A note on natural models[1]

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Abstract

Just a short note on the relation between the natural models of [1] and the presentation of the MLTT in [2].

Followed by a discussion about the relation between natural models and sets of morphisms in a (strict pre-)category.

In [1] a natural model is defined as a category $\mathcal{C}$ with a final object together with a representable natural transformation $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ of presheaves (of sets) on $\mathcal{C}$.

The version of this concept where instead of representable natural transformations one considers represented natural transformations is closely related to the categories with families of Peter Dybjer ([3]). Dybjer's categories with families (CwF) are formulated in the generalized algebraic framework. Their analog in the essentially algebraic framework will be called below by the name D-systems.

1 Two essentially algebraic presentations of natural models.

There are two different, but equivalent, essentially-algebraic presentation natural model defined with "represented" instead of "representable" which I am going to discuss. I will argue that one of these presentations is simpler than the other two. This is especially important for the considerations concerning quotients, and therefore concerning the "judgmental" equality in type systems.

As a guiding principle to the discussion I will use the concept of rank of an operation in an essentially algebraic theory. An operation is said to be of rank 0 if it is everywhere defined. An operation is said to be of rank $n+1$ if its domain of definition involves equations between operations of rank $\leq n$. Note that it is possible to replace a theory with an equivalent one in which all operations will be of rank no more than 1 but such a modification increases the number of sorts. Therefore, if the number of sorts is the highest ranking measure of complexity of a theory with the numbers of operations of various ranks being measures of complexity of lower rankings then we obtain a partial (pre-)ordering on essentially algebraic theories. Algebraic theories with one sort are the simplest ones in this pre-ordering. Then algebraic theories with several sorts. Then essentially algebraic theories with one sort and operations of no more than the first rank etc.

Let us consider now the natural models.

One starts with the standard essentially-algebraic presentation of a category. This presentation has two sorts $C1$ and $C0$, three operations of rank zero $dom, codom : C1 \to C0$ and $id : C0 \to C1$, one operation of rank one $\circ : (f \in C1, g \in C1, codom(f) = dom(g)) \to C1$ and seven equations:

$$dom(id(X)) = X \quad codom(id(X)) = X$$

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$$dom(circ(f,g)) = dom(f) \quad codom(circ(f,g)) = codom(g)$$

$$circ(id(X),f) = f \quad circ(f,id(Y)) = f$$

$$circ(f,circ(g,h)) = circ((circ(f,g),h))$$

Note that we use the notation $circ(f,g)$ for what in many texts is written as $g \circ f$.

We will be interested in the free models of our essentially algebraic theories. A free model of a theory with the set of operation Op and the set of relations Rel is the quotient of the free model of the theory with he set of operations Op and no relations.

If all operations are of rank ≤ 0, i.e. if our theory is an algebraic theory then the canonical homomorphism from the free model of Op to the free model of (Op, Rel) is surjective on all sorts.

If there are some operations of rank 1 or greater than this is not anymore true. However we may find some general patterns in this case as well.

Let us consider first the case when all operations are of ranks 0 and 1. Operations of rank 1 have domains of definitions defined by systems of equations on operations of rank 0.

A category with a final object is obtained by adding to this presentation two operations of rank zero $pt \in C0$ and $sm : C0 \to C1$ and one operation of rank one $cosm : (f \in C1, codom(f) = pt) \to C1$ and four equations

$$dom(sm(X)) = X \quad codom(sm(X)) = pt$$

$$cosm(f) = f \quad cosm(f) = sm(dom(f))$$

A category with a presheaf of sets can be described by the extension of the theory for categories with a sort A, one operation of rank zero $p_A : A \to C0$, one operation of rank one $\phi_A : (f \in C1, a \in A, codom(f) = p_A(a)) \to A$ and three equations

$$p_A(\phi_A(f,a)) = dom(f)$$

$$\phi_A(id(X),a) = a$$

$$\phi_A(circ(f,g),a) = \phi_A(f,\phi_A(g,a))$$

all of rank one. The interpretation is that for $X \in C0$ the presheaf takes X to $(a \in A, p_A(a) = X)$.

