Interpretation of the rules for dependent sums on the C-system defined by a universe category[1]

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Abstract

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1 Introduction

In the syntactic C-systems the Σ-structure, or the structure of dependent sums, requires four operations on the level of raw syntax. An operation $\Sigma(A, x.B)$ with arity $(0, 1)$ - i.e. such that its first argument does not bound any variables and the second argument bounds one variable. This operation is used to represent the dependent sum types and often written $\Sigma\, x : A, B$. Then there is an operation *pair* of arity $(0, 1, 0, 0)$, $(1, 0, 0)$ or $(0, 0)$, depending

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on the amount of typing information that one chooses to include into the term, that is used to represent the object of the dependent sum that is defined by two objects of the argument types. Finally, there are two operations $pr1$ and $pr2$ whose arities vary depending again on the amount of typing information that one choses to include inside the terms of the form $pr1$ and $pr2$ that are used to represent the components of an object of a dependent sum.

John Cartmell has two definitions related to the Σ-structure - definition of a C-system (contextual category) with disjoint unions [1, p.3.2] and definition of the Σ-component of the strong M.L-structure [1, p.3.36]. Cartmell asserts without a proof ([1, p.3.42]) that these two notions are equivalent and indeed it is not difficult to show that this is the case. Cartmell also remarks that by dropping the uniqueness part of the second version of the definition one obtains a notion that directly corresponds to the dependent sums of [3]. We reformulate Cartmell's definition in a slightly more constructive terms and define both the weak Σ-structure corresponding to dependent sums in [3] and the strong Σ-structure, or simply Σ-structure, corresponding to the main concept considered by Cartmell.

We also introduce the notions of a weak and strong Σ'-structures and show their equivalence with weak and strong Σ-structures. This is used later to connect Σ-structures on the C-systems of the form $CC(\mathcal{C}, p)$ with the Σ-structures on the universe p that are defined in Section 3.2.

Note: mention Remark 2.1.10 as an explanation for the complexity of proofs.

Note: This paper, as the papers [7] and [6], is written in the formalization ready style. None of the arguments are omitted as obvious only because they are intuitively obvious to readers who are deeply familiar with a particular mathematical tradition. In this paper we also tried to start paying attention to the strength of the the formal theory required for the formalization of our constructions.

Univalent foundations are based on the use of dependent type theories such as Calculus of Inductive Constructions in combination with the intuition that comes from the univalent models of these theories in ordinary homotopy theory.

The bond that connects univalent foundations to the set-theoretic foundations and ensures that these two approaches are foundations for the same Mathematics is the univalent model itself. This model is a complex mathematical construction that itself requires verification and certification relative to some foundational system or systems. The more simple is this meta-theory in which the model can be formalized and verified the better.

Many of the constructions of this paper are needed to give a rigorous mathematical formulation of this model. Because of this fact we tried to keep them expressed in as simple (from the perspective of proof theory) a language as possible.

2 Σ-structures on C-systems

2.1 Some general results about C-systems

[**sec1**]

Let us start by making some changes to the notations that were introduced in [8]. The new notations that we introduce are consistent with the notations introduced in [2, pp.239-240].

Let CC be a C-system. We will say that an object X is over an object Y if $l(X) \geq l(Y)$ and $Y = ft^{l(X)-l(Y)}(X)$. Note that "is over" and "is above" are well-defined relations on $Ob(CC)$ with "is over" being reflexive and transitive and "is above" being transitive.

If X is over Γ we will write $p(X, \Gamma)$ for the composition of the canonical projections going from X to Γ that was previously denoted $p_{X,n}$ where $n = l(X) - l(\Gamma)$. In particular, if $l(X) = l(\Gamma)$ then $X = \Gamma$ and $p(X, X) = Id_X$.

If X is over Γ and $f : \Gamma' \to \Gamma$ is a morphism we will write $f^*(X)$ for what previously was denoted $f^*(X, n)$ where $n = l(X) - l(\Gamma)$ and

$$q(f, X) : f^*(X) \to X$$

for what was previously denoted by $q(f, X, n)$.

Lemma 2.1.1 *[***2015.06.15.l1***] For any X and f as above $f^*(X)$ is an object over Γ' and that the square*

[**2015.06.11.sq2**]
$$\begin{array}{ccc} f^*(X) & \xrightarrow{q(f,X)} & X \\ {\scriptstyle p(f^*(X),\Gamma')}\downarrow & & \downarrow{\scriptstyle p(X,\Gamma)} \\ \Gamma' & \xrightarrow{f} & \Gamma \end{array} \qquad (1)$$

is a pull-back square.

Proof: It is proved easily by induction on $n = l(X) - l(\Gamma)$ applying the fact that the vertical composition of two pull-back squares is a pull-back square.

Lemma 2.1.2 *[***2015.06.11.l2***] Let X be an object over Γ and $f : \Gamma' \to \Gamma$, $g : \Gamma'' \to \Gamma'$ two morphisms. Then $(g \circ f)^*(X) = g^*(f^*(X))$ and*

$$q(g \circ f, X) = q(g, f^*(X)) \circ q(f, X)$$

Proof: The proof is by induction on $l(X) - l(\Gamma)$ using the axioms of a C-system.

If X, Y are objects over Γ and $f : X \to Y$ is a morphism we will say that f is a morphism over Γ if $f \circ p(Y, \Gamma) = p(X, \Gamma)$. Compositions of morphisms over Γ is easily seen to be a morphism over Γ.

If X, Y are objects over Γ, $a : X \to Y$ is a morphism over Γ and $f : \Gamma' \to \Gamma$ is a morphism then we let

$$f^*(a) : f^*(X) \to f^*(Y)$$

denote the unique morphism over Γ' such that

$$[\mathbf{2015.06.11.eq7}] f^*(a) \circ q(f, Y) = q(f, X) \circ a \tag{2}$$

One verifies easily that one has

$$[\mathbf{2015.06.11.eq2}] f^*(a \circ b) = f^*(a) \circ f^*(b) \tag{3}$$

and for $g : \Gamma'' \to \Gamma'$ one has

$$[\mathbf{2015.06.11.eq3}] g^*(f^*(a)) = (g \circ f)^*(a) \tag{4}$$

One also has that for X over Γ, $p(X, \Gamma)$ is a morphism over Γ and that

$$[\mathbf{2015.06.15.eq3}] f^*(p(X, \Gamma)) = p(f^*(X), \Gamma'). \tag{5}$$

If X is an object over Y and Y is an object over Γ then X is an object over Γ and one has

$$[\mathbf{2015.06.11.eq4}] p(X, \Gamma) = p(X, Y) \circ p(Y, \Gamma) \tag{6}$$

If further $f : \Gamma' \to \Gamma$ is a morphism then one has

$$[\mathbf{2015.06.11.eq5}] f^*(X) = q(f, Y)^*(X) \tag{7}$$

and

$$[\mathbf{2015.06.11.eq6}] q(f, X) = q(q(f, Y), X) \tag{8}$$

The proofs of all of these equations are by induction on $l(X) - l(Y)$.

Lemma 2.1.3 *[**2015.06.15.l2**] [**2015.06.11.l1**] Let X, Z be objects over Γ, Y an object over Z and $f : X \to Z$ a morphism over Z. Then one has:*

$$[\mathbf{2015.06.15.eq1}] g^*(f^*(Y)) = (g^*(f))^*(g^*(Y)) \tag{9}$$

and

$$[\mathbf{2015.06.15.eq2}] g^*(q(f, Y)) = q(g^*(f), g^*(Y)) \tag{10}$$

Proof: We have

$$g^*(f^*(Y)) = q(g, X)^*(f^*(Y)) = (q(g, X) \circ f)^*(Y) = (g^*(f) \circ q(g, Z))^*(Y) =$$

$$(g^*(f))^*(q(g, Z)^*(Y)) = (g^*(f))^*(g^*(Y))$$

where the first and the fifth equations are by (7), the second and the fourth are by (3) and the third equation is by (2). This proves (9) and also proves that the morphisms on the left and the right hand side of (10) have the same domain. The codomain of the morphisms on both sides of (10) is $g^*(Y)$ that is a pull-back with projections $p(g^*(Y), g^*(Z))$ and $q(g, Y)$. It is, therefore sufficient to verify that the compositions of the right and the left hand side morphisms with each of the projections coincide. We have (where we leave matching of the steps with the previously established equations to the reader):

$$g^*(q(f,Y)) \circ p(g^*(Y), g^*(Z)) = g^*(q(f,Y)) \circ g^*(p(Y,Z)) =$$

$$g^*(q(f,Y) \circ p(Y,Z))$$

$$q(g^*(f), g^*(Y)) \circ p(g^*(Y), g^*(Z)) = p((g^*(f))^*(g^*(Y)), g^*(X)) \circ g(f) =$$

$$p(g^*(f^*(Y)), g^*(X)) \circ g^*(f) = g^*(p(f^*(Y), X)) \circ g^*(f) = g^*(p(f^*(Y), X) \circ f) =$$

$$g^*(q(f,Y) \circ p(Y,Z))$$

and

$$g^*(q(f,Y)) \circ q(g,Y) = q(g, f^*(Y)) \circ q(f,Y) = q(q(g,X), f^*(X)) \circ q(f,Y) =$$

$$q(q(g,X) \circ f, Y)$$

$$q(g^*(f), g^*(Y)) \circ q(g,Y) = q(g^*(f), g^*(Y)) \circ q(q(g,Z),Y) = q(g^*(f) \circ q(g,Z), Y) =$$

$$q(q(g,X) \circ f, Y)$$

Lemma is proved.

Some of the previous results can be combined into the following theorem that was mentioned in [2, pp. 240-241] but without a proof.

Given a C-system CC and an object X of CC the set of objects over X and morphisms over X equipped with the length function given by $l_X(Y) = l(Y) - l(X)$ and with all of the other structures of a C-system restricted in the obvious way from CC form a new C-system that we can denote $CC//X$ (to avoid the possible confusion with the category CC/X).

For a morphism $g : X' \to X$, the maps $Y \mapsto g^*(Y)$ and $f \mapsto g^*(f)$ give us maps from the underlying sets of $CC//X$ to the underlying sets of $CC//X'$.

Recall that the detailed definition of a homomorphism of C-systems is given in [5, Definition 3.1].

Theorem 2.1.4 *[***205.06.15.th1***] The maps $f^* : Ob(CC//X) \to Ob(CC//X')$, $f^* : Mor(CC//X) \to Mor(CC//X')$ corresponding to a morphism $f : X' \to X$ define a homomorphism of C-systems.*

Proof: The commutation with the length function is easy to prove. The commutation with the ft map is easy to prove. The commutation with the domain and codomain maps are automatic. The commutation with the identities is easy to prove, the commutation with compositions is (3). The commutation with the p-morphisms is (5). The the commutation with q-morphisms is shown in Lemma 2.1.3. One now applies [5, Lemma 3.4] that shows that the commutation with the s-operations hold automatically.

For any morphism $f : X \to Y$ we let $Sec(f)$ denote the set of sections of f i.e. of morphisms $g : Y \to X$ such that $g \circ f = Id_Y$.

If X is an object over Y we let $Tm(Y, X)$ denote the set $Sec(p(X, Y))$. For $X \in Ob(CC)$ such that $l(X) > 0$ we have $\widetilde{Ob}(X) = Sec(p(X, ft(X))) = Tm(ft(X), X)$.

If X, Y are objects over Γ and f is a morphism over Γ then any section of f is a morphism over Γ. In particular, for any X over Γ and $f : \Gamma' \to \Gamma$ we have a function

$$Tm(\Gamma, X) \to Tm(\Gamma', f^*(X))$$

which we also denote f^*. Note that we have

$$[\mathbf{2015.06.19.eq7}] f^*(s) \circ q(s, f) = f \circ s \tag{11}$$

which is a particular case of (2).

For X over Γ and $f : \Gamma' \to X$ we let $ft(f, \Gamma) : \Gamma' \to \Gamma$ denote the composition $f \circ p(X, \Gamma)$ and $s(f, \Gamma)$ the unique element of $Tm(\Gamma', ft(f, \Gamma)^*(X))$ such that

$$[\mathbf{2015.06.11.eq1}] s(g, \Gamma) \circ q(ft(f, \Gamma), X) = f \tag{12}$$

which can be seen on the diagram

$$\begin{array}{ccccc}
\Gamma' & \xrightarrow{s(f,\Gamma)} & ft(f,\Gamma)^*(X) & \xrightarrow{q(ft(f,\Gamma),X)} & X \\
 & & {\scriptstyle p(ft(f,\Gamma)^*(X),\Gamma')}\downarrow & & \downarrow{\scriptstyle p_{X,\Gamma}} \\
 & & \Gamma' & \xrightarrow{ft(f,\Gamma)} & \Gamma
\end{array}$$

Lemma 2.1.5 *[**2015.05.15.l4**] Let X, Z be objects over Γ, Y an object over Z, $f : X \to Y$ a morphism over Γ and $g : \Gamma' \to \Gamma$ a morphism. Then one has*

$$[\mathbf{2015.06.15.eq6}] g^*(s(f, Z)) = s(g^*(f), g^*(Z)) \tag{13}$$

Proof: Let $h = ft(f, Z)$. Then

$$ft(g^*(f), g^*(Z)) = g^*(f) \circ p(g^*(Y), g^*(Z)) = g^*(f) \circ g^*(p(Y, Z)) = g^*(h)$$

In view of Lemma 2.1.3 the pull-back of the canonical square based on h and Z is the canonical square based on $f^*(h)$ and $f^*(Z)$. In particular, the right hand side and the left hand side morphisms of (13) belong to the same set

$$Tm(g^*(X), g^*(h^*(Y))) = Tm(g^*(X), g^*(h)(g^*(Y)))$$

By definition $s(g^*(f), g^*(Z))$ is the unique element of this set such that

$$s(g^*(f), g^*(Z)) \circ q(g^*(h), g^*(Y)) = g^*(f)$$

therefore we need to check this property for $g^*(s(f,Z))$. We have

$$g^*(s(f,Z)) \circ q(g^*(h), g^*(Y)) = g^*(s(f,Z)) \circ g^*(q(h,Y)) = g^*(s(f,Z) \circ q(h,Y)) = g^*(f)$$

Lemma is proved.

