product $M \otimes_{\mathbf{C}} N$ equipped with the H-module structure via the comultiplication $\Delta: H \to H \otimes H$. In order to equip H-mod with a braiding, H should carry the additional structure namely an element $\mathscr{R} \in H \otimes H$ such that two conditions hold:

(* * *) We have $\Delta'(a)\mathscr{R} = \mathscr{R}\Delta(a)$ for any $a \in H$, where Δ' is the permuted comultiplication $P \circ \Delta$.

(* * **) In $H \otimes H \otimes H$ we have the equalities

$$(\Delta \otimes \mathrm{Id})(\mathscr{R}) = \mathscr{R}_{12}\mathscr{R}_{23}, \qquad (\mathrm{Id} \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$$

where $\mathscr{R}_{12} = \mathscr{R} \otimes 1$, $\mathscr{R}_{23} = 1 \otimes \mathscr{R}$, and $\mathscr{R}_{13}$ is defined similarly; cf. (1-2).

If condition (***) holds, then for any $M, N \in H$-mod one can define an H-morphism $R_{M,N}: M \otimes N \to N \otimes M$ by setting $R_{M,N}(m \otimes n) = \mathscr{R}(n \otimes m)$. The conditions (****) are exact replicas of the two braiding triangles (axioms $(\bullet \otimes (\bullet \otimes \bullet))$, $((\bullet \otimes \bullet) \otimes \bullet)$ of Definition 3.1).

For more examples the reader is referred to [Maj2] and to articles in the reprint volume [Ji].

3.4. The Yang-Baxter hexagon. Let $(\mathscr{A}, R)$ be a braided monoidal category. Consider any three objects A, B, C and form the diagram

$$\begin{tikzcd}
 & B \otimes A \otimes C \arrow[r, "B\otimes R_{A,C}"] & B \otimes C \otimes A \arrow[dr, "R_{B,C}\otimes A"] & \\
A \otimes B \otimes C \arrow[ur, "R_{A,B}\otimes C"] \arrow[dr, "A\otimes R_{B,C}"'] \arrow[urr, dashed] & & & C \otimes B \otimes A \\
 & A \otimes C \otimes B \arrow[r, "R_{A,C}\otimes B"] \arrow[urr, dashed] & C \otimes A \otimes B \arrow[ur, "C\otimes R_{A,B}"'] &
\end{tikzcd}$$

This diagram is called the *Yang-Baxter hexagon.* The commutativity of it can be viewed as an "abstract" version of the Yang-Baxter equation (1-1). The following proposition is well-known **[FY]** and **[JS1-2]**.

3.5. PROPOSITION. *The Yang-Baxter hexagon is commutative for any three objects A, B, C.*

PROOF. The dotted lines in the hexagon decompose it into two triangles of the type $(\bullet \otimes (\bullet \otimes \bullet))$ and a square of naturality, which are all commutative.

3.6. The permutohedron. The vertices of the Yang-Baxter hexagon correspond to all permutations of three letters. We now consider a convex polytope whose vertices correspond to permutation of n letters, $n \geq 2$.

By definition, the $(n-1)$-dimensional *permutohedron* P_n (see **[Bau]** and **[Mi]**) is the convex hull of $n!$ points $(\sigma(1), \cdots, \sigma(n)) \in \mathbf{R}^n$, where σ runs over all the permutations of $\{1, \cdots, n\}$.

It is clear from this definition that P_n lies in the hyperplane

$$\{(x_1, \cdots, x_n) \in \mathbf{R}^n \colon \sum x_i = n(n-1)/2\}$$

and its dimension equals $(n-1)$.

Let S_n be the symmetric group of all permutations of $\{1, \cdots, n\}$.

For any $\sigma \in S_n$ we shall denote by $[\sigma]$ the point $(\sigma^{-1}(1), \cdots, \sigma^{-1}(n)) \in P_n$, where σ^{-1} is the inverse permutation.

The 2-dimensional permutohedron P_3 is the hexagon in Figure 13.

$$\begin{tikzcd}[column sep=small]
 & {[123]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[213]} \arrow[d, dash] & & {[132]} \arrow[d, dash] \\
{[231]} \arrow[dr, dash] & & {[312]} \arrow[dl, dash] \\
 & {[321]} &
\end{tikzcd}$$

Figure 13

The permutohedron P_4 will be drawn in formula (6-2) in §6 below (see also pictures in **[Bau]** and **[Mi]**).

Two vertices $[\sigma], [\tau] \in P_n$ are connected by an edge if and only if the corresponding permutations $[\sigma], [\tau] \in P_n$ are connected by an edge if and only if the corresponding permutations σ, τ are obtained from each other by interchanging two numbers standing in the consecutive positions. This fact is true for any permutohedron P_n, and follows from the description of all the faces of P_n given in **[Mi]** and **[Bau]** which we now recall.

Faces of P_n are in 1-1 correspondence with partitions of the set $\{1, \cdots, n\}$ into a union of several nonintersecting numbered subsets:

$$\{1, \cdots, n\} = C_1 \cup \cdots \cup C_r.$$

The face $\Gamma(C_1, \ldots, C_r)$ has, by definition, the vertices $[\sigma]$ for all the permutations σ which preserve each C_i. It is easy to see that $\Gamma(C_1, \ldots, C_r)$ is linearly isomorphic to the product of smaller permutohedra, namely $P_{|C_1|} \times \cdots \times P_{|C_r|}$ and has codimension r in P_n.

In this description our choice of the notations of vertices is important. If we had introduced the straightforward notation for the vertices, i.e., the vertex $(\sigma(1), \ldots, \sigma(n))$ denoted by (σ), then the fact that two vertices are connected by an edge would mean that the two permutations are obtained from each other by interchanging some two consecutive *numbers*, e.g., 5 and 6, regardless of the position where these numbers stand in the permutation. Thus our chosen notation is "local" and does not make appeal to the order on the permuted symbols $1, \ldots, n$ which makes it possible to choose any other set of symbols.

3.7. Weak Bruhat order on the group S_n. By definition (see, e.g., **[MS2]**) this order $<$ is generated by the preorder $\prec$ where $\sigma \prec \tau$ when the following conditions hold:

(1) σ can be obtained from τ by interchanging two symbols standing in consecutive positions (i.e., $[\sigma]$ and $[\tau]$ are connected by an edge).

(2) The length (i.e., the number of inversions) of τ is greater than the length of σ. In other words, the Bruhat order $<$ is obtained by orienting all edges of P_n in the direction of increasing length.

3.8. Permutohedral diagrams in an (iso-) braided category. Let $\mathscr{A}$ be an isobraided monoidal category and n objects $A_1, \ldots, A_n$ of it. For any permutation $\sigma \in S_n$ denote by A_σ the product $A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$ and put this product at the vertex $[\sigma]$ of the permutohedron P_n. Let $[\sigma], [\tau]$ be two adjacent vertices of P_n. Then the product A_σ differs from A_τ by interchanging two consecutive factors $A_{\tau(i)}$ and $A_{\tau(i+1)}$. Let e be the edge joining $[\sigma]$ and $[\tau]$. Assume that $\sigma < \tau$ with respect to the weak Bruhat order. The braiding gives two morphisms

$$A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(i-1)} \otimes R_{A_{\sigma(i)}, A_{\sigma(i+1)}} \otimes A_{\sigma(i+2)} \otimes \cdots \otimes A_{\sigma(n)} \colon A_\sigma \to A_\tau$$

and

$$A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(i-1)} \otimes R_{A_{\sigma(i+1)}, A_{\sigma(i)}} \otimes A_{\sigma(i+2)} \otimes \cdots \otimes A_{\sigma(n)} \colon A_\tau \to A_\sigma.$$

We denote the first of these morphisms by U_e and the second by V_e. Thus we obtain a diagram in which to every edge of the permutohedron two morphisms are associated. We shall call it the *permutohedral diagram* associated to $A_1, \dots, A_n$ and denote it by $P(A_1, \dots, A_n)$. Given two permuted products A_σ, A_τ, we are interested in how many morphisms from A_σ to A_τ one can construct by composing various U_e, V_e and their inverses. More precisely, to define such a morphism we should specify an edge path on the permutohedron, leading from $[\sigma]$ to $[\tau]$ and make a choice at each edge e of the two morphisms going in the direction of the path. These morphisms are either U_e, V_e^{-1} or U_e^{-1}, V_e, depending on the edge and its direction (here we use the assumption that the braiding morphisms are isomorphisms).

The first fact about $P(A_1, \dots, A_n)$ we want to mention is a generalization of Proposition 3.5 on the commutativity of the Yang-Baxter hexagon. Denote the part of the permutohedral diagram $P(A_1, \dots, A_n)$ formed only by the morphisms U_e, e running over all the edges of P_n, by $P^+(A_1, \dots, A_n)$. Denote the part formed by only the V_e by $P^-(A_1, \dots, A_n)$.

3.9. PROPOSITION. *Let $\mathscr{A}$ be any braided monoidal category and $A_1, \dots, A_n$ any objects of $\mathscr{A}$. Then the diagrams $P^+(A_1, \dots, A_n)$ and $P^-(A_1, \dots, A_n)$ are commutative.*

PROOF. Since P_n is a convex polytope, to prove the commutativity of, say, P_n^+, it suffices to prove the commutativity of each two-dimensional face of P_n. These faces are either hexagons or squares. The commutativity of hexagons follows from Proposition 3.5 and the commutativity of squares follows from the naturality of $\otimes$ (axioms$(\to \otimes \to)$ of a monoidal category).

However, the whole diagram $P(A_1, \dots, A_n)$ is not, in general, commutative. This can be seen already on the example of two objects, say, A and B. Then we have two morphisms $R_{A,B} \colon A\otimes B \to B\otimes A$ and $R_{B,A} \colon B\otimes A \to A\otimes B$ which need not be inverses to each other. In fact, the coherence theorem for braided monoidal categories [FY], [JS1], and [JS2] implies that morphisms of this diagram and their inverses define an action, on each permuted product, of the pure braid group $T(n)$ on n strands. See [Bi] for general background on braid groups.

3.10. Yang-Baxter systems. Let $\mathscr{A}$ be a monoidal category. By a Yang-Baxter system (YB-system) we mean a set of objects $I \subset \mathrm{Ob}\,\mathscr{A}$ and a collection of morphisms

$$R_{A,B}\colon A\otimes B \to B\otimes A, \qquad A, B\in I,$$

such that for any $A, B, C \in I$ the hexagon

$$\begin{tikzcd}
 & B\otimes A\otimes C \arrow[r, "B\otimes R_{A,C}"] & B\otimes C\otimes A \arrow[dr, "R_{B,C}\otimes A"] & \\
A\otimes B\otimes C \arrow[ur, "R_{A,B}\otimes C"] \arrow[dr, "A\otimes R_{B,C}"'] & & & C\otimes B\otimes A \\
 & A\otimes C\otimes B \arrow[r, "R_{A,C}\otimes B"] & C\otimes A\otimes B \arrow[ur, "C\otimes R_{A,B}"'] &
\end{tikzcd}$$

commutes.

For the case $\mathscr{A} = (\mathrm{Vect}, \otimes)$ and I consisting of one vector space V, a YB-system is nothing but a solution of the modified Yang-Baxter equation (1-4) (which is equivalent to the standard form (1-1)). More generally, any solution of the "variable" Yang-Baxter equation (6) gives rise to a YB-system. Namely, let $\mathscr{A}$ be again $(\mathrm{Vect}, \otimes)$. For each element x of the parameter set X, take one copy $V(x)$ of the same vector space V. Define the morphism

$$R_{V(x),V(y)}\colon V(x)\otimes V(y) \to V(y)\otimes V(x)$$

to be given by the operator $P\circ R(x, y)\colon V\otimes V \to V\otimes V$, where P is the permutation. We immediately find that (1-6) is identical to the commutativity of the required hexagon. In physically interesting examples, however, the matrix $R(x, y)$ can have poles for some (x, y) and can be degenerate for some (x, y). The first inconvenience can be amended by multiplying $R(x, y)$ by a scalar function $f(x, y)$, which does not affect the Yang-Baxter equation. However, the degeneracy of $R(x, y)$ cannot be helped in such a way. This is why we have introduced a distinction between notions of a braiding and an isobraiding (Definition 3.1).

Consider, for example, the Yang's solution (1-8):

$$R(x, y) = (x-y)I + \eta P$$

where P is the permutation of $V\otimes V$ and $\eta \in \mathbf{C}$. Since P has eigenvalues ± 1, the matrix $R(x, h)$ will be degenerate for $x - y = \pm\eta$.

Proposition 3.5 implies that if a monoidal category $\mathscr{A}$ is equipped with a braiding then there is a YB-system in $\mathscr{A}$ with the set (class) I being the whole $\mathrm{Ob}\mathscr{A}$ and R_{AB} given by the braiding.

3.11. A braiding from a Yang-Baxter system. Let us associate to any YB-system (I, R) in a monoidal category $\mathscr{A}$ a braided monoidal category $\widetilde{I}$.

Let $\widetilde{\mathscr{A}}$ be the category of formal tensor products of objects of $\mathscr{A}$. By definition, objects of $\widetilde{\mathscr{A}}$ are ordered strings $\langle A_1, \ldots, A_k \rangle$, $A_i \in \mathrm{Ob}\mathscr{A}$, $i \geq 0$ (the empty string $\langle \varnothing \rangle$ is also allowed). For any two such strings $\langle A_1, \ldots, A_k \rangle$, $\langle B_1, \ldots, B_l \rangle$ define

$$\mathrm{Hom}_{\widetilde{\mathscr{A}}}(\langle A_1, \ldots, A_k \rangle, \langle B_1, \ldots, B_l \rangle) = \mathrm{Hom}_{\mathscr{A}}(A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_l).$$

The product of the empty string is set to be the unit object. Define the monoidal structure on $\widetilde{\mathscr{A}}$ to be given on objects by juxtaposition of strings

$$\langle A_1, \ldots, A_k \rangle \otimes \langle B_1, \ldots, B_l \rangle = \langle A_1, \ldots, A_k, B_1 \ldots, B_l \rangle$$

with obvious extension to morphisms (note that $\mathscr{A}$ is assumed strict). In this way we get a new strict monoidal category equivalent to $\mathscr{A}$. The reason for its introduction is the possibility of distinguishing two tensor products in $\mathscr{A}$ which "accidentally" coincide.

Let $\widetilde{I}$ be the smallest monoidal subcategory of $\widetilde{\mathscr{A}}$ containing the objects of the form $\langle A \rangle$, where $A \in I$ and the morphisms $R_{A,B}\colon (A, B) \to (B, A)$. Let $\langle A_1, \ldots, A_k \rangle$, $\langle B_1, \ldots, B_l \rangle$ be two objects of $\widetilde{I}$. Define

$$R_{\langle A_1, \ldots, A_k \rangle, \langle B_1, \ldots, B_l \rangle}\colon (A_1, \ldots, A_k) \otimes (B_1, \ldots, B_l) \\ \to (B_1, \ldots, B_l) \otimes (A_1, \ldots, A_k)$$

as the morphism corresponding to the composition

$$\begin{aligned} A_1 \otimes \cdots \otimes A_k \otimes B_1 \otimes \cdots \otimes B_l &\to A_1 \otimes \cdots \otimes B_1 \otimes A_k \otimes \cdots \otimes B_l \\ \cdots &\to B_1 \otimes A_1 \otimes \cdots \otimes A_k \otimes B_2 \otimes \cdots \otimes B_l \\ \cdots &\to B_1 \otimes B_2 \otimes A_1 \otimes \cdots \otimes A_k \otimes B_3 \otimes \cdots \otimes B_l \\ \cdots &\to B_1 \otimes \cdots \otimes B_l \otimes A_1 \otimes \cdots \otimes A_k. \end{aligned}$$

3.12. THEOREM. *The collection of $R_{\langle A_1, \ldots, A_k \rangle, \langle B_1, \ldots, B_l \rangle}$ is a braiding in the monoidal category $\widetilde{I}$. If all $R_{A,B}$ are isomorphisms, this collection is an isobraiding.*

PROOF. We need to verify the naturality of our "braiding" and the commutativity of the triangles $(\bullet \otimes (\bullet \otimes \bullet))$, $((\bullet \otimes \bullet) \otimes \bullet)$ from Definition 3.1.

The naturality concerns a morphism and an object in $\widetilde{I}$. Note that morphisms in $\widetilde{I}$ are, by definition, generated by elementary morphisms $R_{A,B}$, $A, B \in I$, by means of compositions and tensor products with other objects from $\widetilde{I}$. Therefore to verify the naturality of our "braiding" with respect to the first argument it suffices to show the commutativity of each diagram of the form

$$\begin{tikzcd}
A_1 \cdots A_k B_1 \cdots B_l \arrow[r] \arrow[d] & A_1 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_k B_1 \cdots B_l \arrow[d] \\
B_1 \cdots B_l A_1 \cdots A_k \arrow[r] & B_1 \cdots B_l A_1 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_k .
\end{tikzcd}$$

Both paths on this diagram can be seen as certain edge paths on the positive part $P^+(A_1, \dots, A_k, B_1, \dots, B_l)$ of the permutohedral diagram corresponding to $A_1, \dots, A_k, B_1, \dots, B_l$ (see 3.8). Therefore the assertion follows from the commutativity of this diagram (Proposition 3.9).

