Bloch-Kato Conjecture

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Homology: Let $f : A \to B$, $g : B \to C$ be homomorphisms of abelian groups such that for all $a \in A$ one has $g(f(a)) = 0$. Define $H(f, g)$ as the abelian group with generators $[b]$ for each $b \in B$ such that $g(b) = 0$ and relations $[b_1] + [b_2] = [b_1 + b_2]$ and $[b] + [f(a)] = [b]$.

Group cohomology: Let G be a group. We write the operation in G as $(g_1, g_2) \mapsto g_1 g_2$ and the unit of G as e. A module over G is an abelian group M together with a map $ms : G \times M \to M$ such that for all $g_1, g_2, g \in G$ and $m, m_1, m_2 \in M$ one has $ms(g_1 g_2, m) = ms(g_1, ms(g_2, m))$, $ms(e, m) = m$, $ms(g, m_1 + m_2) = ms(g, m_1) + ms(g, m_2)$ and $ms(g, 0) = 0$.

For a G-module M and a natural number n define $C^n(G, M)$ as the set of maps $G^n \to M$ where $G^0 = pt$ and $G^{n+1} = G^n \times G$. This set has a structure of a abelian group given by $(\phi + \psi)(x) = \phi(x) + \psi(x)$.

Define for each natural number $n \geq 0$ a map $d^n : C^n(G, M) \to C^{n+1}(G, M)$ inductively as follows. First define maps $d^n_i : C^n(G, M) \to C^{n+1}(G, M)$ for $i = 0, \ldots, n + 1$:

- d^n_0 is given by $d^n_0(\phi)(g_1, \ldots, g_{n+1}) = ms(g_1, \phi(g_2, \ldots, g_{n+1}))$,
- for $i = 1, \ldots, n$, d^n_i is given by $d^n_i(\phi)(g_1, \ldots, g_{n+1}) = \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})$,
- d^n_{n+1} is given by $d^n_i(\phi)(g_1, \ldots, g_{n+1}) = \phi(g_1, \ldots, g_n)$.

Now set for $g = (g_1, \ldots, g_{n+1})$:

$$d^n(\phi)(g) = d^n_0(\phi)(g) + (\sum_{i=1}^{n}(-1)^i d^n_i(\phi)(g)) + (-1)^{n+1} d^n_{n+1}(\phi)(g).$$

Lemma 1 *For any G, M, n and $\phi \in C^n(G, M)$ one has $d^{n+1}(d^n(\phi)) = 0$.*

Because of this lemma the construction of $H(d^n, d^{n+1})$ is applicable and one defines:

$$H^0(G, M) = H(0, d^0)$$

$$H^{n+1}(G, M) = H(d^n, d^{n+1})$$

where 0 is the unique homomorphism $0 \to C^0(G, M)$.

Tensor products: Let A, B be abelian groups. We write the operations in A and B as $+$ and units as 0.

The tensor product $A \otimes B$ is the abelian group given by generators $a \otimes b$ where $a \in A$ and $b \in B$ and relations $(a + a') \otimes b = a \otimes b + a' \otimes b$, $a \otimes (b + b') = a \otimes b + a \otimes b'$ and $0 \otimes b = a \otimes 0 = 0$.

Given two modules M and N over G the tensor product $M \otimes N$ of the underlying abelian groups has a module structure given by $ms(g, a \otimes b) = ms(g, a) \otimes ms(g, b)$.

For a natural number n define inductively $M^{\otimes n}$ setting $M^{\otimes 0} = \mathbf{Z}$ where $\mathbf{Z}$ is considered with the trivial action of G and $M^{\otimes(n+1)} = M^{\otimes n} \otimes M$.

Cup product in group cohomology: Let G be a group and M, N be two G-modules. For any two natural numbers n, m define a map

$$sm_{n,m} : C^n(G, M) \times C^m(G, N) \to C^{n+m}(G, M \otimes N)$$

setting

$$sm_{n,m}(\phi, \psi)(g_1, \dots, g_{n+m}) = (-1)^{n+m}(\phi(g_1, \dots, g_n) \otimes ms(g_1 \dots g_n, \psi(g_{n+1}, \dots, g_{n+m})))$$

Lemma 2 *The map $sm_{n,m}$ respects the relations defining $\otimes$ and therefore defines a homomorphism of abelian groups*

$$\smile_{n,m}: C^n(G, M) \otimes C^m(G, N) \to C^{n+m}(G, M \otimes N)$$

Lemma 3 *For any $a \in C^n(G, M)$, $a' \in C^m(G, N)$ one has*

$$d^{n+m}(a \smile_{n,m} a') = d^n(a) \smile_{n+1,m} a' + (-1)^n\, a \smile_{n,m+1} d^m(a')$$

Lemma 4 *For any $a \in C^n(G, M)$, $a' \in C^m(G, N)$ such that $d^n(a) = 0$ and $d^m(a') = 0$ one has $d^{n+m}(a \otimes a') = 0$.*

Lemma 5 *For any $b \in C^n(G, M)$, $a' \in C^m(G, N)$ such that $d^m(a') = 0$ one has*

$$d^n(b) \smile_{n+1,m} a' = d^{n+m}(b \smile_{n,m} a')$$

Lemma 6 *For any $a \in C^n(G, M)$ such that $d^n(a) = 0$ and $b' \in C^m(G, N)$ one has*

$$a \smile_{n,m+1} d^m(b') = (-1)^n d^{n+m}(a \smile_{n,m} b')$$

From these lemmas one deduces easily that the homomorphism $\smile_{n,m}$ defines a homomorphism

$$H^n(G, M) \otimes H^m(G, N) \to H^{n+m}(G, M \otimes N)$$

which we denote by the same symbol $\smile_{n,m}$.

Fields: A field k is a commutative, associative ring with a unit 1_k such that for any $a \in k$ satisfying $a \neq 0$ there exists $b \in k$ such that $ab = 1_k$.

The set of non-zero elements of a field is an abelian group with respect to multiplication and we denote it by k^*.

If $n \in \mathbf{N}$ is a natural number such that $n \cdot 1_k \neq 0$ then n is said to be invertible in k.

For a natural number n we denote by $\mu_n(k)$ the subset of k^* which consists of elements a such that $a^n = 1_k$. This is easily seen to be a subgroup of k^* and in particular an abelian group.

A field is called algebraically closed if for any non-constant polynomial $f(x) \in k[x]$ over k there exists $a \in k$ such that $f(a) = 0$.

Bloch-Kato Conjecture

Let $\bar{k}$ be an algebraically closed field. Let k be a subfield of $\bar{k}$ such that $\bar{k}$ is algebraic over k i.e. such that every element of $\bar{k}$ is a root of non-constant polynomial with coefficients in k. Let q be a natural number which is invertible in k.

Let $Gal(\bar{k}/k)$ be the group of automorphisms of $\bar{k}$ which act trivially on k. This group acts in particular on $\mu_q(\bar{k})$ in such a way that $\mu_q(\bar{k})$ becomes a $Gal(\bar{k}/k)$-module.

For each natural number $n \geq 1$ define the homomorphism of abelian groups

$$bk_n : H^1(Gal(\bar{k}/k), \mu_q(\bar{k}))^{\otimes n} \to H^n(Gal(\bar{k}/k), (\mu_q(\bar{k}))^{\otimes n})$$

inductively by the rule $bk_1(x) = x$ and $bk_{n+1}(x_n \otimes x) = bk_n(x_n) \smile_{n,1} x$.

Theorem 7 ("Bloch-Kato Conjecture") *For any $\bar{k}$, k, q as above and any natural number $n \geq 1$, the map bk_n is surjective.*