A category with two presheaves $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ and a natural transformation $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ is given in addition to the sorts $C0, C1, A, B$ (A for $\mathcal{U}$ and B for $\widetilde{\mathcal{U}}$) and operations p_A, ϕ_A and p_B, ϕ_B as above by an additional operation of rank zero $\partial : B \to A$ (corresponding to π) and two equations:

$$p_A(\partial(b)) = p_B(b)$$

$$\partial(\phi_B(f,b)) = \phi_A(f,\partial(b))$$

The presentability structure on π can be formulated in two different ways.

In both approaches one first specifies for $a \in A$, an element $\partial^*(a) \in B$ and $p(a) \in C1$ such that the square

$$\begin{array}{ccc} dom(p(a)) & \xrightarrow{\partial^*(a)} & \widetilde{\mathcal{U}} \\ {\scriptstyle p(a)}\downarrow & & \downarrow{\scriptstyle \partial} \\ p_A(a) & \xrightarrow{a} & \mathcal{U} \end{array}$$

is a commutative square.

This gives us two operations $\partial^* : A \to B$ and $p : A \to C1$ of rank zero and equations

$$dom(p(a)) = p_B(\partial^*(a)) \quad codom(p(a)) = p_A(a)$$

$$[\mathbf{2014.10.06.eq3}]\phi_A(p(a), a) = \partial(\partial^*(a)) \tag{1}$$

The most straightforward way to say that the square that we obtained is a pull-back square is by specifying a new operation of *rank two*

$$carsq : (a \in A, f \in C1, b \in B, codom(f) = p_A(a) \quad \partial(b) = \phi_A(f, a)) \to C1$$

and equations

$$dom(carsq(a, f, b)) = dom(f) \quad codom(carsq(a, f, b)) = p_B(\partial^*(a))$$

$$circ(carsq(a, f, b), p(a)) = f \quad \phi_B(carsq(a, f, b), \partial^*(a)) = b$$

$$carsq(circ(u, p(a)), \phi_B(u, \partial^*(a))) = u$$

The generalized algebraic presentation of categories with families by Dybjer in [3] and the presentation of the MLTT in [2, Figure 1] are based on this approach. In [3], $carsq(\_, \gamma, a)$ is denoted by $\langle \gamma, a \rangle$ and in [2], $carsq(\_, \sigma, u)$ is denoted by (σ, u).

However as is suggested in [1] and as it is also suggested by the approach to C-systems from [4, Def. 2.2] there is a different way of presenting the representability structure[4].

Instead of the operation $carsq$ of rank 2 one introduces two operations, one of rank 0 and one of rank 1:

$$s : B \to C1 \quad (rank\, 0)$$

$$q : (f \in C1, a \in A, codom(f) = p_A(a)) \to C1 \quad (rank\, 1)$$

and the following equations

$$dom(s(b)) = p_B(b) \quad codom(s(b)) = p_B(\partial^*(\partial(b)))$$

$$[\mathbf{2014.10.04.eq2}]\phi_B(s(b), \partial^*(\partial(b))) = b \tag{2}$$

$$[\mathbf{2014.10.06.eq1}]circ(f, s(b)) = circ(s(\phi_B(f, b)), q(f, \partial(b))) \tag{3}$$

$$dom(q(f, a)) = p_B(\partial^*(\phi_A(f, a))) \quad codom(q(f, a)) = p_B(\partial^*(a))$$

$$[\mathbf{2014.10.04.eq3}]circ(q(f, a), p(a)) = circ(p(\phi_A(f, a)), f) \tag{4}$$

$$q(id(p_A(a)), a) = id(p_B(\partial^*(a)))$$

$$q(circ(g, f), a) = circ(q(g, \phi_A(f, a)), q(f, a))$$

and

$$[\mathbf{2014.10.04.eq1}]circ(s(b), p(\partial(b))) = id(p_B(b)) \tag{5}$$

[4]Note that univalently speaking we are working here with strict pre-categories. For presheaves on pre-categories the representability is a *structure* that becomes a property for presheaves on categories.

$$[\mathbf{2014.10.06.eq2}]circ(s(\partial^*(a)), q(p(a), a)) = id_{p_B(\partial^*(a))} \tag{6}$$

The correspondence between this description and the suggestion of [1] is as follows. As is observed in that paper a natural transformation $\widetilde{\mathcal{U}} \to \mathcal{U}$ is representable if and only if the associated functor F between the Grothendieck categories $\mathcal{B}$, $\mathcal{A}$ whose objects are elements of our B and A has a right adjoint F^*.