4. Monoidal 2-categories

4.1. DEFINITION. Let $\mathscr{A}$ be a strict 2-category. A (*lax*) *monoidal structure on* $\mathscr{A}$ is a collection of the following data:

(1) An object $\mathbf{1} = \mathbf{1}_{\mathscr{A}}$ called the unit object

$(\bullet \otimes \bullet)$ For any two objects $A, B \in \mathscr{A}$ a new object $A \otimes B$, also denoted AB

$(\to \otimes \bullet)$ For any 1-morphism $u\colon A \to A'$ and any object B a 1-morphism $u \otimes B\colon A \otimes B \to A' \otimes B$

$(\bullet \otimes \to)$ For an object A and a morphism $v\colon B \to B'$ a morphism $A \otimes v\colon A \otimes B \to A \otimes B'$

$\left(\substack{\to \\ \downarrow \\ \to} \otimes \bullet\right)$ For any 2-morphism

$$\begin{tikzcd}
A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow T"] & A'
\end{tikzcd}$$

and an object B a 2-morphism

$$\begin{tikzcd}
A \otimes B \arrow[r, bend left, "u \otimes B"] \arrow[r, bend right, "u' \otimes B"'] \arrow[r, phantom, "\Downarrow T \otimes B"] & A' \otimes B
\end{tikzcd};$$

$\left(\bullet \otimes \substack{\to \\ \downarrow \\ \to}\right)$ For any object A and any 2-morphism

$$\begin{tikzcd}
B \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow S"] & B'
\end{tikzcd}$$

a 2-morphism

$$\begin{tikzcd} A\otimes B \arrow[r, bend left, "A\otimes v"{name=U}] \arrow[r, bend right, "A\otimes v'"'{name=D}] & A\otimes B \arrow[from=U, to=D, phantom, "\Downarrow A\otimes S"] \end{tikzcd}$$

$(\bullet\otimes\bullet\otimes\bullet)$ For any three objects A, B, C an isomorphism $a_{A,B,C}: A\otimes(B\otimes C)\to(A\otimes B)\otimes C$

$(1\otimes\bullet)$ For any object A an isomorphism $l_A: \mathbf{1}\otimes A\to A$

$(\bullet\otimes 1)$ For any object A an isomorphism $r_A: A\otimes\mathbf{1}\to A$

$(\to\otimes\to)$ For any two morphisms $u: A\to A'$, $v: B\to B'$ a 2-isomorphism

$$\begin{tikzcd} A\otimes B \arrow[r, "A\otimes v"] \arrow[d, "u\otimes B"'] \arrow[dr, phantom, "\swarrow\otimes_{u,v}"] & A\otimes B' \arrow[d, "u\otimes B'"] \\ A'\otimes B \arrow[r, "A'\otimes v"] & A'\otimes B' \end{tikzcd}$$

$(\to\to\otimes\bullet)$ For any pair of composable morphisms $A\xrightarrow{u}A'\xrightarrow{u'}A''$ and an object B a 2-isomorphism

$$\begin{tikzcd} A\otimes B \arrow[r, "(u'u)\otimes B"] \arrow[d, "u\otimes B"'] \arrow[dr, phantom, "\Uparrow\otimes_{u,u',B}"] & A''\otimes B \\ A'\otimes B \arrow[ur, "u'\otimes B"'] & \end{tikzcd}$$

$(\bullet\otimes\to\to)$ For any object A and any pair of composable morphisms $B\xrightarrow{v}B'\xrightarrow{v'}B''$ a 2-isomorphism

$$\begin{tikzcd} A\otimes B \arrow[r, "A\otimes(v'v)"] \arrow[d, "A\otimes v"'] \arrow[dr, phantom, "\Uparrow\otimes_{A,v,v'}"] & A\otimes B'' \\ A\otimes B' \arrow[ur, "A\otimes v'"'] & \end{tikzcd}$$

$(\bullet\otimes\bullet\otimes\bullet\otimes\bullet)$ For any four objects A, B, C, D a 2-morphism

$$\begin{tikzcd}
 & A\otimes(B\otimes(C\otimes D)) \arrow[dl, "a_{A,B,C\otimes D}"'] \arrow[dr, "A\otimes a_{B,C,D}"] & \\
(A\otimes B)\otimes(C\otimes D) \arrow[d, "a_{A\otimes B,C,D}"'] & \arrow[d, phantom, "\overset{a_{A,B,C,D}}{\Leftarrow}"] & A\otimes((B\otimes C)\otimes D) \arrow[d, "a_{A,B\otimes C,D}"] \\
((A\otimes B)\otimes C)\otimes D & & (A\otimes(B\otimes C))\otimes D. \arrow[ll, "a_{A,B,C}\otimes D"']
\end{tikzcd}$$

$(\to\ \otimes \bullet \otimes\bullet)$ For any morphism $u\colon A \to A'$ and objects B, C a 2-isomorphism

$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "u\otimes(B\otimes C)"'] \arrow[dr, phantom, "\swarrow a_{u,B,C}"] & (A\otimes B)\otimes C \arrow[d, "(u\otimes B)\otimes C"] \\
A'\otimes(B\otimes C) \arrow[r, "a_{A',B,C}"] & (A'\otimes B)\otimes C
\end{tikzcd}$$

$(\bullet\otimes \to \otimes\bullet)$ For any object A, morphism $v\colon B\to B'$, and another object C a 2-isomorphism

$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "A\otimes(v\otimes C)"'] \arrow[dr, phantom, "\swarrow a_{A,v,C}"] & (A\otimes B)\otimes C \arrow[d, "(A\otimes v)\otimes C"] \\
A\otimes(B'\otimes C) \arrow[r, "a_{A,B',C}"] & (A\otimes B')\otimes C
\end{tikzcd}$$

$(\bullet\otimes\bullet\otimes\to)$ For any objects A, B and a morphism $w\colon C\to C'$ a 2-isomorphism

$$\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "a_{A,B,C}"] \arrow[d, "A\otimes(B\otimes w)"'] \arrow[dr, phantom, "\swarrow a_{A,B,w}"] & (A\otimes B)\otimes C \arrow[d, "(A\otimes B)\otimes w"] \\
A\otimes(B\otimes C') \arrow[r, "a_{A,B,C'}"] & (A\otimes B)\otimes C'
\end{tikzcd}$$

$(\bullet\otimes\bullet\otimes 1)$ For any two objects A, B a 2-isomorphism

$$\begin{tikzcd}
A \otimes (B \otimes \mathbf{1}) \arrow[r, "A\otimes r_B"] \arrow[d, "a_{A,B,1}"'] & A \otimes B \\
(A \otimes B) \otimes \mathbf{1} \arrow[ur, "r_{A\otimes B}"'] \arrow[u, phantom, "\Uparrow \rho_{A,B}"]
\end{tikzcd}$$

$(1 \otimes \bullet \otimes \bullet)$ For any two objects A, B a 2-isomorphism

$$\begin{tikzcd}
\mathbf{1} \otimes (A \otimes B) \arrow[r, "l_{A\otimes B}"] \arrow[d, "a_{1,A,B}"'] & A \otimes B \\
(\mathbf{1} \otimes A) \otimes B \arrow[ur, "l_A\otimes B"'] \arrow[u, phantom, "\Uparrow \lambda_{a,b}"]
\end{tikzcd}$$

$(\bullet \otimes 1 \otimes \bullet)$ For any objects A, B a 2-isomorphism

$$\begin{tikzcd}
A \otimes (\mathbf{1} \otimes B) \arrow[r, "A\otimes l_B"] \arrow[d, "a_{A,1,B}"'] & A \otimes B \\
(A \otimes \mathbf{1}) \otimes B \arrow[ur, "r_A\otimes B"'] \arrow[u, phantom, "\Uparrow \mu_{A,B}"]
\end{tikzcd}$$

$(1\otimes \rightarrow)$ For any 1-morphism $u\colon A \rightarrow A'$ a 2-isomorphism

$$\begin{tikzcd}
\mathbf{1} \otimes A \arrow[r, "1\otimes u"] \arrow[d, "l_A"'] \arrow[dr, phantom, "\swarrow l_u"] & \mathbf{1} \otimes A' \arrow[d, "l_{A'}"] \\
A \arrow[r, "u"] & A'
\end{tikzcd}$$

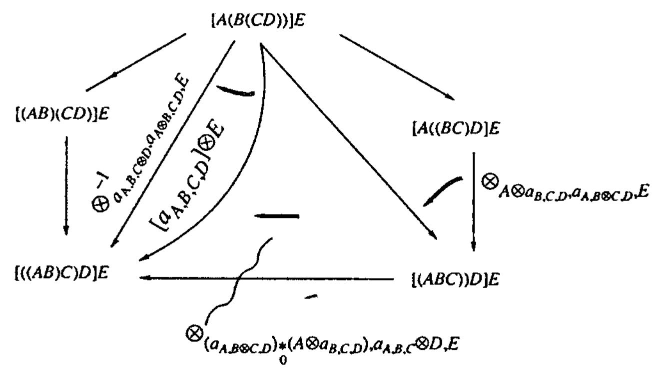
$(\rightarrow \otimes 1)$ For any 1-morphism $u\colon A \rightarrow A'$ a 2-isomorphism

$$\begin{tikzcd}
A \otimes \mathbf{1} \arrow[r, "u\otimes 1"] \arrow[d, "r_V"'] \arrow[dr, phantom, "\swarrow r_u"] & A' \otimes \mathbf{1} \arrow[d, "r_{A'}"] \\
A \arrow[r, "u"] & A'
\end{tikzcd}$$

$(1 \otimes 1)$ A 2-isomorphism

$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1} \arrow[r, bend left, "r_1"] \arrow[r, bend right, "l_1"'] \arrow[r, phantom, "\Downarrow \varepsilon"] & \mathbf{1}
\end{tikzcd}$$

Before listing the conditions on the structure data, let us introduce the modified tensor product $\hat{\otimes}$ which will be used for multiplication of one of the above structure 2-morphisms and an object. For example, $a_{A,B,C,D}\hat{\otimes}E$ will denote the 2-morphism



Note that since $a_{A,B,C,D}$ is a 2-morphism

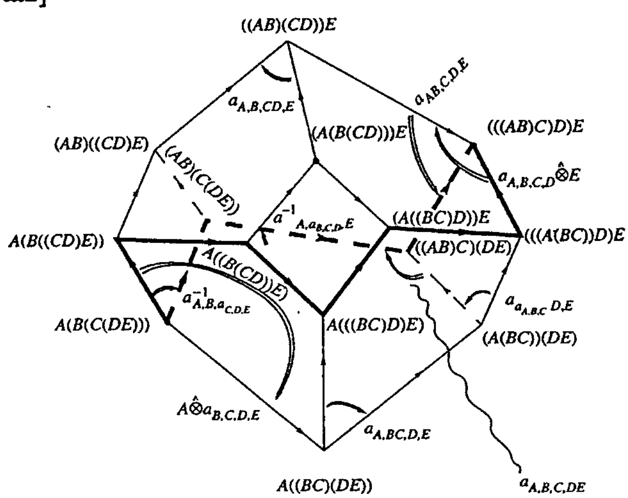
$$(a_{A,B,C}\otimes D)*_0(a_{A,B\otimes C,D})*_0(A\otimes a_{B,C,D})\Rightarrow(a_{A\otimes B,C,D})*_0(a_{A,B,C\otimes D}),$$

the usual product $a_{A,B,C,D}\otimes E$ (in the middle of the preceding picture) is a 2-morphism from the product of the left composition with E to the product of the right composition with E. The 2-morphism $a_{A,B,C,D}\hat{\otimes}E$ acts between $*_0$-composition of the products with E of individual 1-morphisms in the above formula.

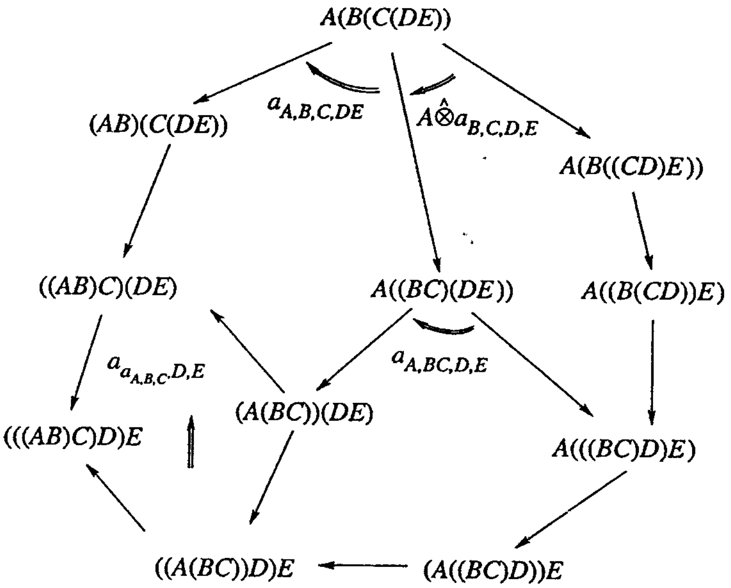
The $\hat{\otimes}$-product for other types of structure 2-morphisms is defined similarly.

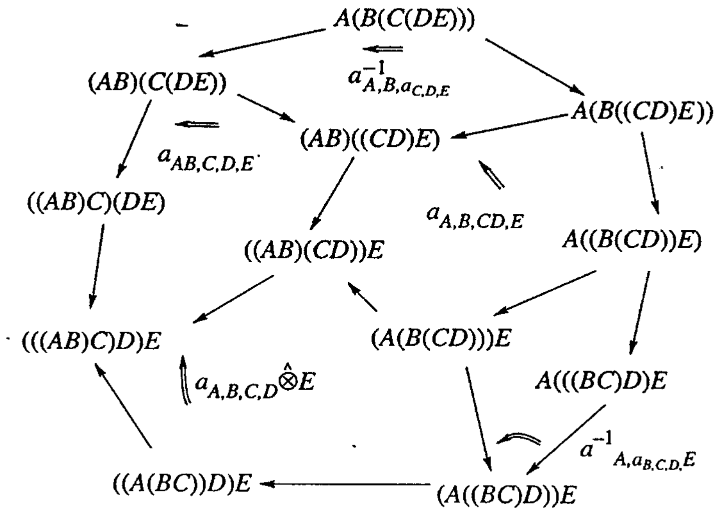
Now the conditions for the structure data in a monoidal 2-category are as follows:

($\bullet\otimes\bullet\otimes\bullet\otimes\bullet\otimes\bullet$) For any five objects A, B, C, D, E the Stasheff polytope **[Sta1 and Sta2]**

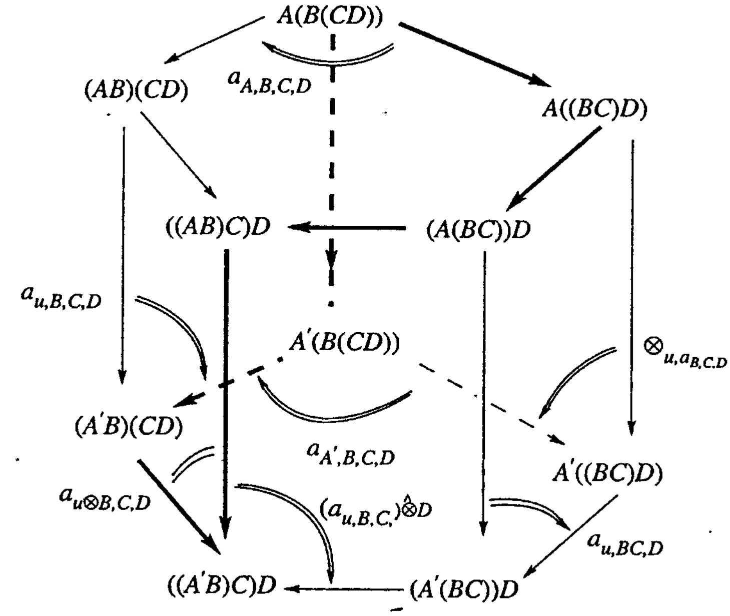


is commutative. As we explained in 2.13, the commutativity of a polytope means that the pastings of its two composable halves coincide. In the present case these halves are





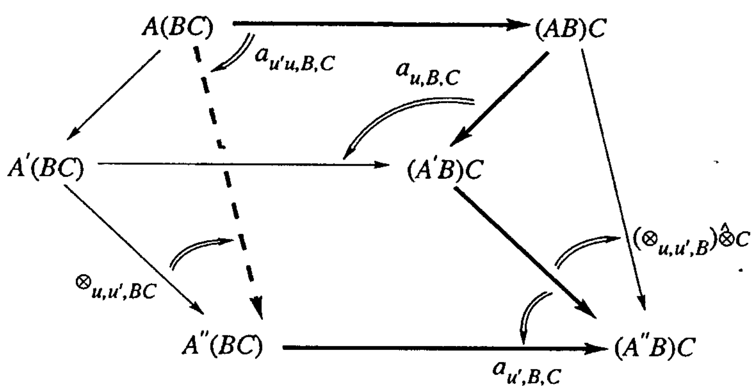
$(\to \otimes \bullet \otimes \bullet \otimes \bullet)$ For any 1-morphism $u\colon A \to A'$ and objects B, C, D the pentagonal prism



is commutative.

$(\bullet\otimes \to \otimes\bullet\otimes\bullet)$, $(\bullet\otimes\bullet\otimes \to \otimes\bullet)$, $(\bullet\otimes\bullet\otimes\bullet\otimes \to)$ Similar prisms corresponding to 1-morphism in the second, third, or fourth factor.

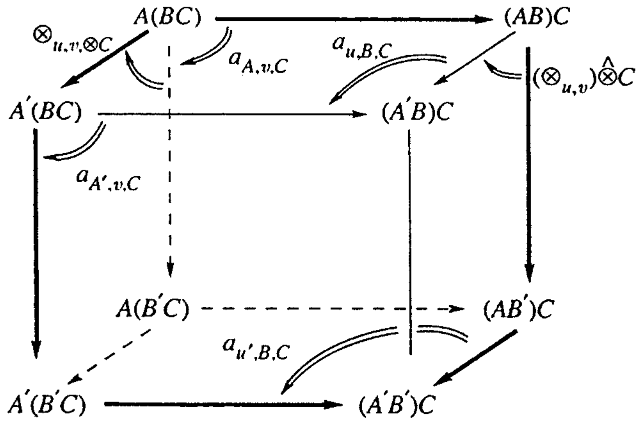
$(\to\to \otimes\bullet\otimes\bullet)$ For any composable pair $A \xrightarrow{u} A' \xrightarrow{u'} A''$ of 1-morphisms and objects B, C the diagram



is commutative.

$(\bullet\otimes \to\to \otimes\bullet)$, $(\bullet\otimes\bullet\otimes \to\to)$ Similar prisms corresponding to a composable pair of 1-morphisms in the second or third factor.

$(\to \otimes \to \otimes\bullet)$ For any two 1-morphisms $u: A \to A'$, $v: B \to B'$ and an object C the cube



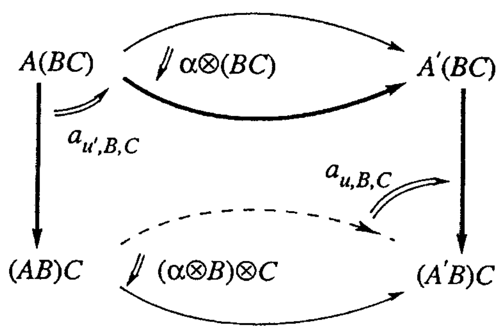
is commutative.

$(\to \otimes \bullet \otimes \to)$, $(\bullet\otimes \to \otimes \to)$ Similar cubes. Left to the reader.

$\left(\underset{\to}{\overset{\to}{\downarrow}} \otimes \bullet \otimes \bullet\right)$ For any 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \end{tikzcd}$$

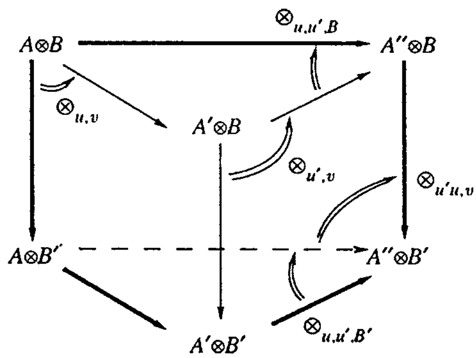
and objects B, C the cylinder



is commutative.

$\left(\bullet \otimes \underset{\to}{\overset{\to}{\downarrow}} \otimes \bullet\right)$, $\left(\bullet \otimes \bullet \otimes \underset{\to}{\overset{\to}{\downarrow}}\right)$ Similar cylinders. Left to the reader.

$(\to\to \otimes \to)$ For any composable pair of 1-morphisms $A \xrightarrow{u} A' \xrightarrow{u'} A''$ and another 1-morphism $v\colon B \to B'$ the triangular prism.



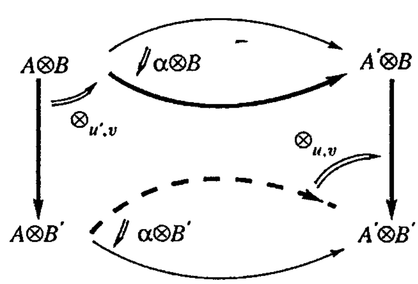
is commutative.

$(\to \otimes \to\to)$ Similar prism. Left to the reader

$\left(\substack{\to\\ \downarrow\\ \to} \otimes \to\right)$ For any 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \end{tikzcd}$$

and a 1-morphism $v\colon B \to B'$ the cylinder



is commutative.

$\left(\to \otimes \substack{\to\\ \downarrow\\ \to}\right)$ Similar cylinder. Left to the reader.

$\left(\substack{\to\\ \downarrow\\ \to\\ \downarrow\\ \to} \otimes \bullet\right)$ If $u \overset{\alpha}{\Rrightarrow} u' \overset{\alpha'}{\Rrightarrow} u''$ is a 1-composable pair of 2-morphisms and B is any object, then

$$(\alpha' *_1 \alpha) \otimes B = (\alpha' \otimes B) *_1 (\alpha \otimes B).$$

$\left(\bullet \otimes \substack{\to\\ \downarrow\\ \to\\ \downarrow\\ \to}\right)$ Left to the reader.

$\left(\substack{\to\\ \downarrow\\ \dashrightarrow} \to \otimes\bullet\right)$ Let

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \arrow[r, "v"] & A'' \end{tikzcd}$$

be a 0-composable pair of a 2-morphism and a 1-morphism, and B an object. Then the 2-morphism

$$
\begin{tikzcd}
A\otimes B \arrow[r, bend left, "u\otimes B"] \arrow[r, bend right, "u'\otimes B"'] \arrow[r, phantom, "\Downarrow \alpha\otimes B"] \arrow[rr, bend left=70] \arrow[rr, bend right=70] \arrow[rr, phantom, bend left=45, "\Downarrow \otimes^{-1}_{u,v,B}"] \arrow[rr, phantom, bend right=45, "\Downarrow \otimes_{u',v,B}"'] & A'\otimes B \arrow[r, "v\otimes B"] & A''\otimes B
\end{tikzcd}
$$

coincides with $(v *_0 \alpha)\otimes B$.