Lemma 2.1.6 *[**2015.06.13.l5**] For X over Γ, $f : \Gamma' \to \Gamma$ and $g : \Gamma'' \to f^*(X)$ one has:*

$$[\mathbf{2015.06.07.eq2}]s(g, \Gamma') = s(g \circ q(f,X), \Gamma) \tag{14}$$

(cf. [8, Definition 2.3(4)]).

Proof: Let $h = ft(g, \Gamma')$. Then

$$ft(g \circ q(f,X), \Gamma) = g \circ q(f,X) \circ p_{X,\Gamma} = g \circ p_{f^*(X),\Gamma'} \circ f = h \circ f$$

The left hand side morphism of (14) is an element of $Tm(\Gamma'', h^*(f^*(X)))$ while the right hand side an element of

$$Tm(\Gamma'', ft(g \circ q(f,X), \Gamma)^*(X)) = Tm(\Gamma'', (h \circ f)^*(X)) = Tm(\Gamma'', h^*(f^*(X)))$$

i.e. the left hand side morphism and the right hand side morphism belong to the same set. By definition $s(g, \Gamma')$ is the only element in $Tm(\Gamma'', h^*(f^*(X)))$ such that

$$s(g, \Gamma') \circ q(h, f^*(X)) = g$$

Therefore, we need to verify that

$$s(g \circ q(f,X), \Gamma) \circ q(h, f^*(X)) = g.$$

Since the codomain of both morphisms is $f^*(X)$ it is sufficient to verify that

$$s(g \circ q(f,X), \Gamma) \circ q(h, f^*(X)) \circ p(f^*(X), \Gamma') = g \circ p(f^*(X), \Gamma')$$

and

$$s(g \circ q(f,X), \Gamma) \circ q(h, f^*(X)) \circ q(f,X) = g \circ q(f,X)$$

For the first equality we have

$$s(g \circ q(f,X), \Gamma) \circ q(h, f^*(X)) \circ p(f^*(X), \Gamma') = s(g \circ q(f,X), \Gamma) \circ p(h^*(f^*(X)), \Gamma'') \circ h =$$

$$g \circ p(f^*(X), \Gamma')$$

For the second one we have

$$s(g \circ q(f,X), \Gamma) \circ q(h, f^*(X)) \circ q(f,X) = s(g \circ q(f,X), \Gamma) \circ q(h \circ f, X) =$$

$$s(g \circ q(f,X), \Gamma) \circ q(ft(g \circ q(f,X), \Gamma), X) = g \circ q(f,X).$$

Lemma is proved.

Lemma 2.1.7 *[**2015.06.13.l6**] Let X be an object over Γ and $s \in Tm(\Gamma, X)$ then*

$$[\mathbf{2015.06.07.eq3}] s(s, \Gamma) = s. \tag{15}$$

Proof: It follows immediately from the fact that $ft(s, \Gamma) = s \circ p(X, \Gamma) = Id_\Gamma$ and (12).

For $\Gamma \in Ob(CC)$ and X over Γ let

$$\delta(X, \Gamma) = s(Id_X, \Gamma)$$

i.e., $\delta(X, \Gamma)$ is the unique objects of $Tm(X, p(X, \Gamma)^*(X))$ such that

$$[\mathbf{2015.06.09.eq2}] \delta(X, \Gamma) \circ q(p(X, \Gamma), X) = Id_X \tag{16}$$

Lemma 2.1.8 *[**2015.06.13.l2**] Let X be an object over Γ and $s \in Tm(\Gamma, X)$. Then one has*

$$s^*(\delta(X, \Gamma)) = s$$

Proof: First of all let us check that both sides of the equality are elements of the same set. We have $s \in Tm(\Gamma, X)$ and

$$s^*(\delta(X, \Gamma)) \in Sec(s^*(p(p(X, \Gamma)^*(X), X))) = Sec(p(s^*(p(X, \Gamma)^*(X)), \Gamma)) = Tm(\Gamma, X)$$

since $s \circ p(X, \Gamma) = Id_\Gamma$.

By definition $s^*(\delta(X, \Gamma))$ is the only element of $Sec(s^*(p(p(X, \Gamma)^*(X), X)))$ such that

$$s^*(\delta(X, \Gamma)) \circ q(s, p(X, \Gamma)^*(X)) = s \circ \delta(X, \Gamma)$$

and therefore we have to check that

$$s \circ q(s, p(X, \Gamma)^*(X)) = s \circ \delta(X, \Gamma)$$

Since the codomain of both morphisms is $p(X, \Gamma)^*(X)$ which is a part of the canonical pull-back square it is sufficient to check that

$$s \circ q(s, p(X, \Gamma)^*(X)) \circ p(p(X, \Gamma)^*(X), X) = s \circ \delta(X, \Gamma) \circ p(p(X, \Gamma)^*(X), X)$$

and

$$s \circ q(s, p(X, \Gamma)^*(X)) \circ q(p(X, \Gamma), X) = s \circ \delta(X, \Gamma) \circ q(p(X, \Gamma), X)$$

For the first equation one calculates easily that both sides equal s. Similar result is obtained for the second equation using the fact that

$$q(s, p(X, \Gamma)^*(X)) \circ q(p(X, \Gamma), X) = q(s \circ p(X, \Gamma), X).$$

Lemma 2.1.9 *[**2015.06.13.l1**] Let X be an object over Γ and $f : \Gamma' \to \Gamma$, then*

$$[\mathbf{2015.06.13.eq1}] f^*(\delta(X, \Gamma)) = \delta(f^*(X), \Gamma') \tag{17}$$

Proof: This is a particular case of Lemma 2.1.5.

Remark 2.1.10 [**2015.06.13.rem1**] Lemma 2.1.9 looks like something that should be true in a more general context than C-systems. After all, δ is just the diagonal morphism and f^* is just the pull-back and we know that the fiber products in the slice categories and in particular the diagonal morphisms there are "preserved" by the pull-backs.

For example one may ask whether Lemma 2.1.9 remains true in the context of categories together with a class of morphisms D for which pull-backs along all other morphisms are chosen and that contains isomorphisms and is closed under compositions (i.e. in the context of display map categories where the class of display maps is closed under compositions).

The morphism $f^*(\delta(X, \Gamma))$ is a section of $f^*(p(p(X, \Gamma)^*(X), X))$ while $\delta(f^*(X), \Gamma')$ is a section of $p(p(f^*(X), \Gamma')^*(f^*(X)), f^*(X))$. The codomain of the first morphism is $f^*(p(X, \Gamma)^*(X))$ and the codomain of the second morphism is $p(f^*(X), \Gamma')^*(f^*(X))$. In a general category as above there is no reason for these two objects to be equal. Moreover, even if the objects happen to be equal there is no reason for the pairs of projection morphisms

$$[\mathbf{2015.06.17.eq1}](f^*(p(p(X, \Gamma)^*(X), X)), f^*(q(p(X, \Gamma), X))) \tag{18}$$

and

$$[\mathbf{2015.06.17.eq2}](p(p(f^*(X), \Gamma')^*(f^*(X)), f^*(X)), q(p(f^*(X), \Gamma'), f^*(X))) \tag{19}$$

to be equal. For example one pair can differ from the other one by an arbitrary automorphism of their common source.

What is true in this more general context is that there is a unique isomorphism between $f^*(p(X, \Gamma)^*(X))$ and $p(f^*(X), \Gamma')^*(f^*(X))$ that transforms the pair (18) to the pair (19) and that this isomorphism also transforms $f^*(\delta(X, \Gamma))$ to $\delta(f^*(X), \Gamma')$. The content of Lemma 2.1.9 is that this isomorphism equals the identity isomorphism of a particular object of CC.

Let Z be an object over Y and Y and object over X. Let

$$Tm(X, Y, Z) = \{s_1, s_2 \,|\, s \in Tm(X, Y),\ s_2 \in Tm(X, s_1^*(Z))\}$$

Define a map

$$tm_{X,Y,Z} : Tm(X, Z) \to Tm(X, Y, Z)$$

by $s \mapsto (s \circ p(Z, Y), s(s, Y))$.

Define a map

$$tm^!_{X,Y,Z} : Tm(X, Y, Z) \to Tm(X, Z)$$

by $(s_1, s_2) \mapsto s_2 \circ q(s_1, Z)$.

Lemma 2.1.11 *[2015.06.07.l3] Let X, Y, Z be as above. Then $tm_{X,Y,Z}$ and $tm^!_{X,Y,Z}$ are mutually inverse bijections.*

Proof: We can see some of the main objects of the lemma on the diagram:

$$\begin{array}{ccccc} X & \xrightarrow{s_2} & s_1^*(Z) & \xrightarrow{q(s_1,Z)} & Z \\ & & {\scriptstyle p(s_1^*(Z),X)}\downarrow & & \downarrow{\scriptstyle p(Z,Y)} \\ & & X & \xrightarrow{s_1} & Y \\ & & & & \downarrow{\scriptstyle p(Y,X)} \\ & & & & X \end{array}$$

Observe first that for $(s_1, s_2) \in Tm(X, Y, Z)$, the composition $s_2 \circ q(s_1, Z)$ is defined and that for $s \in Tm(X, Z)$ we have $s \circ p(Z, Y) \in Tm(X, Y)$ and $s(s, Y) \in Tm(X, (s \circ p(Z, Y))^*(Z))$ so that $(s \circ p(Z, Y), s(s, Y)) \in Tm(X, Y, Z)$.

Next we have

$$tm^!(tm(s)) = tm^!(s \circ p(Z, Y), s(s, Y)) = s(s, Y) \circ q(s \circ p(Z, Y), Z) = s$$

where the last equality is the defining equality of $s(s, Y)$ and

$$tm(tm^!(s_1, s_2)) = tm(s_2 \circ q(s_1, Z)) = (s_2 \circ q(s, Z) \circ p(Z, Y), s(s_2 \circ q(s_1, Z), Y)) =$$

$$(s_2 \circ q(s_1, Z) \circ p(Z, Y), s(s_2, Y)) = (s_2 \circ p(s_1^*(Z), X) \circ s_1, s(s_2, Y)) =$$

$$(s_1, s(s_2, Y)) = (s_1, s_2)$$

where the third equality holds by (14), the fourth by commutativity of the canonical squares, the fifth because $s_2 \circ p(s_1^*(Z), X) = Id_X$ and the last equality holds because of (15).

Lemma 2.1.12 *[2015.06.13.l4] The bijections of Lemma 2.1.11 are natural in X i.e. for $f : X' \to X$ one has*

$$f^*(tm_{X,Y,Z}(s)) = tm_{f^*(X),f^*(Y),f^*(Z)}(f^*(s))$$

$$f^*(tm^!_{X,Y,Z}(s_1, s_2)) = tm^!_{f^*(X),f^*(Y),f^*(Z)}(f^*(s_1), f^*(s_2))$$

Proof: One has

$$f^*(tm_{X,Y,Z}(s)) = (f^*(s \circ p(Z, Y)), f^*(s(s, Y))) = (f^*(s) \circ p(f^*(Z), f^*(Y)), f^*(s(s, Y))) =$$

$$(f^*(s) \circ p(f^*(Z), f^*(Y)), s(f^*(s), f^*(Y))) = tm_{f^*(X),f^*(Y),f^*(Z)}(f^*(s))$$

where the third equality holds by Lemma 2.1.5.

The second property follows formally from the first since $tm^!$ is the inverse bijection of tm.

2.2 Σ-structures on C-systems

In the previous section we proved several results that can be formulated (with the exception of Theorem 2.1.4) directly in the language of C-systems when C-systems are defined, as it is done in [8] using operation $f \mapsto s_f$ instead of the condition that the canonical squares are pull-back squares. All of the dependent sorts that we used, such as $Sec(f)$ or $Tm(X, Y)$ can be treated as mere notations for subsets of tuples of elements from just two sets $Ob(CC)$ and $Mor(CC)$.

Results of this sections concern operations on C-systems i.e. partial multi-argument functions between the sorts of C-systems that satisfy some conditions. As such they can not be reasoned about using the internal language of C-systems.

It is a surprising fact that as we proceed towards the construction of these operations in the C-systems defined by locally-cartesian closed universe categories we will be able to return to the context of an essentially algebraic language - the language of a locally cartesian closed universe category, in reasoning about these operations. Operations themselves will appear as elements or finite combinations of elements of the sorts of the theory that satisfy certain (essentially algebraic) equations while constructions producing operations of one kind from operations of another as (essentially algebraic) formulas in the language.

Definition 2.2.1 *[***2015.06.07.def1***] A 2-to-1 structure S on a C-system CC is a family of functions*

$$S_\Gamma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

given for all $\Gamma \in Ob(CC)$ and such that for any $f : \Gamma' \to \Gamma$ and $B \in Ob_2(\Gamma)$ one has $f^*(S_\Gamma(B)) = S_{\Gamma'}(f^*(B))$.

Example 2.2.2 [**2015.06.07.ex1**] The Π-component of a structure of products of families of types as defined in [4, p.71] or in [7] and the Π-component of a (Π, λ)-structure (also defined in [7]) are examples of 2-to-1 structures. The first component of the weak and strong Σ-structures considered below are also 2-to-1 structures.