The action of F on morphisms is uniquely determined by its action on objects that we have denoted by ∂.

The action of F^* on objects is our ∂^* and its action on morphisms is our q.

The adjunction $FF^* \to 1$ is our p. That p is a morphism in $\mathcal{A}$ is equation (1). The naturality of p is equation (4).

The adjunction $1 \to F^*F$ is our s. That s is a morphism in $\mathcal{B}$ is equation (2). The naturality of s is equation (3).

The adjunction equation $(F \to FF^*F \to F) = 1_F$ is our (5).

The adjunction equation $(F^* \to F^*FF^* \to F^*) = 1_{F^*}$ is our (6).

Definition 1.1 *[**2014.10.14.def1**] A D-system is a model of the essentially algebraic theory with sorts $(C0, C1, A, B)$ and operations $(dom, codom, id, p_A, \phi_A, p_B, \phi_B, \partial, \partial^*, p, q, s)$ and relations that are described above.*

The expression of $carsq$ through s and q is as follows:

$$carsq(a, f, b) = circ(s(b), q(f, a))$$

The expression of s and q through $carsq$ is as follows:

$$s(b) = carsq(\partial(b), id(p_B(b)), b)$$

$$q(f, a) = carsq(a, circ(p(\phi_A(f, a)), f), \partial^*(\phi_A(f, A)))$$

2 Natural models and sets of morphisms

There is a complex relation between natural models over a given (strict, pre-)category $C = (C0, C1, \ldots)$ and subsets in $C1$. The fact that this relation is important is shown by Problem 2.11 below.

Let us say that a *represented* natural transformation of presheaves is a natural transformation $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ together with a representability structure on π that is a function that for any (X, a) where $X \in C0$ and $a \in \mathcal{U}(X)$ gives $(Z(X, a), p(X, a), \partial^*(X, a))$ where $p(X, a) : Z(X, a) \to X$, $\partial^*(X, a) \in \widetilde{\mathcal{U}}(Z(a))$ and such that the square

$$\begin{tikzcd} Z(X,a) \arrow[r, "\partial^*(X,a)"] \arrow[d, "p(X,a)"'] & \widetilde{\mathcal{U}} \arrow[d, "\pi"] \\ X \arrow[r, "a"] & \mathcal{U} \end{tikzcd}$$

is a pull-back square of presheaves.

We will also say that a represented natural transformation is a natural transformation with a representability structure. This is simply a re-saying in a dependent form of the essentially algebraic definitions from the above.

Any natural model over a category C defines a subset in $C1$ that is the image of the operation p. The following lemma is straightforward.

Lemma 2.1 *[**2014.10.08.l4**] The set of morphisms D defined by a natural model is weakly closed under pull-backs, that is, for any $d : Y \to X$ and any $f : X' \to X$ there exists a pull-back square of the form*

$$\text{[2014.01.08.eq1]} \quad \begin{tikzcd} Y' \arrow[r] \arrow[d, "d'"'] & Y \arrow[d, "d"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{7}$$

such that $d' \in D$.

Proof: Straightforward.

Definition 2.2 *[**awodeydef**] A subset $D \subset C1$ is called strongly closed under pull-backs or stable if one has:*

1. *for any $f : X' \to X$ and $d : Y \to X$ where $d \in D$ there exists a pull-back square of the form 7.*
2. *for any pull-back square of the form 7 where $d \in D$ one has $d' \in D$.*

The sets of morphisms defined by natural models usually *are not* stable.

Definition 2.3 *[**2014.10.08.def4**] A represented natural transformation is called stably represented if the set of morphisms defined by it is stable.*

The following examples show that not all representable natural transformations can be stable represented and that the same natural transformation can have both stable and unstable representations.

Example 2.4 [**2014.10.08.ex1**] Let C be the contractible category with two objects: $C0 = \{X_1, X_2\}$ and $C1 = \{f_{11} : X_1 \to X_1, f_{12} : X_1 \to X_2, f_{21} : X_2 \to X_1, f_{22} : X_2 \to X_2\}$. Let $\widetilde{\mathcal{U}} = \mathcal{U} = pt$ be the one point presheaf and $\pi = Id$. This natural transformation has four representability structures and neither of them is stable.