$\left(\rightarrow \begin{smallmatrix}\rightarrow\\ \downarrow\\ \dashrightarrow\end{smallmatrix} \otimes \bullet\right)$, $\left(\bullet \otimes \begin{smallmatrix}\rightarrow\\ \downarrow\\ \dashrightarrow\end{smallmatrix} \rightarrow\right)$, $\left(\bullet\otimes \rightarrow \begin{smallmatrix}\rightarrow\\ \downarrow\\ \dashrightarrow\end{smallmatrix}\right)$ Left to the reader.

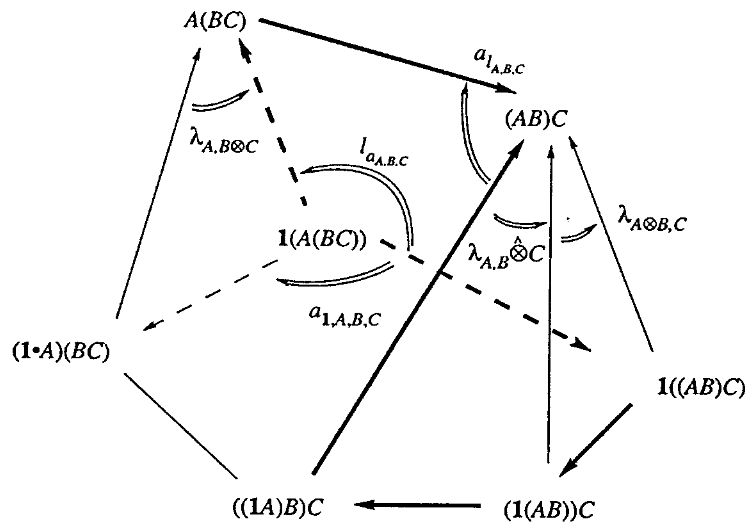
$(\rightarrow\rightarrow\rightarrow\ \otimes\bullet)$ For any composable triple $A \xrightarrow{u} A' \xrightarrow{u'} A'' \xrightarrow{u''} A'''$ of 1-morphisms and an object B the tetrahedron

$$
\begin{tikzcd}
 & A'''\otimes B & \\
A\otimes B \arrow[ur, "\otimes_{u,u''u',B}"] \arrow[rr, dashed] \arrow[dr] \arrow[r, phantom, "\otimes_{u'u,u'',B}"] & & A''\otimes B \arrow[ul] \\
 & A'\otimes B \arrow[uu] \arrow[ur] \arrow[ul, phantom, "\otimes_{u,u',B}"] \arrow[uu, phantom, "\otimes_{u',u'',B}"'] &
\end{tikzcd}
$$

commutes.

$(\bullet\otimes \rightarrow\rightarrow\rightarrow)$ Similarly.

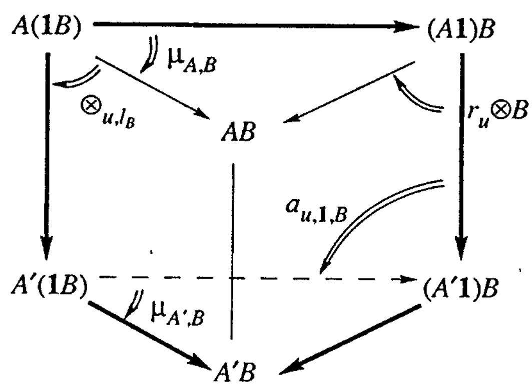
$(1\otimes\bullet\otimes\bullet\otimes\bullet)$ For any three objects A, B, $C \in \mathscr{A}$ the polytope



commutes.

$(\bullet\otimes 1\otimes\bullet\otimes\bullet)$, $(\bullet\otimes\bullet\otimes 1\otimes\bullet)$, $(\bullet\otimes\bullet\otimes\bullet\otimes 1)$ Similarly.
These polytopes can be called Kelly polytopes since their boundaries give the diagrams used by Kelly [Ke 2] in his proof that axiom $(\bullet\otimes 1\otimes\bullet)$ for monoidal categories implies axioms $(\bullet\otimes\bullet\otimes 1)$ and $(1\otimes\bullet\otimes\bullet)$.

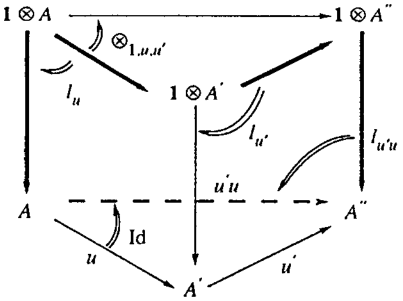
$(\to\otimes 1\otimes\bullet)$ For any 1-morphism $u\colon A\to A'$ and an object B the triangular prism



is commutative.

$(1\otimes\to\otimes\bullet)$, $(\to\otimes\bullet\otimes 1)$, $(\bullet\otimes 1\otimes\to)$, $(1\otimes\bullet\otimes\to)$, $(\bullet\otimes\to\otimes 1)$ Left to the reader.

$(1\otimes\to\to)$ For any composable pair of 1-morphisms $A\xrightarrow{u}A'\xrightarrow{u'}A''$ the prism



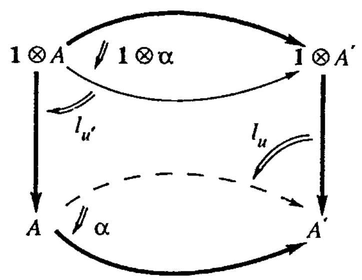
is commutative.

($\to\to\otimes 1$) Similarly.

$\left(1 \otimes \overset{\rightarrow}{\underset{\rightarrow}{\downarrow}}\right)$ For any 2-morphism

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \end{tikzcd}$$

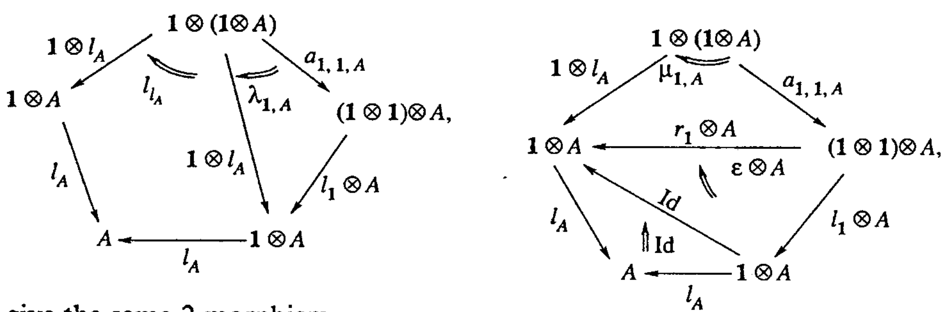
the cylinder



is commutative.

$\left(\overset{\rightarrow}{\underset{\rightarrow}{\downarrow}} \otimes 1\right)$ Similarly.

($1 \otimes 1 \otimes \bullet$) For any object $A \in \mathscr{A}$ the two pastings



give the same 2-morphism.

($1 \otimes \bullet \otimes 1$), ($\bullet \otimes 1 \otimes 1$) Similarly.

4.2. Example. Consider a 2-category with one object. As we have seen (2.10), such 2-categories correspond to monoidal (1-) categories. We denote by $\Omega^{-1}(\mathscr{A})$ the 2-category corresponding to a braided monoidal 1-category $(\mathscr{A}, \otimes)$. Suppose that $\mathscr{A}$ is equipped with an isobraiding R. Introduce on

$\Omega^{-1}\mathscr{A}$ the monoidal structure $\odot$ setting

(1) $pt \odot x = x \odot pt = x$ for any i-morphism x of $\Omega^{-1}\mathscr{A}$, $i = 0, 1, 2$;

(2) for any pair of 1-morphisms $A: pt \to pt$, $B: pt \to pt$ of $\Omega^{-1}\mathscr{A}$ (i.e., objects of $\mathscr{A}$) the 2-morphism

$$\begin{tikzcd} pt \odot pt \arrow[r, "A \odot pt"] \arrow[d, "pt \odot B"'] & pt \odot pt \arrow[d, "pt \odot B"] \arrow[dl, phantom, "\swarrow \odot_{A,B}"] \\ pt \odot pt \arrow[r, "A \odot pt"'] & pt \odot pt \end{tikzcd}$$

of $\Omega^{-1}\mathscr{A}$ corresponds to 1-morphism $R_{B,A}: B \otimes A \to A \otimes B$ in $\mathscr{A}$ given by the braiding;

(3) all other structure 1- and 2-morphisms are identities.

We leave to the reader the verification that this indeed gives a monoidal 2-category. Thus a general notion of a monoidal 2-category can be seen as a "distributed" version of the notion of a braided monoidal category (in the same sense in which a category is a "distributed" semigroup).

4.3. Strict and semistrict monoidal 2-categories. A monoidal 2-category $\mathscr{A}$ is called *strict* if all structure 1- and 2-morphisms are identities. We say that $\mathscr{A}$ is *semistrict* if all structure 1- and 2-morphisms except $\otimes_{u,v}$ are identities. For example, the monoidal 2-category $\Omega^{-1}(\mathscr{A})$ constructed in 4.2 is semistrict.

A strict monoidal 2-category can be delooped, in a way similar to that of 2.10, to a strict 3-category with one object. The notion of a strict n-category for any n was defined in [S]. Various lax versions of the notion of a monoidal 2-category can be regarded as approximations to the would-be notion of a fully lax 3-category.

Unlike the case of monoidal 1-categories it is not always possible to replace a lax monoidal 2-category with an "equivalent" strict one. Indeed, in Example 4.2 we have encoded the braiding in a monoidal 1-category $\mathscr{A}$ as the "laxness" of the monoidal structure on $\Omega^{-1}(\mathscr{A})$ (the data $(\to \otimes \to)$). It is clearly not possible to get rid of the braiding.

The right coherence theorem for monoidal 2-categories is that every lax monoidal 2-category is lax monoidal 2-equivalent to a semistrict one. To even define the notion of monoidal 2-equivalence would require scores of pages. This theorem will be proven in [KV 2]. A still more general coherence theorem was recently announced by Gordon, Power, and Street.

5. 2-vector spaces

5.1. Introduction: 2-matrices. In §2 we encountered, as a basic example of a monoidal category, the category Vect of complex finite-dimensional vector spaces and its coordinatized analog Vect_c, which has the advantage of being a strict monoidal category. Now we are going to construct their 2-categorical analog. More precisely, we are going to construct the 2-category of 2-vector spaces in three versions: 2-Vect, 2-Vect_c, and 2-Vect_{cc}, which differ by the level of coordinatization. The idea of 2-vector spaces was induced by Segal's definition of conformal field theory [**Se, Mo-S**], especially by the concept of modular functor.

By a *2-matrix* we shall mean a "matrix" $\|V_{ij}\|$, $i = 1, \dots, k$, $j = 1, \dots, l$, whose entries V_{ij} are not numbers but finite-dimensional complex vector spaces. By taking dimensions of these vector spaces we obtain a numerical matrix. Similarly, we define a *2-vector* of length k to be just a k-tuple $(V_1, \dots, V_k)$ of vector spaces.

Given a $(k \times l)$ 2-matrix $V = \|V_{ij}\|$ and an $(l \times m)$ 2-matrix $W = \|W_{jp}\|$, we define their 2-matrix product to be the 2-matrix $V * W = Z = \|Z_{ip}\|$, where we set $Z_{ip} = \bigoplus_j (V_{ij} \otimes W_{jp})$, Given a $(k \times l)$ 2-matrix $V = \|V_{ij}\|$ and a 2-vector $E = (E_1, \dots, E_k)$ of length k, we define the 2-vector VE of length l to have components $(VE)_i = \bigoplus_j (V_{ij} \otimes E_j)$.

If V, W, X are three 2-matrices of formats $(k \times l)$, $(l \times m)$, $(m \times n)$, then the 2-matrices $V * (W * X)$ and $(V * W) * X$ are not exactly equal, but their corresponding entries are connected by natural isomorphisms which are obtained from the standard associativity and distributivity isomorphisms for operations $\oplus$, $\otimes$ on vector spaces. Now we define our first version of the 2-category of 2-vector spaces.

5.2. Definition. The (lax) 2-category 2-Vect_c of coordinatized 2-vector spaces has objects $\{n\}$, $n = 0, 1, 2, \dots$. A 1-morphism from $\{m\}$ to $\{n\}$ is an $(m \times n)$ 2-matrix $V = \|V_{ij}\|$. The composition of 1-morphisms is given by the 2-matrix product defined above. A 2-morphism $T \colon V \Rightarrow W$, where $V, W \colon \{m\} \to \{n\}$, is a family of linear operators $T_{ij} \colon V_{ij} \to W_{ij}$. The $*_1$-composition of 2-morphisms is given by the usual composition of linear operators.

We have left to the reader the definition of the $*_1$-composition of a 1-morphism and a 2-morphism as well as the construction of the canonical associativity 2-isomorphism $V * (W * X) \Rightarrow (V * W) * X$. It is rather straightforward to see that these data indeed define a 2-category.

Definition 5.2 is an exact analog of the definition of the category of vector space in terms of matrices (see 2.5). It is natural, therefore, to seek a more "coordinate-free" definition. To arrive at such a definition, note that the notion of a 2-matrix appeals to the structure on the category Vect which we have not yet considered, namely that of a *ring category* given by operations $\oplus$, $\otimes$. This structure was formalized by Kelly [**Ke1**] and Laplaza [**L**]. In the following definitions we employ, as previously, hierogliphical notation.

5.3. Definition. A *ring category* is a category $\mathscr{R}$ equipped with two monoidal structures $\oplus$, $\otimes$ (which include corresponding associativity morphisms $a^{\oplus}_{A,B,C}$, $a^{\otimes}_{A,B,C}$ and unit objects denoted $\mathbf{0}$, $\mathbf{1}$) together with natural isomorphisms

$$u_{A,B}\colon A\oplus B\to B\oplus A,\qquad v_{A,B,C}\colon A\otimes(B\oplus C)\to(A\otimes B)\oplus(A\otimes C),$$
$$w_{A,B,C}\colon (A\oplus B)\otimes C\to(A\otimes C)\oplus(B\otimes C),$$
$$x_A\colon A\otimes\mathbf{0}\to\mathbf{0},\quad y_A\colon \mathbf{0}\otimes A\to\mathbf{0}.$$

These isomorphisms are required to satisfy the following conditions.

$(\bullet\oplus\bullet)$ The isomorphisms $u_{A,B}$ define on $\mathscr{R}$ a structure of a symmetric monoidal category i.e., they form a braiding and $u_{A,B}u_{B,A}=1$.

$(\bullet\otimes(\bullet\oplus\bullet))$ For any objects A, B, C the diagram

$$\begin{tikzcd}
A\otimes(B\oplus C) \arrow[r, "v_{A,B,C}"] \arrow[d, "A\otimes u_{B,C}"'] & (A\otimes B)\oplus(A\otimes C) \arrow[d, "u_{A\otimes B,A\otimes C}"] \\
A\otimes(C\otimes B) \arrow[r, "v_{A,C,B}"] & (A\otimes C)\oplus(A\otimes B)
\end{tikzcd}$$

is commutative.

$((\bullet\oplus\bullet)\otimes\bullet)$ For any objects A, B, C the diagram

$$\begin{tikzcd}
(A\oplus B)\otimes C \arrow[r, "w_{A,B,C}"] \arrow[d, "u_{A,B}\otimes C"'] & (A\otimes C)\oplus(B\otimes C) \arrow[d, "u_{A\otimes C,B\otimes C}"] \\
(B\oplus A)\otimes C \arrow[r, "w_{B,A,C}"] & (B\oplus C)\oplus(A\otimes C)
\end{tikzcd}$$

is commutative.

$((\bullet\oplus\bullet\oplus\bullet)\otimes\bullet)$ For any objects A, B, C, D the diagram

$$\begin{tikzcd}
((A\oplus(B\oplus C))D \arrow[r, "w_{A,B\oplus C,D}"] \arrow[d, "a^{\oplus}_{A,B,C}\otimes D"'] & AD\oplus((B\oplus C)D) \arrow[r, "AD\oplus w_{B,C,D}"] & AD\oplus(BD\oplus CD) \arrow[d, "a^{\oplus}_{AD,BD,CD}"] \\
((A\oplus B)\oplus C)D \arrow[r, "w_{A\oplus B,C,D}"] & (A\oplus B)D\oplus CD \arrow[r, "w_{A,B,D}\oplus CD"] & (AD\oplus BD)\oplus CD
\end{tikzcd}$$

is commutative.

$(\bullet\otimes(\bullet\oplus\bullet\oplus\bullet))$ Similar to the above. Left to the reader.

$(\bullet\otimes\bullet\otimes(\bullet\oplus\bullet))$ For any objects A, B, C, D the diagram

$$\begin{tikzcd}
A(B(C\oplus D)) \arrow[r, "A\otimes v_{B,C,D}"] \arrow[d, "a^{\otimes}_{A,B,C\oplus D}"'] & A(BC\oplus BD) \arrow[r, "v_{A,BC,BD}"] & A(BC)\oplus A(BD) \arrow[d, "a^{\otimes}_{A,B,C}\oplus a^{\otimes}_{A,B,D}"] \\
(AB)(C\oplus D) \arrow[rr, "v_{AB,C,D}"] & & (AB)C\oplus(AB)D
\end{tikzcd}$$

is commutative.

$(\bullet\oplus\bullet)\otimes\bullet\otimes\bullet)$ Similar to the above. Left to the reader.

$(\bullet\otimes(\bullet\otimes\bullet)\otimes\bullet)$ Similar to the above. Left to the reader

$((\bullet\oplus\bullet)\otimes(\bullet\oplus\bullet))$ For any objects A, B, C, D the diagram

$$\begin{tikzcd}
(A\oplus B)(C\oplus D) \arrow[r] \arrow[d] & A(C\oplus D)\oplus B(C\oplus D) \arrow[r] & (AC\oplus AD)\oplus(BC\oplus BD) \arrow[d] \\
(A\oplus B)C\oplus(A\oplus B)D \arrow[d] & & ((AC\oplus AD)\oplus BC)\oplus BD \arrow[d] \\
(AC\oplus BC)\oplus(AD\oplus BD) \arrow[d] & & (AC\oplus(AD\oplus BC))\oplus BD \arrow[dl] \\
((AC\oplus BC)\oplus AD)\oplus BD \arrow[r] & (AC\oplus(BC\oplus AD))\oplus BD &
\end{tikzcd}$$

is commutative (we have omitted the notation for arrows; they are obvious).

$(0\otimes 0)$ The maps $x_0, y_0\colon \mathbf{0}\otimes\mathbf{0}\to\mathbf{0}$ coincide.

$(0\otimes(\bullet\oplus\bullet))$ For any objects A, B the diagram

$$\begin{tikzcd}
\mathbf{0}\otimes(A\oplus B) \arrow[r, "v_{0,A,B}"] \arrow[d, "y_{A\oplus B}"] & (\mathbf{0}\otimes A)\oplus(\mathbf{0}\otimes B) \arrow[d, "y_A\oplus y_B"] \\
\mathbf{0} & \mathbf{0} \arrow[l, "l_0^{\oplus}=r_0^{\oplus}"']
\end{tikzcd}$$

is commutative.

$((\bullet\oplus\bullet)\otimes 0)$ Similar to the above. Left to the reader.

$(0\otimes 1)$ The morphisms $x_1, r_0^{\otimes}\colon \mathbf{0}\otimes\mathbf{1}\to\mathbf{0}$ coincide.

$(1\otimes 0)$ Similarly.

$(0\otimes\bullet\otimes\bullet)$ For any objects A, B the diagram

$$\begin{tikzcd}
\mathbf{0}\otimes(A\otimes B) \arrow[r, "a^{\otimes}_{0,A,B}"] \arrow[d, "y_{A\otimes B}"'] & (\mathbf{0}\otimes A)\otimes B \arrow[d, "y_A\otimes B"] \\
\mathbf{0} & \mathbf{0}\otimes B \arrow[l, "y_B"']
\end{tikzcd}$$

is commutative.

$(\bullet\otimes 0\otimes\bullet)$, $(\bullet\otimes\bullet\otimes 0)$ Similarly.

