Type categories, C-systems, and universe categories[1]

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Abstract

This is the third paper in a series started in [?]. In it we construct a C-system $CC(\mathcal{C}, p)$ starting from a category $\mathcal{C}$ together with a morphism $p : \widetilde{U} \to U$, a choice of pull-back squares based on p for all morphisms to U and a choice of a final object of $\mathcal{C}$. Such a quadruple is called a universe category. We then define universe category functors and construct homomorphisms of C-systems $CC(\mathcal{C}, p)$ defined by universe category functors. As a corollary of this construction and its properties we show that the C-systems corresponding to different choices of pull-backs and final objects are constructively isomorphic.

1 Introduction

The concept of a C-system in its present form was introduced in [?]. The type of the C-systems is constructively equivalent to the type of contextual categories defined by Cartmell in [?] and [?] but the definition of a C-system is slightly different from the Cartmell's foundational definition.

In [?] we constructed for any pair (R, LM) where R is a monad on $Sets$ and LM a left R-module with values in $Sets$ a C-system $CC(R, LM)$. In the particular case of pairs (R, LM) corresponding to signatures as in [?, p.228] or to nominal signatures the regular sub-quotients of $CC(R, LM)$ are the C-systems corresponding to dependent type theories of the Martin-Lof genus.

In this paper we describe another construction that generates C-systems. This time the input data is a quadruple that consists of a category $\mathcal{C}$, a morphism $p : \widetilde{U} \to U$ in this category, a choice of pull-back squares based on p for all morphisms to U and a choice of a final object in $\mathcal{C}$. Such a quadruple is called a universe category. For any universe category we construct a C-system that we denote by $CC(\mathcal{C}, p)$.

We then define the notion of a universe category functor and construct homomorphisms of C-systems of the form $CC(\mathcal{C}, p)$ corresponding to universe category functors. For universe category functors satisfying certain conditions these homomorphisms are isomorphisms. In particular, any equivalence $F : \mathcal{C} \to \mathcal{C}'$ together with an isomorphism $F(p) \cong p'$ (in the category of morphsims) defines a universe category functor whose associated homomorphism of C-systems is an isomorphism.

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To the best of our knowledge it is the only known construction of a C-system from a category level data that transforms equivalences into isomorphisms. Because of this fact we find it important to present both the construction of the C-system and the construction of the homomorphisms defined by universe functors in detail.

To avoid the abuse of language inherent in the use of the Theorem-Proof style of presenting mathematics when dealing with constructions we use the pair of names Problem-Construction for the specification of the goal of a construction and the description of the particular solution.

In the case of a Theorem-Proof pair one usually refers (by name or number) to the statement when using both the statement and the proof. This is acceptable in the case of theorems because the future use of their proofs is such that only the fact that there is a proof but not the particulars of the proof matter.

In the case of a Problem-Construction pair the content of the construction often matters in the future use. Because of this we often have to refer to the construction and not to the problem and we assign in this paper numbers both to Problems and to the Constructions.

Following the approach used in [**?**] we write the composition of morphisms in categories in the diagrammatic order, i.e., for $f : X \to Y$ and $g : Y \to Z$ their composition is written as $f \circ g$. This makes it much easier to translate between diagrams and equations involving morphisms.

The methods of this paper are fully constructive.

We use the word "category" to refer to that which in the univalent formalization may be replaced by the concept of a precategory (see [**?**]). However, due to the invariance of our constructions under equivalences all of them should factor through the Rezk completion. This invariance also makes the use of the word "category" consistent with the practice suggested in the introduction to [**?**].

This paper is based almost entirely on the material of [**?**]. I am grateful to The Centre for Quantum Mathematics and Computation (QMAC) and the Mathematical Institute of the University of Oxford for their hospitality during my work on the previous version of the paper and to the Department of Computer Science and Engineering of the University of Gothenburg and Chalmers University of Technology for its the hospitality during my work on the present version.

2 The canonical presheaf of C-systems on a split type category

Let us recall the following definition (cf. [**?**], [**?**, Def. 2.2.1]).

Definition 2.1 *[***2015.07.09.def1***] A type (pre)category is a collection of data of the form*

1. *A (pre)category $\mathcal{C}$,*

2. *For each $X \in \mathcal{C}$ a collection of objects $Ty(X)$,*

3. *For each $X \in \mathcal{C}$ a map $Ty(X) \to \mathcal{C}/X$ that we will denote as*

$$T \mapsto (p_{X,T} : c_X(T) \to X)$$

We will often write $c(T)$ instead of $c_X(T)$.

4. *For each $f : X' \to X$ in $\mathcal{C}$ a map $f^* : Ty(X) \to Ty(X')$ and for each $T \in Ty(X)$ a morphism $Q(f,T) : c(f^*(T)) \to c(T)$ such that the square*

[**2015.07.09.eq7**]
$$\begin{array}{ccc} c(f^*(T)) & \xrightarrow{Q(f,T)} & c(T) \\ {\scriptstyle p_{X',f^*(T)}}\downarrow & & \downarrow{\scriptstyle p_{X,T}} \\ X' & \xrightarrow{f'} & X \end{array} \qquad (1)$$

is a pull-back square.

