Lawvere theories, relative monads and monads.[1]

Vladimir Voevodsky[2]

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2 Relative monads

The notion of a relative monad is introduced in [1, Def.1, p. 299] and considered in more detail in [2]. In our terminology it would be more natural to call it a relative Kleisli triple (cf. Definition 7.6) but we will keep the original name.

Definition 2.1 *[**2015.12.22.def1**] Let $J : C \to D$ be a functor. A relative monad RR on J is a collection of data of the form*

1. *a function $RR_{Ob} : Ob(C) \to Ob(D)$,*
2. *for each X in C a morphism $\eta(X) : J(X) \to RR_{Ob}(X)$,*
3. *for each X, Y in C and $f : J(X) \to RR_{Ob}(Y)$ a morphism $\rho(f) : RR_{Ob}(X) \to RR_{Ob}(Y)$,*

such that the following conditions hold:

1. *for any $X \in C$, $\rho(\eta(X)) = Id_{RR_{Ob}(X)}$,*
2. *for any $f : J(X) \to RR_{Ob}(Y)$, $\eta(X) \circ \rho(f) = f$,*

[1] *2000 Mathematical Subject Classification: 18D99, 08C99, 03B15 03F50,*

[2] School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu

3. for any $f : J(X) \to RR_{Ob}(Y)$, $g : J(Y) \to RR_{Ob}(Z)$,

$$\rho(f) \circ \rho(g) = \rho(f \circ \rho(g))$$

In what follows we will often write $RR(-)$ instead of $RR_{Ob}(-)$. The following definition repeats [2, Definition 2.2, p.4].

Definition 2.2 *[***2015.12.22.def2***] Let* $J : C \to D$ *be a functor and* $RR = (RR_{Ob}, \eta, \rho)$, $RR' = (RR'_{Ob}, \eta', \rho')$ *be two relative monads on* J*. A morphism* $\phi : RR \to RR'$ *is a function* $\phi : Ob(C) \to Mor(D)$ *that to each* $X \in C$ *assigns a morphism* $\phi(X) : RR_{Ob}(X) \to RR'_{Ob}(X)$ *such that*

1. *for any* $X \in C$ *one has* $\eta'(X) = \eta(X) \circ \phi(X)$,
2. *for any* $f : J(X) \to RR(Y)$ *one has*

$$\rho(f) \circ \phi(Y) = \phi(X) \circ \rho'(f \circ \phi(Y))$$

Lemma 2.3 *[***2015.12.22.l1***] Let* $J : C \to D$ *be a functor and* RR *a relative monad on* J*. Then the function* $X \mapsto Id_{RR(X)}$ *is a morphism of relative monads* $RR \to RR$*.*

Proof: Both conditions of Definition 2.2 are straightforward to prove.

Lemma 2.4 *[***2015.12.22.l2***] Let* $J : C \to D$ *be a functor and* RR, RR', RR'' *be relative monads on* J*. Then if* ϕ *and* ϕ' *are functions* $Ob(C) \to Mor(D)$ *which are morphisms of relative monads* $RR \to RR'$ *and* $RR' \to RR''$ *then the function* $X \mapsto \phi(X) \circ \phi'(X)$ *is a morphism* $RR \to RR''$*.*

Proof: Let $X \in C$ then

$$\eta(X) \circ \phi(X) \circ \phi'(X) = \eta'(X) \circ \phi'(X) = \eta''(X)$$

this proves the first condition of Definition 2.2. To prove the second condition let $f : J(X) \to RR(Y)$ then we have

$$\rho(f) \circ \phi(Y) \circ \phi'(Y) = \phi(X) \circ \rho(f \circ \phi(Y)) \circ \phi'(Y) = \phi(X) \circ \phi'(X) \circ \rho(f \circ \phi(Y) \circ \phi'(Y))$$

Problem 2.5 *[***2015.12.22.prob3***] Let* $J : C \to D$ *be a functor. To construct a category* $RMon(J)$ *of relative monads on* J*.*

Construction 2.6 [2015.12.18.constr3] Applying the same approach as before we obtain category data with the set of objects being the set $RMon(J)$ of relative monads on J, the set of morphisms being the set of triples $((RR, RR'), \phi)$ where RR, RR' are relative monads on J and ϕ is a morphism of relative monads from RR to RR' as given by Definition 2.2, the identity morphisms given by Lemma 2.3 and compositions by Lemma 2.4. It follows immediately from the corresponding properties of morphisms in C that these data satisfies the left and right identity and the associativity axioms forming a category. The set of morphisms from RR to RR' in this category is not equal to the set of morphisms of relative monads but it is in the obvious bijective correspondence with this set and we will use both functions of this bijective correspondence as coercions[3].

[3]When a function $f : X \to Y$ is declared as a *coercion* then every time that one has an expression a that denotes an element of the set X in a position where an element of the set Y is expected it is replaced by $f(a)$

Lemma 2.7 *[**2016.01.03.l5**] Let $\phi : RR \to RR'$ be a morphism of relative monads on $J : C \to D$ such that for all $X \in C$ the morphism $\phi(X) : RR(X) \to RR'(X)$ is an isomorphism. Then ϕ is an isomorphism in the category of relative monads on J.*

Proof: Set $\phi'(X) = (\phi(X))^{-1}$. In view of the definition of the composition of morphisms of relative monads and the identity morphism of relative monads it is sufficient to verify that the family ϕ' is a morphism of relative monads from RR' to RR. That it is the inverse to ϕ is then straightforward to prove.

Let us check the two conditions of Definition 2.2. The equality

$$\eta(X) = \eta'(X) \circ \phi'(X)$$

follows from the equality $\eta'(X) = \eta(X) \circ \phi(X)$ by composing it with $\phi'(X)$ on the right and using the fact that $\phi(X) \circ \phi'(X) = Id_{RR(X)}$.

The second condition is of the form, for any $f' : J(X) \to RR'(Y)$,

$$[\textbf{2016.01.03.eq2}]\rho'(f') \circ \phi'(Y) = \phi'(X) \circ \rho'(f' \circ \phi'(Y)) \quad (1)$$

Applying the second condition of Definition 2.2 for ϕ to $f = f' \circ \phi'(Y)$ and using the equality $\phi'(Y) \circ \phi(Y) = Id_{RR'(Y)}$ we get

$$\rho(f' \circ \phi'(Y)) \circ \phi(Y) = \phi(X) \circ \rho'(f' \circ \phi'(Y) \circ \phi(Y)) = \phi(X) \circ \rho'(f')$$

It remains to compose this equality with $\phi'(Y)$ on the right and $\phi'(X)$ on the left and rewrite the equalities $\phi(Y) \circ \phi'(Y) = Id_{RR(Y)}$ and $\phi'(X) \circ \phi(X) = Id_{RR'(X)}$.

Let us remind the definition of the Kleisli category of a relative monad (see [2, p.8]).

Problem 2.8 *[**2015.12.22.prob1**] Let $J : C \to D$ be a functor and RR be a relative monad on J. To define a category $K(RR)$ that will be called Kleisli category of RR.*

Construction 2.9 [**2015.12.22.constr3**] We set $Ob(K(RR)) = Ob(C)$ and

$$Mor(K(RR)) = \amalg_{X,Y \in Ob(K(RR))} Mor(J(X), RR(Y))$$

We will, as before, identify the set of morphisms in $K(RR)$ from X to Y with $Mor(J(X), RR(Y))$ by means of the obvious bijections.

For $X \in Ob(C)$ we set $Id_{X,K(RR)} = \eta(X)$.

For $f \in Mor(J(X), RR(Y))$, $g \in Mor(J(Y), RR(Z))$ we set $f \circ_{K(RR)} g = f \circ_D \rho(g)$.

Verification of the associativity and the left and right identity axioms of a category are straightforward.

Problem 2.10 *[**2015.12.22.prob2**] Let $J : C \to D$ be a functor and RR be a relative monad on J. To construct a functor $L_{RR} : C \to K(RR)$.*

Construction 2.11 [**2015.12.22.constr4**] We set $L_{Ob} = Id$ and for $f : X \to Y$, $L(f) = J(f) \circ_D \eta(Y)$. Verification of the identity and composition axioms of a functor are straightforward.

The following lemma will be needed below.

Lemma 2.12 *[***2016.01.03.l4b***] Let $u : X \to Y$ in C and $g : J(Y) \to RR(Z)$ in D. Then one has*

$$L_{RR}(u) \circ_{K(RR)} g = J(u) \circ_D g$$

Proof: One has

$$L_{RR}(u) \circ_{K(RR)} g = L_{RR}(u) \circ_D \rho(g) = J(u) \circ_D \eta(Y) \circ_D \rho(g) = J(u) \circ_D g$$

Problem 2.13 *[***2015.12.22.prob4***] Let $J : C \to D$ be a functor and $\phi : RR \to RR'$ a morphism of relative monads on J. To construct a functor $K(\phi) : K(RR) \to K(RR')$ such that $L_{RR} \circ K(\phi) = L_{RR'}$.*

Construction 2.14 [**2015.12.22.constr5**] This construction is not, as far as we can tell, described in [2] and we will do all computations in detail.

We set $K(\phi)_{Ob} = Id$. For $f \in Mor_D(J(X), RR(Y))$ we set

$$K(\phi)(f) = f \circ_D \phi(Y).$$

For the identity axiom of a functor we have

$$K(\phi)(Id_{X,K(RR)}) = K(\phi)(\eta_X) = \eta_X \circ_D \phi(X) = \eta'_X = Id_{X,K(RR')}$$

For the composition axiom, for $f \in Mor_D(J(X), RR(Y))$, $g \in Mor_D(J(Y), RR(Z))$ we have

$$K(\phi)(f \circ_{RR} g) = K(\phi)(f \circ_D \rho(g)) = f \circ_D \rho(g) \circ_D \phi(Z) = f \circ_D \phi(Y) \circ_D \rho'(g \circ_D \phi(Z))$$

and

$$K(\phi)(f) \circ_{RR'} K(\phi)(g) = (f \circ_D \phi(Y)) \circ_{RR'} (g \circ_D \phi(Z)) = f \circ_D \phi(Y) \circ_D \rho'(g \circ_D \phi(Z))$$

The condition $L_{RR} \circ K(\phi) = L_{RR'}$ obviously holds on objects and on morphisms we have for $f \in Mor_C(X, Y)$:

$$(L_{RR} \circ K(\phi))(f) = K(\phi)(L_{RR}(f)) = K(\phi)(J(f) \circ_D \eta(Y)) = J(f) \circ_D \eta(Y) \circ_D \phi(Y) =$$

$$J(f) \circ_D \eta'(Y) = L_{RR'}(f).$$

Construction 2.14 is completed.