$(\bullet\otimes(0\oplus\bullet))$ For any objects A, B the diagram

$$\begin{tikzcd}
A\otimes(\mathbf{0}\oplus B) \arrow[r, "v_{A,0,B}"] \arrow[d, "A\otimes l_B^{\oplus}"'] & (A\otimes\mathbf{0})\oplus(A\otimes B) \arrow[d, "x_A\oplus(A\otimes B)"] \\
A\otimes B & \mathbf{0}\oplus(A\otimes B) \arrow[l, "l^{\oplus}_{A\otimes B}"']
\end{tikzcd}$$

is commutative.

$((0\oplus\bullet)\otimes\bullet)$, $(\bullet\otimes(\bullet\oplus 0))$, $((\bullet\oplus 0)\otimes\bullet)$ Similarly.

Note that our definition of a ring category differs a little from the one given by Laplaza [L] in that we do not assume any kind of commutativity (or braiding) for the multiplication $\otimes$, though we do assume commutativity for the addition.

5.4. Definition. Let $\mathscr{R}$ be a ring category. The lax 2-category $\mathscr{R}$-Mod_c of coordinatized free $\mathscr{R}$-modules has objects $\{n\}$, $n = 0, 1, 2, \ldots$, 1-morphisms matrices with entries from $\mathrm{Ob}\mathscr{R}$, and 2-morphisms matrices of 1-morphisms in $\mathscr{R}$.

5.5. Definition. A ring category $\mathscr{R}$ is called *strict* if both monoidal structures $\oplus$, $\otimes$ are strict and the structure morphisms $v_{A,B,C}$, $w_{A,B,C}$, x_A, y_A (note that $u_{A,B}$ is not included) are identities.

5.6. Example. The monoidal category Vect_c of coordinatized vector spaces introduced in 2.5, can be equipped with a structure of a strict ring category. On objects we set $[m]\oplus[n] = [m+n]$, $[m]\otimes[n] = [mn]$. If $[m]$ is an object and $A\colon [n] \to [p]$ is a morphism i.e., an $(n\times p)$-matrix $\|a_{ij}\|$ then we set

$$[m]\oplus A = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & A \end{pmatrix}, \qquad A\oplus[m] = \begin{pmatrix} A & 0 \\ 0 & \mathbf{1}_m \end{pmatrix},$$

where $\mathbf{1}_m$ is the unit $(m\times m)$ matrix, and

$$[m]\otimes A = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A \end{pmatrix},$$

$$A\otimes[m] = \begin{pmatrix} a_{11}\cdot\mathbf{1}_m & a_{12}\cdot\mathbf{1}_m & \cdots & a_{1p}\cdot\mathbf{1}_m \\ a_{21}\cdot\mathbf{1}_m & a_{22}\cdot\mathbf{1}_m & \cdots & a_{2p}\cdot\mathbf{1}_m \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}\cdot\mathbf{1}_m & a_{n2}\cdot\mathbf{1}_m & \cdots & a_{np}\cdot\mathbf{1}_m \end{pmatrix}$$

The zero and unit objects are [0] and [1].

5.7. Coherence for ring categories. The main result of the coherence theory for ring category developed by Kelly [Kel1] and Laplaza [L] can be stated by saying that every lax ring category $\mathscr{R}$ is equivalent (in a suitable sense) to a strict one $\widetilde{\mathscr{R}}$. The "rectification" $\widetilde{\mathscr{R}}$ can be constructed from $\mathscr{R}$ in a canonical way. Namely, objects of $\widetilde{\mathscr{R}}$ are formal polynomials $\bigoplus(A_{i_1}\otimes\cdots\otimes A_{i_p})$ in objects of $\mathscr{R}$. Morphisms between such polynomials are set to be morphisms in $\mathscr{R}$ between their actual values (evaluated with respect to some preferred bracketing). Operations $\oplus$, $\otimes$ on formal polynomials are defined in an obvious way and are strictly associative and distributive. Addition or multiplication of an object and a morphism requires the use of coherence isomorphisms. We leave these details to the reader. The reader is also referred to the original paper of Laplaza [L]. The following fact is straightforward.

5.8. Proposition. *If $\mathscr{R}$ is a strict ring category then the 2-category $\mathscr{R}$-mod_c is also strict.*

Thus we can obtain a strict version of the 2-category of 2-vector spaces either by rectifying the lax 2-category 2-Vect_c (by using the Mac Lane-Pare theorem, see 2.12) or by considering 2-matrices whose entries are not vector spaces but objects of strict ring categories $\widetilde{\mathrm{Vect}}$ or Vect_c. The latter will be especially useful for us.

5.9. DEFINITION. The strict 2-category $2\text{-}\mathrm{Vect}_{cc}$ of totally coordinatized 2-vector spaces is $\mathrm{Vect}_c\text{-}\mathrm{mod}_c$.

Thus objects of $2\text{-}\mathrm{Vect}_{cc}$ are symbols $\{n\}$, and 1-morphisms $\{m\} \to \{n\}$ are matrices with entries $[a_{ij}]$ where $a_{ij} \in \mathbf{Z}_+$. We can equally well say that 1-morphisms are just matrices $A = \|a_{ij}\|$ with nonnegative integer entries. The composition of 1-morphisms is given by the usual matrix multiplication. A 2-morphism between two integral matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ of the same format is a collection of usual complex matrices T_{ij} of formats $(a_{ij}) \times (b_{ij})$. In other words, this is a rectangular matrix whose entries are rectangular matrices of various formats. We pay so much attention to strict versions of 2-categories since we are eventually interested in a monoidal structure on them, a notion which we have defined only for the strict case.

5.10. 2-Vector calculus and module categories. The multiplication of a 2-matrix by a 2-vector has the following conceptual sense. Note that 2-vectors of length n form a category Vect^n if morphisms are defined componentwise. In other words, Vect^n is just the direct sum of n copies of the category Vect. Given an $(m \times n)$ 2-matrix $V = \|V_{ij}\|$, the multiplication with V defines a functor $\overline{V}\colon \mathrm{Vect}^m \to \mathrm{Vect}^n$. Each collection $T_{ij}\colon V_{ij} \to W_{ij}$ of linear operators between corresponding entries of two 2-matrices V, W of the same format $(m \times n)$ defines a natural transformation between their associated functors $\overline{V}, \overline{W}\colon \mathrm{Vect}^m \to \mathrm{Vect}^m$. In this way we realize our 2-category 2-Vect as a subcategory of the "archetypal" 2-category **Cat**.

However, there is one subtlety involved. Namely, **Cat** is a strict 2-category while our $2\text{-}\mathrm{Vect}_c$ is not. The point is that the previous reasoning defines a *lax* 2-functor $f\colon 2\text{-}\mathrm{Vect}_c \to \mathbf{Cat}$. Let us describe f in more detail.

On objects we have $f(\{n\}) = \mathrm{Vect}^n$, the category of 2-vectors of length n. On 1-morphisms it is given by the correspondence $V \mapsto \overline{V}$ above. However, this composition does not commute with the composition of 1-morphisms. Indeed, let V, W be two 2-matrices of formats $(k \times l)$ and $(l \times m)$. Then the functor $\overline{W} \circ \overline{V}$ takes a 2-vector $(E_1, \dots, E_k)$ into $(F_1, \dots, F_m)$, where $F_p = \bigoplus_i (W_{pi} \otimes (\bigoplus_j V_{ij} \otimes E_j))$, whereas the corresponding component of $\overline{(W * V)}$ is $F'_p = \bigoplus_j ((\bigoplus_i (W_{pi} \otimes V_{ij})) \otimes E_j)$. We take the compatibility 2-isomorphism $(\to\to)$ for the 2-functor f to be the natural isomorphism between F_p and F'_p.

Let us characterize functors of the type $\overline{V}$. To do this, we note that each Vect^n can be seen as a kind of "module" over the "ring" Vect. More precisely, we have the following definition.

5.11. DEFINITION. Let $\mathscr{R}$ be a (lax) ring category. A (*lax*) *left module category over* $\mathscr{R}$ is a symmetric monoidal category $(\mathscr{M}, \oplus, \mathbf{0}_{\mathscr{M}})$ together with a bifunctor $\otimes\colon \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ and natural isomorphisms

$$
\begin{gathered}
a_{A,B,N}\colon A \otimes (B \otimes N) \to (A \otimes B) \otimes N, \\
v_{A,B,N}\colon (A \oplus B) \otimes N \to (A \otimes N) \oplus (B \otimes N), \\
w_{A,M,N}\colon A \otimes (M \oplus N) \to (A \otimes M) \oplus (A \otimes N), \\
l^{\otimes}_M\colon \mathbf{1}_{\mathscr{R}} \otimes M \to M, \\
x_A\colon A \otimes \mathbf{0}_{\mathscr{M}} \to \mathbf{0}_{\mathscr{M}}, \qquad y_M\colon \mathbf{0}_{\mathscr{R}} \otimes M \to \mathbf{0}_{\mathscr{M}},
\end{gathered}
$$

given for any objects $A, B \in \mathscr{R}$, $M, N \in \mathscr{M}$. These isomorphisms are required to satisfy the coherence conditions which are obtained from the conditions for a ring category by "polarization" (setting the last factor to belong to $\mathscr{M}$).

5.12. Definition. Let $\mathscr{R}$ be a strict ring category. An $\mathscr{R}$-module category $\mathscr{M}$ is called *strict* if $(\mathscr{M}, \oplus)$ is strict as a monoidal category and the structure isomorphisms $a_{A,B,N}$, $v_{A,B,N}$, $w_{A,M,N}$, $l^{\otimes}_M$, x_A, y_M are all identities.

5.13. Definition. Let $\mathscr{M}, \mathscr{N}$ be two lax module categories over a lax ring category $\mathscr{R}$. A (lax) $\mathscr{R}$-module functor $F\colon \mathscr{M} \to \mathscr{N}$ is a symmetric monoidal functor (see [Saa]) $(\mathscr{M}, \oplus) \to (\mathscr{N}, \oplus)$ (also denoted by F) together with natural isomorphisms $F_{A,M}\colon F(A \otimes M) \to A \otimes F(M)$ given for any $A \in \mathscr{R}$, $M \in \mathscr{M}$ which satisfy the following conditions:

$(\bullet \otimes \bullet \otimes \bullet)$ For any objects $A, B \in \mathscr{R}$, $M \in \mathscr{M}$ the diagram

$$
\begin{tikzcd}
F(A \otimes (B \otimes M)) \arrow[r] \arrow[d] & A \otimes F(B \otimes M) \arrow[r] & A \otimes (B \otimes F(M)) \arrow[d] \\
F((A \otimes B) \otimes M) \arrow[rr] & & (A \otimes B) \otimes F(M)
\end{tikzcd}
$$

is commutative.

$((\bullet \oplus \bullet) \otimes \bullet)$ For any objects $A, B \in \mathscr{R}$, $M \in \mathscr{M}$ the diagram

$$
\begin{tikzcd}
F((A \oplus B) \otimes M) \arrow[r] \arrow[d] & F((A \otimes M) \oplus (B \otimes M)) \arrow[r] & F(A \otimes M) \oplus F(B \otimes M) \arrow[d] \\
(A \oplus B) \otimes F(M) \arrow[rr] & & (A \otimes F(M)) \oplus (B \otimes F(M))
\end{tikzcd}
$$

is commutative.

$(\bullet \otimes (\bullet \oplus \bullet))$ For any objects $A \in \mathscr{R}$, $M, N \in \mathscr{M}$ the diagram

$$
\begin{tikzcd}
F(A \otimes (M \oplus N)) \arrow[r] \arrow[d] & F((A \otimes M) \oplus (A \otimes N)) \arrow[r] & F(A \otimes M) \oplus F(A \otimes N) \arrow[d] \\
A \otimes F(M \oplus N) \arrow[r] & A \otimes (F(M) \oplus F(N)) \arrow[r] & (A \otimes F(M)) \oplus (A \otimes F(N))
\end{tikzcd}
$$

is commutative.

($1 \otimes \bullet$) For any object $M \in \mathcal{M}$ the diagram

$$\begin{tikzcd} F(\mathbf{1}_{\mathcal{R}} \otimes M) \arrow[r] \arrow[d] & \mathbf{1}_{\mathcal{R}} \otimes F(M) \arrow[dl] \\ F(M) & \end{tikzcd}$$

is commutative

($\bullet \otimes 0$) For any object $A \in \mathcal{R}$ the diagram

$$\begin{tikzcd} F(A \otimes \mathbf{0}_{\mathcal{M}}) \arrow[r] \arrow[d] & & A \otimes F(\mathbf{0}_{\mathcal{M}}) \arrow[d] \\ F(\mathbf{0}_{\mu}) \arrow[r] & \mathbf{0}_{\mathcal{N}} & A \otimes \mathbf{0}_{\mathcal{N}} \arrow[l] \end{tikzcd}$$

is commutative.

($0 \otimes \bullet$) For any object $M \in \mathcal{M}$ the diagram

$$\begin{tikzcd} F(\mathbf{0}_{\mathcal{R}} \otimes M) \arrow[r] \arrow[d] & & \mathbf{0}_{\mathcal{R}} \otimes F(M) \arrow[d] \\ F(\mathbf{0}_{\mu}) \arrow[r] & \mathbf{0}_{\mathcal{N}} & A \otimes \mathbf{0}_{\mathcal{N}} \arrow[l] \end{tikzcd}$$

is commutative.

5.14. Definition. Let $\mathcal{R}$ be a strict ring category and $\mathcal{M}$, $\mathcal{N}$ two strict left $\mathcal{R}$-module categories. An $\mathcal{R}$-module functor $F: \mathcal{M} \to \mathcal{N}$ is called *strict* if it defines a strict monoidal functor of strict monoidal categories $(\mathcal{M}, \oplus) \to (\mathcal{N}, \oplus)$ and the structure morphisms $F_{A,M}$ are identities.

5.15. Definition. Let $\mathcal{R}$ be a (lax) ring category, $\mathcal{M}$, $\mathcal{N}$ two $\mathcal{R}$-module categories, and $F, G: \mathcal{M} \to \mathcal{N}$ two lax $\mathcal{R}$-module functors. A (lax) $\mathcal{R}$-module natural transformation $T: F \Rightarrow G$ is a symmetric monoidal natural transformation (see [Saa]) between underlying monoidal functors $(\mathcal{M}, \oplus) \to (\mathcal{N}, \oplus)$ such that $T_{A\otimes M} = A \otimes T_M: F(M) \to G(M)$ for any $A \in \mathcal{R}$, $M \in \mathcal{M}$.

5.16. Proposition. (a) *Let $\mathcal{R}$ be a lax ring category. Then all lax $\mathcal{R}$-module categories, their lax module functors, and their module natural transformations form a strict 2-category, denote* $\mathcal{R}$-mod.

(b) *Let $\mathcal{R}$ be a strict ring category. Then all the strict $\mathcal{R}$-module categories, their strict $\mathcal{R}$-module functors and their module natural transformations also form a strict 2-category* R-mod^{s}.

Proof. We define compositions, leaving the verifications to the reader. Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three $\mathcal{P}$-module categories, and $F: \mathcal{M} \to \mathcal{N}$, $G: \mathcal{N} \to \mathcal{P}$ two lax $\mathcal{R}$-module functors. Their composition consists of the underlying monoidal functor $(\mathcal{M}, \oplus) \to (\mathcal{P}, \oplus)$ given by the usual composition of the underlying monoidal functors for F and G (see [Saa]), and the structure isomorphisms $(GF)_{A,M}\colon GF(A\otimes M)\to A\otimes GF(M)$ defined as the composition

$$G(F(A\otimes M))\to G(A\otimes F(M))\to A\otimes GF(M).$$

Since module natural transformations are just monoidal natural transformations with additional condition, their composition is defined by the usual rules (see 2.8).

In any 2-category there is a class of 1-morphisms called *equivalences* (cf. [S]) For the 2-category **Cat** they are just equivalence of categories in the usual sense. Considering the 2-category $\mathscr{R}$-mod, we get a notion of a *lax $\mathscr{R}$-module equivalence* of lax $\mathscr{R}$-module categories $\mathscr{M}$, $\mathscr{N}$.

Now we are ready to give a coordinate-free definition of 2-vector spaces.

5.17. DEFINITION. A *2-vector space* is a lax module category $\mathscr{V}$ over the ring category Vect which is module-equivalent to Vect^n for some n. The number n is called the *rank* of $\mathscr{V}$ and is denoted $\mathrm{rk}(\mathscr{V})$. The strict 2-category 2-Vect has as objects 2-vector spaces, as 1-morphisms their Vect-module functors, and as 2-morphisms their module natural transformations.

5.18. Tensor products of 2-vector spaces. Let $\mathscr{V}$, $\mathscr{W}$ be two 2-vector spaces. Define a new Vect-module category $\mathscr{V}\otimes\mathscr{W}$. Its structure data are as follows:

($\bullet$) Objects of $\mathscr{V}\otimes\mathscr{W}$ are formal expressions of the form

$$(V_1\otimes W_i)\otimes\cdots\otimes(V_r\otimes W_r),\qquad r\geq 0,$$

where $V_i\in\mathrm{Ob}\,\mathscr{V}$, $W_i\in\mathrm{Ob}\,\mathscr{W}$. Note that we assume an ordering in the formal direct sum. For brevity we denote objects of $\mathscr{V}\otimes\mathscr{W}$ as $\bigoplus_i(V_i\otimes W_i)$.

($\to$) Define

$$\begin{aligned}\mathrm{Hom}_{\mathscr{V}\otimes\mathscr{W}}&\left(\bigoplus_i(V_i\otimes W_i),\,\bigoplus_i(V_i'\otimes W_i')\right)\\&=\bigoplus_{i,j}\mathrm{Hom}_{\mathscr{V}}(V_i,\,V_j')\otimes_{\mathbf{C}}\mathrm{Hom}_{\mathscr{W}}(W_i,\,W_j'),\end{aligned}$$

where ordering of indices (i,j) on the right-hand side is assumed to be the lexicographical one: $(i,j),(i',j')$ if $i<i'$ or $i=i'$ and $j<j'$.

($\bullet\otimes\bullet$) For any vector space Z define

$$Z\otimes\left(\bigoplus_i(V_i\otimes W_i)\right)=\bigoplus_i((Z\otimes V_i)\otimes W_i).$$

($\bullet\otimes\bullet\otimes\bullet$) Define the structure isomorphism

$$\left(Z_1\otimes Z_2\otimes\left(\bigoplus_i(V_i\otimes W_i)\right)\right)\to(Z_1\otimes Z_2)\otimes\left(\bigoplus_i(V_i\otimes W_i)\right).$$

This should be a certain element h of the Hom-space between objects in the left- and right-hand sides. This Hom-space is, by definition,

$$\bigoplus_{i,j} \operatorname{Hom}(Z_1 \otimes (Z_2 \otimes V_i), (Z_1 \otimes Z_2) \otimes V_j) \otimes \operatorname{Hom}(W_i, W_j).$$

We define h to have nontrivial component only for $i = j$ and the corresponding component to be $a_{Z_1, Z_2, V_i} \otimes \mathrm{Id}_{W_i}$.

We leave to the reader the construction of data related to zero and unit objects and the verification of the axioms of module category.

This definition is so similar to the usual definition of tensor product of vector spaces that it seems to be erroneous. However, we have the following fact.

5.19. PROPOSITION. *$\mathscr{V} \otimes \mathscr{W}$ is a 2-vector space of rank equal to* $\operatorname{rk}(\mathscr{V}) \operatorname{rk}(\mathscr{W})$.

PROOF. We can assume that $\mathscr{V} = \mathrm{Vect}^m$, $\mathscr{W} = \mathrm{Vect}^n$. Let us construct a left module functor F from $\mathscr{V} \otimes \mathscr{W}$ to Vect^{mn}, thinking of objects of Vect^{mn} as $(m \times n)$ 2-matrices $\|Z_{ij}\|$, $i = 1, \dots, m$, $j = 1, \dots, n$. As usual, we label the data for this functor by hierogliphs.

($\bullet$) Let $V = (V_1, \dots, V_m)$, $W = (W_1, \dots, W_n)$ be some objects of $\mathscr{V}$ and $\mathscr{W}$ respectively. We set $F(V \otimes W)$ to be the the 2-matrix $\|V_i \otimes W_j\|$ (here $V \otimes W$ is the formal tensor product which is an object of $\mathscr{V} \otimes \mathscr{W}$). If $V^{(k)} = (V_1^{(k)}, \dots, V_m^{(k)})$, $W^{(k)} = (W_1^{(k)}, \dots, W^{(k)})$, $k = 1, \dots, r$, are r objects of $\mathscr{V}$ and $\mathscr{W}$, then we set $F(\bigoplus_k (V^{(k)} \otimes W^{(k)}))$ to be the 2-matrix whose (i, j)th entry is $\bigoplus_k (V_i^{(k)} \otimes W_j^{(k)})$.