For a C-system CC, $\Gamma \in Ob(CC)$ and $B \in Ob_2(\Gamma)$ we will use the notation $\widetilde{Ob}_{1,1}(B)$ for $Tm(\Gamma, ft(B), B)$. Since for X such that $l(X) > 0$ one has $Tm(ft(X), X) = \widetilde{Ob}(X)$ one can also write

$$\widetilde{Ob}_{1,1}(B) = \{o_1, o_2 | \, o_1 \in \widetilde{Ob}(ft(B)), o_2 \in \widetilde{Ob}(o_1^*(B))\}$$

For $f : \Gamma' \to \Gamma$ we have $Tm(\Gamma', f^*(ft(B)), f^*(B)) = \widetilde{Ob}_{1,1}(f^*(B))$ and therefore the function f^* maps $\widetilde{Ob}_{1,1}(B)$ to $\widetilde{Ob}_{1,1}(f^*(B))$.

Definition 2.2.3 *[***2015.06.07.def2***] A pair-structure r over a 2-to-1 structure Σ on CC is a family of functions*

$$r_B : \widetilde{Ob}_{1,1}(B) \to \widetilde{Ob}(\Sigma(B))$$

given for all Γ and $B \in Ob_2(\Gamma)$ and such that for any $f : \Gamma' \to \Gamma$, $B \in Ob_2(\Gamma)$ and $(o_1, o_2) \in \widetilde{Ob}_{1,1}(B)$ one has

$$f^*(r_B(o_1, o_2)) = r_{f^*(B)}(f^*(o_1), f^*(o_2))$$

We let $Pair(\Sigma)$ *denote the set of pair-structures over* Σ.

Definition 2.2.4 *[**2015.06.07.def3**] A pair′-structure* r' *over a 2-to-1 structure* Σ *on* CC *is a family of morphisms* $r'_B : B \to \Sigma(B)$ *given for all* Γ *and* $B \in Ob_2(\Gamma)$ *and such that* r'_B *is a morphism over* Γ *and for all* $f : \Gamma' \to \Gamma$ *and* $B \in Ob_2(\Gamma)$ *one has*

$$f^*(r'_B) = r'_{f^*(B)}$$

We let $Pair'(\Sigma)$ *denote the set of pair′-structures over* Σ.

Definition 2.2.5 *[**2015.06.07.def2a**] A pair″-structure* r'' *over a 2-to-1 structure* Σ *on* CC *is a family of functions*

$$r''_B : Tm(\Gamma, B) \to Tm(\Gamma, \Sigma(B))$$

given for all Γ *and* $B \in Ob_2(\Gamma)$ *and such that for any* $f : \Gamma' \to \Gamma$, $B \in Ob_2(\Gamma)$ *and* $s \in Tm(\Gamma, B)$ *one has*

$$f^*(r''_B(s)) = r_{f^*(B)}(s)$$

We let $Pair''(\Sigma)$ *denote the set of pair″-structures over* Σ.

Lemma 2.2.6 *[**2015.06.19.l1**] Let* Σ *be a 2-to-1 structure on* CC*. Then the functions*

$$to\_r'' : Pair(\Sigma) \to Pair''(\Sigma)$$

$$from\_r'' : Pair''(\Sigma) \to Pair(\Sigma)$$

of the form

$$[\mathbf{2015.06.19.eq1}] to\_r''(r)_B(s_1, s_2) = r_B(tm^!_{\Gamma,ft(B),B}(s_1, s_2)) \quad (20)$$

$$[\mathbf{2015.06.19.eq2}] from\_r''(r'')_B(s) = r''_B(tm_{\Gamma,ft(B),B})(s) \quad (21)$$

define mutually inverse bijections between $Pair(\Sigma)$ *and* $Pair''(\Sigma)$.

Proof: Lemma 2.1.12 shows that the formulas (20), (21) define maps between $Pair(\Sigma)$ and $Pair''(\Sigma)$. The fact that these maps are mutually inverse follows from Lemma 2.1.11 and, in the case when type theoretic formalization is used, function extensionality.

Next we are going to construct bijections between the sets $Pair'(\Sigma)$ and $Pair''(\Sigma)$.

Lemma 2.2.7 *[**2015.06.19.l2**] Let* $r'' \in Pair''(\Sigma)$*. Then one has:*

1. *for any* $\Gamma \in Ob(CC)$ *and* $B \in Ob_2(\Gamma)$ *the expression*

$$[\mathbf{2015.06.19.eq3}] r''\_to\_r'(r'')_B = r''_{p(B,\Gamma)^*(B)}(\delta(B, \Gamma)) \circ q(p(B, \Gamma), \Sigma(B)) \quad (22)$$

   is a well defined morphism $B \to \Sigma(B)$ *over* Γ, *2. for any $f : \Gamma' \to \Gamma$ one has*

$$f^*(r''\_to\_r'(r'')_B) = r''\_to\_r'(r'')_{f^*(B)}$$

Proof: We have that $\delta(B,\Gamma) \in Tm(B, p(B,\Gamma)^*(B))$ and therefore

$$r''_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma)) \in Tm(B, \Sigma(p(B,\Gamma)^*(B))) = Tm(B, p(B,\Gamma)^*(\Sigma(B)))$$

i.e. $r''_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma))$ is a morphism from B to $p(B,\Gamma)^*(\Sigma(B))$. Therefore its composition with $q(p(B,\Gamma), \Sigma(B)$ is defined and is a morphism $B \to \Sigma(B)$.

The second part of the lemma follows by rewriting from Lemmas 2.1.3, 2.1.9 and the definition of $Pair''$.

Lemma 2.2.7 implies that the formulas (22) define a function

$$r''\_to\_r' : Pair''(\Sigma) \to Pair'(\Sigma)$$

For B as above, $s \in Tm(\Gamma, B)$ and $r' \in Pair'(\Sigma)$ consider

$$r'\_to\_r''(r')_B(s) = s \circ r'_B$$

then $r'\_to\_r''(r')_B$ is a function $Tm(\Gamma, B) \to Tm(\Gamma, \Sigma(B))$ and one verifies immediately that for $f : \Gamma' \to \Gamma$ one has

$$f^*(r'\_to\_r''(r')_B(s)) = r'\_to\_r''(r')_{f^*(B)}(f^*(s))$$

i.e. that r'_to_r'' is a function from $Pair'(\Sigma)$ to $Pair''(\Sigma)$.

Lemma 2.2.8 *[**2015.06.19.l3**] The functions*

$$r''\_to\_r' : Pair''(\Sigma) \to Pair'(\Sigma)$$

$$r'\_to\_r'' : Pair'(\Sigma) \to Pair''(\Sigma)$$

are mutually inverse bijections.

Proof: We have

$$r''\_to\_r'(r'\_to\_r''(r'))_B = (r'\_to\_r''(r'))_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma)) \circ q(p(B,\Gamma), \Sigma(B)) =$$

$$\delta(B,\Gamma) \circ r'_{p(B,\Gamma)^*(B)} \circ q(p(B,\Gamma), \Sigma(B)) = \delta(B,\Gamma) \circ p(B,\Gamma)^*(r'_B) \circ q(p(B,\Gamma), \Sigma(B)) =$$

$$\delta(B,\Gamma) \circ q(p(B,\Gamma), B) \circ r'_B = r'_B$$

where the fourth equality holds by (2) and the last one by (16).

In the opposite direction we have

$$r'\_to\_r''(r''\_to\_r'(r''))_B(s) = s \circ (r''\_to\_r'(r''))_B = s \circ r''_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma)) \circ q(p(B,\Gamma), \Sigma_B) =$$

$$s^*(r''_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma))) \circ q(s, p(B,\Gamma)^*(B)) \circ q(p(B,\Gamma), \Sigma_B) = s^*(r''_{p(B,\Gamma)^*(B)}(\delta(B,\Gamma))) =$$

$$= r''_{s^*(p(B,\Gamma)^*(B))}(s^*(\delta(B,\Gamma))) = r''_B(s)$$

where the third equality holds by (11) and the last equality by Lemma 2.1.8. Lemma is proved.

Definition 2.2.9 *[**2015.06.07.def4**] A destruct structure d over a 2-to-1 structure Σ on CC is a family of functions*

$$d_B : \widetilde{Ob}(\Sigma(B)) \to \widetilde{Ob}_{1,1}(B)$$

given for all Γ, $B \in Ob_2(\Gamma)$ and such that for any $f : \Gamma' \to \Gamma$, $B \in Ob_2(\Gamma)$ and $o \in \widetilde{Ob}(\Sigma(B))$ one has

$$f^*(d_B(o)) = d_{f^*(B)}(f^*(o)))$$

We let $Destr(\Sigma)$ denote the set of destruct structures over Σ.

Definition 2.2.10 *[**2015.06.07.def5**] A destruct′ structure d' over a 2-to-1 structure Σ on CC is a family of morphisms $d'_B : \Sigma(B) \to B$ given for all Γ, $B \in Ob_2(\Gamma)$ and such that d'_B is a morphism over Γ and for any $f : \Gamma' \to \Gamma$ and $B \in Ob_2(\Gamma)$ one has*

$$f^*(d'_B) = d'_{f^*(B)}$$

We let $Destr'(\Sigma)$ denote the set of destruct′ structures over Σ.

Definition 2.2.11 *[**2015.06.19.def1**] A destruct′’ structure d'' over a 2-to-1 structure Σ on CC is a family of functions*

$$d''_B : Tm(\Gamma, \Sigma(B)) \to Tm(\Gamma, B)$$

given for all Γ and $B \in Ob_2(\Gamma)$ and such that for all $f : \Gamma' \to \Gamma$ one has

$$f^*(d''_B(s)) = d''_{f^*B}(f^*(s))$$

We let $Destr''(\Sigma)$ denote the set of destruct′’ structures over Σ.

Lemma 2.2.12 *[**2015.06.19.l4**] Let Σ be a 2-to-1 structure on CC. Then the functions*

$$to\_d'' : Destr(\Sigma) \to Destr''(\Sigma)$$

$$from\_d'' : Destr''(\Sigma) \to Destr(\Sigma)$$

of the form

$$[\mathbf{2015.06.19.eq4}] to\_d''(d)_B(s) = tm^!_{\Gamma,ft(B),B}(d_B(s)) \tag{23}$$

$$[\mathbf{2015.06.19.eq5}] from\_d''(d'')_B(s) = tm_{\Gamma,ft(B),B}(d''_B(s)) \tag{24}$$

define mutually inverse bijections between $Destr(\Sigma)$ and $Destr''(\Sigma)$.

Proof: Lemma 2.1.12 shows that the formulas (23), (24) define maps between $Destr(\Sigma)$ and $Destr''(\Sigma)$. The fact that these maps are mutually inverse follows from Lemma 2.1.11 and, in the case when type theoretic formalization is used, function extensionality.

Lemma 2.2.13 *[**2015.06.19.l5**] Let Σ be as above. Then the formulas*

$$[\mathbf{2015.06.19.eq6}]d''\_t\_d'(d'')_B = d''_{p(\Sigma(B),\Gamma)^*(B)}(\delta(\Sigma(B),\Gamma)) \circ q(p(\Sigma(B),\Gamma),B) \quad (25)$$

and

$$[\mathbf{2015.06.19.eq7a}]d'\_to\_d''(d')_B(s) = s \circ d'_B \quad (26)$$

Define mutually inverse bijections of the form

$$d''\_t\_d' : Destr''(\Sigma) \to Destr'(\Sigma)$$

$$d'\_to\_d'' : Destr'(\Sigma) \to Destr''(\Sigma)$$

Proof: One verifies easily that (25) defines a morphism $\Sigma(B) \to B$. Using Lemmas 2.1.3, 2.1.9 one verifies that the family of morphisms so obtained satisfies the naturality condition for morphisms $f : \Gamma' \to \Gamma$ and therefore defines an element of $Destr'(\Sigma)$.

Similarly one proves that formulas (26) define an element of $Destr''(\Sigma)$. For the claim of the lemma that these morphisms are mutually inverse bijections we have:

$$d''\_to\_d'(d'\_to\_d''(d'))_B = (d'\_to\_d''(d'))_{p(\Sigma(B),\Gamma)^*(B)}(\delta(\Sigma(B),\Gamma)) \circ q(p(\Sigma(B),\Gamma),B) =$$

$$\delta(\Sigma(B),\Gamma) \circ d'_{p(\Sigma(B),\Gamma)^*(B)} \circ q(p(\Sigma(B),\Gamma),B) =$$
$$\delta(\Sigma(B),\Gamma) \circ p(\Sigma(B),\Gamma)^*(d'_B) \circ q(p(\Sigma(B),\Gamma),B) =$$
$$\delta(\Sigma(B),\Gamma) \circ q(p(\Sigma(B),\Gamma),\Sigma(B)) \circ d'_B = d'_B$$

where the fourth equality holds by (2).

In the opposite order of composition we have

$$d'\_to\_d''(d''\_to\_d'(d''))_B(s) = s \circ (d''\_to\_d'(d''))_B =$$

$$s \circ d''_{p(\Sigma(B),\Gamma)^*(B)}(\delta(\Sigma(B),\Gamma)) \circ q(p(\Sigma(B),\Gamma),B) =$$
$$s^*(d''_{p(\Sigma(B),\Gamma)^*(B)}(\delta(\Sigma(B),\Gamma))) \circ q(s, p(\Sigma(B),\Gamma)^*(B)) \circ q(p(\Sigma(B),\Gamma),B) =$$
$$s^*(d''_{p(\Sigma(B),\Gamma)^*(B)}(\delta(\Sigma(B),\Gamma))) = d''_{s^*(p(\Sigma(B),\Gamma)^*(B))}(s^*(\delta(\Sigma(B),\Gamma))) = d''_B(s)$$

where the third equality holds by (11) and the last equality by Lemma 2.1.8.