Lemma 2.15 *[***2016.01.01.l2***] Let $J : C \to D$ be a functor. Then one has:*

1. *for a relative monad RR on J, $K(Id_{RR}) = Id_{K(RR)}$,*
2. *for morphisms $\phi : RR \to RR'$, $\phi' : RR' \to RR''$ of relative monads on J, $K(\phi \circ \phi') = K(\phi) \circ K(\phi')$.*

Proof: The first assertion follows from the right identity axiom for D.

The second assertion follows from the associativity of composition in D.

3 Binary coproducts and finite ordered coproducts in the constructive setting

In the absence of Axiom of Choice (AC) the structure of finite coproducts on a category can not be obtained from an initial object and the structure of binary coproducts. The same, of course, is true for products - the proof of [7, Prop.1, p. 73] essentially depends on the AC. However, binary coproducts allow one to construct finite *ordered* coproducts as described below.

Definition 3.1 *[***2015.12.20.def1***] A binary coproducts structure on a category C is a function that assigns to any pair of objects X, Y of C an object $X \amalg Y$ and two morphisms*

$$ii_0^{X,Y} : X \to X \amalg Y$$

$$ii_1^{X,Y} : Y \to X \amalg Y$$

such that for any object W of C and any two morphisms $f_X : X \to W$, $f_Y : Y \to W$ there exists a unique morphism $\Sigma(f_X, f_Y) : X \amalg Y \to W$ such that

$$ii_0^{X,Y} \circ \Sigma(f_X, f_Y) = f_X$$

$$ii_1^{X,Y} \circ \Sigma(f_X, f_Y) = f_Y$$

Definition 3.2 *[***2015.12.24.def2***] A finite ordered coproduct structure on a category C is a function that for any $m \geq 0$ and any sequence $X = (X_0, \ldots, X_{m-1})$ of objects of C defines an object $\amalg_{i=0}^{m-1} X_i$ and morphisms $ii_i^X : X_i \to \amalg_{i=0}^{m-1} X_i$ such that for any sequence $f_i : X_i \to Y$, $i = 0, \ldots, m-1$ there exists a unique morphism $\Sigma_{i=0}^{m-1} f_i : \amalg_{i=0}^{m-1} X_i \to Y$ such that*

$$[\mathbf{2015.12.24.eq2}] ii_j^X \circ \Sigma_{i=0}^{m-1} f_i = f_j \qquad (2)$$

Note that for $m = 0$ there is a unique sequence of the form $(X_0, \ldots, X_{m-1})$ - the empty sequence, and the corresponding $\amalg_{i=0}^{m-1} X_i$ is an initial object of C.

Problem 3.3 *[***2015.12.24.prob1***] Given a category C with an initial object 0 and a binary coproducts structure to construct a finite ordered coproducts structure on C.*

Construction 3.4 [**2015.12.24.constr1**] By induction on m.

For $m = 0$ one defines $\amalg X_i$ to be 0. The construction of the morphism Σf_i, in this case for the empty set of morphisms f_i, and its properties follow easily from the definition of an initial object.

For $m = 1$ one defines $\amalg X_i = X_0$, $ii_0^X = Id_{X_0}$ and $\Sigma f_i = f_0$. The verification of the conditions is again straightforward.

For the successor one defines

$$\amalg_{i=0}^{m} X_i = (\amalg_{i=0}^{m-1} X_i) \amalg X_m$$

and

$$\Sigma_{i=0}^{m} f_i = \Sigma(\Sigma_{i=0}^{m-1} f_i, f_m)$$

The morphisms ii_i^X for $i = 0, \ldots, m-1$ are given by

$$ii_i^X = ii_i^{X'} \circ ii_0^{\amalg_{i=0}^{m-1} X_i, X_m}$$

where X' is the sequence $(X_0, \ldots, X_{m-1})$, and

$$ii^X_m = ii_1^{\amalg_{i=0}^{m-1} X_i, X_m}$$

To show that $\Sigma_{i=0}^m f_i$ satisfies the condition of Definition 3.4 we have:

1. for $j < m$

$$ii^X_j \circ \Sigma_{i=0}^m f_i = ii^X_j \circ \Sigma(\Sigma_{i=0}^{m-1} f_i, f_m) = ii^{X'}_j \circ ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} \circ \Sigma(\Sigma_{i=0}^{m-1} f_i, f_m) = ii^{X'}_j \circ \Sigma_{i=0}^{m-1} f_i = f_j$$

   where the third equation is from the definition of a binary coproduct,

2. for $j = m$

$$ii^X_m \circ \Sigma_{i=0}^m f_i = ii_1^{\amalg_{i=0}^{m-1} X_i, X_m} \Sigma(\Sigma_{i=0}^{m-1} f_i, f_m) = f_m$$

To show that $f = \Sigma_{i=0}^m f_i$ is a unique morphism satisfying these conditions let g be another morphism such that

$$ii^X_j \circ g = f_j$$

for all $j = 0, \ldots, m$. Both f and g are morphisms from $(\amalg_{i=0}^{m-1} X_i) \amalg X_m$. By the uniqueness condition of Definition 3.1 it is sufficient to show that

$$ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} \circ f = ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} \circ g$$

and

$$ii_1^{\amalg_{i=0}^{m-1} X_i, X_m} \circ f = ii_1^{\amalg_{i=0}^{m-1} X_i, X_m} \circ g$$

To prove the first equality it is sufficient, by the inductive assumption, to prove that

$$ii^{X'}_j \circ ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} \circ f = ii^{X'}_j \circ ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} \circ g$$

for all $j = 0, \ldots, m-1$. This follows from our assumption since

$$ii^{X'}_j \circ ii_0^{\amalg_{i=0}^{m-1} X_i, X_m} = ii^X_j$$

Similarly, the second equality follows from our assumption because

$$ii_1^{\amalg_{i=0}^{m-1} X_i, X_m} = ii^X_m.$$

This completes Construction 3.4.

Lemma 3.5 *[***2016.01.03.l4***] Let C be a category with an initial object 0 and binary coproducts structure $(\amalg, ii_0, ii_1)$. Let $(\amalg', ii'_i)$ be the finite ordered coproducts structure defined on C by Construction 3.4. Then for $X = (X_0, X_1)$ one has*

$$(\amalg')_{i=0}^1 X_i = X_0 \amalg X_1$$

and

$$(ii')^X_0 = ii_0^{X_0, X_1}$$
$$(ii')^X_1 = ii_1^{X_0, X_1}$$

Proof: The proof is by unfolding Construction 3.4 in the case $m = 2$.

Lemma 3.6 *[**2015.12.24.l5**] Given a category C with the finite ordered coproducts structure $(\amalg_i X_i, ii_i^X)$ let $f_i : X_i \to Y$ where $i = 0, \dots, m-1$ and $g : Y \to Z$. Then one has*

$$[\mathbf{2015.12.24.eq4}](\Sigma_i f_i) \circ g = \Sigma_i (f_i \circ g) \tag{3}$$

Proof: By the uniqueness condition of Definition 3.2 it is sufficient to show that for all $i = 0, \dots, m-1$ the precompositions of both sides of (3) with ii_i^X are equal. We have

$$ii_i^X \circ (\Sigma_i f_i) \circ g = f_i \circ g = ii_i^X \circ (f_i \circ g)$$

Lemma 3.7 *[**2015.12.24.l4**] Let C be a category with a finite ordered coproducts structure and $(X_0, \dots, X_{m-1})$ a sequence of objects of C. Then one has*

$$\Sigma_{i=0}^{m-1} ii_i^X = Id_{\amalg_{i=0}^{m-1} X_i}$$

Proof: It follows from the uniqueness part of Definition 3.2.

Definition 3.8 *[**2016.01.01.def1**] Let $(C, \amalg, ii_0, ii_1)$ and $(C', \amalg', ii'_0, ii'_1)$ be two categories with the binary coproducts structure. A functor $G : C \to C'$ is said to strictly respect the binary coproduct structures if for all $X, Y \in C$ one has:*

$$G(X \amalg Y) = G(X) \amalg' G(Y)$$

and

$$G(ii_0^{X,Y}) = (ii'_0)^{X,Y}$$
$$G(ii_1^{X,Y}) = (ii'_1)^{X,Y}$$

Definition 3.9 *[**2016.01.01.def2**] Let $(C, \amalg, ii_i)$ and $(C', \amalg', ii'_i)$ be two categories with finite ordered coproducts structures. A functor $G : C \to C'$ is said to strictly respect the finite ordered coproducts structures if for all $n \in \mathbf{N}$ and all sequences $X = (X_0, \dots, X_{m-1})$ one has*

$$G(\amalg_{i=0}^m X_i) = (\amalg')_{i=0}^{m-1} G(X_i)$$

and for all $i = 0, \dots, m-1$ one has

$$G(ii_i^X) = (ii')_i^{G(X)}$$

Lemma 3.10 *[**2016.01.01.l3**] Let $(C, \amalg, ii_0, ii_1)$ and $(C', \amalg', ii'_0, ii'_1)$ be two categories with the binary coproducts structure and let 0, $0'$ be initial objects in C and C' respectively. Let $G : C \to C'$ be a functor. Then G strictly respects the finite coproduct structure on C and C' defined by the initial object and the binary coproduct structure by Construction 3.4 if and only if one has:*

1. *$G(0) = 0'$,*

2. *G strictly respects the binary coproduct structure.*

Proof: The "only if" part follows from the fact that the initial objects of C and C' defined by the finite ordered coproducts structure of Construction 3.4 are 0 and $0'$ and Lemma 3.5.

The proof of the "if" part is easy by induction on the length of the sequence $X = (X_0, \ldots, X_m)$ of Definition 3.9.

Remark 3.11 [**2016.01.05.rem1**] It is not true in general that a finite ordered coproducts structure is determined by the corresponding initial object and the binary coproducts structure. In particular, the converse of Lemma 3.10 is false - a functor that strictly respects the initial object and the binary coproducts structure defined by a finite ordered coproducts structure need not strictly respect the finite ordered coproducts structure itself.

Lemma 3.12 *[***2016.01.01.l6***] Let $(C, \amalg, ii_i)$ and $(C', \amalg', ii'_i)$ be two categories with finite ordered coproducts structures and $G : C \to C'$ a functor that strictly respect the finite ordered coproducts structures.*

Let $X = (X_0, \ldots, X_{m-1})$ be a sequence of objects of C and $f_i : X_i \to Y$ a sequence of morphisms. Then one has

$$[\mathbf{2016.01.01.eq2}] G(\Sigma_{i=0}^{m-1} f_i) = \Sigma_{i=0}^{m-1} G(f_i) \tag{4}$$

where the Σ on the left is with respect to $(\amalg, ii_i)$ and Σ on the right is with respect to $(\amalg', ii'_i)$.