($\to$) Let

$$V = (V_1, \dots, V_m), \qquad V' = (V_1', \dots, V_m') \in \operatorname{Ob} \mathscr{V},$$
$$W = (W_1, \dots, W_n), \qquad W' = (W_1', \dots, W_n') \in \operatorname{Ob} \mathscr{W}.$$

We have

$$\begin{aligned} \operatorname{Hom}_{\mathscr{V} \otimes \mathscr{W}}(V \otimes W, \; V' \otimes W') &= \operatorname{Hom}_{\mathscr{V}}(V, V') \otimes \operatorname{Hom}_{\mathscr{W}}(W, W') \\ &= \left(\bigoplus_i \operatorname{Hom}_{\mathrm{C}}(V_i, V_i') \right) \otimes \left(\bigoplus_j \operatorname{Hom}_{\mathrm{C}}(W_j, W_j') \right). \end{aligned}$$

On the other hand, we have

$$\operatorname{Hom}_{\mathrm{Vect}^{mn}} F(V \otimes W), F(V' \otimes W')) = \bigoplus_{i,j} \operatorname{Hom}_{\mathrm{C}}(V_i \otimes W_j, V_i' \otimes W_j').$$

We define the action of F on morphisms,

$$\bigoplus (\operatorname{Hom}(V_i, V_i') \otimes \operatorname{Hom}(W_j, W_j')) \to \bigoplus_{i,j} \operatorname{Hom}(V_i \otimes W_j, \; V_i' \otimes W_j'),$$

to be the tensor product of 1-morphisms in the monoidal 1-category (Vect, $\otimes$). This defines the data ($\to$) on morphisms between decomposable objects of

$\mathscr{V} \otimes \mathscr{W}$ and we extend these data to morphisms between arbitrary objects by $\oplus$-additivity.

$(\bullet \otimes \bullet)$ Let Z be a vector space, $V = (V_1, \dots, V_m) \in \mathrm{Ob}\,\mathscr{V}$, $W = (W_1, \dots, W_n) \in \mathrm{Ob}\,\mathscr{W}$. We have $Z \otimes (V \otimes W) = (Z \otimes V) \otimes W$. Therefore $F(Z \otimes (V \otimes W))$ is a 2-matrix whose (i, j)th entry is $(Z \otimes V_i) \otimes W_j$. On the other hand, $Z \otimes F(V \otimes W)$ has at the (i, j)th place $Z \otimes (V_i \otimes W_j)$. We define the structure 1-morphism

$$F_{Z, V\otimes W}\colon F(Z \otimes (V \otimes W)) \to Z \otimes F(V \otimes W)$$

to be given, at the (i, j)th place, by the standard associativity isomorphism a^{-1}_{Z, V_i, W_j} for vector spaces. Then we extend the data to arbitrary objects of $\mathscr{V} \otimes \mathscr{W}$ by $\oplus$-additivity.

We leave the reader the task of proving that these data define a lax Vect-module equivalence $F\colon \mathscr{V} \otimes \mathscr{W} \to \mathrm{Vect}^{mn}$.

5.20. THEOREM. *The strict* 2*-category* 2-Vect *admits a lax monoidal structure* $\otimes$ *which on objects is given by the tensor product of* 2*-vector spaces defined above.*

PROOF. We already have the data $(\bullet \otimes \bullet)$. Let us define the rest of data.

$(\bullet\otimes \to)$ Let $\mathscr{V}$ be a 2-vector space and $G\colon \mathscr{W} \to \mathscr{W}'$ a module functor of 2-vector spaces. We define the module functor $\mathscr{V} \otimes G\colon \mathscr{V} \otimes \mathscr{W} \to \mathscr{V} \otimes \mathscr{W}'$ by the following data:

$(\bullet)$ $V \otimes W \mapsto V \otimes G(W)$, and this extends on arbitrary objects of $\mathscr{V} \otimes \mathscr{W}$ by $\oplus$-additivity.

$(\to)$ Let $V \otimes W$, $X \otimes Y$ be two decomposable objects of $\mathscr{V} \otimes \mathscr{W}$. We define the action of $\mathscr{V} \otimes G$ on $\mathrm{Hom}_{\mathscr{V}\otimes\mathscr{W}}(V \otimes W,\ X \otimes Y)$, i.e., the map

$$\mathrm{Hom}(V, X) \otimes \mathrm{Hom}(W, Y) \to \mathrm{Hom}(V, X) \otimes \mathrm{Hom}(G(W), G(Y))$$

as $\mathrm{Id} \otimes G$.

$(\bullet \otimes \bullet)$ Let Z be a vector space and $V \otimes W$ a decomposable object of $\mathscr{V} \otimes \mathscr{W}$. We have

$$\begin{aligned}(\mathscr{V} \otimes G)(Z \otimes (V \otimes W)) &= (\mathscr{V} \otimes G)((Z \otimes V) \otimes W) \\ &= (Z \otimes V) \otimes G(W) = Z \otimes (V \otimes G(W)) \\ &= Z \otimes (\mathscr{V} \otimes G)(V \otimes W),\end{aligned}$$

and we define the structure morphism $(\mathscr{V} \otimes G)_{Z, V\otimes W}$ to be the identity. This defines $\mathscr{V} \otimes G$.

$(\to \otimes\bullet)$ Let $F\colon \mathscr{V} \to \mathscr{V}'$ be a module functor of 2-vector spaces and $\mathscr{W}$ another 2-vector space. We define the module functor $F \otimes \mathscr{W}\colon \mathscr{V} \otimes \mathscr{W} \to \mathscr{V}' \otimes \mathscr{W}$ by the following data:

$(\bullet)$ $V \otimes W \mapsto F(V) \otimes W$.

($\to$) Let $V \otimes W$, $X \otimes Y$ be decomposable objects of $\mathscr{V} \otimes \mathscr{W}$. We define the action of $F \otimes \mathscr{W}$ on $\mathrm{Hom}_{\mathscr{V}\otimes\mathscr{W}}(V \otimes W, X \otimes Y)$ to be the map

$$\mathrm{Hom}(V, X) \otimes \mathrm{Hom}(W, Y) \xrightarrow{F\otimes \mathrm{Id}} \mathrm{Hom}(F(V), F(X)) \otimes \mathrm{Hom}(W, Y).$$

($\bullet \otimes \bullet$) Let Z be a vector space and $V \otimes W$ as above. We define the morphism $(F \otimes \mathscr{W})_{Z, V\otimes W}$ to be

$$\begin{aligned}(F \otimes \mathscr{W})(Z \otimes (V \otimes W)) &= (F \otimes \mathscr{W})((Z \otimes V) \otimes W) \\ &= F(Z \otimes V) \otimes W \xrightarrow{F_{Z,V}} (Z \otimes F(V)) \otimes W \\ &= Z \otimes ((F \otimes \mathscr{W})(V \otimes W)).\end{aligned}$$

($\to \otimes \to$). Let $F: \mathscr{V} \to \mathscr{V}'$, $G: \mathscr{W} \to \mathscr{W}'$ be two module functors between 2-vector spaces. In the diagram

$$\begin{tikzcd} \mathscr{V} \otimes \mathscr{W} \arrow[r, "\mathscr{V}\otimes G"] \arrow[d, "F\otimes\mathscr{W}"'] & \mathscr{V} \otimes \mathscr{W}' \arrow[d, "F\otimes\mathscr{W}'"] \\ \mathscr{V}' \otimes \mathscr{W} \arrow[r, "\mathscr{V}'\otimes G"] & \mathscr{V}' \otimes \mathscr{W}' \end{tikzcd}$$

both paths give functors which take a decomposable object $V \otimes W$, $V \in \mathrm{Ob}\,\mathscr{V}$, $W \in \mathrm{Ob}\,\mathscr{W}$, into $F(V) \otimes G(W)$. The same will hold for a general object of $\mathscr{V} \otimes \mathscr{W}$, i.e., a formal direct sum of decomposable objects. We define the 2-morphism $\otimes_{F,G}$ to be the identity natural transformation.

($\bullet\otimes\bullet\otimes\bullet$) Let $\mathscr{V}$, $\mathscr{W}$, $\mathscr{X}$ be three 2-vector spaces. Let us define a module functor $a_{\mathscr{V},\mathscr{W},\mathscr{X}}: \mathscr{V} \otimes (\mathscr{W} \otimes \mathscr{X}) \to (\mathscr{V} \otimes \mathscr{W}) \otimes \mathscr{X}$ by the following data:

($\bullet$) Each decomposable object $V \otimes (W \otimes X) \in \mathrm{Ob}(\mathscr{V} \otimes (\mathscr{W} \otimes \mathscr{X}))$ is taken into $(V \otimes W) \otimes X$.

($\to$) Let $V, V' \in \mathrm{Ob}\,\mathscr{V}$, $W, W' \in \mathrm{Ob}\,\mathscr{W}$, $X, X' \in \mathrm{Ob}\,\mathscr{X}$. We have

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{V}\otimes(\mathscr{W}\otimes\mathscr{X})}(V \otimes (W \otimes X), V' \otimes (W' \otimes X')) \\ &\quad = \mathrm{Hom}_{\mathscr{V}}(V, V') \otimes (\mathrm{Hom}_{\mathscr{W}}(W, W') \otimes \mathrm{Hom}_{\mathscr{X}}(X, X'))\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}_{(\mathscr{V}\otimes\mathscr{W})\otimes\mathscr{X}}((V \otimes W) \otimes X, (V' \otimes W') \otimes X') \\ &\quad = (\mathrm{Hom}_{\mathscr{V}}(V, V') \otimes \mathrm{Hom}_{\mathscr{W}}(W, W')) \otimes \mathrm{Hom}_{\mathscr{X}}(X, X'),\end{aligned}$$

and we define the action of $a_{\mathscr{V},\mathscr{W},\mathscr{X}}$ on Hom's to be given by the usual associativity isomorphism for vector spaces.

($\bullet \otimes \bullet$) Let Z be a vector space and V, W, X objects of $\mathscr{V}$, $\mathscr{W}$, $\mathscr{X}$ respectively. We have

$$Z \otimes (V \otimes (W \otimes X)) = (Z \otimes V) \otimes (W \otimes X)$$

and hence

$$\begin{aligned} &a_{\mathscr{V},\mathscr{W},\mathscr{X}}(Z \otimes (V \otimes (W \otimes X))) \\ &\quad = ((Z \otimes V) \otimes W) \otimes X = Z \otimes ((V \otimes W) \otimes X) \\ &\quad = Z \otimes a_{\mathscr{V},\mathscr{W},\mathscr{X}}(V \otimes (W \otimes X)). \end{aligned}$$

So we define the data $(\bullet \otimes \bullet)$ for $a_{\mathscr{V},\mathscr{W},\mathscr{X}}$ to be the identity.

$(\to \otimes \bullet \otimes \bullet)$ Let $F: \mathscr{V} \to \mathscr{V}'$ be a module functor of 2-vector spaces and $\mathscr{W}, \mathscr{X}$ another pair of 2-vector spaces. We need to construct a module transformation

$$\begin{tikzcd} \mathscr{V} \otimes (\mathscr{W} \otimes \mathscr{X}) \arrow[r, "a_{\mathscr{V},\mathscr{W},\mathscr{X}}"] \arrow[d, "F\otimes(\mathscr{W}\otimes\mathscr{X})"'] \arrow[dr, phantom, "\swarrow a_{F,\mathscr{W},\mathscr{X}}"] & (\mathscr{V} \otimes \mathscr{W}) \otimes \mathscr{X} \arrow[d, "F(\otimes\mathscr{W})\otimes\mathscr{X}"] \\ \mathscr{V}' \otimes (\mathscr{W} \otimes \mathscr{X}) \arrow[r, "a_{\mathscr{V}',\mathscr{W},\mathscr{X}}"] & (\mathscr{V}' \otimes \mathscr{W}) \otimes \mathscr{X} \end{tikzcd}$$

Let $V \otimes (W \otimes X)$ be a decomposable object of $\mathscr{V} \otimes (\mathscr{W} \otimes \mathscr{X})$. Both paths in the boundary of the above diagram take this object to $F((V) \otimes V) \otimes X$ so we define the transformation $a_{F,\mathscr{W},\mathscr{X}}$ to be the identity.

$(\bullet\otimes \to \otimes\bullet)$, $(\bullet \otimes \bullet\otimes \to)$ Similarly.

$(\bullet \otimes \bullet \otimes \bullet \otimes \bullet)$ Let $\mathscr{V}, \mathscr{W}, \mathscr{X}, \mathscr{Y}$ be 2-vector spaces. Both paths in the Stasheff pentagon take a decomposable object $V \otimes (W \otimes (X \otimes Y))$ to $((V \otimes W) \otimes X) \otimes Y$, and we define the transformation to be the identity one.

$(\bullet\otimes \to\to)$, $(\to\to \otimes\bullet)$ identities.

$\left(\bullet \otimes \begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\right)$ Let $\mathscr{V}$ be a 2-vector space, and

$$\begin{tikzcd} \mathscr{W}' \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow T"] & \mathscr{W}'' \end{tikzcd}$$

a module transformation. We define the transformation

$$\begin{tikzcd}[column sep=large] \mathscr{V}\otimes\mathscr{W}' \arrow[r, bend left, "\mathscr{V}\otimes G"] \arrow[r, bend right, "\mathscr{V}\otimes G'"'] \arrow[r, phantom, "\Downarrow \mathscr{V}\otimes T"] & \mathscr{V}\otimes\mathscr{W}'' \end{tikzcd}$$

to take a decomposable object $V \otimes W$ into the morphism $V \otimes G(W) \to V \otimes G'(W)$ given by $\mathrm{Id} \otimes T_W$.

$\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix} \otimes \bullet\right)$ Similarly.

Thus we have defined all the data. We leave to the reader the verification of the axioms of Definition 4.1. Theorem 5.20 is proven.

5.21. Monoidal structure on totally coordinatized 2-vector spaces. We mentioned in §4 that any lax monoidal 2-category can be rectified to a semistrict one. For a 2-vector space a semistrict version can be constructed explicitly The underlying 2-category will be the 2-category 2-Vect_{cc} of totally coordinatized 2-vector spaces (i.e., we consider the 2-matrix version, but the entries of 2-matrices are now coordinatized vector spaces coded by their dimensions).

5.22. DEFINITION. The monoidal structure on the strict 2-category 2-Vect_{cc} is defined as follows:

($\bullet \otimes \bullet$) On objects we set $\{m\} \otimes \{n\} = \{mn\}$.

($\bullet \otimes \rightarrow$) Given an object $\{m\}$ and a 1-morphism $B = \|b_{ij}\| \colon \{n\} \to \{n'\}$ (i.e., a matrix with entries from $\mathbf{Z}_+$), the tensor product $\{m\} \otimes B$ is the 1-morphism $\{mn\} \to \{mn'\}$ given by the integral matrix

$$\begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B \end{pmatrix} \qquad (m \text{ times}).$$

$\left(\bullet \otimes \overset{\rightarrow}{\underset{\rightarrow}{\downarrow}}\right)$ Let $\{m\}$ be an object, $B = \|b_{ij}\|$, $B' = \|b'_{ij}\| \colon \{n\} \to \{n'\}$ two 1-morphisms and $T = \|T_{ij}\| \colon B \Rightarrow B'$ a 2-morphism (i.e., a collection of complex matrices T_{ij} of format $(b_{ij} \times b'_{ij})$). The product $\{m\} \otimes T$ is the collection of usual matrices M_{pq}, $1 \le p \le mn$, $1 \le q \le mn'$ defined as follows. Suppose that $p = nd + i$, $q = n'd' + j$, where $1 \le i \le n$, $1 \le j \le n'$. Then $M_{pq} = T_{ij}$ if $d = d'$, and $M_{pq} = 0$ otherwise.

($\rightarrow \otimes \bullet$) Let $A \colon \{m\} \to \{m'\}$ be a 1-morphism given by the integral matrix $\|a_{ij}\|$, and let $\{n\}$ be an object. The product $A \otimes \{n\}$ is the 1-morphism given by the matrix

$$\begin{pmatrix} a_{11} \cdot \mathbf{1}_n & a_{12} \cdot \mathbf{1}_n & \cdots & a_{1m'} \cdot \mathbf{1}_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} \cdot \mathbf{1}_n & a_{m2} \cdot \mathbf{1}_n & \cdots & a_{mm'} \cdot \mathbf{1}_n \end{pmatrix}.$$

$\left(\overset{\rightarrow}{\underset{\rightarrow}{\downarrow}} \otimes \bullet\right)$ Let $A, A' \colon \{m\} \to \{m'\}$ be two 1-morphisms given by matrices a_{ij}, a'_{ij}, and let $S \colon A \Rightarrow A'$ be a 2-morphism (i.e., a collection of complex matrices S_{ij} of format $(a_{ij} \times a'_{ij})$). The tensor product $S \otimes \{n\}$ is the collection of usual matrices N_{pq}, $1 \le p \le mn$, $1 \le q \le m'n$, defined as follows. Suppose $p = dn + i$, $q = d'n + j$, where $1 \le i, j \le n$, $1 \le d \le m$, $1 \le d' \le m'$. Then $N_{pq} = S_{dd'}$ if $i = j$, and $N_{pq} = 0$ otherwise.

($\rightarrow \otimes \rightarrow$) Let $A \colon \{m\} \to \{m'\}$, $B \colon \{n\} \to \{n'\}$ be two 1-morphisms given by matrices $\|a_{ij}\|$ and $\|b_{kl}\|$. Let $A_{ij} = [a_{ij}] = \mathbf{C}^{a_{ij}}$ and $B_{kl} = [b_{kl}] = \mathbf{C}^{b_{kl}}$ be the corresponding coordinatized vector spaces. The right path in the diagram

$$\begin{tikzcd} \{mn\} \arrow[r, "\{m\}\otimes B"] \arrow[d, "A\otimes\{n\}"'] & \{mn'\} \arrow[d, "A\otimes\{n'\}"] \\ \{m'n\} \arrow[r, "\{m'\}\otimes B"] & \{m'n'\} \end{tikzcd}$$

is represented by the 2-matrix of format $(mn) \times (m'n')$ whose entry with number $(im+j, kn+l)$ equals $A_{ik} \otimes B_{jl}$. The left path in the same diagram is represented by a similar matrix whose entries are $B_{jl} \otimes A_{ik}$.

We define the 2-morphism $\otimes_{A,B}$ to be given on each such space by the standard permutation operator in the tensor product.

5.23. THEOREM. 2-Vect_{cc} *is a semistrict* (*see* 4.3) *monoidal* 2-*category.*

The proof is left to the reader.

Note that the data $(\to \otimes \to)$ for 2-Vect (the category of "abstract" 2-vector spaces) are identities, whereas for 2-Vect_{cc} they are not. Thus the definition of 2-Vect_{cc} makes essential use of braiding in Vect. We can replace in this construction Vect by any strict ring category for which $\otimes$ is equipped with a braiding.

6. 2-Braidings and Zamolodchikov equations

6.1. DEFINITION OF A 2-BRAIDING. Let $\mathscr{A}$ be a semistrict monoidal 2-category. A 2-braiding in $\mathscr{A}$ is a collection of the following data:

$(\bullet \otimes \bullet)$ 1-morphisms (not necessarily isomorphisms or equivalences) $R_{A,B}\colon A \otimes B \to B \otimes A$ given for any pair A, B of objects of $\mathscr{A}$.