On the other hand if we take $\widetilde{\mathcal{U}} = \mathcal{U} = pt \amalg pt$ and π to be again the identity natural transformation then π will have both stable and unstable representations.

When we try to explore the correspondence between natural transformations and sets of morphisms further the following difficulty arises: a set of morphisms defines a natural transformation, a *represented* natural transformation defines a set of morphisms, but a set of morphisms by itself does not define a represented natural transformation.

Definition 2.5 *[**2014.10.08.def2**] A pull-back structure on $D \subset C1$ is a a function that to any pair $(f \in C1, d \in D, codom(f) = codom(d))$ assigns a pull-back square of the form*

$$\begin{array}{ccc} dom(p(f,d)) & \xrightarrow{q(f,d)} & dom(d) \\ {\scriptstyle p(f,d)} \downarrow & & \downarrow {\scriptstyle d} \\ dom(f) & \xrightarrow{f} & codom(d) \end{array}$$

Note that the morphisms $p(f,d)$ of Definition 2.5 need not be elements of D.

Problem 2.6 *[**2014.10.08.prob1**] For a subset $D \subset C1$ construct a natural transformation of presheaves $\pi_D : \widetilde{\mathcal{U}}_D \to \mathcal{U}_D$.*

Construction 2.7 [**2014.10.08.constr1**]In the essentially algebraic notation we need to define two sets A_D, B_D and operations p_A, ϕ_A, p_B, ϕ_B and $\partial : B \to A$ as above.

One sets:

$$A_D := (f \in C1, d \in D, codom(f) = codom(d))$$
$$p_A(f,d) = dom(f), \phi_A(g,(f,d)) = (circ(g,f), d)$$
$$B_D := (f \in C1, d \in D, codom(f) = dom(d))$$
$$p_B(f,d) = dom(f), \phi_B(g,(f,d)) = (circ(g,f), d)$$
$$\partial(f,d) = (circ(f,d), d)$$

The verification of the equations is straightforward.

Problem 2.8 *[**2014.10.08.prob3**] For a subset $D \subset C1$ with a pull-back structure $p(f,d), q(f,d)$ construct a represented natural transformation.*

Construction 2.9 [**2014.10.08.constr3**]We take as the natural transformation the one corresponding to D by Construction 2.7. For $a : X \to \mathcal{U}_D$ of the form $(f : X \to X', d : X'' \to X')$ we define $\partial^*(X,a) := dom(p(f,d))$ and $p(X,a) := p(f,d)$:

$$\begin{array}{ccc} dom(p(f,d)) & \xrightarrow{(q(f,d),d)} & \widetilde{\mathcal{U}}_D \\ {\scriptstyle p(f,d)} \downarrow & & \downarrow {\scriptstyle \pi_D} \\ X & \xrightarrow{(f,d)} & \mathcal{U} \end{array}$$

to check that this is a pull-back square use Lemma 2.10 below.

Lemma 2.10 *[**2014.10.08.l1**] Let $D \subset C1$ and $(d : X' \to X) \in D$. Then the square of presheaves*

$$\begin{array}{ccc} X' & \xrightarrow{(id(X'),d)} & \widetilde{\mathcal{U}}_D \\ {\scriptstyle d} \downarrow & & {\scriptstyle \pi_D} \downarrow \\ X & \xrightarrow{(id(X),d)} & \mathcal{U}_D \end{array}$$

is a pull-back square.

Proof: One can easily see that for any $f : Y \to X$ the morphisms from Y to X' over X are the same as morphisms from $Y \to X \to \mathcal{U}_D$ to $\widetilde{\mathcal{U}}_D$ over $\mathcal{U}_D$.

At this point it seems that we should be able to construct for a set of morphisms with a pull-back structure a represented natural transformation and for a represented natural transformation a set of morphisms with a pull-back structure. But it is not so.

Given a represented natural transformation $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$, $p(X, a), \partial^*(X, a)$ we can define D to be the set of $p : Y \to X$ such that there exists $a \in \mathcal{U}(X)$ such that $p(X, a) = p$. But we can not construct a pull-back structure on this set because there may exist more than one a such that $p(X, a) = p$.