A type category is called split if the following conditions hold:

1. *for all X and $T \in Ty(X)$ one has $Id_X^*(T) = T$,*
2. *for all $f' : X'' \to X'$, $f : X' \to X$ and $T \in Ty(X)$ one has $(f')^*(f^*(T)) = (f' \circ f)^*(T)$,*
3. *for all X and $T \in Ty(X)$ one has $Q(Id_X, T) = Id_{c(T)}$,*
4. *for all $f' : X'' \to X'$, $f : X' \to X$ and $T \in Ty(X)$ one has*

$$(f')^*(f^*(T)) = (f' \circ f)^*(T)$$

and

$$Q(f', f^*(T)) \circ Q(f,T) = Q(f' \circ f, T)$$

5. *(*) for all X, $Ty(X)$ is a set.*

Note that the last condition is relevant only in the foundations where not all collections of objects are sets.

If $\mathcal{C}$ is a type (pre)category we will denote by the same letter its underlying category.

Remark 2.2 [**2015.07.09.rem2**] For any type category $\mathcal{C}$ and $X \in \mathcal{C}$ define a family $Fm(X)$ with the base $Ty(X)$ and the fiber over each $T \in Ty(T)$ being the set of sections of $p_{X,T}$, i.e., the morphisms $s : X \to c_X(T)$ such that $s \circ p_{X,T} = Id_X$. If $\mathcal{C}$ is split them Fm is a contravariant functor from $\mathcal{C}$ to the category of families of sets and the collection of data formed by $\mathcal{C}$, Fm and the comprehension structure is a category with families as defined by Dybjer (see [**?**]). If one considers type categories and categories with families as essentially algebraic structures then this construction forms a part of a constructive equivalence from the category of split type categories to the category of CwF's. In the univalent foundations this construction forms a part of a constructive equivalence from the type of split type categories to the type of CwF's, see [**?**].

Let $\mathcal{C}$ be a type category and $X \in \mathcal{C}$. Define by induction on n pairs $(Ob_n(X), int_{n,X})$ where $Ob_n(X)$ are collections of objects and $int_{n,X} : Ob_n(X) \to Ob(\mathcal{C})$ are functions, as follows:

1. $Ob_0(X) = unit$ where $unit$ is the distinguished set with only one point tt and $int_{0,X}$ maps the whole of $Ob_0(X)$ to X.
2. $Ob_{n+1}(X) = \amalg_{A \in Ob_n} Ty(int_{n,X}(A))$ and $int_{n+1,X}(A,T) = c_{int(A)}(T)$.

In what follows we will write int_X or even int instead of $int_{n,X}$ since both n and X can usually be inferred.

Define for each n the function $ft_{n+1} : Ob_{n+1}(X) \to Ob_n(X)$ by the formula $ft_{n+1}(A,T) = A$ and define ft_0 as the identity function of Ob_0.

For each $B = (ft(B), T) \in Ob_{n+1}(X)$ define $p_B : int(B) \to int(ft(B))$ as $p_{int(ft(B)),T}$. For $B \in Ob_0$ define p_B as $Id_{int(B)}$.

For each $A \in Ob_m(X)$, $B = (ft(B), T) \in Ob_{n+1}(X)$ and $f : int(A) \to int(ft(B))$ define $f^*(B) \in Ob_{m+1}(X)$ as

$$f^*(B) = (A, f^*(T))$$

Assume now that $\mathcal{C}$ is split. For each $A \in Ob_m(X)$, $B = (ft(B), T) \in Ob_{n+1}(X)$ and $f : int(A) \to int(ft(B))$ define $q(f,B) : int(f^*(B)) \to int(B)$ as

$$q(f,B) = Q(f,T)$$

Problem 2.3 [**2014.09.18.prob1**] *Let $\mathcal{C}$ be a split type category as above and $X \in \mathcal{C}$. To define a C-system $CC_{\mathcal{C}}(X)$.*

Construction 2.4 [**2014.09.18.constr1**]We set

$$Ob(CC_{\mathcal{C}}(X)) = \amalg_{n \geq 0} Ob_n(X)$$

where $Ob_n(X)$ are the sets introduced above. For $\Gamma = (n, A)$ and $\Gamma' = (n', A')$ in $Ob(CC_{\mathcal{C}}(X))$ we define

$$Hom_{CC_{\mathcal{C}}(X)}(\Gamma, \Gamma') = Hom_{\mathcal{C}}(int_{n,X}(A), int_{n',X}(A'))$$

The identity morphisms and the composition of morphisms are defined as in $\mathcal{C}$. The proofs of the axioms of a category are straightforward. The function $ft : Ob(CC) \to Ob(CC)$ is defined as the sum of functions ft_n defined above. The canonical morphisms $p_{(n,A)}$ are defined as p_A where p_A where defined above. Similarly one defines the morphisms $q(f, (n+1, B))$ as the morphisms $q(f, B)$.

The conditions (1)-(4) that define split type categories show that the structure so defined satisfied the axioms of a C0-system as defined in [**?**, Definition 2.1].

The canonical squares of $CC_{\mathcal{C}}(X)$ are of the form

$$[\mathbf{2015.07.09.eq3}] \quad \begin{array}{ccc} int(f^*(B)) & \xrightarrow{q(f,B)} & int(B) \\ {\scriptstyle p_{f^*(B)}}\downarrow & & \downarrow{\scriptstyle p_B} \\ int(A) & \xrightarrow{f} & int(ft(B)) \end{array} \qquad (2)$$

Unfolding the definitions we see that these are particular cases of squares of the form (1). In particular they are pull-back squares in $\mathcal{C}$.