Proof: Both the left and the right hand side of (4) are morphisms from $\amalg_{i=0}^{m-1} G(X_i)$ to $G(Y)$ according to the Definition 3.9. The right hand side is the unique morphism with these domain and codomain such that for all $i = 0, \ldots, m-1$ its pre-composition with $(ii')_i^{G(X)}$ equals $G(f_i)$. It remains to show that the same property holds for the right hand side. We have

$$(ii')_i^{G(X)} \circ G(\Sigma_{i=0}^{m-1} f_i) = G(ii_i^X) \circ G(\Sigma_{i=0}^{m-1} f_i) = G(ii_i^X \circ \Sigma_{i=0}^{m-1} f_i) = G(f_i)$$

. The lemma is proved.

4 More on the category F

Following [3] we let F denote the category with the set of objects $\mathbf{N}$ and the set of morphisms from m to n being $Fun(stn(m), stn(n))$, where $stn(m) = \{i \in \mathbf{N} \mid i < m\}$ is our choice for the standard set with m elements (cf. [11]).

For $m, n \in \mathbf{N}$ let $ii_0^{m,n} : stn(m) \to stn(m+n)$ and $ii_1^{m,n} : stn(n) \to stn(m+n)$ be the injections of the initial segment of length m and the concluding segment of length n.

Lemma 4.1 *[***2016.01.03.l2***] One has:*

1. *0 is the initial object of F,*
2. *the function*
$$(m, n) \mapsto (m+n, ii_0^{m,n}, ii_1^{m,n})$$
is a binary coproduct structure on F.

Proof: We have $stn(0) = \emptyset$ and there is a unique function from $\emptyset$ to any other set.

The second assertion can be reduced to the case $n = 1$ by induction on n and then proved by direct reasoning involving the details of the set-theoretic definition of a function.

Definition 4.2 *[**2016.01.03.d1**] The binary coproducts structure on F defined by Lemma 4.1 is called the standard binary coproducts structure.*

The finite ordered coproducts structure on F defined by and Lemma 4.1 and Construction 3.4 is called the standard finite ordered coproducts structure.

Example 4.3 *[**2016.01.03.ex1**]* There are binary coproducts structures on F that are different from the standard binary coproducts structure. For example, the function that is equal to the standard binary coproducts structure on all pairs (m, n) other than $(1, 1)$ such that $1 \amalg 1 = 2$, $ii_0^{1,1}(0) = 1$ and $ii_1^{1,1} = 0$ is a binary coproducts structure on F that is not equal to the standard one.

Remark 4.4 [**2016.01.03.rem1**] It is easy to define the concept of a finite coproducts structure on a category. The only non-trivial choice one has to make is which of the definitions of a finite set to use and it is reasonable to define a finite set as a set for which there exists, in the ordinary logical sense, $m \in \mathbf{N}$ and a bijection from $stn(m)$ to this set.

One can show then that it is impossible to construct a finite coproducts structure on F without using the axiom of choice. Indeed, one would have to define, among other things, for each finite set I and a function $X : I \to \mathbf{N}$ the coproduct $\amalg X = \amalg_{i \in I} X(i) \in \mathbf{N}$ and a family of functions

$$ii_i^X : stn(X(i)) \to stn(\amalg X)$$

for $i \in I$ such that for any n the function

$$Fun(stn(\amalg X), stn(n)) \to \prod_{i \in I} Fun(stn(X(i)), stn(\amalg X))$$

defined by this family is a bijection. The latter condition is easily shown to be equivalent to the condition that

$$stn(\amalg X) = \amalg_{i \in I} Im(ii_i^X)$$

One can easily shown also that $\amalg X = \Sigma_{i \in I} X(i)$ where the sum on the right is the usual commutative sum in $\mathbf{N}$. Consider the case when I is a set with 2 elements and $X(i) = 1$ for all $i \in I$. Then $\amalg X = 2$ and $ii_i^X : stn(1) \to stn(2)$ are functions whose images do not intersect and cover $stn(2)$. Then the function $i \mapsto ii_i^X(0)$ is a bijection from I to $stn(2)$, i.e., we have found a canonical bijection from any finite set with 2 elements to $stn(2)$. This amounts to a particular case of the axiom of choice for the proper class of all sets with 2 elements or, if we consider finite coproducts relative to a universe U, for the set of sets with 2 elements in U.

Lemma 4.5 *[**2016.01.03.l3**] Consider F with the standard finite ordered coproducts structure. Then for any $m \in \mathbf{N}$, $n_0, \dots, n_{m-1} \in \mathbf{N}$ one has:*

1. $\amalg_{i=0}^{m-1} n_i = \Sigma_{i=0}^{m-1} n_i$,

2. *for each* $i = 0, \ldots, m-1$ *and* $j = 0, \ldots, k_i - 1$ *one has*

$$ii_i^{(n_0,\ldots,n_{m-1})}(j) = (\Sigma_{l=0}^{i-1} n_l) + j$$

In particular, $ii_i^{(1,\ldots,1)}(0) = i$.

Proof: By induction on m using Construction 3.4.

5 Lawvere theories

Lawvere theories were introduced in [6]. Let us remind an equivalent but more direct definition here.

Definition 5.1 *[***2015.11.24.def1***] A Lawvere theory structure on a category* T *is a functor* $L : F \to T$ *such that the following conditions hold:*

1. L *is a bijection on the sets of objects,*
2. $L(0)$ *is an initial object of* T*,*
3. *for any* $m, n \in \mathbf{N}$ *the square*

$$\begin{array}{ccc} L(0) & \longrightarrow & L(n) \\ \downarrow & & \downarrow L(ii_1^{m,n}) \\ L(m) & \xrightarrow{L(ii_0^{m,n})} & L(m+n) \end{array}$$

is a push-out square.

A Lawvere theory is a pair (T, L) *where* T *is a category and* L *is a Lawvere theory structure on* T*.*

Lemma 5.2 *[***2015.12.24.l3***] A functor* $L : F \to T$ *is a Lawvere structure on* T *if an only if it is bijective on objects,* $L(0)$ *is an initial object of* T *and the function*

$$(X, Y) \to (L(L^{-1}(X) + L^{-1}(Y)), L(ii_0^{L^{-1}(X),L^{-1}(Y)}), L(ii_1^{L^{-1}(X),L^{-1}(Y)}))$$

is a binary coproducts structure on T*.*

Proof: It follows by unfolding definitions and using the equalities $L(L^{-1}(X)) = X$ and $L^{-1}(L(n)) = n$.

Definition 5.3 *[***2016.01.03.def2***] Let* (T, L) *be a Lawvere theory. The binary coproducts structure on* T *defined in Lemma 5.2 is called the standard binary coproducts structure defined by (the Lawvere theory structure)* L*.*

The finite ordered coproducts structure on T *defined by the initial object* $L(0)$ *and the standard binary coproducts structure on* T *by Construction 3.4 is called the standard finite ordered coproducts structure defined by* L*.*

Everywhere below, unless the opposite is explicitly stated, we consider, for a Lawvere theory (T, L) the category T with the standard binary coproduct and finite ordered coproduct structures.

Lemma 5.4 *[**2015.12.24.l5b**] Let (T, L) be a Lawvere theory. Then L strictly respects the standard finite coproduct structures on F and T, i.e., for any $m \in \mathbf{N}$, $n_0, \ldots, n_{m-1} \in \mathbf{N}$ one has:*

1. $\amalg_{i=0}^{m-1} L(n_i) = L(\Sigma_{i=0}^{m-1} n_i)$,
2. *for any $i = 0, \ldots, m-1$,*

$$L(ii_i^{(n_0,\ldots,n_{m-1})}) = ii_i^{(L(n_0),\ldots,L(n_{m-1}))}$$

Proof: Simple by induction on m using the explicit form of Construction 3.4.

Lemma 5.5 *[**2016.01.05.l1**] Let (T, L) be a Lawvere theory and let $u \in Fun(stn(m), stn(n))$. Then one has*

$$L(u) = \Sigma_{i=0}^{m-1} ii_{u(i)}^{(L(1),\ldots,L(1))}$$

Proof: Both sides of the equality are morphisms from $L(m)$ to $L(n)$ in T. Since by Lemma 5.4(1) $L(m)$ is the finite coproduct of the sequence $(L(1), \ldots, L(1))$ to prove that two morphisms from $L(m)$ are equal it is sufficient to prove that their pre-compositions with $ii_i^{(L(1),\ldots,L(1))}$ are equal for all $i = 0, \ldots, m-1$. We have

$$ii_i^{(L(1),\ldots,L(1))} \circ \Sigma_{i=0}^{m-1} ii_{u(i)}^{(L(1),\ldots,L(1))} = ii_{u(i)}^{(L(1),\ldots,L(1))} = L(ii_{u(i)}^{(1,\ldots,1)})$$

and

$$ii_i^{(L(1),\ldots,L(1))} \circ L(u) = L(ii_i^{(1,\ldots,1)}) \circ L(u) = L(ii_i^{(1,\ldots,1)} \circ u)$$

It remains to show that

$$ii_{u(i)}^{(1,\ldots,1)} = ii_i^{(1,\ldots,1)} \circ u$$

in F. Since both sides are functions from $stn(1)$ it is sufficient to prove that their values on 0 are equal. This follows from Lemma 4.5.

Recall that a morphism of Lawvere theories $G : (T, L) \to (T', L')$ is a functor $G : T \to T'$ such that $L \circ G = L'$.

Lemma 5.6 *[**2015.01.01.l4**] Let $G : (T, L) \to (T', L')$ be a morphism of Lawvere theories. Then G strictly respects the binary coproduct structures of Lemma 5.2.*

Proof: It follows by unfolding definitions and using the equalities $L(L^{-1}(X)) = X$ and $L^{-1}(L(n)) = n$.

Lemma 5.7 *[**2016.01.01.l5**] Let $G : (T, L) \to (T', L')$ be a morphism of Lawvere theories. Then G strictly respects the standard ordered finite coproduct structures on T and T'.*

Proof: It follows directly from Lemmas 3.10 and 5.6 and the equality $G(L(0)) = (L \circ G)(0) = L'(0)$.