$(\to \otimes \bullet)$ For any 1-morphism $u\colon A \to A'$ and any object B a 2-morphism

$$\begin{tikzcd} A \otimes B \arrow[r, "u\otimes B"] \arrow[d, "R_{A,B}"'] \arrow[dr, phantom, "\Swarrow R_{u,B}"] & A' \otimes B \arrow[d, "R_{A',B}"] \\ B \otimes A \arrow[r, "B\otimes u"'] & B \otimes A' \end{tikzcd}$$

$(\bullet \otimes \to)$ For any object A and any 1-morphism $v\colon B \to B'$ a 2-morphism

$$\begin{tikzcd} A \otimes B \arrow[r, "A\otimes v"] \arrow[d, "R_{A,B}"'] \arrow[dr, phantom, "\Swarrow R_{A,v}"] & A \otimes B' \arrow[d, "R_{A,B'}"] \\ B \otimes A \arrow[r, "v\otimes A"'] & B' \otimes A \end{tikzcd}$$

$((\bullet \otimes \bullet) \otimes \bullet)$ For any objects A_1, A_2 and B a 2-morphism

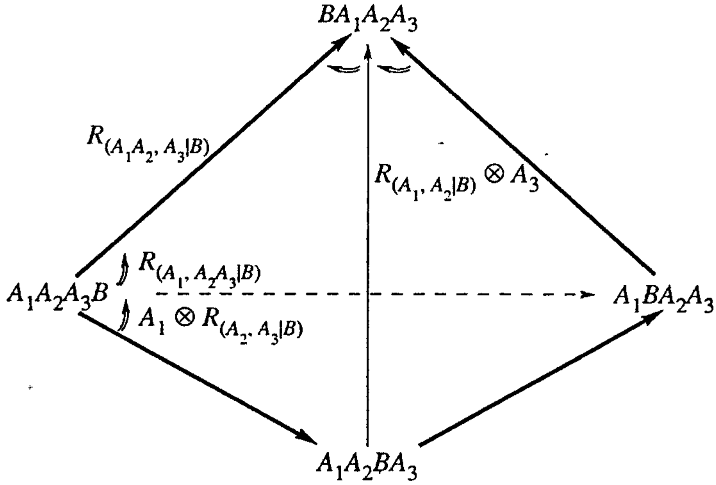
$$\begin{tikzcd} A_1 \otimes A_2 \otimes B \arrow[r, "A_1\otimes R_{A_2,B}"] \arrow[d, "R_{A_1\otimes A_2,B}"'] \arrow[dr, phantom, "\Swarrow R_{(A_1,A_2|B)}"] & A_1 \otimes B \otimes A_2 \arrow[dl, "R_{A_1,B}\otimes A_2"] \\ B \otimes A_1 \otimes A_2 & \end{tikzcd}$$

$(\bullet \otimes (\bullet \otimes \bullet))$ For any objects A, B_1, and B_2 a 2-morphism

$$\begin{tikzcd} A_1 \otimes B_1 \otimes B_2 \arrow[r, "R_{A,B_1}\otimes B_2"] \arrow[d, "R_{A,B_1\otimes B_2}"'] \arrow[dr, phantom, "\Swarrow R_{(A|B_1,B_2)}"] & B_1 \otimes A \otimes B_2 \arrow[dl, "B_1\otimes R_{A,B_2}"] \\ B_1 \otimes B_2 \otimes A & \end{tikzcd}$$

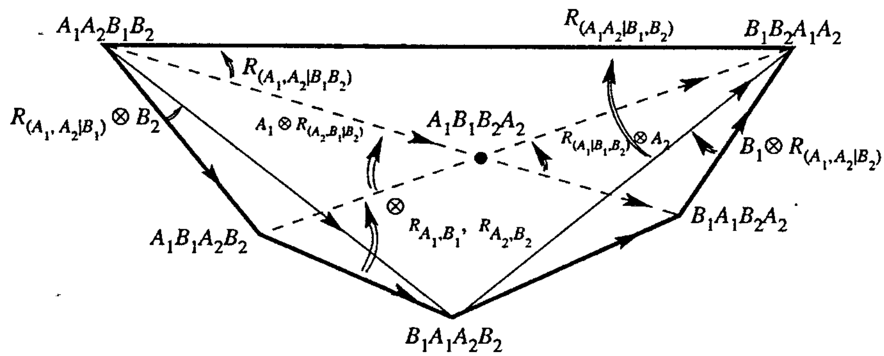
These data should satisfy the following conditions:

$((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ For any objects A_1, A_2, A_3, B the tetrahedron



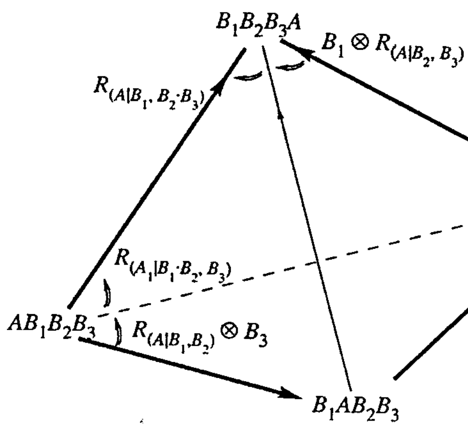
is commutative.

$((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ For any objects A_1, A_2, B_1, B_2 the polytope



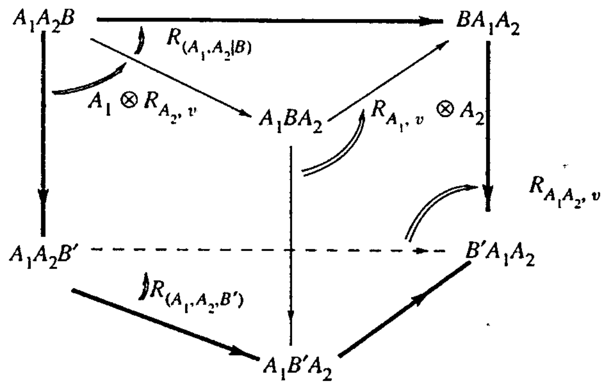
is commutative.

$(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ For any objects A, B_1, B_2, B_3 the tetrahedron



is commutative.

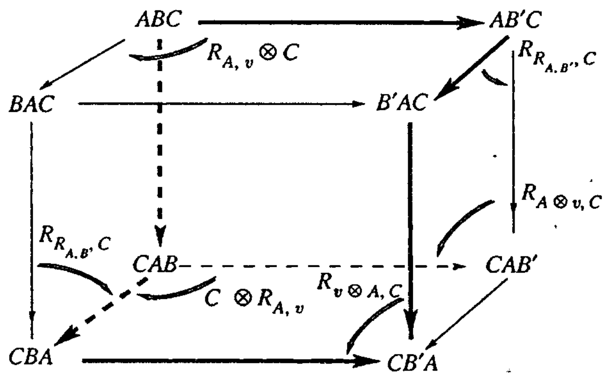
$((\bullet\otimes\bullet)\otimes\to)$ For any two objects A_1, A_2 and a 1-morphism $v: B\to B'$ the triangular prism



is commutative.

$(\to\otimes(\bullet\otimes\bullet)$ Similar prism. Left to the reader.

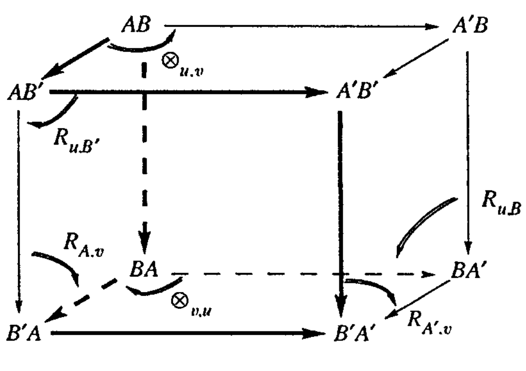
$((\bullet\otimes\to)\otimes\bullet)$ For an object A, a 1-morphism $v: B\to B'$, and an object C the cube



is commutative.

$((\to\otimes\bullet)\otimes\bullet)$, $(\bullet\otimes(\bullet\otimes\to))$, $(\bullet\otimes(\to\otimes\bullet))$ Similar cubes. Left to the reader.

$A'(\to\otimes\to)$ For any two 1-morphisms $u: A\to A'$, $v: B\to B'$ the cube



is commutative.

$\left(\bullet\otimes\overset{\to}{\underset{\to}{\downarrow}}\right)$ For any object A and any 2-morphism

$$\begin{tikzcd} B \arrow[r, bend left=50, "v"{name=U}] \arrow[r, bend right=50, "v'"'{name=D}] & B' \arrow[Rightarrow, from=U, to=D, "\beta"] \end{tikzcd}$$

the cylinder

$$\begin{tikzcd}[column sep=huge, row sep=huge] A\otimes B \arrow[r, bend left=30, ""{name=T1}] \arrow[r, bend right=20, ""{name=T2}] \arrow[d] \arrow[dr, phantom, "R_{A,v'} \qquad R_{A,v}"] & A\otimes B' \arrow[d] \\ B\otimes A \arrow[r, dashed, bend left=20, ""{name=B1}] \arrow[r, bend right=30, ""{name=B2}] & B'\otimes A \arrow[Rightarrow, from=T1, to=T2, "A\otimes\beta"] \arrow[Rightarrow, from=B1, to=B2, "\beta\otimes A"] \end{tikzcd}$$

is commutative.

$\left(\overset{\rightarrow}{\underset{\rightarrow}{\downarrow}} \otimes \bullet\right)$ A similar cylinder for a 2-morphism and an object.

$(\rightarrow\rightarrow \otimes\bullet)$ For any composable pair $A \xrightarrow{u} A' \xrightarrow{u'} A''$ of 1-morphisms and an object B the 2-morphism $R_{u'u,B}$ coincides with the pasting

$$\begin{tikzcd} A\otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow R_{u,B}"] & A'\otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow R_{u',B}"] & A''\otimes B \arrow[d] \\ B\otimes A \arrow[r] & B\otimes A' \arrow[r] & B\otimes A''. \end{tikzcd}$$

$(\bullet\otimes \rightarrow\rightarrow)$ Similarly.

6.2. The resultohedra N_{mn}. The first three polytopes entering in the axioms of 2-braiding (as well as both triangles entering in the axioms of the usual braiding) are members of a bigger family of polytopes which we call *resultohedra* and describe below. These polytopes were first introduced and studied in [GKZ] in connection with the Newton polytope of the resultant of two polynomials in one variable.

Consider $m+n$ letters $A_1, \dots, A_m, B_1, \dots, B_n$. By an (m, n)-shuffle we shall mean a word w in A's and B's such that:

(i) w contains each A_i and each B_j exactly once (i.e., w defines a permutation of $(m+n)$ symbols).

(ii) The order of A's in w, as well as the order of B's, is increasing.

For example, $A_1B_1B_2A_2$ is a (2, 2)-shuffle. Clearly, the number of (m, n)-shuffles is $\binom{m+n}{m}$ since a shuffle w is uniquely determined by an m-element subset in the set $\{1, \dots, m+n\}$ of positions occupied by A's. We regard any shuffle as a sequence of strings of consecutive A's followed by strings of consecutive B's and conversely.

For an (m, n)-shuffle w let $p_0(w)$ be the length of the maximal string of consecutive B's at the beginning of w (so $p_0(w) = 0$ if w starts from A_1). Let also $p_i(w)$, $i = 1, \dots, m$ denote the length of the maximal string if consecutive B's immediately following A_i. In this way we associate to w an integer vector $p(w) = (p_0(w), \dots, p_m(w)) \in \mathbf{Z}^{m+1}$ such that $\sum p_i(w) = n$.

Similarly, for any w as above we define $q_0(w)$ to be the length of the maximal string of consecutive A's at the beginning of w and by $q_j(w)$, $j = 1, \dots, n$ the length of the maximal string of consecutive A's immediately following B_j. In this way we get an integer vector $q(w) = (q_0(w), \dots, q_n(w)) \in \mathbf{Z}^{n+1}$ such that $\sum q_j(w) = m$.

The resultohedron N_{mn} is by definition, the convex hull of the points $(p(w), q(w)) \in \mathbf{Z}^{m+n+2}$ for all the (m, n)-shuffles w. One can give a more conceptual definition. For any $p \geq 0$ consider the p-dimensional simplex $\Delta^p = \{(x_0, \dots, x_p) \in \mathbf{R}_+^{p+1} : \sum x_i = 1\}$. The product $\Delta^m \times \Delta^n$ is embedded, therefore into $\mathbf{R}^{m+n+2}$. There is a well-known triangulation of $\Delta^m \times \Delta^n$ into $\binom{m+n}{n}$ simplices labelled by shuffles (see [Mac3] and [GZ]). The resultohedron $N_{m,n}$ is (up to homothety) just the convex hull of the barycenters of the simplices of this triangulation.

It was proven in [GKZ] that N_{mn} is a convex polytope of dimension $m + n - 1$ and any point $(p(w), q(w))$ is actually a vertex of N_{mn}.

It can be decided, in general, when two vertices $(p(w), q(w))$ and $(p(w'), q(w'))$ of N_{pq} are connected by an edge. The condition is [GKZ] that either w' can be obtained from w (or w can be obtained from w') by the following procedure. Find in one of the shuffles a string (not necessarily maximal) of consecutive A's followed by a string (also not necessarily maximal) of consecutive B's and interchange these strings. In [GKZ] the whole face lattice of N_{mn} was described. In particular, each face of N_{mn} is a product of several smaller resultohedra.

This implies that if $A_1, \dots, A_m, B_1, \dots, B_m$ are not just symbols but objects of some braided monoidal category $\mathscr{A}$ then we can form in $\mathscr{A}$ a diagram of shape N_{mn} by associating to edges the morphisms given by the braiding. Since 2-faces of any resultohedron are either triangles (N_{21} or N_{12}) or squares $N_{11} \times N_{11}$, the axioms of braiding imply that the whole diagram will be commutative.

Similarly, if we work in a braided monoidal 2-category then we can associate 2-morphisms to the 2-faces of the diagram of the shape N_{mn} associated to objects $A_1, \dots, A_m, B_1, \dots, B_n$. The axioms for 2-braiding imply the commutativity of every three-dimensional face of N_{mn} so the whole diagram is commutative.

6.3. Resultohedra and the resultant. Consider two polynomials

$$f(x) = a_0 + \cdots + a_m x^m, \; g(x) = b_0 + \cdots + b_n x^n$$

of degrees m and n in one variable x with indeterminate coefficients. The *resultant* $R(f, g)$ is an integral irreducible polynomial in a_i, b_j such that the vanishing of $R(f, g)$ for some concrete polynomials f, g is equivalent to the fact that these polynomials have a common root. It is given by the classical Sylvester determinant formula [vdW]. We write $R(f, g)$ in the developed form

$$R(f, g) = \sum c_{p_0, \dots, p_m, q_0, \dots, q_n} a_0^{p_0} \cdots a_m^{p_m} b_0^{q_0} \cdots b_n^{q_n} = \sum c_{pq} a^p b^q$$

where $p = (p_0, \dots, p_m) \in \mathbf{Z}^{m+1}$, $q = (q_0, \dots, q_n) \in \mathbf{Z}^{n+1}$. We define the *Newton polytope* $\mathscr{N}_{m,n}$ of $R(f, g)$ as the convex hull in $\mathbf{R}^{m+n+2}$ of the points (p, q) such that $c_{pq} \neq 0$.

It was proven in [GKZ] that the resultohedron $N_{m,n}$ coincides with the Newton polytope $\mathscr{N}_{m,n}$.

On the intuitive level, one can think of objects $A_1, \dots, A_m, B_1, \dots, B_n$ of a braided monoidal 2-category as corresponding to roots $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$ of polynomials f and g. The condition that the resultant does not vanish means that $\alpha_i \neq \beta_j$. A braiding morphism in the product $A_1 \cdots A_m B_1 \cdots B_n$ can be thought of as corresponding to a path in the space of pairs of polynomials going around the zero locus of the resultant (so that roots interchange places).

6.4. The decomposition of a permutohedron into (products of) resultohedra. Let P_n be an $(n-1)$-dimensional permutohedron (see 3.6). We shall construct a polyhedral decomposition $\mathscr{D}(\Gamma)$ of P_n for any face $\Gamma \subset P_n$ of codimension 1. The polyhedra (of full dimension) of the decomposition $\mathscr{D}(\Gamma)$ will be in bijection with the faces (of all dimensions) of Γ. By general description of faces of P_n (see 3.6) the face Γ is defined by a partition of $\{1, \dots, n\}$ into two disjoint parts, $C_1 \cup C_2$. More precisely, Γ has vertices $[\sigma]$), where the permutation σ preserves C_1 and C_2. Let $\Delta \subset \Gamma$ be an arbitrary nonempty face (including $\Delta = \Gamma$). It is defined by a pair of partitions $C_1 = Q_1 \cup \cdots \cup Q_k$, $C_2 = R_1 \cup \cdots \cup R_l$. We say that a permutation $\sigma \in S_n$ is a (Δ, Γ)-shuffle if it is obtained by shuffling R_i among Q_j and then arbitrarily reordering elements inside R_i and Q_j. Denote the set of all (Δ, Γ)-shuffles by $S(\Delta, \Gamma)$.

6.5. THEOREM. (a) *The polytope* $\operatorname{Conv} S(\Delta, \Gamma) \subset P_n$ *is linearly isomorphic to*

$$N_{k,l} \times \prod P_{|Q_i|} \times \prod P_{|R_j|}.$$

(b) *For a given face* Γ, *the polytopes* $\operatorname{Conv} S(\Delta, \Gamma)$ *form a polyhedral decomposition* $\mathscr{D}(\Gamma)$ *of* P_n *(i.e., they cover* P_n *and intersect along common faces).*

One can also permit Γ to have arbitrary dimension, thus obtaining new polyhedral decompositions of P_n, which we do not use here.

The proof of this theorem will be given in a separate paper of the authors devoted expecially to the polyhedral aspects of the theory. In the present

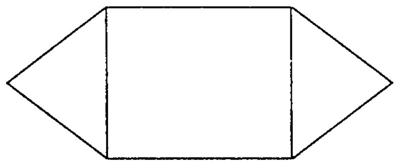


Figure 14

work we need only the particular cases of this theorem corresponding to two- and three-dimensional permutohedra, which can be verified easily. We shall show the arising decompositions explicitly.

6.6. Examples of the decompositions. (a) If Γ is an edge of the hexagon P_3, then the decomposition $\mathscr{D}(\Gamma)$ has the form shown in Figure 14. This type of decomposition was used in the proof of Proposition 3.5.

(b) Let Γ be a hexagonal face of the permutohedron P_4. The decomposition $\mathscr{D}(\Gamma)$ consists of six tetrahedra $N_{1,3}$ (corresponding to the vertices of Γ), six triangular prisms (corresponding to edges of Γ), and a hexagonal prism (corresponding to Γ itself), see Figure 15.

(c) Let Γ be a quadrangular face of the permutohedron P_4. The decomposition $\mathscr{D}(\Gamma)$ consists of four polytopes of type $N_{2,2}$ (corresponding to the vertices of Γ), four triangular prisms (corresponding to edges of Γ), and a cube (corresponding to Γ itself), see Figure 16.

