Bloch-Kato Conjecture

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**Homology:** Let \( f : A \to B, \ g : B \to C \) be homomorphisms of abelian groups such that for all \( a \in A \) one has \( g(f(a)) = 0 \). Define \( H(f, g) \) as the abelian group with generators \([b]\) for each \( b \in B \) such that \( g(b) = 0 \) and relations \([b_1] + [b_2] = [b_1 + b_2] \) and \([b] + [f(a)] = [b]\).

**Group cohomology:** Let \( G \) be a group. We write the operation in \( G \) as \((g_1, g_2) \mapsto g_1 g_2 \) and the unit of \( G \) as \( e \). A module over \( G \) is an abelian group \( M \) together with a map \( m s : G \times M \to M \) such that for all \( g_1, g_2, g \in G \) and \( m, m_1, m_2 \in M \) one has \( m s(g_1 g_2, m) = m s(g_1, m s(g_2, m)) \), \( m s(e, m) = m \), \( m s(g, m_1 + m_2) = m s(g, m_1) + m s(g, m_2) \) and \( m s(g, 0) = 0 \).

For a \( G \)-module \( M \) and a natural number \( n \) define \( C^n(G, M) \) as the set of maps \( G^n \to M \) where \( G^0 = pt \) and \( G^{n+1} = G^n \times G \). This set has a structure of a abelian group given by \((\phi + \psi)(x) = \phi(x) + \psi(x)\).

Define for each natural number \( n \geq 0 \) a map \( d^n : C^n(G, M) \to C^{n+1}(G, M) \) inductively as follows.

First define maps \( d^n_0 : C^n(G, M) \to C^{n+1}(G, M) \) for \( i = 0, \ldots, n + 1 \):

- \( d^n_0 \) is given by \( d^n_0(\phi)(g_1, \ldots, g_{n+1}) = m s(g_1, \phi(g_2, \ldots, g_{n+1})) \),
- for \( i = 1, \ldots, n \), \( d^n_i \) is given by \( d^n_i(\phi)(g_1, \ldots, g_{n+1}) = \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \),
- \( d^n_{n+1} \) is given by \( d^n_{n+1}(\phi)(g_1, \ldots, g_{n+1}) = \phi(g_1, \ldots, g_n) \).

Now set for \( g = (g_1, \ldots, g_{n+1}) \):

\[
d^n(\phi)(g) = d^n_0(\phi)(g) + \sum_{i=1}^{n} (-1)^i d^n_i(\phi)(g) + (-1)^{n+1} d^n_{n+1}(\phi)(g).
\]

**Lemma 1** For any \( G, M, n \) and \( \phi \in C^n(G, M) \) one has \( d^{n+1}(d^n(\phi)) = 0 \).

Because of this lemma the construction of \( H(d^n, d^{n+1}) \) is applicable and one defines:

\[
H^0(G, M) = H(0, d^0)
\]

\[
H^{n+1}(G, M) = H(d^n, d^{n+1})
\]

where \( 0 \) is the unique homomorphism \( 0 \to C^0(G, M) \).

**Tensor products:** Let \( A, B \) be abelian groups. We write the operations in \( A \) and \( B \) as + and units as 0.

The tensor product \( A \otimes B \) is the abelian group given by generators \( a \otimes b \) where \( a \in A \) and \( b \in B \) and relations \((a + a') \otimes b = a \otimes b + a' \otimes b \), \( a \otimes (b + b') = a \otimes b + a \otimes b' \) and \( 0 \otimes b = a \otimes 0 = 0 \).

Given two modules \( M \) and \( N \) over \( G \) the tensor product \( M \otimes N \) of the underlying abelian groups has a module structure given by \( m s(g, a \otimes b) = m s(g, a) \otimes m s(g, b) \).