6 Lawvere theories and relative monads

Let us start by reminding that for any set U there is a category $Sets(U)$ of the following form. The set of objects of $Sets(U)$ is U. The set of morphisms is

$$Mor(Sets(U)) = \cup_{X,Y\in U} Fun(X,Y)$$

Since a function from X to Y is defined as a triple (X, Y, G) where G is the graph subset of this function the domain and codomain functions are well defined on $Mor(Sets(U))$ such that

$$Mor_{Sets(U)}(X,Y) = Fun(X,Y)$$

and a composition function can be defined that restricts to the composition of functions function on each $Mor_{Sets(U)}(X, Y)$. Finally the identity function $U \to Mor(Sets(U))$ is obvious and the collection of data that one obtains satisfies the axioms of a category. This category is called the category of sets in U and denoted $Sets(U)$.

We will only consider the case when U is a universe.

Following [1] we let $Jf : F \to Sets(U)$ denote the functor that takes n to $stn(n)$ and that is the identity on morphisms between two objects (on the total sets of morphisms the morphism component of this functor is the inclusion of a subset).

The category of relative monads on Jf plays a special role and we denote it by $SLW(U)$ and call its objects strict Lawvere theories in U. By simply unfolding definitions we get the following explicit form for the definition of a strict Lawvere theory.

Lemma 6.1 *[***2016.01.01.l1***] A strict Lawvere theory in U is a collection of data of the form:*

1. *for each $n \in \mathbf{N}$ a set $RR(n)$ in U,*
2. *for each $n \in \mathbf{N}$ a function $stn(n) \to RR(n)$,*
3. *for each $m, n \in \mathbf{N}$ and $f : stn(m) \to RR(n)$, a function $\rho(f) : RR(m) \to RR(n)$,*

such that the following conditions hold:

1. *for all $n \in \mathbf{N}$, $\rho(\eta(n)) = Id_{RR(n)}$,*
2. *for all $f : stn(m) \to RR(n)$, $\eta(m) \circ \rho(f) = f$,*
3. *for all $f : stn(k) \to RR(m)$, $g : stn(m) \to RR(n)$, $\rho(f) \circ \rho(g) = \rho(f \circ \rho(g))$.*

The main goal of this section is to provide a construction for the following problem.

Problem 6.2 *[***2016.01.05.prob1***] For a universe U to construct an equivalence between the category $LW(U)$ of Lawvere theories in U and the category $SLW(U)$ of strict Lawvere theories in U.*

The construction will be given in Construction 6.16 below.

Lemma 6.3 *[***2015.12.22.l3***] Let RR be a relative monad on $Jf : F \to Sets(U)$. Then $(K(RR), L_{RR})$ is a Lawvere theory.*

Proof: We need to prove that the pair $(K(RR), L_{RR})$ satisfies conditions of Definition 5.1. The first condition is obvious. The second condition is also obvious since $Fun(stn(0), RR(n))$ is a one point set for any set $RR(n)$. The third condition is straightforward to prove as well since the square

$$\begin{array}{ccc} Fun(stn(m+n), RR(k)) & \xrightarrow{ii_1^{m,n}\circ\_} & Fun(stn(n), RR(k)) \\ {\scriptstyle ii_0^{m,n}\circ\_}\downarrow & & \downarrow \\ Fun(stn(m), RR(k)) & \longrightarrow & Fun(stn(0), RR(k)) \end{array}$$

is a pull-back square for any set $RR(k)$.

Problem 6.4 *[**2016.01.01.prob1**] To construct a functor $RML_U : SLW(U) \to LW(U)$.*

Construction 6.5 [**2015.12.22.def5**] We define the object component of RML setting

$$RML_{Ob}(RR) = (K(RR), L_{RR})$$

It is well defined by Lemma 6.3.

We define the morphism component of RLM setting $RML_{Mor}(\phi) = K(\phi)$. It is well defined by the condition of Problem 2.13.

The identity and composition axioms of a functor follow from Lemma 2.15.

Below we consider, for a Lawvere theory (T, L), the category T with the finite ordered coproducts structure obtained by applying Lemma 5.2 and Construction 3.4.

Problem 6.6 *[**2015.12.22.prob5**] Let U be a universe and (T, L) a Lawvere theory in U. To construct a strict Lawvere theory (RR, η, ρ) in U.*

Construction 6.7 [**2015.12.22.constr6**] We set:

1. $RR(n) = Mor_T(L(1), L(n))$,
2. $\eta(n)$ is the function $stn(n) \to Mor_T(L(1), L(n))$ given by $i \mapsto ii_i^X$ where $X = (L(1), \ldots, L(1))$. This function is well defined because

   $$\amalg_{i=0}^{n-1} L(1) = L(n)$$

   by Lemma 5.4,
3. for $f \in Fun(stn(m), Mor_T(L(1), L(n)))$ we define

   $$\rho(f) \in Fun(Mor_T(L(1), L(m)), Mor_T(L(1), L(n)))$$

   as $g \mapsto g \circ \Sigma_{i=0}^{m-1} f(i)$. This formula is again well-defined in view of Lemma 5.4.

Let us verify the conditions of Lemma 6.1.

For the first condition we have

$$\rho(\eta(n))(g) = g \circ \Sigma_{i=0}^{n-1} \eta(n)(i) = g \circ \Sigma_{i=0}^{n-1} ii_i^{(L(1),\ldots,L(1))} = g \circ Id_{L(n)} = g$$

where the third equality is by Lemma 3.7.

For the second condition let $f \in Fun(stn(m), Mor_T(L(1), L(n)))$. To verify that $\eta(m) \circ \rho(f) = f$ we need to verify that these two functions from $stn(m)$ are equal, i.e., that for each $i = 0, \ldots, m-1$ we have

$$(\eta(m) \circ \rho(f))(i) = f(i)$$

We have

$$(\eta(m) \circ \rho(f))(i) = \rho(f)(\eta(m)(i)) = \rho(f)(ii_i^{(L(1),\ldots,L(1))}) = ii_i^{(L(1),\ldots,L(1))} \circ \Sigma_{j=0}^{m-1} f(j) = f(i)$$

For the third condition let $f \in Fun(stn(k), Mor_T(L(1), L(m)))$ and $g \in Fun(stn(m), Mor_T(L(1), L(n)))$. We need to check that

$$\rho(f) \circ \rho(g) = \rho(f \circ \rho(g))$$

Both sides are functions from $Mor_T(L(1), L(k))$. To verify that they are equal we need to show that for any $h \in Mor_T(L(1), L(k))$ we have

$$(\rho(f) \circ \rho(g))(h) = \rho(f \circ \rho(g))(h)$$

We have

$$(\rho(f) \circ \rho(g))(h) = \rho(g)(\rho(f)(h)) = \rho(g)(h \circ \Sigma_{i=0}^{k-1} f(i)) = h \circ (\Sigma_{i=0}^{k-1} f(i)) \circ (\Sigma_{j=0}^{m-1} g(j))$$

and

$$\rho(f \circ \rho(g))(h) = h \circ (\Sigma_{i=0}^{k-1} (f \circ \rho(g))(i)) = h \circ (\Sigma_{i=0}^{k-1} (\rho(g)(f(i)))) = h \circ (\Sigma_{i=0}^{k-1} (f(i) \circ \Sigma_{j=0}^{m-1} g(j)))$$

The right hand sides of these two expressions are equal by Lemma 3.6. This completes the construction.

We let $LRM(T, L)$ denote the strict Lawvere theory defined in Construction 6.7.

Problem 6.8 *[***2016.01.01.prob2***] Let $G : (T, L) \to (T', L')$ be a morphism of Lawvere theories. To construct a morphism of relative monads $LRM(T, L) \to LRM(T', L')$.*

Construction 6.9 [**2016.01.01.constr2**] We need to construct a family of functions

$$\phi(n) : Mor_T(L(1), L(n)) \to Mor_{T'}(L'(1), L'(n))$$

that satisfies the conditions of Definition 2.2 for $J = Jf$ and relative monads $LRM(T, L) = (RR_{Ob}, \eta, \rho)$ and $LRM(T', L') = (RR'_{Ob}, \eta', \rho')$. Set

$$\phi(n) = G_{L(1),L(n)}$$

since $L' = L \circ G$ these functions have the correct domain and codomain.

For the first condition of Definition 2.2 we need to show that for any $n \in \mathbf{N}$ one has

$$\eta'(n) = \eta(n) \circ G_{L(1),L(n)}$$

Since both sides are functions from $stn(n)$ it is sufficient to show that for all $i = 0, \ldots, n-1$ one has $\eta'(n)(i) = (\eta(n) \circ G_{L(1),L(n)})(i)$. By construction

$$(\eta(n) \circ G_{L(1),L(n)})(i) = G(\eta(n)(I)) = G(ii_i^X)$$

and

$$\eta'(n)(i) = ii_i^{X'}$$

where $X = (L(1), \ldots, L(1))$ and $X' = (L'(1), \ldots, L'(1))$. Therefore we need to show that $G(ii_i^X) = ii_i^{X'}$. This follows from Lemma 5.7.

For the second condition of Definition 2.2 let $f : stn(m) \to Mor_T(L(1), L(n))$. We need to show that

$$\rho(f) \circ \phi(n) = \phi(m) \circ \rho(f \circ \phi(n))$$

Both sides are functions from $Mor_T(L(1), L(m))$ to $Mor_{T'}(L'(1), L'(n))$. To show that they are equal we have to show that for each $g \in Mor_T(L(1), L(m))$ one has

$$(\rho(f) \circ \phi(n))(g) = (\phi(m) \circ \rho'(f \circ \phi(n)))(g)$$

For the left hand side of this equality we have:

$$(\rho(f) \circ \phi(n))(g) = \phi(n)(\rho(f)(g)) = \phi(n)(g \circ \Sigma_{i=0}^{m-1} f(i)) = G(g \circ \Sigma_{i=0}^{m-1} f(i)) = G(g) \circ G(\Sigma_{i=0}^{m-1} f(i)) =$$

$$G(g) \circ \Sigma_{i=0}^{m-1} G(f(i))$$

where the last equality follows from Lemma 3.12.

For the right hand side we have:

$$(\phi(m) \circ \rho'(f \circ \phi(n)))(g) = \rho'(f \circ \phi(n))(\phi(m)(g)) = \rho'(f \circ \phi(n))(G(g)) = G(g) \circ \Sigma_{i=0}^{m-1} (f \circ \phi(n))(i) =$$

$$G(g) \circ \Sigma_{i=0}^{m-1} (\phi(n)(f(i))) = G(g) \circ \Sigma_{i=0}^{m-1} G(f(i))$$

This completes the proof of the second condition of Definition 2.2 and the construction.