Definition 2.2.14 *[**2015.05.28.def1**] Let CC be a C-system. A weak Σ-structure on CC is a triple (Σ, r', d') where Σ is a 2-to-1 structure, d' is a pair$'$ structure over Σ, d' is a destruct$'$ structure over Σ and for all Γ and $B \in Ob_2(\Gamma)$ one has*

$$r'_B \circ d'_B = Id_B$$

A weak Σ-structure is called a strong Σ-structure if in addition for each Γ and B one has

$$[\mathbf{2015.05.28.eq1}]d'_B \circ r'_B = Id_{\Sigma(B)} \quad (27)$$

3 Σ-structures on universe categories

3.1 Some general results about universes in locally cartesian closed categories

Let $\mathcal{C}$ be a category and $(p : \widetilde{U} \to U, p_{X,F}, Q(F))$ a universe in $\mathcal{C}$. For an object X in $\mathcal{C}$ and $n \geq 0$ define a pair $(Ob_n(X), c_n)$ where $Ob_n(X)$ is a set and $c_n : Ob_n \to Ob(\mathcal{C})$ a function, by induction as follows:

1. $Ob_0(X)$ is the standard one point set whose element we denote by pt and $c_0(pt) = X$,
2. $Ob_1(X) = Hom(X, U)$ and $c_1(F) = (X; F)$,
3. $Ob_{n+1}(X) = \amalg_{A \in Ob_n(X)} Hom(c_n(A), U)$ and $c_{n+1}(A, F) = (c_n(A); F)$.

We will write $c(A)$ instead of $c_n(A)$ when no ambiguity is possible. When $A \in Ob_{n+1}(X)$ then A is of the form $(A', F : A' \to U)$ and we let $ft(A)$ denote $A' \in Ob_n(X)$ and $u_1(A)$ denote F. When $A \in Ob_0(X)$ we set $ft(A) = A$ and $u_1(A)$ is undefined.

As before we say that $A \in Ob_n(X)$ is over $B \in Ob_m(X)$ if $n \geq m$ and $B = ft^{n-m}(A)$.

Let further, for $A \in Ob_{n+1}(X)$, p_A be the morphism $p_{c_n(ft(A)),u_1(A)}$ that is a part of the canonical square

$$\textbf{[2015.06.21.eq1]} \qquad \begin{array}{ccc} c_{n+1}(ft(A), u_1(A)) & \xrightarrow{Q(u_1(A))} & \widetilde{U} \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p} \\ c_n(ft(A)) & \xrightarrow{u_1(A)} & U \end{array} \tag{28}$$

For $A \in Ob_n(X)$ and $m \leq n$ we let $p_{A,m} : c_n(A) \to c_{n-m}(ft^m(A))$ denote the composition $p_A \circ p_{ft(A)} \circ \ldots \circ p_{ft^m}(A)$ such that $p_{A,0} = Id_A$ and $p_{A,n}$ is a morphism from $c(A)$ to X.

This let us to define a (pre-)category $CC(X, p)$ with the set of objects $\amalg_{n \geq 0} Ob_n(X)$ and the set of morphisms of the form

$$Mor = \amalg_{n,m \geq 0} \amalg_{A \in Ob_n(X)} \amalg_{B \in Ob_m(X)} Hom_X((c_n(A), p_{A,n}), (c_m(B), p_{m,B}))$$

If $(B, F) \in Ob_{n+1}(X)$ and $a : A \to B$ is a morphism in $CC(X, p)$ then we define the canonical square associated to (B, F) and a as

$$\begin{array}{ccc} c(A, a \circ F) & \xrightarrow{Q(a,F)} & c(B, F) \\ {\scriptstyle p_{(A, a\circ F)}}\downarrow & & \downarrow{\scriptstyle p_{(B,F)}} \\ c(A) & \xrightarrow{a} & c(B) \end{array}$$

i.e., $a^*(B, F)$ is defined as $(A, a \circ F)$ and $q(a, (B, F))$ is defined as $Q(a, F)$.

By [7, Lemma 3.1] the canonical squares are pull-back squares. Therefore, for a morphism $b : A \to (B, F)$ in $CC(X, p)$ there is a unique morphism $s(b) : A \to a^*(B, F)$ such that $s(b) \circ p_{a^*(B,F)} = Id_A$ and $s(b) \circ q(a, (B, F)) = b$. We define the s-operation of a C-system using this construction.

Theorem 3.1.1 *[**2015.06.25.th1**] The structure on the category $CC(X,p)$ given by the obvious length function and the operations pt, ft, p_A, $a^*(A)$ and $q(a,A)$ and $s(a)$ introduced above satisfies the axioms of a C-system.*

Proof: The first six conditions of [8, Definition 2.1] are obvious. The seventh one follows from [7, Lemma 3.2]. The conditions of [8, Definition 2.3] follow from [8, Proposition 2.4] and [7, Lemma 3.1].

If we have chosen a final object in $\mathcal{C}$ such that $CC(\mathcal{C},p)$ is defined and $\Gamma \in Ob(CC(\mathcal{C},p))$ then we should distinguish $Ob_n(\Gamma)$ and $Ob_n(int(\Gamma))$ as these two sets are not equal. However we have the following.

Problem 3.1.2 *[**2015.06.25.prob1**] Let $(\mathcal{C},p,pt)$ be a universe category and $\Gamma \in Ob(CC(\mathcal{C},p))$. To construct a bijection*

$$u_{n,\Gamma} : Ob_n(\Gamma) \to Ob_n(int(\Gamma))$$

such that for any $A \in Ob_n(\Gamma)$ one has

$$[\mathbf{2015.06.25.eq3}] c(u_{n,\Gamma}(A)) = int(A) \tag{29}$$

Construction 3.1.3 [**2015.06.25.constr1**] Let $l = l(\Gamma)$. Then $Ob_n(\Gamma)$ is the subset in $Ob_{n+l}(CC(\mathcal{C},p))$ that consists of A such that $ft^n(A) = \Gamma$.

The proof is by induction on n. For $n = 0$ the bijection is between two one point sets, sending Γ to pt. For $n = 1$, $Ob_1(\Gamma)$ is the set of pairs of the form (A,F) where $F : int(A) \to U$ and $A = \Gamma$ and the bijection is given by $u_{1,\Gamma}(A,F) = F$.

For an element B in $Ob_{n+1}(\Gamma)$ we have $ft(B) \in Ob_n(\Gamma)$, i.e., $B = (A,F)$ where $A \in Ob_n(\Gamma)$. We set $u_{n+1,\Gamma}(A,F) = (u_n(A),F)$. The fact that $u_{n+1,\Gamma}$ is a bijection follows from the inductive assumption that $u_{n,\Gamma}$ is a bijection and that $c(u_{n,\Gamma}(A)) = int(A)$ for all $A \in Ob_n(\Gamma)$.

Remark 3.1.4 [**2015.06.25.rem1**] The bijections $u_{1,\Gamma}$ and $u_{2,\Gamma}$ coincide with the bijections with the same notation constructed in [7, Constructions 3.4, 3.6].

Remark 3.1.5 [**2015.06.25.rem2**] The equation (29) can also be re-written in the following form

$$[\mathbf{2015.06.25.eq4}] int(u_{n,\Gamma}^{-1}(A)) = c_n(A) \tag{30}$$

For a C-system CC one has $Ob_n(CC) = Ob_n(pt_{CC})$ where pt is the final object of CC. Therefore, for $\Gamma = pt_{CC(\mathcal{C},p)}$ the bijections $u_{n,pt}$ define a bijection

$$u_{Ob} : \amalg_{n\geq 0} u_{n,pt} : Ob(CC(\mathcal{C},p)) \to Ob(CC(pt_{\mathcal{C}},p))$$

For $\Gamma \in Ob(CC(\mathcal{C},p))$ we have

$$int(\Gamma) = c(u_{Ob}(\Gamma))$$

and therefore

$$Hom_{CC(\mathcal{C},p)}(\Gamma,\Gamma') = Hom_{CC(pt_{\mathcal{C}},p)}(u_{Ob}(\Gamma), u_{Ob}(\Gamma'))$$

Therefore we obtain a bijection

$$u_{Mor} : Mor(CC(\mathcal{C},p)) \to Mor(CC(pt_{\mathcal{C}},p))$$

Theorem 3.1.6 *[***2015.07.05.th1***] The pair $u = (u_{Ob}, u_{Mor})$ is an isomorphism of C-systems.*

Proof: The proof is very easy and we leave it for the formalized version of the paper.

Remark 3.1.7 [**2015.07.05.rem1**] Observe that the bijection u_{Ob} and, as a consequence, the bijection u_{Mor} is almost the identity bijection. The only reason why $Ob_n(CC(\mathcal{C},p)) \neq Ob_n(CC(pt_{\mathcal{C}},p))$ appears at $n = 1$ when one has:

$$Ob_1(CC(\mathcal{C},p)) = \amalg_{Hom(pt,pt)} Hom(pt,U)$$

and

$$Ob_1(CC(pt_{\mathcal{C}},p)) = Hom(pt,U)$$

One could remove this discrepancy by changing the definition of $CC(X,p)$ or the definition of $CC(\mathcal{C},p)$. However, if we were to change the definition of $CC(X,p)$, the convenient equality of Lemma 3.1.10 would not hold anymore and the fact that the bijections $u_{1,\Gamma}$ and $u_{2,\Gamma}$ coincide with the bijections with the same name introduced previously would disappear as well. Changing the definition of $CC(\mathcal{C},p)$ would mean doing backward changes in all of the papers that depend on this definition. In any case, since the functions $u_{n,\Gamma}$ are not identities if $l(\Gamma) > 0$ it does not seem to be of much importance to try to ensure that they are identities for $l(\Gamma) = 0$.

Problem 3.1.8 *[***2015.06.23.prob1***] Let $f : X' \to X$ be a morphism. To define a function*

$$f^* : Ob_n(X) \to Ob_n(X')$$

and, for each $A \in Ob_n(A)$ a morphism

$$Q(f,A) : c(f^*(A)) \to c(A)$$

Construction 3.1.9 [**2015.06.23.constr1**] We define f^* and $Q(f,A)$ by induction on n as follows:

1. $f^*(pt) = pt$ and $Q(f,pt) = f$,
2. for $F \in Ob_1(X) = Hom(X,U)$ we set

$$f^*(F) = f \circ F$$

$$Q(f,(F)) = Q(f,F)$$

3. for $(A, F) \in Ob_{n+1}(X)$ we set

$$f^*(A, F) = (f^*(A), Q(f, A) \circ F)$$

$$Q(f, (A, F)) = Q(Q(f, A), F)$$

where the notation $Q(f, F)$ was introduced in [7, Section 3] as the unique morphism $(X'; f \circ F) \to (X; F)$ such that

$$[\mathbf{2015.07.09.eq1}]Q(f, F) \circ Q(F) = Q(f \circ F) \tag{31}$$

and

$$[\mathbf{2015.07.09.eq2}]Q(f, F) \circ p_{X,F} = p_{X',f\circ F} \circ f \tag{32}$$

This can be illustrated by the diagram

$$\begin{array}{ccccccc}
c(f^*(A), Q(f, A) \circ F) & = & (c(f^*(A)); Q(f, A) \circ F) & \xrightarrow{Q((f,A),F)} & (c(A); F) & \xrightarrow{Q(F)} & \widetilde{U} \\
{\scriptstyle p_{(f^*(A),Q(f,A)\circ F)}} \downarrow & & {\scriptstyle p_{c(f^*(A)),Q(f,A)\circ F}} \downarrow & & \downarrow {\scriptstyle p_{c(A),F}} & & \downarrow {\scriptstyle p} \\
c(f^*(A)) & = & c(f^*(A)) & \xrightarrow{Q(f,A)} & c(A) & \xrightarrow{F} & U
\end{array}$$

Recall that the functor data $D_p(-, V)$ was introduced in [7, above Problem 3.5].

Lemma 3.1.10 *[**2015.06.25.l8**] For any X one has*

$$[\mathbf{2015.06.25.eq1a}]Ob_2(X) = D_p(X, U) \tag{33}$$

and for any $f : X' \to X$ one has $f^* = D_p(f, U)$.

Proof: These equalities are obtained by unfolding definitions.

In what follows we leave the proofs of special case of $n = 1$ to the formalized version of the paper and only consider the induction from n to $n + 1$ for $n \geq 1$.

Lemma 3.1.11 *[**2015.06.23.l1**] Let $f : X' \to X$ be a morphism. Then one has:*

1. *the functions $f^* : Ob_n(X) \to Ob_n(X')$ commute with ft, i.e., $f^*(ft(A)) = ft(f^*(A))$,*
2. *if B is an object over A in $CC(X, p)$ then $f^*(B)$ is an object over $f^*(A)$ in $CC(X', p)$.*

Proof: The first assertion is easily proved by induction on n. The second is proved from the first by induction on $l(A) - l(B)$.

Lemma 3.1.12 *[**2015.06.23.l2**] Let $A, B \in Ob(CC(X,p))$ and suppose that B is an object over A. Then the square*

$$[\mathbf{2015.06.23.eq2}] \quad \begin{array}{ccc} c(f^*(B)) & \xrightarrow{Q(f,B)} & c(B) \\ {\scriptstyle p(f^*(B),f^*(A))}\downarrow & & \downarrow{\scriptstyle p(B,A)} \\ c(f^*(A)) & \xrightarrow{Q(f,A)} & c(A) \end{array} \tag{34}$$

is a pull-back square in $\mathcal{C}$.