Let int_X be the sum of the functions $int_{n,X}$. Together with the identity maps on the sets of morphisms between two objects it defines a full embedding of the category underlying $CC_{\mathcal{C}}(X)$ to $\mathcal{C}/X$. Since the squares (2) are pull-back squares in $\mathcal{C}$ they are also pull-back squares in $\mathcal{C}/X$ and, being pull-back squares in the image of a full embedding, are also pull-back squares in the source of the embedding, i.e., in $CC_{\mathcal{C}}(X)$. In view of [**?**, Proposition 2.4] this implies that the C0-system $CC_{\mathcal{C}}(X)$ has a unique structure of a C-system.

Remark 2.5 The image of int_X on objects consists of those objects over X for which the morphism to X can be represented as a composition of morphisms of the form $p_{X,T}$. Note that int_X need not be an injection on the sets of objects. For example, if $\mathcal{C}$ is type category whose underlying category is the one point category with $Ob(\mathcal{C}) = unit$ and Ty is the one point presheaf with $Ty(unit) = unit$ then $Ob_{\mathcal{C}}(unit)$ will be isomorphic to the set of natural numbers.

Remark 2.6 [**2015.07.09.rem3**] Recall that for a C-system CC we let $Ob_n(CC)$ denote the subset in $Ob(CC)$ that consists of objects of length n. Note that $Ob_n(CC_{\mathcal{C}}(X)) \neq Ob_n(X)$. Indeed, the elements of the first set are pairs of the form (n, A) where A is an element of the second set. This difference is the reason for some extra notation that we have to use below.

For $n \geq m$ let $ft^m_{n,X} : Ob_n(X) \to Ob_{n-m}(X)$ be the composition $ft_n \circ \ldots \circ ft_{n-m+1}$. We will usually write these functions simply as ft^m. For $A \in Ob_n(X)$ and $A' \in Ob_{n-m}(X)$ such that $A' = ft^m(A)$ we will write

$$p(A, A') : int(A) \to int(A')$$

for the composition of m canonical projections $p_A \circ \ldots \circ p_{ft^{m-1}(A)}$. When $A \in Ob_n(X)$ and $A' \in Ob_{n-m}(X)$ are such that $ft^m(A) = A'$ we will say that A is over A'. The morphisms $p(A, A')$ are defined for all pairs A, A' such that A is over A'. In particular, any A is over $tt \in unit = Ob_0(X)$ so that the morphism

$$p(A, tt) : int(A) \to X$$

is defined.

Let us now construct an extension of the function $X \mapsto CC_{\mathcal{C}}(X)$ from $\mathcal{C}$ to the type of C-systems to a presheaf of C-systems on $\mathcal{C}$.

For $f : X' \to X$ in $\mathcal{C}$ define by induction on n pairs $(f^{\#,n}, Q^n(f,-))$ where $f^{\#,n}$ are functions $f^{\#,n} : Ob_n(X) \to Ob_n(X')$ and $Q^n(f,-)$ are families of morphisms of the form

$$Q^n(f,A) : int_{X'}(f^{\#,n}(A)) \to int_X(A)$$

given for all $A \in Ob_n(X)$:

1. $f^{\#,0}$ is the only map from *unit* to *unit* and $Q^0(f,tt) = f$,
2. for $A = (ft(A),T) \in Ob_{n+1}(X)$ we set

$$f^{\#,n+1}(ft(A),T) = (f^{\#,n}(ft(A)), Q^n(f,ft(A))^*(T))$$

   and

$$Q^{n+1}(f,(ft(A),T)) = Q(Q^n(f,ft(A)),T)$$

In what follows we will omit the index n and write $Q(f,A)$ instead of $Q^n(f,A)$ and $f^{\#}(A)$ instead of $f^{\#,n}(A)$.

Note that by construction, for any $A \in Ob_{n+1}(X)$ we have a square

$$\begin{array}{ccc} int_{X'}(f^{\#}(A)) & \xrightarrow{Q(f,A)} & int_X(A) \\ {\scriptstyle p_{f^{\#}(A)}}\downarrow & & \downarrow{\scriptstyle p_A} \\ int_{X'}(ft(f^{\#}(A))) & \xrightarrow{Q(f,ft(A))} & int(ft(A)) \end{array}$$

which is of the form (1) but not necessarily of the form (2).

Lemma 2.7 *[**2015.07.09.l9**] The functions $f^{\#,n}$ commute with the functions ft i.e. for $A \in Ob_n(X)$ one has*

$$ft(f^{\#,n}(A)) = f^{\#,n}(ft(A))$$

Proof: It is immediate from the construction.

Lemma 2.8 *[**2015.07.09.l8**] Let $A \in Ob_n(X)$, $A' \in Ob_{n-m}(X)$ be such that A is over A'. Let $f : X' \to X$ be a morphism. Then $f^{\#}(A)$ is over $f^{\#}(A')$ and the square*

$$\begin{array}{ccc} int_{X'}(f^{\#}(A)) & \xrightarrow{Q(f,A)} & int_X(A) \\ {\scriptstyle p(f^{\#}(A),f^{\#}(A'))}\downarrow & & \downarrow{\scriptstyle p(A,A')} \\ int_X(f^{\#}(A')) & \xrightarrow{Q(f,A')} & int_X(A') \end{array} \quad [\mathbf{2015.07.09.eq8}] \qquad (3)$$

is a pull-back square.

Proof: The proof is by induction on m using the fact that the square (3) is the vertical composition of m squares of the form (1) and that such squares are pull-back squares.