We let $LRM(\phi)$ or $LRM_{Mor}(\phi)$ denote the morphism of relative monads defined by Construction 6.9

Problem 6.10 *[***2016.01.01.prob3***] For a universe U, to construct a functor*

$$LRM_U : LW(U) \to SLW(U)$$

Construction 6.11 [**2016.01.01.constr5**] We define the object component of LRM as the function defined by Construction 6.7 and the morphism component as the function defined by Construction 6.9.

We need to verify that these two functions satisfy the identity and composition axioms of a functor.

Both follow immediately from the definitions of the identity functor and composition of functors.

Problem 6.12 *[***2016.01.01.prob4***] For any universe U to construct an isomorphism of functors*

$$RML_U \circ LRM_U \to Id_{SLW(U)}.$$

Construction 6.13 [**2016.01.01.constr6**] Let $RR = (RR, \eta, \rho)$ be a strict Lawvere theory in U, i.e., a relative monad on the functor $Jf : F \to Sets(U)$. Let

$$(T, L) = RML_U(RR, \eta, \rho)$$

and

$$(RR', \eta', \rho') = LRM_U(T, L).$$

We need to construct an isomorphism of relative monads

$$\phi_{RR} : (RR', \eta', \rho') \to (RR, \eta, \rho)$$

and show that the family ϕ_{RR} satisfies the naturality axiom of the definition of functor morphism.

We have

$$RR'(n) = Mor_T(L(1), L(n)) = Mor_{K(RR)}(L_{RR}(1), L_{RR}(n)) = Mor_{K(RR)}(1, n) =$$

$$Fun(stn(1), RR(n))$$

and we define $\phi_{RR}(n) : RR'(n) \to RR(n)$ as the obvious bijection given by setting

$$\phi_{RR}(n)(f) = f(0)$$

Let us show that these functions form a morphism of relative monads, i.e., that they satisfy two conditions of Definition 2.2. We should exchange places between the η and η' since we consider a morphism $RR' \to RR$. The first condition becomes

$$\eta(n)(i) = (\eta'(n) \circ \phi_{RR}(n))(i)$$

for any $n \in \mathbf{N}$ and $i = 0, \ldots, n-1$ and the second

$$(\rho'(f) \circ \phi_{RR}(n))(g) = (\phi_{RR}(m) \circ \rho(f \circ \phi_{RR}(n)))(g)$$

for any $f \in Fun(stn(m), RR'(n))$ and $g \in RR'(m)$.

For $n \in \mathbf{N}$ and $i = 0, \ldots, n-1$ we have

$$(\eta'(n) \circ \phi_{RR}(n))(i) = \phi_R R(n)(\eta'(n)(i)) = \phi_{RR}(ii_i^{(L(1),\ldots,L(1))}) = ii_i^{(L(1),\ldots,L(1))}(0) =$$

$$L(ii_i^{(1,\ldots,1)})(0) = L_{RR}(ii_i^{(1,\ldots,1)})(0) = (ii_i^{(1,\ldots,1)} \circ \eta(n))(0) = \eta(n)(ii_i^{(1,\ldots,1)}(0)) = \eta(n)(i)$$

where the fourth equality is by Lemma 5.4 and the eighth equality is by Lemma 4.5.

For the second condition, $f \in Fun(stn(m), RR'(n))$ and $g \in RR'(m)$ we have

$$(\rho'(f) \circ \phi_{RR}(n))(g) = \phi_{RR}(n)(\rho'(f)(g)) = \phi_{RR}(n)(g \circ_T \Sigma_{i=0}^{m-1} f(i)) = (g \circ_T \Sigma_{i=0}^{m-1} f(i))(0)$$

where f is considered as an element of $Fun(stn(m), Mor_T(L(1), L(n)))$ and g as an element of $Mor_T(L(1), L(m))$. Next we have:

$$(g \circ_T \Sigma_{T,i=0}^{m-1} f(i))(0) = (g \circ_{K(RR)} \Sigma_{T,i=0}^{m-1} f(i))(0) = (g \circ \rho(\Sigma_{T,i=0}^{m-1} f(i)))(0) = \rho(\Sigma_{T,i=0}^{m-1} f(i))(g(0))$$

where on the right g is considered as an element of $Fun(stn(1), RR(m))$.

On the other hand we have:

$$(\phi_{RR}(m)\circ\rho(f\circ\phi_{RR}(n)))(g)=\rho(f\circ\phi_{RR}(n))(\phi_{RR}(m)(g))=\rho(f\circ\phi_{RR}(n))(g(0))$$

where on the right g is considered as an element of $Fun(stn(1), RR(m))$.

Let us show that

$$\Sigma_{T,i=0}^{m-1}f(i)=f\circ\phi_{RR}(n),$$

Since both sides are morphisms in T from $L(m)$ to $L(n)$ and it is sufficient to show that for any $j=0,\ldots,m$ one has

$$ii_j^{(L(1),\ldots,L(1))}\circ_T(\Sigma_{T,i=0}^{m-1}f(i))=ii_j^{(L(1),\ldots,L(1))}\circ_T(f\circ\phi_{RR}(n))$$

The left hand side equals $f(j)$. For the right hand side we have

$$ii_j^{(L(1),\ldots,L(1))}\circ_T(f\circ\phi_{RR}(n))=L(ii_j^{(1,\ldots,1)})\circ_T(f\circ\phi_{RR}(n))=L(ii_j^{(1,\ldots,1)})\circ_{K(RR)}(f\circ\phi_{RR}(n))=$$

$$ii_j^{(1,\ldots,1)}\circ f\circ\phi_{RR}(n)$$

where the first equality is by Lemma 5.4 and the third equality is by Lemma 2.12. Both $f(j)$ and $ii_j^{(1,\ldots,1)}\circ f\circ\phi_{RR}(n)$ are elements of $Fun(stn(1), RR(n))$. To prove that they are equal it is sufficient to prove that they coincide on 0. We have:

$$(ii_j^{(1,\ldots,1)}\circ f\circ\phi_{RR}(n))(0)=(f\circ\phi_{RR}(n))(i)=\phi_{RR}(n)(f(i))=f(i)(0)$$

where the first equality is by Lemma 4.5(2).

This completes the proof of the fact that the family of functions ϕ_{RR} is a morphism of relative monads.

Let us show that the family ϕ_{RR} satisfies the naturality axiom of the definition of functor morphism. Let $u: RR_1\to RR_2$ be a morphism of relative monads. Let $(T_i, L_i)=RML(RR_i)$ and $RR'_i=LRM(T_i,L_i)$, $i=1,2$. Let $G=RML(u)$ and $u'=LRM(G)$. We need to show that the square

$$\begin{array}{ccc} RR'_1 & \xrightarrow{u'} & RR'_2\\ \downarrow{\scriptstyle \phi_{RR_1}} & & \downarrow{\scriptstyle \phi_{RR_2}}\\ RR_1 & \xrightarrow{u} & RR_2\end{array}$$

commutes, i.e., that for any $n\in\mathbf{N}$ one has

$$[\mathbf{2016.01.03.eq1}]u'(n)\circ\phi_{RR_2}(n)=\phi_{RR_1}(n)\circ u(n) \quad (5)$$

We have that

$$u'(n)\in Fun(RR'_1(n), RR'_2(n))=Fun(Fun(stn(1), RR_1(n)), Fun(stn(1), RR_2(n)))$$

and

$$u'(n)(f)=(LRM(G)(n))(f)=G_{L_1(1),L_1(n)}(f)=G_{1,n}(f)=f\circ u(n)$$

Both sides of (5) are functions from $Fun(stn(1), RR_1(n))$. Therefore to prove that they are equal we need to prove that their values on any $f\in Fun(stn(1), RR_1(n))$ are equal. We have:

$$(u'(n)\circ\phi_{RR_2}(n))(f)=\phi_{RR_2}(n)(u'(n)(f))=(u'(n)(f))(0)=(f\circ u(n))(0)=u(n)(f(0))$$

and

$$(\phi_{RR_1}(n) \circ u(n))(f) = u(n)(\phi_{RR_1}(n)(f)) = u(n)(f(0)).$$

This completes the proof of the fact that the family ϕ_{RR} is a morphism of functors $RML_U \circ LRM_U \to Id_{SLW(U)}$. That it is an isomorphism follows from the general properties of functor morphisms and Lemma 2.7. This completes Construction 6.12.

Problem 6.14 *[2016.01.03.prob1] For a universe U to construct a functor isomorphism*

$$LRM_U \circ RML_U \to Id_{LW(U)}$$

Construction 6.15 *[2016.01.03.constr1]* Let (T, L) be a Lawvere theory in U. Let

$$(RR, \eta, \rho) = LRM(T, L)$$

and

$$(T', L') = RML(RR, \eta, \rho)$$

We need to construct an isomorphism of Lawvere theories

$$G^{(T,L)} : (T', L') \to (T, L)$$

and show that the family $G^{(T,L)}$ is natural with respect to the morphisms of Lawvere theories $(T_1, L_1) \to (T_2, L_2)$. While constructing $G^{(T,L)}$ we will abbreviate its notation to G.

We have:

$$Ob(T') = Ob(K(RR)) = Ob(F) = \mathbf{N}$$

$$Mor_{T'}(m, n) = Mor_{K(RR)}(m, n) = Fun(stn(m), RR(n)) = Fun(stn(m), Mor_T(L(1), L(n)))$$

We set the object component of G to be the object component of L.

We set the morphism component

$$G_{m,n} : Mor_{T'}(m, n) = Fun(stn(m), Mor_T(L(1), L(n))) \to Mor_T(L(m), L(n)) = Mor_T(m, n)$$

to be of the form:

$$G_{m,n}(f) = \Sigma^{m-1}_{T,i=0} f(i)$$

To show that $G_{m,n}$ is a bijection consider the function in the opposite direction given by, for $u \in Mor_T(m, n)$ and $i = 0, \dots, m-1$

$$G^*_{m,n}(u)(i) = ii_i^{(L(1),\dots,L(1))} \circ u$$

The fact that G and G^* are mutually inverse follows easily from the definition of finite ordered coproducts.