For a natural number \( n \) define inductively \( M^{\otimes n} \) setting \( M^{\otimes 0} = Z \) where \( Z \) is considered with the trivial action of \( G \) and \( M^{\otimes (n+1)} = M^{\otimes n} \otimes M \).
Cup product in group cohomology: Let $G$ be a group and $M$, $N$ be two $G$-modules. For any two natural numbers $n, m$ define a map

$$sm_{n,m} : C^n(G, M) \times C^m(G, N) \to C^{n+m}(G, M \otimes N)$$

setting

$$sm_{n,m}(\phi, \psi)(g_1, \ldots, g_{n+m}) = (-1)^{n+m}(\phi(g_1, \ldots, g_n) \otimes ms(g_1 \ldots g_n, \psi(g_{n+1}, \ldots, g_{n+m})))$$

Lemma 2 The map $sm_{n,m}$ respects the relations defining $\otimes$ and therefore defines a homomorphism of abelian groups

$$\sim_{n,m} : C^n(G, M) \otimes C^m(G, N) \to C^{n+m}(G, M \otimes N)$$

Lemma 3 For any $a \in C^n(G, M)$, $a' \in C^m(G, N)$ one has

$$d^{n+m}(a \sim_{n,m} a') = d^n(a) \sim_{n+1,m} a' + (-1)^n a \sim_{n,m+1} d^m(a')$$

Lemma 4 For any $a \in C^n(G, M)$, $a' \in C^m(G, N)$ such that $d^n(a) = 0$ and $d^m(a') = 0$ one has $d^{n+m}(a \otimes a') = 0$.

Lemma 5 For any $b \in C^n(G, M)$, $a' \in C^m(G, N)$ such that $d^m(a') = 0$ one has

$$d^n(b) \sim_{n+1,m} a' = d^{n+m}(b \sim_{n,m} a')$$

Lemma 6 For any $a \in C^n(G, M)$ such that $d^n(a) = 0$ and $b' \in C^m(G, N)$ one has

$$a \sim_{n,m+1} d^m(b') = (-1)^n d^{n+m}(a \sim_{n,m} b')$$

From these lemmas one deduces easily that the homomorphism $\sim_{n,m}$ defines a homomorphism

$$H^n(G, M) \otimes H^m(G, N) \to H^{n+m}(G, M \otimes N)$$

which we denote by the same symbol $\sim_{n,m}$.

Fields: A field $k$ is a commutative, associative ring with a unit $1_k$ such that for any $a \in k$ satisfying $a \neq 0$ there exists $b \in k$ such that $ab = 1_k$.

The set of non-zero elements of a field is an abelian group with respect to multiplication and we denote it by $k^*$.

If $n \in \mathbb{N}$ is a natural number such that $n \cdot 1_k \neq 0$ then $n$ is said to be invertible in $k$.

For a natural number $n$ we denote by $\mu_n(k)$ the subset of $k^*$ which consists of elements $a$ such that $a^n = 1_k$. This is easily seen to be a subgroup of $k^*$ and in particular an abelian group.

A field is called algebraically closed if for any non-constant polynomial $f(x) \in k[x]$ over $k$ there exists $a \in k$ such that $f(a) = 0$. 
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Let $\bar{k}$ be an algebraically closed field. Let $k$ be a subfield of $\bar{k}$ such that $\bar{k}$ is algebraic over $k$ i.e. such that every element of $\bar{k}$ is a root of non-constant polynomial with coefficients in $k$. Let $q$ be a natural number which is invertible in $k$.

Let $Gal(\bar{k}/k)$ be the group of automorphisms of $\bar{k}$ which act trivially on $k$. This group acts in particular on $\mu_q(\bar{k})$ in such a way that $\mu_q(\bar{k})$ becomes a $Gal(\bar{k}/k)$-module.

For each natural number $n \geq 1$ define the homomorphism of abelian groups

$$bk_n : H^1(Gal(\bar{k}/k), \mu_q(\bar{k}))^{\otimes n} \to H^n(Gal(\bar{k}/k), (\mu_q(\bar{k}))^{\otimes n})$$

inductively by the rule $bk_1(x) = x$ and $bk_{n+1}(x_n \otimes x) = bk_n(x_n) \cdot x_{n,1}$.

**Theorem 7 (“Bloch-Kato Conjecture”)** For any $\bar{k}$, $k$, $q$ as above and any natural number $n \geq 1$, the map $bk_n$ is surjective.