We can still ask the following: given a set of morphisms D with a pull-back structure p, q, consider the represented natural transformation that corresponds to it by Construction 2.9. Let D' be the set of morphisms defined by this represented natural transformation, as above. What conditions should (D, p, q) satisfy so that $D' = D$?

Since D' is simply the set of morphisms of the form $p(f, d)$ the inclusion $D' \subset D$ is equivalent to the condition that $p(f, d) \in D$. In particular this implies that D is weakly closed under pull-backs.

In order to have $D \subset D'$ it is sufficient to require[5] that $p(id, d) = d$.

In particular, if D is a stable set of morphisms then by choosing for each f, d a $p(f, d)$ (which will be automatically in D) and choosing such that $p(id, d) = d$ we obtain a pull-back structure on D such that the corresponding represented natural transformation again gives D.

In this sense, *non-constructively*, any stable set of morphisms "can be obtained from a represented natural transformation".

Note also that a natural model can not be recovered (up to an isomorphism) from the set of maps that it defines.

Problem 2.11 *[***2014.10.14.prob1***] To construct a pre-category C, a natural transformation of presheaves $\partial : B \to A$ over C and two representability structures pqs and pqs' over (C, ∂) such that the corresponding D-systems $D(C, \partial, pqs)$ and $D(C, \partial, pqs')$ are not isomorphic.*

Construction 2.12 [**2014.10.14.constr1**] Let $C_G = (C0 := pt, C1 := G)$ where G is a group. Let $B = A$ be the one point presheaves (i.e. $A = B = C0$) and $\partial = id$.

Then there are G representability structures on (C_G, id). The group of automorphisms of (C_G, id) is G that acts on the set of representability structures in the obvious way. Therefore, there are $G/Aut(G)$ isomorphism classes of D-systems over (C_G, id). In particular, there are two non-isomorphic D-systems over $(C_{\mathbf{Z}/2}, id)$.

Remark 2.13 The interpretation of Construction 2.12 in terms of adjoint functors between the categories of elements is as follows. Let us write B_0G for the (strict pre-)category with one object and G as the monoid of endomorphisms of this object. Two functors $B_0G \to B_0H$ are isomorphic when they differ by an inner automorphism of H.

There are exactly G adjoints to the identity functor $B_0G \to B_0G$. Any element g defines a pair $(F(g), p(g))$ where $F(g)$ is the functor and $p(g) : F- > Id$ the adjunction morphism. The functor $F(g)$ is given by $x \mapsto gxg^{-1}$ and the morphism is given by g itself.

[5] In the classical formalization any p can be modified to satisfy this condition. Constructively, if we already have a p and want to modify it it would require decidability of equality with identity morphisms.

The set of adjoints considered as a category with respect to the obvious choice of morphisms is contractible, i.e., it has G objects and one morphism between any two objects.

This example also shows that if we consider tuples $(C, C, F : C \to C, G : C \to C, p : FG \to Id)$ as essentially algebraic structures then over a given triple (C, C, F) there can be many non-isomorphic tuples (C, C, F, G, p) where p is an adjunction.

3 A generalized algebraic version of the (s,q)-presentation.

The generalized algebraic presentation of Categories with Families, which are equivalent to natural models and D-systems, in [3] uses an operation analogous to our $carsq$ that is introduced by the rule:

$$\frac{\Delta, \Gamma : Context \quad A : Type(\Gamma) \quad \gamma : \Delta \to \Gamma \quad a : \Delta \vdash A[\gamma]}{\langle \gamma, a \rangle : \Delta \to \Gamma; A}$$

together with four equations. The presentation of Dybjer also uses operations p and q. We will continue writing p for p from [3] since p there is the equivalent of our p and will denote the $q = q(A)$ from [3] by $\partial^*(A)$ since it is the equivalent of our ∂^*. Note also that it is related to δ of B-systems (see [5]). We will also use the Dybjer's notation for composition writing $f \circ g$ for the arrow g followed by the arrow f.