Let $A_i \in Ob_{m_i}(X)$, $i = 1, 2$, $a : int(A_1) \to int(A_2)$ a morphism over X and $f : X' \to X$ a morphism in $\mathcal{C}$. By Lemma 2.8 the squares

$$[\mathbf{2015.07.09.eq11}] \quad \begin{array}{ccc} int(f^{\#}(A_i)) & \xrightarrow{Q(f,A_i)} & int(A_i) \\ {\scriptstyle p(f^{\#}(A_i),tt)}\downarrow & & \downarrow{\scriptstyle p(A_i,tt)} \\ X' & \xrightarrow{f} & X \end{array} \tag{4}$$

are pull-back squares. Therefore, there exists a unique morphism $f^{\#}(a) : int(f^{\#}(A_1)) \to int(f^{\#}(A_2))$ over X' such that

$$[\mathbf{2015.07.10.eq4}] f^{\#}(a) \circ Q(f, A_2) = Q(f, A_1) \circ a \tag{5}$$

Lemma 2.9 *[**2015.07.09.l10**] Let $f : X' \to X$ be a morphism. Then one has:*

1. *for $A \in Ob_n(X)$ one has $f^{\#}(Id_{int(A)}) = Id_{int(f^{\#}(A))}$,*
2. *for $A_i \in Ob_{m_i}(X)$, $i = 1, 2, 3$, $a : int(A_1) \to int(A_2)$ and $a' : int(A_2) \to int(A_3)$ one has $f^{\#}(a \circ a') = f^{\#}(a) \circ f^{\#}(a')$.*

Proof: This is an easy exercise using the fact that the squares (4) are pull-back squares.

Lemma 2.10 *[**2015.07.09.l11**] Let $f : X' \to X$ be a morphism and $A \in Ob_n(X)$. Then $f^{\#}(p_A) = p_{f^{\#}(A)}$.*

Proof: If $n = 0$ then $p_A = Id_X$ and the result follows from Lemma 2.9. Let $A = (ft(A), T) \in Ob_{n+1}(X)$.

By definition $f^{\#}(p_A)$ is the unique morphism $int(f^{\#}(A)) \to int(f^{\#}(ft(A)))$ over X' and such that

$$f^*(p_A) \circ Q(f, ft(A)) = Q(f, A) \circ p_A$$

The morphism $p_{f^{\#}(A)}$ is of the form $int(f^{\#}(A)) \to int(ft(f^{\#}(A)))$ and it is a morphism over X'. We have $ft(f^{\#}(A)) = f^{\#}(ft(A))$ by Lemma 2.7. It remains to verify that

$$p_{f^{\#}(A)} \circ Q(f, ft(A)) = Q(f, A) \circ p_A$$

This is a particular case of the commutativity of (3).

Lemma 2.11 *[**2015.07.09.l12**] Let $f : X' \to X$ be a morphism in $\mathcal{C}$, $A \in Ob_m(X)$, $B = (ft(B), T) \in Ob_{n+1}(X)$ and $a : int(A) \to int(ft(B))$ a morphism. Then one has*

$$[\mathbf{2015.07.09.eq12}] f^{\#}(a^*(B)) = (f^{\#}(a))^*(f^{\#}(B)) \tag{6}$$

and

$$[\mathbf{2015.07.09.eq13}] f^{\#}(q(a, B)) = q(f^{\#}(a), f^{\#}(B)) \tag{7}$$

Proof: For the proof it will be convenient to consider the square

$$\begin{array}{ccc} int(f^{\#}(A)) & \xrightarrow{Q(f,A)} & int(A) \\ {\scriptstyle f^{\#}(a)}\downarrow & & \downarrow{\scriptstyle a} \\ int(f^{\#}(ft(B))) & \xrightarrow{Q(f,ft(B))} & int(ft(B)) \end{array} \quad (8)$$

[**2015.07.10.eq2**]

which commutes because of the defining relation (5) of morphisms $f^{\#}(-)$.

Recall from the construction that for any $(C,T) \in Ob_{i+1}(X)$ we have

$$f^{\#}(C,T) = (f^{\#}(C), Q(f,C)^*(T))$$

$$Q(f,(C,T)) = Q(Q(f,C),T)$$

Let $B = (ft(B),T)$. For the proof of (6) we now have

$$f^{\#}(a^*(ft(B),T)) = f^{\#}(A, a^*(T)) = (f^{\#}(A), (Q(f,A)\circ a)^*(T))$$

and

$$f^{\#}(a)^*(f^{\#}(ft(B),T)) = f^{\#}(a)^*(f^{\#}(ft(B)), Q(f,ft(B))^*(T)) =$$
$$(f^{\#}(A), (f^{\#}(a)\circ Q(f,ft(B)))^*(T))$$

and the right hand sides are equal because of the commutativity of (8).

To prove (7) recall that $f^{\#}(q(a,B))$ is the unique morphism from $int(f^{\#}(a^*(B)))$ to $int(f^{\#}(B))$ over X' such that

$$f^{\#}(q(a,B))\circ Q(f,B) = Q(f,a^*(B))\circ q(a,B)$$

The morphism $q(f^{\#}(a), f^{\#}(B))$ is a morphism from $int(f^{\#}(a)^*(f^{\#}(B)))$ to $int(f^{\#}(B))$ and since we have shown that (6) holds we conclude that it has the same domain and codomain as $f^{\#}(q(a,B))$. Also $q(f^{\#}(a), f^{\#}(B))$ is a morphism over X'. It remains to show that

$$q(f^{\#}(a), f^{\#}(B))\circ Q(f,B) = Q(f,a^*(B))\circ q(a,B)$$

We have

$$q(f^{\#}(a), f^{\#}(B))\circ Q(f,B) = q(f^{\#}(a), f^{\#}(ft(B),T))\circ Q(f,(ft(B),T)) =$$

$$q(f^{\#}(a), (f^{\#}(ft(B)), Q(f,ft(B))^*(T)))\circ Q(f,(ft(B),T)) =$$

$$Q(f^{\#}(a), Q(f,ft(B))^*(T))\circ Q(Q(f,ft(B)),T) = Q(f^{\#}(a)\circ Q(f,ft(B)),T)$$

On the other hand

$$Q(f,a^*(B))\circ q(a,B) = Q(f,a^*(ft(B),T))\circ q(a,(ft(B),T)) = Q(f,(A,a^*(T)))\circ Q(a,T) =$$

$$Q(Q(f,A),a^*(T))\circ Q(a,T) = Q(Q(f,A)\circ a,T)$$

and the right hand sides are equal because of the commutativity of (8). This completes the proof of Lemma 2.11.