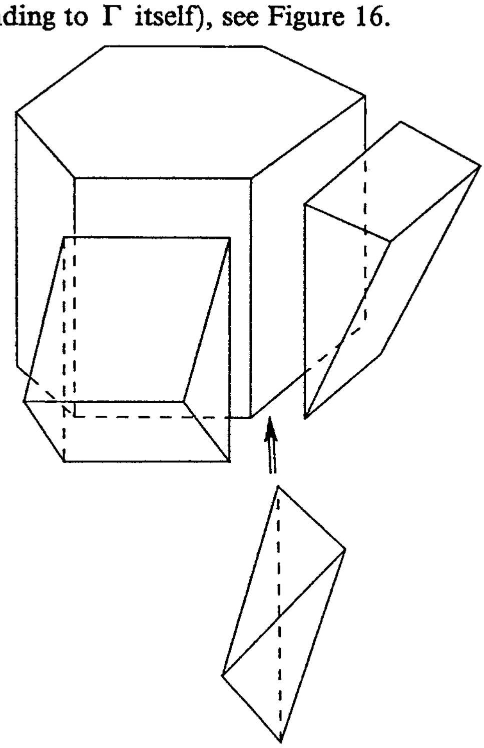


Figure 15

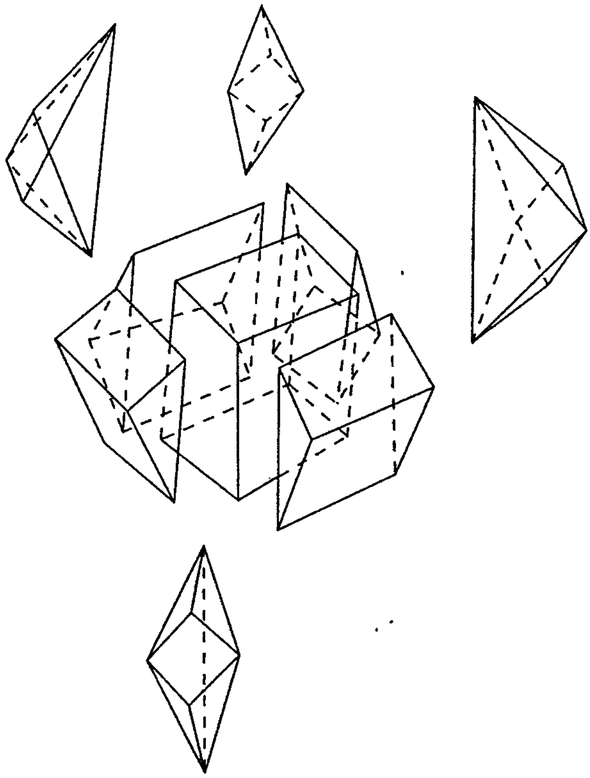


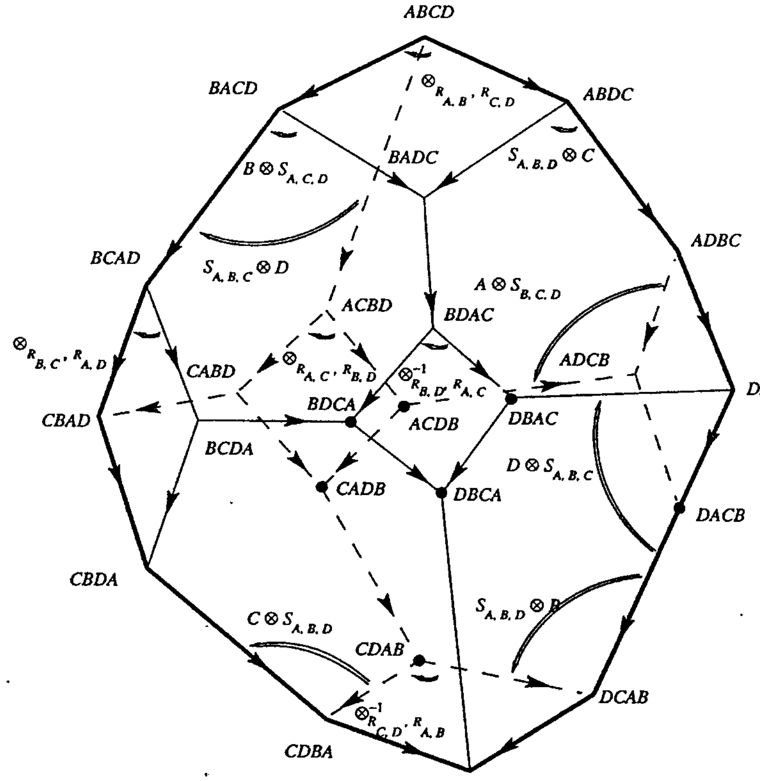
Figure 16

6.7. Zamolodchikov systems. Let $\mathscr{A}$ be a semistrict monoidal 2-category. A Zamolodchikov system (Z-system, for short) in $\mathscr{A}$ is a collection I of objects of $\mathscr{A}$, a family of 1-morphisms $R_{A,B}: A\otimes B \to B\otimes A$ given for any two objects $A, B \in I$ and a family of 2-morphisms

(6-1)
$$\begin{tikzcd}[column sep=small]
 & A\otimes B\otimes C \arrow[dl, "R_{A,B}\otimes C"'] \arrow[dr, "A\otimes R_{B,C}"] & \\
B\otimes A\otimes C \arrow[d, "B\otimes R_{A,C}"'] & & A\otimes C\otimes B \arrow[d, "R_{A,C}\otimes B"] \\
B\otimes C\otimes A \arrow[dr, "R_{B,C}\otimes A"'] & \arrow[u, phantom, "\overset{S_{A,B,C}}{\Rightarrow}"] & C\otimes A\otimes B \arrow[dl, "C\otimes R_{A,B}"] \\
 & C\otimes B\otimes A &
\end{tikzcd}$$

given for any three objects $A, B, C, \in I$. These data should satisfy the following condition: for any four objects A, B, C, D the permutohedral diagram

(6-2)



is commutative.

6.8. Examples. (a) Let $\mathscr{A}$ be the monoidal 2-category of 2-vector spaces (see §5). More precisely, we set $\mathscr{A}$ to be $2\text{-}\mathrm{Vect}_{cc}$, the most coordinatized version of this category (Definition 5.22). Let I consist of one object $\{1\}$ of 2-Vect (the one-dimensional 2-vector space; in the module-categorical interpretation of 2-Vect (Definition 5.17) it corresponds to Vect as a module category over itself). Let us see what a Zamolodchikov system on the set I is. We have $\{1\}\otimes\{1\} = \{1\}$. Therefore a morphism $R_{\{1\},\{1\}}\colon \{1\}\otimes\{1\} \to \{1\}\otimes\{1\}$ is just one vector space V. Since $\{1\}\otimes\{1\}\otimes\{1\} = \{1\}$, both 1-morphisms given by composition of right or left boundary paths in the hexagon

$$
\begin{tikzcd}[column sep=small]
 & \{1\}\otimes\{1\}\otimes\{1\} \arrow[dl] \arrow[dr] & \\
\{1\}\otimes\{1\}\otimes\{1\} \arrow[d] & & \{1\}\otimes\{1\}\otimes\{1\} \arrow[d] \\
\{1\}\otimes\{1\}\otimes\{1\} \arrow[dr] & & \{1\}\otimes\{1\}\otimes\{1\} \arrow[dl] \\
 & \{1\}\otimes\{1\}\otimes\{1\} &
\end{tikzcd}
$$

are given by the vector space $V \otimes V \otimes V$ (see 5.1). Therefore a 2-morphism $S_{\{1\},\{1\},\{1\}}$ filling the above hexagon is nothing but a linear operator

$$S: V \otimes V \otimes V \to V \otimes V \otimes V.$$

The condition that S indeed defines a Z-system (i.e., that the permutohedral diagram above is commutative) amounts to the following equation for operators in $V^{\otimes 6}$:

$$(6\text{-}3) \qquad P_{34}S_{456}S_{234}P_{12}P_{45}S_{234}S_{456} = S_{123}S_{345}P_{23}P_{56}S_{345}S_{123}P_{34}.$$

Let us compare this with the Zamolodchikov tetrahedra equations (1-10) corresponding to coloring of segments of plane configurations. After the pairs (12), (13), (14), (23), (24), (34) are numbered as 1, 2, 3, 4, 5, 6, Equation (1-10) takes the form

$$(6\text{-}4) \qquad S_{456}S_{236}S_{135}S_{124} = S_{124}S_{135}S_{236}S_{456}.$$

It is straightforward to see that if S satisfies (6-3) then the operator $S' = S \circ P_{13}$, where P_{13} is the permutation of the first and the third factors, satisfies (6-4), and vice versa.

(b) Let $S(z_1, z_2, z_3) \in \mathrm{End}(V \otimes V \otimes V)$ be a solution of the parametric Zamolodchikov equation (1-11) where parameters z_i vary in some set X. We associate to such a solution a Z-system as follows. We again take the monoidal 2-category $\mathscr{A}$ to be 2-Vect. For any $z \in X$ choose a copy $\{1\}(z)$ of the one-dimensional 2-vector space $\{1\}$. For any two values of parameters $z_1, z_2 \in X$ define the 1-morphism $R_{\{1\}(x_1),\{1\}(x_2)}$ to be the vector space V. For any three values of parameters z_1, z_2, z_3 we define the 2-morphism $S_{\{1\}(z_1),\{1\}(z_2),\{1\}(z_3)}$ to be given by the linear operator

$$S(z_1, z_2, z_3) \circ P_{13}: V \otimes V \otimes V \to V \otimes V \otimes V.$$

The equivalence of Equation (1-11) and the commutativity of permutohedra (6-2) are established similarly to the previous example.

For the concrete physical setting of [Z1] and [Z2] this recipe amounts to taking a copy of the 2-vector space $\{1\}$ for each direction of affine planes in $\mathbf{R}^3$, i.e., for each point of the real projective plane $\mathbf{R}P^2$.

(c) As we have seen in 4.2, any braiding in a monoidal 1-category $(\mathscr{A}, \otimes)$ gives rise to a monoidal structure on the 2-category $\Omega^{-1}(\mathscr{A})$ with one object, associated to $(\mathscr{A}, \otimes)$. A Zamolodchikov system in $\Omega^{-1}(\mathscr{A})$ consists of

(1) A 1-morphism $R_{pt,pt}: pt \odot pt \to pt \odot pt$ in $\Omega^{-1}(\mathscr{A})$, i.e., an object $V \in \mathscr{A}$

(2) A 2-morphism $S_{pt,pt,pt}$ in $\Omega^{-1}(\mathscr{A})$ which corresponds to a (1-) morphism $V \otimes V \otimes V \to V \otimes V \otimes V$ in $\mathscr{A}$, which we also denote by S.

The commutativity of the permutohedron (6-2) amounts to the following equation on endomorphisms of $V^{\otimes 6}$ (which differs from (6-3) by replacing permutations with braidings):

$$R_{34}S_{456}S_{234}R_{12}R_{45}R_{234}S_{456} = S_{123}S_{345}R_{23}R_{56}S_{345}S_{123}R_{34}. \tag{6-5}$$

Here $R\colon V\otimes V \to V\otimes V$ is the braiding in $\mathscr{A}$. Equation (6-5) can be called the *coupled Zamolodchikoiv-Yang-Baxter equation.* It was proposed also by Lawrence [Law].

6.9. Tetrahedra equations as the "vectorization" of triangle equations. Let us see what are more general Z-systems in the category 2-Vect. Let us keep the assumption that I contains just one element, but allow this element to be an arbitrary (coordinatized) 2-vector space $\{n\}$. A 1-morphism $R = R_{\{n\},\{n\}}\colon \{n\}\otimes\{n\} \to \{n\}\otimes\{n\}$ is nothing but a collection of vector spaces $\mathscr{R}_{ij}^{kl}$ indexed by four numbers $i, j, k, l \in \{1, \dots, n\}$. In other words, $\mathscr{R}$ is an "R-matrix" in the usual sense, but its entries are not numbers but vector spaces. Now a 2-morphism $S = S_{\{n\},\{n\},\{n\}}$ is a collection of isomorphisms of usual vector spaces

$$S_{i,j,k}^{i',j',k'}\colon \bigoplus_{a,b,c}(\mathscr{R}_{ij}^{ca}\otimes\mathscr{R}_{ka}^{lj'}\otimes\mathscr{R}_{cl}^{j'k'}) \to \bigotimes_{a,b,c}(\mathscr{R}_{ik}^{cb}\otimes\mathscr{R}_{bj}^{k'a}\otimes\mathscr{R}_{ac}^{i'j'}) \tag{6-6}$$

given for any $i, j, k, i', j', k' \in \{1, \dots, n\}$.

Note that for the existence of such isomorphisms it is necessary that dimensions of the space in the right- and left-hand sides coincide for every i, j, k, i', j', k'. In other words, the "numerical" R-matrix $\|R_{ij}^{kl}\| = \|\dim\mathscr{R}_{ij}^{kl}\|$ should satisfy the usual Yang-Baxter equations. When these equations are satisfied, we can look for a system of isomorphisms which form a Z-system, i.e., make the permutohedral diagram (6-2) commute. We shall not write the corresponding version of the Zamolodchikov equations in full detail, since this can be done automatically by "unraveling" the permutohedron. Instead we summarize the results of the above digression:

1. Any solution $\|R_{ij}^{kl}\|$, $1 \le i, j, k, l \le n$, of the Yang-Baxter equation whose entries are natural numbers can provide a setup for formulating Zamolodchikov tetrahedra equations. The usual setting of §1 corresponds to the trivial solution with $n = 1$ and $R_{11}^{11} = m$ a fixed natural number.

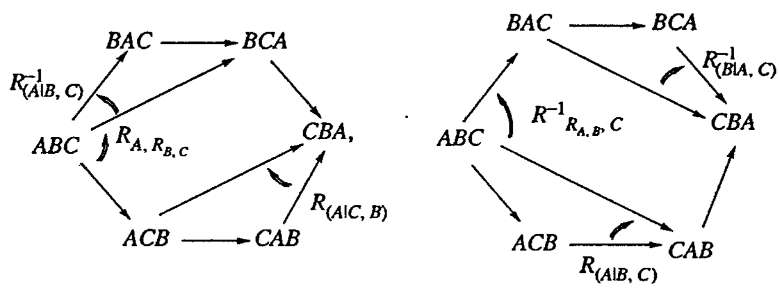
2. More precisely, given a solution R_{ij}^{kl} as above, we consider vector spaces $\mathscr{R}_{ij}^{kl}$ of dimensions R_{ij}^{kl}. The unknowns of the Zamolodchikov equations associated to R_{ij}^{kl} are isomorphisms $S_{i,j,k}^{i',j',k'}$, "materializing" the Yang-Baxter equations for R_{ij}^{kl}. The equations themselves are certain natural compatibility conditions on these isomorphisms.

We can modify this approach to take into account solutions of YB with not necessarily integer entries. More precisely, given an arbitrary solution $\|R_{ij}^{kl}\|$, $R_{ij}^{kl} \in \mathbf{C}$, we can look for a collection $(\mathscr{R}_{ij}^{kl}, T_{ij}^{kl})$ of vector spaces $\mathscr{R}_{ij}^{kl}$ and linear operators $T_{ij}^{kl}\colon \mathscr{R}_{ij}^{kl} \to \mathscr{R}_{ij}^{kl}$ such that $\mathrm{tr}(T_{ij}^{kl}) = R_{ij}^{kl}$. In other words, we replace the vector space with finite-dimensional $\mathbf{C}[T]$-modules. Such modules form a monoidal category with respect to the usual tensor multiplication over $\mathbf{C}$ and the action of T given by $T(a\otimes b) = T(a)\otimes T(b)$. The trace of T is multiplicative with respect to this tensor product and additive with respect to the direct sum. Thus the setup for any R-matrix is the following:

> Find n^4 finite-dimensional $\mathbf{C}[T]$-modules $\mathscr{R}_{ij}^{kl}$ such that $\mathrm{tr}(T|_{\mathscr{R}_{ij}^{kl}}) = R_{ij}^{kl}$ and n^6 isomorphisms of $\mathbf{C}[T]$-modules $S_{ijk}^{i'j'k'}$ of the form (6-6) making the permutohedral diagram (6-2) commute.

6.10. A Zamolodchikov system from a 2-braiding. Let $\mathscr{A}$ be a braided monoidal 2-category. We are going to define two Z-systems $(R_{A,B}, S^+_{A,B,C})$ and $(R_{A,B}, S^-_{A,B,C})$ on the same set of objects $I = \mathrm{Ob}\,\mathscr{A}$.

The 1-morphisms $R_{A,B}$ in both systems are given by the braiding. The 2-morphisms $S^+_{A,B,C}$ and $S^-_{A,B,C}$ are defined respectively by pastings



6.11. THEOREM. *The collections $(R_{A,B}, S^+_{A,B,C})$ and $(R_{A,B}, S^-_{A,B,C})$ are Z-systems on the collection of objects $I = \mathrm{Ob}\,\mathscr{A}$.*

PROOF. Consider the vertex $ABCD$ of the permutohedron (6-2). It is contained in three facets, two hexagonal and one quadrangular. For Γ one of the two hexagonal faces, the decomposition $\mathscr{D}(\Gamma)$ of the permutohedron constructed in Theorem 6.5 (see also Example 6.6(c)) shows that S^+ and S^- are indeed Zamolodchikov systems.

In the proof of Theorem 6-11 we have not used the decomposition of the permutohedron corresponding to a quadrangular face. Taking it into account results in an additional relation between S^+ and S^-.

In a subsequent paper [KV3] we will show that this relation permits one to define, for any objects $A_1, \dots, A_n$ of a braided monoidal 2-category $\mathscr{A}$, an action of the Manin-Schechtman higher braid group $T(2, n)$ (defined in [MS 2]) by 2-automorphisms on any braiding 1-morphism $A_1 \otimes \cdots \otimes A_n \to A_n \otimes \cdots \otimes A_1$.

6.12. The 2-category of formal tensor products. Our aim in the rest of this section is to construct, for any Z-system (I, R, S) in a (semistrict) monoidal 2-category $\mathscr{A}$, a certain braided monoidal 2-category $\widetilde{I}$. As in the 1-categorical case (3.14), we will define $\widetilde{I}$ to consist of formal tensor products of objects from I, and again we need to distinguish two tensor products which accidentally coincide. To this end we perform the following construction.

Let $\mathscr{A}$ be a semistrict monoidal 2-category. The tensor product of several objects $A_1, \ldots, A_n$ of $\mathscr{A}$ will be denoted by $A_1 \cdots A_n$ (without brackets since we assume semistrictness). The same for a product of several objects and one 1- or 2-morphism. Define a new monoidal 2-category $\widetilde{\mathscr{A}}$ by the following data.

($\bullet$) By definition, objects of $\widetilde{\mathscr{A}}$ are formal symbols $\langle A_1, \ldots, A_k \rangle$, where A_i are objects of $\mathscr{A}$ and $k \geq 0$. We also include the symbol $\langle \varnothing \rangle$ or $\langle\ \rangle$, corresponding to the empty set of objects.

We use the following notation, sometimes without explanation. A typical object of $\widetilde{\mathscr{A}}$ is denoted by a tilded letter, $\widetilde{A} = \langle A_1, \ldots, A_k \rangle$. An untilded letter denotes the product $A = A_1 \cdots A_k$ of objects constituting $\widetilde{A}$.

($\rightarrow$) Let $\widetilde{A} = \langle A_1, \ldots, A_k \rangle$, $\widetilde{B} = \langle B_1, \ldots, B_l \rangle$ be two objects of $\widetilde{\mathscr{A}}$, and $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_l)$ the corresponding objects of $\mathscr{A}$. To define the set $1\text{-}\mathrm{Hom}_{\widetilde{\mathscr{A}}}(\widetilde{A}, \widetilde{B})$ we first define the set of what we call *elementary* 1-morphisms from $\widetilde{A}$ to $\widetilde{B}$ to be

$$\mathrm{Hom}^{\mathrm{el}}_{\widetilde{\mathscr{A}}}(\widetilde{A}, \widetilde{B}) = \mathrm{Hom}_{\mathscr{A}}(A, B).$$

We extend the use of the tilde to morphisms; i.e., if $f\colon A \to B$ is a 1-morphism in $\mathscr{A}$, then by $\tilde{f}\colon \widetilde{A} \to \widetilde{B}$ we denote the corresponding 1-morphism in $\widetilde{\mathscr{A}}$.

We define 1-morphisms in $\mathscr{A}$ to be formal compositions of elementary 1-morphisms, i.e.,

$$\mathrm{Hom}_{\widetilde{\mathscr{A}}}(\widetilde{A}, \widetilde{B}) = \coprod_{r \geq 0} \coprod_{\widetilde{C}^{(1)}, \ldots, \widetilde{C}^{(r)}} \mathrm{Hom}^{\mathrm{el}}_{\widetilde{\mathscr{A}}}(\widetilde{A}, \widetilde{C}^{(1)}) \times \cdots \times \mathrm{Hom}^{\mathrm{el}}_{\widetilde{\mathscr{A}}}(\widetilde{C}^{(r)}, B).$$

1-morphisms in $\widetilde{\mathscr{A}}$ are denoted $\langle\langle \tilde{f}_1, \ldots, \tilde{f}_r \rangle\rangle$, where $\tilde{f}_1, \ldots, \tilde{f}_r$ is a composable sequence of elementary 1-morphisms. Such a composite morphism will sometimes be denoted $\tilde{f} = \langle\langle \tilde{f}_1, \ldots, \tilde{f}_r \rangle\rangle$, and by f we denote the composition $f_1 \cdots f_r$ in $\mathscr{A}$.

The composition of 1-morphisms in $\widetilde{\mathscr{A}}$ is given by the formal concatenation.