Let us show that G is a functor. For the composition axiom, let $f \in Mor_{T'}(k, m)$, $g \in Mor_{T'}(m, n)$, then

$$G_{k,m}(f) \circ_T G_{m,n}(g) = (\Sigma^{k-1}_{T,i=0} f(i)) \circ_T (\Sigma^{m-1}_{T,j=0} g(j)) = \Sigma^{k-1}_{T,i=0}(f(i) \circ_T (\Sigma^{m-1}_{T,j=0} g(j)))$$

and

$$G_{k,n}(f \circ_{T'} g) = \Sigma^{k-1}_{T,i=0}((f \circ \rho(g))(i)) = \Sigma^{k-1}_{T,i=0}(\rho(g)(f(i))) = \Sigma_{T,i=0}(f(i) \circ_T (\Sigma^{m-1}_{j=0} g(j)))$$

where the last equality is by Construction 6.7(3).

For the identity axiom, let $n \in \mathbf{N}$ then

$$G_{n,n}(Id_{T',m}) = G_{n,n}(\eta(m)) = \Sigma^{m-1}_{T,i=0}(\eta(m)(i)) = \Sigma^{m-1}_{T,i=0}(ii_i^{(L(1),\dots,L(1))}) = Id_{T,L(m)}$$

where the first equality is by Construction 2.9, the third one is by Construction 6.7(2) and the third one is by Lemma 3.7.

To prove that G is a morphism of Lawvere theories we have to show that $L' \circ G = L$. On objects the equality is obvious. To show that it holds on morphisms let $u \in Fun(stn(m), stn(n))$. Then

$$(L' \circ G)(u) = G(L'(u)) = \Sigma^{m}_{T,i=0}L'(u)(i) = \Sigma^{m}_{T,i=0}L_{RR}(u)(i) = \Sigma^{m}_{T,i=0}(u \circ \eta(n))(i) =$$

$$\Sigma^{m}_{T,i=0}\eta(n)(u(i)) = \Sigma^{m}_{T,i=0}ii_{u(i)}^{(L(1),\dots,L(1))} = L(u)$$

where the fourth equality is by Construction 2.11 and the sixth one is by Construction 6.7(2) and the seventh one is by Lemma 5.5.

This completes the construction of the Lawvere theory morphisms $G^{(T,L)}$.

It remains to show that they are natural with respect to morphisms of Lawvere theories. Let $H : T_1 \to T_2$ be such a morphism. Let $(RR_i, \eta_i, \rho_i) = LRM(T_i, L_i)$ for $i = 1, 2$, $(T'_i, L'_i) = RML(RR_i, \eta_i, \rho_i)$, $\phi = LRM(H)$ and $H' = RML(\phi)$.

Since $(L'_i)_{Ob} = Id_{\mathbf{N}}$ and $L'_1 \circ H' = L'_2$ we have that $(H')_{Ob} = Id_{\mathbf{N}}$.

For $m, n \in \mathbf{N}$ and

$$f \in Mor_{T'_1}(m, n) = Fun(stn(m), Mor_{T_1}(L_1(1), L_1(n)))$$

we have

$$H'(f) = RML(\phi)(f) = K(\phi)(f) = f \circ \phi(n) = f \circ LRM(H)(n) = f \circ H_{L_1(1),L_1(n)}$$

where the third equality is by Construction 2.14 and the fifth equality is by Construction 6.9.

We need to show that the square

$$\begin{array}{ccc} T'_1 & \xrightarrow{H'} & T'_2 \\ {\scriptstyle G^{(T_1,L_1)}}\downarrow & & \downarrow{\scriptstyle G^{(T_2,L_2)}} \\ T_1 & \xrightarrow{H} & T_2 \end{array}$$

commutes.

For the object components, since $(G^{(T_i,L_i)})_{Ob} = (L_i)_{Ob}$ it means that for all $n \in \mathbf{N}$ one has

$$L_2(H'(n)) = H(L_1(n)),$$

i.e., that $L_2(n) = H(L_1(n))$ which follows from the fact that H is a morphism of Lawvere theories.

For the morphism component it means that for all $m, n \in \mathbf{N}$ and $f \in Fun(stn(m), Mor_{T_1}(L_1(1), L_1(n)))$ one has

$$G^{(T_2,L_2)}(H'(f)) = H(G^{(T_1,L_1)}(f)),$$

For the left hand side we have:

$$G^{(T_2,L_2)}(H'(f)) = G^{(T_2,L_2)}(f \circ H_{L_1(1),L_1(n)}) = \Sigma^{m-1}_{T_2,i=0}(f \circ H_{L_1(1),L_1(n)})(i) = \Sigma^{m-1}_{T_2,i=0}(H(f(i)))$$

For the right hand side we have:

$$H(G^{(T_1,L_1)}(f)) = H(\Sigma_{T_1,i=0}^{m-1} f(i)) = \Sigma_{T_2,i=0}^{m-1}(H(f(i)))$$

where the second equality is by Lemmas 5.7 and 3.12.

This completes the proof that the constructed family of Lawvere theories morphisms $G^{(T,L)}$ is a morphism of functors and with it completes Construction 6.15.

We can now provide a construction for Problem 6.2.

Construction 6.16 [**2016.01.05.constr1**] A functor RML_U from $SLW(U)$ to $LW(U)$ is provided by Construction 6.5. A functor LMR_U from $LW(U)$ to $SLW(U)$ is provided by Construction 6.11. A functor isomorphism $RML_U \circ LRM_U \to Id_{SLW(U)}$ is provided by Construction 6.13. A functor isomorphism $LRM_U \circ RML_U \to Id_{LW(U)}$ is provided by Construction 6.15.

Remark 6.17 [**2016.01.05.rem2**] The composition $RML_U \circ LRM_U$ is just slightly off from being *equal* to the identity functor on $SLW(U)$. This can be achieved by a modification to the functor LRM by setting the family of sets $LRM(T, L)$ to be given by $Mor_T(L(1), L(n))^m$ where for a set X and $m \in \mathbf{N}$ one defines X^m inductively as $X^0 = stn(1)$, $X^1 = X$ and $X^{n+1} = X^n \times X$. However, this modified version of LRM is not a particular case of a general construction that works for all relative monads as our LRM is.

7 Kleisli triples and monads

Let us remind the standard definition of a monad (see e.g. [7, p.133]).

Definition 7.1 *[**2015.12.18.def1**] Let C be a category. A monad $\mathbf{R}$ on C is a triple (R, η, μ) where:*

1. *$R : C \to C$ is a functor,*
2. *$\eta : Id_C \to R$ is a functor morphism,*
3. *$\mu : R \circ R \to R$ is a functor morphism,*

such that the following conditions hold:

1. *for all $X \in C$ one has $R(\mu_X) \circ \mu_X = \mu_{R(X)} \circ \mu_X$,*
2. *for all $X \in C$ one has $R(\eta_X) \circ \mu_X = Id_{R(X)}$,*
3. *for all $X \in C$ one has $\eta_{R(X)} \circ \mu_X = Id_{R(X)}$.*

Definition 7.2 *[**2015.12.10.def1**] A morphism of monads $\phi : \mathbf{R}_1 \to \mathbf{R}_2$ is a function $X \mapsto \phi_X$ from $Ob(C)$ to $Mor(C)$ such that $\phi_X : R_1(X) \to R_2(X)$ and one has:*

1. *ϕ is a morphism of functors, i.e., for any $f : X \to Y$ one has $R_1(f) \circ \phi_Y = \phi_X \circ R_2(f)$,*

2. $(\phi * \phi) \circ \mu_2 = \mu_1 \circ \phi$ *where* $*$ *is the horizontal composition of functor morphisms,*

3. $\eta_1 \circ \phi = \eta_2$.

Lemma 7.3 *[***2015.12.10.l1***] Let* $\phi_1 : R_1 \to R_2$, $\phi_2 : R_2 \to R_3$ *be two morphisms of monads. Then the functor morphism* $\phi_1 \circ \phi_2$ *is a morphism of monads.*

Proof: For the first condition of Definition 7.2 it follows from the fact that composition of functor morphisms is a functor morphism.

For the second condition we have

$$((\phi_1 \circ \phi_2) * (\phi_1 \circ \phi_2)) \circ \mu_3 = (\phi_1 * \phi_1) \circ (\phi_2 * \phi_2) \circ \mu_3 = (\phi_1 * \phi_1) \circ \mu_2 \circ \phi_2 = \mu_1 \circ \phi_1 \circ \phi_2$$

where the first equality is the so called 2-dimensional associativity of functor morphisms compositions.

For the third condition we have

$$\eta_1 \circ (\phi_1 \circ \phi_2) = \eta_2 \circ \phi_2 = \eta_3$$

Problem 7.4 *[***2015.12.18.prob1***] For a category* C *to construct a category* $Mon(C)$ *of monads on* C.

Construction 7.5 [**2015.12.18.constr1**] It is easy to prove that the identity functor morphism of the functor underlying a monad is a morphism of monads. The associativity of composition and the left and right identity axioms follow in a straightforward way from the corresponding properties of the composition of functor morphisms.

One can now define a category $Mon(C)$ whose set of objects is the set of monads on a category C and morphisms are iterated pairs $((R_1, R_2), \phi)$ where R_1, R_2 are the domain and codomain monads of the morphism and ϕ is a morphism of monads $R_1 \to R_2$. Again we will use the obvious bijections from the set of such objects to the set of morphisms of monads as a coercion in both direction which let us not to mention these bijections explicitly (cf. Construction 2.6).

Relative monads on the identity functor $Id_C : C \to C$ has long been considered in the literature. In [9] they are called Kleisli triples on C and we will use this name below.

Definition 7.6 *[***205.11.14.def1***] A Kleisli triple* $K = (K_{Ob}, \eta, \rho)$ *on a category* C *is a relative monad on the identity functor* $Id_C : C \to C$

It turns out that Kleisli triples are equivalent to monads see, e.g., [4, p.219]. We want to have a precise statement of this equivalence. In what follows we often write $K(-)$ instead of $K_{Ob}(-)$.

Definition 7.7 *[***2015.12.16.def1***] A morphism* $\phi : K \to K'$ *of Klesili triples is a morphism of relative monads.*

Definition 7.8 *[***2015.12.22.def3***] Let* C *be a category. The category* $KT(C)$ *of Kleisli triples on* C *is the category of relative monads on the identity functor of* C.

Problem 7.9 *[**2015.12.16.prob1**] For a category C to construct an isomorphism of categories between $KT(C)$ and $Mon(C)$.*

We will first construct a functor $MK : Mon(C) \to KT(C)$, then a functor KM in the opposite direction and then will prove that they are mutually inverse isomorphisms of categories.