Problem 2.12 *[***2015.07.10.prob1***] Let $\mathcal{C}$ be a split type category as above. Let $f : X' \to X$ be a morphism. To construct a homomorphism of C-systems*

$$f^{\#} : CC_{\mathcal{C}}(X) \to CC_{\mathcal{C}}(X')$$

Construction 2.13 We define $f^{\#}_{Ob}$ as the sum of functions $f^{\#,n}$ constructed above. For $\Gamma = (n, A)$ and $\Gamma' = (n', A')$ in $Ob(CC_{\mathcal{C}}(X))$ and $a : int(A) \to int(A')$ over X we define $f^{\#}_{Mor}(a)$ as $f^{\#}(a)$ constructed above.

The fact that $f^{\#}$ commute with the length function is obvious. The fact that it commutes with the ft function follows from Lemma 2.7. That $f^{\#}_{Ob}$ and $f^{\#}_{Mor}$ form a functor follows from Lemma 2.9. That $f^{\#}_{Mor}$ satisfies the p-morphism condition follows from Lemma 2.10. That it satisfies the q-morphism condition follows from Lemma 2.11. This shows that $f^{\#}_{Ob}$ and $f^{\#}_{Mor}$ satisfy the first five conditions of Definition **??** and therefore by Lemma **??** they form a homomorphism of C-systems.

Lemma 2.14 *[***2015.07.10.l1***] Let $\mathcal{C}$ be as above. Let $g : X'' \to X'$ and $f : X' \to X$ be two morphisms. Then for $A \in Ob_n(X)$ one has*

$$g^{\#}(f^{\#}(A)) = (g \circ f)^{\#}(A)$$

and

$$Q(g, f^{\#}(A)) \circ Q(f, A) = Q(g \circ f, A)$$

Proof: The proof is by induction on n. For $n = 0$ the statement is obvious. Let $A = (ft(A), T) \in Ob_{n+1}(X)$. Then one has

$$g^{\#}(f^{\#}(ft(A), T)) = g^{\#}(f^{\#}(ft(A)), Q(f, ft(A))^*(T)) =$$

$$(g^{\#}(f^{\#}(ft(A))), Q(g, f^{\#}(A))^*(Q(f, ft(A))^*(T))) = ((g \circ f)^{\#}(ft(A)), Q((g \circ f, A)^*(T))) =$$
$$(g \circ f)^{\#}(ft(A), T)$$

where the third equality is by inductive assumption.

Lemma 2.15 *[***2015.07.10.l1***] Let $\mathcal{C}$ be as above. Let $g : X'' \to X'$ and $f : X' \to X$ be two morphisms. Let $A_i \in Ob_{n_i}(X)$, $i = 1, 2$ and let $a : int(A_1) \to int(A_2)$ be a morphism over X. Then one has*

$$g^{\#}(f^{\#}(a)) = (g \circ f)^{\#}(a)$$

Proof:

???

3 Universe categories and categories with families

Definition 3.1 *[**2009.11.1.def1**] Let $\mathcal{C}$ be a category. A universe structure on a morphism $p : \widetilde{U} \to U$ in $\mathcal{C}$ is a mapping that assigns to any morphism $f : X \to U$ in $\mathcal{C}$ a pull-back square*

$$\begin{array}{ccc} (X;f) & \xrightarrow{Q(f)} & \widetilde{U} \\ {\scriptstyle p_{X,f}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & U \end{array}$$

A universe in $\mathcal{C}$ is a morphism p together with a universe structure on it.

In what follows we will write $(X; f_1, \ldots, f_n)$ for $(\ldots((X; f_1); f_2)\ldots; f_n)$.

Example 3.2 [**2015.04.06.ex1**] Let G be a group. Consider the category BG with one object pt whose monoid of endomorphisms is G. Recall that any commutative square where all four arrows are isomorphisms is a pull-back square. Let $p : pt \to pt$ be the unit object of G. Then a universe structure on p can be defined by specifying, for every $g : pt \to pt$, of the horizontal morphism $Q(g)$ in the corresponding canonical square. There are no restrictions on the choice of $Q(g)$ since for any such choice one can take the vertical morphism to be $Q(g)g^{-1}$ obtaining a pull-back square. Therefore, the set of universe structures on p is G^G. The automorphisms of BG are given by $Aut(G)$ (with two automorphisms being isomorphic as functors if they differ by an inner automorphisms of G). Therefore, there are $(G^G)/Aut(G)$ isomorphism classes of categories with universes with the underlying category BG and the underlying universe morphism being $Id : pt \to pt$. Note that in this case all auto-equivalences of the category are automorphisms and so simply saying that we will consider universes up to an equivalence of the underlying category does not change the answer. To have, as is suggested by category-theoretic intuition, no more than one universe structure on a morphism one needs to consider categories with universes up to equivalences of categories with universes and then one has the obligation to prove that the constructions that are supposed to produce objects such as C-systems map equivalences of categories with universes to isomorphisms. In the case of the main construction of this paper it is achieved in Lemma 4.4 and with respect to universe category functors of a somewhat wider class than the class of universe category equivalences.