Proof: By induction over $l(B) - l(A)$ and using the fact that the vertical composition of two pull-back squares is a pull-back square it remains to show that the square

$$\begin{array}{ccc} c(f^*(B)) & \xrightarrow{Q(f,B)} & c(B) \\ {\scriptstyle p(f^*(B),ft(f^*(B)))}\downarrow & & \downarrow{\scriptstyle p(B,ft(B))} \\ c(ft(f^*(B))) & \xrightarrow{Q(f,ft(B))} & c(ft(B)) \end{array}$$

is a pull-back square. If $l(B) = 0$ then it is obvious. If $l(B) > 0$ then this square is of the form of the middle square in the diagram in Construction 3.1.9 that is a pull-back square.

Problem 3.1.13 *[**2015.06.23.prob2**] Let $f : X' \to X$ be a morphism and let $a : A \to B$ be a morphism in $CC(X,p)$. To define a morphism $f^*(a) : f^*(A) \to f^*(B)$ in $CC(X',p)$.*

Construction 3.1.14 [**2015.06.23.constr2**] Morphisms of the form $f^*(A) \to f^*(B)$ in $CC(X',p)$ are, by definition, morphisms of the form $b : c(f^*(A)) \to c(f^*(B))$ in $\mathcal{C}$ such that $b \circ p(f^*(B), pt) = p(f^*(A), pt)$. By Lemma 3.1.12 the square

$$[\mathbf{2015.06.23.eq3}] \quad \begin{array}{ccc} c(f^*(B)) & \xrightarrow{Q(f,B)} & c(B) \\ {\scriptstyle p(f^*(B),pt)}\downarrow & & \downarrow{\scriptstyle p(B,pt)} \\ X' & \xrightarrow{f} & X \end{array} \tag{35}$$

is a pull-back square. Therefore there is a unique morphism $f^*(a) : c(f^*(A)) \to c(f^*(B))$ over X' such that

$$[\mathbf{2015.06.23.eq4}] f^*(a) \circ Q(f,B) = Q(f,A) \circ a \tag{36}$$

This completes the construction.

Lemma 3.1.15 *[**2015.06.23.l5**] Let $f : X' \to X$ be a morphism and $a : A \to B$, $b : B \to C$ be a pair of morphisms in $CC(X,p)$. Then one has $f^*(a \circ b) = f^*(a) \circ f^*(b)$.*

Proof: The sets of morphisms in $CC(X',p)$ are defined as the sets of morphisms in $\mathcal{C}$ that satisfy the condition of being morphisms over X'. Therefore it is sufficient to prove this equation as an equation between morphisms in $\mathcal{C}$. As a morphism in $\mathcal{C}$, $f^*(a \circ b)$ is defined as the unique morphism $c(f^*(A)) \to c(f^*(C))$ which is a morphism over X' relative to the morphisms $p(f^*(A), pt)$ and $p(f^*(C), pt)$ and such that

$$f^*(a \circ b) \circ Q(f, C) = Q(f, A) \circ (a \circ b)$$

The composition $f^*(a) \circ f^*(b)$ is easily proved to be a morphism over X'. Therefore it remains to verify that

$$f^*(a) \circ f^*(b) \circ Q(f, C) = Q(f, A) \circ (a \circ b)$$

We have

$$f^*(a) \circ f^*(b) \circ Q(f, C) = f^*(a) \circ Q(f, B) \circ b = Q(f, A) \circ a \circ b$$

Lemma is proved.

Lemma 3.1.16 *[**2015.06.23.l5a**] Let $f : X' \to X$ be a morphism and B an object over A in $CC(X, p)$. Then*

$$f^*(p(B, A)) = p(f^*(B), f^*(A))$$

Proof: By definition, $f^*(p(B, A))$ is the unique morphism $c(f^*(B)) \to c(f^*(A))$ over X' such that $f^*(p(B, A)) \circ Q(f, A) = Q(f, B) \circ p(B, A)$. Therefore we need to show that $p(f^*(B), f^*(A)) \circ Q(f, A) = Q(f, B) \circ p(B, A)$ which is equivalent to the commutativity of the square (34).

Lemma 3.1.17 *[**2015.06.23.l6**] Let $f : X' \to X$ be a morphism and $A, B \in Ob(CC(X, p))$. Then for a morphism $a : A \to B$ in $CC(X, p)$ and $F : B \to U$ one has*

$$f^*(q(a, (B, F))) = q(f^*(a), f^*(B, F))$$

Proof: Let $Y = c(A)$, $Z = c(B)$. Let further $Y' = c(f^*(A))$ and $Z' = c(f^*(B))$ and Then $c(a^*(B, F)) = (Y, a \circ F)$ and $q(a, (B, F)) = Q(a, F)$ as can be seen on the following diagram

$$\begin{array}{ccccc} (Y; a \circ F) & \xrightarrow{Q(a,F)} & (Z; F) & \xrightarrow{Q(F)} & \widetilde{U} \\ {\scriptstyle p_{a^*(B,F)}}\downarrow & & \downarrow{\scriptstyle p_{B,F}} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{a} & Z & \xrightarrow{F} & U \end{array}$$

The pull-back by f^* of the left hand side square of this diagram is

$$\begin{array}{ccc} (Y'; Q(f, A) \circ a \circ F) & \xrightarrow{f^*(q(a,(B,F)))} & (Z'; Q(f, B) \circ F) \\ {\scriptstyle f^*(p_{a^*(B,F)})}\downarrow & & {\scriptstyle f^*(p_{B,F})}\downarrow \\ Y' & \xrightarrow{f^*(a)} & Z' \end{array}$$

The morphism $q(f^*(a), f^*(B,F))$ on the other hand is a part of the square

$$\begin{array}{ccc}
(Y'; f^*(a) \circ Q(f,B) \circ F) & \xrightarrow{q(f^*(a), f^*(B,F))} & (Z'; Q(f,B) \circ F) \\
{\scriptstyle f^*(p_{a^*(B,F)})} \downarrow & & {\scriptstyle f^*(p_{B,F})} \downarrow \\
Y' & \xrightarrow{f^*(a)} & Z'
\end{array}$$

We have $f^*(a) \circ Q(f,B) = Q(f,A) \circ a$ by (36) that shows that the domains of $f^*(q(a,(B,F))$ and $q(f^*(a), f^*(B,F))$ coincide. We also have

$$[\mathbf{2015.06.23.eq6}] q(f^*(a), f^*(B,F)) = Q(f^*(a), Q(f,B) \circ F) \tag{37}$$

by definition of canonical squares in $CC(X', p)$. It remains to show that

$$f^*(q(a,(B,F)) = Q(f^*(a), Q(f,B) \circ F)$$

By definition, $f^*(q(a,(B,F)))$ is the only morphism from $c(f^*(a^*(B,F)))$ to $c(f^*(B,F))$ over X' such that

$$f^*(q(a,(B,F)) \circ Q(f,(B,F)) = Q(f, a^*(B,F)) \circ q(a,B,F) = Q(f, a^*(B,F)) \circ Q(a,F)$$

We know that $q(f^*(a), f^*(B,F))$ is a morphism over X'. In view of (37) it remains to show that

$$Q(f^*(a), Q(f,B) \circ F) \circ Q(f,(B,F)) = Q(f, a^*(B,F)) \circ Q(a,F)$$

We have

$$Q(f^*(a), Q(f,B) \circ F) \circ Q(f,(B,F)) = Q(f^*(a), Q(f,B) \circ F) \circ Q(Q(f,B), F) =$$

$$Q(f^*(a) \circ Q(f,B), F) = Q(Q(f,A) \circ a, F)$$

where the first equality holds by Construction 3.1.9, the second one by [7, Lemma 3.2] and the third one by (36). On the other hand

$$Q(f, a^*(B,F)) \circ Q(a,F) = Q(f,(A, a \circ F)) \circ Q(a,F) = Q(Q(f,A), a \circ F) \circ Q(a,F) =$$

$$Q(Q(f,A) \circ a, F)$$

where the first equality holds by the construction of a^* in $CC(X,p)$, the second by Construction 3.1.9 and the third by [7, Lemma 3.2]. Lemma is proved.

Theorem 3.1.18 *[***2015.06.23.th1***] Let $f : X' \to X$ be a morphism. Then the functions $f^* : Ob(CC(X,p)) \to Ob(CC(X',p))$ and $f^* : Mor(CC(X,p)) \to Mor(CC(X',p))$ constructed in Constructions 3.1.9 and 3.1.14 form a homomorphism of C-systems that we also denote f^*.*

Proof: The function f^* on objects commutes with the length function and with ft by construction. The function on morphisms satisfies the property that $f^*(p_A) = p_{f^*(A)}$ by Lemma 3.1.16. It is also very easy to prove that f^* takes identity morphisms to identity morphisms. That f^* commutes with compositions follows from Lemma 3.1.15 and the final remaining axiom of a homomorphism of C-systems follows from Lemma 3.1.17.

Lemma 3.1.19 *[2015.06.23.l7] Let $g : X'' \to X'$ and $f : X' \to X$ be two morphisms then one has:*

1. *for any $A \in Ob_n(X)$ one has $(g \circ f)^*(A) = g^*(f^*(A))$ and*

$$[\mathbf{2015.06.25.eq1}]Q(g \circ f, A) = Q(g, f^*(A)) \circ Q(f, A) \tag{38}$$

2. *for any $a : A \to B$ in $Mor(CC(X, p))$ one has $(g \circ f)^*(a) = g^*(f^*(a))$.*

Proof: The proof of the first assertion (goal) is by induction on n. For $n = 0$ the assertion is obvious. Let $(A, F) \in Ob_{n+1}(X)$. Then one has

$$(g \circ f)^*(A, F) = ((g \circ f)^*(A), Q(g \circ f, A) \circ F) = (g^*(f^*(A)), Q(g, f^*(A)) \circ Q(f, A) \circ F) =$$

$$g^*((f^*(A), Q(f, A) \circ F)) = g^*(f^*(A, F))$$

where the first equation is by Construction 3.1.9, the second one by inductive assumption and the third and fourth ones again by Construction 3.1.9.

For the equation (38) one has

$$Q(g \circ f, (A, F)) = Q(Q(g \circ f, A), F) = Q(Q(g, f^*(A)) \circ Q(f, A), F) =$$

$$Q(Q(g, f^*(A)), Q(f, A) \circ F) \circ Q(Q(f, A), F) = Q(Q(g, f^*(A)), Q(f, A) \circ F) \circ Q(f, (A, F))$$

Where the first equality holds by Construction 3.1.9, the second by the inductive assumption, the third by [7, Lemma 3.2] and the fourth again by Construction 3.1.9. On the other hand:

$$Q(g, f^*(A, F)) = Q(g, (f^*(A), Q(f, A) \circ F)) = Q(Q(g, f^*(A)), Q(f, A) \circ F)$$

where both equalities hold by Construction 3.1.9. This completes the proof of the first assertion of the lemma.

To prove the second assertion recall that $(g \circ f)^*(a)$ is defined in Construction 3.1.14 as the unique morphism over X'' such that

$$(g \circ f)^*(a) \circ Q(g \circ f, B) = Q(g \circ f, A) \circ a$$

Since $g^*(f^*(a))$ is by construction a morphism over X'' it remains to verify that

$$g^*(f^*(a)) \circ Q(g \circ f, B) = Q(g \circ f, A) \circ a$$

We have

$$g^*(f^*(a)) \circ Q(g \circ f, B) = g^*(f^*(a)) \circ Q(g, f^*(B)) \circ Q(f, B) = Q(g, f^*(A)) \circ f^*(a) \circ Q(f, B) =$$

$$Q(g, f^*(A)) \circ Q(f, A) \circ a = Q(g \circ f, a) \circ a$$

where the first equality is by (38), the second and the third by (36) and the fourth by (38).

This completes the proof of the lemma.

Let now $\mathcal{C}$ be a locally cartesian closed category. Recall that we let $I_p(V)$ denote the object

$$I_p(V) = \underline{Hom}_U((\widetilde{U}, p), (U \times V, pr_1))$$

and $prI_p(V)$ denote the canonical morphism $p \triangle pr_1$ from this object to U.

When $V = U$, this morphism defines an element A_1 of $Ob_1(I_p(U))$ and $c(A_1) = (I_p(U); prI_p(U))$. Note that A_1 is just a notation for $prI_p(U)$.

Next, we have the evaluation morphism

$$ev^{(\widetilde{U},p)}_{(U\times V, pr1)} : (I_p(V), prI_p(V)) \times_U (\widetilde{U}, p) \to U \times V$$

and in [7] we let

$$st_p(V) : (I_p(V); prI_p(V)) \to V$$

denote the composition $\iota_{prI_p(V)} \circ ev^{(\widetilde{U},p)}_{(U\times V, pr1)} \circ pr_2$ where for $F : X \to U$,

$$\iota_F = \langle p_{X,F}, Q(F)\rangle : (X; F) \to (X, F) \times_U (\widetilde{U}, p)$$

is the obvious isomorphism.

For $V = U$ the morphism $st_p(U)$ defines an element

$$[\mathbf{2015.07.03.eq1}] A_2 = (prI_p(U), st_p(U)) \qquad (39)$$

in $Ob_2(I_p(U))$ such that $ft(A_2) = A_1$ and $c(A_2) = (I_p(U); prI_p(U), st_p(U))$.

In [7, Construction 3.9] we introduced bijections

$$\eta^!_{X,V} : Hom(X, I_p(V)) \to D_p(X, V)$$

natural in X and V. In particular, $\eta^!_{X,U}$ are natural in X bijections from $Hom(X, I_p(U))$ to $D_p(X, U)$. By Lemma 3.1.10 we have $D_p(-, U) = Ob_2(-)$ as functor data.