The equivalents of s and q in this generalized algebraic presentation are as follows:

$$\frac{\Gamma : Context \quad A : Type(\Gamma) \quad a : \Gamma \vdash A}{s(A, a) : \Gamma \to \Gamma; A}$$

$$\frac{\Delta, \Gamma : Context \quad f : \Delta \to \Gamma \quad A : Type(\Gamma)}{q(f, A) : (\Delta, A[f]) \to (\Gamma; A)}$$

The equations needed in this presentation are as follows:

$$A[p][s] = A$$

$$\partial^*(A)[s(A, a)] = a$$

$$s(A, a) \circ f = q(f, A) \circ s(A[f], a[f])$$

$$p(A) \circ q(f, A) = f \circ p(A[f])$$

$$q(id(\Gamma), A) = id(\Gamma; A)$$

$$q(f \circ g, A) = q(f, A) \circ q(g, A[f])$$

$$p(A) \circ s(A, a) = id(\Gamma)$$

$$q(p(A), A) \circ s(\partial^*(A)) = id(\Gamma; A)$$

The expression for $\langle \_, \_ \rangle$ through s and q is as follows:

$$\langle \gamma, a \rangle = q(\gamma, A) \circ s(A[\gamma], a)$$

The expression of s and q through $\langle \_, \_ \rangle$ is as follows:

$$s(A, a) = \langle id(\Gamma), a \rangle$$

$$q(f, A) = \langle \gamma \circ p(A[\gamma]), \partial^*(A[\gamma]) \rangle$$

4 The Π-structure on a D-system.

The Π-structure is defined in [3, p.126] in terms of the generalized algebraic description and the operation $\langle \_, \_ \rangle$ which is the equivalent of our *carsq*. However, the equations that are involved in the specification of the Π-structure become simpler when reformulated in terms of the operations s and q.

Definition 4.1 *[***2014.11.03.def1***] The Π-structure on a D-system is given by operations*

$$\Pi : (a \in A, a' \in A, p_A(a') = p_B(\partial^*(a))) \to A \quad (rank\, 1)$$

$$\lambda : (a \in A, b \in B, p_B(b) = p_B(\partial^*(a))) \to B \quad (rank\, 1)$$

$$app : (b \in B, a' \in A, b' \in B, p_A(a') = p_B(\partial^*(\partial(b'))), p_B(b) = \Pi(\partial(b'), a')) \to B \quad (rank\, 2)$$

and equations:

$$p_A(\Pi(a, a')) = p_A(a)$$

$$p_B(\lambda(a, b)) = \Pi(a, \partial(b))$$

$$p_B(app(b, a', b')) = \phi_A(s(b'), a')$$

and

$$\phi_A(f, \Pi(a, a')) = \Pi(\phi_A(f, a), \phi_A(q(f, a), a'))$$

$$\phi_B(f, \lambda(a, b)) = \lambda(\phi_A(f, a), \phi_B(q(f, a), b))$$

$$\phi_B(f, app(b, a', b')) = app(\phi_B(f, b), \phi_A(q(f, \partial(b')), a'), \phi_B(f, b'))$$

and

$$app(\lambda(a, b), a', b') = \phi_B(s(b'), b)$$

$$\lambda(a, app(\phi_B(p(A), b), \phi_A(q(p(a), a), a'), \partial^*(a))) = b$$

The generalized-algebraic presentations of this structure looks as follows:

Definition 4.2 *[***2014.11.03.def2***] A Π-structure on (s, q)-GA-CwF is given by operations (just as in [3, p.126]):*

$$\frac{\Gamma : Context \quad A : Type(\Gamma) \quad B : Type(\Gamma; A)}{\Pi(A, B) : Type(\Gamma)}$$

$$\frac{\Gamma : Context \quad A : Type(\Gamma) \quad B : Type(\Gamma; A) \quad b : \Gamma; A \vdash B}{\lambda(A, b) : \Gamma \vdash \Pi(A, B)}$$

$$\frac{\Gamma : Context \quad A : Type(\Gamma) \quad B : Type(\Gamma; A) \quad c : \Gamma \vdash \Pi(A, B) \quad a : \Gamma \vdash A}{app(c, B, a) : \Gamma \vdash B[s(A, a)]}$$

and equations:

$$\Pi(A, B)[f] = \Pi(A[f], B[q(f, A)])$$

$$\lambda(A, b)[f] = \lambda(A[f], b[q(f, A)])$$

$$app(c, B, a)[f] = app(c[f], B[q(f, A)], a)$$

and

$$app(\lambda(A, b), B, a) = b[s(A, a)]$$

$$\lambda(A, app(c[p(A)], B[q(p(A), A)], \partial^*(A)) = c$$

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