For $f : X \to U$ as in Definition 3.1 and $a : Y \to X$, $b : Y \to \widetilde{U}$ such that $a \circ F = b \circ p$ let $a * b$ be the unique morphism $Y \to (X; F)$ such that

$$(a * b) \circ p_{X,F} = a$$

$$(a * b) \circ Q(F) = b$$

Definition 3.3 *[**2015.03.21.def2**] A universe category is a triple $(\mathcal{C}, p, pt)$ where $\mathcal{C}$ is a category, $p : \widetilde{U} \to U$ is a morphism in $\mathcal{C}$ with a universe structure on it and pt is a final object in $\mathcal{C}$.*

4 Functoriality of $CC(\mathcal{C}, p)$.

Definition 4.1 *[**2015.03.21.def1**] Let $(\mathcal{C}, p, pt)$ and $(\mathcal{C}', p', pt')$ be universe categories. A functor of universe categories from $(\mathcal{C}, p, pt)$ to $(\mathcal{C}', p', pt')$ is a triple $(\Phi, \phi, \widetilde{\phi})$ where $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor and $\phi : \Phi(U) \to U'$, $\widetilde{\phi} : \Phi(\widetilde{U}) \to \widetilde{U}'$ are morphisms such that:*

1. *Φ takes the canonical pull-back squares based on p to pull-back squares,*
2. *Φ takes pt to a final object of $\mathcal{C}'$,*
3. *the square*

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

is a pull-back square.

Let

$$(\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p, pt) \to (\mathcal{C}', p', pt')$$

be a functor of universes categories. Let $Ob_n = Ob_n(\mathcal{C}, p)$ and $Ob'_n = Ob_n(\mathcal{C}', p')$. Let int and int' be the corresponding functions to $\mathcal{C}$ and $\mathcal{C}'$.

Denote by ψ the isomorphism $\psi : pt' \to \Phi(pt)$. Define, by induction on n, pairs (H_n, ψ_n) where $H_n : Ob_n \to Ob'_n$ and ψ_n is a family of isomorphisms of the form

$$\psi_n(A) : int'(H_n(A)) \to \Phi(int(A))$$

given for all $A \in Ob_n$. We set:

1. for $n = 0$, H_0 is the unique map from a one point set to a one point set and $\psi_0(A) = \psi$,
2. $H_{n+1}(A, F) = (H_n(A), \psi_n(A) \circ \Phi(F) \circ \phi)$ and

$$\psi_{n+1}(A, F) : (int(H_n(A)); \psi_n(A) \circ \Phi(F) \circ \phi) \to \Phi(int(A, F))$$

is the unique morphism such that the diagram

[2009.10.26.eq2]

$$\begin{array}{ccccccc} int'(H_{n+1}(A, F)) & \xrightarrow{\psi_{n+1}(A,F)} & \Phi(int(A, F)) & \xrightarrow{\Phi(Q(F))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle p_{H_{n+1}(A,F)}}\downarrow & & \downarrow{\scriptstyle \Phi(p_{(A,F)})} & & \downarrow{\scriptstyle \Phi(p)} & & \downarrow{\scriptstyle p'} \\ int'(H_n(A)) & \xrightarrow{\psi_n(A)} & \Phi(int(A)) & \xrightarrow{\Phi(F)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array} \quad (9)$$

commutes and

$$\psi_{n+1}(A, F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = Q(\psi_n(A) \circ \Phi(F) \circ \phi)$$

Note that the existence and uniqueness of $\psi_{n+1}(A,F)$ follows from the fact that the right hand side squares of (9) are pull-back squares as a corollary of the definition of a universe category functor and the fact that the canonical square for the morphism $\psi_n(A)\circ\Phi(F)\circ\phi$ commutes.

Moreover since the outer square of (9) is a pull-back square, the left-most square commutes and the two right hand side squares are pull-back squares we conclude that the left hand side square is a pull-back square. In combination with the inductive assumption that $\psi_n(A)$ is an isomorphism this implies that $\psi_{n+1}(A,F)$ is an isomorphism.

Problem 4.2 [**2014.09.18.prob2**] *Let*

$$(\Phi,\phi,\widetilde{\phi}):(\mathcal{C},p,pt)\to(\mathcal{C}',p',pt')$$

be a functor of universes categories. To define a homomorphism of C-systems $H=H(\Phi,\phi,\widetilde{\phi})$ from $CC(\mathcal{C},p)$ to $CC(\mathcal{C}',p')$.

Construction 4.3 [**2014.09.18.constr2**]We define H_{Ob} as the sum of the functions H_n that were constructed above. To define H_{Mor} on morphisms we use the fact that morphisms $\psi(A)$ are isomorphisms and for

$$f\in Hom_{CC}((n,A),(n',A'))=Hom_{\mathcal{C}}(int(A),int(A'))$$

we set

$$[\mathbf{2009.10.26.eq6}]H_{Mor}(f)=\psi(A)\circ\Phi(f)\circ\psi(A')^{-1} \tag{10}$$

To show that (H_{Ob},H_{MoRr}) is a homomorphism of C-systems it is sufficient, In view of Lemma **??**, to verify the first five conditions of Definition **??**.