Problem 7.10 *[**2015.11.14.prob1**] To construct a functor $MK : Mon(C) \to KT(C)$.*

Construction 7.11 [**2015.11.14.constr1**]We first construct a function $MK = MK_{Ob}$ from monads on C to Kleisli triples on C. Given a monad $\mathbf{R} = (R, \eta, \mu)$ we define the corresponding Kleisli triple as the triple (R_{Ob}, η, ρ) where

$$\rho(f) = R_{Mor}(f) \circ \mu(Y)$$

and

$$\eta(X) = \eta_X$$

Verification of the equations is simple.

Let $\phi : R \to R'$ be a morphism of monads. We define the corresponding morphism of Kleisli triples as the same function $Ob(C) \to Mor(C)$ but denote it $X \mapsto \phi(X)$ instead of $X \mapsto \phi_X$. Let us verify the equalities of Definition 7.7.

The first condition of this definition is the third condition of Definition 7.2.

For the second condition consider $f : X \to R(Y)$. We need to prove that

$$\rho(f) \circ \phi(Y) = \phi(X) \circ \rho'(f \circ \phi(Y))$$

We have

$$[\mathbf{2015.12.17.eq1}]\rho(f)\circ\phi(Y) = R(f)\circ\mu_Y\circ\phi_Y = R(f)\circ(\phi*\phi)_Y\circ\mu'_Y = R(f)\circ\phi_{R(Y)}\circ R'(\phi_Y)\circ\mu'_Y \quad (6)$$

where the equality

$$(\phi * \phi)_Y = \phi_{R(Y)} \circ R'(\phi_Y)$$

follows from the general properties of functor morphisms. The chain of equalities (6) now continues as follows

$$R(f) \circ \phi_{R(Y)} \circ R'(\phi_Y) \circ \mu'_Y = \phi_X \circ R'(f) \circ R'(\phi_Y) \circ \mu'_Y$$

On the other hand

$$\phi(X) \circ \rho'(f \circ \phi(Y)) = \phi_X \circ R'(f \circ \phi_Y) \circ \mu'_Y = \phi_X \circ R'(f) \circ R'(\phi_Y) \circ \mu'_Y$$

This show that we obtained a function from morphisms of monads to morphisms of Kleisli triples.

The fact that this function takes the identity morphism to the identity morphism and commutes with composition is easy to prove taking into account that it does not change the underlying function R_{Ob}.

Problem 7.12 *[**2015.11.14.prob2**] To construct a functor $KM : KT(C) \to Mon(C)$.*

Construction 7.13 *[2015.11.14.constr2]* We first construct a function on objects that we also write as KM. Let $K = (K_{Ob}, \eta, \rho)$ be a Kleisli triple on C. To define the functor R underlying the monad $KM(K)$ we take $R_{Ob} = K_{Ob}$ and define R_{Mor} by the rule

$$R_{Mor}(f) = \rho(f \circ \eta(Y)).$$

Verification of functor axioms is simple.

To define η of the monad we set $\eta_X = \eta(X)$. We need to prove that it is a morphism of functors, i.e., that for any $f : X \to Y$ one has

$$\eta_X \circ R_{Mor}(f) = f \circ \eta_Y$$

We have:

$$\eta_X \circ R_{Mor}(f) = \eta_X \circ \rho(f \circ \eta(Y)) = f \circ \eta(Y)$$

where the second equality is by the second condition of Definition 7.6.

To define μ we set

$$\mu_X = \rho(Id_{K(X)})$$

To prove that it is a morphism of functors we need to show that for any $f : X \to Y$ one has

$$\mu_X \circ R_{Mor}(f) = R_{Mor}(R_{Mor}(f)) \circ \mu_Y$$

We have

$$\mu_X \circ R_{Mor}(f) = \rho(Id_{K(X)}) \circ \rho(f \circ \eta(Y)) = \rho(Id_{K(X)} \circ \rho(f \circ \eta(Y))) = \rho(\rho(f \circ \eta(Y)))$$

and

$$R_{Mor}(R_{Mor}(f)) \circ \mu_Y = R_{Mor}(\rho(f \circ \eta(Y))) \circ \rho(Id_{K(Y)}) = \rho(\rho(f \circ \eta(Y)) \circ \eta(K(Y))) \circ \rho(Id_{K(Y)}) =$$

$$\rho(\rho(f \circ \eta(Y)) \circ \eta(K(Y)) \circ \rho(Id_{K(Y)})) = \rho(\rho(f \circ \eta(Y)) \circ Id_{K(Y)}) = \rho(\rho(f \circ \eta(Y)))$$

where the fourth equality is by the second condition of Definition 7.6.

It remains to prove three remaining conditions of Definition 7.1.

For the first one we have:

$$R_{Mor}(\mu_X) \circ \mu_X = \rho(\mu_X \circ \eta(K(X))) \circ \mu_X = \rho(\rho(Id_{K(X)}) \circ \eta(K(X))) \circ \rho(Id_{K(X)}) =$$

$$\rho(\rho(Id_{K(X)}) \circ \eta(K(X)) \circ \rho(Id_{K(X)})) = \rho(\rho(Id_{K(X)}))$$

and

$$\mu_{K(X)} \circ \mu_X = \rho(Id_{K(K(X))}) \circ \rho(Id_{K(X)}) = \rho(Id_{K(K(X))} \circ \rho(Id_{K(X)})) = \rho(\rho(Id_{K(X)}))$$

For the second one we have

$$R_{Mor}(\eta_X) \circ \mu_X = \rho(\eta(X) \circ \eta(K(X))) \circ \rho(Id_{K(X)}) = \rho(\eta(X) \circ \eta(K(X)) \circ \rho(Id_{K(X)})) =$$

$$\rho(\eta(X) \circ Id_{K(X)}) = \rho(\eta(X)) = Id_{K(X)}$$

For the third one we have

$$\eta_{K(X)} \circ \mu_X = \eta_{K(X)} \circ \rho(Id_{K(X)}) = Id_{K(X)}$$

We have proved that $KM(K) = ((K, R_{Mor}), \eta, \mu)$ is a monad.

Let $\phi(X) : K_{Ob}(X) \to K'_{Ob}(X)$ be a morphism of Kleisli triples. Let us show that the same family of morphisms, which we will denote ϕ_X instead of $\phi(X)$, is also a morphism of monads $KM(K) \to KM(K')$, i.e., that the conditions of Definition 7.2 hold.

For the first condition we need to check that for any $f : X \to Y$ the equality

$$R_{Mor}(f) \circ \phi_Y = \phi_X \circ R'_{Mor}(f)$$

We have

$$R_{Mor}(f) \circ \phi_Y = \rho(f \circ \eta(Y)) \circ \phi(Y) = \phi(X) \circ \rho'(f \circ \eta(Y)) = \phi_X \circ R'_{Mor}(f)$$

For the second condition consider $X \in C$. We need to show that

$$(\phi * \phi)_X \circ \mu'_X = \mu_X \circ \phi_X$$

We have:

$$(\phi * \phi)_X \circ \mu'_X = \phi(K(X)) \circ R'_{Mor}(\phi(X)) \circ \rho'(Id_{K'(X)}) =$$
$$\phi(K(X)) \circ \rho'(\phi(X) \circ \eta(K'(X))) \circ \rho'(Id_{K'(X)}) = \phi(K(X)) \circ \rho'(\phi(X) \circ \eta(K'(X)) \circ \rho'(Id_{K'(X)})) =$$
$$\phi(K(X)) \circ \rho'(\phi(X)) = \rho(Id_{K(X)}) \circ \phi(X) = \mu_X \circ \phi_X$$

Where the first equality follows from the general properties of functor morphisms and the fifth equality follows from the second condition of Definition 7.7.

The third condition of Definition 7.1 follows immediately from the first condition of Definition 7.7.

We have constructed a function from morphisms of Kleisli triples to morphisms of monads. It remains to verify that together with the function on objects it satisfies axioms of a functor. This is straightforward since the underlying function of objects and the underlying family of morphisms in C remain the same.

Lemma 7.14 *[***2015.11.14.l1***] One has*

$$MK \circ KM = Id_{Mon(C)}$$

$$KM \circ MK = Id_{KT(C)}$$

Proof: Two monads are equal when the underlying functors are equal on objects and morphisms of C and families of morphisms η and μ are equal. Given a monad $\mathbf{R} = (R, \eta, \mu)$ we have for $KM(MK(\mathbf{R}))$:

1. $KM(MK(\mathbf{R}))_{Ob} = MK(\mathbf{R})_{Ob} = R_{Ob}$,
2. for a morphism $f : X \to Y$ we have

$$KM(MK(R))_{Mor}(f) = \rho_{MK(R)}(f \circ \eta(Y)) = R_{Mor}(f \circ \eta(Y)) \circ \mu(Y) =$$
$$f \circ \eta(Y) \circ \mu(Y) = f \circ Id_Y = f$$

3. The family of morphisms η is not changed by the functors KM and MK and we conclude that $\eta_{KM(MK(\mathbf{R}))} = \eta_{\mathbf{R}}$.

4. For the family of morphisms $\mu(X)$ we have

$$\mu_{KM(MK(\mathbf{R})),X} = \rho_{MK(\mathbf{R})}(Id_{R(X)}) = R_{Mor}(Id_{R(X)}) \circ \mu_{\mathbf{R},X} = \mu_{\mathbf{R},X}$$

We conclude that $MK \circ KM = Id$ on objects of $Mon(C)$.

Two morphisms of monads are equal when the underlying functions $Ob(C) \to Mor(C)$ are equal. Since functors MK and KM do not change this function we conclude that $MK \circ KM = Id$ on morphisms as well.

Two Kleisli triples are equal when the corresponding functions on objects are equal and the families of morphisms η and ρ are equal. Given a Kleisli triple $K = (K_{Ob}, \eta, \rho)$ we have for $MK(KM(K))$:

1. for $X \in C$, $KM(MK(K))_{Ob}(X) = K_{Ob}(X)$,
2. for $X \in C$ the η of the triple is the same as the η of the monad,
3. for a morphism $f : X \to K(Y)$ we have

$$\rho(f) = KM(K)_{Mor}(f) \circ \mu_{KM(K),Y} = \rho(f \circ \eta(Y)) \circ \rho(Id_{K(X)}) = \rho(f \circ \eta(Y) \circ \rho(Id_{K(X)})) =$$

$$\rho(f \circ Id_{K(X)}) = \rho(f)$$

Two morphisms of Kleisli triples are equal when the corresponding functions $Ob(C) \to Mor(C)$ are equal. As for the composition in the opposite direction we conclude that $KM \circ MK = Id$ on morphisms since both functors do not change this function.

We can now provide construction for Problem 7.9.