$\left(\begin{smallmatrix}\rightarrow\\ \downarrow\\ \dashrightarrow\end{smallmatrix}\right)$ Let $\tilde{f}$, $\tilde{g}$ be two 1-morphisms in $\widetilde{\mathscr{A}}$ with common source and target and f, g the corresponding 2-morphisms in $\mathscr{A}$. We define

$$2\text{-}\mathrm{Hom}_{\widetilde{\mathscr{A}}}(\tilde{f}, \tilde{g}) = 2\text{-}\mathrm{Hom}_{\mathscr{A}}(f, g).$$

Again, if $T: f \Rightarrow g$ is a 2-morphism in $\mathscr{A}$, we denote by $\tilde{T}: \tilde{f} \Rightarrow \tilde{g}$ the corresponding 2-morphisms in $\widetilde{\mathscr{A}}$. The 0- and 1-compositions of 2-morphisms are defined in an obvious way.

It is easy to see that in this way we obtain a strict 2-category $\widetilde{\mathscr{A}}$ (which is 2-equivalent to $\mathscr{A}$, but we do not need this). Let us introduce on $\widetilde{\mathscr{A}}$ a monoidal structure by means of the following data:

(1) The object $1_{\widetilde{\mathscr{A}}}$ is the empty sequence $\langle\varnothing\rangle$.

$(\bullet\otimes\bullet)$ The product $\langle A_1, \ldots, A_k\rangle\otimes\langle B_1, \ldots, B_l\rangle$ is defined to be $\langle A_1, \ldots, A_k, B_1, \ldots, B_l\rangle$.

$(\rightarrow\otimes\bullet)$ Let $\tilde{f}: \tilde{A} \to \tilde{B}$ be an elementary 1-morphism and $\tilde{B}$ an object in $\widetilde{\mathscr{A}}$. The 1-morphism $\tilde{f}\otimes\tilde{B}$ is set to be the elementary 1-morphism corresponding to the 1-morphism $f\otimes B: A\otimes B \to A'\otimes B$ in $\mathscr{A}$ (with the above conventions on tildas). We extend the definition of the data $(\rightarrow\otimes\bullet)$ to arbitrary 1-morphisms (which are formal compositions of elementary ones) by setting

$$\langle\langle\tilde{f}_1, \ldots, \tilde{f}_r\rangle\rangle\otimes\tilde{B} = \langle\langle(\tilde{f}_1\otimes\tilde{B}), \ldots, (\tilde{f}_r\otimes\tilde{B})\rangle\rangle.$$

$(\bullet\otimes\rightarrow)$, $\left(\underset{\dashrightarrow}{\overset{\rightarrow}{\downarrow}}\otimes\bullet\right)$, $\left(\bullet\otimes\underset{\dashrightarrow}{\overset{\rightarrow}{\downarrow}}\right)$ Similarly.

$(\bullet\otimes\bullet\otimes\bullet)$, $(1\otimes\bullet)$, $(\bullet\otimes 1)$ The associativity, left and right unity 1-morphisms in $\widetilde{\mathscr{A}}$ are set to be identities.

$(\rightarrow\otimes\rightarrow)$ Let $\tilde{A} = \langle A_1, \ldots, A_k\rangle$, $\tilde{A}' = \langle A'_1, \ldots, A'_{k'}\rangle$, $\tilde{B} = \langle B_1, \ldots, B_l\rangle$, $\tilde{B}' = \langle B'_1, \ldots, B'_{l'}\rangle$ be objects of $\widetilde{\mathscr{A}}$, and $\tilde{f}: \tilde{A} \to \tilde{A}'$, $\tilde{g}: \tilde{B} \to \tilde{B}'$ elementary 1-morphisms. We define the required 2-morphism $\bigotimes_{\tilde{f},\tilde{g}}$ to correspond to the 2-morphism $\bigotimes_{f,g}$ in $\mathscr{A}$ where $f: A \to A'$, $g: B \to B'$ are the 1-morphisms in $\mathscr{A}$ corresponding to $\tilde{f}$, $\tilde{g}$.

We extend this definition to nonelementary 1-morphisms by concatenation. Let $\tilde{A}_1 \xrightarrow{\tilde{f}_r} \tilde{A}_2 \xrightarrow{\tilde{f}_{r-1}} \cdots \xrightarrow{\tilde{f}_1} \tilde{A}_{r+1}$ and $\tilde{B}_1 \xrightarrow{\tilde{g}_s} \tilde{B}_2 \xrightarrow{\tilde{g}_{s-1}} \cdots \xrightarrow{\tilde{g}_1} \tilde{B}_{s+1}$ be two composable strings of elementary 1-morphisms in $\widetilde{\mathscr{A}}$. We define $\otimes_{\langle\langle\tilde{f}_1,\ldots,\tilde{f}_r\rangle\rangle,\langle\langle\tilde{g}_1,\ldots,\tilde{g}_s\rangle\rangle}$ to be the pasting of the obvious rectangular diagram consisting of rs squares $\bigotimes_{\tilde{f}_i,\tilde{g}_j}$.

All the other data for $\widetilde{\mathscr{A}}$ are set to be identities. This obviously makes sense due to the semistrictness of $\mathscr{A}$.

We verify at once that $\widetilde{\mathscr{A}}$ is a semistrict monoidal 2-category.

6.13. A 2-braiding from a Z-system. Let (I, R, S) be a Z-system in a semistrict monoidal 2-category $\mathscr{A}$. Let $\tilde{I}$ be the smallest monoidal sub-2-category in $\widetilde{\mathscr{A}}$ containing elementary objects $\langle A\rangle$, $A \in I$, 1-morphisms $R_{A,B}$, and 2-morphisms $S_{A,B,C}$. Loosely speaking, 2-morphisms of $\tilde{I}$ are all 2-morphisms which can be constructed from structure 2-morphisms $\otimes_{f,g}$ and also $R_{A,B}$, $S_{A,B,C}$ by tensor products, 0- and 1-composition.

Let us introduce on the monoidal 2-category $\tilde{I}$ a braiding which will also be denoted R.

($\bullet \otimes \bullet$) Let $\widetilde{A} = \langle A_1, \ldots, A_k \rangle$, $\widetilde{B} = \langle B_1, \ldots, B_l \rangle$ be two objects of $\widetilde{I}$. We define the 1-morphism $R_{\widetilde{A},\widetilde{B}}: \widetilde{A} \otimes \widetilde{B} \to \widetilde{B} \otimes \widetilde{A}$ to correspond to the 1-morphism in $\mathscr{A}$ given by the composition

(6-7)
$$\begin{aligned} A_1 \cdots A_k B_1 \cdots B_l &\to A_1 \cdots A_{k-1} B_1 A_k B_2 \cdots B_l \to \cdots \to B_1 A_1 \cdots A_k B_2 \cdots B_l \\ &\to \cdots \to B_1 B_2 A_1 \cdots A_k B_3 \cdots B_l \\ &\quad\vdots \\ &\to \cdots \to B_1 B_2 \cdots B_l A_1 \cdots A_k. \end{aligned}$$

($\to \otimes \bullet$) For any 1-morphism $\tilde{u}: \widetilde{A} \to \widetilde{A'}$ in $\widetilde{I}$ and any object $\widetilde{B} \in \widetilde{I}$ we need to construct a 2-morphism

$$\begin{tikzcd} \widetilde{A} \otimes \widetilde{B} \arrow[r, "\tilde{u} \otimes \widetilde{B}"] \arrow[d, "R_{\widetilde{A},\widetilde{B}}"'] \arrow[dr, phantom, "R_{\tilde{u},\widetilde{B}}"] & \widetilde{A'} \otimes \widetilde{B} \arrow[d, "R_{\widetilde{A'},\widetilde{B}}"] \\ \widetilde{B} \otimes \widetilde{A} \arrow[r, "\widetilde{B} \otimes \tilde{u}"] & \widetilde{B} \otimes \widetilde{A'}. \end{tikzcd}$$

We define $R_{\tilde{u},\widetilde{B}}$ for the case when $\tilde{u}$ is an elementary 1-morphism, i.e., it corresponds to a 1-morphism of $\mathscr{A}$ of the form

$$A_1 \cdots A_k \to A_1 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_k.$$

For such $\tilde{u}$ the 2-morphism $R_{\tilde{u},\widetilde{B}}$ is defined to correspond to the 2-morphism in $\mathscr{A}$ given by the pasting

$$\begin{tikzcd} AB_1 \cdots B_l \arrow[r] \arrow[d] \arrow[dr, phantom, "U_1"] & A'B_1 \cdots B_l \arrow[d] \\ B_1 A B_2 \cdots B_l \arrow[r] \arrow[d] & B_1 A' B_2 \cdots B_l \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ B_1 \cdots B_{l-1} A B_l \arrow[r] \arrow[d] \arrow[dr, phantom, "U_{l-1}"] & B_1 \cdots B_{l-1} A' B_l \arrow[d] \\ B_1 \cdots B_l A \arrow[r] & B_1 \cdots B_l A' \end{tikzcd}$$

where $A = A_1 \cdots A_k$, $A' = A_1 \cdots A_{i-1} A_{i+1} A_i A_{i+2} \cdots A_k$, and

$$U_j = B_1 \otimes \cdots \otimes B_{l-j-1} \otimes V_j \otimes B_{l-j+1} \otimes \cdots \otimes B_l$$

where V_j is given by the pasting

$$\begin{tikzcd}[column sep=large]
A_1 \cdots A_i A_{i+1} \cdots A_k B_j \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Swarrow"] & & A_1 \cdots A_{i+1} A_i \cdots A_k B_j \arrow[d] \\
A_1 \cdots A_i A_{i+1} \cdots A_{k-1} B_j A_k \arrow[rr, "\bigotimes_{A_1 \cdots A_{i-1} \otimes R_{A_i, A_{i+1}} \otimes A_{i+2} \cdots A_{k-1}, R_{A_k, B_j}}"] \arrow[d] & & A_1 \cdots A_{i+1} A_i \cdots A_{k-1} B_j A_k \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
A_1 \cdots A_i A_{i+1} B_j A_{i+2} \cdots A_k \arrow[rr, "A_1 \cdots A_{i_1} \otimes S^{-1}_{A_i, A_{i+1} B_j} \otimes A_{i+2} \cdots A_k"'] \arrow[d] \arrow[ddrr, phantom, "\Swarrow"] & & A_1 \cdots A_{i+1} A_i B_j A_{i+2} \cdots A_k \arrow[d] \\
A_1 \cdots A_i B_j A_{i+1} A_{i+2} \cdots A_k \arrow[d] & & A_1 \cdots A_{i+1} B_j A_i A_{i+2} \cdots A_k \arrow[d] \\
A_1 \cdots A_{i-1} B_j A_i A_{i+1} A_{i+2} \cdots A_k \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Swarrow"] & & A_1 \cdots A_{i-1} B A_{i+1} A_i A_{i+2} \cdots A_k \arrow[d] \\
A_1 \cdots A_{i-2} B_j A_{i-1} A_i A_{i+1} \cdots A_k \arrow[rr, "A_1 \cdots A_{i-2} \otimes \bigotimes_{R_{A_{i-1}, B_j}, R_{A_i, A_{i+1}}} \otimes A_{i+2} \cdots A_k"] \arrow[d] & & A_1 \cdots A_{i-1} B_j A_{i+1} A_{i+2} \cdots A_k \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
B_j A_1 \cdots A_k \arrow[rr] & & B_j A_1 \cdots A_{i+1} A_i \cdots A_k
\end{tikzcd}$$

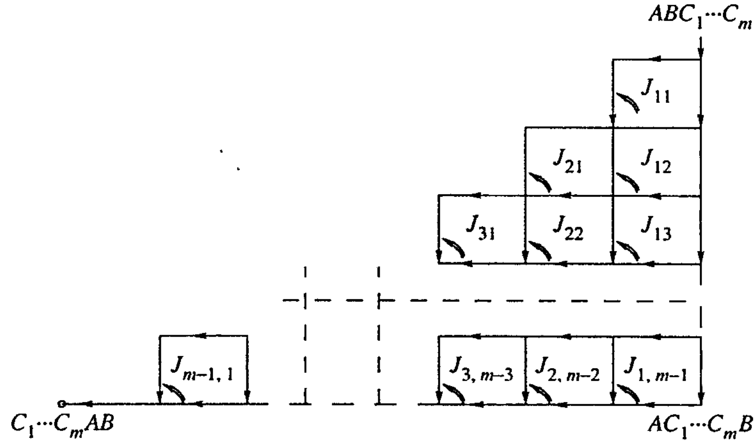
(one hexagon and $(k-2)$ squares). Since the composition of 1-morphisms in $\widetilde{\mathscr{A}}$ is formal, we define in this way an unambiguous 2-morphism $\otimes_{\tilde{u}, \widetilde{B}}$ for any $\tilde{u}$ by concatenation.

$(\bullet \otimes \rightarrow)$. The definition is absolutely similar to that of $(\rightarrow \otimes \bullet)$. We leave it to the reader.

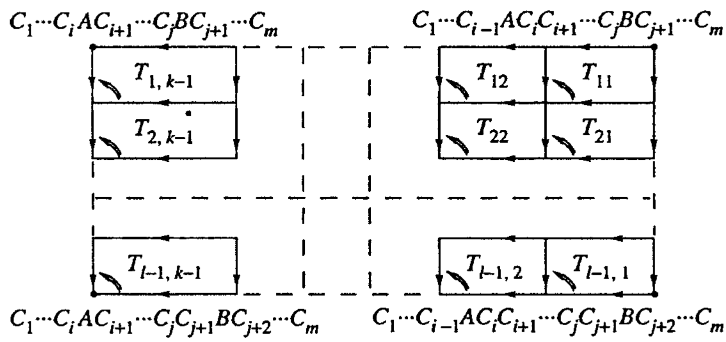
$((\bullet \otimes \bullet) \otimes \bullet)$. Let $\widetilde{A} = \langle A_1, \ldots, A_k \rangle$, $\widetilde{B} = \langle B_1, \ldots, B_l \rangle$, $\widetilde{C} = \langle C_1, \ldots, C_m \rangle$ be objects of $\widetilde{I}$. We need to construct a 2-morphism

$$\begin{tikzcd}
\widetilde{A} \otimes \widetilde{B} \otimes \widetilde{C} \arrow[r, "\widetilde{A} \otimes R_{\widetilde{B}, \widetilde{C}}"] \arrow[d, "R_{\widetilde{A} \otimes \widetilde{B}, \widetilde{C}}"'] \arrow[dr, phantom, "\overset{R_{(\widetilde{A}, \widetilde{B} | \widetilde{C})}}{\Leftarrow}"] & \widetilde{A} \otimes \widetilde{C} \otimes \widetilde{B} \arrow[dl, "R_{\widetilde{A}, \widetilde{C}} \otimes \widetilde{B}"] \\
\widetilde{C} \otimes \widetilde{A} \otimes \widetilde{B} &
\end{tikzcd}$$

We define it to correspond to the 2-morphism in $\mathscr{A}$ given by the pasting



where $A = A_1 \cdots A_k$, $B = B_1 \cdots B_l$, and the 2-morphism J_{ij} is given by the pasting



Here $T_{pq} = T_{pq}^{ij}$ is the 2-morphism of naturality $\otimes_{f,g}$ corresponding to

$$f = (C_1 \cdots C_{i-1} A_1 \cdots A_{k-p}) \otimes R_{A_{k-p+1}, C_i} \otimes (A_{k-p+2} \cdots A_k C_{i+1} \cdots C_j),$$
$$g = (B_1 \cdots B_{l-q}) \otimes R_{B_{l-q+1}, C_{j+1}} \otimes (B_{l-q+2} \cdots B_l C_{j+2} \cdots C_m).$$

$(\bullet \otimes (\bullet \otimes \bullet))$. Let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ be as before. We need to construct a 2-morphism

$$\begin{tikzcd}
\widetilde{A} \otimes \widetilde{B} \otimes \widetilde{C} \arrow[r, "R_{\widetilde{A},\widetilde{B}} \otimes \widetilde{C}"] \arrow[d, "R_{\widetilde{A},\widetilde{B}\otimes\widetilde{C}}"'] \arrow[dr, phantom, "\overset{R_{(\widetilde{A}|\widetilde{B},\widetilde{C})}}{\Leftarrow}"] & \widetilde{B} \otimes \widetilde{A} \otimes \widetilde{C} \arrow[dl, "\widetilde{B} \otimes R_{\widetilde{A},\widetilde{C}}"] \\
\widetilde{B} \otimes \widetilde{C} \otimes \widetilde{A} &
\end{tikzcd}$$

It follows from the definition of $R_{\widetilde{A},\widetilde{B}\otimes\widetilde{C}}$ that the boundary of this triangle is a commutative diagram of 1-morphisms. We define $R_{(\widetilde{A}|\widetilde{B},\widetilde{C})}$ to correspond to the identity 2-morphism in $\mathscr{A}$.

6.14. THEOREM. *The data defined above indeed form a braiding in the monoidal 2-category $\widetilde{I}$.*

PROOF. For any n objects $A_1, \ldots, A_n$ from I we can construct a permutohedral diagram similar to the diagram $P^+(A_1, \ldots, A_n)$ defined in 3.8 for a braided monoidal 1-category. Namely, we put in the vertices of the permutohedron P_n all the permuted products of A_i, and associate to edges the morphisms induced by R-morphisms of our Z-system. In addition, 2-faces of P_n are either hexagons or squares. We fill each hexagon with the 2-morphism given by the S-part of the Z-system and every square with the 2-morphism of naturality of the tensor product (datum $(\to\otimes\to)$ in $\mathscr{A}$). The axioms of monoidal 2-category and the definition of a Z-system imply that every 3-face of the permutohedron is commutative. This implies that the whole permutohedron is commutative. In other words, if γ and δ are two monotone edge paths in P_n with common beginnings and ends and T, U are two membranes (composable pasting schemes which are unions of 2-faces) bounding $\gamma\cup\delta$, then the pastings of T and U give the same 2-morphism from the 1-morphism represented by γ to that represented by δ.

Now let us note that the verification of any axiom of the braiding for $\widetilde{I}$ is reduced to comparison of the pastings of two membranes in a permutohedral diagram as above. Indeed, for an axiom involving only objects this is so by definition. To prove the axioms involving 1- or 2-morphisms it suffices to consider the case when these morphisms are elementary, i.e., come from the braiding. In this case the boundary of any diagram in question again will be represented as a sphere in some permutohedron and the assertion follows. Theorem 6.14 is proven.

6.15. COROLLARY. *Let V be a complex vector space and $S \in \mathrm{End}(V\otimes V\otimes V)$ a solution of the Zamolodchikov equation* (1-10). *For any $m \geq 2$ let $W_m = V^{\otimes(m^2)}$. Then there is a natural operator $S_m \in \mathrm{End}(W_m\otimes W_m\otimes W_m)$ also satisfying the Zamolodchikov equation. Its matrix elements are polynomials in matrix elements of S.*

In other words, we can construct "tensor powers" of a solution of Zamolodchikov equations. The existence of such powers is well known for Yang-Baxter equations. In our case, however, we get only m^2th powers.

PROOF. After replacing S by $S' = S\circ P_{13}$ (see 6.8) we get a Z-system in the monoidal 2-category 2-Vect. The set I for this system consists of one object $\{1\}$, and the braiding 1-morphism $R_{\{1\},\{1\}}\colon \{1\}\otimes\{1\}\to\{1\}\otimes\{1\}$ is the vector space V. Let $\widetilde{I}$ be the braided monoidal 2-category associated to (I, V, S) by Theorem 6.14. Consider the object $X = \langle\{1\}, \ldots, \{1\}\rangle$ (m times) of the 2-category $\widetilde{I}$. Formula (6-7) defined the braiding 1-morphism $R_{X,X}\colon X \otimes X \to X \otimes X$. The definitions imply that $R_{X,X}$ is the tensor product of m^2 copies of the space V. Now pastings in 6.10 define 2-morphisms

$$S_m^{\pm}\colon R_{X,X} *_0 R_{X,X} *_0 R_{X,X} \to R_{X,X} *_0 R_{X,X} *_0 R_{X,X}.$$

These are just endomorphisms of $W_m \otimes W_m \otimes W_m$. Theorem 6.11 implies that each of them forms a Z-system. In addition, reasoning with the permutohedron similar to that used in the proof of Theorem 6.14 shows that $S_m^+ = S_m^-$. So we get one solution of the Zamolodchikov equation.

An explicit formula for S_m can be obtained by unraveling diagrams in 6.13.

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