Lemma 3.1.20 *[**2015.06.25.l9**] Let $f : X \to I_p(U)$ be a morphism. Then $\eta^!_{X,U}(f) = f^*(A_2)$.*

Proof: The naturality in X means that for any $g : Y \to X$ one has

$$\eta^!_{Y,U}(g \circ f) = g^*(\eta^!_{X,U}(f))$$

In particular,

$$\eta^!(f) = \eta^!(f \circ Id_{I_p(U)}) = f^*(\eta^!(Id_{I_p(U)}))$$

Unfolding the definition of $\eta^!$ given in [7, Construction 3.9] we see that

$$\eta^!(Id_{I_p(U)}) = (Id \circ prI_p(U), Q(Id, prI_p(U)) \circ st_p(U))$$

therefore $\eta^!(Id_{I_p(U)}) = A_2$ by rewriting using the fact that $Q(Id_X, F) = Id_{(X;F)}$ and the unity axiom of the category structure.

Recall from [7, Construction 3.12] that for $\Gamma \in Ob(CC(\mathcal{C}, p))$ we let

$$\mu_\Gamma : Ob_2(\Gamma) \to Hom(int(\Gamma), I_p(U))$$

denote the composition $u_{2,\Gamma} \circ \eta_\Gamma$ where η_Γ is the inverse to $\eta^!_{int(\Gamma),U}$.

Lemma 3.1.21 *[2015.06.25.l1] For any Γ and $B \in Ob_2(\Gamma)$ one has*

$$c(\mu_\Gamma(B)^*(A_2)) = int(B)$$

Proof: We have

$$c(\mu_\Gamma(B)^*(A_2)) = c(\eta_\Gamma(u_{2,\Gamma}(B))^*(A_2)) = c(\eta^!_{int(\Gamma),U}(\eta_\Gamma(u_{2,\Gamma}(B)))) =$$

$$c(u_{2,\Gamma}(B)) = int(B)$$

where the second equality holds by Lemma 3.1.20 and the fourth one by (29).

3.2 Σ-structures on a universe in a locally cartesian closed category

[**sec2**] In what follows $\mathcal{C}$ is a locally cartesian closed category and p is a universe in $\mathcal{C}$.

Definition 3.2.1 *[2015.07.05.def1] A 2-to-1 structure on p is a morphism $I_p(U) \to U$ or, equivalently, an element in $Ob_1(I_p(U))$.*

Definition 3.2.2 *[2015.05.28.def2] A weak Σ-structure on p is a collection of data of the following form:*

1. *a 2-to-1 structure $\Sigma p \in Ob_1(I_p(U))$,*
2. *a morphism $rp : A_2 \to \Sigma p$ in $CC(I_p(U), p)$,*
3. *a morphism $dp : \Sigma_p \to A_2$ in $CC(I_p(U), p)$.*

such that $rp \circ dp = Id$. If in addition one has $dp \circ rp = Id$ then this Σ-structure is called a strong Σ-structure on p.

Problem 3.2.3 *[2015.05.28.prob1] Given a weak (resp. strong) Σ-structure on p to construct a weak (resp. strong) Σ-structure on $CC(\mathcal{C}, p)$.*

Construction 3.2.4 [**2015.05.28.constr1**] The formula

$$[\mathbf{2015.07.05.eq3}]\Sigma(B) = u^{-1}_{1,\Gamma}(\mu_\Gamma(B)^*(\Sigma p)) \tag{40}$$

defines a map

$$\Sigma : Ob_2(\Gamma) \to Ob_1(\Gamma)$$

for any Γ. As was verified in [7, Construction 4.3] maps constructed in this way are compatible with the base change along maps $\Gamma' \to \Gamma$.

By (30) we have

$$[\mathbf{2015.06.27.eq1}]int(\Sigma(B)) = c_{1,int(\Gamma)}(\mu(B)^*(\Sigma p)) \quad (41)$$

On the other hand

$$[\mathbf{2015.06.27.eq2}]int(B) = c_{1,int(\Gamma)}(\mu(B)^*(A_2)) \quad (42)$$

by Lemma 3.1.21. Both rp and dp are morphisms in $CC(I_p(U), p)$. Applying to them homomorphism of C-systems $\mu(B)^*$ constructed in Theorem 3.1.18 and using equations (41) and (42) we obtain morphisms

$$r' : int(B) \to int(\Sigma(B))$$

$$d' : int(\Sigma(B)) \to int(B)$$

over $int(\Gamma)$ that can be seen as morphisms $r' : B \to \Sigma(B)$ and $d' : \Sigma(B) \to B$ in $CC(\mathcal{C}, p)$. Since $\mu(B)^*$ is a homomorphism of C-systems it in particular commutes with composition of morphisms and the equality $rp \circ dp = Id$ implies the equality $r' \circ d' = Id$. Similarly for a strong Σ-structure the equality $dp \circ rp = Id$ implies $d' \circ r' = Id$.

It remains to verify that the morphisms $r' = r'_B$ and $d' = d'_B$ are stable under the pull-back along morphisms $f : \Gamma' \to \Gamma$. Let us do it for the morphisms r'_B. We have

$$f^*(r'_B) = f^*(\mu_\Gamma(B)^*(rp)) = (f \circ \mu_\Gamma(B))^*(rp) = \mu_{\Gamma'}(f^*(B))^*(rp) = r_{f^*(B)}$$

where the second equality holds by Lemma 3.1.19 and the third equality holds because the bijections μ are natural in Γ (see [7, Construction 3.12]).

4 Functoriality of Σ-structures

4.1 General comments on universe category functors

Let us recall that a functor of universe categories is a triple

$$\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi}) : (\mathcal{C}, p) \to (\mathcal{C}', p')$$

where $\Phi : \mathcal{C} \to \mathcal{C}'$ is a functor and ϕ and $\widetilde{\phi}$ are morphisms that are elements of a pull-back square of the form

$$\begin{array}{ccc} \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

and such that Φ takes the canonical squares of $(\mathcal{C}, p)$ to pull-back squares and $\Phi(pt)$ is a final object of $\mathcal{C}$. A functor of universe categories defines a homomorphism of C-systems

$$H = H(\mathbf{\Phi}) : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

constructed in [5, Construction 3.3].

Problem 4.1.1 *[**2015.06.27.prob1**] To construct, for all $n \geq 0$ and all $X \in \mathcal{C}$ a pair $(H_{X,n}, \gamma_X)$ where*

$$H_{X,n} : Ob_n(X) \to Ob_n(\Phi(X))$$

is a function and for any $A \in Ob_n(X)$,

$$\gamma_X(A) : c_{\Phi(X),n}(H_{X,n}(A)) \to \Phi(c_{X,n}(A))$$

an isomorphism in $\mathcal{C}'$.

Construction 4.1.2 [**2015.06.27.constr1**] Construction is by induction on n:

1. for $n = 0$ we set $H_{X,0}(pt) = pt$ and $\gamma_{X,pt} = Id_{\Phi(X)}$,
2. for $n = 1$ we set $H_{X,1}(F) = \Phi(F) \circ \phi$ and let

   $$\gamma_X(F) : (\Phi(X); H_X(F)) \to \Phi(X; F)$$

   to be the unique morphism such that

   $$\gamma_X(F) \circ \Phi(p_{X,F}) = p_{\Phi(X),\Phi(F)\circ\phi}$$

   $$\gamma_X(F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = Q(\Phi(F) \circ \phi)$$

   as can be seen on the diagram

   $$\begin{array}{ccccccc} (\Phi(X); H_X(F)) & \xrightarrow{\gamma_X(F)} & \Phi(X;F) & \xrightarrow{\Phi(Q(F))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle p_{H_X(F)}}\downarrow & & {\scriptstyle \Phi(p_F)}\downarrow & & {\scriptstyle \Phi(p)}\downarrow & & \downarrow{\scriptstyle p'} \\ \Phi(X) & = & \Phi(X) & \xrightarrow{\Phi(F)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array}$$

3. for the successor of n we set

   $$H_{X,n+1}(A, F) = (H_{X,n}(A), \gamma_X(A) \circ \Phi(F) \circ \phi)$$

   To simplify the notation we will write H_X instead of $H_{X,n}$ since n can be uniquely inferred from the argument. We let $\gamma_X(A, F)$ to be the unique morphism such that

   [**2015.06.27.eq3**] $$\gamma_X(A, F) \circ \Phi(p_{(A,F)}) = p_{H_X(A,F)} \circ \gamma_X(A) \qquad (43)$$

   [**2015.06.27.eq4**] $$\gamma_X(A, F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = Q(\gamma_X(A) \circ \Phi(F) \circ \phi) \qquad (44)$$

The corresponding diagram looks as follows:

[**2015.07.06.eq1**]
$$\begin{array}{ccccccc} c(H_X(A,F)) & \xrightarrow{\gamma_X(A,F)} & \Phi(c(A,F)) & \xrightarrow{\Phi(Q(F))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\ {\scriptstyle p_{H_X(A,F)}}\downarrow & & \downarrow{\scriptstyle \Phi(p_{(A,F)})} & & \downarrow{\scriptstyle \Phi(p)} & & \downarrow{\scriptstyle p'} \\ c(H_X(A)) & \xrightarrow{\gamma_X(A)} & \Phi(c(A)) & \xrightarrow{\Phi(F)} & \Phi(U) & \xrightarrow{\phi} & U' \end{array} \qquad (45)$$

That the equations (43) and (44) make sense and that there exists a unique morphism $\gamma_X(A,F)$ that satisfies these equations follows from the diagram (45). The two right hand side squares in this diagram are pull-back squares by definition of a universe category functor. Therefore, their union is a pull-back square. This implies the uniqueness of $\gamma_X(A,F)$. The existence follows from the commutativity of the canonical square for $\gamma_X(A) \circ \Phi(F) \circ \phi$.

Next, observe that the left hand square of the diagram is also a pull-back square. Indeed, the external square is the canonical square for $\gamma_X(A) \circ \Phi(F) \circ \phi$ and therefore a pull-back square and as we already observed the composition of the two right hand side squares is a pull-back square. Therefore, the left hand side square is a pull-back square.

By the inductive assumption, $\gamma_X(A)$ is an isomorphism and we conclude that $\gamma_X(A,F)$ is an isomorphism as the pull-back of an isomorphism.

Also, we leave the proofs of the special case $n = 1$ for the formalized version of the paper considering only the inductive steps from n to $n+1$ when $n \geq 1$.

Problem 4.1.3 *[***2015.06.27.prob2***] To construct for all $X \in \mathcal{C}$ and all $a : A \to B$ in $CC(X,p)$ a morphism*

$$H_X(a) : H_X(A) \to H_X(B)$$

in $CC(\Phi(X),p')$.

Construction 4.1.4 [**2015.06.27.constr2**] We set

$$H_X(a) = \gamma_X(A) \circ \Phi(a) \circ \gamma_X(B)^{-1}$$

such that we obtain a commutative square

[2015.07.01.eq1]
$$\begin{array}{ccc} c(H_X(A)) & \xrightarrow{\gamma_X(A)} & \Phi(c(A)) \\ {\scriptstyle H_X(a)}\downarrow & & \downarrow{\scriptstyle \Phi(a)} \\ c(H_X(B)) & \xrightarrow{\gamma_X(B)} & \Phi(c(B)) \end{array} \qquad (46)$$

Lemma 4.1.5 *[***2015.06.27.l1***] The maps H_X on objects and morphisms form functor from the underlying (pre-)category of the C-system $CC(X,p)$ to the underlying (pre-)category of the C-system $CC(\Phi(X),p')$.*

Proof: One clearly has

$$H_X(Id_A) = Id_{H_X(A)}$$

It remains to show that H_X commutes with compositions. Indeed, for $a : A \to B$ and $b : B \to C$ one has

$$H_X(a \circ b) = \gamma_X(A) \circ \Phi(a \circ b) \circ \gamma_X(C)^{-1} = \gamma_X(A) \circ \Phi(a) \circ \gamma_X(B)^{-1} \circ \gamma_X(B) \circ \Phi(b) \circ \gamma_X(C)^{-1} =$$

$$H_X(a) \circ H_X(b)$$

Lemma 4.1.6 *[**2015.06.27.l2**] For $(A, F) \in Ob_{n+1}(X)$ one has*

$$H_X(p_{(A,F)}) = p_{H_X(A,F)}$$

Proof: We have

$$H_X(p_{(A,F)}) = \gamma_X(A,F) \circ \Phi(p_{c(A),F}) \circ \gamma_X(A)^{-1} = p_{c(H_X(A)),\gamma_X(A)\circ\Phi(F)\circ\phi} \circ \gamma_X(A) \circ \gamma_X(A)^{-1} =$$

$$p_{H_X(A,F)}$$

where the second equality holds by (43).