Construction 7.15 [**2015.12.18.constr4**] We take the functors MK and KM of Constructions 7.11 and 7.13 as the morphism components of the required isomorphisms. Lemma 7.14 shows that they are mutually inverse. This completed the construction for Problem 7.9.

8 Relative monads and monads

For a set U, we let Mnd_U denote the category $Mon(Sets(U))$.

Problem 8.1 *[***2015.12.10.prob1***] To construct a functor $ML_U : Mnd_U \to LW(U)$.*

Construction 8.2 [**2015.12.10.constr1**] Let U be a universe. First we need a function from the set of objects of Mnd_U to the set of objects of $LW(U)$ which will become the object part of our functor. Let $\mathbf{R} = (R, \eta, \mu)$ be a monad on $Sets(U)$. We need to construct a category T and a functor $L : F \to T$ that satisfy the conditions of Definition 5.1.

We take for objects of T the set $\mathbf{N}$ of natural numbers and for morphisms $m \to n$ the set of morphisms $Fun(stn(m), R(stn(n)))$ from $stn(n)$ to $stn(m)$ in the Kleisli category $K(\mathbf{R})$ of $\mathbf{R}$. The composition is the composition in the Kleisli category and the identity morphisms are the identity morphisms in the Kleisli category. The functor L is identity on objects and takes $f : stn(m) \to stn(n)$ to $f \circ \eta_{stn(n)}$ on morphisms.

For the proofs we will need to remind the construction of the Kleisli category (see [5],[8, around Th. 3.18][4]). For a monad $\mathbf{R}$ on a category C the set of objects of $K(C)$ is the set of objects of C. One then defines $Hom_{K(C)}(X,Y) = Mor_C(X, R(Y))$ and defines $Mor(K(C))$ using the same approach that we have already used several times above. The composition for $f : X \to R(Y)$ and $g : Y \to R(Z)$ is given by

$$f \circ g = f \circ R(g) \circ \mu_Z$$

The identity morphism for X is given by η_X. For any $C, \mathbf{R}$ one defines a functor $CK : C \to K(C)$ that is identity on objects and that maps a morphism $f : X \to Y$ in C to $f \circ \eta_Y \in Mor_C(X, R(Y))$.

The fact that this composition and these identity morphisms define a category implies that our composition together and identity morphisms define a category structure on T. The fact that CK is a functor implies that our L is a functor. It is also easy to prove it directly from the axioms of a monad.

We now have T and L defined by $\mathbf{R} \in Mnd_U$. It remains to prove that they satisfy the conditions of Definition 5.1. The first condition is obvious. The second condition is also obvious since $Fun(stn(0), R(stn(n)))$ is a one point set for any set $R(stn(n))$. The third condition is straightforward to prove as well since the square

$$\begin{array}{ccc} Fun(stn(m+n), R(stn(k))) & \xrightarrow{ii_1^{m,n} \circ \_} & Fun(stn(n), R(stn(k))) \\ {\scriptstyle ii_0^{m,n} \circ \_}\downarrow & & \downarrow \\ Fun(stn(m), R(stn(k))) & \longrightarrow & Fun(stn(0), R(stn(k))) \end{array}$$

is a pull-back square for any set $R(stn(k))$. This completes the construction of ML_U on objects.

Let $\phi : R_1 \to R_2$ be a morphism of monads on $Sets(U)$. We define a morphism of Lawvere theories $G = ML_U(\phi) : T_1 \to T_2$, where T_1, T_2 are the categories corresponding to $\mathbf{R}_1, \mathbf{R}_2$ according to the construction described above, as follows. We let $\circ_i$, $i = 1,2$ denote the compositions in T_1 and T_2 and $\circ$ denote the composition of functions between sets. On objects G is the identity. On morphisms, for $f \in Fun(stn(m), R_1(stn(n)))$ we set

$$[\mathbf{2015.12.10.eq2}] G(f) = f \circ \phi_{stn(n)} \tag{7}$$

Let us check that G is a functor. For the identity morphisms we have

$$G(Id_{1,n}) = \eta_{1,stn(n)} \circ \phi_{stn(n)} = \eta_{2,stn(n)} = Id_{2,n}$$

For the composition, when $f \in Fun(stn(k), R_1(stn(m)))$ and $g \in Fun(stn(m), R_1(stn(n)))$ we have

$$[\mathbf{2015.12.10.eq1}] G(f \circ_1 g) = G(f \circ R_1(g) \circ \mu_{1,stn(n)}) = f \circ R_1(g) \circ \mu_{1,stn(n)} \circ \phi_{stn(n)} =$$
$$f \circ R_1(g) \circ (\phi * \phi)_{stn(n)} \circ \mu_{2,stn(n)} \tag{8}$$

and

$$G(f) \circ_2 G(g) = f \circ \phi_{stn(m)} \circ R_2(g \circ \phi_{stn(n)}) \circ \mu_{2,stn(n)} = f \circ \phi_{stn(m)} \circ R_2(g) \circ R_2(\phi_{stn(n)}) \circ \mu_{2,stn(n)}$$

[4]Kleisli works with what we today would call a co-monad instead of monads. Manes works with monads but he is more concerned with the construction of a monad from the data that defines Kleisli category.

By the general properties of functor morphisms we have

$$(\phi * \phi)_n = \phi_{R_1(stn(n))} \circ R_2(\phi_{stn(n)})$$

Therefore (8) continues as

$$f \circ R_1(g) \circ (\phi * \phi)_{stn(n)} \circ \mu_{2,stn(n)} = f \circ R_1(g) \circ \phi_{R_1(stn(n))} \circ R_2(\phi_{stn(n)}) \circ \mu_{2,stn(n)}$$

Since ϕ is a morphism of functors (natural transformation) we have

$$R_1(g) \circ \phi_{R_1(stn(n))} = \phi_{stn(m)} \circ R_2(g)$$

which finishes the proof that G commutes with compositions and with it the proof that G is a functor.

To show that G is a morphism of Lawvere theories we need to check that $L_1 \circ G = L_2$. On objects it is obvious. On morphisms, for $f \in Fun(stn(m), R_1(stn(n)))$, we have:

$$(L_1 \circ G)(f) = G(L_1(f)) = G(f \circ \eta_{stn(n)}) = f \circ \eta_{1,stn(n)} \circ \phi_{stn(n)} = f \circ \eta_{2,stn(n)} = L_2(f)$$

This completes the construction of the object and morphism components of ML_U. It remains to prove the axioms of a functor.

For the identity morphism axiom we need to check that given $\phi = Id_R$ the corresponding functor $G : T \to T$ constructed above is the identity functor. Its object component is identity for any ϕ. That its morphism component is identity follows immediately from (7).

For the composition morphism consider $\phi_1 : R_1 \to R_2$ and $\phi_2 : R_2 \to R_3$. Let G_1, G_2 be the functors constructed from ϕ_1, ϕ_2 and G the functor constructed from $\phi_1 \circ \phi_2$. That $G = G_1 \circ G_2$ on objects is obvious. For the morphism component and $f \in Fun(stn(m), R_1(stn(n)))$ we have:

$$G(f) = f \circ (\phi_{1,stn(n)} \circ \phi_{2,stn(n)})$$

and

$$(G_1 \circ G_2)(f) = G_2(G_1(f)) = G_2(f \circ \phi_{1,stn(n)}) = f \circ \phi_{1,stn(n)} \circ \phi_{2,stn(n)}$$

This completes the proof that ML_U is a functor and with it completes Construction 8.2.

Problem 8.3 *[***2015.12.10.prob2***] Let U be a universe. To construct a functor $LM_U : LW(U) \to Mnd_U$.*

Constructing a solution to Problem 8.3 turns out to be much more difficult than constructing a solution to Problem 8.1. We start with several results that we will be using in the construction.

9 Radditive functors

In [10, p.213] we defined the concept of radditive functors on categories with finite coproducts. We will now re-define radditive functors as functors satisfying a certain property on categories with an initial object and the structure of binary coproducts. In the presence of the AC the new definition is equivalent to the old one. In the absence of the AC it is, as we will show, a more general one.

Definition 9.1 *[**2015.12.20.def2**] Let C be a category with an initial object 0 and a structure of binary coproducts $(\amalg, ii_0^{*,*}, ii_1^{*,*}, \Sigma(-,-))$. Let U be a universe. A radditive functor Φ from C to $Sets(U)$ is a contravariant functor such that one has:*

1. *$\Phi(0)$ is a one-point set,*
2. *for any $X, Y \in C$ the function $a \mapsto (F(ii_0^{X,Y}), F(ii_1^{X,Y}))$ from $F(X \amalg Y)$ to $F(X) \times F(Y)$ is a bijection.*

Let us show that this definition is more general than the definition of [10], i.e., that if C is a category with finite coproducts and Φ is a radditive functor in the sense of [10, p.213] then we can construct a structure of binary coproducts on C and with respect to this structure Φ is a radditive functor in the sense of definition 9.1.

We first remind the definition of the finite coproducts structure. For this we will need to fix a definition of a finite set, which we define as a set I such that there exists $n \in \mathbf{N}$ and a bijection $\phi : I \to stn(n)$. Note that we can assume that ϕ is chosen in proofs of statements but not in constructions of objects since the latter is done in predicate logic by proving statements of the form "there exists a unique".

Definition 9.2 *[**2015.12.20.def3**] A finite coproducts structure on a category C is, for any finite set I and a function $X : I \to Ob(C)$, an object Y, denoted $\amalg_{i\in I} X(i)$, and a function $ii^X : I \to Mor(C)$ such that the following conditions hold:*

1. *for all $i \in I$ one has $ii^X(i) : X(i) \to Y$,*
2. *for all $Z \in C$ and all families of elements $f_i \in Mor_C(X(i), Z)$ there exists a unique element g, denoted by $\amalg_{i\in I} f_i$, in $Mor_C(Y, Z)$ such that for all $j \in I$ one has $ii^X(i) \circ g = f_i$.*

One often writes X_i instead of $X(i)$ and ii_i^X instead of $ii^X(i)$. In the case of $I = \emptyset$, Definition 9.2 specifies an object of C that is usually denoted by 0 or $\emptyset$ and called the initial object of the structure.

One also observes easily that any category with finite coproducts structure can be given a structure of binary coproducts in an obvious way.

Lemma 9.3 *[**2015.12.20.l1**] Let C be a category with a finite coproducts structure and let Φ be a functor that is radditive with respect to the corresponding initial object and the binary coproducts structure. Then for any finite set I the function*

$$\prod_{i\in I} F(ii^X(i)) : F(\amalg_{i\in I} X_i) \to \prod_{i\in I} F(X_i)$$

is a bijection.

Proof:

10 Summary

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