It is clear that H_{Ob} respects the length function and the ft maps.

The fact that this construction gives a functor i.e. satisfies the unity and composition axioms is straightforward.

The fact that it takes the canonical projections to canonical projections is equivalent to the commutativity of the left hand side square in (9).

It remains to show that it satisfies the q-morphisms condition. Consider a canonical square of the form (**??**). Its image is a square of the form

$$[\mathbf{2009.10.26.eq4}]\quad \begin{array}{ccc} (G'_1,\ldots,G'_n,G'_{n+1}) & \xrightarrow{H(q(f))} & (F'_1,\ldots,F'_{m+1})\\ {\scriptstyle H(p_G)}\downarrow & & \downarrow{\scriptstyle H(p_F)}\\ (G'_1,\ldots,G'_n) & \xrightarrow{H(f)} & (F'_1,\ldots,F'_m) \end{array} \tag{11}$$

We already know that the vertical arrows are canonical projections. We have to show that $G'_{n+1}=int(H(f))\circ F'_{m+1}$ and

$$[\mathbf{2009.10.26.eq8}]H(q(f))\circ Q(F'_{m+1})=Q(H(f)\circ F'_{m+1}) \tag{12}$$

By (**??**) we have

$$G'_{n+1} = \psi_{(G_1,\dots,G_n)} \circ \Phi(F_{m+1}f) \circ \phi$$
$$F'_{m+1} = \psi_{(F_1,\dots,F_m)} \circ \Phi(F_{m+1}) \circ \phi$$

and by (10)

$$H(f) = \psi_{(G_1,\dots,G_n)} \circ \Phi(f) \circ \psi^{-1}_{(F_1,\dots,F_m)}$$
$$H(q(f)) = \psi_{(G_1,\dots,G_n,F_{m+1}f)} \circ \Phi(q(f)) \circ \psi^{-1}_{(F_1,\dots,F_{m+1})}$$

Therefore the relation $G'_{n+1} = H(f) \circ F'_{m+1}$ follows immediately and the relation (12) follows by application of (**??**).

Lemma 4.4 *[***2014.09.18.l1***] Let $(\Phi, \phi, \widetilde{\phi})$ be as in Problem 4.2 and let H be the corresponding solution of Construction 4.3. Then if Φ is a full embedding and ϕ and $\widetilde{\phi}$ are isomorphisms then H is an isomorphism of C-systems.*

Proof: Straightforward.

Lemma 4.4 implies in particular that considered up to a canonical isomorphism $CC(\mathcal{C}, p)$ depends only on the equivalence class of the pair $(\mathcal{C}, p)$ i.e. that our construction maps the type of pairs $(\mathcal{C}, p)$ to the type of C-systems.

Let us describe now a construction which shows that any C-system is isomorphic to a C-system of the form $CC(\mathcal{C}, p)$.

Problem 4.5 *[***2014.09.18.prob3***] Let CC be a C-system. Construct a universe category $(\mathcal{C}, p)$ and an isomorphism $CC \cong CC(\mathcal{C}, p)$.*

Construction 4.6 *[***2014.09.18.constr3***]*Denote by $PreShv(CC)$ the category of contravariant functors from the category underlying CC to $Sets$.

Let Ty be the functor which takes an object $\Gamma \in CC$ to the set

$$Ty(\Gamma) = \{\Gamma' \in CC \,|\, ft(\Gamma') = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $\Gamma' \mapsto f^*\Gamma'$. It is a functor due to the composition and unity axioms for f^*. Let Tm be the functor which takes an object Γ to the set

$$Tm(\Gamma) = \{s \in \widetilde{CC} \,|\, ft\,\partial(s) = \Gamma\}$$

and a morphism $f : \Delta \to \Gamma$ to the map $s \mapsto f^*(s)$ where $f^*(s)$ (or $f^*(s, 1)$ in the notation of [**?**]) is the pull-back of the section s along f. Let further $p : Tm \to Ty$ be the morphism which takes s to $\partial(s)$. It is well defined as a morphisms of families of sets and forms a morphism of presheaves since $\partial(f^*(s)) = f^*(\partial(s))$.

Let us construct an isomorphism $CC \cong CC(PreShv(CC), p)$.

In what follows we identify objects of CC with the corresponding representable presheaves and, for a presheaf F and an object Γ, we identify morphisms $\Gamma \to F$ in $PreShv(CC)$ with $F(\Gamma)$. Recall that for $X \in CC$ such that $l(X) > 0$ we let $\delta(X) : X \to p^*_X(X)$ denote the section of $p_{p^*_X(X)}$ given by the diagonal.

Lemma 4.7 *[**2009.12.28.l1**] Let $\Gamma' \in Ob(CC)$ and let $\Gamma = ft(\Gamma')$. Then the square*

[**2009.12.28.eq2**]
$$\begin{array}{ccc} \Gamma' & \xrightarrow{\delta(\Gamma')} & Tm \\ {\scriptstyle p_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle p} \\ \Gamma & \xrightarrow{\Gamma'} & Ty \end{array} \qquad (13)$$

is a pull-back square.