Lemma 4.1.7 *[**2015.06.27.l3**] For $a : A \to B$ one has*

$$[\mathbf{2015.06.27.eq5}]H_X(a^*(B,F)) = H_X(a)^*(H_X(B,F)) \tag{47}$$

and

$$[\mathbf{2015.06.27.eq6}]H_X(q(a,(B,F))) = q(H_X(a), H_X(B,F)) \tag{48}$$

Proof: In the proof we will write H instead of H_X. We have

$$H(a^*(B,F)) = H((A, a \circ F)) = (H(A), \gamma_X(A) \circ \Phi(a \circ F) \circ \phi)$$

On the other hand

$$H(a)^*(H(B,F)) = (\gamma_X(A) \circ \Phi(a) \circ \gamma_X(B)^{-1})^*(H(B), \gamma_X(B) \circ \Phi(F) \circ \phi) =$$

$$(H(A), \gamma_X(A) \circ \Phi(a) \circ \gamma_X(B)^{-1} \circ \gamma_X(B) \circ \Phi(F) \circ \phi) = (H(A), \gamma_X(A) \circ \Phi(a \circ F) \circ \phi)$$

This proves equation (47). Consider now the equation (48). Since $\gamma_X(B,F)$ is an isomorphism it is sufficient to show that the compositions of both sides with this morphism are the same. The codomain of both compositions is $\Phi(c(B,F))$, which is a part of the diagram

$$\begin{array}{ccccc}
\Phi(c(B,F)) & \xrightarrow{\Phi(Q(F))} & \Phi(\widetilde{U}) & \xrightarrow{\widetilde{\phi}} & \widetilde{U}' \\
{\scriptstyle \Phi(p_{(B,F)})}\downarrow & & \downarrow{\scriptstyle \Phi(p)} & & \downarrow{\scriptstyle p'} \\
\Phi(c(B)) & \xrightarrow{\Phi(F)} & \Phi(U) & \xrightarrow{\phi} & U'
\end{array}$$

The two squares of this diagram are pull-back squares and therefore so is the external square and $\Phi(c(B,F))$ is a fiber product with projections $\Phi(Q(F)) \circ \widetilde{\phi}$ and $\Phi(p_{c(B),F})$. Therefore, to prove (48) it is sufficient to prove that both sides agree after further composition with the projections, i.e., we need to prove two equalities

$$H(q(a,(B,F))) \circ \gamma_X(B,F) \circ \Phi(p_{(B,F)}) =$$

$$[\mathbf{2015.07.01.eq2}]q(H(a), H(B,F)) \circ \gamma_X(B,F) \circ \Phi(p_{(B,F)}) \tag{49}$$

and

$$H(q(a,(B,F))) \circ \gamma_X(B,F) \circ \Phi(Q(F)) \circ \widetilde{\phi} =$$

$$[\mathbf{2015.07.01.eq3}]q(H(a), H(B,F)) \circ \gamma_X(B,F) \circ \Phi(Q(F)) \circ \widetilde{\phi} \tag{50}$$

Observe further that the square

$$\begin{array}{ccc} c(H(B,F)) & \xrightarrow{\gamma(B,F)} & \Phi(c(B(F))) \\ {\scriptstyle p_{H(B,F)}}\downarrow & & \downarrow{\scriptstyle \Phi(p_{(B,F)})} \\ c(H(B) & \xrightarrow{\gamma(B)} & \Phi(c(B)) \end{array}$$

commutes as a particular case of square (46) for $a = p_{(B,F)}$ since

$$H(p_{(B,F)}) = p_{H(B,F)}$$

The square

$$\begin{array}{ccc} c(H(a^*(B,F))) & \xrightarrow{H(q(a,(B,F)))} & c(H(B,F)) \\ {\scriptstyle p_{H(a^*(B,F))}}\downarrow & & \downarrow{\scriptstyle p_{H(B,F)}} \\ c(H(A)) & \xrightarrow{H(a)} & c(H(B)) \end{array}$$

also commutes which can be proved by rewriting using the fact that H commutes with compositions and satisfies the p-morphism axiom (Lemmas 4.1.5 and 4.1.6).

Therefore we have

$$H(q(a,(B,F))) \circ \gamma_X(B,F) \circ \Phi(p_{(B,F)}) = H(q(a,(B,F))) \circ p_{H(B,F)} \circ \gamma_X(B) =$$

$$p_{H(a^*(B,F))} \circ H(a) \circ \gamma_X(B) = q(H(a), H(B,F)) \circ p_{H(B,F)} \circ \gamma_X(B) =$$
$$q(H(a), H(B,F)) \circ \gamma_X(B,F) \circ \Phi(p_{(B,F)})$$

which proves (49).

Next we have

$$H(q(a,(B,F))) \circ \gamma_X(B,F) \circ \Phi(Q(F)) \circ \widetilde{\phi} = \gamma(a^*(B,F)) \circ \Phi(q(a,(B,F))) \circ \Phi(Q(F)) \circ \widetilde{\phi} =$$

$$\gamma(a^*(B,F)) \circ \Phi(Q(a,F)) \circ \Phi(Q(F)) \circ \widetilde{\phi} =$$
$$\gamma(a^*(B,F)) \circ \Phi(Q(a \circ F)) \circ \widetilde{\phi} = \gamma(A, a \circ F) \circ \Phi(Q(a \circ F)) \circ \widetilde{\phi} =$$
$$= Q(\gamma(A) \circ \Phi(a \circ F) \circ \phi) = Q(\gamma(A) \circ \Phi(a) \circ \Phi(F) \circ \phi)$$

where the first equality holds by (46), the second holds since $q(a,(B,F)) = Q(a,F)$ by definition, the third hold by (31) and the fifth one by (44).

On the other hand

$$q(H(a), H(B,F)) \circ \gamma_X(B,F) \circ \Phi(Q(F)) \circ \widetilde{\phi} =$$

$$Q(H(a), \gamma(B) \circ \Phi(F) \circ \phi) \circ \gamma_X(B,F) \circ \Phi(Q(F)) \circ \widetilde{\phi} =$$
$$Q(H(a), \gamma(B) \circ \Phi(F) \circ \phi) \circ Q(\gamma(B) \circ \Phi(F) \circ \phi) = Q(H(a) \circ \gamma(B) \circ \Phi(F) \circ \phi)$$

where the first equality holds by definition of q-morphisms in $CC(\Phi(X), p')$, the second by (44) and the third by definition of $Q(-, F)$.

It remains to use the fact that

$$\gamma(A) \circ \Phi(a) = H(a) \circ \gamma(B)$$

by (46). The proof of Lemma 4.1.7 is completed.

Theorem 4.1.8 *[**2015.06.27.th1**] The maps H_X between the sets of objects and morphisms of the categories $CC(X, p)$ and $CC(\Phi(X), p')$ defined in Constructions (4.1.2) and (4.1.4) form a homomorphism of C-systems*

$$H_X : CC(X, p) \to CC(\Phi(X), p')$$

Proof: In view of [5, Lemma 3.4] it is sufficient to check the first five conditions of [5, Definition 3.1]. The first two are satisfied by construction. That H_X is a functor is proved in Lemma 4.1.5. That it satisfied the p-morphisms condition is proved in Lemma 4.1.6. That it satisfies the q-morphisms condition is proved in Lemma 4.1.7.

Recall that in [7, Construction 5.2] we defined functions

$$\mathbf{\Phi}^2 : D_p(X, V) \to D_{p'}(\Phi(X), \Phi(U))$$

and that in Lemma 3.1.10 we have shown that

$$D_p(X, U) = Ob_2(X)$$

Lemma 4.1.9 *[**2015.07.03.l1**] For a universe category functor $\mathbf{\Phi}$, $X \in \mathcal{C}$ and $A \in Ob_2(X)$ one has*

$$H_X(A) = \mathbf{\Phi}^2(A) \circ \phi$$

Proof: We have $A = (F_1, F_2)$ where $F_1 : X \to U$ and $F_2 : (X; F_1) \to U$. By Construction 4.1.2 we have

$$H_X(F_1, F_2) = (H_X(F_1), \gamma_X(F_1) \circ \Phi(F_2) \circ \phi) = (\Phi(F_1) \circ \phi, \gamma_X(F_1) \circ \Phi(F_2) \circ \phi)$$

On the other hand

$$\Phi^2(F_1, F_2) \circ \phi = D_{p'}(\Phi(X), \phi)(\Phi^2(F_1, F_2)) = D_{p'}(\Phi(X), \phi)(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2)) =$$

$$(\Phi(F_1) \circ \phi, \iota \circ \Phi(F_2) \circ \phi)$$

It remains to observe that for $F : X \to U$ the morphism $\gamma_X(F)$ of Construction 4.1.2 is equal to the morphism ι of [7, Construction 5.2] in view of their definitions as unique morphisms satisfying the same pair of equations.

Lemma 4.1.10 *[***2015.07.05.l5***] Let* **Φ** *be a universe category functor and let* $H : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$ *be the corresponding homomorphism of C-systems. Then for any* $n \geq 0$ *and* $\Gamma \in Ob_n(CC(\mathcal{C}, p))$ *the square of sets*

[**2015.07.09.eq5**]
$$\begin{array}{ccc} Ob_n(\Gamma) & \xrightarrow{u_{n,\Gamma}} & Ob_n(int(\Gamma)) \\ {\scriptstyle H_{n,\Gamma}}\downarrow & & \downarrow{\scriptstyle H_{n,int(\Gamma)}} \\ Ob_n(H(\Gamma)) & \xrightarrow{u_{n,H(\Gamma)}} & Ob_n(\Phi(int(\Gamma))) \end{array} \qquad (51)$$

commutes

Proof: Let $CC = CC(\mathcal{C}, p)$, $CC' = CC(\mathcal{C}', p')$, $CC_{pt} = CC(pt_{\mathcal{C}}, p)$ and $CC'_{pt} = CC(pt_{\mathcal{C}'}, p')$. The proof is by induction on n.

For $n = 0$ the sets in the diagram are 1-point sets and therefore the diagram commutes.

For $n = 1$, an element in $Ob_1(\Gamma)$ is an element A in $Ob_{1+l}(CC)$ such that $ft(A) = \Gamma$. By definition of $CC(\mathcal{C}, p)$ in [5, Construction 2.5], the set $Ob_{1+l}(CC)$ is the set of pairs (Δ, F) where $\Delta = ft(\Delta, F)$ and $F : int(\Delta) \to U$ is a morphism. Therefore the set $Ob_1(\Gamma)$ equals to the set of pairs of the form (Γ, F) where $F : int(\Gamma) \to U$. The map u_1 is defined in Construction 3.1.3 by the formula $u_1(\Gamma, F) = F$. The map $H_{1,X}$ is defined in Construction 4.1.2 by the formula $H_{1,X}(F) = \Phi(F) \circ \phi$. The map $H_{1,\Gamma}$ is defined in [**?**, Construction 3.8] by the formula

Therefore the commutativity of (51) follows from the equalities

$$H_{1,int(\Gamma)}(u_1(\Gamma, F)) = H_{1,int(\Gamma)}(F) = F \circ \phi$$

and

Lemma 4.1.11 *[***2015.07.05.l6***] Let* **Φ** *be a universe category functor and let* $f : X' \to X$ *be a morphism. Then for any* $n \geq 0$ *the square of sets*

[**2015.07.09.eq3**]
$$\begin{array}{ccc} Ob_n(X) & \xrightarrow{f^*} & Ob_n(X') \\ {\scriptstyle H_{n,X}}\downarrow & & \downarrow{\scriptstyle H_{n,X'}} \\ Ob_n(\Phi(X)) & \xrightarrow{\Phi(f)^*} & Ob_n(\Phi(X')) \end{array} \qquad (52)$$

commutes and for any $A \in Ob_n(X)$*, the diagram of morphisms in* $\mathcal{C}'$*:*

[**2015.07.09.eq4**]
$$\begin{array}{ccccc} c(\Phi(f)^*(H_X(A))) & = & c(H_{X'}(f^*(A))) & \xrightarrow{\gamma_{X'}(f^*(A))} & \Phi(c(f^*(A))) \\ {\scriptstyle q(\Phi(f),H(A))}\downarrow & & & & \downarrow{\scriptstyle \Phi(Q(f,A))} \\ c(H_X(A)) & \xrightarrow{\gamma_X(A)} & \Phi(c(A)) & = & \Phi(c(A)) \end{array} \qquad (53)$$

commutes.

Proof: The proof is by induction on n.

For $n = 0$ the first square consists of one point sets and therefore commutes. The second diagram is of the form

$$\begin{array}{ccccc} \Phi(X') & = & \Phi(X') & \xrightarrow{Id_{\Phi(X')}} & \Phi(X') \\ \Phi(f)\downarrow & & & & \Phi(f)\downarrow \\ \Phi(X) & \xrightarrow{Id_{\Phi(X)}} & \Phi(X) & = & \Phi(X) \end{array}$$

which commutes.

For $n = 1$ we have $Ob_1(X) = Hom(X, U)$ and for $F \in Hom(X, U)$ we have

$$H_{X'}(f^*(F)) = H_{X'}(f \circ F) = \Phi(f \circ F) \circ \phi$$

and

$$\Phi(f)^*(H_X(F)) = \Phi(f)^*(\Phi(F) \circ \phi) = \Phi(f) \circ \Phi(F) \circ \phi$$

which shows that the first square commutes. The second digram, when we open up all definitions, takes the form

$$\begin{array}{ccc} (\Phi(X'); \Phi(f \circ F) \circ \phi) & \xrightarrow{\gamma_{X'}(f \circ F)} & \Phi(X'; f \circ F) \\ Q(\Phi(f), \Phi(F) \circ \phi)\downarrow & & \downarrow\Phi(Q(f,F)) \\ (\Phi(X); \Phi(F) \circ \phi) & \xrightarrow{\gamma_X(F)} & \Phi(X; F) \end{array}$$

Since $\Phi(X, F)$ is a fiber product with the projections $\Phi(F) \circ \widetilde{\phi}$ and $\Phi(p_{X,F})$ it is sufficient to check that the two paths become equal after composition with each of the projections. We have

$$\gamma_{X'}(f \circ F) \circ \Phi(Q(f, F)) \circ \Phi(F) \circ \widetilde{\phi} = \gamma_{X'}(f \circ F) \circ \Phi(Q(f \circ F)) \circ \widetilde{\phi} = Q(\Phi(f \circ F) \circ \phi)$$

and

$$Q(\Phi(f), \Phi(F) \circ \phi) \circ \gamma_X(F) \circ \Phi(F) \circ \widetilde{\phi} = Q(\Phi(f), \Phi(F) \circ \phi) \circ Q(\Phi(F) \circ \phi) = Q(\Phi(f \circ F) \circ \phi)$$

For the second projection we have

$$\gamma_{X'}(f \circ F) \circ \Phi(Q(f, F)) \circ \Phi(p_{X,F}) = \gamma_{X'}(f \circ F) \circ \Phi(p_{X',f \circ F}) \circ \Phi(f) = p_{\Phi(X'),\Phi(f \circ F) \circ \phi} \circ \Phi(f)$$

and

$$Q(\Phi(f), \Phi(F) \circ \phi) \circ \gamma_X(F) \circ \Phi(p_{X,F}) = Q(\Phi(f), \Phi(F) \circ \phi) \circ p_{\Phi(X),\Phi(F) \circ \phi} = p_{\Phi(X'),\Phi(f \circ F) \circ \phi} \circ \Phi(f)$$

Let us show now the inductive step from n to $n + 1$ for $n \geq 1$. Let $(A, F) \in Ob_{n+1}$. Then one has

$$H_{X'}(f^*(A, F)) = H_{X'}(f^*(A), Q(f, A) \circ F) =$$
$$(H_X(f^*(A)), \gamma_{X'}(f^*(A)) \circ \Phi(Q(f, A) \circ F) \circ \phi)$$

On the other hand

$$\Phi(f)^*(H_X(A,F)) = \Phi(f)^*(H_X(A), \gamma_X(A)\circ\Phi(F)\circ\phi) =$$

$$(\Phi(f)^*(H_X(A)), Q(\Phi(f), H_X(A))\circ\gamma_X(A)\circ\Phi(F)\circ\phi)$$

In view of the inductive assumption it remains to check that

$$\gamma_{X'}(f^*(A))\circ\Phi(Q(f,A)\circ F)\circ\phi = Q(\Phi(f), H_X(A))\circ\gamma_X(A)\circ\Phi(F)\circ\phi$$

which follows from the commutativity of the second digram for n.

Let us prove commutativity of the second diagram for $n+1$, i.e., of the diagram

$$\begin{array}{ccccc} c(\Phi(f)^*(H_X(A,F))) & = & c(H_{X'}(f^*(A,F))) & \xrightarrow{\gamma_{X'}(f^*(A,F))} & \Phi(c(f^*(A,F))) \\ \downarrow{\scriptstyle Q(\Phi(f),H_X(A,F))} & & & & \downarrow{\scriptstyle \Phi(Q(f,(A,F)))} \\ c(H_X(A,F)) & \xrightarrow{\gamma_X(A,F)} & \Phi(c(A,F)) & = & \Phi(c(A,F)) \end{array}$$

where $A\in Ob_n(X)$ and $F: c(A)\to U$. Unfolding the definitions we get the digram

$$\begin{array}{ccc} (\Phi(f)^*(H_X(A)); Q(\Phi(f), H_X(A))\circ g) & \xrightarrow{\gamma_{X'}(f^*(A), Q(f,A)\circ F)} & \Phi(c(f^*(A)); Q(f,A)\circ F) \\ \downarrow{\scriptstyle Q(Q(\Phi(f),H_X(A)),g)} & & \downarrow{\scriptstyle \Phi(Q(Q(f,A),F))} \\ (c(H_X(A)); g) & \xrightarrow{\gamma_X(A,F)} & \Phi(c(A,F)) \end{array}$$

where $g = \gamma_X(A)\circ\Phi(F)\circ\phi$. The object $\Phi(c(A,F))$ is a fiber product with projections $\Phi(Q(F))\circ\widetilde{\phi}$ and $\Phi(p_{(A,F)})$. Therefore it is sufficient to show that the compositions of the two paths in the square with each of these morphisms are equal. We have

$$\gamma_{X'}(f^*(A), Q(f,A)\circ F)\circ\Phi(Q(Q(f,A),F))\circ\Phi(Q(F))\circ\widetilde{\phi} =$$

$$\gamma_{X'}(f^*(A), Q(f,A)\circ F)\circ\Phi(Q(Q(f,A)\circ F))\circ\widetilde{\phi} = Q(\gamma_{X'}(f^*(A))\circ\Phi(Q(f,A)\circ F)\circ\phi) =$$
$$Q(\gamma_{X'}(f^*(A))\circ\Phi(Q(f,A))\circ\Phi(F)\circ\phi)$$

where the first equality is by (31) and second one by (44), and

$$Q(Q(\Phi(f), H_X(A)), g)\circ\gamma_X(A,F)\circ\Phi(Q(F))\circ\widetilde{\phi} =$$

$$Q(Q(\Phi(f), H_X(A)), \gamma_X(A)\circ\Phi(F)\circ\phi)\circ Q(\gamma_X(A)\circ\Phi(F)\circ\phi) =$$
$$Q(Q(\Phi(f), H_X(A))\circ\gamma_X(A)\circ\Phi(F)\circ\phi) = Q(\gamma_{X'}(f^*(A))\circ\Phi(Q(f,A))\circ\Phi(F)\circ\phi)$$

where the first equality is by definition of g and by (44), the second one by (31) and the third equality is by the inductive assumption.

For the composition with the second projection we have:

$$\gamma_{X'}(f^*(A,F))\circ\Phi(Q(f,(A,F))\circ\Phi(p_{(A,F)}) = \gamma_{X'}(f^*(A,F))\circ\Phi(p_{f^*(A,F)})\circ\Phi(Q(f,A)) =$$

$$p_{H_X(f^*(A,F))} \circ \gamma_{X'}(f^*(A)) \circ \Phi(Q(f,A)) = p_{H_X(f^*(A,F))} \circ Q(\Phi(f), H_X(A)) \circ \gamma_X(A)$$

where the first equality is by (32), the second one by (43) and the third one is by the inductive assumption.

On the other hand:

$$Q(\Phi(f), H_X(A,F)) \circ \gamma_X(A,F) \circ \Phi(p_{(A,F)}) = Q(\Phi(f), H_X(A,F)) \circ p_{H_X(A,F)} \circ \gamma_X(A) =$$

$$p_{\Phi(f)^*(H_X(A,F))} \circ Q(\Phi(f), H_X(A)) \circ \gamma_X(A) = p_{H_X(f^*(A,F))} \circ Q(\Phi(f), H_X(A)) \circ \gamma_X(A)$$

where the first equation is by (43), the second by the commutativity of the square (34) and the third by the commutativity of (52). This completes the proof of Lemma 4.1.11.

When $\mathcal{C}$ and $\mathcal{C}'$ are locally cartesian closed categories we defined (see [7, above Lemma 6.2]) for any $\mathbf{\Phi}$ a morphism

$$\xi_{\mathbf{\Phi}} : \Phi(I_p(U)) \to I_{p'}(U')$$

by the formula

$$\xi_{\mathbf{\Phi}} = \chi_{\mathbf{\Phi}}(U) \circ I_{p'}(\phi)$$

where

$$\chi_{\mathbf{\Phi}}(V) = \eta'(\mathbf{\Phi}^2(\eta^!(Id_{I_p(V)})))$$

by [7, Construction 5.6] and where

$$\eta^! : Hom(X, I_p(V)) \to D_p(X, V)$$

is defined in [7, Construction 3.9] and η is the inverse to $\eta^!$.

Lemma 4.1.12 *[***2015.07.03.l2***] One has*

$$H_{I_p(U)}(A_2) = \xi^*_{\mathbf{\Phi}}(A'_2)$$

Proof: We have

$$H_{I_p(U)}(A_2) = \mathbf{\Phi}^2(A_2) \circ \phi$$

by Lemma 4.1.9. On the other hand

$$\xi^*_{\mathbf{\Phi}}(A'_2) = \eta^!(\xi_{\mathbf{\Phi}}) = \eta^!(\eta'(\mathbf{\Phi}^2(\eta^!(Id_{I_p(U)}))) \circ I_{p'}(\phi)) =$$

$$\eta^!(\eta'(\mathbf{\Phi}^2(A_2)) \circ I_{p'}(\phi)) = \eta^!(\eta'(\mathbf{\Phi}(A_2) \circ \phi)) = \mathbf{\Phi}(A_2) \circ \phi$$

where the first and the third equalities holds by Lemma 3.1.20 and the fourth by [7, Problem 3.8(1)]. This completes the proof of Lemma 4.1.12.

4.2 Homomorphisms of C-systems compatible with Σ-structures.

Let CC_1, CC_2 be two C-systems.

Definition 4.2.1 [**2015.07.05.def4**] *Let S_1, S_2 be 2-to-1 structures on CC_1 and CC_2 respectively. A homomorphism $H : CC_1 \to CC_2$ is said to be compatible with S_1, S_2 if for all $\Gamma \in CC_1$, $B \in Ob_2(\Gamma)$ one has*

$$H_{1,\Gamma}(S_{1,\Gamma}(B)) = S_{2,H(\Gamma)}(H_{2,\Gamma}(B))$$

where $H_{n,\Gamma} : Ob_n(\Gamma) \to Ob_n(H(\Gamma))$ is defined by H.

Let further (Σ_1, r'_1, d'_1) and (Σ_2, r'_2, d'_2) be weak Σ-structures on CC_1 and CC_2 respectively.

Definition 4.2.2 [**2015.07.05.def5**] *A homomorphism of C-systems $H : CC_1 \to CC_2$ is said to be compatible with (Σ_1, r'_1, d'_1) and (Σ_2, r'_2, d'_2) if it is compatible with Σ_1 and Σ_2 and for any $\Gamma \in CC_1$ and $B \in Ob_2(\Gamma)$ one has*

$$H(r'_{1,B}) = r'_{2,H_\Gamma(B)}$$

$$H(d'_{1,B}) = d'_{2,H_\Gamma(B)}$$

4.3 Universe category functors compatible with Σ-structures.

In what follows $\mathbf{\Phi} = (\Phi, \phi, \widetilde{\phi})$ is a universe category functor from a universe category $(\mathcal{C}, p)$ to a universe category $(\mathcal{C}', p')$ and both $\mathcal{C}$ and $\mathcal{C}'$ are equipped with locally cartesian closed structures.

Definition 4.3.1 [**2015.07.05.def2**] *Let $\Sigma p \in Ob_1(I_p(U))$ be a 2-to-1 structure on p and $\Sigma p' \in Ob_1(I_{p'}(U'))$ a 2-to-1 structure on p'. The functor Φ is called compatible with Σp and $\Sigma p'$ if*

$$[\mathbf{2015.07.05.eq1}] H_{I_p(U)}(\Sigma p) = \xi^*_{\mathbf{\Phi}}(\Sigma p') \quad (54)$$

Note that both sides of (54) are elements of $CC(\Phi(I_p(U)), p')$.

Lemma 4.3.2 [**2015.07.05.l2**] *Assume that $\mathbf{\Phi}$ is compatible with $\Sigma p, \Sigma p'$ then the homomorphism of C-systems $H = H_{\mathbf{\Phi}}$ is compatible with Σ, Σ' where Σ and Σ' are obtained from Σp and $\Sigma p'$ respectively by formula (40).*

Proof: We need to check the condition of Definition 4.2.1 i.e. that for all $\Gamma \in CC(\mathcal{C}, p)$, $B \in Ob_2(\Gamma)$ one has

$$H_{1,\Gamma}(\Sigma_\Gamma(B)) = \Sigma'_{H(\Gamma)}(H_{2,\Gamma}(B))$$

where

$$\Sigma_\Gamma(B) = u^{-1}_{1,\Gamma}(\mu_\Gamma(B)^*(\Sigma p))$$

$$\Sigma'_{\Gamma'}(B') = u^{-1}_{1,\Gamma'}(\mu_{\Gamma'}(B')^*(\Sigma p'))$$

We have

$$H_{1,\Gamma}(\Sigma_\Gamma(B)) = H_{1,\Gamma}(u^{-1}_{1,\Gamma}(\mu_\Gamma(B)^*(\Sigma p))) =$$

Definition 4.3.3 [**2015.07.05.def3**] *Let $(\Sigma p, rp, dp)$ be a weak Σ-structure on p and $(\Sigma p', rp', dp')$ a weak Σ-structure on p'. We say that $\mathbf{\Phi}$ is compatible with $(\Sigma p, rp, dp)$ and $(\Sigma p', rp', dp')$ respectively if it is compatible with Σp and $\Sigma p'$ and one has*

$$H_{I_p(U)}(rp) = \xi^*_{\mathbf{\Phi}}(rp')$$

$$H_{I_p(U)}(dp) = \xi^*_{\mathbf{\Phi}}(dp')$$

$\mathbf{\Phi}$ is said to be compatible with strong Σ-structures if it is compatible with the corresponding weak Σ-structures.

Note that Definition 4.3.3 makes sense because

$$H_{I_p(U)}(rp) : H_{I_p(U)}(A_2) \to H_{I_p(U)}(\Sigma p)$$

and

$$\xi^*_{\mathbf{\Phi}}(rp') : \xi^*_{\mathbf{\Phi}}(A'_2) \to \xi^*_{\mathbf{\Phi}}(\Sigma p)$$

and we have

$$H_{I_p(U)}(\Sigma p) = \xi^*_{\mathbf{\Phi}}(\Sigma p)$$

from the compatibility with $(\Sigma p, \Sigma p')$ condition and

$$H_{I_p(U)}(A_2) = \xi^*_{\mathbf{\Phi}}(A'_2)$$

by Lemma 4.1.12. A similar argument applies to the second equality.

Theorem 4.3.4 [**2015.06.27.th1**] *In the notations and assumptions of Definition 4.3.3, the homomorphism of C-systems*

$$H : CC(\mathcal{C}, p) \to CC(\mathcal{C}', p')$$

is compatible with the weak Σ-structures on these C-systems defined by the weak Σ-structures $(\Sigma p, rp, dp)$ and $(\Sigma p', rp', dp')$ on p and p' by Construction 3.2.4.

Proof:

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