Proof: We have to show that for any $\Delta \in CC$ the obvious map

$$Hom(\Delta, \Gamma') \to Hom(\Delta, \Gamma) \times_{Ty(\Delta)} Tm(\Delta)$$

is a bijection. Let $f_1, f_2 : \Delta \to \Gamma'$ be two morphisms such that their images under (13) coincide i.e. such that $f_1 \circ p_{\Gamma'} = f_2 \circ p_{\Gamma'}$ and $f_1^*(\delta(\Gamma')) = f_2^*(\delta(\Gamma)')$. These two conditions are equivalent to saying, in the notation of [**?**], that $ft(f_1) = ft(f_2)$ and $s_{f_1} = s_{f_2}$. This implies that $f_1 = f_2$. Let $f : \Delta \to \Gamma$ be a morphism and $s \in Tm(\Delta)$ a section such that $ft(\partial(s)) = f^*(\Gamma')$. Then the composition $s \circ q(f, \Gamma')$ is a morphism $f' : \Delta \to \Gamma'$ such that $f' \circ p_{\Gamma'} = f$. We also have

$$(f')^*(\delta(\Gamma')) = s^*(q(f, \Gamma')^*(\delta(\Gamma'))) = s$$

which proves that (13) is surjective.

To construct the required isomorphism we now choose a universe structure on p such that the pull-back squares associated with morphisms from representable objects are squares (13). The isomorphism is now obvious.

Example 4.8 We can use Construction 2.4 to produce a C-system from a pre-category C with a final object pt and fiber products. This example was inspired by a question from an anonymous referee of [**?**]. Here we have to use the word "pre-category" since this construction, unlike all other constructions of this paper, is not invariant under equivalences.

Given a pre-category C with a final object and fiber products consider the category $PreShv(C)$ of presheaves of sets on C. Let U be the presheaf that takes X to the set of all pairs of morphisms (f, g) such that $f : X \to Y$ and $g : Z \to Y$. The functoriality is defined by compositing f. Similarly let $\widetilde{U}$ be the presheaf that takes X to the set of all pairs of morphisms (f', g) such that $f' : X \to Z$, $g : Z \to Y$ and functoriality is again through composition of f'. There is a morphism $p : \widetilde{U} \to U$ that takes (f', g) to $(f' \circ g, g)$. A square

$$\begin{array}{ccc} X' & \xrightarrow{(f', g')} & \widetilde{U} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{(f, g)} & U \end{array}$$

commutes if $g' = g$ and $u \circ f = f' \circ g'$. It is a pull-back square if the square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pull-back square. In particular, if C has pull-backs then the C-system $CC(PreShv(C), p)$ is well defined.

Note that this construction is not invariant under equivalences in C. If C is replaced by an equivalent but not an isomorphic category the morphism p will be replaced by a morphism that is not isomorphic to it.

On the other hand the change in the choice of pull-backs without a change in C will lead to the change of the C-system by a constructively isomorphic one,

Example 4.9 [**2015.06.15.ex1**] An important example of a C-system of the form $CC(\mathcal{C}, p)$ is "the" C-system Fam of families of sets considered in [**?**] and [**?**]. The definition of Fam in [**?**][p.238] as well as the preceding it discussion in [**?**]p.232 is somewhat incomplete in that the notion of "a set" and moreover the notion of "a family of sets" are taken as being uniquely determined by some previous agreement that is never explicitly referred to.

To define Fam as a C-system of the form $CC(\mathcal{C}, p)$ we need two "universes" (e.g. Grothendieck universes) U and U_1 in our set theory such that U_1 is an element of U. One then defines the category $Sets(U)$ of small sets as the category whose set of objects is U and such that for $X, Y \in U$ the set of morphisms from X to Y in $Sets(U)$ is the set of functions from X to Y in the ambient set theory. This category will contain U_1 as an object and also, because of the closure conditions that U satisfies, it will contain as an object the set $\widetilde{U}_1$ of pairs (X, x) where $X \in U_1$ and $x \in X$. Since morphisms in $Sets$ are the same as functions in the ambient set theory we also get $p_{U_1} : \widetilde{U}_1 \to U_1$ that takes (X, x) to X. Using the standard construction of pull-backs in sets we obtain a universe structure on p. Now we can define:

$$Fam = Fam(U, U_1) := CC(Sets(U), p_{U_1})$$

The explicit definition given in [**?**] avoids the use of the second universe (universe U in our notations) by constructing the same category "by hand". In our approach we have to use U but the resulting category does not depend on U. Indeed, if our set theory assumes two universes U and U' such that both contain U_1 as an element then one can show that

$$[\mathbf{2015.06.15.eq1}] CC(Sets(U), p_{U_1}) = CC(Sets(U'), p_{U_1}) \tag{14}$$

where equality of categories means in particular that their sets of objects are equal as sets. Because of this one can denote this category as $Fam(U_1)$.

Definition 4.10 *[***2009.12.27.def1***] Let CC be a C-system. A universe model of CC is a pair of a universe category $(\mathcal{C}, p)$ and a C-system homomorphism $CC \to CC(\mathcal{C}, p)$.*

Conjecture [**2009.12.27.prop1**] Let $\mathcal{C}$ be a category, CC be a C-system and $M : CC \to \mathcal{C}$ a functor such that $M(pt_{CC})$ is a final object of $\mathcal{C}$ and M maps distinguished squares of CC to pull-back squares of $\mathcal{C}$. Then there exists a universe $p_M : \widetilde{U}_M \to U_M$ in $PreShv(\mathcal{C})$ and a C-system homomorphism $M' : CC \to CC(PreShv(\mathcal{C}), p_M)$ such that the square

$$\begin{array}{ccc} CC & \xrightarrow{M} & \mathcal{C} \\ \downarrow{\scriptstyle M'} & & \downarrow \\ CC(PreShv(\mathcal{C}), p_M) & \xrightarrow{int} & PreShv(C) \end{array}$$

where the right hand side vertical arrow is the Yoneda embedding, commutes up to a functor isomorphism.